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A METHOD OF CALCULATING

THE PROPULSIVE EFFICIENCY OF A HELICOPTER

A THESIS

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A METHOD OF CALCULATING
THE PROPULSIVE EFFICIENCY OF A HELICOPTER

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LIST OF SYMBOLS

Symbol	Definition
a	Mean slope of the lift curve of the blade element
a_o	Coning angle
a_1	Lateral cyclic pitch
Δa_1	Increment of lateral cyclic pitch
A_o	Mean blade angle
ΔA_o	Increment of mean blade angle
b	Number of rotor blades
b_1	Longitudinal cyclic pitch
Δb_1	Increment of longitudinal cyclic pitch
c	Blade chord at radius r
c_{d_o}	Profile drag coefficient
c_1	Lift coefficient
C_{M_X}	Rolling moment coefficient
C_{M_Y}	Pitching moment coefficient
C_Q	Rotor torque coefficient
ΔC_Q	Increment of rotor torque
C_T	Rotor thrust coefficient
ΔC_T	Increment of rotor thrust coefficient
C_X	Rotor X-force coefficient
ΔC_X	Increment of rotor X-force coefficient
D_f	Total resultant drag force acting parallel to and in the direction of relative wind
ΔD_f	Increment of resultant drag force

f_a	Frontal area of helicopter
F_X	Component of the resultant force acting along the intersection of the XZ-plane and the tip path plane
K	Constants
Q	Rotor torque
r	Radius of blade element
R	Radius of blade tip
T	Rotor thrust acting perpendicular to tip path plane
U	Resultant component of velocity normal to blade axis
v_a	Freestream inflow velocity ratio, $\frac{V \sin \alpha_R}{\Omega R}$
Δv_a	Increment of freestream inflow velocity ratio
v_i	Non-dimensional induced velocity (Equation (A-2))
Δv_i	Increment of non-dimensional induced velocity
V	Velocity along flight path
w	Slope of the longitudinal variation of induced velocity (Equation (A-4))
Δw	Increment of change in slope
W	Weight of the helicopter
x	Non-dimensional radius measured along blade axis
y	Slope of lateral variation of induced velocity (Equation (A-3))
α	Blade element angle of attack measured from line of zero lift
α_R	Angle of attack of tip path plane, positive for up-flow through rotor
$\Delta\alpha_R$	Increment of change in angle of attack due to added increment of change
γ	Element of blade circulation
$\Delta\gamma$	Increment of element of blade circulation

Γ Circulation of blade element at radius r and aximuth angle Ψ

$\Delta\Gamma$ Increment of circulation of blade element at radius r and aximuth angle Ψ

δ Value of c_d at $c_1 = 0$

η Efficiency of the rotor system of the helicopter

θ Blade pitch angle

θ_1 Blade twist from root to tip, positive for lower angle at the tip

θ_y Angle between horizontal and rotor tip path plane, positive for clockwise or rearward tilt of the tip path plane

μ Inplane velocity ratio, $\dfrac{V \cos \alpha_R}{\Omega R}$

ρ Mass density of air

σ_n Defined as $\dfrac{1}{\pi R} \displaystyle\int_{x_1}^{x_2} c x^{n-1}\, dx$

φ Blade element inflow angle

φ_c Angle between flight path and horizontal, positive for descending flight

Ψ Aximuth angle of blade axis measured about Z-axis from X-axis

Ω Mean angular velocity of rotor blade about Z-axis

SUMMARY

The propulsive efficiency of the helicopter rotor system has in the past been obtained from flight data. This study presents a theoretical solution to an equation which expresses a propulsive efficiency as arbitrarily defined for a helicopter rotor system and which is derived from two-dimensional blade element theory.

The propulsive efficiency, as defined in the mathematical presentation, is expressed as the ratio of an incremental increase of drag to a corresponding incremental increase of shaft power as the limit of the drag increment approaches zero.

The development of an expression which would express the increment of power required in terms of the increment of drag added was made possible by utilizing accepted performance equations which have been published by earlier authors. Some of the advantages obtained by using these equations are that they permit the use of large angles and simplified inplane force and moment equations. The exact equation expressing the incremental increase of shaft power is restricted to the assumptions which condition the accepted performance equations.

The efficiency equation can be useful in calculating the efficiency of any conventional single rotor system. By calculating the efficiency of many similar systems, with only one or two rotor parameters changed in each system, it is believed that a highly effective and efficient rotor can be developed prior to building a prototype model.

A sample problem is included to reveal the mechanics of the equation. The efficiency factors calculated by no means validate the theoretical solution; however, they do compare very favorably with solutions obtained from using available flight data.

CHAPTER I

INTRODUCTION

It is the intent of this study to mathematically derive an expression from which a propulsive efficiency as arbitrarily defined for a helicopter rotor system can be calculated by means of a theoretical solution.

The propulsive efficiency is defined as the ratio of an incremental increase of drag to a corresponding incremental increase in shaft power as the limit of the drag increment approaches zero. Given any curve that represents the efficiency of a rotor system, any point on this curve can be expressed in terms of the power required versus drag force imposed on the system. Such a curve is represented in Figure 1. Obviously, any change in the drag force would change the rotor efficiency. Furthermore, for any change in drag force, the required change in power required can be calculated in terms of the unknown drag increment. When taking the limit of this drag increase it can be proven mathematically that as the limit of the drag increment approaches zero, the incremental change in efficiency likewise approaches zero. Requiring the drag increment to approach zero places the helicopter in a stabilized flight configuration and establishes the resultant drag-power ratio as a fixed constant. This constant represents the efficiency of the rotor.

The approach in calculating the change in power required utilizes a performance solution by Castles (1).[1] Some of the advantages that this

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[1]Numbers represent order of listing in Bibliography.

solution offers are that it permits the use of large angles and simplified inplane force and moment equations. The assumption is made that the induced velocity remains constant and that it is distributed in a triangular pattern along the blade radii with the mean thrust center located at the three-quarter radius. A further assumption is made that the profile drag coefficient is accurately represented by the first two terms of an even power series.

A typical force system is represented in Figure 2. Using this force system as an initial flight condition an incremental element of drag is added to the existing drag force. With the establishment of the requirement that equilibrium is to be maintained throughout the time interval that the element of drag is added a new force system is formulated and is represented in Figure 3. These two systems of equations and the use of the accepted performance solution make possible the development of the efficiency equation.

CHAPTER II

ANALYSIS

System of Equations

The efficiency equation has been defined as the ratio of an incremental increase of drag to a corresponding incremental increase in shaft power in the limit that $C_{d_p} \rightarrow 0$. In equation form this is represented by Equation (1).

$$\eta = \lim_{\Delta C_{d_p} \rightarrow 0} \frac{f_1(\Delta C_{d_p})}{g_1(\Delta C_{d_p})} \tag{1}$$

where the drag increment is represented by

$$f_1(\Delta C_{d_p}) = \frac{1}{2}\rho\pi R^2(\Omega R)^2\left\{(V\Delta C_{d_p}) + (C_{d_p}\ \Delta V)\right\} \tag{2}$$

and the power increment by

$$g_1(\Delta C_{d_p}) = \rho\pi R^5\Omega^3\Delta C_Q \tag{3}$$

when

R = radius of blade

Ω = mean angular velocity of rotor blade

V = freestream velocity

ΔC_{d_p} = drag increment

ΔC_Q = power increment .

The equation expressing the increment of shaft power is developed from performance equations derived by Castles (1). Those equations needed for the derivation of the efficiency equation are included without proof$_a$.[1]

Equations expressing the incremental changes to the various rotor parameters are developed as required.

Equation for Change in Angle of Attack

Consider that a single rotor has a balanced force system, which is illustrated in Figure 2. Further assume that the helicopter is in a level unaccelerated flight configuration. These two assumptions are not necessary to obtain a valid solution; however, they simplify the system of equations immensely.

The static force equations in the X and Z direction can be derived from this balanced system, and are represented by equations (4) and (5), respectively.

$$T \sin \alpha_R + F_X \cos \alpha_R = -D_f \tag{4}$$

$$T \cos \alpha_R - F_X \sin \alpha_R = W \tag{5}$$

where,

T = resultant rotor thrust acting perpendicular to the tip path plane

F_X = component of the resultant force acting along the intersection of the XZ-plane and the tip path plane

D_f = total resultant drag force acting parallel to and in direction of relative wind = $\frac{1}{2} \rho V^2 f_a$

W = weight of the helicopter

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[1]Subscripts indicate supplemental material is provided in Appendix; listings are alphabetical.

f_a = frontal area of helicopter

and

$$\alpha_R = (\theta_y + \varphi_c)$$

$$\sin(-\alpha_R) = -\sin\alpha_R$$

$$\cos(-\alpha_R) = +\cos\alpha_R$$

When an increment of drag is added to the system represented in Figure 2 the force structure becomes unbalanced until sufficient power is added to overcome the added increment of drag. This new force system is represented in Figure 3. The new force equations are represented by Equations (6) and (7), respectively.

$$[T + \Delta T][\cos\alpha_R \cos\Delta\alpha_R - \sin\alpha_R \sin\Delta\alpha_R] +$$
$$[F_X + \Delta F_X][-\sin\alpha_R \cos\Delta\alpha_R - \cos\alpha_R \sin\Delta\alpha_R] = W \qquad (6)$$

$$[T + \Delta T][\sin\alpha_R \cos\Delta\alpha_R + \cos\alpha_R \sin\Delta\alpha_R] +$$
$$[F_X + \Delta F_X][\cos\alpha_R \cos\Delta\alpha_R - \sin\alpha_R \sin\Delta\alpha_R] = -D_f - \Delta D_f \qquad (7)$$

where

Δ = increment of change to each variable in the force system and,

$$\sin -(\alpha_R + \Delta\alpha_R) = -\sin\alpha_R \cos\Delta\alpha_R - \cos\alpha_R \sin\Delta\alpha_R$$

$$\cos -(\alpha_R + \Delta\alpha_R) = \cos\alpha_R \cos\Delta\alpha_R - \sin\alpha_R \sin\Delta\alpha_R$$

The unknowns which require a solution are the change in thrust, ΔT, and the change in rotor angle, $\Delta\alpha_R$. The change in rotor angle of attack is represented by Equation $(B\text{-}10)_b$.

$$\Delta\alpha_R = \frac{-\Delta D_f \cos\alpha_R + \Delta F_X}{T - \Delta D_f \sin\alpha_R}$$

$$- \frac{[\Delta D_f \cos\alpha_R + \Delta F_X]^2 \, [F_X + \Delta F_X]}{[T - \Delta D_f \sin\alpha_R]^3} \tag{B-10}$$

Expressing Equation (B-10) in coefficient form, neglecting small quantities of second and higher orders, the equation is reduced to the following expression:

$$\Delta\alpha_R = \frac{-\Delta C_{d_p} \cos\alpha_R + \Delta C_X}{2C_T - \Delta C_{d_p} \sin\alpha_R} \tag{8}$$

where,

$$\Delta C_{d_p} = \frac{\Delta D_f}{\frac{1}{2}\rho\pi R^2 (\Omega R)^2}$$

$$\Delta C_X = \frac{\Delta F_X}{\frac{1}{2}\rho\pi R^2 (\Omega R)^2}$$

$$C_T = \frac{T}{\rho\pi R^2 (\Omega R)^2}$$

Equation for Change in Required Thrust

The equation expressing the increase in required thrust is likewise derived from Equations (4), (5), (6), and (7). The change in thrust due to the added increment of drag is represented by Equation $(C\text{-}6)_c$.

$$\Delta T = \frac{-[T\Delta D_f \sin\alpha_R + \Delta D_f \Delta F_X \cos\alpha_R + \Delta D_f F_X \cos\alpha_R + F_X \Delta F_X + \Delta F_X{}^2]}{T + \Delta D_f \sin\alpha_R}$$

$$-\left[\frac{T^2 \Delta D_f^2 \sin^2\alpha_R + T\Delta D_f^2 F_X \sin\alpha_R \cos\alpha_R + \Delta D_f^2 F_X^2 \cos^2\alpha_R}{(T + \Delta D_f \sin\alpha_R)^3}\right] \quad \text{(C-6)}$$

Expressing Equation (C-6) in coefficient form, and neglecting second and higher order terms, the ΔT equation is reduced to

$$\Delta C_T = \frac{-\Delta C_{d_p}[2C_T \sin\alpha_R + C_X \cos\alpha_R] + C_X \Delta C_X}{4C_T + 2\Delta C_{d_p} \sin\alpha_R} \quad (9)$$

when,

$$\Delta C_T = \frac{\Delta T}{\rho\pi R^2 (\Omega R)^2}$$

<u>Equation for Change in Component of Rotor Force Coefficient C_X</u>

Equation (A-31) provides a solution for obtaining the component of rotor force acting along the intersection of the XZ-plane and the tip path plane. This equation is shown below for further development.

$$C_X = -\frac{b}{2\pi}\int_0^{2\pi}\int_{x_1}^{x_2} \frac{dC_{XY}}{dx} \sin\Psi \, d\Psi \, dx \quad \text{(A-31)}$$

where $x = \frac{r}{R}$

When the drag disturbance is applied to the rotor system Equation (A-31) is expanded to

$$C_X + \Delta C_X = -\frac{b}{2\pi}\int_0^{2\pi}\int_{x_1}^{x_2} \frac{dC_{XY} + d\Delta C_{XY}}{dx} \sin\Psi \, d\Psi \, dx \quad (10)$$

In the limit as ΔC_{d_p} approaches zero the function is assumed to be linear;

therefore,

$$\Delta C_X = -\frac{b}{2\pi}\int_0^{2\pi}\int_{x_1}^{x_2} \frac{d\Delta C_{XY}}{dx} \sin\Psi \, d\Psi \, dx \tag{11}$$

The complexity of obtaining a solution to Equation (11) overshadows the obvious fact that the ΔC_X force is definitely a very small quantity and could be neglected without introducing any appreciable error into the final efficiency equation. Therefore, for the purpose of obtaining a solution to Equations (8) and (9) the ΔC_X force is assumed to be zero. It will be shown in the sample calculation that this assumption was justifiable.

