



# Spectras of interacting particles on quantum graphs

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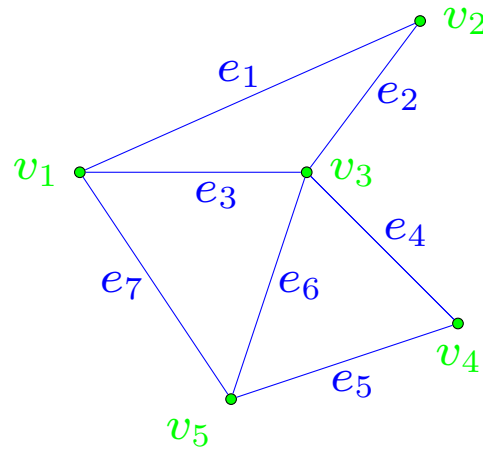
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Based on joint work with J. Kerner, G. Garforth

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## Quantum graphs

Quantum graph: *one* particle moving along edges of a finite, metric graph.



A graph with  $V = 5$  vertices and  $E = 7$  edges

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The (simplest) one-particle Hamiltonian is a Laplacian, describing a free particle on the graph. It acts on the edge- $e$  component as

$$(-\Delta_1 \psi)_e(x_e) = -\frac{\partial^2 \psi_e}{\partial x_e^2}(x_e) ,$$

plus (self-adjoint) boundary conditions in the vertices.

There is a close analogy to Laplacians on manifolds. Many *spectral properties* are known, including

- Eigenvalue count follows a Weyl law
- Eigenvalue correlations (empirically) follow RMT predictions
- Trace formulae
- Periodic orbit correlations (heuristically) leading to eigenvalues correlations
- Inverse problems, isospectrality
- Nodal domains of eigenfunctions

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Many-particle quantum systems on graphs are, in comparison, still less studied.

Our goals:

- Construct models with two-particle interactions
- Study basic spectral properties: discreteness of spectra, Weyl law etc.
- Identical particles: bosons, fermions
- (Bose-Einstein condensation)
- Secular equation, eigenvalues, spectral statistics

Other topics include:

- Many-particle statistics (Harrison, Keating, Robbins, Sawicki 2010-13)
- Non-linear (Schrödinger or Gross-Pitaevskii) equations (Adami et al 2010-13)
- Anderson localisation (Sabri 2012-13)
- Trace formulae
- Other types of operators: Schrödinger, Pauli, Dirac

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## Basic constructions

One-particle Hilbert space

$$\mathcal{H}_1 = L^2(\Gamma) = \bigoplus_{e=1}^E L^2(0, l_e) = \{ \psi = (\psi_1, \dots, \psi_E); \psi_e \in L^2(0, l_e) \} ,$$

and similarly Sobolev spaces  $H^m(\Gamma)$ . Boundary values (in edge ends),

$$\begin{aligned} \psi_{bv} &= (\psi_1(0), \dots, \psi_E(0), \psi_1(l_1), \dots, \psi_E(l_E))^T \\ \psi'_{bv} &= (\psi'_1(0), \dots, \psi'_E(0), -\psi'_1(l_1), \dots, -\psi'_E(l_E))^T . \end{aligned}$$

Linear maps on the space  $\mathbb{C}^{2E}$  of boundary values:

- Projector  $P_1$ ,
- $L_1$  self-adjoint on  $\text{ran } P_1^\perp$ .

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Self-adjoint realisations of the Laplacian are in one-to-one correspondence to closed, semi-bounded quadratic forms.

**Theorem** [Kuchment 04]

*The quadratic form*

$$Q_{P_1, L_1}^{(1)}[\psi] = \sum_{e=1}^E \int_0^{l_e} |\psi'_e(x)|^2 dx - (\psi_{bv}, L_1 \psi_{bv})_{\mathbb{C}^{2E}} ,$$

*with domain*

$$\mathcal{D}_{Q^{(1)}} = \{ \psi \in H^1(\Gamma); P_1 \psi_{bv} = 0 \} ,$$

*is associated with the one-particle Laplacian on the domain*

$$\mathcal{D}_1(P_1, L_1) = \{ \psi \in H^2(\Gamma); P_1 \psi_{bv} = 0 \text{ and } P_1^\perp \psi'_{bv} + L_1 P_1^\perp \psi_{bv} = 0 \} .$$

