GLOBAL APPROXIMATIONS TO SOLUTIONS
OF ORDINARY INITIAL VALUE PROBLEMS

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GLOBAL APPROXIMATIONS TO SOLUTIONS

OF ORDINARY INITIAL VALUE PROBLEMS

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I dedicate this dissertation to my parents, who made my college education possible, and to my wife, Rosita, for her patience and support.

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SUMMARY

We use the theory of collectively compact operators to investigate implicit methods for the construction of global approximations to the solutions of nonlinear ordinary initial value problems. We obtain new methods by taking into account certain characteristics of the differential equation, and by using a weight function to deal with difficult behavior. The approach unifies and generalizes from the point of view of functional analysis several methods which have appeared recently in the literature, such as projection and block methods. Sufficient conditions are given for the existence and uniqueness of the approximate solution in a neighborhood of the solution and for the convergence of Newton's method. Error bounds are derived. Finally, some aspects of stability for first and second-order equations are studied, and numerical examples are given to compare some of the methods and to illustrate the rates of convergence.

CHAPTER I

INTRODUCTION

In recent years there has been an interest in the construction of piecewise polynomial approximations to the solutions of ordinary differential equations, with the benefit of derivative approximations.

We present here a wide class of implicit one-step methods for the construction of more general global approximations to the solution of the single differential equation

$$y^{(s)}(x) = f(x, y(x), \ldots, y^{(m)}(x)) \;,$$

$$0 \leq x \leq b, \tag{1.1}$$

with initial conditions

$$y^{(i)}(0) = g_i \;, \quad 0 \leq i \leq s-1. \tag{1.2}$$

These methods include as special cases the collocation schemes of Russell and Shampine [43], deBoor and Swartz [16], and several of the more general projection methods of Wittenbrink [59], as applied to (1.1), (1.2). Some of Butcher's [8] and Ehle's [20] implicit Runge-Kutta formulae based on special quadratures, Hulme's [27], [28] Galerkin methods and a subclass of the block implicit methods of Shampine and Watts [49], [57] have been shown before to be equivalent to collocation and hence are included in our methods. These block methods were also studied by Williams and deHoog [58]. In addition we point out that Callender's [9] schemes are equivalent to collocation, as are the deficient spline methods of Micula [41].

Some other important special cases are the Hermite methods of Loscalzo [39], which contain the implicit one-step ($k = 1$) formulae of Lambert and Mitchell [36]; some of Ehle's [19] L-acceptable methods, which include Enright's [21] formula for $k = 1$ (or Liniger and Willoughby's [37] formula for $a = b = 1/3$); and a subclass of the generalized implicit Runge-Kutta formulae of Kastlunger and Wanner [34]. The quintic spline scheme of Hung [29] is also included, as are a subclass of the quadrature methods of Cooper [11] and the natural spline implicit block methods of Andria, Byrne and Hill [2].

Most of the methods mentioned above apply only to first-order problems and were not discussed within the framework of functional analysis. On the other hand, [16], [43], [59] deal with more general nth order boundary value problems, using the theory of projection methods. Our main theory on existence, convergence and error bounds for the approximate solution is based mostly on the collectively compact operator theory of Anselone [3], and the operators we consider are not necessarily projections. Many of the main results of [16], [43], [59], restricted to (1.1), (1.2) are generalized in several directions. Many of these generalizations should carry over to systems of initial value problems and boundary value problems since the corresponding integral equations have essentially the same properties as the integral equations considered here.

Equation (1.1) can be transformed into a first-order system, but constructing global vector approximations requires solving larger systems of algebraic equations, since each vector component is not necessarily related to the others through differentiation. This is specially true if s is much larger than m in (1.1). Dealing directly with (1.1), we construct a function y_n such that $y_n^{(i)}$ approximates $y^{(i)}$ for several values of i, where y is the solution of (1.1), (1.2); moreover, we can take advantage of some special characteristics which (1.1) may have, such as slow-varying partial derivatives of f.

In Chapter II, we introduce the general methods and present some notation which will be used throughout.

In Chapter III, we show how to construct families of operators P_n satisfying the basic properties (2.6a) - (2.6c), starting with an operator Q defined on a subspace of C[0,1]. Several examples are considered, such as Hermite interpolating operators and local spline approximating operators of Lyche and Schumaker [40]. Useful error bounds for the operators P_n are presented.

Chapter IV treats the linear problem. A general result, Theorem 4.2 is proved and is then applied in conjunction with the operators P_n of Example 3.1 to illustrate the type of bounds one can obtain. There is also a discussion on the numerical treatment of the approximate solutions.

Chapter V extends the results of Chapter IV to the nonlinear problem. The main result here is Theorem 5.2, which also establishes the validity of Newton's method in finding the approximate solutions. Again, some applications are given in conjunction with the operators P_n of Example 3.1, and higher rates of convergence are obtained for special choices of such operators, following some basic ideas of deBoor and Swartz [16].

In Chapter VI, we relate our methods to the methods mentioned at the beginning of this chapter. In some cases, we point out how to obtain new error bounds or how to improve existing ones.

A unified treatment of A-stability for methods based on Hermite interpolating operators is presented in Chapter VII, with particular emphasis on their relationship to Padé approximations to the exponential e^z. Among the new results, we show that methods based on a generalization of Gaussian points due to Stancu and Stroud [52] are A-stable. We also introduce the concept of B-stability for methods which approximate the solution of the equation

$$y''(x) = -\lambda^2 y(x), \quad 0 \leq x \leq b$$

subject to

$$y(0) = y_o, \quad y'(0) = y'_o \quad ,$$

and present same B-stable methods.

Finally, Chapter VIII contains numerical results of some of our methods, illustrating rates of convergence and improvements over other methods.

CHAPTER II

AN EQUIVALENT PROBLEM AND ITS APPROXIMATION

Consider the initial value problem (1.1), (1.2), with $0 \leq m \leq s-1$. We will assume that f is a real-valued function continuous in D,

$$D = [o,b] \times R^{m+1}, \quad R = (-\infty, \infty) ,$$

and that f satisfies a uniform Lipschitz condition

$$|f(x, u_o, \ldots, u_m) - f(x, v_o, \ldots, v_m)| \leq M \sum_{k=o}^{m} |u_k - v_k| , \qquad (2.1)$$

in D. Then it is well known that (1.1), (1.2) has a unique solution $y \in C^s[o, b]$. We will need to assume stronger differentiability conditions on f in most of our analysis; the assumptions are stated where they apply.

If (2.1) is only satisfied in a subset E of D, one can usually define a continuous extension f* of f which satisfies (2.1) in all of D. Let y* be the solution of (1.1), (1.2) with f replaced by f*. If $(x, y^*(x), \ldots, y^{*(m)}(x)) \in E$ for each $x \in [o, b]$, then y* is a solution of (1.1), (1.2). The global approximations given by any of the methods presented here can be used to determine if $(x, y^*(x), \ldots, y^{*(m)}(x)) \in E$.

To simplify the notation, we define

$$f(x; y) = f(x, y(x), \ldots, y^{(m)}).$$

Before we consider the numerical methods, we will transform (1.1), (1.2) into an equivalent problem. Let a_k, $o \leq k \leq m$ be arbitrary constants, and define the operator L by

$$(Ly)(x) = y^{(s)}(x) - \sum_{k=o}^{m} a_k \, y^{(k)}(x) \; .$$

Equation (1.1) is equivalent to

$$(Ly)(x) = g(x;\, y) \equiv f(x;\, y) - \sum_{k=o}^{m} a_k \, y^{(k)}(x), \qquad 0 \le x \le b \; . \qquad (2.2)$$

Let $G(x,\, t) \equiv v(x - t)$, $o \le t \le x \le b$, where v is the solution of $Lv = o$, $v^{(i)}(o) = o$, $o \le i \le s-2$, $v^{(s-1)}(o) = 1$. Then the solution y of (2.2), (1.2) satisfies

$$y(x) = \sum_{k=1}^{s} \alpha_k \phi_k(x) + \int_o^x G(x,\, t)\, g(t;\, y)\, dt \; , \qquad o \le x \le b \; ,$$

where

$$\sum_{k=1}^{s} \alpha_k \phi_k(x) \quad \text{is the solution of } Lu = o \text{ and } (1.2).$$

Now suppose that $g(x;\, y) = w(x)\, H(x;\, y)$, where $w \in C[o,\, b]$ and H is sufficiently smooth. The equivalent problem is

$$y(x) = \sum_{k=1}^{s} \alpha_k \phi_k(x) + \int_o^x G(x,\, t)\, w(t)\, H(t;\, y)\, dt \; , \quad o \le x \le b \; . \quad (2.3)$$

Equation (2.3) is set in the Banach space $C^{s-1}[o,\, b]$, where $C^r[o,\, b]$ is the set of all r-times continuously differentiable functions in $[o,\, b]$ with norm

$$||g||_r = \sup_{o \le x \le b} \sum_{i=o}^{r} |g^{(i)}(x)| \; .$$

The approximate methods will consist of replacing equation (2.3) by a perturbed equation, also in $C^{s-1}[o, b]$, which in turn is equivalent to an algebraic system of equations. Let $\{\Delta_n\}$ be a sequence of partitions of $[o, b]$ given by

$$\Delta_n: \quad 0 = x_o < x_1 < x_2 < \ldots < x_n = b \ , \tag{2.4}$$

and let $|\Delta_n| = \max_j \Delta x_j \to o$ as $n \to \infty$, where $\Delta x_j = x_j - x_{j-1}$. We say that $\{\Delta_n\}$ is quasi-uniform if $\max_j |\Delta_n|/\Delta x_j \leq A$ for some constant $A > 0$.

Define $C_n^r[o, b]$ as the set of all real-valued functions g on $[o, b]$, such that $g \in C^r(x_{j-1}, x_j)$, $1 \leq j \leq n$, and such that $g^{(i)}(x_{j-1}^+)$ and $g^{(i)}(x_j^-)$ exist for $o \leq i \leq r$, $1 \leq j \leq n$. A seminorm on $C_n^r[o, b]$ is given by

$$||g||_{r,n} = \max_{1\leq j\leq n} \ \sup_{x_{j-1}<x<x_j} \ \sum_{i=o}^{r} |g^{(i)}(x)| \ .$$

The approximate solution y_n to the solution y of (2.3) is defined by

$$y_n(x) = \sum_{k=1}^{s} \alpha_k^n \phi_k(x) + \int_o^x G(x, t)w(t)\ P_n(H(\cdot; y_n))(t)\ dt \ ,$$

$$o \leq x \leq b \ , \tag{2.5}$$

where α_k^n is an approximation to α_k, and $\{P_n\}$ is a family of linear operators satisfying

$$P_n: \ C^{q_1}[o, b] \to C_n^o[o,b] \ , \quad \text{for some integer } q_1 \text{ independent of } n; \tag{2.6a}$$

$$||P_n g||_{o,n} \leq C_1 ||g||_{q_1} \quad \text{for all } g \in C^{q_1}[o, b],\ C_1 \text{ independent of } n; \tag{2.6b}$$

$$||P_n g - g||_{o,n} \to o \quad \text{as} \quad n \to \infty , \quad \text{for each } g \in C^{q_1}[o, b] . \qquad (2.6c)$$

Even though the analysis will be carried out for equations (2.3) and (2.5), in practice the solution y_n of (2.5) can be found in a step-by-step process, since for $j = 1, 2, \ldots, n$,

$$y_n(x) = \sum_{k=1}^{s} \alpha_{k,j}^n \phi_k(x) + \int_{x_{j-1}}^{x} G(x, t)w(t) P_n(H(.; y_n))(t)\, dt ,$$

$$x_{j-1} \leq x \leq x_j \quad , \qquad (2.7)$$

where

$\sum_{k=1}^{s} \alpha_{k,j}^n \phi_k(x)$ is the solution of $Lu = o$, $u^{(i)}(x_{j-1}) = y_n^{(i)}(x_{j-1})$,

$o \leq i \leq s-1$.

One can show that any function $V_n \in C^{s-1}[o, b]$ which satisfies (2.7) for all $j = 1, 2, \ldots, n$, must satisfy (2.5).

In the next chapter we consider more specific families of operators P_n, for which we can describe in more detail the process of solving (2.7).

The arbitrary constants a_k in the definition of L will not affect the asymptotic rate of convergence of y_n to y, which depends on the smoothness of the function H and the family $\{P_n\}$. The function w could be chosen to produce a smoother H, and sometimes one can conveniently pick the a_k's to incorporate into y_n some of the properties of y. For example we could let

$$a_k = \frac{\partial f}{\partial y^{(k)}} (o, y(o), \ldots, y^{(m)}(o))$$

for those k for which $\partial f/\partial y^{(k)}(x, y(x), \ldots, y^{(m)}(x))$ does not vary much in [o, b]. Then the methods would be exact when applied to homogeneous linear differential equations with constant coefficients.

The theory is developed on the assumption that the a_k's and w are constant throughout the numerical process. If a change is wanted after y_n has been computed in an interval $[o, b_1]$, one has to consider a new initial value problem in $[b_1, b]$. However, we will see that the asymptotic error is the same in $[o, b_1]$ and $[b_1, b]$.

Two more remarks concerning notation: The constant C will be used as a generic constant throughout; π_r is the set of all polynomials of degree $\leq$ r.

CHAPTER III

CONSTRUCTION OF SEQUENCES OF OPERATORS

It is possible to construct operators P_n satisfying (2.6a) - (2.6c) starting with an operator Q defined on $C^{q_1}[o, 1]$. Define the norm $||.||$ by

$$||G|| = \sup_{o \le t \le 1} |G(t)| ,$$

and the modulus of continuity of a function G on [o, 1] by

$$\omega(G, k) = \sup \{ |G(u) - G(v)| : |u - v| \le k \} .$$

Lemma 3.1

Suppose Q is an operator defined on $C^{q_1}[o, 1]$ satisfying

(i) $(QG)(t) = \sum_{i=o}^{s_1} \lambda_i(G)\, \tilde{p}_i(t) , \quad o \le t \le 1 , \quad s_1 < \infty$

where $\{\tilde{p}_i\}_{i=o}^{s_1}$ is a basis for a subspace X_1 of C[o, 1], and $\{\lambda_i\}_{i=o}^{s_1}$ is a set of bounded linear functionals mapping $C^{q_1}[o, 1]$ into R, independent over X_1;

(ii) there are constants C, α such that for all $G \in C^{q_1}[o, 1]$,

$$||QG - G|| \le C \ \omega(G^{(q_1)}, \alpha) .$$

If $g \in C^{q_1}[o, b]$, define the operators P_n by

$$(P_n g)(x) = (QG_j)\left(\frac{x-x_{j-1}}{\Delta x_j}\right) = \sum_{i=o}^{s_1} \lambda_i(G_j)\, \tilde{p}_i\left(\frac{x-x_{j-1}}{\Delta x_j}\right) , \quad x_{j-1} < x < x_j ,$$

$$1 \le j \le n ,$$

with $G_j(t) = g(x_{j-1} + t\,\Delta x_j)$, and let $(P_n g)(x_j)$, $o \le j \le n$, be the average of the left and right-hand limits. Then $\{P_n\}$ satisfies (2.6a) - (2.6c).

Proof

P_n clearly satisfies (2.6a). By the boundedness of λ_i we have

$$\sup_{x_{j-1}<x<x_j} |(P_n g)(x)| \le C \left[\sup_{o\le t\le 1} \sum_{i=o}^{q_1} |G_j^{(i)}(t)| \right] \left[\sum_{i=o}^{s_1} \sup_{o\le t\le 1} |\tilde{p}_i(t)| \right]$$

$$\le C \sup_{x_{j-1}\le x\le x_j} \sum_{i=o}^{q_1} |g^{(i)}(x)|(\Delta x_j)^i \le C \,||g||_{q_1},$$

where C is independent of j. This implies (2.6b). To verify (2.6c), notice that by (ii),

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)(x)| = \sup_{o<t<1} |(QG_j - G_j)(t)| \le C\ \omega(G_j^{(q_1)}, \alpha)$$

$$\le C\ (\Delta x_j)^{q_1}\ \omega(g^{(q_1)}, \alpha\Delta x_j) \le C\ \omega(g^{(q_1)}, \alpha|\Delta n|),$$

where C is independent of j. Hence $||P_n g - g||_{o,n} \le C\ \omega(g^{(q_1)}, \alpha|\Delta n|)$, and since $g^{(q_1)}$ is continuous and $|\Delta_n| \to o$ as $n \to \infty$, (2.6c) follows.

We now give some examples of operators Q which will give rise to the numerical methods, since they satisfy (2.6a) - (2.6c).

