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ENDS OF TRAVEL, LOCKING POSITIONS, AND "UNCERTAIN MOTION" IN SPATIAL MECHANISMS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Eure Merwinn Jenkins, Jr.

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

December, 1967

ENDS OF TRAVEL, LOCKING POSITIONS, AND "UNCERTAIN MOTION"

IN SPATIAL MECHANISMS

Approved:

Chairm

Date approved by Chairman: 29. viii. 67

ACKNOWLEDGMENTS

The author wishes to express sincere appreciation to all of those who have made this work possible. Special thanks are extended to Dr. F. R. E. Crossley, the author's faculty advisor, and to Professor K. H. Hunt, Dean of Engineering of Monash University in Melbourne, Australia, without whose assistance and guidance the work could not have been completed.

Special thanks are also extended to Nancy B. Jenkins, the author's wife, for her patient understanding and confidence during this period of her husband's education.

Finally, tha author wishes to thank the National Science Foundation for its financial support from NSF grant number GK-1203.

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ABSTRACT

The object of this thesis is to determine the most appropriate method for predicting ends of travel, locking positions, and points of "uncertainty" in the gross motion of simple spatial mechanisms. It involves a survey of known techniques, as well as the introduction of a new method for determining such conditions. The new method, an analysis of the curve of intersection of surfaces generated by points in a spatial mechanism, is also related in such a manner as to render it useful in determining position relationships in spatial mechanisms.

NOMENCLATURE

R --- a revolute pair.

P --- a prismatic pair.

S --- a screw pair,

C --- a cylindrical pair.

G --- a ball or global pair.

F --- a plane-on-plane pair.

Q' and Q" --- points in a link of an open kinematic chain which trace trace surfaces in space.

Q --- the center of a G pair in a mechanism in which Q' and Q" are coincident.

τ --- translational displacement.

ω --- angular displacement.

h-value --- the ratio of τ to ω which is related to a screwing motion.

X, Y & Z --- major axes of a coordinate system.

x, y & z --- coordinates in a three-dimensional coordinate system.

$(\ )_{ij}$ --- a kinematic pair which connects body i to body j

$\vec{i}, \vec{j}, \& \vec{k}$ --- unit vectors in the x, y, and z directions, respectively.

$\$_{ij}$ --- the permanent screw axis for the kinematic pair which connects body i to body j.

$\vec{a}_{ij}$ --- a unit vector along $\$_{ij}$.

$\vec{b}_{ij}$ --- a unit vector along a line from Q which intersects and is perpendicular to $\vec{a}_{ij}$.

$\vec{S}_{ij}$ --- a vector parallel to $\vec{a}_{ij}$.

T --- a point on $\vec{a}_{ij}$.

$\vec{T}$--- the position vector for T with respect to the origin of the coordinate system.

$\vec{R}(y)$ --- the vector equation for the curve of intersection of two surfaces; a function of the independent variable y.

$\vec{U}(y)$ --- $\vec{R}(y) - \vec{T}$.

$\vec{c}$--- a unit vector in the xy - plane.

∇ --- a vector operator defined to be $\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$\vec{N}_A$--- $\nabla(A)$, the gradient of the surface A

$\vec{V}_\ell(y)$ --- a vector which is a function of the independent variable y, and which, when evaluated at a point Q by $\vec{R}(y)$, has the same direction as the velocity of the point Q.

CHAPTER I

INTRODUCTION

Historical Background

A spatial mechanism is any mechanism which provides motion in a three dimensional space. Therefore, all mechanisms are of the spatial type. Historically, however, mechanisms have been divided into two main categories; 1) all those mechanisms which provide motion in a family of parallel planes, called "planar" mechanisms, and 2) all other mechanisms which provide any other type of motion, and are called "spatial" mechanisms. An example of a special class of spatial mechanism is the spherical mechanism, i.e. any mechanism which provides motion on the surface of a family of concentric spheres.

The history of mechanisms is as old as recorded time. Prior to the mid - 16th century, mechanisms seem to have been designed primarily on a basis of geometry and intuition. The intricacy of some of the earlier mechanisms is amazing, considering the limited design techniques which were available to the designers of those times. With the development of the science of kinematics, more and more knowledge was amassed which dealt primarily with planar mechanisms, and, in particular, the four-bar planar linkages. Gradually, as a broader basis of understanding developed, design concepts and procedures became far more complex than the mechanisms being designed. However, the founders of kinematics were primarily intellectuals interested in the development of theory, not in the mundane application of theories already developed, and, therefore, the generation of theories continued.

As the world entered the industrial age, requirements for speed and accuracy in mechanisms brought about many changes in machine design techniques. Graphical methods, because they were slow and lacked extreme accuracy, were gradually replaced by mathematical methods, and machine, or mechanism, design slowly, but surely, became an exercise in mathematics, instead of a process of visualization of physical phenomena. Design criteria changed, for the most part, from "Why?" and "How?", to "How fast?", and "How much?"

Although the greatest portion of the work being done in kinematics today emphasizes the "How much?", "How fast?" point of view, in the past 25 years there has been a reaction taking place which seeks to return to the questions of "Why?" and "How?"; particularly in the area of spatial mechanisms. Once these questions, "Why?" and "How?", are answered for this more general class of mechanism, perhaps the answers to the "How fast?", "How much?" questions will become more obvious in a physical, rather than a theoretical, sense.

Mechanical Background

Physically, and in completely general terms, a spatial mechanism consists of a number of links, bodies, joined in various manners to form loop configurations. There may be more than one loop in a spatial mechanism, however, only those mechanisms which exhibit a single-loop configuration will be considered in this thesis. Further, only those single-loop mechanisms which are constrained by one input, i.e. which display a total mobility of one, will be discussed.

The links of any spatial mechanism are connected by joints, called kinematic pairs. There are a variety of these kinematic pairs, and they are divided into two main categories; lower pairs, and higher pairs. Lower pairs provide surface contact between the two links which they connect, while higher pairs provide line or point contact. If two links are connected by a lower pair, the pattern of motion created by points in one body, as it undergoes relative motion with respect to its connected neighbor, is exactly the same, regardless of which link is considered to be the reference link. For example, the revolute pair (Fig. 1-a) is a lower pair which restricts any arbitrary point Q, in body # 2, to move along a circular path, in a single plane which is perpendicular to the pair axis, about the pair axis. Likewise, if body # 2 is taken to be the reference link, any point Q', in body # 1, is restricted to move on a circular path, in a single plane which is perpendicular to the pair axis. On the other hand, if two links are connected by a higher pair, the pattern of motion created by a point in one body, moving relative to its connected neighbor, varies according to which link is considered to be the reference link. For example, if a ball of radius r is constrained to move within a gutter, a hollow cylinder, of radius r and length ℓ any arbitrary point Q, in the ball, can be located anywhere within the volume of a cylinder of radius q; $0 \le q \le r$, and length ℓ. However, if the ball is considered fixed; any arbitrary point Q', in the gutter, can be located anywhere within the volume of a sphere of

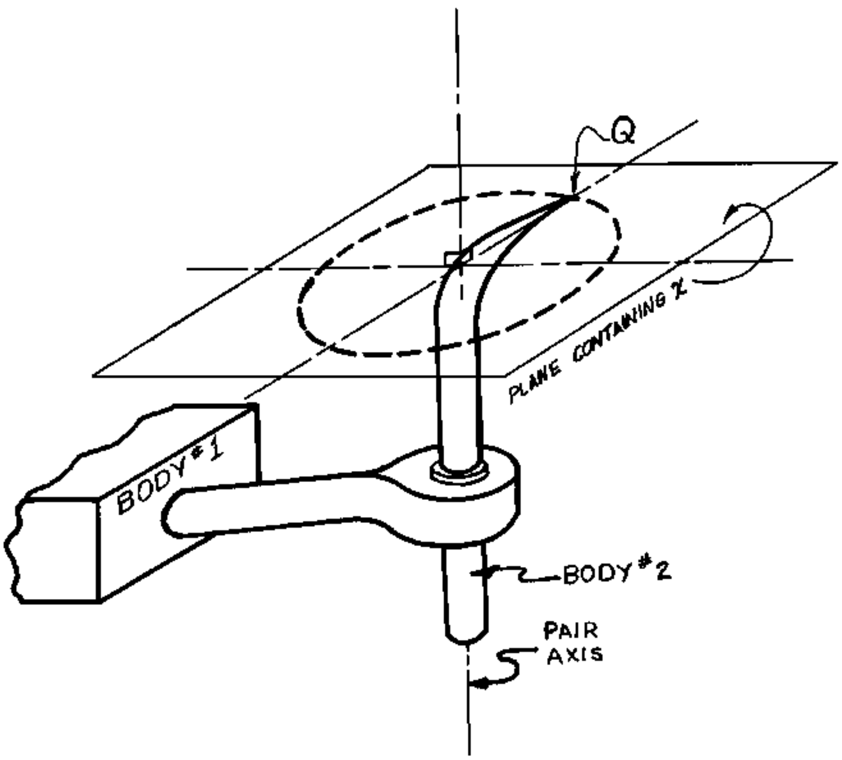


Figure 1-a. Revolute (R) Pair.

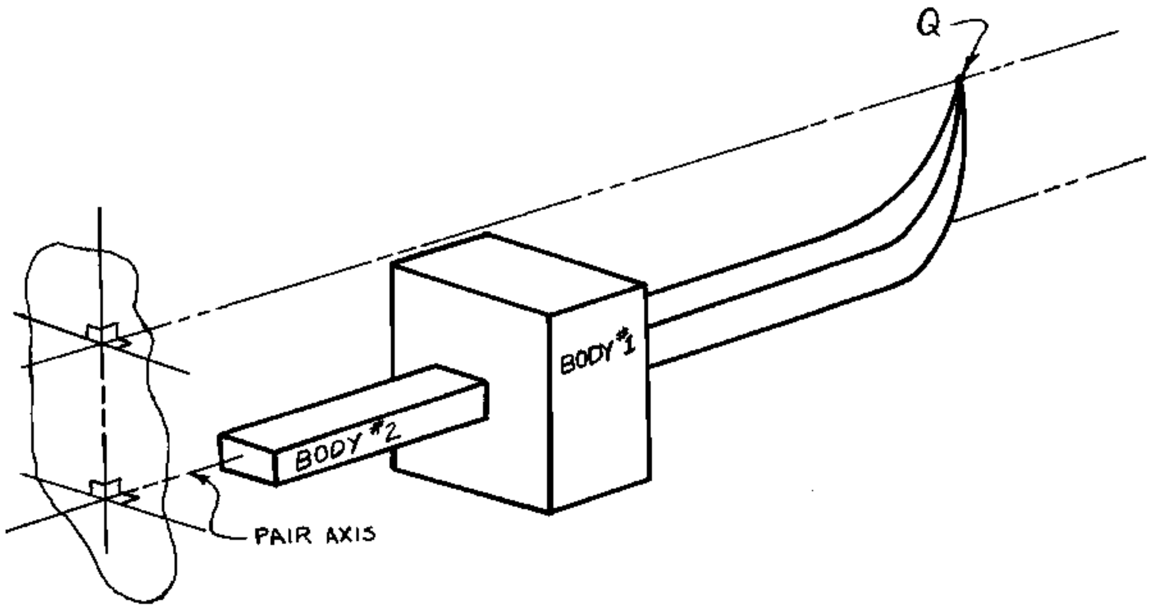


Figure 1-b. Prismatic (P) Pair.

radius q; $\frac{\ell}{2} \leq q \leq \ell$.

Kinematic pairs may be sub-categorized according to the degrees of relative freedom which they provide between the two connected links. In a Cartesian, three-dimensional, coordinate system there is a possibility of six degrees of relative freedom; i.e. rotations about each of the three axes of the system, and translations along each of these axes. Where there are six degrees of relative freedom between two bodies, the two bodies are not connected.

There are six types of lower pairs; 1. revolute pairs, which are designated at R pairs; 2. prismatic, or P pairs; 3. screw, or S pairs; 4. cylindrical, or C pairs; 5. ball, global, or G pairs; and 6. plane-on-plane, or F pairs. Each of these pairs provides a particular degree, or combination of degrees, or relative freedom which allows a particular type of relative motion between the two links which it connects. The R pair, since it allows only a relative rotation about its axis, is a one-degree-of-relative-freedom pair.

A P pair allows only a translation along its axis, and is, therefore, also a one-degree-of-relative-freedom pair. Consider two bodies connected by a P pair (Fig. 1-b): any arbitrary point Q, in body # 2, is restricted to move along a line parallel to the pair axis.

An S pair allows a translation along, and a rotation about its axis, but there is a constant ratio, the "pitch" or h-value, of the translation τ to the rotation ω. Therefore, an S pair is also a one-degree-of-relative-freedom pair. If two bodies are connected by an

S pair (Fig. 1-c), then any arbitrary point Q, in body # 2, is restricted to move on a constant radius, constant pitch helix along, and about, the pair axis. If, as one special case, the pitch, $h = \tau/\omega$, of an S pair is zero, then $\tau = 0$, and the S pair becomes an R pair. If the pitch becomes infinite, then $\omega = 0$, and the S pair becomes a P pair. The motion of any body in space can be instantaneously described by a rotation about, and a translation along some screw axis, designated by the symbol $\$_{ij}$, where i indicates the reference body with respect to which the motion is measured, and j indicates the body in motion. The pair axes of those pairs so far described are generally referred to as permanent screw axes, and will be so designated.

A C pair allows a rotation about its axis and a translation along its axis. Since there is no constant ratio between the translation and the rotation, the C pair is a two-degrees-of-relative-freedom pair. If two bodies are connected by a C pair (Fig. 1-d), then any arbitrary point Q, in body # 2, is restricted to move on the surface of a right circular cylinder whose axis of revolution corresponds to $\$_{12}$.

A G pair allows rotation about each of its three axes, and is, therefore, a three-degrees-of-relative-freedom pair. If two bodies are connected by a G pair (Fig. 1-e), any arbitrary point Q, in body # 2, is restricted to move on the surface of a sphere. Notice that the point Q may be moved to any position on the sphere by rotating body # 2 about only two axes. That is to say that only two degrees of relative freedom are required for the point Q to generate a sphere with respect to body # 1. However, a third degree of relative freedom provided by

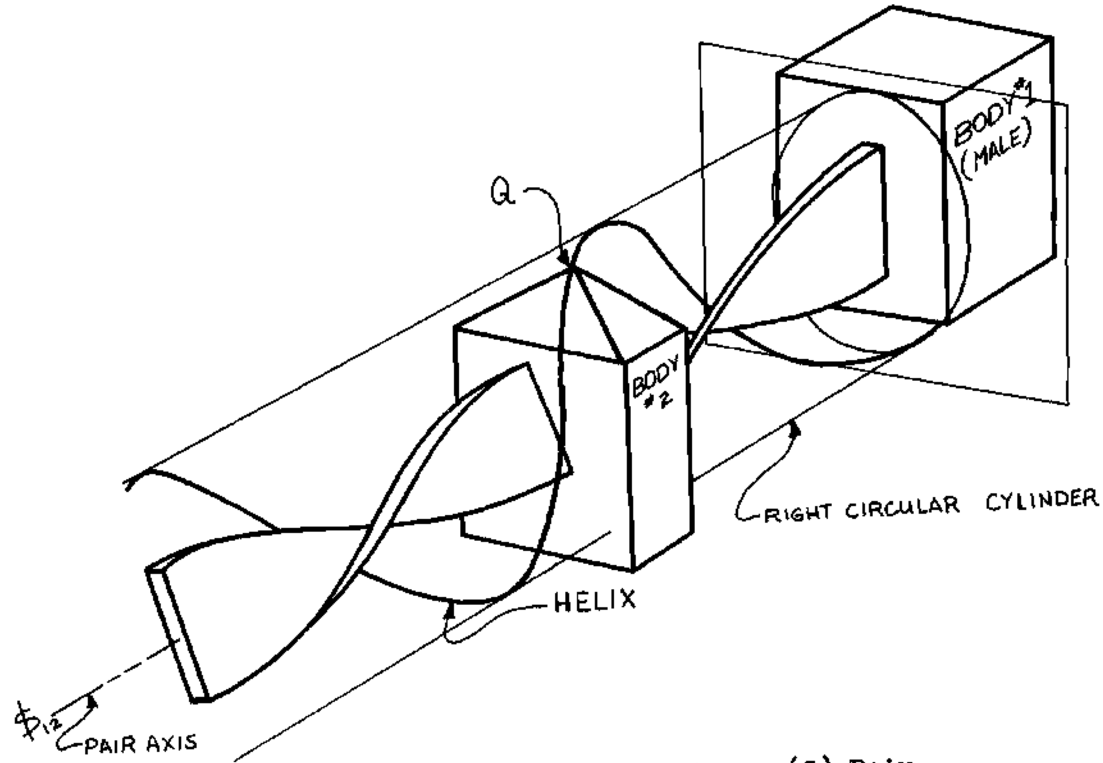


Figure 1-c. Screw (S) Pair.