<u>Equations for Change in Freestream Velocity Components</u>

All velocity components are illustrated in Figure 4. The inplane velocity ratio, μ, is changed so slightly by the addition of the drag element that it is assumed to be constant. The inplane velocity ratio is represented by

$$\mu = \frac{V \cos \alpha_R}{\Omega R} \tag{A-6}$$

Adding the increment of drag expands Equation (A-6) to

$$\mu + \Delta\mu = \frac{(V + \Delta V) \cos (\alpha_R + \Delta\alpha_R)}{\Omega R} \tag{12}$$

Neglecting second order terms and removing μ from Equation (12), $\Delta\mu$ reduces to

$$\Delta\mu = \frac{\Delta V \cos \alpha_R - V\Delta\alpha_R \sin \alpha_R}{\Omega R} = 0 \tag{13}$$

Rewriting Equation (13), ΔV can be solved for directly in terms of the change in angle of attack.

$$\Delta V = V \Delta\alpha_R \tan \alpha_R \tag{14}$$

The change in the freestream inflow velocity ratio is solved for in the manner prescribed above. Representing v_a by

$$v_a = \frac{V \sin \alpha_R}{\Omega R} \tag{A-7}$$

Adding the increment of drag expands the v_a equation to

$$v_a + \Delta v_a = \frac{(V + \Delta V) \sin (\alpha_R + \Delta\alpha_R)}{\Omega R} \tag{15}$$

Neglecting second order terms and substituting Equation (14) for ΔV reduces Equation (15) to

$$\Delta v_a = \frac{V \Delta\alpha_R}{\Omega R \cos \alpha_R} = - \frac{V \Delta C_{d_p}}{\Omega R \left[2C_T - \Delta C_{d_p} \sin \alpha_R \right]} \tag{16}$$

where

$$\cos^2 \alpha_R + \sin^2 \alpha_R = 1$$

<u>Equation for Change in Average Blade Angle per Revolution</u>

In considering a solution for the change in the average blade angle it is necessary to first consider the original thrust equation, (A-8), represented by

$$\frac{2C_T}{ab\sigma_3} = \left(A_o + \frac{3}{4}\theta_1\right) K_1 - \theta_1 K_2 + v_a K_4 - v_i + \frac{1}{2} y K_3 - a_1 K_3 \tag{A-8}$$

and the rolling moment equation (A-10), which is represented by

$$\frac{2C_{MX}}{ab\sigma_4} = 2A_oK_5 + 2\theta_1 \left(\frac{3}{4}\frac{\sigma_3}{\sigma_4} - 1\right)\mu - a_1K_6 + v_aK_7 - v_iK_5 + y \qquad (A\text{-}10)$$

where the K factors are represented by

$$K_1 = 1 + \frac{1}{2}\mu^2\frac{\sigma_1}{\sigma_3} \qquad (17)$$

$$K_2 = \frac{\sigma_4}{\sigma_3} + \frac{1}{2}\mu^2\frac{\sigma_2}{\sigma_3} \qquad (18)$$

$$K_3 = \mu\frac{\sigma_2}{\sigma_3} \qquad (19)$$

$$K_4 = \frac{\sigma_2}{\sigma_3} \qquad (20)$$

$$K_5 = \mu\frac{\sigma_3}{\sigma_4} \qquad (21)$$

$$K_6 = 1 + \frac{3}{4}\mu^2\frac{\sigma_2}{\sigma_4} \qquad (22)$$

$$K_7 = \mu\frac{\sigma_2}{\sigma_4} \qquad (23)$$

$$K_8 = 1 + \frac{1}{4}\mu^2\frac{\sigma_2}{\sigma_4} \qquad (24)$$

when

$$\sigma_n = \frac{1}{R\pi}\int_{x_1}^{x_2} cx^{n-1}\,dx \qquad (A\text{-}9)$$

By adding the increment of the required change of thrust to Equation (A-8) and solving for the total thrust required, Equation (25) is readily obtained:

$$\frac{2[C_T + \Delta C_T]}{ab\sigma_3} = [(A_o + \Delta A_o) + \frac{3}{4}\theta_1] K_1 - \theta_1 K_2 - (a_1 + \Delta a_1)K_3$$

$$+ (v_a + \Delta v_a) K_4 + \frac{1}{2} y K_3 - v_i \qquad (25)$$

When the basic parameters are removed from Equation (25) the desired change in thrust is expressed by

$$\frac{2\Delta C_T}{ab\sigma_3} = \Delta A_o K_1 - \Delta a_1 K_3 + \Delta v_a K_4 \qquad (26)$$

Similarly, by adding the respective parameter increments to the rolling moment equation, Equation (27) is obtained.

$$\frac{2[C_{M_X} + \Delta C_{M_X}]}{ab\sigma_4} = 2K_5(A_o + \Delta A_o) + 2\theta_1 \left(\frac{3}{4}\frac{\sigma_3}{\sigma_4} - 1\right)$$

$$- (a_1 + \Delta a_1)K_6 + (v_a + \Delta v_a) K_7 - v_i K_5 + y \qquad (27)$$

Again removing the original parameters from Equation (27) and remembering the ΣM about the mast for the air forces will equal zero. Equation (28) is developed.

$$2K_5 \Delta A_o - \Delta a_1 K_6 + \Delta v_a K_7 = 0 \qquad (28)$$

The desired change of ΔA_o is obtained from a simultaneous solution of Equations (26) and (28).

$$\Delta A_o = \frac{\dfrac{2\Delta C_T K_6}{ab\sigma_3} - \Delta v_a (K_4 K_6 - K_7 K_3)}{K_1 K_6 - 2K_5 K_3} \tag{29}$$

where

$$\Delta C_T = \frac{- 2\Delta C_{d_p} C_T \sin \alpha_R - \Delta C_{d_p} C_X \cos \alpha_R}{4C_T + 2\Delta C_{d_p} \sin \alpha_R} \tag{9}$$

and

$$\Delta v_a = \frac{- V\Delta C_{d_p}}{\Omega R[2C_T - \Delta C_{d_p} \sin \alpha_R]} \tag{16}$$

Equations for Change in Cyclic Pitch

The lateral angular component of the cyclic pitch is also obtained from a simultaneous solution of Equations (26) and (28).

$$\Delta a_1 = \frac{\Delta v_a (2K_4 K_5 - K_1 K_7) - \dfrac{4\Delta C_T K_5}{ab\sigma_3}}{2K_3 K_5 - K_1 K_6} \tag{30}$$

By applying the incremental changes to the air force pitching moment equation the change in the longitudinal component of the cyclic pitch angle can be determined. Expressing first the basic moment equation by

$$\frac{2C_{M_Y}}{ab\sigma_4} = - b_1 (1 + \frac{1}{4} \mu^2 \frac{\sigma_2}{\sigma_4}) + a_o \mu \frac{\sigma_3}{\sigma_4} + w \tag{A-15}$$

Then by adding the incremental changes to the variables and remembering that in steady flight the ΣM_Y about the mast equals zero, Equation (A-15) becomes

$$\frac{2(C_{M_Y} + \Delta C_{M_Y})}{ab\sigma_4} = 0 = -(b_1 - \Delta b_1)\, K_8 + (a_o - \Delta a_o)\, K_5 + w \qquad (31)$$

Subtracting Equation (A-15) from Equation (31) the increment of the longitudinal component can be solved for directly. That is,

$$\Delta b_1 = \frac{\Delta a_o K_5}{K_8} \qquad (32)$$

Since any change in the coning angle would be a small quantity of the second order, the change in the longitudinal component would also be of the second order. Hence, Equation (32) is reduced to

$$\Delta b_1 \simeq 0 \qquad (33)$$

Equation for Change in Power Required

Having derived all the equations for expressing the incremental changes created by the added drag disturbance the required increment of power necessary to maintain equilibrium can now be calculated. This is best accomplished by writing the rotor torque equation.

$$C_Q = -\frac{b}{2(2\pi)} \int_0^{2\pi} \int_{x_1}^{x_2} \left\{ \frac{dC_{XY}}{dx} \right\} x \, dx \, d\Psi \qquad (A\text{-}30)$$

Equation (A-30) represents the power required to maintain the steady-state condition represented in Figure 2. After the arbitrary increment of drag has been applied, the power required can be represented by

$$C_Q + \Delta C_Q = -\frac{b}{2(2\pi)} \int_0^{2\pi} \int_{x_1}^{x_2} \frac{d(C_{XY} + \Delta C_{XY})}{dx} \; x \; dx \; d\Psi \tag{34}$$

where in the limit as $\Delta C_{d_p} \longrightarrow 0$, Equation (34) can be assumed to be linear. This permits that part of the power equation required solely to balance the increment of drag to be removed from Equation (34).

$$\Delta C_Q = \frac{-\,b}{2(2\pi)} \int_0^{2\pi} \int_{x_1}^{x_2} \frac{d(\Delta C_{XY})}{dx} \; x \; dx \; d\Psi \tag{35}$$

The total increment of force acting along the chord of the airfoil and in the plane of rotation is represented by$_d$:

$$\Delta C_{XY} = \frac{a}{\pi R} \int_{x_1}^{x_2} c \left\{ [\frac{U \sin \varphi}{\Omega R}][\frac{2\,\Delta\Gamma}{ac\Omega R}] + [\;\Delta(\frac{U \sin \varphi}{\Omega R})(\frac{2\Gamma}{ac\Omega R})] \right\} dx$$

$$- \frac{\delta_o}{\pi R} \int_{x_1}^{x_2} c \left\{ v_a \; \Delta v_a - v_i \; \Delta v_a x \right\} dx$$

$$- \frac{\epsilon a^2}{2\pi R} \int_{x_1}^{x_2} 2c \left\{ \frac{2\Gamma}{ac\Omega R} \right\} \left\{ \frac{2\Delta\Gamma}{ac\Omega R} \right\} dx \tag{D-8}$$

where,

$$\frac{U \sin \varphi}{\Omega R} = v_a - v_i x \tag{A-32}$$

$$\frac{2\Gamma}{ac\Omega R} = \Upsilon_o + \Upsilon_1 x + \Upsilon_2 x^2 + (\Upsilon_3 + \Upsilon_4 x) \sin\Psi + (\Upsilon_5 + \Upsilon_6 x) \cos\Psi$$

$$+ \Upsilon_7 \sin 2\Psi + \Upsilon_8 \cos 2\Psi \tag{A-16}$$

$$\frac{2\Delta\Gamma}{ac\Omega R} = \Delta\Upsilon_o + \Delta\Upsilon_1 x + \Delta\Upsilon_2 x^2 + (\Delta\Upsilon_3 + \Delta\Upsilon_4 x)\ \sin\Psi$$

$$+ (\Delta\Upsilon_5 + \Delta\Upsilon_6 x)\ \cos\Psi + \Delta\Upsilon_7\ \sin 2\Psi + \Delta\Upsilon_8 \cos 2\Psi \quad (36)$$