Example 3.1

Hermite interpolation. Consider a partition D_o of [o, 1] given by

$$D_o:\quad o \le \gamma_1 < \gamma_2 < \dots < \gamma_p \le 1 ,\quad p \ge 2 . \tag{3.1}$$

Let r_i be a nonnegative integer, $1 \leq i \leq p$, and let $q = \max_i r_i$. For $G \in C^q[o, 1]$ define Q_1G to be the polynomial in $\pi_{\tilde{p}-1}$ such that

$$(Q_1G)^{(i)}(\gamma_k) = G^{(i)}(\gamma_k), \quad 1 \leq k \leq p, \quad o \leq i \leq r_k,$$

where $\tilde{p} = p + \sum_{i=1}^{p} r_i$. We can write

$$(Q_1G)(t) = \sum_{k=1}^{p} \sum_{i=o}^{r_k} G^{(i)}(\gamma_k)\, \ell_{k,i}(t), \quad o \leq t \leq 1,$$

where

$$\ell_{k,i} \in \pi_{\tilde{p}-1} \quad \text{and} \quad \ell_{k,i}^{(v)}(\gamma_u) = \delta_{iv}\delta_{ku}, \quad 1 \leq u \leq p, \quad o \leq v \leq r_u.$$

Here, δ_{iv} is the Kronecker delta. An explicit expression for $\ell_{k,i}$ is found in [51]. If we let

$$\gamma_{ij} = x_{j-1} + \gamma_i \Delta x_j, \quad 1 \leq i \leq p, \quad 1 \leq j \leq n, \tag{3.2}$$

then

$$(P_ng)(x) = \sum_{k=1}^{p} \sum_{i=o}^{r_k} (\Delta x_j)^i\, g^{(i)}(\gamma_{kj})\, \ell_{k,i}\left(\frac{x-x_{j-1}}{\Delta x_j}\right), \quad x_{j-1}<x<x_j. \tag{3.3}$$

Clearly, Q_1 satisfies (i) of Lemma 3.1 with $q_1 = q$. We have been unable to find in the literature error bounds of the form (ii) of Lemma (3.1) with $q_1 = q$ for general Hermite interpolation. Most bounds given assume that $G \in C^{\tilde{p}}[o, 1]$; Birkhoff, Schultz and Varga [6] give error bounds for the case $p = 2$, but they require $G^{(q)}$ to be absolutely continuous. We will derive our error bounds based on the following results of Jackson [32, pp. 15-17].

Theorem 3.1

If $G \in C[o,1]$, then for each $N = 1, 2, 3, \ldots$, there exists a polynomial $p_N \in \pi_N$ such that $||G - p_N|| \leq C\, \omega(G, 1/N)$, where C is independent of N and G. If $G \in C^v[o, 1]$ for some $v \geq 1$, then for each $N > v - 1$ there exists a polynomial $p_N \in \pi_N$ such that $||G - p_N|| \leq C\, ||G^{(v)}||/N^v$, where C is independent of N and G.

Lemma 3.2

If $G \in C^q[o, 1]$ then

$$||(Q_1G - G)^{(i)}|| \leq C\, \omega(G^{(q)}, \frac{1}{\tilde{p}-q-1}) , \quad o \leq i \leq q ,$$

where C is independent of G. If $G \in C^{(q+u)}[o, 1]$ for some u such that $1 \leq u \leq \tilde{p}-q-1$, then

$$||(Q_1G - G)^{(i)}|| \leq C\, ||G^{(q+u)}||, \quad o \leq i \leq q,$$

where C is independent of G.

Proof

Suppose $r_k = q \geq 1$. Define an operator $\tilde{Q}: C[o, 1] \rightarrow \pi_{\tilde{p}-q-1}$ as follows: if $z \in C[o, 1]$, let

$$S(t) = \int_{\gamma_k}^{t} \frac{(t-u)^{q-1}}{(q-1)!} z(u)du , \quad o \leq t \leq 1.$$

Then define $\tilde{Q}z = (Q_1S)^{(q)}$. One verifies from this definition that for any function $G \in C^q[o, 1]$,

$$\tilde{Q}(G^{(q)}) = (Q_1G)^{(q)} , \qquad (3.4)$$

since if

$$G^*(t) \equiv G(t) - \sum_{i=o}^{q-1} G^{(i)}(\gamma_k)\ (t-\gamma_k)^i/i! = \int_{\gamma_k}^{t} (t-u)^{q-1}/(q-1)!\ G^{(q)}(t)\ dt,$$

then $\tilde{Q}(G^{(q)}) = (Q_1G^*)^{(q)} = (Q_1G)^{(q)}$, because Q_1 preserves polynomials of degree $\leq q-1$. If v is any polynomial in $\pi_{\tilde{p}-q-1}$, then $\tilde{Q}v = (Q_1V)^{(q)} = v$, where

$$V(t) = \int_{\gamma_k}^{t} \frac{(t-u)^{q-1}}{(q-1)!} v(t)\ dt .$$

Thus,

$$(Q_1G - G)^{(q)} = (Q_1G^*)^{(q)} - G^{(q)} + v - \tilde{Q}v = v - G^{(q)} - (Q_1(V-G^*))^{(q)} .$$

However, there is a constant C independent of G and v such that

$$||(Q_1(V-G^*))^{(q)}|| \leq C \max_{\substack{1\leq k\leq p \\ o\leq i\leq r_k}} |(V-G^*)^{(i)}(\gamma_k)| \leq C \max_{o\leq i\leq q} ||(V-G^*)^{(i)}||$$

$$\leq C\ ||v - G^{(q)}|| .$$

Therefore

$$||(Q_1\ G-G)^{(q)}|| \leq C\ ||v-G^{(q)}|| \quad \text{for any} \quad v \in \pi_{\tilde{p}-q-1} .$$

This inequality and Theorem 3.1 applied to $G^{(q)}$ imply that if $G \in C^q[o, 1]$, then

$$||(Q_1G-G)^{(q)}|| \leq C\ \omega(G^{(q)}, \frac{1}{\tilde{p}-q-1}) ,$$

and if $G \in C^{(q+u)}[o, 1]$, $1 \leq u \leq \tilde{p} - q - 1$, then

$$||(Q_1G-G)^{(q)}|| \leq C\ ||G^{(q+u)}|| .$$

Now these results are combined with

$$(Q_1G-G)^{(q-i-1)}(t) = \int_{\gamma_K}^{t} (Q_1G-G)^{(q-i)}(u)\ du , \quad o \leq i \leq q-1 ,$$

to obtain the final inequalities for the case $q \geq 1$. If $q = o$, then

$$Q_1 G - G = v - G + Q_1(v - G)$$

for any $v \in \pi_{\tilde{p}-1}$, so that $||Q_1 G - G|| \leq C \, ||v - G||$, where C is independent of G and v. The rest of the proof is again an application of Theorem 3.1.

It will be useful to have error bounds as in Lemma 3.2, but for derivatives higher than q.

Lemma 3.3

If $G \in C^{\tilde{p}}[o, 1]$, then

$$||(Q_1 G - G)^{(i)}|| \leq C \, ||G^{(\tilde{p})}|| \, , \quad o \leq i \leq \tilde{p} - 1 \, ,$$

where C is independent of G.

Proof

By Rolle's Theorem we see that $(Q_1 G)^{(i)}$ is a Hermite interpolant of degree $\tilde{p} - i - 1$ of $G^{(i)}$ in $[o, 1]$, $o \leq i \leq \tilde{p} - 1$. Since $G^{(i)} \in C^{\tilde{p}-i}[o, 1]$ then by the standard error bound for Hermite interpolation [30, p. 256] we have the result of the lemma.

Lemma 3.4

Let $\{P_n\}$ be the sequence of operators obtained from Q_1. Then there is a constant C independent of n such that

(i) for all $g \in C^q[o, b]$,

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)(x)| \leq C(\Delta x_j)^q \, \omega(g^{(q)}, \frac{\Delta x_j}{\tilde{p} - q - 1}) \, ;$$

(ii) for all $g \in C^{q+u}[o, b]$, where $1 \leq u \leq \tilde{p} - q - 1$,

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)^{(i)}(x)| \le C\ (\Delta x_j)^{q+u-i} \sup_{x_{j-1}<x<x_j} |g^{(q+u)}(x)| ,$$

$$o \le i \le q \quad ;$$

(iii) for all $g \in C^{\tilde{p}}[o, b]$,

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)^{(i)}(x)| \le C\ (\Delta x_j)^{\tilde{p}-i} \sup_{x_{j-1}<x<x_j} |g^{(\tilde{p})}(x)| ,$$

$$o \le i \le \tilde{p} - 1 .$$

Proof

The proof follows easily from the inequality

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)^{(i)}(x)| \le \frac{||(Q_1 G_j - G_j)^{(i)}||}{(\Delta x_j)^i} ,$$

and from Lemmas 3.2 and 3.3 since

$$\omega(G_j^{(q)}, \frac{1}{\tilde{p}-q-1}) \le (\Delta x_j)^q\ \omega(g^{(q)}, \frac{\Delta x_j}{\tilde{p}-q-1})$$

and

$$||G_j^{(q+u)}|| \le (\Delta x_j)^{q+u} \sup_{x_{j-1}\le x\le x_j} |g^{(q+u)}(x)| .$$

Example 3.2

When we construct methods based on the operator Q_1 of Example 3.1, we will have to restrict q so that $0 \le q \le s - 1 - m$. One way to get around this difficulty is to pick a less general partition D_1 of [o, 1] given by

$$D_1 : \quad 0 = \gamma_1 < \gamma_2 < \dots < \gamma_p = 1,\ p \ge 2 , \tag{3.5}$$

and to restrict r_i so that

$$0 \leq r_p - r_1 \leq 1 \quad \text{and} \quad 0 \leq r_i \leq r_p = q \geq 1, \quad 2 \leq i \leq p-1 . \qquad (3.6)$$

In practice one uses the operator Q_1 with D_1 and r_i as above and with any $q \geq 1$, but the theory is carried out in a transformed problem using the operator $\tilde{Q}$: $C[o, 1] \to \pi_{\tilde{p}-q-1}$ defined as follows:
for $z \in C[o, 1]$, let

$$S(t) = \int_o^t \frac{(t-u)^{q-1}}{(q-1)!} z(u)\, du , \quad o \leq t \leq 1 .$$

Then define $\tilde{Q}z = (Q_1 S)^{(q)}$, where Q_1 is the Hermite interpolating operator of Example 3.1 with the restrictions (3.5) and (3.6). We will show that $\tilde{Q}$ satisfies the hypothesis of Lemma 3.1 with $q_1 = 0$. By definition,

$$(\tilde{Q}z)(t) = \sum_{k=1}^{p} \sum_{i=o}^{r_k} S^{(i)}(\gamma_k)\, \ell_{k,i}^{(q)}(t) , \quad o \leq t \leq 1 ,$$

hence, (i) of Lemma 3.1 holds. Finally, $\tilde{Q}z - z = (Q_1 S - S)^{(q)}$, so (ii) of Lemma (3.1) follows from the first inequality of Lemma 3.2. Thus the operators $\tilde{P}_n$ obtained from $\tilde{Q}$ satisfy (2.6a) - (2.6c).

We will not need error bounds for $\tilde{P}_n g - g$ similar to those of Lemma 3.4. Instead it will be convenient to have some results on the relationship between the operators $\tilde{P}_n$ of this example and the operators P_n of Example 3.1.

<u>Lemma 3.5</u>

Let Q_1 be the operator of Example 3.1, but with the restrictions (3.5) and (3.6), and let $\tilde{Q}$ be the operator of Example 3.2. Then for any $G \in C^q[o, 1]$,

$$\int_o^t \frac{(t-u)^{s-1}}{(s-1)!} (Q_1 G)(u)\, du = \sum_{i=o}^{q-1} G^{(i)}(o) \frac{t^{i+s}}{(i+s)!}$$

$$+ \int_o^t \frac{(t-u)^{s+q-1}}{(s+q-1)!} \tilde{Q}(G^{(q)})(u)\, du \quad .$$

Proof

Just as in the proof of (3.4), one can show that $(Q_1 G)^{(q)} = \tilde{Q}(G^{(q)})$. Hence,

$$(Q_1 G)(t) = \sum_{i=o}^{q-1} G^{(i)}(o) \frac{t^i}{i!} + \int_o^t \frac{(t-u)^{q-1}}{(q-1)!} \tilde{Q}(G^{(q)})(u)\, du \, ,$$

$$0 \le t \le 1 \quad .$$

The conclusion of the lemma is obtained integrating both sides of this equation s times.

Lemma 3.6

Let $\{P_n\}$ and $\{\tilde{P}_n\}$ be the sequences of operators obtained from the operators Q_1 and $\tilde{Q}$ of Lemma 3.5, respectively. Then for $g \in C^q[o, b]$,

$$\int_{x_{j-1}}^x \frac{(x-t)^{s-1}}{(s-1)!} (P_n g)(t)\, dt = \sum_{i=o}^{q-1} g^{(i)}(x_{j-1}) \frac{(x-x_{j-1})^{i+s}}{(i+s)!}$$

$$+ \int_{x_{j-1}}^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(g^{(q)})(t)\, dt \, ,$$

$$x_{j-1} \le x \le x_j \quad ,$$

and

$$\int_o^x \frac{(x-t)^{s-1}}{(s-1)!} (P_n g)(t)\, dt = \sum_{i=o}^{q-1} g^{(i)}(o) \frac{x^{i+s}}{(i+s)!}$$

$$+ \int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(g^{(q)})(t)\, dt ,$$

$$o \leq x \leq b \quad .$$

Proof

Let $G_j(t) = g(x_{j-1} + t\,\Delta x_j)$, $o \leq t \leq 1$. From the definition of $\tilde{P}_n$ we obtain

$$\tilde{P}_n(g^{(q)})(x) = \frac{1}{(\Delta x_j)^q} \tilde{Q}(G_j^{(q)}) \left(\frac{x - x_{j-1}}{\Delta x_j}\right) , \quad x_{j-1}<x<x_j \; . \tag{3.7}$$

Also, from the definition of P_n and since $(Q_1 G_j)^{(q)} = \tilde{Q}(G_j^{(q)})$,

$$(P_n g)^{(q)}(x) = \frac{1}{(\Delta x_j)^q} (Q_1 G_j)^{(q)} \left(\frac{x - x_{j-1}}{\Delta x_j}\right) = \tilde{P}_n(g^{(q)})(x) \quad ,$$

$$x_{j-1}<x<x_j \quad . \tag{3.8}$$

We now have, with $u = (t - x_{j-1})/\Delta x_j$, $z = (x - x_{j-1})/\Delta x_j$, and by Lemma (3.5) that

$$\int_{x_{j-1}}^x \frac{(x-t)^{s-1}}{(s-1)!} (P_n g)(t)\, dt = (\Delta x_j)^s \int_o^z \frac{(z-u)^{s-1}}{(s-1)!} (P_n g)(x_{j-1} + u\,\Delta x_j)\, du$$

$$= (\Delta x_j)^s \int_o^z \frac{(z-u)^{s-1}}{(s-1)!} (Q_1 G_j)(u)\, du = (\Delta x_j)^s \left[\sum_{i=o}^{q-1} G_j^{(i)}(o) \frac{z^{i+s}}{(i+s)!} \right.$$

$$+ \int_o^z \frac{(z-u)^{s+q-1}}{(s+q-1)!} \tilde{Q}(G_j^{(q)})(u)\, du \Bigg] = \sum_{i=o}^{q-1} g^{(i)}(x_{j-1}) \frac{(x-x_{j-1})^{i+s}}{(i+s)!}$$

$$+ \int_o^z \frac{(z-u)^{s+q-1}}{(s+q-1)!} (\Delta x_j)^{s+q} \tilde{P}_n(g^{(q)})(x_{j-1}+u\Delta x_j)\, du$$

$$= \sum_{i=o}^{q-1} g^{(i)}(x_{j-1}) \frac{(x-x_{j-1})^{i+s}}{(i+s)!} + \int_{x_{j-1}}^{x} \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(g^{(q)})(t)\, dt ,$$

$$x_{j-1} \le x \le x_j ,$$

where we have also used (3.7).