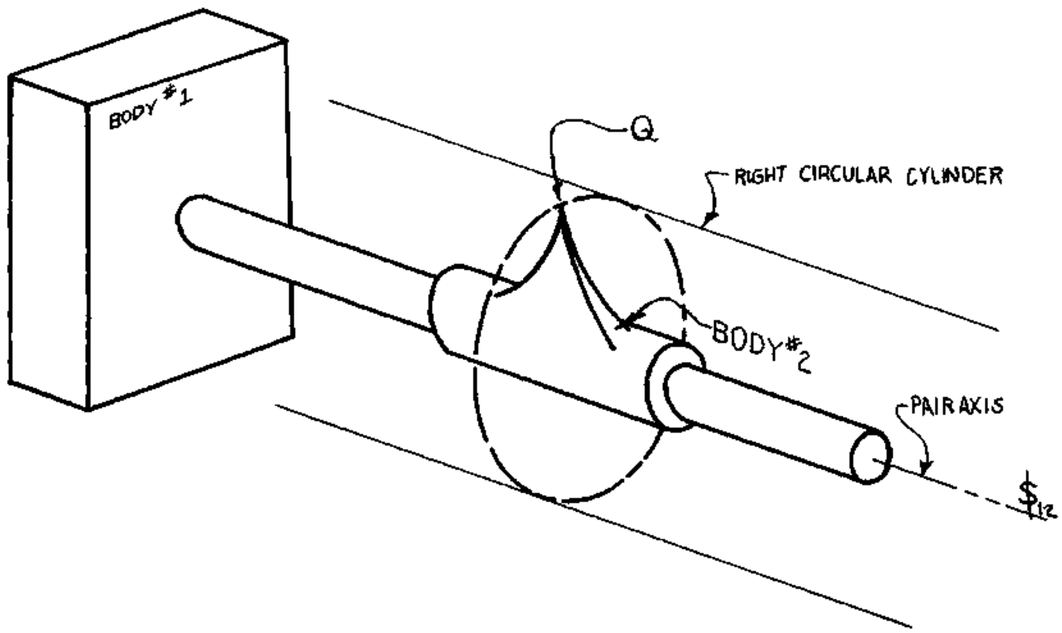


Figure 1-d. Cylindrical (C) Pair.

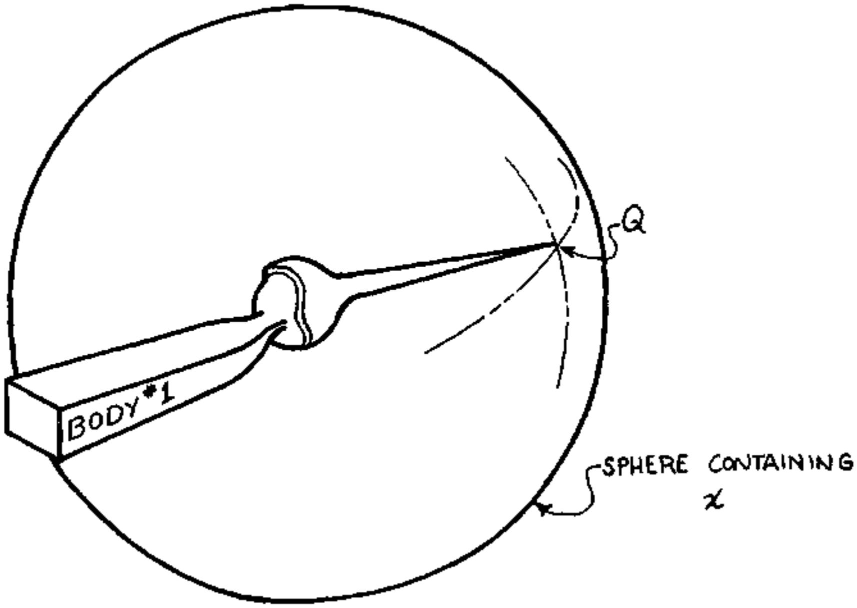


Figure 1-e. Ball or Globe (G) Pair.

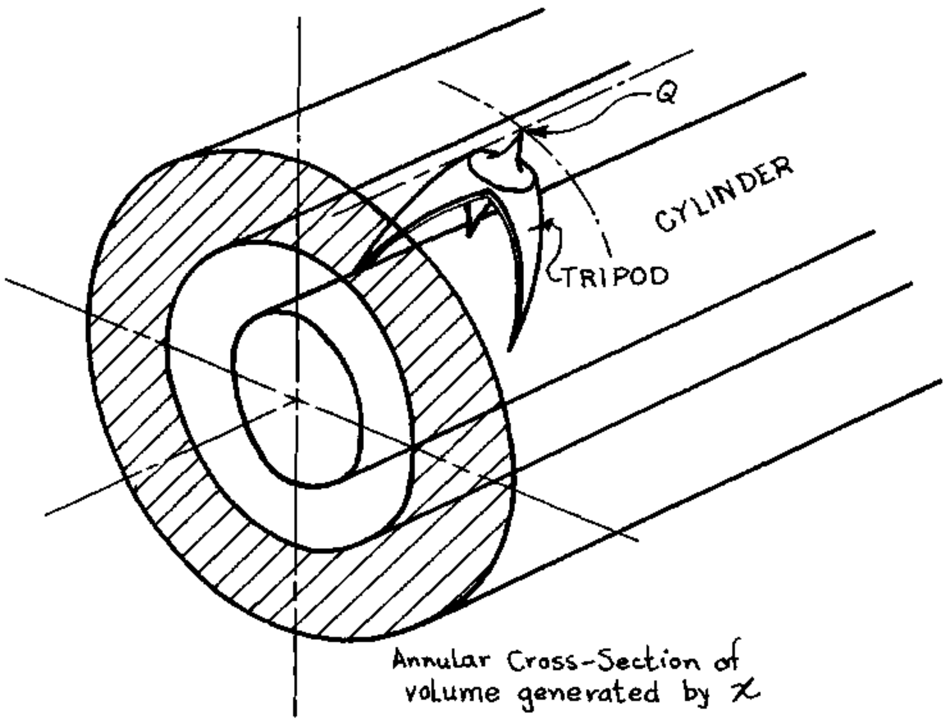


Figure 1-f. Higher Pair (Three Degrees of Relative Freedom).

the G pair can be seen by considering a rotation of body # 2 about an axis which passes through the center of the G pair and the point Q. Such a rotation does not alter the position of Q relative to body # 1, but it does alter the position of all points in body # 2, not on this new axis, with respect to body # 1.

An F pair is a limiting case of the G pair, i.e. the radius of the sphere becomes infinite. It allows translations along two axes and a rotation about an axis perpendicular to the planes of contact, and, therefore, is a three-degrees-of-relative-freedom pair.

An example of a higher pair which provides three degrees of relative freedom is a tripod resting on a cylinder (Fig. 1-f). This pair allows a translation along, and a rotation about the cylinders axis, and a rotation about an axis made up of all points equidistant from the three points of contact between the cylinder and the tripod. This last rotation allows any arbitrary point Q, in the tripod, to be raised and lowered with respect to the surface of the cylinder. Therefore, this pair allows any point Q, in the tripod, to "generate the volume", i.e. to be able to reach every point with an annular cross-section, as shown in the figure. The generation of a volume depends for its existence on the provision of three independent degrees of relative freedom by the pair or pairs. Three independent dimensions are required to locate Q with respect to the cylinder, and three such dimensions define a volumetric space.

In general, any kinematic pair which provides one degree of relative freedom allows the generation of a curve; any pair which provides two degrees of relative freedom allows the generation of a surface; and any pair which provides three degrees of relative freedom, with the exceptions of the G and F pairs, allows the generation of a volume, by any point in one body relative to its connected neighbor.

In order for these curves, surfaces, and volumes to be generated, the link containing the arbitrary point Q, the generating link, must undergo gross motion; this term is here used to indicate that the generating link moves through the totality of its possible, relative position range with respect to its connected neighbor.

Futhermore, the physical orientation of these pairs and the sequence in which they appear in a mechanism govern not only how one link moves relative to its connected neighbor, but also how, and if, the entire mechanism moves.

Problem Definition

It is necessary, when determining whether or not a mechanism is satisfactory for a particular application, to consider all of the mechanism's physical characteristics. Does the mechanism perform the desired task with reasonable accuracy? Will the mechanism operate properly in the available space? Will the mechanism ever reach a "locking position"? Is there a point in the mechanism's operation where it may diverge from its desired function; i.e. is it possible for a link in a mechanism to follow two intersecting paths for the same configuration of the mechanism?

Obviously, the physical dimensions of the mechanisms components indicate an approximate volume for the space required to contain the mechanism. However, in order to define the required volume more accurately, it is necessary to be able to predict the limiting positions for components of the mechanism. These limiting positions, called ends of travel, occur simultaneously with zero velocities for certain of the mechanism's components, and are defined as positions where there exists a zero velocity component, either translational or angular, for points in a body along or about the pair axis between the body and its connected neighbor.

"Locking-positions" occur when the force acting on a link has a direction which corresponds concurrently to an impossible direction of motion for the link. For example, if a force is applied to a body connected to its neighbor by a P pair in such a direction as to be perpendicular to the pair axis, there can be no motion between the connected bodies as a result of this force. Notice that only one condition need be satisfied to create a locking position for a one-degree-of-relative-freedom pair, while two conditions must be satisfied simultaneously to create a locking position for a two-degrees-of-relative-freedom pair; e.g. in order to lock body # 2 (Fig. 1-d) to body # 1, a force must be applied in such a manner as to act along a line which both intersects and is perpendicular to $\$_{12}$. In a closed-loop mechanism, locking positions depend upon which link is the driving link and which is the driven link. In a planar, four-bar, slider-crank mechanism, for example, if the crank is the driving link, there are no locking positions, unless the connecting rod becomes perpendicular to the path of the slider. However, if the slider is the driving link, there are always two locking positions; one for each of the slider's ends of travel, usually called "dead centers". If the linkage is at rest with the slider at an end of travel, the slider can not be moved from its position by applying force directly to the slider; a fact well known to the designers of steam locomotives.

A mechanism may, moreover, have points of "uncertainty" in its operating range; i.e. it may be possible for a point in a mechanism to move on two paths which share one or more common points for the same configuration of the mechanism. If the point paths of a point in a mechanism share common points, "branch points", these points are "points of uncertainty". When a point in a mechanism lies at rest at a "branch" point, the point in the mechanism may, as the linkage is set into motion, move on any path which emanates from the "branch" point, depending upon the direction of the motivating force; thus the name, point of "uncertainty". Once the mechanism is in motion, the dynamics of the problem generally eliminates the problem as to which of the paths the point must follow. The designer must determine thelocations of points of "uncertainty" so as to be able to avoid having to attempt to start the mechanism from such a position.

The object of this work is to present a method for obtaining the necessary information, i.e. ends of travel, locking positions, and points of "uncertainty" for spatial mechanisms, and to show that this information can be determined from a purely physical point of view involving the concept of surfaces generated by points in a mechanism. Operational accuracy will not be discussed in this thesis.

CHAPTER II

RELATIVE MOTION OF THREE CONNECTED LINKS

Thus far only two links and one kinematic pair have been considered. It is now necessary that surfaces and volumes which may be generated by points in three links with two kinematic pairs be considered. For convenience, the three links will be referred to as bodies # 1, # 2, and # 3. Body # 1 will always be the reference link, body # 2 will be the coupling link, and body # 3 will contain the generating point Q. The symbol $(\ )_{ij}$ will be used to define the kinematic pair and its location. For example, $(R)_{12}$ indicates the revolute pair which connects body # 1 to body # 2. Unless specifically stated, the permanent screw axes, $\$_{ij}$, of the two pairs will be randomly oriented and, therefore, will not intersect.

There are three main categories of surfaces which may be generated by two active links and two kinematic pairs; 1. cylinders, 2. hyperboloids of one sheet, and 3. tori. Other more complex surfaces may be generated by points in such kinematic chains, but only in the light that they are surfaces which bound generated volumes. Therefore, all "pure" surfaces generated by points in kinematic chains will be of one of the three specified classes.

The combination of $(P)_{12}$ and $(R)_{23}$ allows any point Q in body # 3, (See Fig. 2-a), to generate the surface of an elliptical cylinder. If the permanent axes $\$_{12}$ and $\$_{23}$ are parallel, the cylinder has a circular cross-section in any plane perpendicular to $\$_{12}$. In a linkage,

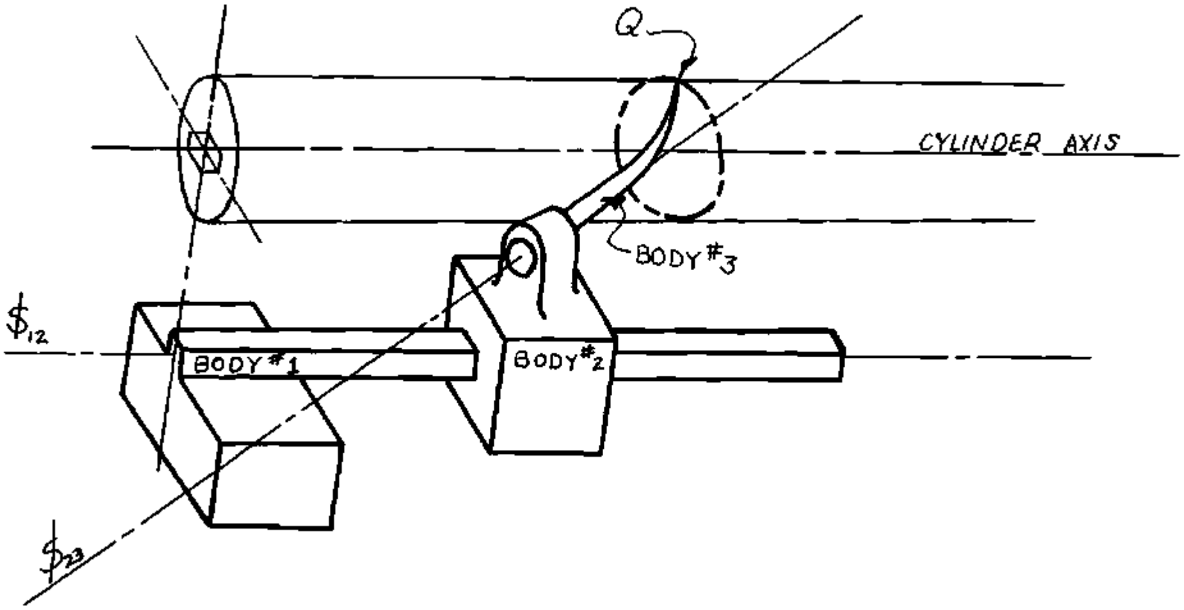


Figure 2-a. Elliptic Cylinder; $(P)_{12}$, $(R)_{23}$ Combination.