$$\frac{\Delta U\ \sin\varphi}{R} \approx \Delta v_a \quad (37)$$

$$\Upsilon_o = v_a - \frac{1}{2}\, a_1\, \mu \quad (A\text{-}17)$$

$$\Upsilon_1 = A_o + \frac{3}{4}\,\theta_1 - v_i \quad (A\text{-}18)$$

$$\Upsilon_2 = -\,\theta_1 \quad (A\text{-}19)$$

$$\Upsilon_3 = (A_o + \frac{3}{4}\,\theta_1)\ \mu \quad (A\text{-}20)$$

$$\Upsilon_4 = -\,(a_1 - y - \theta_1\,\mu) \quad (A\text{-}21)$$

$$\Upsilon_5 = -\,a_o\,\mu \quad (A\text{-}22)$$

$$\Upsilon_6 = b_1 - w \quad (A\text{-}23)$$

$$\Upsilon_7 = \frac{1}{2}\, b_1\,\mu \quad (A\text{-}24)$$

$$\Upsilon_8 = \frac{1}{2}\, a_1\,\mu \quad (A\text{-}25)$$

and

$$\Delta\Upsilon_o = \Delta v_a - \frac{1}{2}\,\Delta a_1\,\mu \quad (38)$$

$$\Delta\Upsilon_1 = \Delta A_o \quad (39)$$

$$\Delta\gamma_2 = 0 \tag{40}$$

$$\Delta\gamma_3 = \Delta A_o \mu \tag{41}$$

$$\Delta\gamma_4 = -\Delta a_1 \tag{42}$$

$$\Delta\gamma_5 = -\mu\Delta a_o = 0 \tag{43}$$

$$\Delta\gamma_6 = \Delta b_1 = 0 \tag{44}$$

$$\Delta\gamma_7 = \frac{1}{2}\mu\Delta b_1 = 0 \tag{45}$$

$$\Delta\gamma_8 = \frac{1}{2}\mu\Delta a_1 \tag{46}$$

Differentiating Equation (D-8) with respect to x gives

$$\frac{d\Delta C_{XY}}{dx} = \frac{ac}{\pi R}\left[(v_a - v_i x)\left(\frac{2\Delta\Gamma}{ac\Omega R}\right)\right] + \frac{ac}{\pi R}\left[\left(\frac{2\Gamma}{ac\Omega R}\right)(\Delta v_a)\right]$$

$$- \frac{\delta_o c}{\pi R}\left[v_a \Delta v_a - v_i \Delta v_a x\right] - \frac{\varepsilon a^2 c}{\pi R}\left[\left(\frac{2\Gamma}{ac\Omega R}\right)\left(\frac{2\Delta\Gamma}{ac\Omega R}\right)\right] \tag{47}$$

The increment of power required, represented by Equation (34), can now be expressed in terms of the blade circulation distribution.

$$\Delta C_Q = -\frac{b}{2(2\pi)}\int_0^{2\pi}\int_{x_1}^{x_2}\left\{\frac{ac}{\pi R}\left[(v_a - v_i x)\left(\frac{2\Delta\Gamma}{ac\Omega R}\right)\right]\right.$$

$$+ \frac{ac}{\pi R}\left[\left(\frac{2\Gamma}{ac\Omega R}\right)(\Delta v_a)\right] - \frac{\delta_o c}{\pi R}\left[v_a \Delta v_a - v_i \Delta v_a x\right]$$

$$- \frac{\epsilon a^2 c}{\pi R} \left[\left(\frac{2\Gamma}{ac\Omega R} \right) \left(\frac{2\Delta\Gamma}{ac\Omega R} \right) \right] \Big\} \; x \; dx \; d\Psi \tag{48}$$

The amount of additional torque required to offset the added increment of drag is obtained from the double integration of Equation (48) and is represented by Equation $(49)_e$.

$$\Delta C_Q = - \frac{b\sigma_5}{2} \left(C_1 \Delta\Upsilon_o + C_2 \Delta v_a + C_3 \Delta\Upsilon_1 - C_4 \Delta\Upsilon_3 - C_5 \Delta\Upsilon_4 - C_6 \Delta\Upsilon_8 \right) \tag{49}$$

where,

$$C_1 = 2.5av_a - 1.667av_i - 2.5\epsilon a^2\Upsilon_o - 1.667\epsilon a^2\Upsilon_1 - 1.25\epsilon a^2\Upsilon_2 \tag{50}$$

$$C_2 = 2.5a\Upsilon_o + 1.667a\Upsilon_1 + 1.25a\Upsilon_2 - 2.5\delta_o v_a + 1.667\delta_o v_i \tag{51}$$

$$C_3 = 1.667av_a - 1.25av_i - 1.667\epsilon a^2\Upsilon_o - 1.25\epsilon a^2\Upsilon_1 - \epsilon a^2\Upsilon_2 \tag{52}$$

$$C_4 = 1.25\epsilon a^2\Upsilon_3 + .833\epsilon a^2\Upsilon_4 \tag{53}$$

$$C_5 = .833\Upsilon_3\epsilon a^2 + .625\epsilon a^2\Upsilon_4 \tag{54}$$

$$C_6 = 1.25\epsilon a^2\Upsilon_8 \tag{55}$$

Equation for the Propulsive Efficiency of the Rotor System

The efficiency equation has been defined as

$$\eta = \lim_{\Delta C_{d_p} \to 0} \frac{f_1(\Delta C_{d_p})}{g_1(\Delta C_{d_p})} \tag{1}$$

Substituting Equations (2) and (3) into Equation (1) gives

$$\eta = \lim_{\Delta C_{d_p} \to 0} \frac{\frac{1}{2\Omega R}\left[V\Delta C_{d_p} + C_{d_p}\,\Delta V\right]}{\Delta C_Q} \tag{56}$$

Substituting Equation (14) and neglecting small quantities of the second order the numerator of Equation (56) becomes

$$\frac{1}{2\Omega R}\left[V\Delta C_{d_p} + C_{d_p}\,\Delta V\right] = \frac{V\Delta C_{d_p}}{2\Omega R}\left[\frac{2C_T - C_{d_p}\sin\alpha_R}{2C_T - \Delta C_{d_p}\sin\alpha_R}\right] \tag{57}$$

then

$$\eta = \lim_{\Delta C_{d_p} \to 0} \frac{\frac{V\Delta C_{d_p}}{2\Omega R}\left[\frac{2C_T - C_{d_p}\sin\alpha_R}{2C_T - \Delta C_{d_p}\sin\alpha_R}\right]}{\Delta C_Q} \tag{58}$$

Multiplying the numerator and denominator by the expression, $4C_T^2 - \Delta C_{d_p}^{\ 2}\sin^2\alpha_R$, gives

$$\eta = \lim_{\Delta C_{d_p} \to 0} \frac{\frac{V\Delta C_{d_p}}{2\Omega R}\left[2C_T - C_{d_p}\sin\alpha_R\right]\left[2C_T + \Delta C_{d_p}\sin\alpha_R\right]}{\Delta C_Q\left[4C_T^{\ 2} - \Delta C_{d_p}^{\ 2}\sin^2\alpha_R\right]} \tag{59}$$

where,

$$f_2(\Delta C_{d_p}) = \frac{(4C_T^{\ 2} - \Delta C_{d_p}^{\ 2}\sin^2\alpha_R)\; f_1(\Delta C_{d_p})}{2\Omega R} \tag{60}$$

and

$$g_2(\Delta C_{d_p}) = (4C_T{}^2 - \Delta C_{d_p}{}^2 \sin^2 \alpha_R)\ \Delta C_Q \tag{61}$$

Equation (59) can now be tested to determine if an indeterminate condition exists when ΔC_{d_p} equals zero. Since the equation takes on an indeterminate solution when the limit is reached it becomes necessary to exercise L'Hospital's Rule.

Differentiating the numerator of Equation (59) with respect to the drag increment yields:

$$f_2'(\Delta C_{d_p}) = \frac{V}{2\Omega R}\ (4C_T{}^2 + 4C_T \sin\alpha_R\ \Delta C_{d_p} - 2C_T C_{d_p} \sin\alpha_R - 2C_{d_p} \sin^2\alpha_R \Delta C_{d_p}) \tag{62}$$

Differentiating the denominator of Equation (59) with respect to the drag increment yields$_f$:

$$\begin{aligned} g_2'(\Delta C_{d_p}) &= 2P_1(C_T + \Delta C_{d_p} \sin \alpha_R) - 2P_2(C_T + \Delta C_{d_p} \sin \alpha_R) \\ &+ 2P_3(C_T - \Delta C_{d_p} \sin \alpha_R) + 2P_4(C_T - \Delta C_{d_p} \sin \alpha_R) \\ &- 2P_5(C_T + \Delta C_{d_p} \sin \alpha_R) \end{aligned} \tag{F-9}$$

where

$$P_1 = \frac{b\sigma_5}{2}\ [C_1 + C_2]\ [\tfrac{V}{\Omega R}] \tag{F-4}$$

$$P_2 = \frac{b\sigma_5}{2}\ [\tfrac{1}{2}\mu C_1 - C_5 + \tfrac{1}{2}\mu C_6][\tfrac{V}{\Omega R}]\ \left[\frac{2K_4K_5 - K_1K_7}{2K_3K_5 - K_1K_6}\right] \tag{F-6}$$

$$P_3 = \frac{b\sigma_5}{2}\left[\frac{1}{2}\mu C_1 - C_5 + \frac{1}{2}\mu C_6\right]\left[\frac{4K_5}{ab\sigma_3}\right]\left[\frac{C_T \sin\alpha_R + \frac{1}{2}C_X\cos\alpha_R}{2K_3K_5 - K_1K_6}\right] \quad \text{(F-6)}$$

$$P_4 = \frac{b\sigma_5}{2}\left[C_3 - \mu C_4\right]\left[\frac{2K_6}{ab\sigma_3}\right]\left[\frac{C_T \sin\alpha_R + \frac{1}{2}C_X\cos\alpha_R}{K_6K_1 - 2K_5K_3}\right] \quad \text{(F-7)}$$

$$P_5 = \frac{b\sigma_5}{2}\left[C_3 - \mu C_4\right]\left[\frac{V}{\Omega R}\right]\left[\frac{K_4K_6 - K_7K_3}{K_6K_1 - 2K_5K_3}\right] \quad \text{(F-8)}$$

A check of Equations (62) and (F-9) reveals that the indeterminate condition no longer exists. Therefore, the efficiency of rotor system can be represented by

$$\eta = \lim_{\Delta C_{d_p}\to 0} \frac{f_2'(\Delta C_{d_p})}{g_2'(\Delta C_{d_p})} = \frac{\left(\frac{V}{2\Omega R}\right)(2C_T - C_{d_p}\sin\alpha_R)}{P_1 - P_2 + P_3 + P_4 - P_5} \quad (63)$$

CHAPTER III

APPLICATION OF EFFICIENCY EQUATION

Procedure

The efficiency of the rotor design given below is calculated using the equations presented in this study. A known rotor head design, namely the HU-1A US Army Helicopter, was used to give a means of comparison with a helicopter for which flight data were readily available.

Rotor Data

The rotor data used in the HU-1A US Army Helicopter conformed to MIL-H-8501 dated 5 November 1952.