To prove the second part of the lemma, let $x \varepsilon [o, b]$ and suppose $x \varepsilon [x_{\ell-1}, x_\ell]$. Then by (3.8),

$$\int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(g^{(q)})(t)\, dt = \sum_{i=1}^{\ell-1} \left[\int_{x_{i-1}}^{x_i} \frac{(x-t)^{s+q-1}}{(s+q-1)!} (P_n g)^{(q)}(t)\, dt \right]$$

$$+ \int_{x_{\ell-1}}^{x} \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(g^{(q)})(t)\, dt \quad .$$

But integrating by parts and since $(P_n g)^{(r)}(x_u) = g^{(r)}(x_u)$, $o \le r \le q-1$, $u = i-1, i$, we have

$$\sum_{i=1}^{\ell-1} \int_{x_{i-1}}^{x_i} \frac{(x-t)^{s+q-1}}{(s+q-1)!} (P_n g)^{(q)}(t)\, dt = \sum_{i=1}^{\ell-1} \left[\sum_{r=o}^{q-1} g^{(r)}(x_i) \frac{(x-x_i)^{s+r}}{(s+r)!} \right.$$

$$\left. - g^{(r)}(x_{i-1}) \frac{(x-x_{i-1})^{s+r}}{(s+r)!} + \int_{x_{i-1}}^{x_i} \frac{(x-t)^{s-1}}{(s-1)!} (P_n g)(t)\, dt \right]$$

$$= \sum_{r=o}^{q-1} \left[g^{(r)}(x_{\ell-1}) \frac{(x - x_{\ell-1})^{s+r}}{(s+r)!} - g^{(r)}(o) \frac{x^{s+r}}{(s+r)!} \right]$$

$$+ \int_o^{x_{\ell-1}} \frac{(x - t)^{s-1}}{(s - 1)!} (P_n g)(t)\, dt = \int_o^x \frac{(x - t)^{s-1}}{(s - 1)!} (P_n g)(t)\, dt$$

$$- \int_{x_{\ell-1}}^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(g^{(q)})(t)\, dt - \sum_{r=0}^{q-1} g^{(r)}(o) \frac{x^{s+r}}{(s+r)!} \quad .$$

The last equality is obtained from the first part of the lemma. The second result of the lemma is now evident.

Example 3.3

Natural spline interpolation. Let D_2 be a uniform partition of [o, 1] given by

$$D_2: \; o = \gamma_o < \gamma_1 < \gamma_2 < \ldots < \gamma_p = 1 ,$$

where

$$\gamma_i = \frac{i}{p} \quad , \qquad o \leq i \leq p \quad , \quad \text{and} \quad p \geq 1 \quad .$$

Let $S_{2\ell-1}$ [o, 1] be the set of all functions v satisfying

$$v \in C^{2\ell-2} [o, 1] \quad ; \tag{3.9a}$$

$$v \in \pi_{2\ell-1} \text{ in } (\gamma_i, \gamma_{i+1}) \quad , \qquad o \leq i \leq \tilde{p} - 1 \quad ; \tag{3.9b}$$

and if $\ell > 1$, also

$$\begin{cases} v^{(\ell)}(o) = v^{(\ell+1)}(o) = \ldots = v^{(2\ell-2)}(o) = o \\ v^{(\ell)}(1) = v^{(\ell+1)}(1) = \ldots = v^{(2\ell-2)}(1) = o \quad . \end{cases} \tag{3.9c}$$

$S_{2\ell-1}$ [o, 1] is called the set of natural splines of degree $2\ell - 1$ having the knots $\gamma_o, \gamma_1, \ldots, \gamma_p$. We introduce a linear operator Q_3: $C[o, 1] \to S_{2\ell-1}$ [o, 1] by defining for $G \in C[o, 1]$ a function $Q_3G \in S_{2\ell-1}$ [o, 1] such that

$$(Q_3G)(\gamma_i) = G(\gamma_i) \quad , \quad o \le i \le p \quad .$$

As described in [47], Q_3G exists and is unique if $1 \le \ell \le p + 1$, which we assume is satisfied. We can write

$$(Q_3G)(t) = \sum_{i=o}^{p} G(\gamma_i)\ T_i(t) \quad , \quad o \le t \le 1 \quad ,$$

where

$$T_i \in S_{2\ell-1}[o, 1] \quad \text{and} \quad T_i(\gamma_j) = \delta_{ij} \quad .$$

If $\ell = 1$, Q_3 represents interpolation by continuous piecewise linear functions, and if $\ell = p + 1$, then $S_{2\ell-1}$ [o, 1] = π_p, i.e., Q_3 represents Lagrange interpolation. In general, Q_3G is a g-spline type II interpolate of G studied in [48], but the error bounds given there are not in the form we require.

Lemma 3.7

Suppose $1 < \ell \le p + 1$. Then if $G \in C[o, 1]$,

$$||Q_3G - G|| \le C\ \omega(G, \frac{1}{\ell - 1}) \quad ,$$

where C is independent of G. If also $G \in C^u[o, 1]$ for some u such that $1 \le u \le \ell - 1$, then

$$||Q_3G - G|| \le C\ ||G^{(u)}|| \quad ,$$

where C is independent of G. Moreover, if $G \in C^{\ell}[o, 1]$, then

$$||(Q_3G - G)^{(i)}|| \leq ||G^{(\ell)}|| , \quad o \leq i \leq \ell - 1 .$$

Proof

From the definition of Q_3 it is clear that $Q_3v = v$ for $v \in \pi_{\ell-1}$. Hence

$$||Q_3G - G|| = ||Q_3(G - v) - (G - v)|| \leq C||G - v|| \quad ,$$

for some C independent of G. The first two inequalities of the lemma follow directly from Theorem 3.1. If $G \in C^{\ell}[o, 1]$, we obtain the result, adapting some ideas of [1], as follows: let $x \in [o, 1]$; then in $\ell - 1$ consecutive subintervals containing x there is some ξ_x^i such that

$$(Q_3G - G)^{(i)}(\xi_x^i) = o \ , \qquad o \leq i \leq \ell - 1 \quad ,$$

by Rolle's theorem.

Hence,

$$|(Q_3\ G - G)^{(\ell-1)}(x)| = |\int_{\xi_x^{\ell-1}}^{x} (Q_3G - G)^{(\ell)}(t)\,dt| \leq \left[\int_o^1 [(Q_3G - G)^{(\ell)}(t)]^2\,dt\right]^{\frac{1}{2}} ,$$

and similarly,

$$||(Q_3G - G)^{(i)}|| \leq \left[\int_o^1 [(Q_3G - G)^{(\ell)}(t)]^2\,dt\right]^{\frac{1}{2}} , \quad o \leq i \leq \ell - 1 .$$

But by the first integral relation for g-splines (see for example [48, Thm. 16]),

$$||(Q_3G - G)^{(i)}|| \leq \left[\int_o^1 [G^{(\ell)}(t)]^2 dt\right]^{\frac{1}{2}} \leq ||G^{(\ell)}|| , \quad o \leq i \leq \ell - 1 ,$$

which concludes the proof of the lemma.

Lemma 3.8

Let P_n be the sequence of operators obtained from Q_3, with $1 < \ell \leq p + 1$. Then there is a constant C independent of n such that

(i) for all $g \varepsilon C[o, 1]$,

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)(x)| \leq C\ \omega(\frac{\Delta x_j}{\ell - 1}) \quad ;$$

(ii) for all $g \varepsilon C^u[o, 1]$, where u satisfies $1 \leq u \leq \ell - 1$,

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)(x)| \leq C\ (\Delta x_j)^u \sup_{x_{j-1}\leq x\leq x_j} |g^{(u)}(x)| \; ;$$

(iii) for all $g \varepsilon C^\ell[o, 1]$,

$$\sup_{x_{j-1}<x<x_j} |(P_n g - g)^{(i)}(x)| \leq C \Big[(\Delta x_j)^{\ell - i}$$

$$\sup_{x_{j-1}\leq x\leq x_j} |g^{(\ell)}(x)| \Big] , \quad o \leq i \leq \ell - 1 \; .$$

Proof

The proof is similar to that of Lemma 3.4, using Lemma 3.7.

Example 3.4

Local spline approximating operators. As an example where the operator Q is not necessarily a projection, we will use a subclass of the explicit polynomial spline operators of Lyche and Schumaker [40, ex. 3.4]. Let D_3 be a partition of [o, 1],

$$D_3 \colon o = \gamma_o < \gamma_1 < \ldots < \gamma_p = 1 \quad .$$

Let k be an integer, $k \geq 1$. Denote by S_k the set of all functions

$G \in C^{k-2}[o, 1]$ such that $G \in \pi_{k-1}$ in (γ_i, γ_{i+1}), $o \leq i \leq p-1$. S_k is the set of smooth polynomial splines of degree $k - 1$ with simple knots at γ_i.

To define a basis for S_k, extend D_3 to a nondecreasing sequence $\{\gamma_i\}_{1-k}^{p+k-1}$, with $\gamma_i < \gamma_{i+k}$. With $G_k(t; x) = (t - x)_+^{k-1}$, define

$$N_{i,k}(x) = (\gamma_{i+k} - \gamma_i)\ G_k(\cdot; x)[\gamma_i, \ldots, \gamma_{i+k}], \quad 1-k \leq i \leq p-1 ,$$

where $G_k(\cdot; x)[\gamma_i, \ldots, \gamma_{i+k}]$ is the kth order divided difference of $G_k(t; x)$ with respect to t.

Fix an integer ℓ, $1 \leq \ell \leq k$, and for each $i = 1-k, \ldots, p-1$, let $\{t_{iu}\}_{u=1}^{\ell}$ be distinct numbers in $[o, 1] \cap [\gamma_i, \gamma_{i+k}]$. The operator Q_4: $C[o, 1] \rightarrow S_k$ is defined by

$$(Q_4 G)(t) = \sum_{i=1-k}^{p-1} \sum_{u=1}^{\ell} a_{iu}\ G[t_{i1}, \ldots, t_{iu}]\ N_{i,k}(t), \quad o \leq t \leq 1 ,$$

with $a_{i1} = 1$ and

$$a_{iu} = \frac{\sum_{v=o}^{u-1} (-1)^v\ \mathrm{sym}_v(t_{i1}, \ldots, t_{iu-1})\ \mathrm{sym}_{u-1-v}(\gamma_{i+1}, \ldots, \gamma_{i+k-1})}{\binom{k-1}{u-1-v}} ,$$

$$2 \leq u \leq \ell .$$

Here $\mathrm{sym}_i(x_1, \ldots, x_r)$ is defined implicitely by

$$(x + x_1) \ldots (x + x_r) = \sum_{v=1}^{r+1} \mathrm{sym}_{r-v+1}(x_1, \ldots, x_r)\ x^{v-1} .$$

The error bounds that we need are readily obtained from [40, Thm. 5.3].

Lemma 3.9

Suppose $1 \leq \ell \leq k$. If $G \in C[o, 1]$, then are constants C, α independent of G such that

$$||Q_4 G - G|| \leq C\, \omega(G, \alpha) \quad .$$

If $G \in C^u[o, 1]$ for some u satisfying $1 \leq u \leq \ell$, then

$$||(Q_4 G - G)^{(i)}|| \leq C\, ||G^{(u)}|| \,, \qquad o \leq i \leq k - 1 \quad ,$$

where C is independent of G.

Bounds such as those of Lemma 3.8 are easily obtained, but we omit them since they will not be applied here.

Example 3.5

Moments. Let Q_5: $C[o, 1] \to \pi_{s_1}$, $s_1 \geq 1$, be given by

$$(Q_5 G)(t) = \sum_{i=o}^{s_1} \lambda_i(G)\, \tilde{p}_i(t) \,, \qquad o \leq t \leq 1 \quad ,$$

where

$$\lambda_i(G) = \int_o^1 t^i\, G(t)\, dt, \quad o \leq i \leq s_1 \quad ,$$

and $\lambda_i(\tilde{p}_k) = \delta_{ik}$. Since $Q_5\, v = v$ for all $v \in \pi_{s_1}$, then for $G \in C[o, 1]$,

$$||Q_5 G - G|| = ||Q_5(G - v) - (G - v)|| \leq C\, ||G - v|| \quad ,$$

where C is independent of G. By Theorem 3.1,

$$||Q_5 G - G|| \leq C\, \omega(G, \frac{1}{s_1}) \quad .$$

Since we will not refer to Q_5 anymore, we omit results like those of Lemma 3.7 and Lemma 3.8, which are easy to obtain.

CHAPTER IV

THE LINEAR PROBLEM

In this chapter we consider problem (1.1), (1.2) with

$$f(x;\ y) = r(x) + \sum_{k=o}^{m} b_k(x) y^{(k)}(x) \quad , \tag{4.1}$$

where r, $b_k \in C[o,\ b]$. The exact solution y satisfies (2.3), which for the linear problem we write as

$$y(x) = F(x) + \int_o^x G(x,\ t)\ w_1(t)\ \tilde{H}(t;\ y)\ dt, \quad o \leq x \leq b\ , \tag{4.2}$$

with

$$F(x) = \sum_{k=1}^{s} \alpha_k \phi_k(x) + \int_o^x G(x,\ t) r(t)\ dt \quad ,$$

$$\tilde{H}(x;\ y) = \sum_{k=o}^{m} c_k(x)\ y^{(k)}(x) \quad ,$$

and w_1, c_k defined by $w_1(x)\ c_k(x) = b_k(x) - a_k$, $o \leq k \leq m$. We assume that w_1, $c_k \in C[o,\ 1]$.

The approximate problem for the linear case is not (2.5), but

$$y_n(x) = F_n(x) + \int_o^x G(x,\ t)\ \ w_1(t)\ P_n(\tilde{H}(\cdot;\ y_n))(t)\, dt \quad , \quad o \leq x \leq b \tag{4.3}$$

with

$$F_n(x) = \sum_{k=1}^{s} \alpha_k^n\ \phi_k(x) + \int_o^x G(x,\ t)(P_n r)(t)\ dt \quad ,$$

where α_k^n is an approximation to α_k.

It is possible to use a different family of operators P_n in the definition of F_n. Such operators would only have to satisfy $||P_n r - r||_{o,n} \to o$ to obtain convergence of y_n to y, if $\alpha_k^n = \alpha_k$.

In practice, one finds y_n step-by-step from

$$y_n(x) = F_{n,j}(x) + \int_{x_{j-1}}^{x} G(x, t)\ w_1(t)\ P_n(\tilde{H}(\cdot;\ y_n))(t)\ dt\ ,$$

$$x_{j-1} \le x \le x_j\ , \tag{4.4}$$

with

$$F_{n,j}(x) = \sum_{k=1}^{s} \alpha_{k,j}^n\ \phi_k(x) + \int_{x_{j-1}}^{x} G(x,\ t)(P_n r)(t)\ dt\ ,$$

and $\alpha_{k,j}^n$ as in (2.7).

If $w \equiv 1$ and all $a_k = o$, the rate of convergence of y_n to y will depend on the smoothness of r and the b_k. If the b_k are smoother than r, one can transform problem (1.1), (1.2), with f as in (4.1), into another which has a smoother non-homogeneous term; this transformation comes from a regularization of the corresponding integral equation. Let

$$z(x) = y(x) - \int_o^x \frac{(x-t)^{s-1}}{(s-1)!}\, r(t)\ dt - \sum_{i=o}^{s-1} y^{(i)}(o)\ \frac{x^i}{i!}\ , \qquad o \le x \le b\ .$$

Then z is the solution of

$$z^{(s)}(x) = \sum_{k=o}^{m} b_k(x) \left[\int_o^x \frac{(x-t)^{s-1-k}}{(s-1-k)!}\, r(t)\ dt + \sum_{i=k}^{s-1} y^{(i)}(o)\ \frac{x^{i-k}}{(i-k)!}\right]$$

$$+ \sum_{k=o}^{m} b_k(x)\ z^{(k)}(x)\ , \qquad o \le x \le b\ ,$$

subject to $z^{(i)}(o) = o$, $o \le i \le s-1$.

In general, a quadrature will have to be used to deal with the integrals involving r. Once an approximation z_n to z has been obtained, the approximation y_n to y is defined thorugh the equation relating z to y.

In operator notation we write (4.2), (4.3) as

$$(I - K)y = F \quad , \tag{4.5}$$

$$(I - K_n)y_n = F_n \tag{4.6}$$

with

$$(Ku)(x) = \int_o^x G(x, t)\, w_1(t)\, \tilde{H}(t; u)\, dt, \quad o \leq x \leq b \quad , \tag{4.7}$$

$$(K_n u)(x) = \int_o^x G(x, t)\, w_1(t)\, P_n(\tilde{H}(\cdot; u))(t)\, dt, \quad o \leq x \leq b \ . \tag{4.8}$$

As we stated before, (4.5) and (4.6) are set in the Banach space $C^{s-1}[o, b]$.

Existence of Approximate Solutions and Error Bounds

It turns out that $||K - K_n||_{s-1} \not\to o$ as $n \to \infty$, in general, so that we cannot use the classical analysis based on Banach's theorem. However, $||K_n g - Kg||_{s-1} \to o$ as $n \to \infty$ for each $g \in C^{s-1}[o, b]$. This strong convergence, plus the fact that $\{K_n\}$ is a collectively compact family of operators if $\{P_n\}$ satisfies (2.6a) - (2.6c), will allow us to use the theory of Anselone [3] for the required analysis.