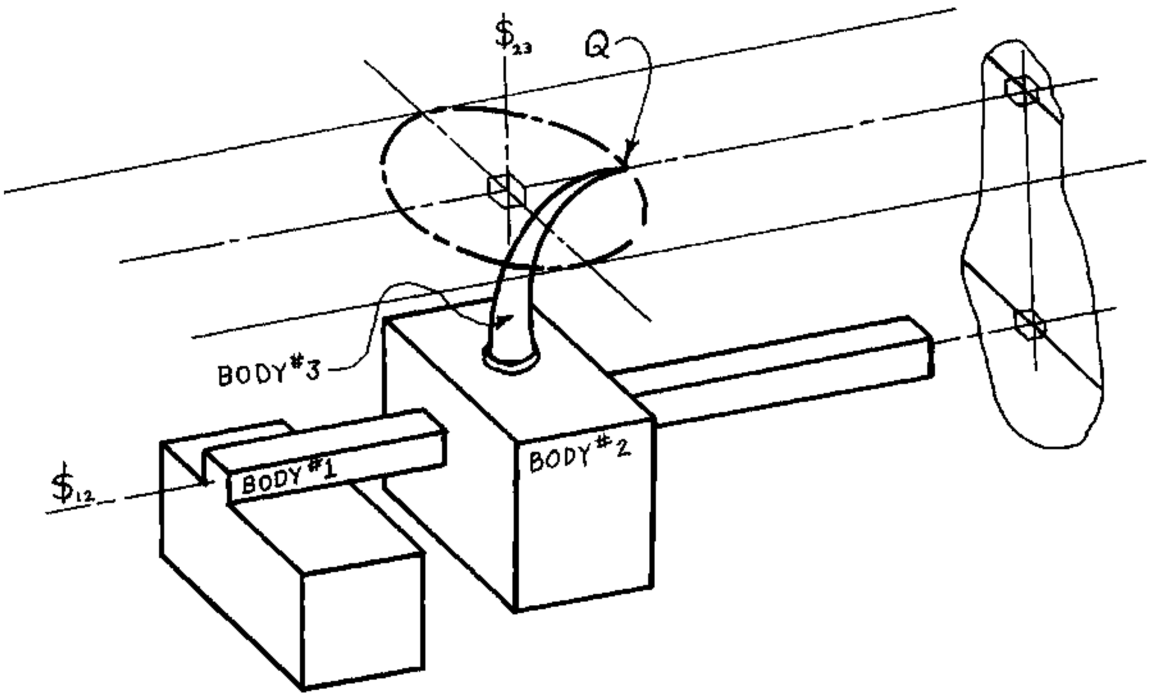


Figure 2-b. Plane; $(P)_{12}$, $(R)_{23}$ Combination.

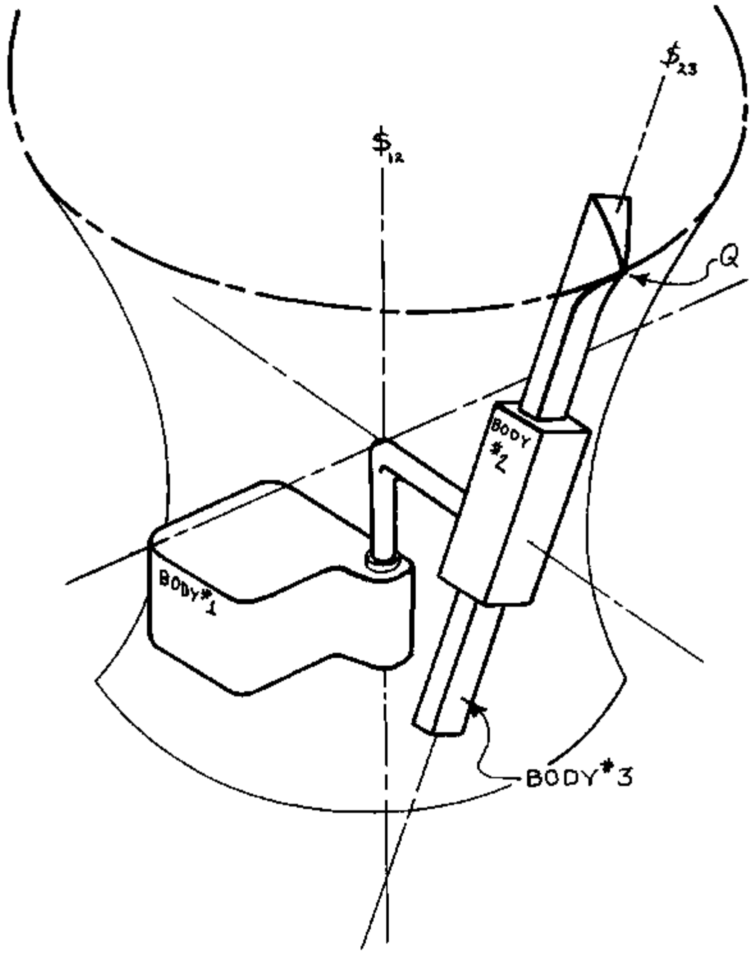


Figure 3-a. Hyperboloid of Revolution; $(R)_{12}$, $(P)_{23}$ Combination.

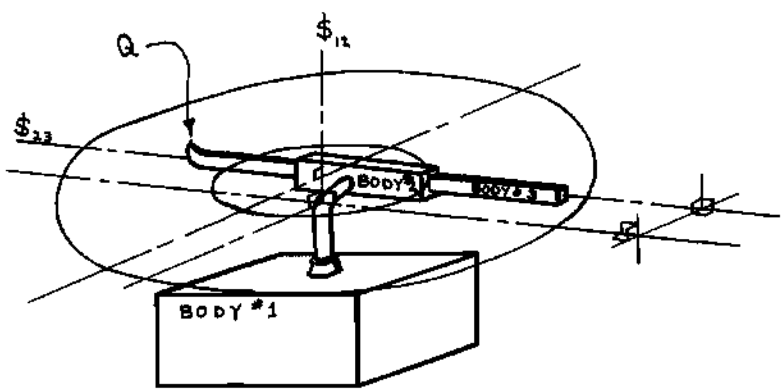


Figure 3-b. Annular Plane; $(R)_{12}$, $(P)_{23}$ Combination.

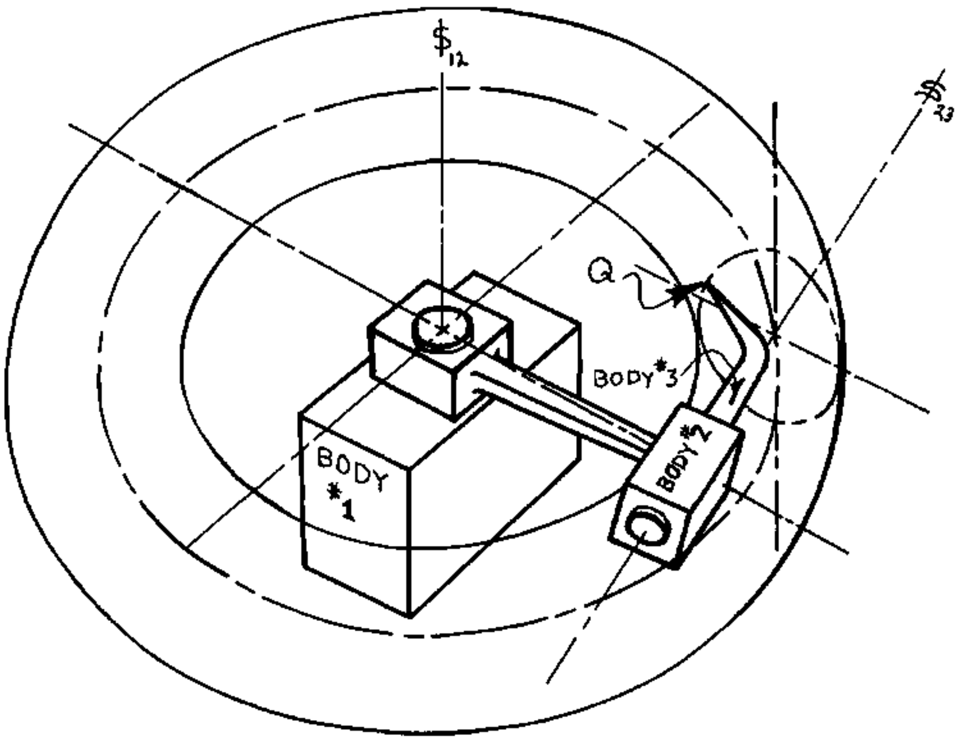


Figure 4-a. Toroidal Surface; $(R)_{12}$, $(R)_{23}$ Combination.

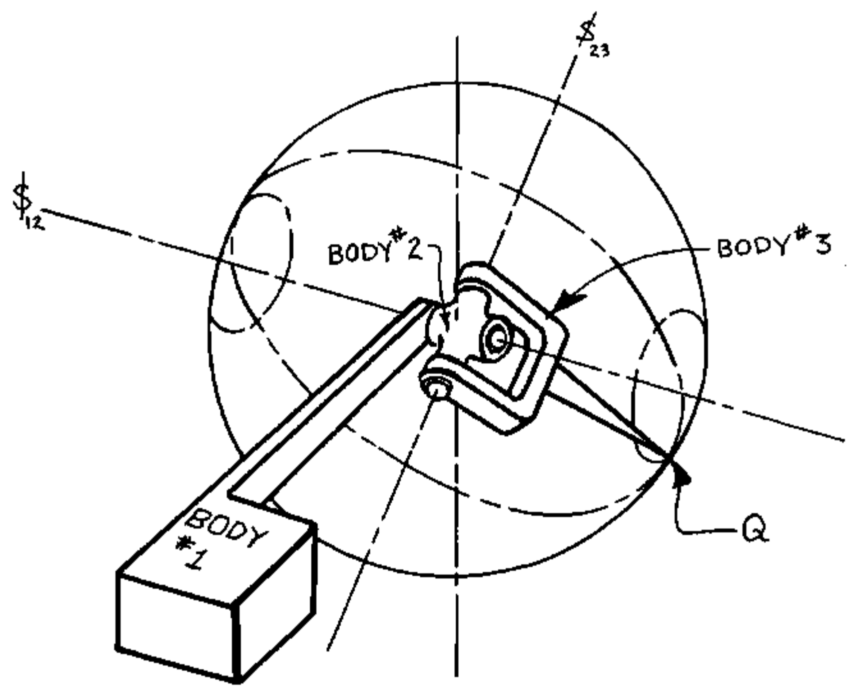
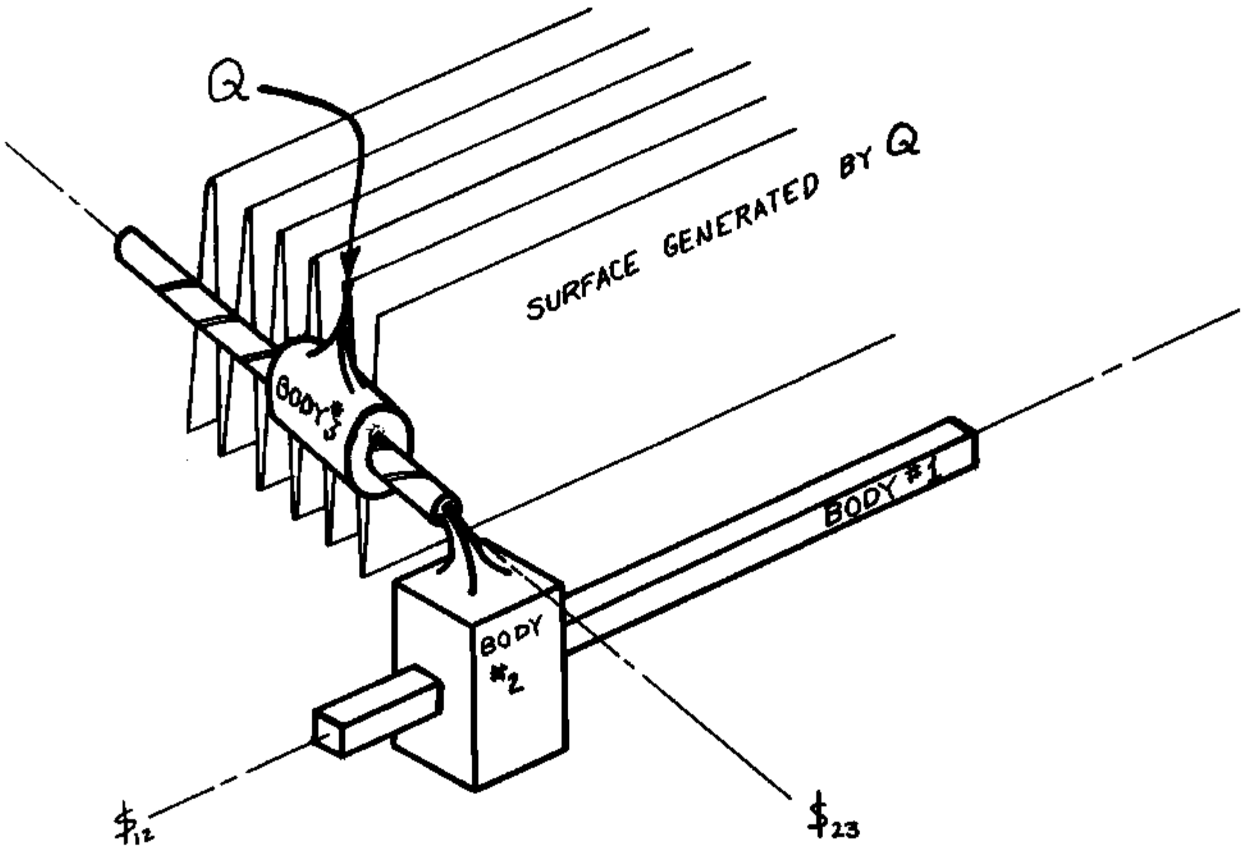


Figure 4-b. Sphere; $(R)_{12}$, $(R)_{23}$ Combination.


Q
SURFACE GENERATED BY Q
BODY #3
BODY #1
BODY #2
$12
$23

this same result can be obtained by employing a C pair whose axis is coincident with $\$_{23}$; thus using one less link. If $\$_{12}$ is perpendicular to $\$_{23}$, then the point Q in body # 3, (See Fig. 2-b), generates a plane to which the axis $\$_{12}$ of $(P)_{12}$ is parallel.

The combination of $(R)_{12}$ and $(P)_{23}$ allows any point Q in body # 3, (See Fig. 3-a), to generate the surface of a hyperboloid of one sheet whose axis coincides with $\$_{12}$. If $\$_{12}$ and $\$_{23}$ are parallel, the surface degenerates to a right circular cylinder, and, again, can be replaced by a C pair. However, in this case, the axis of the C pair must be coincident to $\$_{12}$. When $\$_{12}$ is perpendicular to $\$_{23}$ the generated surface becomes an annular plane which is perpendicular to $\$_{12}$.

The combination of $(R)_{12}$ and $(R)_{23}$ allows any point Q in body # 3 (See Fig. 4-2) to generate the surface of a torus whose major axis coincides with $\$_{12}$ and whose mean radius is the perpendicular distance from the center of the circle generated by Q with respect to body # 2 to $\$_{12}$; a measurement which is a constant of body # 2. If $\$_{12}$ is parallel to $\$_{23}$, the point Q in body # 3 generates an annular plane similar to that plane shown in figure 3-b. If the length of body # 2 is reduced to zero by letting the centers of the circles generated by Q about $\$_{23}$ and Q', a point in body # 2, about $\$_{12}$ becomes coincident, the surface becomes spherical, (See Fig. 4-b), and, during the gross motion of the active links, is generated twice.

The combination of $(P)_{12}$ and $(P)_{23}$ allows any point Q in body # 3, (See Fig. 5) to generate the surface of a plane, regardless of the relative orientation of $\$_{12}$ and $\$_{23}$.

If an S pair is introduced in combination with any other one-degree-of-relative-freedom pair, in any sequence, the nature of the generated surface becomes highly complex. As one example, figure 6 shows a corrugated sheet generated by a $(P)_{12}$, $(S)_{23}$ orthogonal set. Although these surfaces may be described verbally, the application of such surfaces introduces an undesirable degree of initial difficulty, and, therefore, will not be considered in further detail in this work.

Notice that if body # 3 is provided with three degrees of relative freedom with respect to body # 1, the point Q in body # 3 is capable of reaching any point in a specified volume, instead of in a specific surface. The characteristics of the volume are determined in much the same manner as those for the preceding surfaces. (F and G pairs are exceptions in this case.) Because volumes create complex analysis problems, they will not be discussed at length in the remainder of this paper. However, it is worthwhile to note that their existence and characteristics may provide a necessary key to the development of rules for spatial mechanisms which would be comparable to Grashof's rules for planar, four-bar linkages.

In part, the purpose of this work is to show that the existence of, and the characteristics of the patterns of motion provided by these basic kinematic pairs, taken singly, or in series, provide a powerful tool for the investigation of the gross motion of spatial mechanisms.