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## Scattering approach

Eigenvalues  $k^2$  of the Laplacian as zeros of a secular determinant:

$$\det(\mathbb{1} - U(k)) = 0,$$

where  $U(k) = S(k)T(k)$  is a unitary  $2E \times 2E$  matrix with

$$S(k) = -(P + L + ikP^\perp)^{-1}(P + L + ikP^\perp) \quad \text{and} \quad T(k) = \begin{pmatrix} 0 & e^{ikl} \\ e^{ikl} & 0 \end{pmatrix}$$

- Laplacian with Neumann b.c.: Kottos, Smilansky (1997)
- Laplacian with general self adjoint b.c.: Kostrykin, Schrader (2006)
- Schrödinger operators  $-\Delta + V$ : Rueckriemen, Smilansky (2012), JB, Egger, Rueckriemen (2015)

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## Many-particle systems

From  $\mathcal{H}_1$  one constructs the  $N$ -particle Hilbert space

$$\mathcal{H}_N = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_1 ,$$

so that  $N$ -particle states are functions  $\Psi = (\psi_{e_1 \dots e_N})$  with

$$\psi_{e_1 \dots e_N} \in L^2(D_{e_1 \dots e_N}) , \quad \text{where } D_{e_1 \dots e_N} = [0, l_{e_1}] \times \cdots \times [0, l_{e_N}] .$$

In the following we shall restrict ourselves to  $N = 2$  and  $E = 1$ , i.e., two particles on an interval. The configuration space then is  $D = (0, l) \times (0, l)$ .

We construct rigorous versions of  $\delta$ -type contact interactions,

$$H = -\Delta_2 + \alpha(x) \delta(x - y) .$$



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The contact interactions require jump conditions on the normal derivative across the diagonal  $x = y$ . In more detail:

Dissect the square  $D$  as  $D^* = D_- \cup D_+$  along  $x = y$  and introduce functions

$$\psi^\pm : D_\pm \rightarrow \mathbb{C}, \quad \text{such that} \quad \psi(x, y) = \begin{cases} \psi^+(x, y), & x > y \\ \psi^-(x, y), & x < y \end{cases}.$$

Use boundary values

$$\psi_{bv}(\mathbf{y}) = \begin{pmatrix} \psi^-(0, y) \\ \psi^+(l, y) \\ \psi^+(y, 0) \\ \psi^-(y, l) \\ \psi^+(y, y) \\ \psi^-(y, y) \end{pmatrix} \quad \text{and} \quad \psi'_{bv}(\mathbf{y}) = \begin{pmatrix} \psi_x^-(0, y) \\ -\psi_x^+(l, y) \\ \psi_y^+(y, 0) \\ -\psi_y^-(y, l) \\ (\psi_x^+ - \psi_y^+)(y, y) \\ (\psi_x^- - \psi_y^-)(y, y) \end{pmatrix}$$

along the six sides of the triangles  $D_-$  and  $D_+$ .

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Split the space of boundary values as

$$\mathbb{C}^6 = V_{vertex} \oplus V_{contact} .$$

Choose 'non-interacting'  $P_{vertex}$  and  $L_{vertex}$ , as well as

$$P_{contact}(y) = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} , \quad L_{contact}(y) = -\frac{1}{2} \alpha(y) \mathbb{1}_2 ,$$

and set up a quadratic form,

$$Q_{P,L}^{(2)}[\psi] = \langle \nabla \psi, \nabla \psi \rangle_{L^2(D^*)} - \int_0^l \langle \psi_{bv}(y), L(y) \psi_{bv}(y) \rangle_{\mathbb{C}^6} dy ,$$

with domain

$$\mathcal{D}_{Q^{(2)}} = \{ \psi \in H^1(D^*); P(y) \psi_{bv}(y) = 0 \forall y \in [0, l] \} .$$

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This means that  $\psi(x, y)$  is continuous across  $x = y$ , but the normal derivative jumps by  $\alpha$ ,

$$\psi_x^+ - \psi_y^+ - \alpha\psi^+ = \psi_x^- - \psi_y^-.$$

**Theorem** [JB, Kerner 2013]

*The quadratic form  $Q_{P,L}^{(2)}$  on the domain  $\mathcal{D}_{Q^{(2)}}$  is closed and semi-bounded. The associated self-adjoint operator is the Laplacian  $-\Delta_2$  on the domain*

$$\mathcal{D}_2(P, L) := \{\psi \in H^2(D); P(y)\psi_{bv}(y) = 0 \text{ and } P^\perp(y)\psi'_{bv}(y) + L(y)P^\perp(y)\psi_{bv}(y) = 0\}.$$

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A basic spectral property of the operators is the following.