Gross Weight	6600 lbs.
Airfoil	0015
Effective Flat Plate Area (f_a)	20 ft^2
R	22 ft.
c	1.25 ft.
ΩR	700 ft/sec
$\sigma = \frac{bc}{\pi R}$ =	.0368
a_o	3 degrees = .0524 rad.
θ_1	- 12° = - .2094 rad.
$V_{normal\ cruise}$	100 knots = 168.9 ft/sec
a	2π
b	2

r_1	1.75'
r_2	22'

Outline Presentation

<u>Step 1</u>: Determine Drag Force Coefficient

$$D_f = \frac{1}{2}\rho V^2 f_a = 680 \text{ lbs.}$$

$$C_{d_p} = D_f/(\frac{1}{2}\rho\pi\Omega^2 R^4) = .000767$$

<u>Step 2</u>: Determine Flight Path Inclination

$$\alpha_R = (\varphi_c + \theta_y)$$

$$\varphi_c = \tan^{-1}\frac{0}{168.9} = 0$$

From Equation (A-33)

$$\tan\theta_y = \frac{-D_f\cos\varphi_c}{W - D_f\sin\varphi_c} = -\frac{680}{6600} = -.1031$$

$$\alpha_R = -5.88°$$

<u>Step 3</u>: Calculate Axial Velocity Ratio

$$v_a = \frac{168.9\sin(-5.88°)}{700} = -.0247$$

<u>Step 4</u>: Calculate Inplane Velocity Ratio

From Equation (A-6)

$$\mu = \frac{V\cos\alpha_R}{\Omega R} = .240$$

Step 5: Calculate Thrust Coefficient

From Equation (A-34)

$$C_T = .00376$$

Step 6: Calculate Induced Velocity

From Equation (A-2)

$$v_i = .01245$$

From Equation (A-3)

$$y = .00598$$

From Equation (A-4)

$$w = .00915$$

Step 7: Calculate Sigma Factors

From Equation (A-9)

$$\sigma_1 = .0165$$

$$\sigma_2 = .00898$$

$$\sigma_3 = .00603$$

$$\sigma_4 = .00452$$

$$\sigma_5 = .00362$$

Step 8: Calculate Rotor Force Acting Along X Axis

From Equation (A-32)

$$C_X = -.0000634$$

Step 9: Calculate Collective Pitch

From Equation (A-12)

$$A_o = .1761$$

Step 10: Calculate Cyclic

From Equation (A-13)

$$a_1 = .0965$$

From Equation (A-15)

$$b_1 = .0257$$

Step 11: Calculate K Factors

From Equation (17)

$$K_1 = 1.079$$

From Equation (18)

$$K_2 = .793$$

From Equation (19)

$$K_3 = .357$$

From Equation (20)

$$K_4 = 1.489$$

From Equation (21)

$$K_5 = .320$$

From Equation (22)

$$K_6 = 1.086$$

From Equation (23)

$$K_7 = .476$$

From Equation (24)

$$K_8 = 1.029$$

<u>Step 12</u>: Calculate C Constants

From Equation (50)

$$C_1 = -.575$$

From Equation (51)

$$C_2 = 1.142$$

From Equation (52)

$$C_3 = -.398$$

From Equation (53)

$$C_4 = .00262$$

From Equation (54)

$$C_5 = .000725$$

From Equation (55)

$$C_6 = .00455$$

Step 13: Calculate P Constants

From Equation (F-4)

$$P_1 = .000495$$

From Equation (F-5)

$$P_2 = .0000281$$

From Equation (F-6)

$$P_3 = -.00000219$$

From Equation (F-7)

$$P_4 = .00001808$$

From Equation (F-8)

$$P_5 = .000531$$

Step 14: Calculate Efficiency

From Equation (63)

$$\eta = \frac{\frac{1}{2}\,(.241)\,[2(.00376) - (.000768)(-\,.1)]}{.000495 - .0000281 - .00000219 + .00001808 + .000531}$$

$$= \frac{.000915}{.001018} = 90\%$$

CHAPTER IV

CONCLUSIONS

1. The equation for the rotor efficiency, which is based on the blade element theory, is derived in coefficient form and can be applied to any single rotor configuration.

2. While the efficiency equation is presently developed for the single rotor, it is believed that no appreciable error would be introduced if the same method as used to calculate the efficiency of the single rotor was likewise applied to the tandem rotor design.

3. The efficiency of the HU-1A helicopter was calculated to be ninety per cent. This compares favorably with the available flight data.

4. Optimum use of the efficiency equation would be achieved with the assistance of a digital computer. By varying the rotor parameters which are constant for a particular rotor configuration, it is believed that through a large number of iterations, a highly efficient rotor system could be developed.

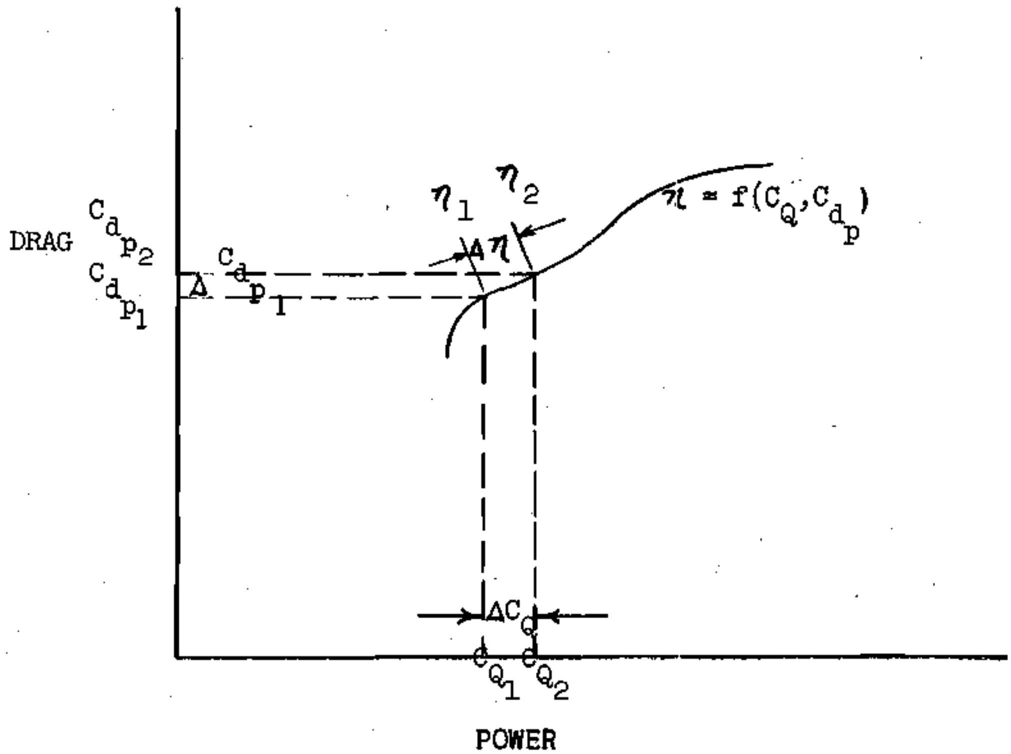
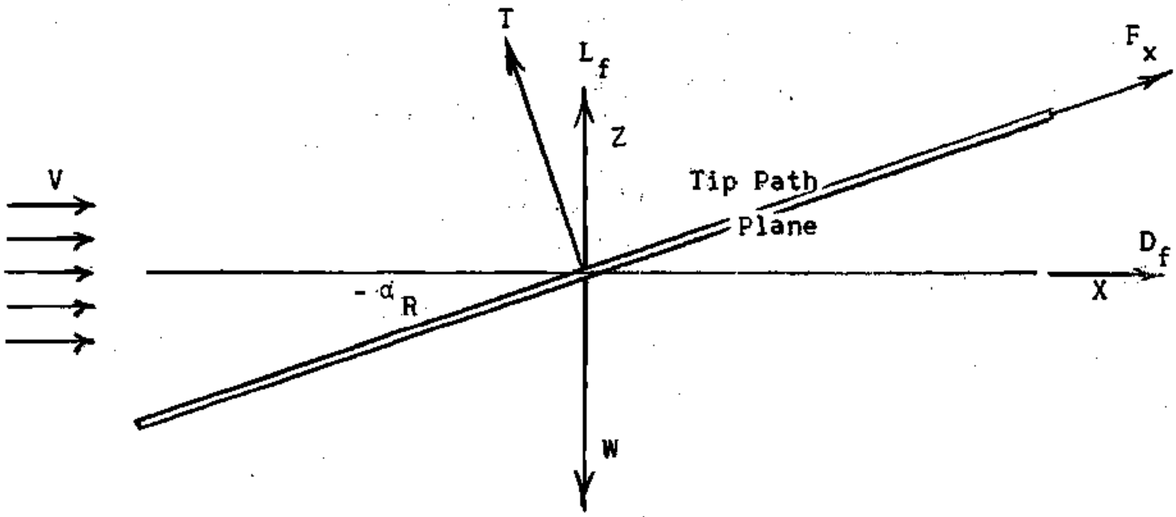


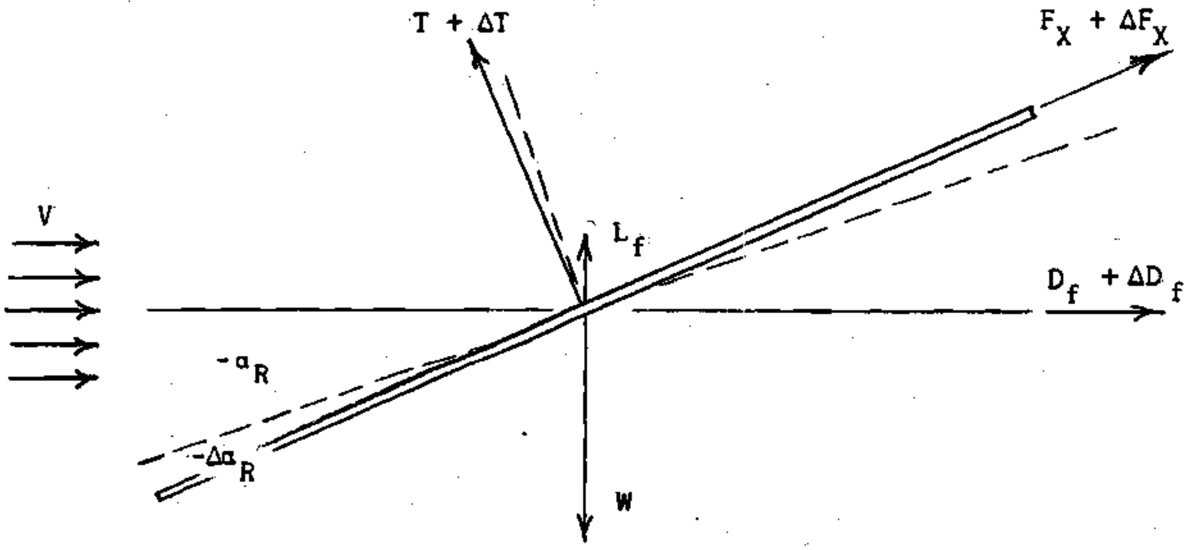
Fig. 1 Drag Power Graph of Efficiency Equation



$$T \sin \alpha_R + F_X \cos \alpha_R = - D_f \qquad \text{(X-Direction)}$$

$$T \cos \alpha_R + F_X \sin \alpha_R = W \qquad \text{(Z-Direction)}$$

Fig. 2 Balanced Force System Before Drag Disturbance



$$(T + \Delta T)(\sin \alpha_R \cos \Delta\alpha_R + \sin \alpha_R \sin \Delta\alpha_R) \quad \text{(X-Direction)}$$

$$+ (F_X + \Delta F_X)(\cos \alpha_R \cos \Delta\alpha_R - \sin \alpha_R \sin \Delta\alpha_R) = - (D_f + \Delta D_f)$$

$$(T + \Delta T)(\cos \alpha_R \cos \Delta\alpha_R - \sin \alpha_R \sin \Delta\alpha_R) \quad \text{(Z-Direction)}$$

$$+ (F_X + \Delta F_X)(- \sin \alpha_R \cos \Delta\alpha_R - \cos \alpha_R \sin \Delta\alpha_R) = W$$