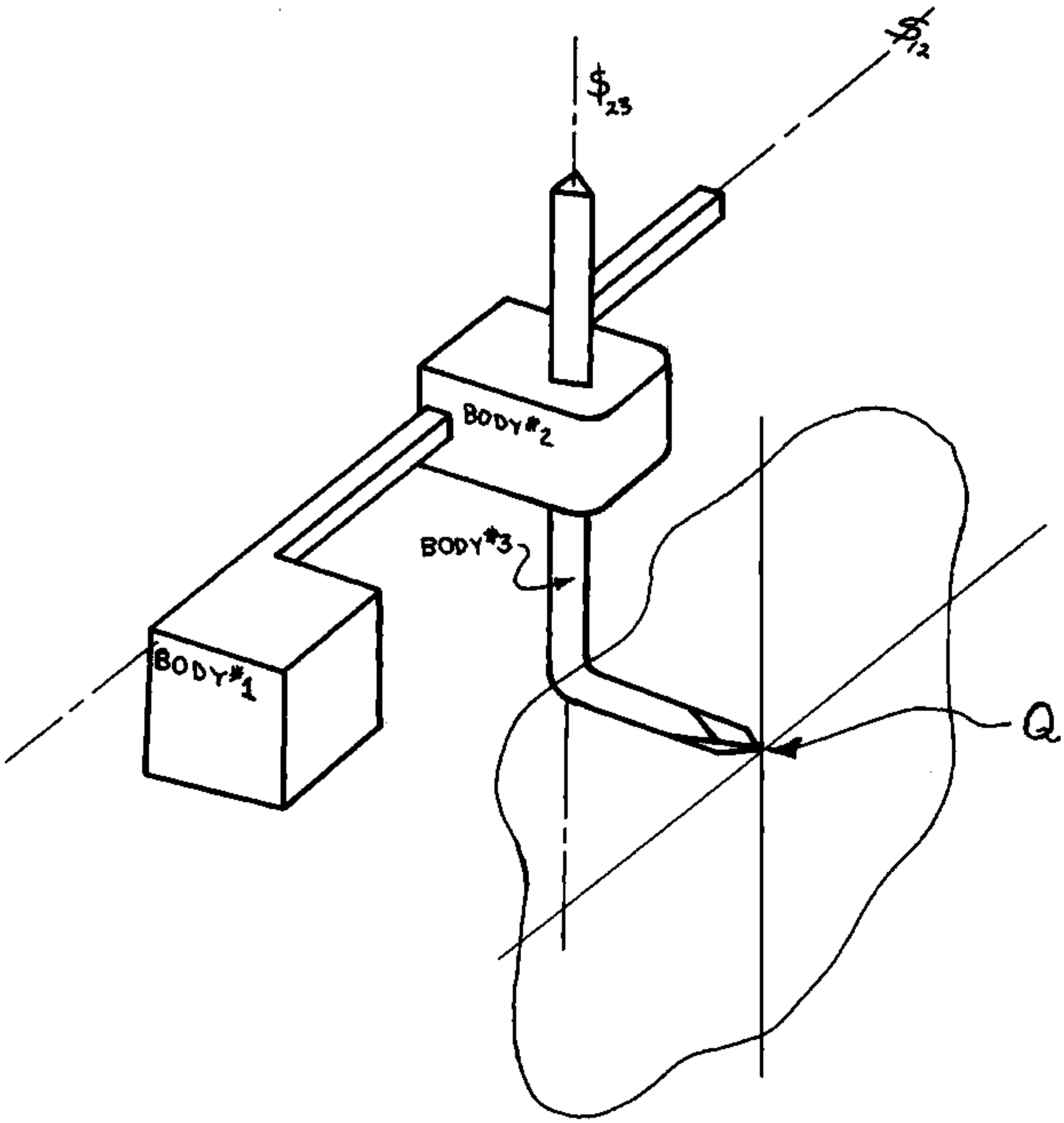


Figure 6. $(P)_{12}$, $(S)_{23}$ Combination.

CHAPTER III

PATTERNS OF MOTION IN SPATIAL MECHANISMS

Information presented in the previous chapters indicates that any link which displays two degrees of relative freedom with respect to some reference link contains some arbitrary point which is capable of generating a "pure" surface relative to the reference link. By the word "pure", it is meant that the generating point must always remain within the surface it generates, as contrasted with the case when the point generates a volume, and of course also the bounding surface of that volume. Suppose now that there are two bodies, A and B, which are kinematically connected to the same reference link such that each of the bodies, A and B, exhibits two degrees of relative freedom with respect to the reference link; then each of the two bodies, A and B, contains an arbitrary point, Q' and Q" respectively, each of which may generate a "pure" surface with respect to the reference link. The surfaces generated by Q' and Q" may be any combination of two of the "pure" surfaces described in Chapters I and II (Table I.) Suppose next that these two surfaces intersect, and that body A can be connected to body B by some "bridge joint" such that Q' in A and Q" in B are constrained to be coincident. A single-loop mechanism is thus created. Suppose finally that this mechanism has a total mobility of one, and is set into motion; the "bridge joint" containing Q with whidh Q' and Q" are coincident, must, since Q' and Q" may only exist in their respective surfaces, move such that the point Q remains in both surfaces

Table 1. Possible Combinations of Surface and Their Generating Kinematic Chains

Q'		Q"	
Kinematic Chain	Surface	Kinematic Chain	Surface
$(C)_{12}$	Right Circular Cylinder	$(C)_{12}$	Rt. Circular Cylinder
		$(P)_{12}$ - $(R)_{23}$	Elliptic Cylinder or Plane
		$(R)_{12}$ - $(P)_{23}$	Hyperboloid of 1 sheet, Rt.Cyl. or Annular Plane
		$(R)_{12}$ - $(P)_{23}$	Torus, Sphere, or Annular Plane
		$(P)_{12}$ - $(P)_{23}$	Plane
		$(F)_{12}$	Plane
		$(G)_{12}$	Sphere
$(P)_{12}$ - $(R)_{23}$	Elliptic Cylinder or Plane	$(P)_{12}$ - $(R)_{12}$	Elliptic Cylinder or Plane
		$(R)_{12}$ - $(P)_{12}$	Hyperboloid, Rt. Circular Cylinder, or Annular Plane
		$(R)_{12}$ - $(R)_{23}$	Torus, Sphere or Annular Plane
		$(P)_{12}$ - $(P)_{23}$	Plane
		$(F)_{12}$	Plane
$(R)_{12}$ - $(P)_{23}$	Hyperboloid Annular Plane, or Right Cir. Cylinder	$(R)_{12}$ - $(R)_{23}$	Hyp., Ann. Plane, Rt. Circular Cylinder
		$(R)_{12}$ - $(R)_{23}$	Torus, Sphere, Annular Plane
		$(P)_{12}$ - $(P)_{23}$	Plane
		$(F)_{12}$	Plane
		$(G)_{12}$	Sphere
$(R)_{12}$ - $(R)_{23}$	Torus, Sphere, Annular Plane	$(R)_{12}$ - $(R)_{23}$	Torus, Sphere, Annular Plane
		$(P)_{12}$ - $(P)_{23}$	Plane
		$(F)_{12}$	Plane
		$(G)_{12}$	Sphere

Table 1. (Cont.) Possible Combinations of Surface and Their Generating Kinematic Chains

Q'		Q"	
Kinematic Chain	Surface	Kinematic Chain	Surface
$(F)_{12}$	Plane	$(F)_{12}$	Plane
		$(G)_{12}$	Sphere
$(G)_{12}$	Sphere	$(G)_{12}$	Sphere
$(P_{12} - (P)_{23}$	Plane	$(P)_{12} - (P)_{23}$	Plane
		$(F)_{12}$	Plane
		$(G)_{12}$	Sphere

simultaneously. The path of the point Q, in space, must, therefore, be the curve(s) of intersection of the surfaces generated by Q' and Q", respectively.

If this mechanism is to exhibit a total mobility of one, then, according to the Kutzbach criterion for the mobility of single-loop mechanisms, there must be a total of seven relative degrees of freedom within the kinematic pairs of the mechanisms loop. There are two degrees of relative freedom between each of the two bodies, A and B, and the reference link; therefore, the "bridge joint" must provide three degrees of relative freedom between bodies A and B. Moreover, since Q' and Q" must remain coincident in Q, none of the three degrees of relative freedom provided by the "bridge joint" may be translations between bodies A and B; the "bridge joint" must be a G pair.

Consider, for example, the open kinematic chain $(R)_{12}$- $(P)_{23}$ - $(P)_{34}$- $(R)_{45}$ in which body # 3 is the reference link, and bodies # 1 and # 5 are not connected. As bodies # 1 and # 2 undergo gross motion with respect to body # 3, an arbitrary point Q' in body # 1 generates the surface of an elliptical cylinder whose axis is parallel to $\$_{23}$. Likewise, as bodies # 4 and # 5 undergo gross motion with respect to body # 3, an arbitrary point Q", in body # 5, generates the surface of an elliptical cylinder whose axis is parallel to $\$_{43}$. If these two surfaces intersect, then Q' and Q" can be considered to be coincident at a point Q on the curve(s) of intersection of the two surfaces; a G pair, whose center is defined to be at Q, can be used to connect bodies # 1 and # 5, thus closing the chain into a loop and creating an $(R)_{12}$- $(P)_{23}$- $(P)_{34}$- $(R)_{45}$- $(G)_{51}$ mechanism. Furthermore, in this mechanism, with body # 3 as the reference link (frame), the curve of intersection of the generated surfaces defines the path of the center of the G pair, Q, as the mechanism moves. Obviously, if the curve of intersection between the two surfaces can be shown to exist only at a single point; i.e. if the two surfaces are externally tangent, then the generating links can only be connected to create a structure. Also, if the two surfaces are coincident over some finite region, the mechanism will operate with a total mobility of two when Q is in this region.

Once the concept of surfaces being generated by points in mechanical links is accepted, it becomes possible to visualize such surfaces being generated by points in spatial mechanisms. In order to facilitate such visualization, it is necessary that the most appropriate reference link be located. In the case of those mechanisms which contain five or less links and at least one G pair, the procedure for locating the reference link is relatively simple. Suppose that a $(C)_{12}$- $(C)_{23}$- $(G)_{31}$ mechanism is to be analysed (Fig. 7). If the G pair were disconnected, then the point Q' in body # 1 could generate a right circular cylinder with respect to body # 2. Likewise, the point Q" in body # 3 could generate a right circular cylinder with respect to body # 2. Body # 2, therefore, is the most appropriate reference link for analyzing this mechanism in the context of generated surfaces.

If the mechanisms which have five or less links but do not contain a G pair are considered, the selection of a reference link becomes more difficult. However, if the Kutzbach criterion for mobility

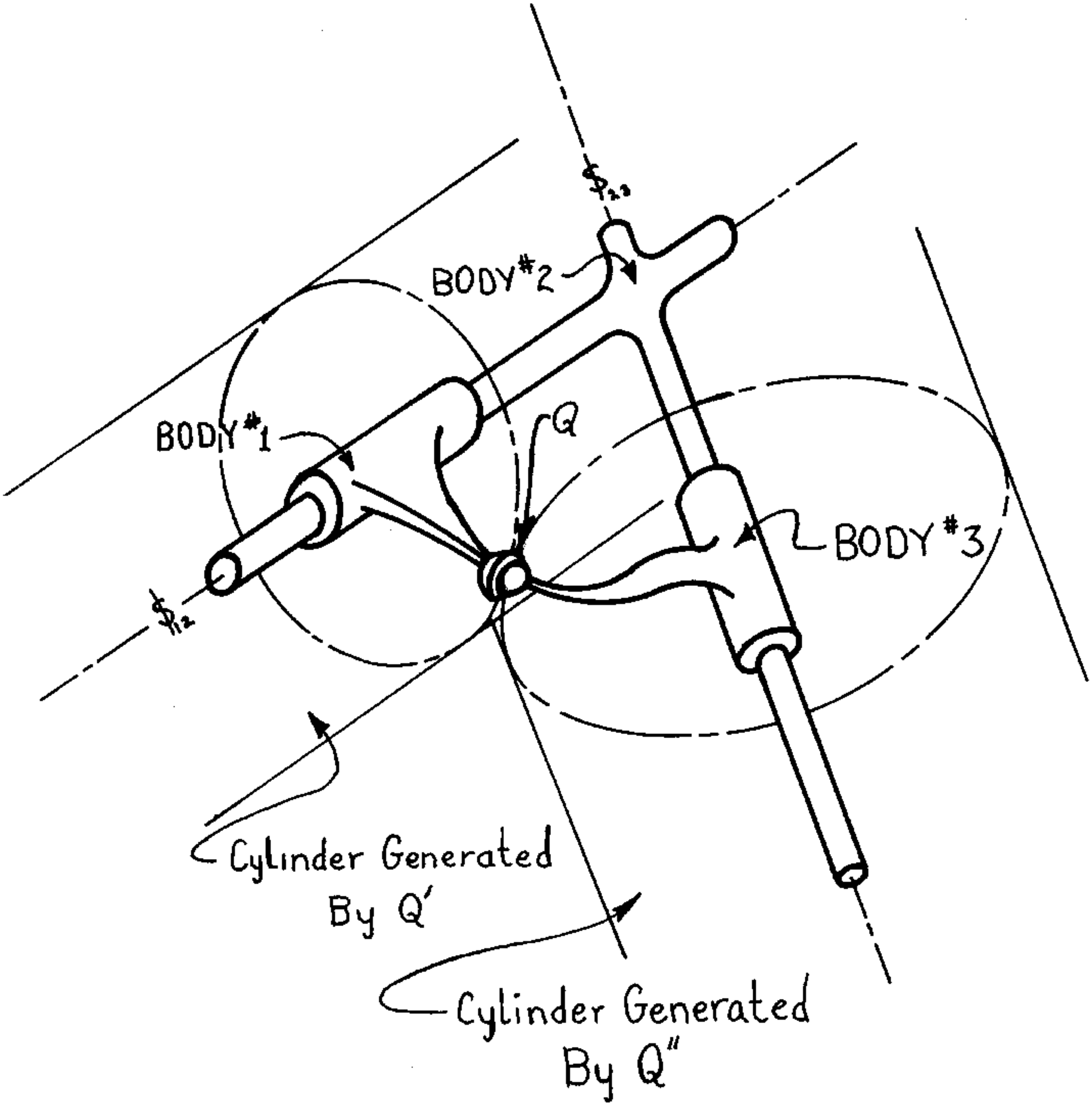


Figure 7. $(C)_{12}$ - $(C)_{23}$ - $(G)_{31}$ Mechanism

is satisfied in such cases, there will be kinematic chains which contain volume generating points, with respect to some reference link, while another kinematic chain, in the same mechanism contains points which generate surfaces with respect to the same reference link. For this reason, such mechanisms can not be considered in the context of generated surfaces.

The term "gross motion", as used earlier to define the total possible range of positions for the links in a kinematic chain, may be applied to any spatial mechanism. If each link in a mechanism has moved through its entire sequence of positions, the range of which is determined by the dimensions and configuration of the remaining links of the mechanism and their connecting pairs, the mechanism has undergone gross motion.

In order for the G pair in the $(C)_{12}$- $(C)_{23}$- $(G)_{31}$ mechanism (Fig. 7) to generate a curve of intersection of the two right circular cylinders, on which the G pair must move, the mechanism must undergo gross motion. It is possible that there may be more than one closed curve of intersection between two surfaces generated by kinematic chains of a mechanism, (e.g. a cylinder intersecting a torus (Fig. 8) may have as many as four closed curves of intersection). However, once a point on one of these curves is taken as one position for the G pair, the configuration of the mechanism is set.