The following lemma [17, pp. 344-345] is basic for the compactness proofs.

Lemma 4.1

A subset A of $C^{s-1}[o, b]$ has compact closure if and only if

(i) A is bounded;

(ii) For every $\varepsilon > o$ there exists $\delta > o$ such that for all $u, v \in [o, b]$

with $|u - v| < \delta$ and all $g \in A$, it is true that $|g^{(s-1)}(u) - g^{(s-1)}(v)| < \varepsilon$.

Lemma 4.2

If w_1, $c_k \in C[o, b]$ then the operator K in (4.7) is a compact linear operator from $C^{s-1}[o, b]$ into $C^{s-1}[o, b]$.

Proof

K is clearly linear and defined on $C^{s-1}[o, b]$, since $o \leq m \leq s-1$. Also,

$$(Ku)^{(s-1)}(x) = \int_o^x \frac{\partial^{s-1} G(x, t)}{\partial x^{s-1}} w_1(t) \tilde{H}(t; y)\, dt \quad \varepsilon\ C[o, 1].$$

Let $B = \{z \in C^{s-1}[o, b]: \ ||z||_{s-1} \leq 1\}$ be the unit ball in $C^{s-1}[o, b]$ and let $KB = \{g \in C^{s-1}[o, b]: g = Kz \text{ for some } z \in B\}$.

If $g \in KB$, then

$$||g||_{s-1} = ||Kz||_{s-1} \leq \sup_{o \leq x \leq b} \sum_{i=o}^{s-1} \sum_{k=o}^{m} \int_o^b \left| \frac{\partial^i G(x, t)}{\partial x^i} w_1(t)\, c_k(t) \right| dt ,$$

so KB satisfies (i) of Lemma 4.1. Now let $\varepsilon > o$, $u, v \in [o, b]$, $g \in KB$. Then, with an application of the Mean Value Theorem

$$|g^{(s-1)}(u) - g^{(s-1)}(v)| = |(Kz)^{(s-1)}(u) - (Kz)^{(s-1)}(v)|$$

$$= \Big| \int_o^u \frac{\partial^s G(\xi, t)}{\partial x^s} (u - v)\, w_1(t)\, \tilde{H}(t; z)\, dt$$

$$- \int_u^v \frac{\partial^{s-1} G(v, t)}{\partial x^{s-1}} w_1(t)\, \tilde{H}(t; z)\, dt \Big| ,$$

where $u < \xi < v$.

Hence $|g^{(s-1)}(u) - g^{(s-1)}(v)| \leq C\, |u - v|$, with C independent of u, v, g. From this inequality, (ii) of Lemma 4.1 is immediate, and therefore K is compact.

Lemma 4.3

Let $\{P_n\}$ be a sequence of linear operators satisfying (2.6a) - (2.6c), and suppose $c_k \in C^{q_1}[o, b]$, $o \le k \le m$, where $o \le q_1 \le s - 1 - m$. Then the sequence of operators $\{K_n\}$ defined in (4.8) satisfies

(i) $K_n: \; C^{s-1}[o, b] \to C^{s-1}[o, b]$.

(ii) $||K_n g - Kg||_{s-1} \to o$ as $n \to \infty$ for any $g \in C^{s-1}[o, b]$.

(iii) $\{K_n\}$ is collectively compact, i.e., $\tilde{B} \equiv \bigcup_{n=1}^{\infty} K_n B$, where B is the unit ball in $C^{s-1}[o, b]$, has compact closure.

Proof

Property (i) is easily established as in the proof of Lemma 4.2. Let $g \in \tilde{B}$. Then $g = K_n z$ for some n and some $z \in B$. Hence,

$$||g||_{s-1} = ||K_n z||_{s-1} = \sup_{o \le x \le b} \sum_{i=o}^{s-1} \left| \int_o^x \frac{\partial^i G(x, t)}{\partial x^i} \, w_1(t) \, P_n(\tilde{H}(\cdot; z))(t) \, dt \right|$$

$$\le ||P_n(\tilde{H}(\cdot; z))||_{o,n} \sup_{o \le x \le b} \sum_{i=o}^{s-1} \int_o^x \left| \frac{\partial^i G(x, t)}{\partial x^i} \, w_1(t) \right| dt$$

$$\le C \, ||\tilde{H}(\cdot; z)||_{q_1} \le C \quad ,$$

where C is independent of n and g. The last two inequalities follow from (2.6b), $o \le q_1 \le s - 1 - m$ and $||z||_{s-1} \le 1$. Therefore $\tilde{B}$ is bounded.

Now let $\varepsilon > o$, $u, v \in [0, b]$, and $g = K_n z \in \tilde{B}$. Then, much as in the proof of Lemma 4.2.

$$|g^{(s-1)}(u) - g^{(s-1)}(v)| \le C \, |u - v| \, ||P_n \tilde{H}(\cdot; z)||_{o,n}$$

$$\le C \, |u - v| \, ||\tilde{H}(\cdot; z)||_{q_1} \le C \, |u - v| \quad ,$$

where C is independent of u, v, g. Once again, (ii) of Lemma 4.1 follows, and so $\tilde{B}$ has compact closure. To complete the proof, let $g \in C^{s-1}[o, b]$. Then

$$||K_n g - Kg||_{s-1} \leq C \; ||P_n \tilde{H}(\cdot;\, g) - \tilde{H}(\cdot;\, g)||_{o,n} \quad ,$$

where C is independent of n. Since $\tilde{H}(x;\, g) \in C^{q_1}[o, b]$, (ii) is implied by (2.6c).

The next theorem is a collection of several results of Anselone [3, Thm 1.6, Corol. 1.9], on which we will base the main results of this chapter.

Theorem 4.1

Let X be a Banach space and $K: X \to X$ be a compact linear operator such that $(I - K)^{-1}$ exists. Let $\{K_n\}$ be a sequence of linear operator satisfying

(i) $K_n: X \to X$;

(ii) $||K_n g - Kg|| \to o$ as $n \to \infty$ for each $g \in X$;

(iii) $\{K_n\}$ is collectively compact.

Then

1) $||(K_n - K)K_n|| \to o$ and $||(K_n - K)K|| \to o$ as $n \to \infty$;

2) there exists $N > o$ such that for all $n \geq N$, the operators $(I - K_n)^{-1}$ exist and are uniformly bounded.

Lemma 4.4

Suppose w_1, $c_k \in C[o, b]$ and let K be as in (4.7). Then $(I - K)^{-1}$ exists and is a bounded operator from $C^{s-1}[o, b]$ into $C^{s-1}[o, b]$. In fact,

$$||(I - K)^{-1}||_{s-1} \leq 1 + M_1 b e^{M_1 b} \quad ,$$

where

$$\max_{o \le k \le m} \sup_{o \le t \le x \le b} \sum_{i=o}^{s-1} \left| \frac{\partial^i G(x, t)}{\partial x^i} w_1(t) c_k(t) \right| \le M_1 .$$

Proof

According to the Fredholm alternative for compact operators, we need to show that for every $v \in C^{s-1}[o, b]$ there is some $u \in C^{s-1}[o, b]$ such that

$$(I - K) u = v .$$

Let M_2 be a constant satisfying

$$\sum_{k=o}^{m} \int_o^b |v^{(k)}(t)| \, dt \le M_2 .$$

We then have

$$\sum_{i=o}^{s-1} |(Kv)^{(i)}(x)| = \sum_{i=o}^{s-1} \left| \int_o^x \frac{\partial^i G(x,t)}{\partial x^i} w_1(t) \sum_{k=o}^{m} c_k v^{(k)}(t) \, dt \right|$$

$$\le M_1 M_2 ,$$

$$\sum_{i=o}^{s-1} |(K^2 v)^{(i)}(x)| \le M_1 \int_o^x \sum_{k=o}^{m} |(Kv)^{(k)}(t)| \, dt \le M_1^2 M_2 x ,$$

and in general

$$\sum_{i=o}^{s-1} |(K^u v)^{(i)}(x)| \le M_1^u M_2 \frac{x^{u-1}}{(u-1)!} , \quad o \le x \le b, \quad u = 1, 2, 3, \ldots .$$

For each $i = o, 1, \ldots, s - 1$, let

$$R_i(x) = v^{(i)}(x) + \sum_{u=1}^{\infty} (K^u v)^{(i)}(x) ,$$

where each series converges uniformly on [o, b] since $(K^u v)^{(i)}(x)$ is dominated by $M_1^u M_2 \dfrac{x^{u-1}}{(u-1)!}$. Moreover, $R_i(x) = R_o^{(i)}(x)$. Now

$$KR_o = Kv + \sum_{u=1}^{\infty} K^{u+1}v = \sum_{u=1}^{\infty} K^u v = R_o - v \quad ,$$

that is, $(I - K)R_o = v$. Hence $(I - K)^{-1}$ exists and is bounded. Finally, if $||v||_{s-1} \leq 1$, then

$$||(I-K)^{-1}v||_{s-1} = ||v + \sum_{u=1}^{\infty} K^u v||_{s-1} \leq ||v||_{s-1} + ||\sum_{u=1}^{\infty} K^u v||_{s-1}$$

$$\leq 1 + M_1 b e^{M_1 b} \quad .$$

The next theorem is the most important of this chapter. It can be used subsequently in conjunction with the operators of Chapter III to obtain practical results about specific methods.

Theorem 4.2

Consider equations (4.5), (4.6) with $w_1 \in C[o, b]$ and r, $c_k \in C^{q_1}[o, b]$, for some q_1 such that $o \leq q_1 \leq s - m - 1$. Suppose $\{P_n\}$ is a sequence of linear operators satisfying (2.6a) - (2.6c), and let y be the solution of equation (4.5). Then there exists N > o such that for all $n \geq N$, the operators $(I - K_n)^{-1}$ exist and are uniformly bounded. For each $n \geq N$, the solution of y_n of (4.6) satisfies

$$||y - y_n||_{s-1} \leq ||(I - K_n)^{-1}||_{s-1} \; ||F_n - F + (K_n - K)y||_{s-1} \quad ,$$

and

$$||y - y_n||_{s-1} \leq ||(I - K)^{-1}||_{s-1} \; ||F_n - F + (K_n - K)y_n||_{s-1} \; .$$

Proof

By Lemmas 4.2, 4.3 and 4.4, the hypotheses of Theorem 4.1 are satisfied. The error estimates follow from

$$(I - K_n)(y_n - y) = F_n - F + K_n y - Ky$$

and

$$(I - K)(y_n - y) = F_n - F + K_n y_n - K y_n \quad .$$

More useful error estimates are given in the following corollary. Notice that the last estimate, together with Lemma 4.4 gives a totally explicit a-posteriori error bound in terms of how closely y_n satisfies the original differential equation

Corollary 4.1

Assume all the hypothesis of Theorem 4.2 and suppose the α_k^n in (4.3) are chosen so that $\sum_{k=1}^{s} \alpha_k^n \phi_k$ is the solution of $Lu = o$, $u^{(i)}(o) = g_i^n$, $o \le i \le s - 1$, where the g_i^n are arbitrary numbers. Let $\varepsilon_n = \max_{o \le i \le s-1} \{|y^{(i)}(o) - g_i^n|\}$. Then there exists a constant C independent of n such that

$$||y - y_n||_{s-1} \le C(\varepsilon_n + ||P_n r - r||_{o,n} + ||P_n \tilde{H}(\cdot; y) - \tilde{H}(\cdot; y)||_{o,n})$$

and

$$||y - y_n||_{s-1} \le ||(I - K)^{-1}||_{s-1} (\varepsilon_n + ||P_n r - r||_{o,n}$$

$$+ ||P_n \tilde{H}(\cdot; y_n) - \tilde{H}(\cdot; y_n)||_{o,n}) \quad .$$

However, if $w_1 \equiv 1$ then

$$||y - y_n||_{s-1} \leq C(\varepsilon_n + ||P_n Ly - Ly||_{o,n})$$

and

$$||y - y_n||_{s-1} \leq ||(I - K)^{-1}||_{s-1} (\varepsilon_n + ||y_n^{(s)} - f(\cdot; y_n)||_{o,n}) .$$

Proof

By construction, $\sum_{k=1}^{s} (\alpha_k - \alpha_k^n)\phi_k$ is the unique solution of $Lu = 0$,

$$u^{(i)}(o) = y^{(i)}(o) - g_i^n \quad , \quad o \leq i \leq s - 1 .$$

Hence $\max_{1\leq k\leq s} \{|\alpha_k - \alpha_k^n|\} \leq C\, \varepsilon_n$, where C is a constant independent of n, and

$$||F_n - F||_{s-1} \leq C(\varepsilon_n + ||P_n r - r||_{o,n}) .$$

Since

$$||(K_n - K)z||_{s-1} \leq C \; ||P_n\tilde{H}(\cdot; z) - \tilde{H}(\cdot; z)||_{o,n}$$

for any $z \in C^{s-1}[o, b]$, the first two inequalities of the corollary follow from the theorem. If $w_1 \equiv 1$, one just observes that

$$F_n - F + (K_n - K)y = \sum_{k=1}^{s} (\alpha_k^n - \alpha_k)\phi_k + \int_o^x G(x, t)\, (P_n Ly - Ly)(t)\, dt$$

and

$$F_n - F + (K_n - K)y_n = \sum_{k=1}^{s} (\alpha_k^n - \alpha_k)\phi_k + \int_o^x G(x, t)\, (y_n^{(s)} - f(\cdot; y_n))(t)\, dt,$$

and uses the theorem again.

Corollary 4.2

Assume all the hypothesis of Corollary 4.1, but let $g_i^n = y^{(i)}(o)$,

$o \le i \le s-1$, $n = 1, 2, 3, \ldots$. Then $||y - y_n||_{s-1} \to o$ as $n \to \infty$.

Proof

The conclusion follows from Corollary (4.1) and (2.6c), since

$$r,\ \tilde{H}(\cdot;\ y) \in C^{q_1}[o,\ b] \quad .$$

Numerical Solution of the Approximate Linear Problem

The solution y_n of the approximate problem (4.3) is found step-by-step, solving equation (4.4) at each step. Equation (4.4) is equivalent to an algebraic system of linear equations for the operators P_n of Chapter III. There are several ways to obtain such systems; the size of the system can vary drastically with the approach used.

Let P_n be obtained from an operator Q satisfying the hypothesis of Lemma 3.1, and suppose d_{ij} is defined by

$$d_{ij}(x) = \int_{x_{j-1}}^{x} G(x,t)\, w_1(t)\, \tilde{p}_i\left(\frac{t - x_{j-1}}{\Delta x_j}\right) dt \quad , \qquad o \le i \le s_1 \ , \qquad 1 \le j \le n \ . \tag{4.9}$$

Then we see that y_n in (4.4) has the form

$$y_n(x) = F_{n,j}(x) + \sum_{i=o}^{s_1} b_{ij}\, d_{ij}(x) \ , \qquad x_{j-1} \le x \le x_j \tag{4.10}$$

Notice that $F_{n,j}(x)$ is known explicitly in the interval $[x_{j-1}, x_j]$ and that the integral defining d_{ij} can be evaluated exactly if for example $w_1 \equiv 1$ and $\tilde{p}_i$ is a polynomial, which is usually the case. If $w_1 \not\equiv 1$, the choice of w_1 may depend on whether or not d_{ij} can be found explicitly.

Operating with L on both sides of (4.4) and (4.10) and combining the results, we obtain

$$w_1(x)\left[\sum_{i=o}^{s_1} [b_{ij} - \lambda_i(\tilde{H}(x_{j-1} + (\cdot)\Delta x_j;\ y_n))]\ \tilde{p}_i(\frac{x - x_{j-1}}{\Delta x_j})\right] = 0 \quad ,$$

$$x_{j-1} < x < x_j \quad .$$

Assuming now that w_1 has at most a countable number of zeroes in [o, b], and by the independence of the $\tilde{p}_i$, it follows that the b_{ij} satisfy the $(1 + s_1)x(1 + s_1)$ system

$$b_{ij} - \sum_{u=o}^{s_1} b_{uj}\ z_{iu} = w_i \quad , \qquad o \leq i \leq s_1 \quad , \tag{4.11}$$

where

$$w_i = \lambda_i(\sum_{k=o}^{m} (c_k\ F_{n,j}^{(k)})(x_{j-1} + (\cdot)\Delta x_j)) \quad ,$$

and

$$z_{iu} = \lambda_i(\sum_{k=o}^{m} (c_k\ d_{uj}^{(k)})(x_{j-1} + (\cdot)\Delta x_j)) \quad .$$

Thus, if y_n is a solution of (4.4) in the form (4.10), then (4.11) is satisfied.