Fig 3 Balanced Force System After Drag Disturbance

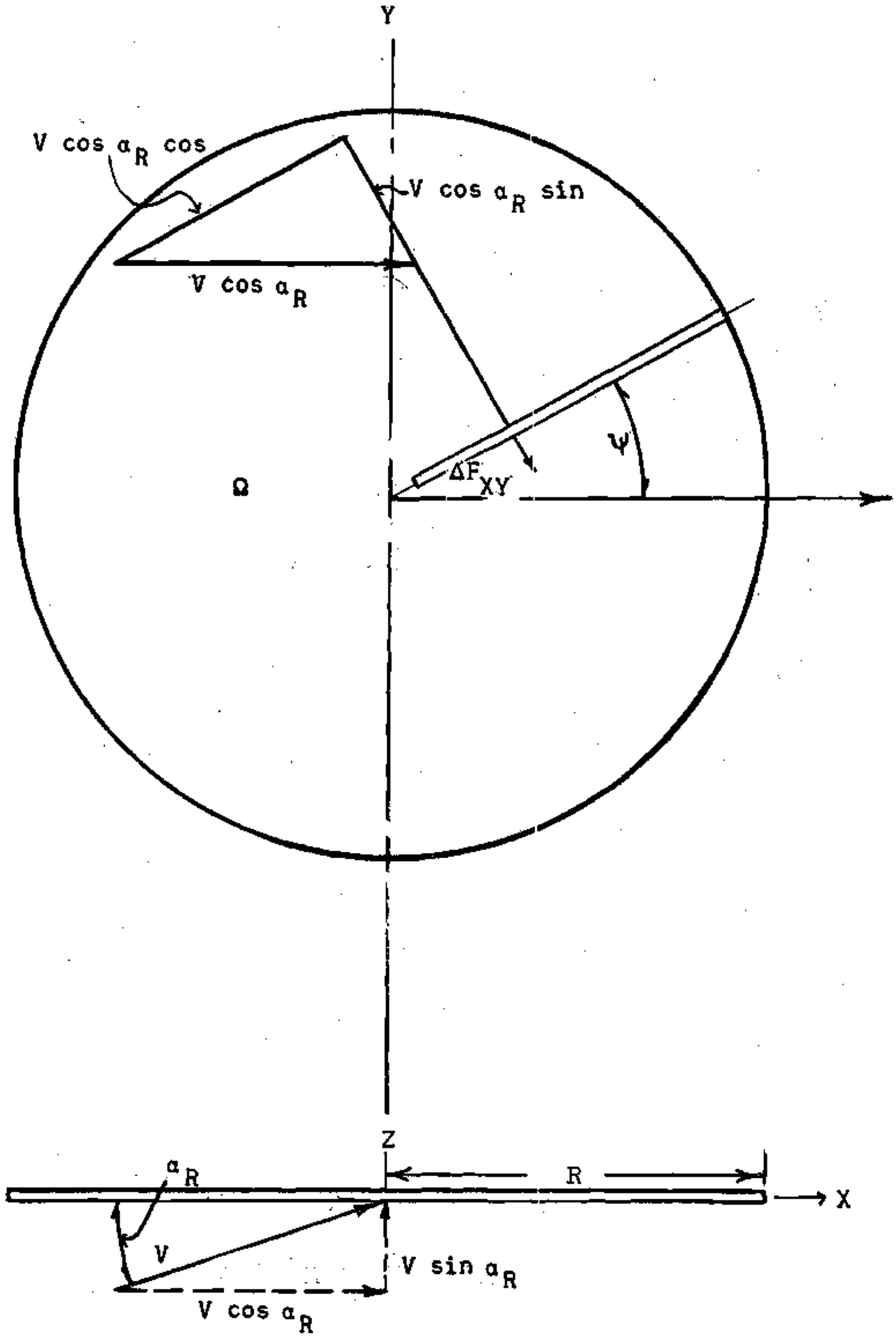


Fig. 4 Velocity Components at Blade Element

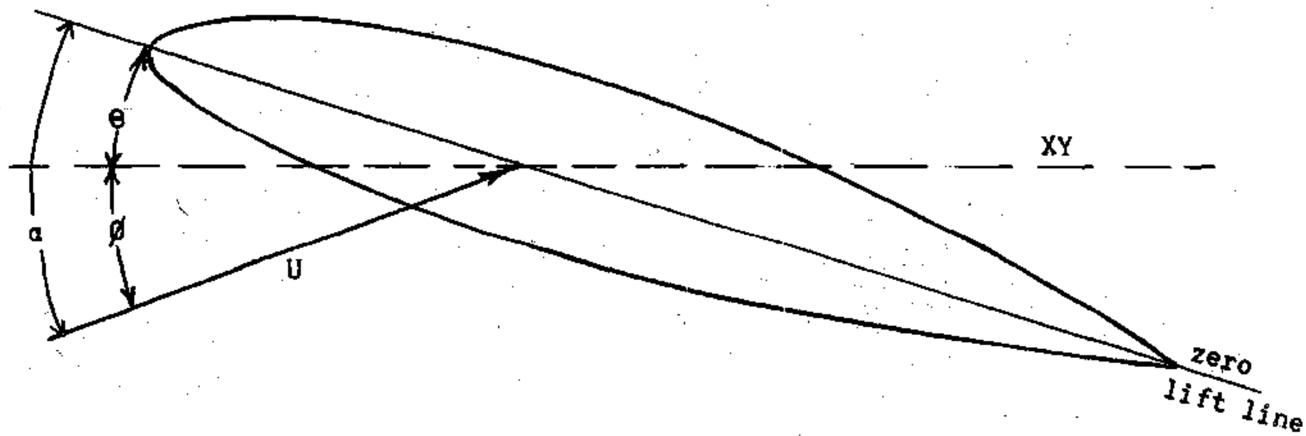


Fig. 5 Geometry of Blade Element

APPENDICES

APPENDIX A

PERFORMANCE EQUATIONS EXTRACTED FROM HELICOPTER PERFORMANCE ESTIMATION (1)

Equation for Induced Velocity

$$\frac{V_i}{\Omega R} = v_i x - yx \sin\Psi + wx \cos\Psi \qquad \text{(A-1)}$$

where

$$v_i = \frac{\frac{3}{4} C_T}{(1 - \mu^2)\sqrt{(v_a - .75v_i)^2 + \mu^2}} \qquad \text{(A-2)}$$

$$y = 2v_i \mu \qquad \text{(A-3)}$$

$$w = v_i \left\{ (1 - 1.8\mu^2)\sqrt{1 + \left(\frac{v_a - .75v_i}{\mu}\right)^2} - \sqrt{\left(\frac{v_a - .75v_i}{\mu}\right)^2} \right\} \qquad \text{(A-4)}$$

Coefficient of Rotor Thrust

$$C_T = \frac{T}{\rho \pi \Omega^2 R^4} \qquad \text{(A-5)}$$

Speed Ratio or Inplane Velocity Ratio

$$\mu = \frac{V \cos \alpha_R}{\Omega R} \qquad \text{(A-6)}$$

Freestream Inflow Velocity Ratio

$$v_a = \frac{V \sin \alpha_R}{\Omega R} \tag{A-7}$$

Thrust Equation

$$\frac{2C_T}{ab\sigma_3} = (A_o + .75\theta_1)\left\{1 + \frac{1}{2}\mu^2 \frac{\sigma_1}{\sigma_3}\right\} - \theta_1\left\{\frac{\sigma_4}{\sigma_3} + \frac{1}{2}\mu^2 \frac{\sigma_2}{\sigma_3}\right\}$$

$$- a_1 \mu \frac{\sigma_2}{\sigma_3} + v_a \frac{\sigma_2}{\sigma_3} - v_i + \frac{1}{2} y \mu \frac{\sigma_2}{\sigma_3} \tag{A-8}$$

and the blade area parameters

$$\sigma_n = \frac{1}{\pi R}\int_{x_1}^{x_2} c x^{n-1}\, dx \tag{A-9}$$

where,

$$x_1 = \frac{r_1}{R}$$

$$x_2 = \frac{r_2}{R}$$

Rolling Moment Equation

$$\left(\frac{2C_{M_X}}{ab\sigma_4}\right) = 2A_o \mu \frac{\sigma_3}{\sigma_4} + 2\theta_1 \left(\frac{3}{4}\frac{\sigma_3}{\sigma_4} - 1\right)\mu - a_1\left(1 + \frac{3}{4}\mu^2 \frac{\sigma_2}{\sigma_4}\right)$$

$$+ v_a \mu \frac{\sigma_2}{\sigma_4} - v_i \mu \frac{\sigma_3}{\sigma_4} + y \tag{A-10}$$

Pitching Moment Equation

$$\left(\frac{2C_{M_Y}}{ab\sigma_4}\right) = -\, b_1\left(1 + \frac{1}{4}\mu^2 \frac{\sigma_2}{\sigma_4}\right) + a_o\mu \frac{\sigma_3}{\sigma_4} + w \tag{A-11}$$

Collective Pitch Equation

$$A_o = \frac{\left\{\bar{\alpha} - \frac{3}{8}\theta_1 \mu^2 - \frac{3}{2} v_a + v_i\left(1 - \frac{3}{2}\mu^2\right)\right\}\left(1 + \frac{3}{2}\mu^2\right) + (3v_a + v_i)\mu^2}{\left(1 + \frac{3}{2}\mu^2\right)^2 - 4\mu^2} \tag{A-12}$$

Lateral Cyclic Equation

$$a_1 = \frac{\frac{8}{3}\left\{\bar{\alpha} - \frac{3}{8}\theta_1 \mu^2 - \frac{9}{4} v_a\left(1 + \frac{1}{2}\mu^2\right) + \frac{3}{4} v_i\left(1 - \frac{5}{2}\mu^2\right)\right\}\mu}{\left(1 + \frac{3}{2}\mu^2\right)^2 - 4\mu^2} \tag{A-13}$$

where,

$$\bar{\alpha} = A_o\left(1 + \frac{3}{2}\mu^2\right) + \frac{3}{8}\theta_1 \mu^2 - \frac{3}{2} a_1 \mu + \frac{3}{2} v_a - v_i\left(1 - \frac{3}{2}\mu^2\right) \tag{A-14}$$

Longitudinal Cyclic Equation

$$b_1 = \frac{\frac{4}{3} a_o \mu + w}{1 + \frac{1}{2}\mu^2} \tag{A-15}$$

Exact Blade Circulation Distribution

$$\frac{2\Gamma}{ac\Omega R} = \Upsilon_o + \Upsilon_1 x + \Upsilon_2 x^2 + (\sin\Psi)(\Upsilon_3 + \Upsilon_4 x)$$

$$+ (\cos\Psi)(\Upsilon_5 + \Upsilon_6 x) + (\sin 2\Psi)\,\Upsilon_7 + (\cos 2\Psi)\Upsilon_8 \tag{A-16}$$

-----

where,

$$\Upsilon_o = v_a - \frac{1}{2} a_1 \mu \qquad \text{(A-17)}$$

$$\Upsilon_1 = A_o + \frac{3}{4} \theta_1 - v_i \qquad \text{(A-18)}$$

$$\Upsilon_2 = - \theta_1 \qquad \text{(A-19)}$$

$$\Upsilon_3 = (A_o + \frac{3}{4} \theta_1) \mu \qquad \text{(A-20)}$$

$$\Upsilon_4 = - (a_1 - y - \theta_1 \mu) \qquad \text{(A-21)}$$

$$\Upsilon_5 = - a_o \mu \qquad \text{(A-22)}$$

$$\Upsilon_6 = b_1 - w \qquad \text{(A-23)}$$

$$\Upsilon_7 = \frac{1}{2} b_1 \mu \qquad \text{(A-24)}$$

$$\Upsilon_8 = \frac{1}{2} a_1 \mu \qquad \text{(A-25)}$$

Inplane Component of Force Equation

$$C_{XY} = (C_{XY})_a - (C_{XY})_{\delta_o} - (C_{XY})_\varepsilon \qquad \text{(A-26)}$$

where,

$$\frac{(C_{XY})_a}{a} = \frac{1}{\pi R} \int_{x_1}^{x_i} c \, \frac{U \sin \varphi}{(\Omega R)} \, \frac{2\Gamma}{(ac\Omega R)} \, dx \qquad \text{(A-27)}$$

$$\frac{(C_{XY})_{\delta_o}}{\delta_o} = \frac{1}{\pi R}\int_{x_1}^{x_i} c\ (\frac{U\cos\varphi}{\Omega R})^2 \frac{1}{\cos\varphi}\,dx \qquad \text{(A-28)}$$

$$\frac{(C_{XY})_{\varepsilon}}{\varepsilon a^2} = \frac{1}{2\pi R}\int_{x_1}^{x_i} c\ (\frac{2\Gamma}{ac\Omega R})^2 \cos\varphi\ dx \qquad \text{(A-29)}$$

Rotor Torque Equation

$$\frac{2C_Q}{b} = -\frac{1}{2\pi}\int_0^{2\pi}\int_{x_1}^{x_2} \frac{dC_{XY}}{dx}\ x\,d\Psi\,dx \qquad \text{(A-30)}$$

Component of Rotor Force Acting Along X Axis Equation

$$C_X = -\frac{b}{2\pi}\int_0^{2\pi}\int_{x_1}^{x_2} \frac{dC_{XY}}{dx}\sin\Psi\ d\Psi\ dx \qquad \text{(A-31)}$$

$$C_X = \left(\frac{\bar{a}}{1-\mu^2}\right)(2v_a - \frac{10}{3}v_i)\,\mu + \frac{\delta_o}{a}\,(3\mu) - \frac{8}{3}\,\varepsilon a\left(\frac{\bar{a}}{1-\mu^2}\right)^2\mu \qquad \text{(A-32)}$$

where δ_o is assumed to be approximately .008.