Obviously, since a point in a mechanism which exhibits a total mobility of one may only describe a single curve in space, when a point in a mechanism having a total mobility of one given a particular position, every other point in the mechanism is uniquely defined. Therefore, if

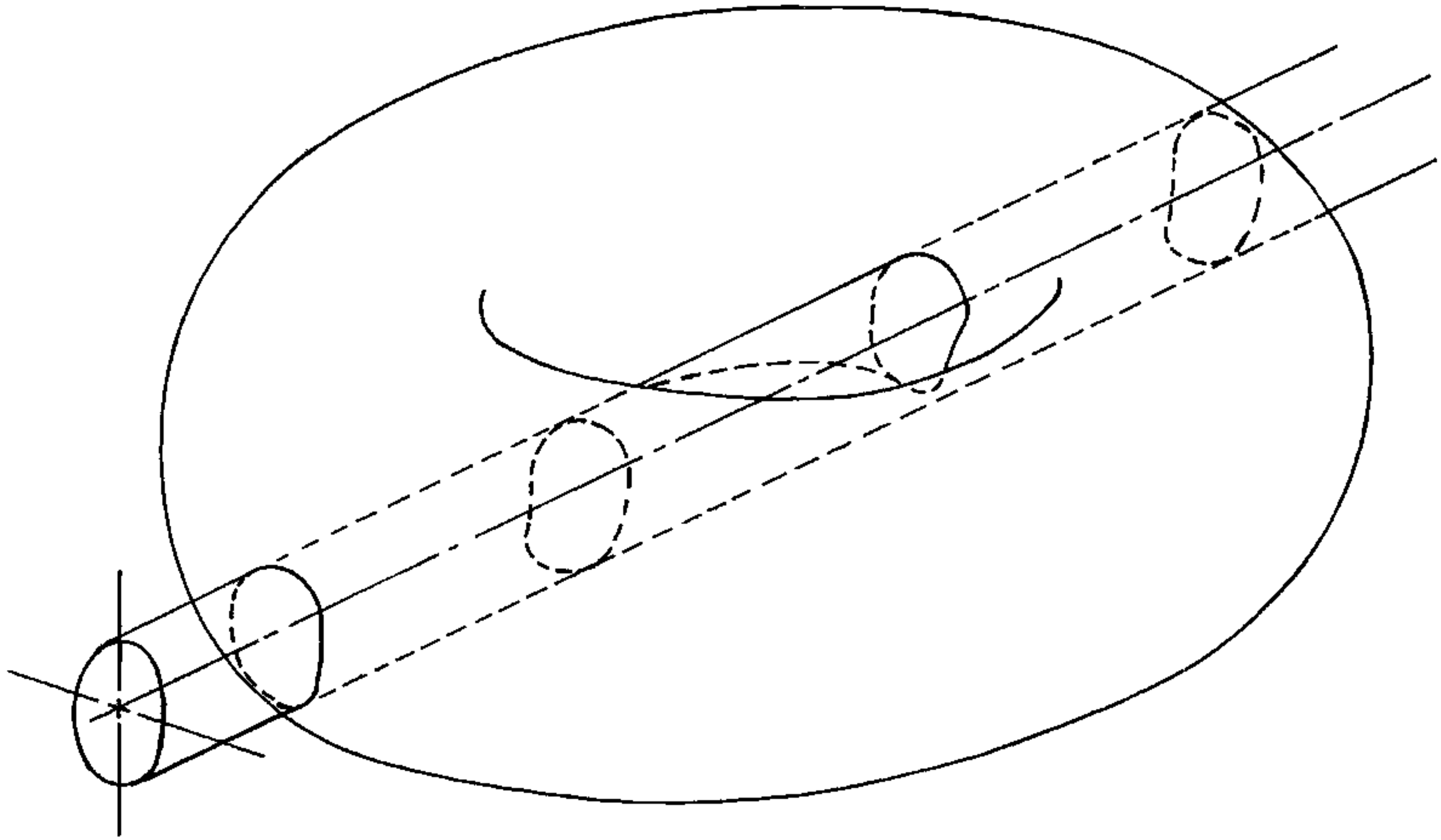


Figure 8. Toroidal Surface Intersected by a Cylinder (4 Curves of Intersection).

the gross motion of a single point in a mechanism is completely defined by a specific curve, the gross motion of the entire mechanism is completely defined. Moreover, since it both defines the gross motion of a point in a mechanism, and is readily available by means of simple algebraic manipulation, the curve of intersection of two generated surfaces in a mechanism is most appropriate for defining the properties of the gross motion of the generating mechanism.

CHAPTER IV

THE CURVE OF INTERSECTION

The curve of intersection of two surfaces is that curve, or those curves, made up of all points which simultaneously reside in both surfaces. Algebraically, the curve of intersection is the simultaneous solution of the equations of the two surfaces. Naturally, a meaningful solution of the equations can be obtained if, and only if, the two surfaces are defined in the same coordinate system. The choice of coordinate systems is completely arbitrary; the system may be oriented in any manner, and it may be defined in any convenient notation.

Since the three main categories of surfaces which may be generated by points in a mechanism, i.e. tori, hyperboloids of one sheet, and cylinders, are relatively simple and may be expressed in rectilinear coordinates in a Cartesian, three-dimensional, coordinate system, such a system will be used throughout this paper. The orientation of the coordinate system will depend upon the nature of the surfaces which appear in the problem. It has been found that the orientation of the system is best chosen such that the more complex of the two surfaces may be expressed in its simplest algebraic form. Once the more complex surface is situated in the coordinate system, the less complex surface is oriented such that its algebraic expression is simplified as much as possible. Consider the case where the center of a G pair in a mechanism generates the curve of intersection between a torus and a cylinder. Since the torus is the more complex surface, it is best to choose the center of the torus as the origin of the coordinate system. Moreover, the Z axis of the coordinate system is best chosen to coincide with the axis of revolution of the torus. The X and Y axes of the coordinate system are then oriented such that the direction of the Y axis, or the X axis, which ever is more convenient, is perpendicular to the axis of the cylinder. Such orientation can always be achieved, and, in general, will simplify the algebraic equations of the two surfaces as much as is possible.

For another example, if the G pair of a mechanism generates the curve of intersection between a torus and a hyperboloid of one sheet, then the coordinate system is best chosen in the following manner. Since the torus has a fourth degree, algebraic equation and the hperboloid has a second degree, algebraic equation, the torus is the more complex; therefore, the coordinate system is best oriented such that its origin corresponds to the center of the torus and its Z axis to the axis of revolution of the torus. This having been accomplished, the X and Y axes are best rotated until the Y axis is perpendicular to the axis of rotation of the hyperboloid. The equations for the surfaces, thus oriented in the coordinate system, are in their simplest possible forms.

Equations for Surfaces

The following equations define the three main categories of surfaces which may be generated by points in a mechanism. As the equations indicate, the surfaces are oriented such that they are symmetrically disposed about the origin and the major axes of the coordinate system in which they are defined. The Z axis of the system corresponds to the major axis of the surface.

Elliptic Cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where:

a = minor (major) radius

b = major (minor) radius

z is an arbitary value.

Note:

If a = b, the cylinder becomes a right circular cylinder of radius a.

Hyperboloid of One Sheet

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1$$

where:

a = minimum cross-sectional (throat) radius

$c = a(\cot \gamma)$

γ = the angle between the axis of revolution and the line generator of the hyperboloid.

Note:

If a = 0, the hyperboloid degenerates to a conical surface, and, if $\gamma = 0$, the hyperboloid degenerates to a right circular cylinder. If both of these conditions occur simultaneously the surface degenerates to a single line; the axis of the hyperboloid.

Torus (Circular Cross-Section)

$$4r_1^2 (x^2 + y^2) = (x^2 + y^2 + z^2 + r_1^2 - r_2^2)^2$$

where:

r_1 = mean radius of the torus

r_2 = cross-sectional radius of the torus

Note:

This equation is derived by rotating the circle $(y - r_1)^2 + z^2 = r_2^2$ about the z axis. The same method may be used to derive the equation of a torus generated by rotating a skew circle about the z axis (See Appendix A).

The Curves of Intersection of Surfaces

Since the curve(s) of intersection of two surfaces is the locus of points which satisfy the equations of both surfaces, points on the curve of intersection must satisfy one equation which is the sum of the two surface equations.

If $f(x, y, z) = 0$ is the equation of one surface, and if $g(x, y, z) = 0$ is the equation for another surface in the same coordinate system, then $f(x, y, z) + g(x, y, z) = 0$ is an equation for the curve of intersection of the two surfaces. However, in this form, the equation is of little practical use. Since the curve of intersection is, in a physical sense, a position curve, a more functional form of the equation is a vectorial representation. A position vector $\vec{R}$, such that $\vec{R}$ emanates from the origin of the coordinate system, and such that the tip of $\vec{R}$ always resides in the curve of intersection, can be used to define the curve. The transformation from the algebraic equations of the surfaces to the vector equation for the position vector equation for the position vector $\vec{R}$, which defines the curve(s) of intersection between two surfaces, can be obtained in the following manner.

Given that:

1. $f(x, y, z) = 0$ defines one surface
2. $g(x, y, z) = 0$ defines another surface

Then:

1. Solve both equations for the same variable.

$$F(x, y) = z \ , \ G(x, y) = z$$

2. Subtract the second equation from the first.

$$F(x, y) - G(x, y) = 0 = h(x, y)$$

3. Solve the resulting equation for one variable as a function of the remaining variable.

$$H(y) = x$$

Note:

Since the most complex equation involves fourth degree variables, this process can be accomplished by employing standard algebraic procedure.

4. Express the two dependent variables as functions of the independent variable.

$$x = H(y) \ , \ z = F(H(y), y) = G(H(y), y)$$

5. Define the position vector $\vec{R}$ as a function of the independent variable.

$$\vec{R}(y) \equiv H(y)\,\vec{i} + y\vec{j} + F(H(y), y)\,\vec{k}$$

where:

$\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the x, y, and z directions, respectively.

Note:

The coefficients of the $\vec{i}$, $\vec{j}$, and $\vec{k}$ unit vectors which make up $\vec{R}(y)$ are the coordinates x, y, and z, respectively, which define the point Q(x, y, z) on the curve of intersection for any value of the independent variable; only the real values of H(y) and F(H(y), y) are of interest.

Curve Analysis

The object of a detailed analysis of the curve of intersection is to locate end of travel, locking positions, and points of "uncertainty" in the gross motion of the generating mechanism. Since these conditions, as defined, require certain restrictions on the components of the velocity of a point in a body projected on the pair axis of the pair which connects the body to its neighbor, it is necessary that, once a point is determined, at least the direction of the velocity of the point be determined. Since the curve of intersection lies on two surfaces, the velocity of any point on the curve must lie along a line which is tangent to both surfaces. The direction of this line, for any point on the curve of intersection, can be found in the following manner.

Given that:

1. $f(x, y, z) = 0$ defines one surface.
2. $g(x, y, z) = 0$ defines another surface in the same coordinate system.
3. $\vec{R}(y) = H(y)\,\vec{i} + y\vec{j} + F(H(y), y)\,\vec{k}$ defines the curve of intersection between $f(x, y, z) = 0$ and $g(x, y, z) = 0$.
4. $Q(x, y, z)$ is a point on the curve defined by $\vec{R}(y)$.
5. $\nabla \equiv \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$, a vector operator such that

$$\nabla\,[f(x, y, z)] = \frac{\partial[f(x, y, z)]}{\partial z}\vec{i} + \frac{\partial[f(x, y, z)}{\partial y}\vec{j}$$

$$+ \frac{\partial[f(x, y, z)]}{\partial z}\vec{k},$$

a vector which is normal to the surface $f(x, y, z) = 0$.

Let:

a) $\nabla[f(x, y, z)]$ be denoted by the vector $\vec{N}_f$, and

b) $\nabla[g(x, y, z)]$ be denoted by the vector $\vec{N}_g$.

Then:

1. Evaluate $\vec{N}_g$ and $\vec{N}_f$ at the desired position $Q(x, y, z)$ on the curve defined by $\vec{R}(y)$.

Note:

$$\vec{N}_g = \vec{N}_g(y) \text{ and } \vec{N}_f = \vec{N}_f(y)$$

2. Calculate the vector cross product of $\vec{N}_g(y)$ and $\vec{N}_f(y)$

$$\vec{N}_f(y) \times \vec{N}_g(y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ N_f(y)_i & N_f(y)_j & N_f(y)_k \\ N_g(y)_i & N_g(y)_j & N_g(y)_k \end{vmatrix}$$

Let:

$\vec{N}_f(y) \times \vec{N}_g(y) \equiv \vec{V}_\ell(y)$, a vector which is tangent to $f(x, y, z) = 0$ and $g(x, y, z) = 0$ at $Q(x, y, z)$ on the curve defined by $\vec{R}(y)$ (Fig. 9).

Note:

The sense of $\vec{V}_\ell(y)$ is not important to the results of the analysis of the curve, in the context of this paper; hence it is inconsequential whether $\vec{N}_f(y) \times \vec{N}_g(y)$ or $\vec{N}_g(y) \times N_f(y)$ is used to define it.

With this "velocity"vector $\vec{V}_\ell(y)$, it is now possible to state the conditions for ends of travel, the locking positions, and the points of "uncertainty" in the gross motion of the generating mechanism in vector equations.

Points of "Uncertainty"

Points of "uncertainty" occur when $\vec{N}_f(y) \times \vec{N}_g(y) = 0$; for then $\vec{N}_f(y)$ and $\vec{N}_g(y)$ are parallel when evaluated at some $Q(x, y, z)$ on the curve defined by $\vec{R}(y)$. This condition arises when the two surfaces are internally tangent to one another. If the surfaces are externally tangent, then the curve exists only at a single point.

Ends of Travel

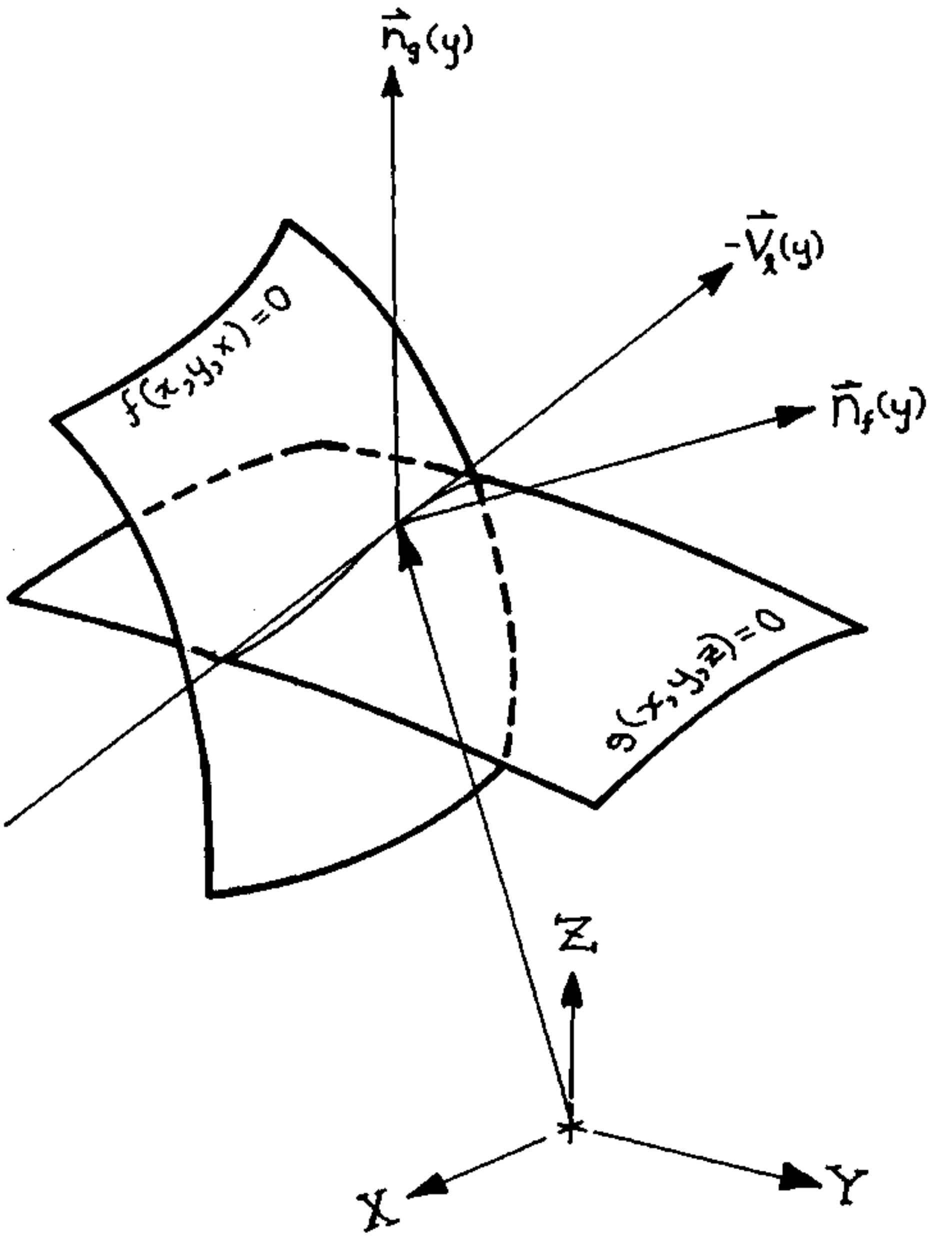


Figure 9. Cross-Product of Vector Quantities.