We can show that conversely, a solution of (4.11) produces a solution of (4.4) through (4.10). One should notice that

$$d_{uj}^{(k)}(x_{j-1} + t\Delta x_j) = \int_{x_{j-1}}^{x_{j-1}+t\Delta xj} \frac{\partial^k G(x_{j-1} + t\Delta x_j,\ u)}{\partial x^k}\ w_1(u)\ \tilde{p}_i(\frac{u - x_{j-1}}{\Delta x_j})\ du$$

$$= (\Delta x_j) \int_o^t \frac{\partial^k G(x_{j-1} + t\Delta x_j,\ x_{j-1} + v\Delta x_j)}{\partial x^k}\ w_1(x_{j-1} + v\Delta x_j)\ \tilde{p}_i(v)\ dv$$

$$= O((\Delta x_j)^{s-k}) \quad ,$$

since $\frac{\partial^k G(x, t)}{\partial x^k}$ has a zero of multiplicity $s - 1 - k$ at $x = t$. Therefore $z_{iu} = O(|\Delta_n|^{s-m})$, so that (4.11) has a unique solution for n sufficiently large, which could be obtained by simple iteration.

Since we will use the operators P_n of Example 3.1 frequently, it is convenient to give equations (4.11) explicitly for this case. The function y_n has the form

$$y_n(x) = F_{n,j}(x) + \sum_{k=1}^{p} \sum_{i=o}^{r_k} b_{kij}\, d_{kij}(x) \quad , \quad x_{j-1} \leq x \leq x_j \quad , \qquad (4.12)$$

where

$$d_{kij}(x) = \int_{x_{j-1}}^{x} G(x, t)\, w_1(t)\, \ell_{k,i}\left(\frac{t - x_{j-1}}{\Delta x_j}\right) dt \quad .$$

If w_1 has at most a countable number of zeroes in [o, b], then

$$b_{kij} - (\Delta x_j)^i \sum_{v=1}^{p} \sum_{\ell=o}^{r_v} b_{v\ell j}\left(\sum_{u=o}^{m} c_u\, d_{v\ell j}^{(u)}\right)^{(i)}(\gamma_{kj}) = (\Delta x_j)^i \left(\sum_{u=o}^{m} c_u F_n^{(u)}\right)^{(i)}(\gamma_{kj}),$$

$$1 \leq k \leq p, \qquad o \leq i \leq r_k \quad , \qquad 1 \leq j \leq n \qquad (4.13)$$

Returning to the general case, suppose P_n is obtained from a projection Q, i.e., $Q^2 = Q$, but let $w_1 \equiv 1$. Equation (4.4) implies

$$(Ly_n)(x_{j-1} + t\Delta x_j) = \sum_{i=o}^{s_1} \lambda_i((r + \tilde{H}(\cdot;\, y_n))(x_{j-1} + (\cdot)\Delta x_j))\, \tilde{p}_i(t)$$

$$= Q((r + \tilde{H}(\cdot;\, y_n))(x_{j-1} + (\cdot)\Delta x_j)(t) \quad , \qquad o < t < 1 \; .$$

Operating with Q on both sides of this equation (interpreting $(Ly_n)\,(x_{j-1} + t\Delta x_j)$ as defined in all of $[o, 1]$ by its limiting values) we obtain

$$\lambda_i((Ly_n)(x_{j-1} + (\cdot)\Delta x_j)) = \lambda_i((r + \tilde{H}(\cdot;\, y_n))(x_{j-1} + (\cdot)\Delta x_j))$$

or

$$\lambda_i(y_n^{(s)}(x_{j-1} + (\cdot)\Delta x_j)) = \lambda_i((r + \sum_{k=o}^{m} b_k y_n^{(k)})(x_{j-1} + (\cdot)\Delta x_j)) \ , \quad o \leq i \leq s_1. \tag{4.14}$$

Equations (4.14) refer directly to the original differential equation. The constants a_k are of course taken into account in the form of y_n given by (4.10). If, for example, P_n is as in Example 3.1, then (4.14) says that y_n satisfies the original differential equation and the r_k-times differentiated original differential equation at γ_{kj}, $1 \leq k \leq p$; if in addition each $a_k = 0$, then it is clear that $y_n \in C^{s-1}[o, b]$ is just a polynomial of degree $\leq \tilde{p} + s - 1$ in $[x_{j-1}, x_j]$, whose coefficients can be found from (4.14) and the continuity requirements.

Instead of finding the coefficients of y_n as we have shown above, it may be possible, and sometimes more convenient, to find values of y_n and its derivatives at certain points in $[x_{j-1}, x_j]$; these values then determine y_n uniquely in the subinterval. We illustrate the method for the operators of Example 3.1, restricted so that $\gamma_1 = 0$, $\gamma_p = 1$, $0 \leq r_p - r_1 \leq 1$, $0 \leq r_i \leq r_p = q$, $q \geq 1$. Suppose $w_1 \in C^{q-1}[o, b]$. From (4.4), we can write

$$y_n(x) = F_{n,j}(x) + \sum_{k=1}^{p} \sum_{i=o}^{r_k} (\Delta x_j)^i (\sum_{u=o}^{m} c_u\, y_n^{(u)})^{(i)} (\gamma_{kj})\, d_{kij}(x) \ ,$$

$$x_{j-1} \leq x \leq x_j \quad , \tag{4.15}$$

with d_{kij} as in (4.12). Hence $y_n(x)$ is determined uniquely in $[x_{j-1}, x_j]$

by the values $y_n^{(v)}(\gamma_{kj})$, $1 \leq k \leq p$, $o \leq v \leq m + r_k$. Some of these values may not be present in (4.15), depending on the functions c_u. If $y_n^{(v)}(\gamma_{kj})$ appears in the right-hand side of (4.15), one differentiates both sides of (4.15) v times and evaluates at γ_{kj}, obtaining a linear system of equations for these values. The maximum size of the system is $p(m + 1) + \sum_{i=1}^{p} r_i$, whereas system (4.13) has maximum size $p + \sum_{i=1}^{p} r_i$. Of course these numbers are the same for the important case m = 0. However, the number of unknowns $y_n^{(v)}(\gamma_{kj})$ can be reduced considerably if for example $r_k \geq s - m$, $1 \leq k \leq p$. The reason is that then we can express $y_n^{(s+u)}(\gamma_{kj})$, $o \leq u \leq m - s + r_k$, in terms of $y_n^{(u)}(\gamma_{kj})$, $0 \leq u \leq s - 1$, using the equation

$$(Ly_n)(x) = (P_n r)(x) + w_1(x)\; P_n(\sum_{k=o}^{m} c_k\, y_n^{(k)})(x), \quad x_{j-1} \leq x \leq x_j \quad ,$$

obtained from (4.4), since $y_n \in C^{s+q-1}[o, b]$ and $0 \leq m - s + r_i \leq q - 1$. In addition, $y_n^{(i)}(\gamma_{1j})$, $o \leq i \leq s - 1$ is known by continuity. Therefore the maximum number of unknowns drops to $(p - 1)s$, quite an improvement over the size of system (4.11). A reduction in size is always possible if $r_k \geq s - m$ for at least some k. This approach allows the methods to be easily set up as discrete methods, especially if $\gamma_1 = 0$, $\gamma_2 = 1$. Another advantage is that we can construct "predictor-corrector" methods in the nonlinear case, as we shall see in Chapter V.

In Chapter VI we will point out how different choices of P_n lead to and extend methods which have been discussed in recent years.

Applications of the Theory to the Linear Problem

We conclude this Chapter with some theorems which are typical of the results one can obtain from Theorem 4.2 and Corollary 4.1.

Theorem 4.3

Consider the initial value problem (1.1), (1.2) with f given by (4.1). Let q be an integer satisfying $o \leq q \leq s - 1 - m$, and let a_k, $o \leq k \leq m$, and g_i^n, $o \leq i \leq s - 1$, be arbitrary constants. Suppose $w_1(x)\ c_k(x) = b_k(x) - a_k$, $o \leq x \leq b$, $o \leq k \leq m$, where each $c_k \in C^q[o, b]$, and where $w_1 \in C[o, b]$ and has at most a countable number of zeroes in $[o, b]$, and define

$$\tilde{H}(x; y) = \sum_{k=o}^{m} c_k(x)\ y^{(k)}(x) \quad .$$

Suppose also that $r \in C^q[o, b]$.

Let $\{\Delta_n\}$ be a sequence of partitions of $[o, b]$ given by (2.4), with $|\Delta_n| \to o$ as $n \to \infty$, and let $\{P_n\}$ be the sequence of operators of Example 3.1.

Then there exists $N > o$ such that for each $n \geq N$, there is a unique function $y_n \in C^{s-1}[o, b]$ of the form (4.12) such that $y_n^{(i)}(o) = g_i^n$, $o \leq i \leq s - 1$, and whose coefficients satisfy (4.13). If y is the solution of (1.1), (1.2), the following estimates apply for $n \geq N$:

$$\text{(i)} \quad ||(y_n - y)^{(i)}||_o \leq C[\varepsilon_n + |\Delta_n|^q \left(\omega(r^{(q)}, \frac{\Delta_n}{\tilde{p} - q - 1}\right)$$

$$+ \omega(\tilde{H}^{(q)}, \frac{\Delta_n}{\tilde{p} - q - 1}))] \ , \qquad o \leq i \leq s - 1 \quad ,$$

where C is a constant independent of n, and

$$\varepsilon_n = \max_{o \leq i \leq s-1} \{|y^{(i)}(o) - g_i^n|\} \quad .$$

(ii) If in addition, $\varepsilon_n \leq C|\Delta_n|^{m_1}$ where C and m_1 are constants independent of n, and r, $\tilde{H}(\cdot; y) \in C^{q+u}[o, b]$ for some u, $1 \leq u \leq \tilde{p} - q$, then

$$||(y_n - y)^{(i)}||_o \leq C|\Delta_n|^{\min(m_1, q+u)}, \quad o \leq i \leq s - 1 ,$$

where C is independent of n.

In particular, if $u = \tilde{p} - q$, and if $w_1 \in C^{\tilde{p}-1}[o, b]$ and $\{\Delta_n\}$ is quasi-uniform, then

$$||(y_n - y)^{(s+i)}||_{o,n} \leq C|\Delta_n|^{\min(m_1, \tilde{p}) - i}, \quad o \leq i \leq \tilde{p} - 1 ,$$

where C is independent of n.

Proof

Choose α_k^n in (4.3) so that $F_n^{(i)}(o) = g_i^n$, $o \leq i \leq s - 1$. As shown in Example 3.1, $\{P_n\}$ satisfies (2.6a) - (2.6c) with $q_1 = q$. By Theorem 4.2, there exists N > o such that equation (4.3) has a unique solution y_n for $n \geq N$. In the previous section we have seen that y_n has the form (4.12) and its coefficients satisfy (4.13). Corollary 4.1 and Lemma 3.4 give the bounds for $||(y_n - y)^{(i)}||_o$, $o \leq i \leq s - 1$. To obtain the final estimate of the theorem, apply L to both sides of (4.2) and (4.4), subtract the resulting equations and differentiate i times. Write the result as

$$(y_n - y)^{(s+i)}(x) = (P_n r - r)^{(i)}(x) + [w_1 P_n(\tilde{H}(\cdot; y_n) - \tilde{H}(\cdot; y))]^{(i)}(x)$$
$$+ [w_1(P_n\tilde{H}(\cdot; y) - \tilde{H}(\cdot; y))]^{(i)}(x) + \sum_{k=o}^{m} a_k (y_n - y)^{(k+i)}(x) ,$$
$$x_{j-1} < x < x_j , \quad o \leq i \leq \tilde{p} - 1 . \tag{4.16}$$

The first and third terms in the right-hand side of (4.16) are $O(|\Delta_n|^{\tilde{p}-i})$ by Lemma 3.4. To bound the second term, notice that

$$(\tilde{H}(\cdot;y_n)-\tilde{H}(\cdot;y))^{(u)}(x)=(\sum_{k=o}^{m}c_k(y_n-y)^{(k)})^{(u)}(x) = O(|\Delta_n|^{\min(\tilde{p},m_1)}) ,$$

$$o \leq u \leq q \quad ,$$

since $||(y_n-y)^{(i)}||_o = O(|\Delta_n|^{\min(\tilde{p},m_1)})$, $o \leq i \leq s-1$, and $m+q \leq s-1$. Hence from (3.3) we see that the second term is $O(|\Delta_n|^{\min(\tilde{p},m_1)-i})$. If $o \leq i \leq s-m-1$, the fourth term is $O(|\Delta_n|^{\min(\tilde{p},m_1)})$. Therefore, the last inequality of (ii) is satisfied for $o \leq i \leq s-m-1$. If $s-m-1 < i \leq \tilde{p}-1$, the first, second, and third terms in the right-hand side of (4.16) are as before, but the fourth term is at least $O(|\Delta_n|^{\min(\tilde{p},m_1)-i})$. Again, the last inequality of (ii) is satisfied. This completes the proof.

Higher rates of convergence will be obtained in our treatment of the nonlinear problem for particular choices of the points γ_{kj}.

Some of the hypothesis of Theorem 4.3 can be weakened. For example, if r, $\tilde{H}(\cdot;y) \in C_n^{q+u}[o,b]$ for all n, where $1 \leq u \leq \tilde{p}-q$, then the estimates in (ii) of the theorem still hold. That is, a finite number of jump discontinuities in $r^{(q+u)}$ and $(\tilde{H}(\cdot;y))^{(q+u)}$ do not decrease the rates if those points belong to every partition Δ_n. The reason is that if $g \in C_n^{q+u}[o,b]$ then (ii) and (iii) of Lemma 3.4 still hold.

Since very often $w_1 \equiv 1$, we give next another version of Theorem 4.3 which takes this into account.

<u>Theorem 4.4</u>

Assume all the hypothesis of Theorem 4.3, but let $w_1 \equiv 1$. Then there exists $N > o$ such that for each $n \geq N$ there is a unique function $y_n \in C^{s-1}[o, b]$ of the form

$$y_n(x) = \sum_{k=1}^{s} \alpha_{k,j}^{n} \phi_k(x) + \sum_{k=1}^{p} \sum_{i=o}^{r_k} b_{kij} d_{kij}(x) , \quad x_{j-1} \leq x \leq x_j ,$$

which satisfies $y_n^{(i)}(o) = g_i^n$, $o \leq i \leq s-1$ and

$$y_n^{(s+v)}(\gamma_{kj}) = (r + \sum_{k=o}^{m} b_k y_n^{(k)})^{(v)}(\gamma_{kj}) , \quad 1 \leq k \leq p, \quad o \leq v \leq r_k ,$$

$$1 \leq j \leq n ,$$

where γ_{kj} is given by (3.2).

If y is the solution of (1.1), (1.2), the following estimates apply for $n \geq N$:

(i) $||(y_n - y)^{(i)}||_o \leq C(\varepsilon_n + |\Delta_n|^q \omega((Ly)^{(q)}, \frac{|\Delta_n|}{\tilde{p} - q - 1}))$, $o \leq i \leq s-1$,

where C is independent of n.

(ii) If the hypothesis on $r, \tilde{H}$ in (ii) of Theorem 4.3 are replaced by $y \in C^{s+q+u}[o, b]$, $1 \leq u \leq \tilde{p} - q$, the estimates there hold.

<u>Proof</u>

The only changes in the proof of Theorem 4.3 are due to (4.14) and the last inequality of Corollary 4.1.

So far we have required that $o \leq q \leq s - 1 - m$, since this condition is necessary in Theorem 4.2. Adapting an idea of Wittenbrink [59] to our approach, there is a way around this difficulty if $w_1 \equiv 1$, $a_k = o$, $o \leq k \leq m$ and we use the partition D_1 of (3.5) and the r_i of (3.6) in the definition of P_n in Example 3.1.

Theorem 4.5

Consider the initial value problem (1.1), (1.2) with f given by (4.1), where r, $b_k \in C^q[o, b]$, $o \leq k \leq m$, for some integer $q \geq 1$. Let D_1 be a partition of [o, 1] given by (3.5), and let r_i, $1 \leq i \leq p$, satisfy (3.6). Suppose $\{\Delta_n\}$ is a sequence of partitions of [o, b] given by (2.4) with $|\Delta_n| \to o$ as $n \to \infty$, and define γ_{kj} by (3.2). Let $\tilde{p} = p + \sum_{i=1}^{p} r_i$, and suppose y is the solution of (1.1), (1.2).