Tip Path Plane in Level Flight

$$\tan\theta_y \approx -\frac{D_f\cos\varphi_c}{W - D_f\sin\varphi_c} \qquad \text{(A-33)}$$

Rotor Thrust Coefficient

$$C_T = \frac{W - D_f\sin\varphi_c + F_X\sin\theta_y}{\rho\pi\Omega^2R^4\cos\theta_y} \qquad \text{(A-34)}$$

APPENDIX B

SOLUTION TO EQUATION EXPRESSING CHANGE IN ANGLE OF ATTACK

Given Equations (4), (5), (6), and (7):

Equation (6) represents the re-balanced force system along the Z axis.

$$(T + \Delta T)(\cos \alpha_R \cos \Delta\alpha_R - \sin\alpha_R \sin \Delta\alpha_R)$$

$$+ (F_X + \Delta F_X)(- \sin \alpha_R \cos \Delta\alpha_R - \cos \alpha_R \sin \Delta\alpha_R) = W \quad (6)$$

Equation (7) represents the re-balanced force system along the X axis.

$$(T + \Delta T)(\sin \alpha_R \cos \Delta\alpha_R + \cos \alpha_R \sin \Delta\alpha_R)$$

$$+ (F_X + \Delta F_X)(\cos \alpha_R \cos \Delta\alpha_R - \sin \alpha_R \sin \Delta\alpha_R) = -D_f - \Delta D_f \quad (7)$$

Equation (5) represents the initial force system along the Z axis.

$$T \cos \alpha_R + F_X(- \sin \alpha_R) = W \quad (5)$$

Equation (4) represents the initial force system along the X axis.

$$T \sin \alpha_R + F_X \cos \alpha_R = - D_f \quad (4)$$

Using small angle approximations, whereby

$$\cos \Delta\alpha_R \doteq 1$$

$$\sin \Delta\alpha_R = \Delta\alpha_R$$

Equation (B-1) is obtained by subtracting Equation (5) from Equation (6).

$$\Delta T \cos \alpha_R - T\Delta\alpha_R \sin \alpha_R - \Delta\alpha_R \Delta T \sin \alpha_R - \Delta F_X \sin \alpha_R$$

$$- \Delta F_X \, \Delta\alpha_R \cos \alpha_R - F_X \, \Delta\alpha_R \cos \alpha_R = 0 \qquad \text{(B-1)}$$

Equation (B-2) is obtained by subtracting Equation (4) from Equation (7).

$$T\Delta\alpha_R \cos \alpha_R + \Delta T \sin \alpha_R + \Delta T \Delta\alpha_R \cos \alpha_R - F_X \Delta\alpha_R \sin \alpha_R$$

$$+ \Delta F_X \cos \alpha_R - \Delta F_X \Delta\alpha_R \sin \alpha_R = - \Delta D_f \qquad \text{(B-2)}$$

Solving Equation (B-1) and (B-2) separately for ΔT yields Equations (B-3) and (B-4).

$$\Delta T = \frac{T\Delta\alpha_R \sin \alpha_R + \Delta F_X \sin \alpha_R + \Delta F_X \Delta\alpha_R \cos \alpha_R + F_X \Delta\alpha_R \cos \alpha_R}{\cos \alpha_R - \Delta\alpha_R \sin \alpha_R} \qquad \text{(B-3)}$$

$$\Delta T = \frac{F_X \Delta\alpha_R \sin \alpha_R - T\Delta\alpha_R \cos \alpha_R - \Delta F_X \cos \alpha_R + \Delta F_X \Delta\alpha_R \sin \alpha_R - \Delta D_f}{\sin \alpha_R + \Delta\alpha_R \cos \alpha_R} \qquad \text{(B-4)}$$

Equating (B-3) to (B-4)

$$T\Delta\alpha_R \sin^2 \alpha_R + \Delta F_X \sin^2 \alpha_R + \Delta F_X \Delta\alpha_R \cos \alpha_R \sin \alpha_R$$

$$+ F_X \Delta\alpha_R \cos \alpha_R \cos \alpha_R \sin \alpha_R + T\Delta\alpha_R^2 \sin \alpha_R \cos \alpha_R$$

$$+ \Delta F_X \Delta\alpha_R \cos \alpha_R \sin \alpha_R + \Delta F_X \Delta\alpha_R^2 \cos^2 \alpha_R + F_X \Delta\alpha_R^2 \cos^2 \alpha_R =$$

$$= F_X \Delta\alpha_R \sin\alpha_R \cos\alpha_R - T\Delta\alpha_R \cos^2\alpha_R - \Delta F_X \cos^2\alpha_R$$

$$+ \Delta F_X \Delta\alpha_R \sin\alpha_R \cos\alpha_R - \Delta D_f \cos\alpha_R - F_X \Delta\alpha_R^{\,2} \sin^2\alpha_R$$

$$+ T\Delta\alpha_R^{\,2} \cos\alpha_R \sin\alpha_R + \Delta F_X \Delta\alpha_R \cos\alpha_R \sin\alpha_R$$

$$- \Delta F_X \Delta\alpha_R^{\,2} \sin^2\alpha_R + \Delta D_f \Delta\alpha_R \sin\alpha_R \qquad \text{(B-5)}$$

and using the simple trigonometric relation where

$$\sin^2\alpha + \cos^2\alpha = 1$$

Equation (B-5) is simplified to

$$T\Delta\alpha_R + \Delta F_X + \Delta F_X \Delta\alpha_R^{\,2} + F_X \Delta\alpha_R^{\,2} + \Delta D_f(\cos\alpha_R - \Delta\alpha_R \sin\alpha_R) = 0 \qquad \text{(B-6)}$$

Rewriting Equation (B-6) in the following form gives the quadratic equation:

$$(\Delta F_X + F_X)\Delta\alpha_R^{\,2} + (T - \Delta D_f \sin\alpha_R)\Delta\alpha_R + (\Delta D_f \cos\alpha_R + \Delta F_X) = 0 \qquad \text{(B-7)}$$

The change in α_R can now be solved for directly.

$$\Delta\alpha_R = \frac{-\left[T - \Delta D_f \sin\alpha_R\right]}{2(F_X + \Delta F_X)}$$

$$\pm \frac{\left[(T - \Delta D_f \sin\alpha_R)^2 - 4(\Delta D_f \cos\alpha_R + \Delta F_X)(\Delta F_X + F_X)\right]^{\frac{1}{2}}}{2(F_X + \Delta F_X)} \qquad \text{(B-8)}$$

Using the binomial expansion,

$$(1 - x)^{\frac{1}{2}} = 1 - \frac{1}{2} x - \frac{1}{8} x^2 - \frac{1}{16} x^3$$

the expression is reduced to

$$\Delta\alpha_R = \frac{- (T - \Delta D_f \sin \alpha_R) + T - \Delta D_f \sin \alpha_R}{2(F_X + \Delta F_X)}$$

$$- \frac{2 \dfrac{(\Delta D_f \cos \alpha_R + \Delta F_X)(\Delta F_X + F_X)}{R - \Delta D_f \sin \alpha_R}}{2(F_X + \Delta F_X)}$$

$$- \frac{\frac{16}{8} \left[\dfrac{[\Delta D_f \cos \alpha_R + \Delta F_X)(\Delta F_X + F_X)]^2}{(F - \Delta D_f \sin \alpha_R)^3}\right]}{2(F_X + \Delta F_X)} \tag{B-9}$$

Rearranging and neglecting small quantities

$$\Delta\alpha_R = \frac{- \Delta D_f \cos \alpha_R + \Delta F_X}{T - \Delta D_f \sin \alpha_R} - \frac{(\Delta D_f \cos \alpha_R + \Delta F_X)^2 (F_X + \Delta F_X)}{(T - \Delta D_f \sin \alpha_R)^3} \tag{B-10}$$

APPENDIX C

SOLUTION FOR EQUATION EXPRESSING INCREMENTAL INCREASE IN ROTOR THRUST

Given Equations (B-1) and (B-2)

$$\Delta T \cos \alpha_R - T\Delta\alpha_R \sin \alpha_R - \Delta T\Delta\alpha_R \sin \alpha_R - \Delta F_X \sin \alpha_R$$

$$- \Delta F_X\Delta\alpha_R \cos \alpha_R - F_X\Delta\alpha_R \cos \alpha_R = 0 \qquad \text{(B-1)}$$

$$T\Delta\alpha_R \cos \alpha_R + \Delta T \sin \alpha_R + \Delta T\Delta \alpha_R \cos \alpha_R - F_X\Delta\alpha_R \sin \alpha_R$$

$$+ \Delta F_X \cos \alpha_R - \Delta F_X \Delta\alpha_R \sin \alpha_R = - \Delta D_f \qquad \text{(B-2)}$$

Solving Equations (B-1) and (B-2) separately for $\Delta\alpha_R$ yields:

$$\Delta\alpha_R = \frac{-\Delta F_X \sin \alpha_R + \Delta T \cos \alpha_R}{T \sin \alpha_R + \Delta T \sin \alpha_R + \Delta F_X \cos \alpha_R + F_X \cos \alpha_R} \qquad \text{(C-1)}$$

$$\Delta\alpha = \frac{-\Delta D_f - \Delta T \sin \alpha_R - \Delta F_X \cos \alpha_R}{T \cos \alpha_R + \Delta T \cos \alpha_R - F_X \sin \alpha_R - \Delta F_X \sin \alpha_R} \qquad \text{(C-2)}$$

Equating (C-1) to (C-2)

$$- T\Delta F_X \sin \alpha_R \cos \alpha_R - \Delta F_X\Delta T \sin \alpha_R \cos \alpha_R + F_X\Delta F_X\sin^2\alpha_R$$

$$+ \Delta F_X^2 \sin^2 \alpha_R + T\Delta T \cos^2 \alpha_R + \Delta T^2 \cos^2 \alpha_R - F_X\Delta T \sin \alpha_R \cos \alpha_R$$

$$- \Delta T\Delta F_X \sin \alpha_R \cos \alpha_R = - T\Delta D_f \sin \alpha_R - \Delta D_f\Delta T \sin \alpha_R$$

$$- \Delta D_f \Delta F_X \cos \alpha_R - \Delta D_f F_X \cos \alpha_R - T\Delta T \sin^2 \alpha_R - \Delta T^2 \sin^2 \alpha_R$$

$$- \Delta T \Delta F_X \sin \alpha_R \cos \alpha_R - \Delta T F_X \cos \alpha_R \sin \alpha_R - T \Delta F_X \sin \alpha_R \cos \alpha_R$$

$$- \Delta F_X \Delta T \sin \alpha_R \cos \alpha_R - \Delta F_X{}^2 \cos^2 \alpha_R - F_X \Delta F_X \cos^2 \alpha_R \qquad \text{(C-3)}$$

Equation (C-3) can easily be reduced to

$$\Delta T^2 + (T + \Delta D_f \sin \alpha_R)\Delta T + (T\Delta D_f \sin \alpha_R + \Delta D_f \Delta F_X \cos \alpha_R$$

$$+ F_X \Delta D_f \cos \alpha_R + F_X \Delta F_X + \Delta F_X{}^2) = 0 \qquad \text{(C-4)}$$

Solving the equation for ΔT gives

$$\Delta T = - \frac{T + \Delta D_f \sin \alpha_R}{2}$$

$$\pm \left\{ \frac{[(T + \Delta D_f \sin \alpha_R)^2 - 4(T\Delta D_f \sin \alpha_R + \Delta D_f DF_X \cos \alpha_R + \Delta D_f F_X \cos \alpha_R + F_X \Delta F_X + \Delta F_X{}^2)]^{\frac{1}{2}}}{2} \right\}$$

$$\text{(C-5)}$$

The equation for ΔT can be further reduced by using the binomial expansion

$$\Delta T = \frac{- [T\Delta D_f \sin \alpha_R + \Delta D_f \Delta F_X \cos \alpha_R + \Delta D_f F_X \cos \alpha_R + F_X \Delta F_X + \Delta F_X{}^2]}{T + \Delta D_f \sin \alpha_R}$$

$$- \left[\frac{T^2 \Delta D_f^2 \sin^2 \alpha_R + T\Delta D_f^2 F_X \sin \alpha_R \cos \alpha_R + \Delta D_f^2 F_X{}^2 \cos^2 \alpha_R}{(T + \Delta D_f \sin \alpha_R)^3} \right] \qquad \text{(C-6)}$$

APPENDIX D

DERIVATION OF THE ΔC_{XY} EQUATION

Given the equation for the inplane component of force

$$C_{XY} = (C_{XY})_a - (C_{XY})_{\delta_o} - (C_{XY})_\varepsilon \qquad \text{(A-26)}$$

where

$$(C_{XY})_a = \frac{a}{\pi R} \int_{x_1}^{x_2} c\left[\frac{U \sin \varphi}{\Omega R}\right]\left[\frac{2\Gamma}{ac\Omega R}\right] dx \qquad \text{(A-27)}$$

$$(C_{XY})_{\delta_o} = \frac{\delta_o}{\pi R} \int_{x_1}^{x_2} c\left[\frac{U \cos \varphi}{\Omega R}\right]^2 \frac{1}{\cos \varphi} dx \qquad \text{(A-28)}$$

$$(C_{XY})_\varepsilon = \frac{\varepsilon a^2}{2\pi R} \int_{x_1}^{x_2} c \left[\frac{2\Gamma}{ac\Omega R}\right]^2 \cos \varphi \, dx \qquad \text{(A-29)}$$

Adding the increment to Equation (A-27) expands the portion due to lift to

$$(C_{XY} + \Delta C_{XY})_a = \frac{a}{\pi R} \int_{x_1}^{x_2} c\left[\frac{U \sin \varphi}{\Omega R} + \Delta\left(\frac{U \sin \varphi}{\Omega R}\right)\right]\left[\frac{2\Gamma + 2\Delta\Gamma}{ac\Omega R}\right] dx \qquad \text{(D-1)}$$

Subtracting Equation (A-27) from Equation (D-1) gives (D-2)

$$(\Delta C_{XY})_a = \frac{a}{\pi R}\int_{x_1}^{x_2} c\ [(\frac{U\ \sin\ \phi}{\Omega R})(\frac{2\Delta\Gamma}{a c\Omega R}) + (\frac{2\Gamma}{a c\Omega R})(\Delta\ \frac{U\ \sin\ \phi}{\Omega R})]\ dx \qquad \text{(D-2)}$$

By using the above procedure for the profile drag similar expressions are obtained for Equations (A-28) and (A-29).