**Theorem** [JB, Kerner 2013]

*The operator  $(-\Delta_2, \mathcal{D}_2(P, L))$  has a compact resolvent. In particular, it possesses a discrete spectrum and the eigenvalue count follows a Weyl law,*

$$\#\{n \in \mathbb{N}; \lambda_n \leq \lambda\} \sim \frac{\mathcal{L}^2}{4\pi} \lambda, \quad \lambda \rightarrow \infty.$$

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## Identical particles

For two particles the bosonic/fermionic projector is

$$(\Pi_{B/F}\psi)(x, y) = \frac{1}{2}(\psi(x, y) \pm \psi(y, x)) .$$

- It can be arranged that  $[H_2, \Pi_B] = 0$ , hence the operator has a bosonic version  $H_{2,B}$ .
- There is an immediate generalisation to arbitrary compact graphs.
- The operator can be promoted to an operator for  $N$  bosons,

$$H_{N,B} = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{i < j} \alpha(x_i) \delta(x_i - x_j) .$$

This yields an extension of the **Lieb-Liniger model** from a circle to a metric graph.

*Hardcore*-limit  $\alpha \rightarrow \infty$ : Dirichlet conditions at  $x_i = x_j$  (**Tonks-Girardeau gas**).

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## Solvable models

(This is recent work with George Garforth: arXiv:1609.00828v1)

For one particle, eigenfunctions are of the form

$$a e^{ikx} + b e^{-ikx}$$

on each edge. This is used to produce the finite-dimensional secular determinant. For two-particles, the general form is

$$\int_{\mathbb{R}^2} a(k_1, k_2) e^{i(k_1 x_1 + k_2 x_2)} dk_1 dk_2 .$$

However, in some cases a **Bethe ansatz** for the eigenfunctions,

$$\sum_{k_1, k_2} A(k_1, k_2) e^{i(k_1 x_1 + k_2 x_2)}$$

(finite sum) will be possible. Such cases are called solvable.

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Early examples are:

- $\delta$ -interacting particles on a circle: Lieb, Liniger (1963)
- $\delta$ -interacting particles on an interval: Yang (1967), Gaudin (1971)
- $\tilde{\delta}$ -interacting particles on a star graph with two infinite edges Caudrelier, Crampe (2007)

The  $\tilde{\delta}$ -interaction on the star graph with two infinite edges is formally given as

$$\alpha(\delta(x_1 - x_2) + \delta(x_1 + x_2)).$$

It acts whenever  $x_1 = \pm x_2$ , i.e., when both particles are located the same distance away from the vertex, either on the same or on different edges.

A similar interaction can be defined on any graph and can be made rigorous in close analogy to the  $\delta$ -interactions. When  $e, e'$  are two edges emanating from the same vertex, then

$$\psi_{ee'}^+ = \psi_{e'e}^-$$

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across  $x = y$ , and

$$\partial_x \psi_{ee'}^+ - \partial_y \psi_{ee'}^+ - \alpha \psi_{ee'}^+ = \partial_x \psi_{e'e}^- - \partial_y \psi_{e'e}^-.$$

The Bethe ansatz then is of the form

$$\sum_{\sigma \in \mathcal{W}_2} A_\sigma e^{i(k_{\sigma(1)}x_1 + k_{\sigma(2)}x_2)},$$

where  $\mathcal{W}_2$  is a finite group (of order 8) generated by  $I, T, R$  with relations

1.  $TT = I$ ,
2.  $RR = I$ ,
3.  $TRTR = RTRT$ .