Then there exists N > o such that for all $n \geq N$ there exists a unique function $y_n \in C^{s+q-1}[o, b]$ which is a polynomial of degree $\tilde{p} + s - 1$ in each subinterval $[x_{j-1}, x_j]$ and which satisfies $y_n^{(i)}(o) = y^{(i)}(o)$, $o \leq i \leq s - 1$ and

$$y_n^{(s+v)}(\gamma_{kj}) = (r + \sum_{k=o}^{m} b_k y_n^{(k)})(\gamma_{kj}) , \quad 1 \leq k \leq p , \quad o \leq v \leq r_k ,$$

$$1 \leq j \leq n .$$

If $y \in C^{\tilde{p}+s}[o, b]$, there is a constant C independent of n such that

$$||(y - y_n)^{(i)}||_o \leq C|\Delta_n|^{\tilde{p}-q+1} , \qquad o \leq i \leq s + q - 1 .$$

In addition, if $\{\Delta_n\}$ is quasi-uniform, then

$$||(y - y_n)^{(s+i)}||_{o,n} \leq C|\Delta_n|^{\tilde{p}-i} , \qquad q \leq i \leq \tilde{p} - 1 .$$

where C is independent of n.

Proof

The solution y of (1.1), (1.2) is also the solution of the problem

$$y^{(s+q)}(x) = (r + \sum_{k=o}^{m} b_k y^{(k)})^{(q)}(x) , \qquad o \leq x \leq b , \tag{4.17}$$

subject to

$$y^{(i)}(o) = g_i \ , \qquad o \le i \le s + q - 1 \ , \tag{4.18}$$

where

$$g_i = (r + \sum_{k=o}^{m} b_k y^{(k)})^{(i-s)}(o) \ , \quad s \le i \le s+q-1 \ . \tag{4.19}$$

Let K and $\tilde{K}$ be operators defined by

$$(Ku)(x) = \int_o^x \frac{(x-t)^{s-1}}{(s-1)!} \sum_{k=o}^{m} b_k(t)\, u^{(k)}(t)\, dt \ ,$$

$$(\tilde{K}u)(x) = \int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} (\sum_{k=o}^{m} b_k u^{(k)})^{(q)}(t)\, dt, \quad o \le x \le b \ ,$$

and let

$$F(x) = \sum_{i=o}^{s-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s-1}}{(s-1)!} r(t)\, dt \ ,$$

$$\tilde{F}(x) = \sum_{i=o}^{s+q-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} r^{(q)}(t)\, dt, \quad o \le x \le b \ .$$

The integral equations equivalent to (1.1), (1.2) and (4.17), (4.18) are

$$(I - K)y = F \ , \tag{4.20}$$

and

$$(I - \tilde{K})y = \tilde{F} \ , \tag{4.21}$$

respectively. The approximate integral equations are

$$(I - K_n)y_n = F_n \ , \tag{4.22}$$

and

$$(I - \tilde{K}_n)\tilde{y}_n = \tilde{F}_n \quad , \tag{4.23}$$

respectively, where K_n and F_n are obtained from K and F using the operators P_n of Example 3.1 restricted by (3.5), (3.6); $\tilde{K}_n$ and $\tilde{F}_n$ are obtained from $\tilde{K}$ and $\tilde{F}$ using the operators $\tilde{P}_n$ of Example 3.2. That is,

$$F_n(x) = \sum_{i=o}^{s-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s-1}}{(s-1)!} (P_n r)(t)\, dt \quad ,$$

$$\tilde{F}_n(x) = \sum_{i=o}^{s+q-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n(r^{(q)}(t))\, dt,$$

and corresponding expressions for K_n and $\tilde{K}_n$.

Theorem 4.2 (with $q_1 = 0$ and s replaced by s + q) applies to equations (4.21) and (4.23). Hence there exists N > 0 such that for all $n \geq N$, (4.23) has a unique solution $\tilde{y}_n$, and there is a constant C independent of n such that

$$\begin{aligned} ||y - \tilde{y}_n||_{s+q-1} &\leq C\ ||\tilde{F}_n - \tilde{F} + \tilde{K}_n y - \tilde{K}y||_{s+q-1} \\ &= C\ ||\int_o^{(\cdot)} \frac{(\cdot - t)^{s+q-1}}{(s+q-1)!} [\tilde{P}_n(y^{(s+q)}) - y^{(s+q)}](t)\, dt||_{s+q-1} \\ &= C\ ||\int_o^{(\cdot)} \frac{(\cdot - t)^{s-1}}{(s-1)!} [P_n(y^{(s)}) - y^{(s)}]\,(t)\, dt||_{s+q-1} \\ &\leq C\ |\Delta_n|^{\tilde{p}-q+1} \quad , \end{aligned}$$

where we have also used Lemmas 3.6 and 3.4.

Now by (4.19), Lemma 3.6 and the fact that $\tilde{y}_n$ satisfies (4.18), we have

$$\tilde{y}_n(x) = \tilde{F}_n(x) + \tilde{K}_n\tilde{y}_n(x)$$

$$= \sum_{i=o}^{s+q-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!} \tilde{P}_n((r + \sum_{k=o}^{m} b_k \tilde{y}_n^{(k)})^{(q)})(t)\, dt$$

$$= \sum_{i=o}^{s-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s-1}}{(s-1)!} P_n(r + \sum_{k=o}^{m} b_k \tilde{y}_n^{(k)})(t)\, dt = F_n + K_n\tilde{y}_n \quad ,$$

that is, the solution $\tilde{y}_n$ of (4.23) is a solution of (4.22). Conversely, one shows that a solution y_n of (4.22) is a solution of (4.23). This equivalency implies all the conclusion of the theorem, except that last inequality.

To establish this inequality, we start from (4.16), but with $a_k = 0$, $w_1 \equiv 1$, which we write as

$$(y_n - y)^{(s+i)}(x) = [P_n(\sum_{k=o}^{m} b_k(y_n - y)^{(k)})]^{(i)}(x)$$

$$+ [P_n(y^{(s)}) - y^{(s)}]^{(i)}(x) \; , \quad x_{j-1} < x < x_j$$

$$q - 1 \leq i \leq \tilde{p} - 1 \quad . \tag{4.24}$$

In particular, if $i = q - 1$, then by the first estimate of the theorem and Lemma 3.4,

$$\sup_{x_{j-1}<x<x_j} (P_n(\sum_{k=o}^{m} b_k(y_n - y)^{(k)}))^{(q-1)}(x) = 0(|\Delta_n|^{\tilde{p}-q+1}) \quad .$$

Applying now Lemma 4.5 below to $(P_n(\sum_{k=o}^{m} b_k(y_n - y)^{(k)}))^{(q-1)}$ in each subinterval $[x_{j-1}, x_j]$ we have

$$\sup_{x_{j-1}<x<x_j} \left|\left(P_n\left(\sum_{k=o}^{m} b_k (y_n - y)^{(k)}\right)\right)^{(q-1+u)}(x)\right| = O(|\Delta_n|^{\tilde{p}-q+1-u}) ,$$

$$o \leq u \leq \tilde{p} - q \quad . \tag{4.25}$$

Finally, from (4.24), (4.25) and Lemma (3.4) we obtain

$$||(y_n - y)^{(s+v)}||_{o,n} \leq C\, |\Delta_n|^{\tilde{p}-v} , \quad q \leq v \leq \tilde{p} - 1 .$$

Lemma 4.5

If p_r is a polynomial of degree r, then

$$\sup_{a\leq x\leq b} |p_r'(x)| \leq \frac{2r^2}{b - a} \sup_{a\leq x\leq b} |p_r(x)| \quad .$$

Proof

This is Markov's inequality [54].

Even though Theorem 4.5 establishes the convergence of the method, the error estimates given there are not the best possible in general for the lower derivatives. We can improve the estimates for the important case m = 0 using the next lemma.

Lemma 4.6

Suppose $u \in C[o, b]$ and

$$o \leq u(x) \leq A + B \int_o^x u(t)\, dt , \quad o \leq x \leq b \quad ,$$

for some positive constants A, B. Then $u(x) \leq Ae^{Bx}$, $o \leq x \leq b$.

Proof

This is a special case of Gronwall's inequality. Let $W(x) = A + B \int_o^x u(t)\, dt$. Then $W'(x) = Bu(x) \leq BW(x)$, so $(W(x)e^{-Bx})' \leq 0$,

$o \le x \le b$. Therefore $W(x)e^{-Bx} \le W(o) = A$ and $u(x) \le W(x) \le Ae^{Bx}$.

Corollary 4.3

Consider problem (1.1), (1.2) with f given by (4.1) but with $m = 0$. Assume all the hypothesis of Theorem 4.5, and in addition let $\{\Delta_n\}$ be quasi-uniform. Then if $r, b_o \in C^{\tilde{p}}[o, b]$, there is a constant C independent of n such that

$$||(y - y_n)^{(i)}||_o \le C\ |\Delta_n|^{\tilde{p}} \quad , \quad o \le i \le s \quad ,$$

and

$$||(y - y_n)^{(s+i)}||_{o,n} \le C\ |\Delta_n|^{\tilde{p}-i} \quad , \quad o \le i \le \tilde{p} - 1 \quad .$$

However,

$$\max_{o \le j \le n} \{|(y - y_n)^{(s+i)}(x_j)|\} \le C\ |\Delta_n|^{\tilde{p}} \quad , \quad o \le i \le r_1 \quad .$$

Proof

Since

$$y_n(x) - y(x) = \int_o^x \frac{(x - t)^{s-1}}{(s - 1)!}\ [P_n(r + b_o y_n)(t) - (r + b_o y_n)(t)$$

$$+ (b_o(y_n - y))(t)]\ dt \quad , \tag{4.26}$$

then

$$|y_n(x) - y(x)| \le C_1\ ||P_n(r + b_o y_n) - (r + b_o y_n)||_{o,n}$$

$$+ C_2 \int_o^x |y_n(t) - y(t)|\ dt \ , \quad o \le x \le b \quad ,$$

for some appropriate positive constants C_1, C_2. By Lemma 4.6,

$$|y_n(x) - y(x)| \le C_1 \; ||P_n(r+ b_o y_n) - (r + b_o y_n)||_{o,n} \; e^{D_2 b} , \quad o \le x \le b \; .$$

By Lemma 3.4, there is a constant C independent of n such that

$$||(P_n(r + b_o y_n) - (r + b_o y_n))^{(i)}||_{o,n} \le C\left[|\Delta_n|^{\tilde{p}-i} \right.$$

$$\left. ||(r + b_o y_n)^{\tilde{p}}||_{o,n}\right] , \quad o \le i \le \tilde{p} - 1 \; . \qquad (4.27)$$

But by Theorem 4.5, $||y_n^{(i)}||_{o,n}$ is uniformly bounded for each $i = 0, 1, \ldots, \tilde{p}$, so

$$||y - y_n||_o \le C \; |\Delta_n|^{\tilde{p}} \quad , \qquad (4.28)$$

where C is independent of n.

Differentiating both sides of (4.26) up to $s + \tilde{p} - 1$ times and using (4.27), (4.28) and the uniform boundedness of $||y_n^{(i)}||_{o,n}$, $o \le i \le \tilde{p}$, one obtains the first two estimates of the corollary. The last estimate follows from the first estimate and

$$(y_n - y)^{(s+i)}(x_j) = (P_n(b_o y_n))^{(i)}(x_j) - (b_o y)^{(i)}(x_j) = (b_o y_n)^{(i)}(x_j)$$

$$- (b_o y)^{(i)}(x_j) = (b_o(y_n - y))^{(i)}(x_j) \; ,$$

$$o \le i \le r_1 \quad , \qquad o \le j \le n \quad .$$

CHAPTER V

THE NONLINEAR PROBLEM

The solution y of (1.1), (1.2) satisfies (2.3) which we write in operator form as

$$(I - K)y = F \quad , \tag{5.1}$$

with

$$(Ku)(x) = \int_o^x G(x, t)\, w(t)(Tu)(t)\, dt, \qquad o \leq x \leq b \ , \tag{5.2}$$

where

$$(Tu)(x) = H(x; u) \quad \text{and} \quad F(x) = \sum_{k=1}^{s} \alpha_k \phi_k(x) \quad .$$

Similarly, the approximate equation (2.5) becomes

$$(I - K_n)y_n = F_n \quad , \tag{5.3}$$

with

$$(K_n u)(x) = \int_o^x G(x, t)\, w(t)\, P_n(Ty_n)(t)\, dt, \qquad o \leq x \leq b \ , \tag{5.4}$$

and

$$F_n(x) = \sum_{k=1}^{s} \alpha_k^n \phi_k(x) \quad .$$

To obtain existence and convergence results, we will need to use the first and second Fréchet derivatives of K and K_n. Hence we will assume that

$$H \in C^2(\eta) \quad , \tag{5.5}$$

where

$$\eta = \{(x, z_o, z_1, \ldots, z_m) : o \leq x \leq b, \quad |z_k - y^{(k)}(x)| \leq \delta \ ,$$

$$o \leq k \leq m \quad , \quad \delta > o\} \tag{5.6}$$

is a neighborhood of the exact solution y.

The analysis is based on a variation of a Kantorovich theorem which appears in [3]. We will obtain sufficient conditions for (5.3) to have a unique solution in a neighborhood of the exact solution y, and for Newton's method to converge when applied to (5.3).

Lemma 5.1

Let K be the operator of (5.2) with $w \in C[o, b]$ and H satisfying (5.5). Then K: $C^{s-1}[o, b] \to C^{s-1}[o, b]$ has first and second Fréchet derivatives at y given by

$$(K'(y)u)(x) = \int_o^x G(x, t)\, w(t)\, (T'(y)u)(t)\, dt \quad , \tag{5.7}$$

$$(K''(y)\, uv)(x) = \int_o^x G(x, t)\, w(t)\, (T''(y)uv)(t)\, dt \quad , \tag{5.8}$$

where

$$(T'(y)u)(t) = \sum_{i=o}^{m} \frac{\partial H(t; y)}{\partial z_i} u^{(i)}(t) \tag{5.9}$$

and

$$(T''(y)uv)(t) = \sum_{i=o}^{m} \sum_{k=o}^{m} \frac{\partial^2 H(t; y)}{\partial z_k \partial z_i} v^{(k)}(t) u^{(i)}(t) \quad . \tag{5.10}$$

Proof

The expressions for $T'(y)$, $T''(y)$ where T: $C^{s-1}[o, b] \to C[o, b]$ are well known. The expressions for $K'(y)$, $K''(y)$ follow easily from their definition.

Lemma 5.2

Let $\{K_n\}$ be the operators of (5.4), where $w \in C[o, b]$, $\{P_n\}$ satisfies (2.6a) - (2.6c) and $H \in C^{q_1+2}(\eta)$. Then for each n, $K_n : C^{s-1}[o, b] \to C^{s-1}[o, b]$ has first and second derivatives at y given by

$$(K_n'(y)u)(x) = \int_o^x G(x, t)\, w(t)\, P_n(T'(y)u)(t)\, dt \quad , \tag{5.11}$$

$$(K_n''(y)uv)(x) = \int_o^x G(x, t)\, w(t)\, P_n(T''(y)uv)(t)\, dt \quad , \tag{5.12}$$

with $T'(y)$, $T''(y)$ as in Lemma 5.1.

Proof

For the case of the first derivative we need to show that

$$\frac{\left|\left|\int_o^{(\cdot)} G(\cdot, t) w(t)\, P_n(T(y+h) - Ty - T'(y)\, h)\,(t)\, dt\right|\right|_{s-1}}{||h||_{s-1}} \to o$$

as $||h||_{s-1} \to o$.

Because of (2.6b), it is sufficient to show that

$$\frac{||T(y+h) - Ty - T'(y)h||_{q_1}}{||h||_{s-1}} \to o \quad \text{as} \quad ||h||_{s-1} \to o \quad ,$$

i.e., that T has the same derivative when considered as a mapping from $C^{s-1}[o, b]$ into $C^{q_1}[o, b]$.

Define for any function $g \in C'(\eta)$ the expression

$$R(g; h)(x) = g(x; y+h) - g(x; y) - \sum_{i=o}^{m} \frac{\partial g(x; y)}{\partial z_i} h^{(i)}(x) \quad ,$$

where

$$g(x;\ y) \equiv g(x,\ y,\ \ldots,\ y^{(m)}) \quad ,$$

and notice that by the Mean Value Theorem,

$$\frac{||R(g;\ h)||_o}{||h||_{s-1}} \to o \quad \text{as} \quad ||h||_{s-1} \to o \ .$$

Now if $g \varepsilon C^2(\eta)$,

$$(R(g;\ h))'(x) = R(\frac{\partial g}{\partial x};\ h)(x) + \sum_{i=o}^{m} R(\frac{\partial g}{\partial z_i};\ h)(x)(y^{(i+1)}(x) + h^{(i+1)}(x))$$

$$+ \sum_{i=o}^{m} \sum_{k=o}^{m} \frac{\partial^2 g(x;\ y)}{\partial z_i \partial z_k} h^{(k)}(x)\ h^{(i+1)}(x) \quad .$$

Hence if $g \varepsilon C^2(\eta)$,

$$\frac{||(R(g;\ h))'||_o}{||h||_{s-1}} \to o \quad \text{as} \quad ||h||_{s-1} \to o \ .$$

In general, one shows by induction that

$$\frac{||(R(g;\ h))^{(i)}||_o}{||h||_{s-1}} \to o \quad \text{as} \quad ||h||_{s-1} \to o$$

if $g \varepsilon C^{q_1+1}(\eta)$, $i = 0, 1, \ldots, q_1$. (Recall $m + q_1 \leq s - 1$).