Let:

$\vec{a}$ = a unit vector along a permanent screw axis.

T = a point on the line of $\vec{a}$.

$\vec{T}$ = the position vector of T in the coordinate system.

$\vec{U} = \vec{R}(y) - \vec{T}$

$\vec{b} = \dfrac{\vec{U} - (\vec{U} \cdot \vec{a})\,\vec{a}}{|\vec{U} - (\vec{U} \cdot \vec{a})\,\vec{a}\,|}$, a unit vector from Q(x, y, z) along a line which intersects and is perpendicular to the line of $\vec{a}$.

then:

1. $\vec{a} \cdot \vec{V}_{\ell}(y) = 0$ defines an end of translational travel or dwell for the point Q(x, y, z) with respect to the axis which coincides with the line of $\vec{a}$.
2. $\vec{a} \times \vec{b} \cdot \vec{V}_{\ell}(y) = 0$ defines an end of angular travel or dwell of the point Q(x, y, z) relative to the axis which coincides with the line of $\vec{a}$.

Locking Positions

Locking positions depend upon the characteristics of the kinematic pairs which are present in a mechanism. There are three types of motion, pure rotation, pure translation, and a combination of the two, which may occur in any mechanism; therefore, locking conditions for each of the types of motion must be considered.

3. If $\vec{b} \times \vec{V}_{\ell}(y) = 0$, then the velocity acts in a line coincident with $\vec{b}$; this condition allows neither rotation about, nor translation along a screw axis. When a C pair appears in a mechanism, this condition must be satisfied, if the C pair is to lock the motion of the mechanism.

4. If $\vec{a} \cdot \vec{V}_{\ell}(y) = 0$, then the velocity acts in a plane which is perpendicular to $\vec{a}$; this condition allows no translation along the line of $\vec{a}$. Therefore, if $\vec{a}$ coincides with the axis of a P pari, this is a locking position for the P pair.
5. If $\vec{a} \times \vec{V}_{\ell}(y) = 0$, then the velocity acts in the direction of the line of $\vec{a}$; this condition allows no rotation about $\vec{a}$. Therefore, if $\vec{a}$ coincides with the axis of an R pair, this is a locking position for the R pair.

Notice that the equations for $\vec{R}$, Q, $\vec{U}$, $\vec{b}$, $\vec{V}_{\ell}$ and the expressions for the conditions of locking, "uncertainty", and ends of travel contain only the independent variable as an unknown. The independent variable may be solved for algebraically, by trail-and-error, or by any method which seems expedient.

Even though a detailed analysis of the curve(s) of intersection of two generated surfaces is necessary to obtain exact positions which agree with ends of travel, locking positions, and points of "uncertainty", a great deal of general information can be obtained from a brief look at the total curve(s). Primarily, the total curve defines the possible gross motion of a point in the mechanism which lies in two bodies of the mechanism, for every configuration of the mechanism. From the position information included in the curve, it is possible, by using computer methods, to predict the position of every link in the mechanism at any point in the mechanism's cycle. Secondly, it is possible to predict the class of motion, i.e. oscillatory or continuous, which is provided by the generating mechanism. Moreover, it is necessary only to consider the physical appearance of the curve(s) relative to the generated surfaces to determine the possible classes of motion which may be produced by the mechanism.

Consider, for example, an $(R)_{12}$ - $(P)_{22}$ - $(G)_{34}$ - $(c)_{41}$ mechanism in which body # 1 is the reference link; the kinematic chain consisting of bodies # 1, # 2, and # 3, when the G pair is disconnected, allows the generation of a hyperboloid of one sheet by a point Q" in body # 3 relative to body # 1; the kinematic chain consisting of bodies # 1 and # 4 allows the generation of a right circular cylinder by a point Q' in body # 4 relative to body # 1 and the point Q, the combination of Q' and Q " in their coincident state, is the center of the G pair which generates the curve of intersection of the two surfaces as the mechanism undergoes gross motion. Because the curve of intersection of the two surfaces is a function of the dimensions of the surfaces, the dimensions of the active links in their respective, generating kinematic chains, and the relative orientation of the two surfaces, the dimensions of the reference link and the angles which locate $\$_{12}$ and $\$_{14}$, it is not necessary to consider the dimensions of the mechanism, per se.

(a) If the curve of intersection between the hyperboloid and the right circular cylinder appears as one continuous curve on either surface such that there are no "branch" points in the curve, then no link in the mechanism can achieve a 360° rotation (Fig. 10). Therefore, only a double oscillation (a double-rocker-motion) may be produced by this mechanism, unless the axis of the cylinder is parallel to a line generator of the hyperboloid.

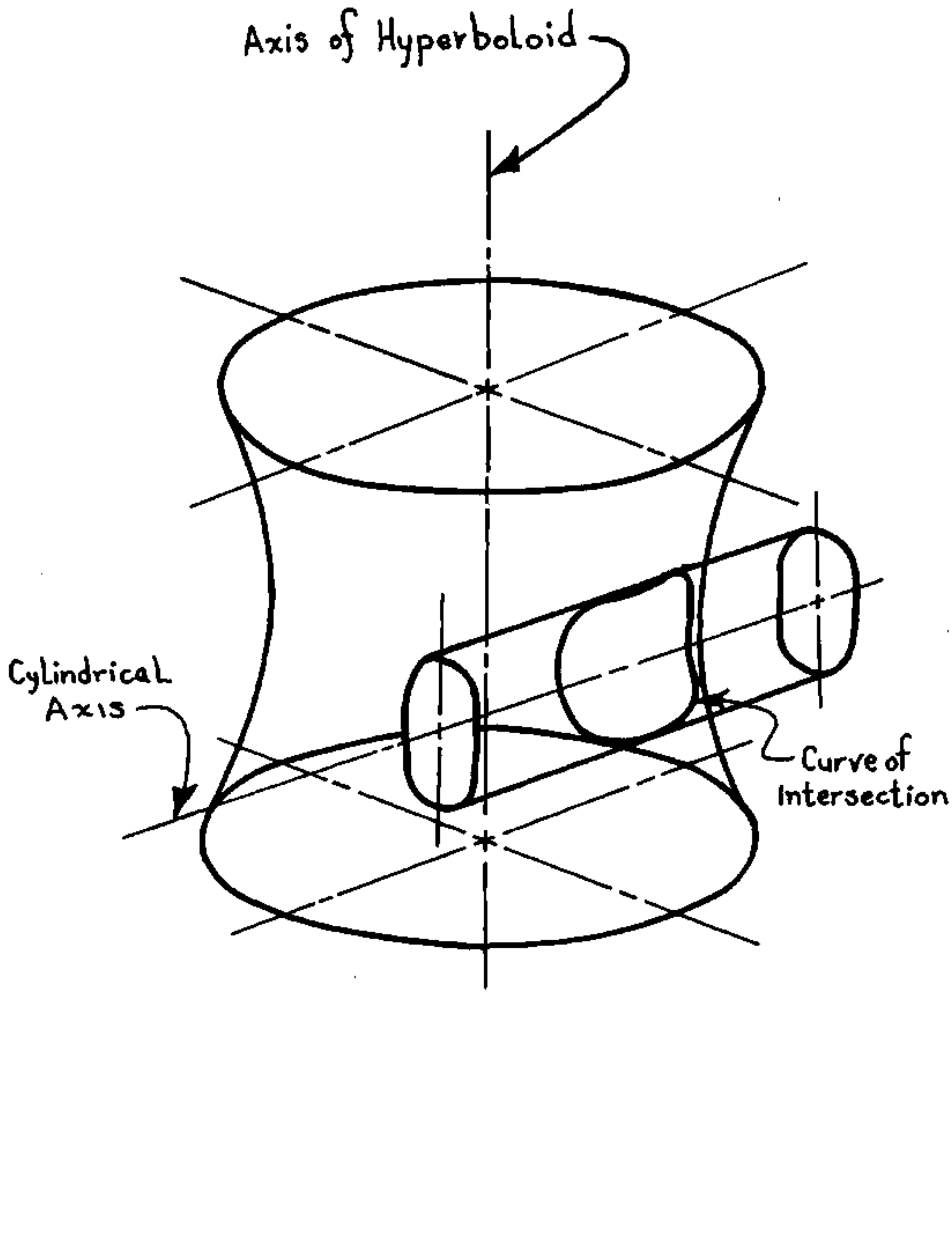


Figure 10. Surface Orientation for Double-Rocker Motion; Type A.

(b) If the intersection of the hyperboloid and the right circular cylinder appears as two continuous curves such that neither shares a point with the other, then one of the links in the mechanism can achieve a 360° rotation (Fig. 11). Therefore, only the combination of one oscillation and one continuous rotation may be produced by this mechanism, unless the angle between the cylindrical axis and the axis of the hyperboloid is less than the angle between the line generator of the hyperboloid and the axis of the hyperboloid.

(c) If the intersection of the hyperboloid and the right circular cylinder appears as one continuous curve which includes a single "branch" point, then one link in the mechanism may achieve a 360° rotation; thus a "crank-rocker" motion may be produced by this mechanism (Fig. 12).

(d) If the intersection of the hyperboloid and the right circular cylinder appears as one continuous curve which contains two "branch" points, then there are two links in the mechanism which can achieve 360° rotations; thus it is a "double-crank". Notice that if this conditions arises, the axis of the cylinder must intersect the axis of the hyperboloid.

If, in case (a), the axis $\$_{14}$ is parallel to a line generator of the hyperboloid, then the radius of the cylinder can negotiate a 360° rotation, and a "crank-rocker" results. In case (b), if the angle between the cylindrical axis and the hyperboloid's axis is less than the angle between the line generator of the hyperboloid and the axis of

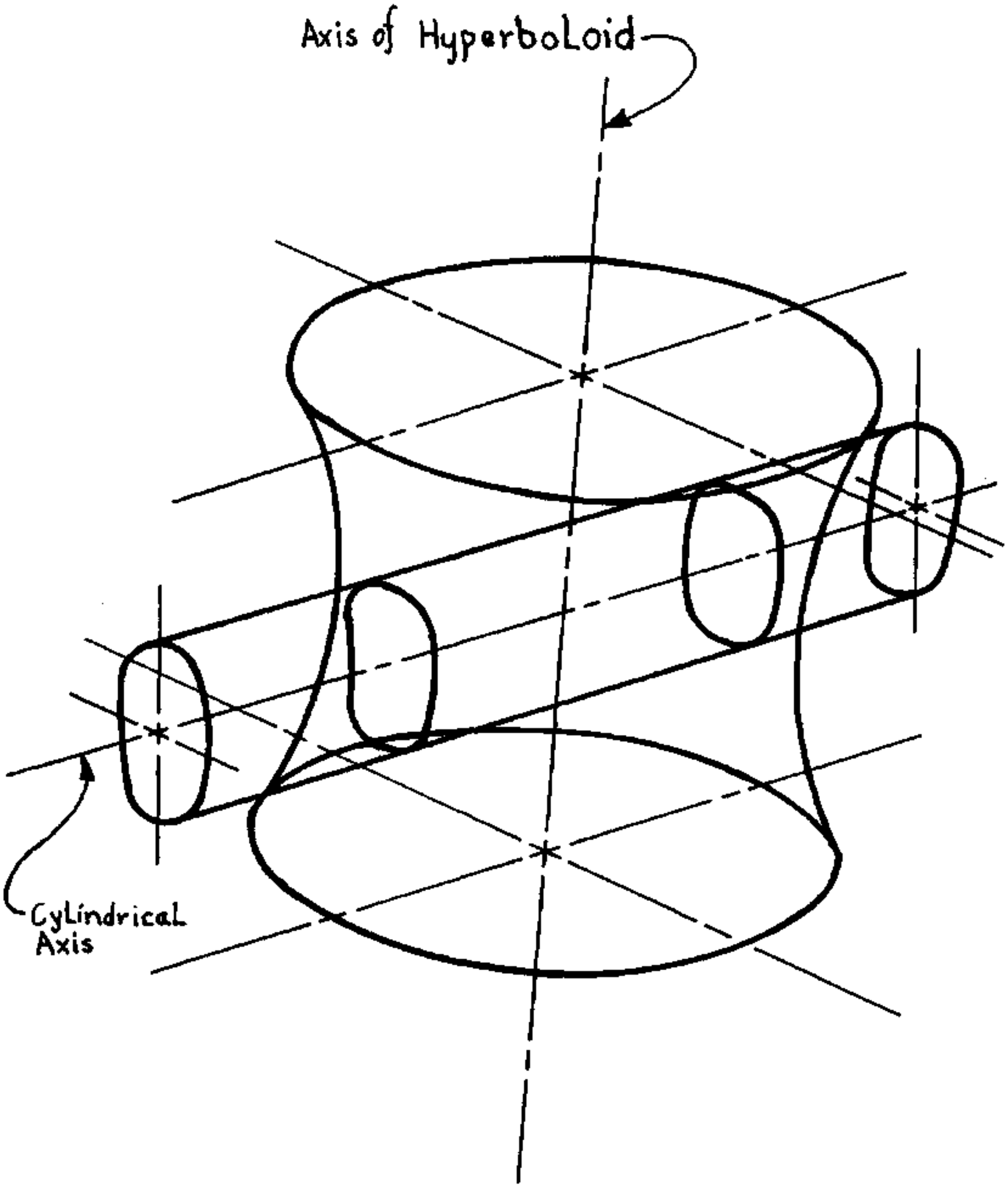


Figure 11. Surface Orientation for Crank-Rocker Motion; Type B.

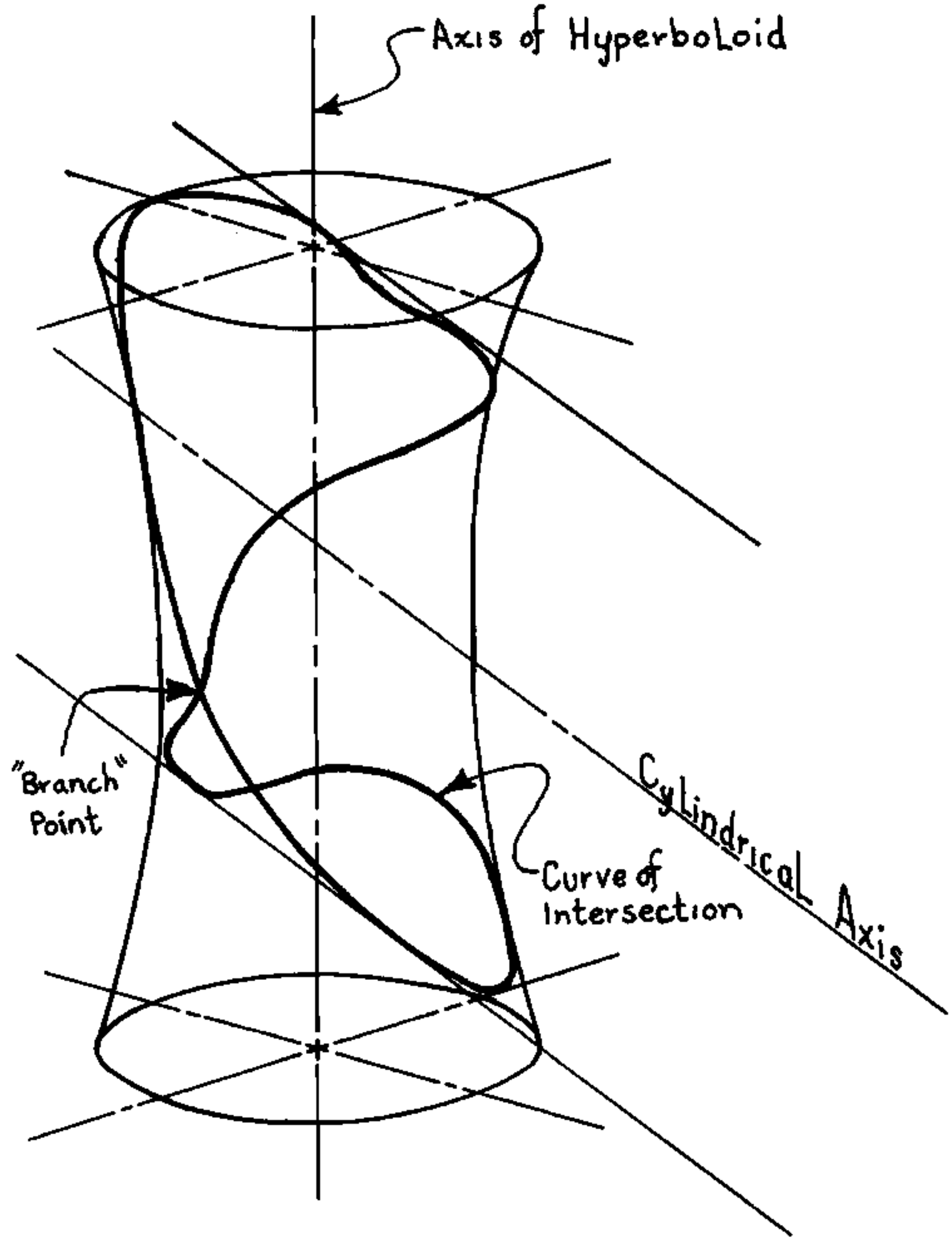


Figure 12. Surface Oreintation for Crank-Rocker Motion; Type C.

the hyperboloid, both the C pair and the R pair may achieve 360° rotations, if $\$_{14}$ passes through the throat area of the hyperboloid, or if the radius of the cylinder is greater than the perpendicular distance between $\$_{12}$ and $\$_{14}$, plus the throat radius of the hyperboloid.