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We applied this in two examples:

- Equilateral star graph with DFT central scattering matrix
- Tetrahedron with rationally independent edge lengths

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In both types of examples we found a secular equation,

$$\det (1 - U(k_1, k_2)) = 0,$$

for the eigenvalues  $k_1^2 + k^2$ , where

$$U(k_1, k_2) = E(k_2)Y(k_2 - k_1)(\mathbb{1}_2 \otimes S(k_2) \otimes \mathbb{1}_{2E})Y(k_1 + k_2),$$

and

$$Y(k) = \frac{1}{k + i\alpha} \begin{pmatrix} -i\alpha & k \\ k & -i\alpha \end{pmatrix} \otimes \boldsymbol{\alpha} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (\mathbb{1}_{E^2} - \boldsymbol{\alpha})\mathbb{T}_{E^2}$$

$$E(k) = \mathbb{1}_{4E} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{e}^{ikl};$$

$\mathbb{T}_{E^2}$  is a permutation matrix.

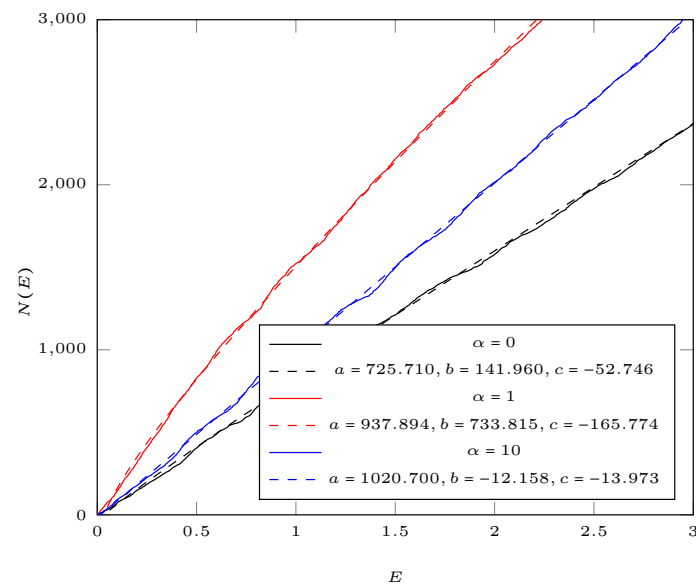
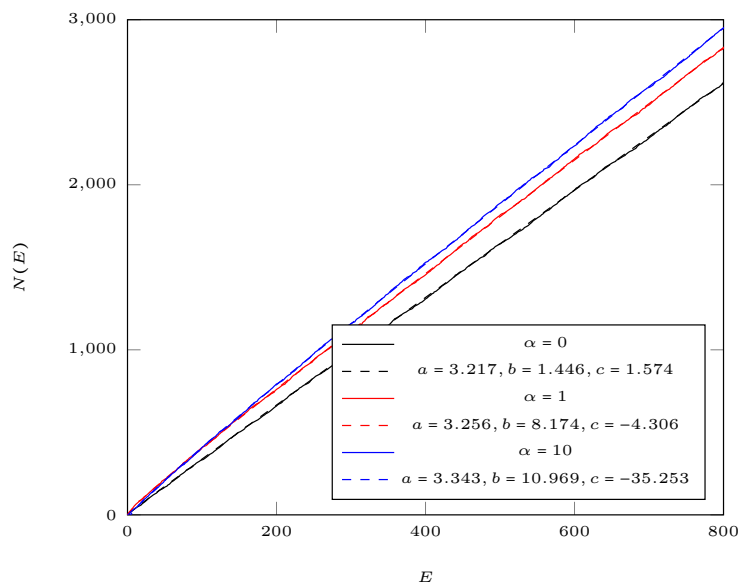


Abbildung 1: Eigenvalue counting function  $N(E)$  for two bosons on a 9-edge equilateral star

Abbildung 2: Eigenvalue counting function  $N(E)$  for two bosons on a tetrahedron

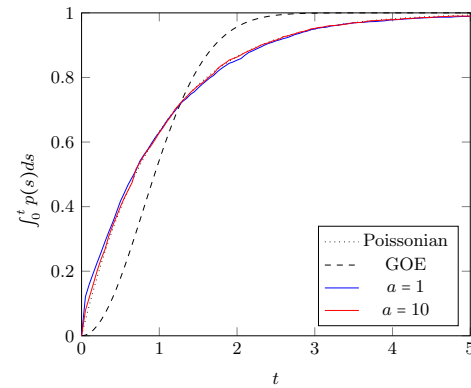


Abbildung 3: Integrated level spacings distributions for systems of two bosons on a tetrahedron (first 3000 eigenvalues).