

Computational Methods for the H^∞ Control of Distributed Systems

Allen R. Tannenbaum
Department of Electrical Engineering
University of Minnesota
Minneapolis, Minnesota 55455

1 Introduction

In this note, we will discuss some explicit design examples using a frequency domain (*skew Toeplitz*) approach in the H^∞ optimization of distributed systems. We will concentrate on the computational aspects of this methodology which allows one to reduce infinite dimensional design problems to finite dimensional matrix and polynomial operations.

First, we will give a very general outline of what is involved in skew Toeplitz theory. This is based on the papers [1], [4], [5], [11], [12], [14] to which we refer the interested reader for all the details. Our main point will be to show how the the solution of the H^∞ optimization problem for distributed plants may derived from a finite system of linear equations called the *singular system*.

We will then apply this theory to a weighted two-block design for unstable plant models with delay. (One such model comes from the control of the X-29; see [3].) This is based on the joint work [3], [12].

Finally, we will discuss a mixed sensitivity design for a flexible beam modelled by the Euler-Bernoulli equation with Kelvin-Voigt damping. We also include a delay in our model. Both the collocated and non-collocated cases will be considered. This discussion is based on the paper [9].

This work was supported in part by grants from the National Science Foundation DMS-8811084, by the Air Force Office of Scientific Research AFOSR-90-0024, and by the Army Research Office DAAL03-91-G-0019.

2 Brief Review of Skew Toeplitz Theory

In this section, we will give the relevant aspects of the skew Toeplitz computational method as applied to the two block problem for unstable distributed plants. In fact, we will show that several two block H^∞ -minimization problems reduce to the computation of the norm of a certain skew Toeplitz operator. For applications of skew Toeplitz theory to other types of H^∞ problems (one block and four block); see [1], [4] [5], and [11]. For a general discussion on the two block problem see [2]. The discussion given here is a summary of that given in Ozbay-Smith-Tannenbaum [12].

We begin with some notation. The Hardy spaces H^2 and H^∞ are defined on the unit disc in the standard way. We denote

$$\begin{aligned} \tilde{H}^\infty &:= \{f \in H^\infty : \overline{f(\bar{z})} = f(z)\} \\ R\tilde{H}^\infty &:= \{\text{rational functions in } \tilde{H}^\infty\} \end{aligned}$$

We consider the standard feedback configuration with plant $P = \frac{G_n}{G_d}$, with $G_n \in \tilde{H}^\infty$, $G_d \in R\tilde{H}^\infty$. We assume that (i) $G_n = m_n G_{n0}$, where $m_n \in \tilde{H}^\infty$ is inner (arbitrary) and $G_{n0} \in \tilde{H}^\infty$ is outer, and (ii) G_n , G_d have no common zeros in the closed unit disc. We also write $G_d = m_d G_{d0}$ where $m_d \in R\tilde{H}^\infty$ is inner and $G_{d0} \in R\tilde{H}^\infty$ is outer. Under these assumptions there exist $X \in R\tilde{H}^\infty$ and $Y \in \tilde{H}^\infty$ such that

$$XG_n + YG_d = 1. \quad (1)$$

The set of all controllers which stabilize the plant can now be written in the form

$$C = \frac{X + QG_d}{Y - QG_n}$$

for some $Q \in \tilde{H}^\infty$. Now let $S := (1 + PC)^{-1}$ and note that

$$S = 1 - XG_n - QG_n G_d. \quad (2)$$

In [12], we show that the computation of

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2 (S - 1) \end{bmatrix} \right\|$$

where $W_1, W_2 \in R\tilde{H}^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in R\tilde{H}^\infty$ may be reduced to computing the norm of the "skew Toeplitz" operator

$$A := \begin{bmatrix} P_{H(m_v)} (W_0(S) - \tilde{W}_0(S)m(S)) \\ G_0(S) \end{bmatrix}, \quad (3)$$

where $S : H^2 \rightarrow H^2$ denotes the unilateral shift, $H(m_v) := H^2 \ominus m_v H^2$ and $P_{H(m_v)}$ the orthogonal projection onto $H(m_v)$, for m, m_v inner functions associated to the plant and weighting filters, and where W_0, \tilde{W}_0, G_0 are rational H^∞ functions computed from the plant and weighting filters. This reduction is true for plants with *arbitrary* outer parts. We can do a similar type of reduction for the following 2-block minimization problem in case the outer part of the numerator of the plant is rational. (See [12] for details.) Find

$$\mu = \inf_{\text{stabilizing } C} \left\| \begin{bmatrix} W_1 S \\ W_2 C S \end{bmatrix} \right\| \quad (4)$$

where $W_1, W_2 \in R\tilde{H}^\infty$ are given weighting functions with $W_1^{-1}, W_2^{-1} \in R\tilde{H}^\infty$.