Rewriting Equation (A-28), the constant term of the profile drag can be expressed by

$$(C_{XY})_{\delta_o} = \frac{\delta_o}{\pi R}\int_{x_1}^{x_2} c[(x + \mu\ \sin\Psi)^2 + \frac{1}{2}\ (v_a - v_i x)^2]\ dx \qquad \text{(D-3)}$$

then,

$$(C_{XY} + \Delta C_{XY})_{\delta_o} = \frac{\delta_o}{\pi R}\int_{x_1}^{x_2} c\left\{[(x + \mu\ \sin\Psi)^2] + \right.$$

$$\left.\frac{1}{2}\ [(v_a + \Delta v_a) - v_i x]^2\right\} dx \qquad \text{(D-4)}$$

The variable part is derived in like manner:

$$(C_{XY} + \Delta C_{XY})_{\varepsilon} = \frac{\varepsilon a^2}{2\pi R}\int_{x_1}^{x_2} c\ [\frac{2\Gamma + 2\Delta\Gamma}{a c\Omega R}]^2\ dx \qquad \text{(D-5)}$$

Assuming that $\cos\ \phi \approx 1$ introduces a very small conservative error, and in the limit that ΔC_{d_p} approaches zero, this assumption does not affect the efficiency solution. Therefore

$$(C_{XY} + \Delta C_{XY})_{\varepsilon} = \frac{\varepsilon a^2}{2\pi R}\int_{x_1}^{x_2} c\ [\frac{2\Gamma + 2\Delta\Gamma}{a c\Omega R}]\ dx \qquad \text{(D-6)}$$

then

$$(\Delta C_{XY})_{\varepsilon} = \frac{\varepsilon a^2}{2\pi R} \int_{x_1}^{x_2} c \left[(2)\left(\frac{2\Gamma}{ac\Omega R}\right)\left(\frac{2\,\Gamma}{ac\Omega R}\right) + \left(\frac{2\,\Gamma}{ac\Omega R}\right)^2\right] dx \qquad \text{(D-7)}$$

Neglecting small quantities

$$(\Delta C_{XY})_T = \frac{a}{\pi R}\int_{x_1}^{x_2} c\left[\left(\frac{U\ \sin\ \varphi}{\Omega R}\right)\left(\frac{2\Delta\Gamma}{ac\Omega R}\right)\right.$$

$$\left. + \left(\frac{2\Gamma}{ac\Omega R}\right)\left(\Delta\left(\frac{U\sin\varphi}{\Omega R}\right)\right)\right]dx - \frac{\delta_o}{\pi R}\int_{x_1}^{x_2} c(v_a - v_i x)\Delta v_a\ dx$$

$$- \frac{\varepsilon a^2}{\pi R} \int_{x_1}^{x_2} c\left[\left(\frac{2\Gamma}{ac\Omega R}\right)\left(\frac{2\Delta\Gamma}{ac\Omega R}\right)\right] dx \qquad \text{(D-8)}$$

APPENDIX E

INTEGRATION OF THE POWER REQUIRED EQUATION

Substituting Equations (38) through (46) into Equation (48) gives

$$
\begin{aligned}
\Delta C_Q = - \frac{b}{2(2\pi)} \int_0^{2\pi} \int_{x_1}^{x_2} \Bigg\{ & \frac{ac}{\pi R} \Big[v_a (\Delta\Upsilon_o + \Delta\Upsilon_1 x + \Delta\Upsilon_2 x^2 \\
& + (\Delta\Upsilon_3 + \Delta\Upsilon_4 x) \sin\Psi + (\Delta\Upsilon_5 + \Delta\Upsilon_6 x) \cos\Psi + \Delta\Upsilon_7 \sin 2\Psi \\
& + \Delta\Upsilon_8 \cos 2\Psi) - v_i x (\Delta\Upsilon_o + \Delta\Upsilon_1 x + \Delta\Upsilon_2 x^2 + \Delta\Upsilon_3 \sin\Psi \\
& + \Delta\Upsilon_4 x \sin\Psi + \Delta\Upsilon_5 \cos\Psi + \Delta\Upsilon_6 x \cos\Psi + \Delta\Upsilon_7 \sin 2\Psi \\
& + \Delta\Upsilon_8 \cos 2\Psi) + \Delta v_a (\Upsilon_o + \Upsilon_3 \sin\Psi + \Upsilon_5 \cos\Psi + \Upsilon_7 \sin 2\Psi \\
& + \Upsilon_8 \cos 2\Psi) + \Delta v_a (\Upsilon_1 + \Upsilon_4 \sin\Psi + \Upsilon_6 \cos\Psi) x + \Delta v_a \Upsilon_2 x^2 \Big] \\
& - \frac{\delta_o c}{\pi R} \left[v_a \Delta v_a - v_i \Delta v_a x \right] - \frac{\epsilon a^2 c}{\pi R} \Big[(\Upsilon_o + \Upsilon_1 x + \Upsilon_2 x^2 \\
& + \Upsilon_3 \sin\Psi + \Upsilon_4 x \sin\Psi + \Upsilon_5 \cos\Psi + \Upsilon_6 x \cos\Psi + \Upsilon_7 \sin 2\Psi \\
& + \Upsilon_8 \cos 2\Psi)(\Delta\Upsilon_o + \Delta\Upsilon_1 x + \Delta\Upsilon_2 x^2 + \Delta\Upsilon_3 \sin\Psi + \Delta\Upsilon_4 x \sin\Psi \\
& + \Delta\Upsilon_5 \cos\Psi + \Delta\Upsilon_6 x \cos\Psi + \Delta\Upsilon_7 \sin 2\Psi + \Delta\Upsilon_8 \cos 2\Psi \Big] \Bigg\} x\,dx\,d\Psi \qquad \text{(E-1)}
\end{aligned}
$$

Collecting like terms, Equation (E-1) is reduced to

$$\Delta C_Q = -\frac{b}{2(2\pi)}\int_0^{2\pi}\int_{x_1}^{x_2}\Big\{\frac{ac}{\pi R}[v_a\Delta\Upsilon_o + v_a\Delta\Upsilon_3 \sin\Psi + v_a\Delta\Upsilon_5 \cos$$

$$+ v_a\Delta\Upsilon_7 \sin 2\Psi + v_a\Delta\Upsilon_8 \cos 2\Psi + \Delta v_a\Upsilon_o + \Upsilon_3 \sin\Psi\,\Delta v_a$$

$$+ \Upsilon_5 \cos\Psi\,\Delta v_a + \Upsilon_7 \sin 2\Psi\,\Delta v_a + \Upsilon_8 \cos 2\Psi\Delta v_a] + \frac{ac}{\pi R}[v_a\Delta\Upsilon_1$$

$$+ v_a\Delta\Upsilon_4 \sin\Psi + v_a\Delta\Upsilon_6 \cos\Psi - v_i\Delta\Upsilon_o - v_i\Delta\Upsilon_3 \sin\Psi$$

$$- v_i\Delta\Upsilon_5 \cos\Psi - v_i\Delta\Upsilon_7 \sin 2\Psi - v_i\Delta\Upsilon_8 \cos 2\Psi + \Delta v_a\Upsilon_1$$

$$+ \Delta v_a\Upsilon_4 \sin\Psi + \Delta v_a\Upsilon_6 \cos\Psi]x + \frac{ac}{\pi R}[v_a\Delta\Upsilon_2 - v_i\Delta\Upsilon_1$$

$$- v_i \sin\Psi\,\Delta\Upsilon_4 - v_i\Delta\Upsilon_6 \cos\Psi + \Delta v_a\Upsilon_2]x^2 + \frac{ac}{\pi R}[-v_i\Delta\Upsilon_2]\,x^3$$

$$- \frac{\delta_o c}{\pi R}[v_a\Delta v_a] - \frac{\delta_o c}{\pi R}[-v_i\Delta v_a]x - \frac{\varepsilon a^2 c}{\pi R}[(\Upsilon_o + \Upsilon_3 \sin\Psi$$

$$+ \Upsilon_5 \cos\Psi + \Upsilon_7 \sin 2\Psi + \Upsilon_8 \cos 2\Psi)(\Delta\Upsilon_o + \Delta\Upsilon_3 \sin\Psi + \Delta\Upsilon_5 \cos\Psi$$

$$+ \Delta\Upsilon_7 \sin 2\Psi + \Delta\Upsilon_8 \cos 2\Psi)] - \frac{\varepsilon a^2 c}{\pi R}[(\Upsilon_o + \Upsilon_3 \sin\Psi + \Upsilon_5 \cos\Psi$$

$$+ \Upsilon_7 \sin 2\Psi + \Upsilon_8 \cos 2\Psi)(\Delta\Upsilon_1 + \Delta\Upsilon_4 \sin\Psi + \Delta\Upsilon_6 \cos\Psi)$$

$$+ (\Upsilon_1 + \Upsilon_4 \sin\Psi + \Upsilon_6 \cos\Psi)(\Delta\Upsilon_o + \Delta\Upsilon_3 \sin\Psi + \Delta\Upsilon_5 \cos\Psi$$

$$+ \Delta\Upsilon_7 \sin 2\Psi + \Delta\Upsilon_8 \cos 2\Psi)]x - \frac{\varepsilon a^2 c}{\pi R}[(\Upsilon_o + \Upsilon_3 \sin\Psi + \Upsilon_5 \cos\Psi$$

$$+ \gamma_7 \sin 2\Psi + \gamma_8 \cos 2\Psi)(\Delta\gamma_2) + (\gamma_1 + \gamma_4 \sin\Psi + \gamma_6 \cos\Psi)(\Delta\gamma_1$$

$$+ \Delta\gamma_4 \sin\Psi + \Delta\gamma_6 \cos\Psi) + (\gamma_2)(\Delta\gamma_o + \Delta\gamma_3 \sin\Psi + \Delta\gamma_5 \cos\Psi$$

$$+ \Delta\gamma_7 \sin 2\Psi + \Delta\gamma_8 \cos 2\Psi)]x^2 - \frac{\varepsilon a^2 c}{\pi R}[\gamma_1 + \gamma_4 \sin\Psi + \gamma_6 \cos\Psi)(\Delta\gamma_2)$$

$$+ (\gamma_2)(\Delta\gamma_1 + \Delta\gamma_4 \sin\Psi + \Delta\gamma_6 \cos\Psi]x^3 - \frac{\varepsilon a^2 c}{\pi R}[(\gamma_2)(\Delta\gamma_2)]x^4\Big\} \, x dx d\Psi \quad \text{(E-2)}$$