Therefore,

$$\frac{||T(y + h) - Ty - T'(y)h||_{q_1}}{||h||_{s-1}} = \frac{||R(H;\ h)||_{q_1}}{||h||_{s-1}} \leq \frac{\sum_{i=o}^{q_1} ||(R(H;\ h))^{(i)}||_o}{||h||_{s-1}} \to o$$

as $||h||_{s-1} \to o$.

For the case of the second derivative, we must show that

$$\sup_{||u||_{s-1}\leq 1} \frac{||\int_o^{(\cdot)} G(\cdot,t)w(t)\, P_n((T'(y+h)-T'(y)-T''(y)h)u)(t)dt||_{s-1}}{||h||_{s-1}} \to o$$

as $||h||_{s-1} \to o$.

It is sufficient to show that

$$\sup_{||u||_{s-1}\leq 1} \frac{||(T'(y+h) - T'(y) - T''(y)h)u||_{q_1}}{||h||_{s-1}} \to o \quad \text{as} \quad ||h||_{s-1} \to o.$$

But

$$(T'(y+h) - T'(y) - T''(y)h)u(x) = \sum_{i=o}^{m} R(\frac{\partial H}{\partial z_i};\ h)(x)u^{(i)}(x)\ ,$$

hence the result will follow if

$$\frac{||(R(\frac{\partial H}{\partial z_i};\ h))^{(\ell)}||_o}{||h||_{s-1}} \to o \qquad \text{as } ||h||_{s-1} \to o$$

for $o \leq i \leq m$, $o \leq \ell \leq q_1$, which is the case since $H \in C^{q_1+2}(\eta)$.

A simple consequence of these results is:

Lemma 5.3

Assume all the hypothesis of Lemmas 5.1 and 5.2. Then $K'(y)$ satisfies the conclusion of Lemmas 4.2 and 4.4 and the operators $\{K_n'(y)\}$ satisfy (i) - (iii) of Lemma 4.3.

Proof

Since $\frac{\partial H}{\partial z_i}(\cdot;y) \in C^{q_1}[o, b]$, $o \leq i \leq m$, the proof parallels those of Lemmas 4.2 and 4.3.

The next theorem is as basic to this chapter as Theorem 4.1 was to Chapter IV.

Theorem 5.1 [3, Thm. 6.5]

Let S be an operator on a Banach space X, $z \in X$, and define the operator R on X by $Rx = (I - S)x - z$, where I is the identity operator. Let M be a bounded linear operator on X, and $x_o \in X$.

Suppose $S'(x_o)$ is compact and that

(1) $(I - M)^{-1}$ is a bounded linear operator on X such that

$$||(I - M)^{-1}|| \leq \beta \; ;$$

(2) $||Rx_o|| \leq d$;

(3) $||(M - S'(x_o))\, Rx_o|| \leq d_o$;

(4) $||(M - S'(x_o))\, S'(x_o)|| \leq d_1 < 1/\beta$;

(5) $||S'(u) - S'(v)|| \leq \gamma||u - v||$ on $B(x_o, r) \equiv \{u: ||u - x_o|| \leq r\}$;

(6) $||(M - S'(x_o))(S'(u) - S'(v))|| \leq d_2\, ||u - v||$ on $B(x_o, r)$;

(7) $h = \dfrac{\beta^2(d + d_o)(\gamma + d_2)}{(1 - \beta\, d_1)^2} \leq \dfrac{1}{2}$

(8) $r_o = \dfrac{\beta(d + d_o)\, w_2(h)}{1 - \beta\, d_1} \leq r$, where $w_2(h) = \dfrac{1 - \sqrt{1 - 2h}}{h}$, $o < h \leq \frac{1}{2}$;

$w_2(o) = 1$.

Then there exists a unique $x^* \in B(x_o, r_o)$ such that $Rx^* = 0$. The Newton iterates x_i are defined and $||x_i - x^*|| \to o$ as $i \to \infty$.

Theorem 5.1 is applied to equation(5.3) to obtain:

Theorem 5.2

Consider equations (5.1) and (5.3), with $w \in C[o, b]$, $H \in C^{q_1+2}(\eta)$ for some q_1 such that $o \leq q_1 \leq s - m - 1$, and suppose $\{P_n\}$ is a sequence of linear operators satisfying (2.6a) - (2.6c). Suppose also that f satisfies (2.1) and let y be the unique solution of (1.1), (1.2). Finally, let $\max_{1\le k\le s} \{|\alpha_k^n - \alpha_k|\} \to o$ as $n \to \infty$. Then there is some $N > 0$ such that for all $n \ge N$ there exists r_n such that (5.3) has a unique solution $y_n \in B(y, r_n) = \{z \in C^{s-1}[o, b]: ||z - y||_{s-1} \le r_n\}$. The Newton iterates $y_{n,i}$ are defined in $B(y, r_n)$ for $n \ge N$, and $||y_{n,i} - y_n||_{s-1} \to o$ as $i \to \infty$.

In addition, there is a constant C independent of n such that

$$||y - y_n||_{s-1} \le C||F - F_n + Ky - K_n y||_{s-1} \le C\{||\sum_{k=1}^{s} (\alpha_k^n - \alpha_k)\phi_k||_{s-1}$$

$$+ ||P_n H(\cdot; y) - H(\cdot; y)||_{s-1}\} .$$

<u>Proof</u>

By Lemma (5.3), $(I - K'(y))^{-1}$ exists and is a bounded linear operator on $C^{s-1}[o, b]$. Let $||(I - K'(y))^{-1}|| \le \beta$. Also from Lemma 5.3, we see that $K'(y)$ and $\{K_n'(y)\}$ satisfy the hypothesis of Theorem 4.1, hence

$$||(K'(y) - K_n'(y)) K_n'(y)||_{s-1} \to o \quad \text{as} \quad n \to \infty .$$

Let $R_n u \equiv (I - K_n)u - F_n$. Then there is a constant C independent of n such that

$$||R_n y||_{s-1} = ||F - F_n + Ky - K_n y||_{s-1} \le C\{||\sum_{k=1}^{s} (\alpha_k - \alpha_k^n)\phi_k||_{s-1}$$

$$+ ||P_n H(\cdot; y) - H(\cdot; y)||_{o,n}\} \quad ,$$

and

$$||(K'(y) - K_n'(y))\, R_n y||_{s-1} \leq C||R_n y||_{s-1}.$$

The second inequality is a consequence of the pointwise convergence of $K_n'(y)$ to $K'(y)$ and follows from the Banach-Steinhaus theorem.

Therefore, $||R_n y||_{s-1} \to o$ and $||(K'(y) - K_n'(y))\, R_n y||_{s-1} \to o$ as $n \to \infty$. Now

$$||K_n'(u) - K_n'(v)||_{s-1} \leq ||u - v||_{s-1} \sup_{o \leq t \leq 1} ||K_n''(v + t(u - v))||_{s-1}$$

if $u, v \in B(y,\delta)$, by the Mean Value Theorem. The number δ is as in (5.6). If $||u_1||_{s-1} \leq 1$, $||v_1||_{s-1} \leq 1$ and $w_t = v + t(u - v)$ for each $t \in [o, 1]$, then

$$||K_n''(w_t)u_1v_1||_{s-1} \leq C||P_n T''(w_t)u_1v_1||_{o,n} \leq C||T''(w_t)u_1v_1||_{q_1} \leq C,$$

where C is independent of t, v_1, u_1, and n, since $w_t \in B(y,\delta)$ and the partial derivatives of H are uniformly bounded in the region η. Thus

$$||K_n'(u) - K_n'(v)||_{s-1} \leq C_1||u - v||_{s-1} \quad \text{for all } u, v \in B(y,\delta),$$

with C_1 independent of n.

This also implies

$$||(K'(y) - K_n'(y))(K_n'(u) - K_n'(v))||_{s-1} \leq C||K_n'(u) - K_n'(v)||_{s-1}$$

$$\leq C_2||u - v||_{s-1}, \quad \text{for all } u, v \in B(y,\delta),$$

with C_2 independent of n.

Now choose N large enough that for $n \geq N$

$$d_n \equiv ||(K'(y) - K_n'(y))\, K_n'(y)||_{s-1} \leq \frac{1}{2\beta} \quad ,$$

$$h_n \equiv \frac{\beta^2[||(K'(y) - K_n'(y))\, R_n y||_{s-1} + ||R_n y||_{s-1}]\,(C_1 + C_2)}{(1 - \beta\, d_n)^2} \leq \frac{1}{2}$$

and

$$r_n \equiv \frac{\beta[||(K'(y) - K_n'(y))\, R_n y||_{s-1} + ||R_n y||_{s-1}]\, w_2(h_n)}{1 - \beta\, d_n} \leq \delta .$$

By Theorem 5.1 we conclude that for $n \geq N$, $(I - K_n)\, u - F_n = 0$ has a unique solution $y_n \in B(y, r_n)$, and that Newton's method is defined in $B(y, r_n)$ and the iterates converge to y_n.

Finally,

$$||y - y_n||_{s-1} \leq r_n \leq C||R_n y||_{s-1} \leq C\{||\sum_{k=1}^{s} (\alpha_k - \alpha_k^n)\phi_k||_{s-1}$$

$$+ ||P_n\, H(\cdot;\, y) - H(\cdot;\, y)||_{o,n}\} \quad ,$$

where C is independent of n, which concludes the proof.

Corollary 5.1

Assume all the hypothesis of Theorem 5.2, and suppose $\sum_{k=1}^{s} \alpha_k^n \phi_k$ is the solution of $Lu = 0$, $u^{(i)}(o) = g_i^n$, $o \leq i \leq s - 1$, where the g_i^n are arbitrary numbers. Let $\varepsilon_n = \max_{o \leq i \leq s-1} \{|y^{(i)}(o) - g_i^n|\}$. Then there exists a constant C independent of n such that

$$||y - y_n||_{s-1} \leq C(\varepsilon_n + ||P_n\, H(\cdot;\, y) - H(\cdot;\, y)||_{o,n}) \quad .$$

In particular, if $w \equiv 1$,

$$||y - y_n||_{s-1} \leq C(\varepsilon_n + ||P_n Ly - Ly||_{o,n}) \quad .$$

Proof

One proceeds as in the proof of Corollary 4.1. If $w \equiv 1$, then $Ly = H(\cdot;\, y)$.

Corollary 5.2

Assume all the hypothesis of Corollary 5.1, but let $g_i^n = y^{(i)}(o)$, $o \leq i \leq s - 1$, $n = 1, 2, 3, \ldots$. Then $||y - y_n||_{s-1} \to o$ as $n \to \infty$.

Proof

The convergence follows from Corollary 5.1 and (2.6c), since

$$H(\cdot;\, y) \in C^{q_1}[o,\, b].$$

Numerical Solution of the Approximate Nonlinear Problem

The Newton iterates for the solution of equation (5.3) are of the form

$$y_{n,r+1}(x) = \sum_{k=1}^{s} \alpha_k^n \phi_k(x) + \int_o^x G(x,\, t)\; w(t)\; P_n(H_{n,r})(t)\, dt \;,\quad o \leq x \leq b, \qquad (5.13)$$

where $H_{n,r}(x) = (Ty_{n,r} - T'(y_{n,r})(y_{n,r} - y_{n,r+1}))(x)$.

For each fixed r, the solution of (5.13) can be found step-by-step, solving at each step

$$y_{n,r+1}(x) = \sum_{k=1}^{s} \alpha_{k,j}^n \phi_k(x) + \int_{x_{j-1}}^x G(x,\, t)\; w(t)\; P_n(H_{n,r})(t)\; dt \;, \qquad (5.14)$$

$$x_{j-1} \leq x \leq x_j \quad ,$$

where $\sum_{k=1}^{s} \alpha_{k,j}^{n}\phi_k$ is the solution of Lu = 0, subject to $u^{(i)}(x_{j-1}) = y_{n,r+1}^{(i)}(x_{j-1})$, $o \leq i \leq s - 1$.

In practice, however, we iterate with equation (5.14) until we obtain y_n to a desired accuracy in $[o, x_1]$. The initial estimate can be taken to be a truncated Taylor series. Then an extrapolation of y_n to $[x_1, x_2]$ can be taken as initial estimate, and we iterate with (5.14) to obtain y_n in $[x_1, x_2]$. The process is then repeated until y_n is obtained in $[o, b]$.

Finding $y_{n,r+1}$ in (5.14) amounts to solving a linear system of equations of the form (4.11). All comments of Chapter IV concerning the solution of equation (4.4) apply to equation (5.14) with appropriate changes in notation. For later reference, notice that for the operators P_n of Chapter III, the solution y_n of (5.3) given by (2.7) has the form

$$y_n(x) = \sum_{k=1}^{s} \alpha_{k,j}^{n}\phi_k(x) + \sum_{i=o}^{s_1} b_{ij}\, d_{ij}(x), \qquad x_{j-1} \leq x \leq x_j \, , \quad (5.15)$$

where d_{ij} is given by (4.9). Following the steps that led to (4.11), we find that if w has at most a countable number of zeroes in $[o, b]$, the b_{ij} satisfy the $(1 + s_1) \times (1 + s_1)$ nonlinear systems

$$b_{ij} = \lambda_i(H(x_{j-1} + (\cdot)\Delta x_j;\, y_n)) \quad , \qquad o \leq i \leq s_1 \, . \qquad (5.16)$$

Conversely, each solution of (5.16) determines a solution of (2.7) through (5.15). If P_n are the operators of Example 3.1, then y_n has the form

$$y_n(x) = \sum_{k=1}^{s} \alpha_{k,j}^{n}\phi_k(x) + \sum_{k=1}^{p} \sum_{i=o}^{r_k} b_{kij}\, d_{kij}(x), \quad x_{j-1} \leq x \leq x_j \quad , \qquad (5.17)$$

with d_{kij} as in (4.12). The systems (5.16) become in this case

$$b_{kij} = (\Delta x_j)^i \, (H(\cdot;\, y_n))^{(i)}(\gamma_{kj}) \,, \tag{5.18}$$

$$1 \le k \le p \,, \qquad o \le i \le r_k \quad .$$

As an alternate approach, from (5.17) and (5.18) we can write

$$y_n(x) = \sum_{k=1}^{s} \alpha^n_{k,j}\phi_k(x) + \sum_{k=1}^{p} \sum_{i=o}^{r_k} (\Delta x_j)^i \, (H(\cdot;\, y_n))^{(i)}(\gamma_{kj}) \, d_{kij}(x) \,,$$

$$x_{j-1} \le x \le x_j \quad . \tag{5.19}$$

Then we can obtain a nonlinear system whose unknowns are the values $y_n^{(v)}(\gamma_{kj})$ appearing on the right-hand side of (5.19). As in Chapter IV, the number of unknowns can be reduced if for example $r_i \ge s - m$, $1 \le i \le p$. In this formulation, the methods can be used as "predictor-corrector" methods by extrapolating y_n from $[x_{j-2}, x_{j-1}]$ to $[x_{j-1}, x_j]$ to predict the values $y_n^{(v)}(\gamma_{kj})$, then using (5.19) to correct them. Such methods have been studied in [39] for $s = 1$, $a_k = 0$, $w \equiv 1$, $p = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$.

Returning to the general case, suppose P_n is obtained from a projection Q, but let $w \equiv 1$. Then following the steps leading to (4.14) we find that y_n satisfies in each subinterval

$$\lambda_i(y_n^{(s)}(x_{j-1} + (\cdot)\Delta x_j)) = \lambda_i(f(x_{j-1} + (\cdot)\Delta x_j;\, y_n)), \quad o \le i \le s_1, \tag{5.20}$$

which refers directly to the original differential equation.

Applications of the Theory to the Nonlinear Problem

Most of this section is concerned with some general theorems which illustrate the results one can obtain from Theorem 5.2 and Corollary 5.1.