In [12], the following result is proven:

Theorem 1 Let n denote the maximum of the MacMillan degrees of the weighting filters W_1 and W_2 , and let ℓ denote the number of unstable poles of the plant P . Then the singular values of A may be derived from an explicitly computable system of $3n + 2\ell$ linear equations (the "singular system.") Moreover, from this system the corresponding singular vectors may also be found.

The singular system of equations is written down in [12]. It is based on the previous singular system derived in [4]. The computation of the *maximal* singular value and the associated singular vectors of A , then allows us to find the optimal performance μ of our original control problem and the corresponding optimal compensator.

3 Delay Systems

In the skew Toeplitz methodology, we prefer to work in the unit disc and then transform the solution to the right half plane via a linear fractional transformation. Hence, we will begin with a more general design in the unit disc D , and then at the end specialize to the delay case. Again, details of the computations which follow below can be found in [12].

Consider a plant $P(z) = m(z)/m_d(z)$, where m is arbitrary inner (possibly infinite dimensional) and m_d is a first order Blaschke function:

$$m_d(z) = \frac{z - a}{1 - az}$$

with $a \in D$ real and $m(a)$ real. The Bezout identity for this system is

$$Xm + Ym_d = 1,$$

so we can choose $X(z) = 1/m(a)$, constant. In this case the sensitivity and the complementary sensitivity are

$$S(z) = 1 - m(z)/m(a) - m(z)m_d(z)Q(z),$$

$$T = 1 - S(z) = m(z)/m(a) + m(z)m_d(z)Q(z),$$

where Q is the free parameter coming from the Youla parametrization of all stabilizing controllers. In the unweighted mixed sensitivity, S and T , minimization problem we want to find

$$\mu = \inf_{Q \in \tilde{H}^\infty} \left\| \begin{bmatrix} 1 - m(z)/m(a) \\ m(z)/m(a) \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} m_v(z)Q \right\|_\infty$$

where $m_a(z) = m(z)m_d(z)$. By employing inner/outer factorizations for the constant matrix $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$, the above can be reduced to

$$\mu = \inf_{Q \in H^\infty} \left\| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 - 2m(z)/m(a) - m_a(z)Q \\ 1 \end{bmatrix} \right\|_\infty.$$

Therefore

$$\mu = \sqrt{\frac{1 + \mu_1^2}{2}},$$

where

$$\mu_1 = \inf_{Q \in H^\infty} \|1 - 2m(z)/m(a) - m_a(z)Q\|_\infty.$$

So the problem is reduced to computing μ_1 , and the corresponding optimal interpolant.

By the Commutant Lifting Theorem we have that $\mu_1 = \|1 - 2m(\mathbf{T})/m(a)\|$, with $\mathbf{T} = \mathbf{P}_{H(m_a)} \mathbf{S}|_{H(m_a)}$. To compute the norm we form the singular-value/singular-vector equation

$$(\rho^2 \mathbf{I} - (\mathbf{I} - 2m(\mathbf{T})^*/m(a))(\mathbf{I} - 2m(\mathbf{T})/m(a))) u = 0 \quad (5)$$

where ρ^2 is a singular value with corresponding singular vector $u \in H(m_a)$. From this, we can find that (see [12] for the details)

$$\mu_1^2 = \left(\frac{2}{m(a)^2} - 1\right) + \frac{2}{m(a)^2} \sqrt{1 - m(a)^2}.$$

Consequently, for this example the optimal mixed sensitivity performance level $\mu = \sqrt{(1 + \mu_1^2)/2}$ can be computed as

$$\mu = \frac{1}{|m(a)|} \sqrt{1 + \sqrt{1 - m(a)^2}}.$$

The optimal sensitivity and controller can be computed to be

$$S_{opt}(z) = \frac{1 - m(z)m(a)(\mu^4/\mu_1^4)}{1 - m(z)m(a)(\mu^2/\mu_1^2)}$$

and

$$\begin{aligned} C_{opt}(z) &= \left(\frac{1}{S_{opt}} - 1\right) \frac{m_d(z)}{m(z)} \\ &= m_d(z) \frac{m(a)(\mu^2 - 1)\mu^2/\mu_1^2}{1 - m(z)m(a)\mu^4/\mu_1^4}. \end{aligned}$$

An important particular case of the above example is a plant (in continuous time) with a delay and one unstable pole:

$$P(s) = e^{-hs} \frac{\sigma s + 1}{\sigma s - 1}, \quad h, \sigma > 0.$$

After transforming the data to the unit disc with the conformal map $z = \frac{s-1}{s+1}$, we find that

$$m(z) = e^{h \frac{z+1}{z-1}}, \quad m_d(z) = \frac{z-a}{1-az},$$

with $a = (1 - \sigma)/(1 + \sigma)$. Then $m(a) = e^{-h/\sigma}$ and hence

$$\mu = e^{h/\sigma} \sqrt{1 + \sqrt{1 - e^{-2h/\sigma}}}.$$

It is interesting to note that as $h \rightarrow \infty$, and/or $\sigma \rightarrow 0$, the best achievable performance increases exponentially, as expected.