Closure

The application of the "generated surface method", to those mechanisms to which it applies, reduces the components of the problem from five or less links and five or less pairs to two surfaces. Once these surfaces and their relative orientation are completely defined, then the dimensions of the links and the sequence of the pairs in a mechanism which is capable of generating the two surfaces are completely defined.

The mechanism may consist of any two kinematic chains (Table 1) which are capable of generating the desired surfaces with respect to the same reference link, and therefore the designer has some latitude in his choice of joints, except that, as always, the end links of the chains are connected, one to the other, by a G pair.

There may be more than one curve of intersection between the generated surfaces, each of which indicates a distinct configuration for the generating mechanism. However, once the G pair is connected the center of the G pair, Q, is restricted to move on a particular curve of intersection between the two generated surfaces as the mechanism undergoes gross motion. It may be possible, depending upon the orientation of the two surfaces, to alter the gross motion of the mechanism radically, or subtly, by choosing alternative configurations for the mechanism.

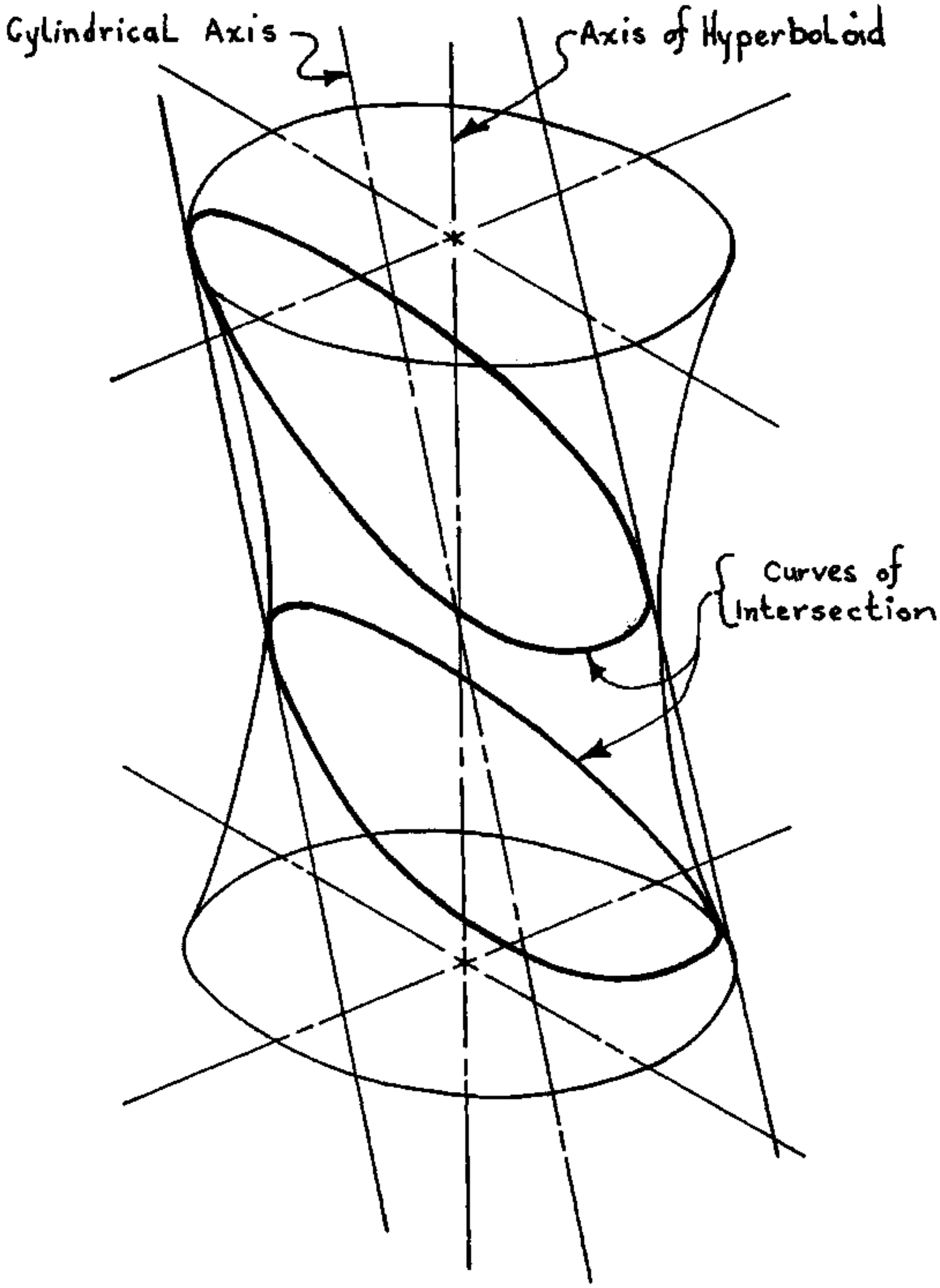


Figure 13. Surface Orientation for Double-Crank Motion; Type D.

CHAPTER V

EXAMPLES

General Problem

Consider an $(R)_{12}$ - $(P)_{23}$ - $(G)_{34}$ - $(C)_{41}$ mechanism (Fig. 14) in which body # 1 is the reference link. Body # 2 is shown as the common perpendicular between $\$_{12}$ and $\$_{23}$, and has a length "a". The angle between $\$_{12}$ and $\$_{23}$ is γ, where $0 < \gamma < 90°$. The angle between the axis $\$_{14}$ and the xy - plane is ξ. The axis $\$_{14}$ is perpendicular to the Y axis and pierces the yz - plane at the point (C_o, d_o).

If the G pair is disconnected, then any point Q' in body # 3 is capable of generating a hyperboloid of one sheet with respect to body # 1. Also, when the G pair is disconnected, any point Q" in body # 4 is capable of generating a right circular cylinder with respect to body # 1. The hyperboloid is the more complex of the two surfaces, and therefore is symmetrically disposed about the origin and axes of the coordinate system. The equation for this hyperboloid is;

$$\frac{x^2 + y^2}{a^2} - \frac{z^2 \tan^2 \gamma}{a^2} = 1$$

or

1. $$\cot \gamma (x^2 + y^2 - a^2)^{\frac{1}{2}} = z$$

If the perpendicular distance from $\$_{14}$ to Q, the center of the G pair, is "r", then the equation for the cylinder by Q" in body # 4 is:

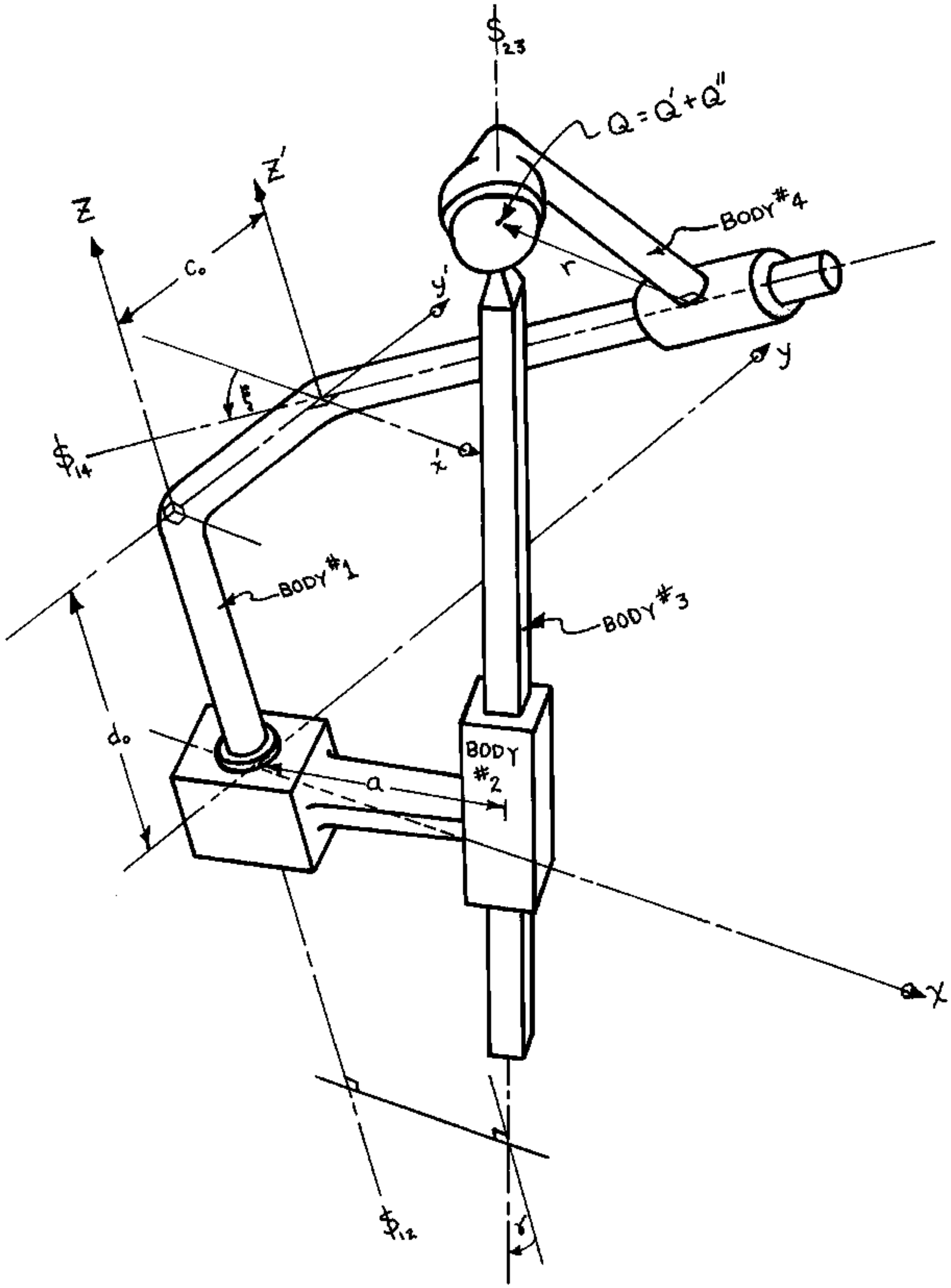


Figure 14. R - P - G - C Mechanism.

$$\frac{[(z - (d_o + x \tan \xi)]^2}{r^2 \cos^2 \xi} + \frac{[y - c_o]^2}{r^2} = 1$$

or

2. $$\cos\xi[r^2 - (y - c_o)^2]^{\frac{1}{2}} + d_o + x \tan \xi = z$$

By solving these two equations simultaneously, the equation for the path of Q, the combineation of Q' and Q" in their coincident state, is obtained.

$$\cot \gamma(x^2 + y^2 - a^2)^{\frac{1}{2}} = \cos \xi[r^2 - (y - c_o)^2]^{\frac{1}{2}} + d_o + x \tan \xi$$

from which:

$$x = \frac{-k_2 \pm \sqrt{k^2 - 4k_1 k_3}}{2k_1}$$

where:

$$k_1 = 1 - \frac{\tan^2 \xi}{\cot^2 \gamma}$$

$$k_2 = - \frac{2\tan \xi}{\cot^2 \gamma}[\ d_o + (r^2 - [y - c_o]^2)^{\frac{1}{2}}]$$

$$k_3 = y^2 - a^2 + \frac{\cos^2 \xi}{\cot^2 \gamma}[(y - c_o)^2 - r^2] - \frac{2d_o(r^2 - [y - c_o]^2)^{\frac{1}{2}} + d_o^2}{\cot^2 \gamma}$$

Then, substituting for x in equation 1:

$$z = \cot\gamma \left[\left(\frac{-k_2 \sqrt{k_2^2 - 4k_1k_3}}{2k_1} \right)^2 + y^2 - a^2 \right]^{\frac{1}{2}}$$

Let:

$$x = f_1(y) \;,\; z = f_2(y)$$

Then:

3. $$\vec{R}(y) = f_1(y)\,\vec{i} + y\vec{j} + f_2(y)\vec{k}$$

Therefore, the path of Q is completely defined as a function of the independent variable y.

Having defined the path of Q as a function of y, it next becomes necessary to determine the direction of the velocity of any point Q, on the path, as a function of y. If $f = f(x, y, z)$ is the algebraic expression for the hyperboloid, then the gradient of f is ∇f, or:

4. $$N_h = \nabla f = \frac{2x}{a^2}\vec{i} + \frac{2y}{a^2}\vec{j} - \frac{2z\tan^2\gamma}{a^2}\vec{k}$$

Similarly the gradient for the cylinder, $\vec{N}_c$, is:

5. $$\vec{N}_c = \frac{-2\tan\xi\,[z - (d_o + \tan\xi)]}{r^2\cos^2\xi}\vec{i} + \frac{2(y - c_o)}{r^2}\vec{j}$$

$$+ \frac{2[z - (d_o + x\tan\xi)]}{r^2\cos^2\xi}\vec{k}$$

By substituting $f_1(y)$ and $f_2(y)$ for x and z, respectively, $\vec{N}_n$ and $\vec{N}_c$ become functions of the independent variable y.

$$6. \qquad \vec{N}_h = \frac{2f_1(y)}{a^2}\,\vec{i} + \frac{2y}{a^2}\,\vec{j} + \frac{-(2\tan^2\gamma)f_2(y)}{a^2}\,\vec{k}$$

$$7. \quad \vec{N}_c = \frac{-2\tan\xi[f_2(y) - (d_o + f_1(y)\tan\xi)]}{r^2\cos^2\xi}\,\vec{i} + \frac{2(y - c_o)}{r^2}\,\vec{j}$$

$$+ \frac{2[f_2(y) - (d_o + f_1(y)\tan\xi)]}{r^2\cos^2\xi}\,\vec{k}$$

Now the vector cross product of $\vec{N}_h(y)$ and $\vec{N}_c(y)$ gives the "velocity" vector $\vec{V}_\ell(y)$.

8.

$$\vec{V}_{\ell}(y) = \vec{N}_h \times \vec{N}_c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1(y) & y & \left\{(-\tan^2\gamma)f_2(y)\right\} \\ \left\{-\tan\xi[f_2(y)-d_o-f_1(y)\tan\xi]\right\} & \left\{(y-c_o)\cos^2\xi\right\} & \left\{f_2(y)-d_o-f_1(y)\tan\xi\right\} \end{vmatrix}$$

By multiplying $\vec{N}_h$ by a factor $a^2/2$, and by multiplying $\vec{N}_c$ by a factor $r^2\cos^2\xi/2$, the magnitudes of the two vectors were reduced, in order to simplify the matrix multiplication necessary to determine $\vec{V}_\ell(y)$. However, since only the direction of $\vec{V}_\ell(y)$ is of importance, the alteration of its magnitude will not affect the results of the problem.

$$9.\quad \vec{V}_\ell(y) = \left\{y\left[f_2(y)-d_o-f_1(y)\tan\xi\right] + f_2(y)\tan^2\gamma\cos^2\xi(y-c_o)\right\}\vec{i}$$

$$+\left\{-y\tan\xi\,[f_2(y)-d_o-f_1(y)\tan\xi] - f_1(y)[f_2(y)-d_o-f_1(y)\tan\xi]\right\}\vec{j}$$

$$+\left\{f_1(y)\,[(y-c_o)\cos^2\xi] + y\tan\xi\,[f_2(y)-d_o-f_1(y)\tan\xi]\right\}\vec{k}$$

The direction of $\vec{V}_\ell(y)$ corresponds to the direction of the velocity of any point on the path of Q on the curve defined by $\vec{R}(y)$, for corresponding values of y.