Remembering that $\Delta\gamma_2 = 0$ and implementing the sigma factor equation, (A-9), Equation (E-2) becomes

$$\Delta C_Q = -\frac{b}{2(2\pi)} \int_0^{2\pi} \Big\{ a\sigma_2 [v_a\Delta\gamma_o + v_a \sin\Psi \,\Delta\gamma_3 + v_a \cos\Psi \,\Delta\gamma_5$$

$$+ v_a \sin 2\Psi\Delta\gamma_7 + v_a \cos 2\Psi\Delta\gamma_8 + \gamma_o\Delta v_a + \gamma_3 \sin\Psi\Delta v_a + \gamma_5 \cos\Psi\Delta v_a$$

$$+ \gamma_7 \sin 2\Psi\Delta v_a + \gamma_8 \cos 2\Psi\Delta v_a] + a\sigma_3[v_a\Delta\gamma_1 + v_a \sin\Psi \,\Delta\gamma_4$$

$$+ v_a \cos\Psi \,\Delta\gamma_6 - v_i\Delta\gamma_o - v_i \sin\Psi\Delta\gamma_3 - v_i \cos\Psi\Delta\gamma_5 - v_i \sin 2\Psi\Delta\gamma_7$$

$$- v_i \cos 2\Psi\Delta\gamma_8 + \gamma_1\Delta v_a + \sin\Psi\gamma_4 \,\Delta v_a + \gamma_6 \cos\Psi\Delta v_a]$$

$$+ a\sigma_4 [- v_i\Delta\gamma_1 - v_i \sin\Psi\Delta\gamma_4 - v_i \cos\Psi\Delta\gamma_6 + \gamma_2\Delta v_a] - \delta_o[v_a\Delta v_a]\sigma_2$$

$$+ \delta_o\sigma_3 [v_i\Delta v_a] - \varepsilon a^2\sigma_2 [(\gamma_o + \gamma_3 \sin\Psi + \gamma_5 \cos\Psi + \gamma_7 \sin 2\Psi$$

$$+ \gamma_8 \cos 2\Psi)(\Delta\gamma_o + \sin\Psi\Delta\gamma_3 + \cos\Psi\Delta\gamma_5 + \sin 2\Psi\Delta\gamma_7 + \cos 2\Psi\Delta\gamma_8]$$

$$- \varepsilon a^2 \sigma_3[(\gamma_o + \gamma_3 \sin\Psi + \gamma_5 \cos\Psi + \gamma_7 \sin 2\Psi + \gamma_8 \cos 2\Psi)(\Delta\gamma_1$$

$$+ \sin\Psi\Delta\gamma_4 + \cos\Psi\Delta\gamma_6) + (\gamma_1 + \gamma_4 \sin\Psi + \gamma_6 \cos\Psi)(\Delta\gamma_o$$

$$+ \sin\Psi\Delta\gamma_3 + \cos\Psi\Delta\gamma_5 + \sin 2\Psi\Delta\gamma_7 + \cos 2\Psi\Delta\gamma_8]$$

$$- \varepsilon a^2\sigma_4[(\gamma_1 + \gamma_4 \sin\Psi + \gamma_6 \cos\Psi)(\Delta\gamma_1 + \Delta\gamma_4 \sin\Psi + \Delta\gamma_6 \cos\Psi)$$

$$+ (\gamma_2)(\Delta\gamma_o + \sin\Psi\Delta\gamma_3 + \cos\Psi\Delta\gamma_5 + \sin 2\Psi\Delta\gamma_7 + \cos 2\Psi\Delta\gamma_8]$$

$$- \varepsilon a^2\sigma_5 [(\gamma_2)(\Delta\gamma_1 + \sin\Psi\Delta\gamma_4 + \cos\Psi\Delta\gamma_6)] \quad d\Psi \qquad \text{(E-3)}$$

Final integration of Equation (E-3) reduces it to Equation (E-4)

$$\Delta C_Q = -\frac{b}{2(2\pi)} \quad a\sigma_2[v_a\Delta\gamma_o(2\ ) + \gamma_o\Delta v_a\ (2\pi)] + a\sigma_3[v_a\Delta\gamma_1(2\pi)$$

$$- v_i\Delta\gamma_o(2\pi) + \gamma_1\Delta v_a(2\pi)] + a\sigma_4[\gamma_2\Delta v_a(2\pi) - v_i\Delta\gamma_1(2\pi)]$$

$$- \delta_o\sigma_2[v_a\Delta v_a(2\pi) + \delta_o\sigma_3[v_i\Delta v_a(2\pi)] - \varepsilon a^2\sigma_2[\gamma_o\Delta\gamma_o(2\pi)$$

$$+ \gamma_3\Delta\gamma_3(\pi) + \gamma_5\Delta\gamma_5(\pi) + \gamma_7\Delta\gamma_7(\pi) + \gamma_8\Delta\gamma_8(\pi)]$$

$$- \varepsilon a^2\sigma_3[\gamma_o\Delta\gamma_1(2\pi) + \gamma_3\Delta\gamma_4(\pi) + \gamma_5\Delta\gamma_6(\pi) + \gamma_1\Delta\gamma_o(2\pi)$$

$$+ \gamma_4\Delta\gamma_3(\pi) + \gamma_6\Delta\gamma_5(\pi)] - \varepsilon a^2\sigma_4[\gamma_1\Delta\gamma_1(2\pi) + \gamma_4\Delta\gamma_4(\pi)$$

$$+ \gamma_6\Delta\gamma_6(\pi) + \gamma_2\Delta\gamma_o(2\pi)] - \varepsilon a^2\sigma_5[\gamma_2\Delta\gamma_1(2\pi)] \qquad \text{(E-4)}$$

where,

$$\int_0^{2\pi} \sin^2\psi \; d\psi = \pi \qquad \text{(E-5)}$$

$$\int_0^{2\pi} \cos^2\psi \; d\psi = \pi \qquad \text{(E-6)}$$

$$\int_0^{2\pi} \cos\psi \; \sin\psi \; d\psi = 0 \qquad \text{(E-7)}$$

$$\int_0^{2\pi} \cos 2\psi \; \sin\psi \; d\psi = 0 \qquad \text{(E-8)}$$

$$\int_0^{2\pi} \cos\psi \; \sin 2\psi \, d\psi = 0 \qquad \text{(E-9)}$$

To assist in simplifying Equation (E-4) the assumption is made that

$$\frac{\sigma_a}{\sigma_b} = \frac{b}{a} \qquad \text{(E-10)}$$

This assumption is true for constant chord blades and if $x_1 = 0$, $x_2 = 1$.

Substituting in Equations (E-5) through (E-10) the increment of power required can be reduced to

$$\Delta C_Q = -\frac{b\sigma_5}{2} \Big\{ 2.5a\, v_a\, \Delta\gamma_o + 2.5a\gamma_o \Delta v_a + 1.667 a v_a \Delta\gamma_1$$

$$- 1.667 a v_i \Delta\gamma_o + 1.667\gamma_1 \Delta v_a + 1.25 a\gamma_2 \Delta v_a - 1.25 a v_i \Delta\gamma_1$$

$$- 2.5\delta_o v_a \Delta v_a + 1.667\delta_o v_i \Delta v_a - 2.5\varepsilon a^2 \gamma_o \Delta\gamma_o - 1.25\varepsilon a^2 \gamma_3 \Delta\gamma_3$$

$$- 1.25\varepsilon a^2 \gamma_5 \Delta\gamma_5 - 1.25\varepsilon a^2 \gamma_7 \Delta\gamma_7 - 1.25\varepsilon a^2 \gamma_8 \Delta\gamma_8 - 1.667\varepsilon a^2 \gamma_o \Delta\gamma_1$$

$$- .833\varepsilon a^2 \gamma_3 \Delta\gamma_4 - .833\varepsilon a^2 \gamma_5 \Delta\gamma_6 - 1.667\varepsilon a^2 \gamma_1 \Delta\gamma_o - .833\varepsilon a^2 \gamma_4 \Delta\gamma_3$$

$$- .833\varepsilon a^2 \gamma_6 \Delta\gamma_5 - 1.25\varepsilon a^2 \gamma_1 \Delta\gamma_1 - .625\varepsilon a^2 \gamma_4 \Delta\gamma_4 - .625\varepsilon a^2 \gamma_6 \Delta\gamma_6$$

$$- 1.25\varepsilon a^2 \gamma_2 \Delta\gamma_o - \varepsilon a^2 \gamma_2 \Delta\gamma_1 \Big\} \qquad \text{(E-11)}$$

Rewriting Equation (E-11) the change in power required is expressed in final form.

$$\Delta C_Q = - \frac{b\sigma_5}{2} \Big\{ [2.5av_a - 1.667av_i - 2.5\varepsilon a^2 \gamma_o - 1.667\varepsilon a^2 \gamma_1$$

$$- 1.25\varepsilon a^2 \gamma_2] \Delta\gamma_o + [2.5a\gamma_o + 1.667a\gamma_1 + 1.25a\gamma_2 - 2.5\delta_o v_a$$

$$+ 1.667\delta_o v_i] \Delta v_a + [1.667av_a - 1.25av_i - 1.667\varepsilon a^2 \gamma_o - 1.25\varepsilon a^2 \gamma_1] \Delta\gamma_1$$

$$- [1.25\varepsilon a^2 \gamma_3 + .833\varepsilon a^2 \gamma_4] \Delta\gamma_3 - [.833\varepsilon a^2 \gamma_3 + .625\varepsilon a^2 \gamma_4] \Delta\gamma_4$$

$$- [1.25\varepsilon a^2 \gamma_5 + .833\varepsilon a^2 \gamma_6] \Delta\gamma_5 - [.833\varepsilon a^2 \gamma_5 + .625\varepsilon a^2 \gamma_6] \Delta\gamma_6$$

$$- 1.25\varepsilon a^2 \gamma_7 \Delta\gamma_7 - 1.25\varepsilon a^2 \gamma_8 \Delta\gamma_8 \Big\} \qquad \text{(E-12)}$$

APPENDIX F

SIMPLIFICATION OF EQUATION (64)

Equation (64) is represented by

$$g_2(\Delta C_{d_p}) = (4C_T^2 - \Delta C_{d_p}^{\ 2} \sin^2 \alpha_R)\Delta C_Q \tag{64}$$

When Equations (39) through (46) and Equation (49) are substituted into Equation (64) the equation can be reduced to

$$g_2(\Delta C_{d_p}) = (4C_T^2 - \Delta C_{d_p}^{\ 2} \sin^2 \alpha_R)(-\frac{b\sigma_5}{2})(C_1 \Delta v_a - \frac{1}{2} C_1 \mu \Delta a_1$$

$$+ C_2 \Delta v_a + C_3 \Delta A_o - C_4 \mu \Delta A_o + C_5 \Delta a_1 - \frac{1}{2} C_6 \mu \Delta a_1 \tag{F-1}$$

Regrouping like terms

$$g_2(\Delta C_{d_p}) = (4C_T^2 - \Delta C_{d_p}^{\ 2} \sin^2 \alpha_R)(-\frac{b\sigma_5}{2}) [(C_1 + C_2) \Delta v_a$$

$$- (\frac{1}{2} \mu C_1 - C_5 + \frac{1}{2} \mu C_6) \Delta a_1 + (C_3 - \mu C_4) \Delta A_o] \tag{F-2}$$

When Equations (9), (16), (29), and (30) are substituted into Equation (F-2), $g_2(\Delta C_{d_p})$ can be expressed solely in terms of the increment of drag.

$$g_2(\Delta C_{d_p}) = P_1 \Delta C_{d_p} \; (2C_T + \Delta C_{d_p} \; \sin \alpha_R)$$

$$- P_2 \Delta C_{d_p} \; (2C_T + \Delta C_{d_p} \; \sin \alpha_R) + P_3 \Delta C_{d_p} \; (2C_T - \Delta C_{d_p} \; \sin \alpha_R)$$

$$+ P_4 \Delta C_{d_p} (2C_T - \Delta C_{d_p} \; \sin \alpha_R) - P_5 \Delta C_{d_p} \; (2C_T + \Delta C_{d_p} \; \sin \alpha_R) \qquad \text{(F-3)}$$

where,

$$P_1 = \frac{b\sigma_5}{2} [C_1 + C_2][\frac{V}{\Omega R}] \qquad \text{(F-4)}$$

$$P_2 = \frac{b\sigma_5}{2}\left[\frac{1}{2} \mu C_1 - C_5 + \frac{1}{2} \mu C_6\right] [\frac{V}{\Omega R}] \left[\frac{2K_4 K_5 - K_1 K_7}{2K_3 K_5 - K_1 K_6}\right] \qquad \text{(F-5)}$$

$$P_3 = \frac{b\sigma_5}{2} [\frac{1}{2} \mu C_1 - C_5 + \frac{1}{2} \mu C_6] \left[\frac{4K_5}{ab\sigma_3}\right] \left[\frac{C_T \sin \alpha_R + \frac{1}{2} C_X \cos \alpha_R}{2K_3 K_5 - K_1 K_6}\right] \qquad \text{(F-6)}$$

$$P_4 = \frac{b\sigma_5}{2} [C_3 - \mu C_4] \left[\frac{2K_6}{ab\sigma_3}\right] \left[\frac{C_T \sin \alpha_R + \frac{1}{2} C_X \cos \alpha_R}{K_6 K_1 - 2K_5 K_3}\right] \qquad \text{(F-7)}$$

$$P_5 = \frac{b\sigma_5}{2} [C_3 - \mu C_4] [\frac{V}{\Omega R}] \left[\frac{K_4 K_6 - K_7 K_3}{K_6 K_1 - 2K_5 K_3}\right] \qquad \text{(F-8)}$$

Equation (F-3) differentiates to

$$g_2'(\Delta C_{d_p}) = 2P_1 (C_T + \Delta C_{d_p} \; \sin \alpha_R) - 2P_2 (C_T + \Delta C_{d_p} \; \sin \alpha_R)$$

$$+ 2P_3 (C_T - \Delta C_{d_p} \; \sin \alpha_R) + 2P_4 (C_T - \Delta C_{d_p} \; \sin \alpha_R)$$

$$- 2P_5 (C_T + \Delta C_{d_p} \; \sin \alpha_R) \qquad \text{(F-9)}$$

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