

5-List-Coloring Graphs on Surfaces

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Joint work with Robin Thomas.

Definition

We say a graph G is k -list-colorable if for every list assignment L , where $|L(v)| \geq k$, there exists an L -coloring of G , that is a proper coloring ϕ of G where $\phi(v) \in L(v)$.

Theorem (Thomassen)

Every planar graph is 5-list-colorable.

Voigt: Best possible.

Definition

Let G be graph embedded in a surface, the *edge-width* of G , denoted by $ew(G)$ is the length of a shortest non-contractible cycle.

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Theorem (Devos, Kawarabayashi, Mohar)

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Definition

We say a graph G is *k-critical* if G is not $(k - 1)$ -colorable but every proper subgraph of G is.

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For every surface S , there exists a linear-time algorithm to decide if a graph embedded in S is 5-colorable.

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Theorem (P., Thomas)

If G is a 6-critical graph embedded in a surface of genus g , then $|V(G)| = O(g)$.

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For every surface S , there exists a linear-time algorithm to decide if a graph embedded in S is 5-list-colorable.

Theorem (Albertson)

If G is a planar graph and $X \subset V(G)$, $d(u, v) \geq 4$ for all $u \neq v \in X$, then every 5-coloring of X extends to a 5-coloring of G .

Albertson and Hutchinson extended this result to graphs embedded on a surface of genus g with $ew(G) \geq 2^{\Omega(g)}$.

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Albertson conjectured that this generalizes to list-coloring:

Extending Pre-colored Subgraphs

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Theorem (Dvorak, Lidicky, Mohar, P.)

$\exists D$ such that the following holds: Let G be a planar graph and L a 5-list-assignment for G . If $X \subset V(G)$, $d(u, v) \geq D$ for all $u \neq v \in X$, then every L -coloring of X extends to an L -coloring of G .

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Theorem (Dvorak, Lidicky, Mohar)

$\exists D$ such that: If G can be drawn in the plane such that every pair of crossings is distance D apart, then G is 5-list-colorable.

Theorem (P., Thomas)

\forall surface S and $k \geq 0$, $\exists D = O(\log k + \log g)$ such that: $\forall G$ embedded in S , $ew(G) \geq D$ and L a 5-list-assignment, if $\mathcal{F} = \{C_1, C_2, \dots\}$ is a collection of facial cycles such that

- $\forall i, |C_i| \leq k$,
- $\forall i \neq j, d(C_i, C_j) \geq D$,
- $\forall i, N_D(C_i)$ is planar,

and ϕ is an L -coloring of \mathcal{F} such that $\forall i, \phi(C_i)$ extends to $N_D(C_i)$, then ϕ extends to an L -coloring of G .

Corollary

If G is embedded in a surface of genus g with $ew(G) \geq \Omega(\log g)$, then G is 5-list-colorable.

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Proof

Let $k = 0$ and $\mathcal{F} = \emptyset$.

Moreover, there exist graphs with $\Omega(\log g)$ edge-width that embed in a surface of genus g ; namely, Ramanujan graphs.

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Corollary

Let S be a surface of genus g . If G is a 6-list-critical graph embedded in S , then $|V(G)| = O(g \log g)$.

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Proof

Suppose we have a collection \mathcal{F} of cycles with $\sum_{C \in \mathcal{F}} |C| \leq g \log g$. Apply Theorem with $k = g \log g$. Either

- two cycles in \mathcal{F} have distance less than $D = O(\log k + \log g) = O(\log g)$, or,
- there exists a non-contractible cycle of length at most $D = O(\log g)$, or,
- some cycle in \mathcal{F} has a non-planar D -neighborhood.

Corollary

If G is a graph embedded on a surface of genus g , $ew(G) \geq \Omega(\log g)$ and $X \subset V(G)$, where $d(u, v) \geq \Omega(\log g)$ for all $u \neq v \in X$, then every 5-coloring of X extends to a 5-coloring of G .

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If G is a graph embedded on a surface of genus g , $ew(G) \geq \Omega(\log g)$ and $X \subset V(G)$, where $d(u, v) \geq \Omega(\log g)$ for all $u \neq v \in X$, then every 5-coloring of X extends to a 5-coloring of G .

Proof

Let $k = 1$.

Corollary

If G is a graph embedded on a surface of genus g , $ew(G) \geq \Omega(\log g)$ and $X \subset V(G)$, where $d(u, v) \geq \Omega(\log g)$ for all $u \neq v \in X$, then every 5-coloring of X extends to a 5-coloring of G .

Corollary

\forall surface S of genus g : If G can be drawn in a surface of genus such that every pair of crossings is distance $D = O(\log g)$ apart and $ew(G) \geq \Omega(\log g)$, then G is 5-list-colorable.

Linear Bounds

One important tool is having a linear bound.

Definition

Let G be a plane graph, $S \subset V(G)$ and L a 5-list-assignment. We say G is S -critical if for every proper subgraph H of G , $C \subseteq H$, there exists an L -coloring of H which does not extend to G .

Lemma

Let G be a plane graph, C its outer cycle, and L a 5-list-assignment. If G is C -critical, then $|V(G)| = O(|V(C)|)$.

Corollary

If G is C -critical, then $d(v, C) \leq O(\log |V(C)|)$ for all $v \in V(G)$.

Outline of Cylinder Proof

The other important idea is generalizing Thomassen's proof of 5-list-colorability of planar graphs.

Theorem

Let G be a plane graph, C its outer cycle and $P = p_1p_2$ is a path of length one in C . If L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G \setminus C)$, $|L(v)| \geq 3$ for all $v \in V(C \setminus P)$ and there exists an L -coloring of P , then there exists an L -coloring of G .

We generalize this to the cases when (characterizing the structure of non- L -colorable graphs):

- Precolored Paths of any length
- Two Precolored Vertices far apart
- Two Precolored Paths far apart
- Many Precolored Paths far apart

Happy Birthday Robin!