In connection with the control of the X-29, in [3] we study the unstable delay system

$$P(s) = \frac{e^{-hs}}{\sigma s - 1}, \quad h, \sigma > 0. \quad (6)$$

In the computation just performed above, we considered an unweighted mixed sensitivity design. In [3] for the plant (6), we work out a controller weighting two block design which is much more realistic. The choice of weights in this problem is very important, for example, a particular choice leads us to the robustness optimization in the gap metric, [7], [10], which is essentially a 1-block problem. The issues are described in [3] where all the details may be found.

4 Flexible Beams

In this section, we would like to briefly sketch the application of the skew Toeplitz computational technique to flexible beam design. This is based on Lenz *et al* [9] which contains all the details.

We first consider an Euler-Bernoulli beam with free ends and Kelvin-Voigt damping with damping coefficient $\epsilon > 0$, uniform mass density ρ of 1, length 1, and stiffness parameter $EI = 1$. We denote the deflection of the beam at time t and location x along the length of the beam by $w(x, t)$. We suppose that a transverse force $-u(t)$ is applied at the end of the beam, $x = 1$. Then the beam dynamics are described by the linear PDE

$$\frac{\partial^4 w}{\partial x^4} + \epsilon \frac{\partial^5 w}{\partial x^4 \partial t} + \frac{\partial^2 w}{\partial t^2} = 0,$$

with boundary conditions

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2}(0, t) + \epsilon \frac{\partial^3 w}{\partial x^2 \partial t}(0, t) &= 0, & \frac{\partial^2 w}{\partial x^2}(1, t) + \epsilon \frac{\partial^3 w}{\partial x^2 \partial t}(1, t) &= 0 \\ \frac{\partial^3 w}{\partial x^3}(0, t) + \epsilon \frac{\partial^4 w}{\partial x^3 \partial t}(0, t) &= 0, & \frac{\partial^3 w}{\partial x^3}(1, t) + \epsilon \frac{\partial^4 w}{\partial x^3 \partial t}(1, t) &= u(t). \end{aligned}$$

In [9], we consider two cases for the measurement of the beam. In the first case, we take the measurement to be $y(t) = w(1, t)$, where $w(1, t)$ is the deflection at the same end of the beam as the applied force, $-u(t)$, and in the second case we take $y(t) = w(0, t)$ the deflection at the opposite end of the beam.

One can work out explicit transfer function models in both cases, and then use skew Toeplitz theory to do optimal two block designs. One can even add a pure time delay to the system. We have also considered finite dimensional approximations of the resulting infinite dimensional optimal controllers. Again space limitations prevent us from writing down all the formulas here.

References

- [1] Bercovici, H., C. Foias, and A. Tannenbaum, "On skew Toeplitz operators," *Operator Theory: Advances and Applications* 32 (1988), pp. 21-43.
- [2] Doyle, J. C., B. Francis, and A. Tannenbaum, *Feedback Control Theory*, to be published by MacMillan, New York, 1991.
- [3] Enns, D., H. Özbay and A. Tannenbaum, "Abstract model and controller design for an unstable aircraft," to appear in *AIAA Journal of Guidance, Control and Dynamics*.
- [4] Foias, C. and A. Tannenbaum, "On the four block problem, II: the singular system" *Operator Theory and Integral Equations* 11 (1988), pp. 726-767.
- [5] Foias, C., A. Tannenbaum, and G. Zames, "Some explicit formulae for the singular values of a certain Hankel operators with factorizable symbol," *SIAM J. Math. Analysis* 19 (1988), pp. 1081-1091.
- [6] Francis, B. A., *A Course in H^∞ Control Theory*, Lecture Notes in Control and Information Sciences, vol. 88, Springer Verlag, 1987.
- [7] Georgiou, T. T. and M. C. Smith, "Optimal robustness in the gap metric," *IEEE Trans. Automatic Control* 35 (1990), pp. 673-686.
- [8] Lenz, K., H. Özbay, A. Tannenbaum, J. Turi, and B. Morton, "Frequency domain analysis and robust control design for an ideal flexible beam," to appear in *Automatica*.
- [9] McFarlane, D. and K. Glover, *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*, LNCIS vol. 138, Springer Verlag, New York, 1990.
- [10] Özbay, H. and A. Tannenbaum, "A skew Toeplitz approach to the H^∞ optimal control of multivariable distributed systems," *SIAM J. Control and Optimization*, 28 (1990) pp. 653-670.
- [11] Özbay, H., M. C. Smith and A. Tannenbaum, "Mixed sensitivity optimization for unstable infinite dimensional systems," submitted for publication in *SIAM J. Control and Optimization*. A short version of the paper, under the title "Controller design for unstable distributed plants," appears in the proceedings of the American Control Conference, May 1990, San Diego, CA, pp. 1583-1588.
- [12] Zames, G. and S. K. Mitter, "A note on essential spectrum and norms of mixed Hankel - Toeplitz operators," *Systems and Control Letters* 10 (1988), pp. 159-165.