In order to determine ends of travel for the point Q on the curve defined by $\vec{R}(y)$ with respect to any permanent screw axis, it is necessary to define a unit vector along each of the permanent axes of the mechanism.

Let:

$\vec{a}_{14}$ = a unit vector along $\$_{14}$

10. $\vec{a}_{14} = (\cos\xi)\,\vec{i} + (\sin\xi)\,\vec{k}$

Let:

$\vec{a}_{12}$ = a unit vector along $\$_{12}$

11. $\vec{a}_{12} = \vec{k}$

It is also necessary that a vector from Q(x, y, z) on the curve defined by $\vec{R}(y)$ which intersects and is perpendicular to the screw axes be known.

The points $(0, c_o, d_o)$ and $(0, 0, 0)$ are points on the lines of $\vec{a}_{14}$ and $\vec{a}_{12}$, respectively, which are used to determine the directions of $\vec{b}_{14}$ and $\vec{b}_{12}$ are:

12. $$b_{14} = \frac{\left[\vec{R}(y) - (c_o\vec{j} + d_o\vec{k}\right] - \left[\vec{R}(y) - c_o\vec{j} - d_o\vec{k}) \cdot \vec{a}_{14}\right]}{\left|\left[\vec{R}(y) - (c_o\vec{j} + d_o\vec{k})\right] - \left[(\vec{R}(y) - c_o\vec{j} - d_o\vec{k}) \cdot \vec{a}_{14}\right]\right|}$$

13. $$\vec{b}_{12} = \frac{\vec{R}(y) - (\vec{R}(y) \cdot \vec{a}_{12})\,\vec{a}_{12}}{|\vec{R}(y) - (\vec{R}(y) \cdot \vec{a}_{12})\,\vec{a}_{12}\,|}$$

With equations 9-13 it is now possible to determine the points of "uncertainty", ends of travel, and locking positions, as defined in section Curve Analysis in vector equations, for all the links of the mechanism. Notice that equations relating body # 3 to body # 1 are not included. The ends of $\vec{N}_c$ has a zero $\vec{j}$ component, or, if this condition does not arise, with the ends of angular travel for body # 4.

Problem 1

Consider a mechanism of the type described in section "General Problem" whose dimensions are as follows:

$$a = 3'', \ r = 8'', \ c_o = 4'', \ d_o = 6'', \ \gamma = 15°, \ \xi = 10°$$

Equations Dealing With the Hyperboloid of One Sheet are:

$$(3.73)(x^2 + y^2 - 9)^{\frac{1}{2}} = z$$

$$\vec{N}_n = \frac{2x}{a^2}\vec{i} + \frac{2y}{a^2}\vec{j} - \frac{2z\tan^2\gamma}{a^2}\vec{k}$$

$$\vec{a}_{12} = \vec{k}$$

Equations Dealing With the Cylinder are:

$$(0.985)[48 + 8y - y^2]^{\frac{1}{2}} + 6 + (0.176)x = z$$

$$\vec{N}_c = \left\{\frac{-(0.176)(z - 6 - [0.176]x)}{(0.970)(32)}\right\}\vec{i} + \left\{\frac{y - 4}{32}\right\}\vec{j}$$

$$+ \left\{\frac{z - 6 - [0.176]x}{(0.970)(32)}\right\}\vec{k}$$

$$\vec{a}_{14} = (0.998)\,\vec{i} + (0)\,\vec{j} + (0.174)\,\vec{k}$$

Equations Dealing With the Curves of Intersection

$$x = \tfrac{1}{2}\left\{(0.025)\left[6 + (48 + 8y - y^2)^{\frac{1}{2}}\right] \pm \left[315.2 + 38.96\,y\right.\right.$$

$$\left.\left. - 8.86\,y^2 + 15.55(48 + 8y - y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}$$

$$z = (0.985)(48 + 8y - y^2)^{\frac{1}{2}} + 6 + (0.088)\Big\{(0.025)\Big[6 + (48 + 8y - y^2)^{\frac{1}{2}}\Big]$$

$$\pm \Big[315.2 + 38.96y - 8.86y - 8.86y^2 + 15.55(48 + 8y - y^2)^{\frac{1}{2}}\Big]^{\frac{1}{2}}\Big\}$$

$$\vec{V}_{\ell}(y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{2}\left\{0.025\,[6 + (48 + 8y - y^2)^{\frac{1}{2}}] \pm \left[315.2 + 38.96\,y - 8.86y^2 + 15.55(48 + 8y - y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\} & y & \frac{1}{13.88}\left\{(0.985)\left[48 + 8y - y^2\right]^{\frac{1}{2}} + 6 + 0.088\left\{0.025\,[6 + (48 + 8y - y^2)^{\frac{1}{2}}] \pm \left[315.2 + 38.96y - 8.86y^2 + 15.55\,(48 + 8y - y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}\right\} \\ (-0.175)(48 + 8y - y^2)^{\frac{1}{2}}\,(0.985) & (0.970)(y - 4) & (0.985)(48 + 8y) - y^2)^{\frac{1}{2}} \end{vmatrix}$$

If there are points of "uncertainty" in this mechanism, there must be some value y_o such that $\vec{V}_\ell(y) = 0$, and $x(y_o)$ and $z(y_o)$ are real values. There is no such y_o in this case; therefore, there are no points of "uncertainty" in the gross motion of this mechanism.

A brief sketch of the orientation of the surfaces generated by Q' and Q" in bodies # 3 and # 4, respectively, indicates that the motion produced by this mechanism is of the "crank-rocker" type in which body # 2 can undergo a 360° rotation. Therefore, it is necessary to consider ends of travel and locking positions for only bodies # 3 and # 4.

Consider first the ends of translational travel for body # 3 with respect to body # 2. Since any point in body # 2 must describe a circle in an xy-plane which is fixed with respect to body # 1, the ends of travel for body # 3 with respect to body # 2 must also be the ends of translational travel for body # 3 with respect to body # 1, and must, therefore, satisfy the condition that $\vec{V}_\ell(y) \cdot \vec{a}_{12} = 0$.

$$\vec{V}_\ell(y) \cdot \vec{a}_{12} = 0.970\ x(y - 4) + 0.176y\ (z - 6 - 0.176x) = 0$$

There are two points, of interest for the configuration of this mechanism, for which the coordinates of Q satisfy this relation; Q(-0.168, - 3.700, 8.107), and Q(3.936, 2.975, 14.730).

$$\vec{V}_\ell(-3.7) \cdot \vec{a}_{12} = 0.970(-0.168)(-3.7 - 4) + 0.176\ (-3.7)[8.107 - 6 - 0.176(-0.168)] \simeq 0$$

The ends of translational travel for body # 4 with respect to body # 1 are defined by the following equation.

$$\vec{V}_{\ell}(y) \cdot \vec{a}_{14} = 0,$$

or

$$0 = y\ (z - 6 - 0.176x) + 0.970(y - 4.0)\left[z\ \frac{0.970}{13.88}\ + x(0.175)(0.985)\right]$$

There are two points, of interest for the configuration of this mechanism, for which the coordinates of Q satisfy this relation; Q(-4.42, 0.05, 12.08), and Q(4.75, 0.75, 14.04).

$$\vec{V}_{\ell}(0.75) = (2.215)\ \vec{i} + (-35.170)\ \vec{j} + (-14.025)\ \vec{k}$$

$$\vec{a}_{14} = (0.985)\ \vec{i} + (0.175)\ \vec{k}$$

$$\vec{V}_{\ell}(0.75) \cdot \vec{a}_{14} = (2.215)(0.985) + (-14.025)(0.175) \simeq 0$$

The total relative translation of body # 4 with respect to body # 1 is a displacement of 9.86", measured along $\$_{14}$.

Moreover, since these two ends of travel are the only possible positions, for this configuration of the mechanism, at which $\vec{V}_{\ell}(y)$ is perpendicular to $\$_{14}$, these must also be the only possible locations for locking positions, for body # 4 with respect to body # 1. However, $\vec{b}_{14} \times \vec{V}_{\ell}(y)$ is not zero at either position. Therefore, the C pair never locks.

$$\vec{b}_{14} = (-0.159)\ \vec{i} + (-0.419)\ \vec{j} + (0.895)\ \vec{k}$$

$$\vec{V}_{\ell}(0.75) = (2.215)\,\vec{i} - (35.170)\,\vec{j} - (14.025)\,\vec{k}$$

$$\vec{b}_{14} \times \vec{V}_{\ell}(0.75) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (-0.159) & (-0.419) & (0.895) \\ (2.215) & (-35.170) & (-14.025) \end{vmatrix} \neq 0$$

The ends of angular travel for bodies # 2 and # 3 with respect to body # 1 are not defined since both rotate continuously about $\$_{12}$. The ends of angular travel for body # 4 with respect to body # 1, however, are defined by the following equations.

$$\vec{a}_{14} \times \vec{b}_{14} \cdot \vec{V}_{\ell}(y) = 0,$$

or

$$0 = -\left\{(x + 0.176y)(0.175x - 0.985[z - 6])(z - 6 - 0.176x)\right\}$$

$$+ 0.970y(y - 4)\left\{0.985x - z \quad \frac{0.175}{13.88}\right\}$$

There are two points of interest, for this configuration of the mechanism, which satisfy this condition; Q(0.152, - 3.705, 8.146), and Q(-0.664, 4.700, 13.852).

$$\vec{b}_{14} = (-0.046)\,\vec{i} + (-0.965)\,\vec{j} + (0.257)\,\vec{k}$$

$$\vec{V}_{\ell}(-3.705) = (12.237)\,\vec{i} + (1.704)\,\vec{j} + (2.518)\,\vec{k}$$

$$\vec{a}_{14} \times \vec{b}_{14} \cdot \vec{V}_{\ell}(y) = \begin{vmatrix} (0.985) & 0 & (0.175) \\ (-0.046) & (-0.965) & (0.257) \\ (12.237) & (1.704) & (2.518) \end{vmatrix} \simeq 0$$

The angular travel of body # 4 is, therefore confined between 85° and 165° measured from the +y axis about $\$_{14}$.

There is one other possibility for a locking position for this mechanism; the prismatic pair between bodies # 2 and # 3 may lock. In order to predict this condition, $\vec{a}_{23}$ must be known. It may be obtained in the following manner.

1. Let $\vec{c}$ be a unit vector in the xy-plane.
2. Determine the coefficients of the $\vec{i}$ and $\vec{j}$ components of $\vec{c}$ such that;

$$\vec{R}(y) \cdot \vec{c} = 1$$

3. Determine the coefficients for a vector $\vec{S}_{23} = \vec{R}(y) - \vec{a}c$, where "a" is the throat radius for the hyperboloid of one sheet.
4. $\vec{S}_{23}$ is a vector parallel to $\vec{a}_{23}$.

Having obtained $\vec{S}$, it is now necessary to locate the position for the point Q on the curve of intersection at which $\vec{V}_{\ell}(y) \cdot \vec{S} = 0$, in order to find locking positions for body # 3 with respect to body # 2. The point Q(-0.980, - 3.600, 8.288) is such a point.

$$\vec{R}(y) = (-0.980)\,\vec{i} + (-3.600)\,\vec{j} + (8.288)\,\vec{k}$$

$$\vec{c} = -\vec{i} + (0)\,\vec{j}$$

$$\vec{S}_{23} = \vec{R}(y) - a\vec{c} = (2.020)\,\vec{i} + (-3.600)\,\vec{j} + (8.288)\,\vec{k}$$

$$\vec{V}_{\ell}(-3.6) = (-13.3)\,\vec{i} + (3.99)\,\vec{j} + (5.56)\,\vec{k}$$

$$\vec{V}_{\ell}(-3.6) \cdot \vec{S} = (2.020)(-13.3) + (-3.6)(3.99) + (5.56)(8.288)$$

$$\vec{V}_{\ell}(-3.6) \cdot \vec{S} \simeq 0$$

When the center of the G pair is located at Q(-0.98), - 3.60, 8.29), the velocity of the center acts along a line which is contained in a plane which is perpendicular to $\vec{a}_{23}$. Therefore, a locking condition exists when the center of the G pair reaches this point.

Problem 2

Consider now a mechanism of the previous type which has the following dimensions;

$a = 3.00$, $r = 8.00$, $c_o = 5.00$, $d_o = 0.00$, $\gamma = 15.00°$, $\xi = 0.00°$

These dimensions indicate that when the point Q resides on the -Y axis, the hyperboloid of one sheet is internally tangent to the cylinder at this point, there should be a point of "uncertainty". Therefore, the velocity of the center of the G pair at Q(0, -3, 0) must be zero.

$$\vec{N}_h(-3) = (0)\,\vec{i} + (-3)\,\vec{j} + (0)\,\vec{k}$$

$$\vec{N}_c(-3) = (0)\,\vec{i} + (-6.79)\,\vec{j} + (0)\,\vec{k}$$

$$\vec{V}_\ell(-3) = \vec{N}_c(-3) \times \vec{N}_h(-3) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -6.79 & 0 \\ 0 & -3 & 0 \end{vmatrix} = (0)\,\vec{i} + (0)\,\vec{j} + (0)\,\vec{k}$$

The condition that $\vec{V}_\ell(y) = 0$ when Q resides at a point of "uncertainty" is therefore satisfied.

CHAPTER VI

CONCLUSIONS

The proposed method for obtaining ends of travel, locking positions, and points of "uncertainty" in the gross motion of spatial mechanisms is well suited for application in mechanisms comprised of five or less links and containing at least one G pair.

Very good approximations for the position of the G pair which corresponds to ends of travel, locking positions, and points of "uncertainty" for components of such mechanisms may be obtained easily by trial and error methods.

The simplicity of the mathematics involved in this method results in an easy understanding of the method, and an easy adapting of the method for computerized analysis.

If an S pair is present in the mechanism, the initial complexity of the problem increases.

The characteristics of the curve(s) of intersection, i.e. the closed curves, and the number of "branch" points, of the generated surfaces lead directly to a physical explanation of the type of input-output motions which can be produced by the generating mechanism.

APPENDIX

Derivation for the Equation of a Torus Created by Rotating a Skew Circle About a Remote Axis

Given: There is a circle of radius r_2 whose center lies on the Y axis at $y = r_1$, and which lies in a plane which contains the Y axis, and has been rotated some angle α about the Y axis.

Reg'd: The equation for the surface of the torus generated as the circle is rotated about the Z axis.

Derivation:

1. The circle is the curve of intersection between the plane $x = z\tan\alpha$ and the sphere $x^2 + (y - r_1)^2 + z^2 = r_2^2$.

   $\therefore$ The equation for y as a function of z which satisfies the conditions for points on the circle is;

$$y = r_1 \pm \sqrt{r_2^2 - z^2\sec^2\alpha}$$

2. As the circle is rotated about the Z axis, any point $(\bar{x}, \bar{y}, k)$ must remain some distance h from the Z axis.

$$h^2 = \bar{x}^2 + \bar{y}^2$$

3. Solving the equation for the sphere for the term $(x^2 + y^2)$;

$$x^2 + y^2 = r_2^2 - r_1^2 + 2r_1y - z^2$$

Since $\bar{x}$ and $\bar{y}$ have no particular relationship, and were taken randomly as points on the circle,

$$h^2 = r_2^2 - r_1^2 + 2r_1 y - z^2$$

4. Substituting from step 1 to find h as a function of z,

$$h^2 = r_2^2 + r_1^2 = z^2 \pm 2r_1 \sqrt{r_2^2 - z^2 \sec^2 \alpha}$$

5. However, from step 3, $h^2 = x^2 + y^2$, therefore

$$x^2 + y^2 = r_2^2 + r_1^2 - z^2 \pm 2r_1 \sqrt{r_2^2 - z^2 \sec^2 \alpha}$$

is the equation for the surface of the torus created by rotating the circle about the Z axis.

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