Theorem 5.3

Consider the initial value problem (1.1), (1.2) where f satisfies (2.1). Let q be an integer with $o \le q \le s - 1 - m$, and let a_k, $o \le k \le m$, and g_i^n, $o \le i \le s - 1$, be arbitrary constants. Let $\{\Delta_n\}$ be a sequence of partitions of [o, b] given by (2.4) with $|\Delta_n| \to o$ as $n \to \infty$, and $\{P_n\}$ be the sequence of operators of Example 3.1. Suppose

$$\max_{o \le i \le s-1} \{|y^{(i)}(o) - g_i^n|\} \le C\ |\Delta_n|^{m_1} \quad ,$$

where C and m_1 are independent of n. Suppose also that

$$f(x;\ y) - \sum_{k=o}^{m} a_k y^{(k)}(x) = w(x)\ H(x;\ y) \ , \quad o \le x \le b \quad ,$$

where $w \in C[o, b]$ and has at most a countable number of zeroes in [o, b], and where $H \in C^{q+2}(\eta)$, η given in (5.6), with y the solution of (1.1), (1.2). Then there is some N > 0 such that for all $n \ge N$, there exists $r_n \le \delta$ such that there is a unique function $y_n \in B(y, r_n)$ of the form (5.17) which satisfies $y_n^{(i)}(o) = g_i^n$, $o \le i \le s - 1$, and whose coefficients satisfy (5.18). The Newton iterates (5.13) are well defined in $B(y, r_n)$ and converge to y_n.

There is a constant C independent of n such that for $n \ge N$,

$$||(y_n - y)^{(i)}||_o \le C\ |\Delta_n|^{\min(m_1, q+2)} , \quad o \le i \le s - 1 \ . \tag{5.21}$$

If in addition $H(\cdot;\ y) \in C^{q+u}[o, b]$ for some u, $3 \le u \le \tilde{p} - q$, then

$$||(y_n - y)^{(i)}||_o \le C\ |\Delta_n|^{\min(m_1, q+u)}, \quad o \le i \le s-1 . \qquad (5.22)$$

In particular, if $u = \tilde{p} - q$, $w \in C^{\tilde{p}-1}[o, b]$ and $\{\Delta_n\}$ is quasi-uniform, then

$$||(y_n - y)^{(s+i)}||_{o,n} \le C\ |\Delta_n|^{\min(m_1, \tilde{p})-i}, \quad o \le i \le \tilde{p} - 1 . \ (5.23)$$

Proof

Construct equations (5.1), (5.3) and choose α_k^n so that $(F_n)^{(i)}(o) = g_i^n$, $o \le i \le s-1$. By Theorem 5.2, equation (5.3) has a unique solution $y_n \in B(y, r_n)$ for $n \ge N$, and Newton's method converges. The comments of the previous section leading to (5.18) show that y_n has the form (5.17) and its coefficients satisfy (5.18), and that there is only one such function in $C^{s-1}[o, b]$.

Corollary 5.1 and Lemma 3.4 imply the estimates for $||(y_n - y)^{(i)}||_o$, $o \le i \le s-1$. To obtain the final estimate we start from

$$L(y - y_n)(x) = w(x)\ [H(x; y) - P_n(H(\cdot; y_n))(x)] ,$$

$$x_{j-1} < x < x_j ,$$

and obtain

$$(y - y_n)^{(s+i)}(x) = [w(H(\cdot; y) - P_n(H(\cdot; y)))]^{(i)}(x)$$

$$+ [w(P_n(H(\cdot; y) - H(\cdot; y_n)))]^{(i)}(x) + \sum_{k=o}^{m} a_k[y^{(k+i)}(x) - y_n^{(k+i)}(x)],$$

$$x_{j-1} < x < x_j, \quad o \le i \le \tilde{p} - 1 . \qquad (5.24)$$

The first term in the right-hand side of (5.24) is $O(|\Delta_n|^{\tilde{p}-i})$ by Lemma 3.4. We claim that

$$\sup_{x_{j-1}\le x\le x_j} |(H(\cdot;\, y) - H(\cdot;\, y_n))^{(u)}(x)| = O(|\Delta_n|^{\min(\tilde{p},m_1)}) \;, \quad (5.25)$$

$$o \le u \le q \quad .$$

To see this, define

$$R_1(H)(x) \equiv H(x;\, y_n) - H(x;\, y) \quad ,$$

and notice that by the Mean Value Theorem

$$\sup_{x_{j-1}\le x\le x_j} |R_1(H)(x)| \le C \max_{o\le i\le s-1} \{||(y - y_n)^{(i)}||_o\} \quad ,$$

where C is a uniform bound on the first partials of H in η.

Thus using the bounds on $||(y - y_n)^{(i)}||_o$ we have

$$\sup_{x_{j-1}\le x\le x_j} |H(x;\, y_n) - H(x;\, y)| = O(|\Delta_n|^{\min(\tilde{p},m_1)}) \quad .$$

Now

$$(R_1(H))'(x) = R_1(\frac{\partial H}{\partial x})(x) + \sum_{i=o}^{m} R(\frac{\partial H}{\partial z_i})(x)\; y^{(i+1)}(x)$$

$$+ \sum_{i=o}^{m} \frac{\partial H(x;\, y_n)}{\partial z_i} (y_n^{(i+1)}(x) - y^{(i+1)}(x)) \quad ,$$

hence

$$\sup_{x_{j-1}\le x\le x_j} |(R_1(H))'(x)| \le C \max_{o\le i\le s-1} \{||(y - y_n)^{(i)}||_o\} = O(|\Delta_n|^{\min(\tilde{p},m_1)}).$$

In general, differentiating $R_1(H)$ up to q times and using a uniform bound on all the partial derivatives up to order $q + 1$ of H in η, we obtain (5.25), since $m + q \leq s - 1$.

Returning to the second term in the right-hand side of (5.24), we see that it is $O(|\Delta_n|^{\min(\tilde{p},m_1)-i})$. The rest of the proof parallels that of Theorem 4.3.

Higher rates for special choices of points γ_{kj} in Example 3.1 will be obtained later using the following lemma.

Lemma 5.4

Assume all the hypothesis of Theorem 5.3, with $g_i^n = y^{(i)}(o)$, $o \leq i \leq s - 1$. In addition, suppose $H \in C^{m_2}(\eta)$ and $w \in C^{m_3}[o, b]$, where for some nonnegative integer r, $m_2 = \max\{\tilde{p}, \tilde{p} + r + 1 + m - s\}$, and $m_3 = \max\{0, \tilde{p} + r + 1 + m - s\}$. Then for all n sufficiently large the equation

$$(I - K_n'(y))\ R_n y = F_n + K_n y - K_n'(y)\ y \tag{5.26}$$

has a unique solution $R_n y$, where K_n, F_n are as in the proof of Theorem 5.3. In addition,

$$\sup_{x_{j-1}<x<x_j} |(y - R_n y)^{(i)}(x)| \leq C \left(\frac{|\Delta_n|}{\Delta x_j}\right)^{\tilde{p}} , \quad o \leq i \leq \tilde{p} + r + 1 + m \tag{5.27}$$

where C is independent of n, and

$$(y_n - y)^{(i)}(x) = (R_n y - y)^{(i)}(x) + O(|\Delta_n|^{2\tilde{p}}) , \tag{5.28}$$

$$o \leq x \leq b , \qquad o \leq i \leq s - 1 .$$

Proof

By Lemma 5.3 and Theorem 4.1, $(I - K_n'(y))^{-1}$ exists and is uniformly bounded for n sufficiently large. By Theorem 5.3, equation (5.3) has a unique solution $y_n \in B(y, r_n)$ for n sufficiently large. We can write

$$y_n = F_n + K_n y_n = F_n + K_n y + K_n' (y)(y_n - y) + E(y_n) \quad ,$$

where $E(y_n)$ satisfies

$$||E(y_n)||_{s-1} \leq C \; ||y_n - y||_{s-1}^2 \tag{5.29}$$

C independent of n, by a form of the Second Mean Value Theorem. Hence,

$$y_n = (I - K_n'(y))^{-1} \; (F_n + K_n y - K_n'(y) \; y + E(y_n)) \quad ,$$

and by (5.26),

$$y_n - y = R_n y - y + (I - K_n'(y))^{-1} \; E(y_n) \quad . \tag{5.30}$$

This implies

$$||(y_n - y) - (R_n y - y)||_{s-1} \leq C \; ||y - y_n||_{s-1}^2 \leq C \; |\Delta_n|^{2\tilde{p}} \tag{5.31}$$

by (5.22) and (5.29).

Rewriting (5.30) and taking the ith derivative we have, by (2.6b) and (5.31),

$$(y_n - y)^{(i)}(x) - (R_n y - y)^{(i)}(x)$$

$$= \int_0^x \frac{\partial^i G(x, t)}{\partial x^i} \; w(t) \; P_n(T'(y)(y_n - y - (R_n y - y)))(t) \; dt + (E(y_n))^{(i)}(x),$$

$$\leq C \, ||T'(y)(y_n - y - (R_n y - y))||_q + ||E(y_n)||_{s-1}$$

$$\leq C \, ||y_n - y||_{s-1}^2 \leq C \, |\Delta_n|^{2\tilde{p}}, \quad o \leq x \leq b \, , \quad o \leq i \leq s-1 \, .$$

Therefore (5.28) is satisfied. Now (5.22) and (5.28) give

$$\sup_{x_{j-1}<x<x_j} |(R_n y - y)^{(i)}(x)| \leq C \, |\Delta_n|^{\tilde{p}} \quad , \quad o \leq i \leq s-1 \, , \quad (5.32)$$

with C independent of n. If $\tilde{p} + r + 1 + m - s < o$, then (5.32) implies (5.27). So suppose $\tilde{p} + r + 1 + m - s \geq o$. Operating with L on both sides of (5.26) we have

$$L(R_n y)(x) = w(x) \, P_n[Ty - T'(y)(y - R_n y)](x) \, , \qquad (5.33)$$

$$x_{j-1} < x < x_j \quad .$$

Now subtract equation (5.33) from $Ly = wTy$ to obtain

$$(y - R_n y)^{(s+i)}(x) = [w(Ty - P_n \, Ty)]^{(i)}(x) + [w \, P_n(T'(y)(y - R_n y))]^{(i)}(x)$$

$$+ \sum_{k=o}^{m} a_k (y - R_n y)^{(k+i)}(x) \, , \quad x_{j-1} < x < x_j \, , \quad (5.34)$$

$$o \leq i \leq \tilde{p} + r + 1 + m - s \quad .$$

If $i = 0, 1, \ldots, \tilde{p} - 1$, by Lemma 3.4 and (5.32), there is C independent of n such that

$$\sup_{x_{j-1}<x<x_j} |(w(Ty - P_n \, Ty))^{(i)}(x)| \leq C \, |\Delta_n|^{\tilde{p}-i} \quad ,$$

and

$$\sup_{x_{j-1}<x<x_j} |(wP_n(T'(y)(y - R_n y)))^{(i)}(x)| \leq C \frac{|\Delta_n|^{\tilde{p}}}{(\Delta x_j)^i} .$$

Therefore from (5.34),

$$\sup_{x_{j-1}<x<x_j} |(y - R_n y)^{(s+i)}(x)| \leq C \frac{|\Delta_n|^{\tilde{p}}}{(\Delta x_j)^i} \leq C \frac{|\Delta_n|^{\tilde{p}}}{(\Delta x_j)^{\tilde{p}-1}} , \quad o \leq i \leq \tilde{p} - 1 , \tag{5.35}$$

where C is independent of n.

If $i = \tilde{p}, \tilde{p} + 1, \ldots, \tilde{p} + r + 1 + m - s$, then by (5.32), (5.35) and since $(P_n g)^{(i)} \equiv o$ for any $g \in C^q[o, b]$, equation (5.34) implies there is some C independent of n such that

$$\sup_{x_{j-1}<x<x_j} |(y - R_n y)^{(s+i)}(x)| \leq C \frac{|\Delta_n|^{\tilde{p}}}{(\Delta x_j)^{\tilde{p}}} , \quad \tilde{p} \leq i \leq \tilde{p} + r + 1 + m - s. \tag{5.36}$$

The inequality (5.27) is obtained from (5.32), (5.35) and (5.36).

If $w \equiv 1$, we can restate Theorem 5.3 as:

<u>Corollary 5.3</u>

Assume all the hypothesis of Theorem 5.3, but with $w \equiv 1$. Then there is some $N > 0$ such that for all $n \geq N$, there exists $r_n \leq \delta$ such that there is a unique function $y_n \in B(y, r_n)$ of the form (5.17) which satisfies $y_n^{(i)}(o) = g_i^n$, $o \leq i \leq s - 1$, and

$$y_n^{(s+v)}(\gamma_{kj}) = (f(\cdot; y_n))^{(v)}(\gamma_{kj}) , \quad 1 \leq k \leq p , \quad o \leq v \leq r_k , \quad 1 \leq j \leq n.$$

All the error estimates of Theorem 5.3 apply.

Proof

The only change in the proof of Theorem 5.3 is due to (5.20).

Once again we have required $o \leq q \leq s-1-m$, but as in the linear problem, we can use larger values of q if $w \equiv 1$, $a_k = 0$, $o \leq k \leq m$, and if we use the partition D_1 of (3.5) and the r_i of (3.6) in the definition of P_n in Example 3.1.

Theorem 5.4

Consider the initial value problem (1.1), (1.2) where f satisfies (2.1). Suppose $f \in C^{q+2}(\eta)$, where η is given in (5.6) and $q \geq 1$ is some integer, and let y be the solution of (1.1), (1.2). Let D_1 be a partition of [o, 1] given by (3.5) and let r_i, $1 \leq i \leq p$, be integers satisfying (3.6). Define $\tilde{p} = p + \sum_{i=1}^{p} r_i$.

Suppose $\{\Delta_n\}$ is a sequence of partitions of [o, b] given by (2.4), with $|\Delta_n| \to o$ as $n \to \infty$, and define γ_{kj} by (3.2). Then there is some $N > o$ such that for all $n \geq N$ there exists $r_n \leq \delta$ such that there is a unique function $y_n \in C(y, r_n) \equiv \{z \in C^{s+q-1}[o, b]: ||z - y||_{s+q-1} \leq r_n\}$, which is a polynomial of degree $\tilde{p} + s - 1$ in each subinterval $[x_{j-1}, x_j]$ and which satisfies $y_n^{(i)}(o) = y^{(i)}(o)$, $o \leq i \leq s-1$ and

$$y_n^{(s+v)}(\gamma_{kj}) = (f(\cdot;\, y_n))^{(v)}(\gamma_{kj}) \ , \quad 1 \leq k \leq p \ , \quad o \leq v \leq r_k, \quad 1 \leq j \leq n.$$

The Newton iterates given by

$$y_{n,r+1}(x) = \sum_{i=o}^{s-1} g_i \frac{x^i}{i!} + \int_o^x \frac{(x-t)^{s-1}}{(s-1)!} P_n(H_{n,r})(t)\, dt, \quad o \leq x \leq b, \quad (5.37)$$

with $H_{n,r}$ as in (5.13) and

where P_n is the operator of Example 3.1 restricted by (3.5), (3.6), are defined in $C(\tilde{y}, r_n)$ and converge to y_n.

If $y \in C^{p+s}[o, b]$, there is a constant C independent of n such that

$$||(y - y_n)^{(i)}||_o \leq C \; |\Delta_n|^{\tilde{p}-q+1} \quad , \quad o \leq i \leq s + q - 1 \; .$$

In addition if $\{\Delta_n\}$ is quasi-uniform, then

$$||(y - y_n)^{(s+i)}||_{o,n} \leq C \; |\Delta_n|^{\tilde{p}-i} \; , \quad q \leq i \leq \tilde{p} - 1 \; .$$

Proof

Most of the main steps of the proof are like those in the proof of Theorem 4.5. The solution y of (1.1), (1.2) is also the solution of

$$y^{(s+q)}(x) = \tilde{f}(x; y) \equiv (f(\cdot; y))^{(q)}(x) \; , \quad o \leq x \leq b, \tag{5.38}$$

subject to

$$y^{(i)}(o) = g_i \quad , \quad o \leq i \leq s + q - 1 \; , \tag{5.39}$$

where

$$g_i = (f(\cdot; y))^{(i-s)}(o) \quad , \quad s \leq i \leq s + q - 1 \; . \tag{5.40}$$

Let K, $\tilde{K}$ be defined by

$$(Ku)(x) = \int_o^x \frac{(x - t)^{s-1}}{(s - 1)!} (Tu)(t) \; dt \quad ,$$

$$(\tilde{K}u)(x) = \int_o^x \frac{(x - t)^{s+q-1}}{(s + q - 1)!} (\tilde{T}u)(t) \; dt \quad ,$$

where

$$(Tu)(x) = f(x;\ u) \quad \text{and} \quad (\tilde{T}u)(x) = \tilde{f}(x;\ u), \quad o \leq x \leq b.$$

Also let

$$F(x) = F_n(x) = \sum_{i=o}^{s-1} g_i \frac{x^i}{i!} \quad ,$$

and

$$\tilde{F}(x) = \tilde{F}_n(x) = \sum_{i=o}^{s+q-1} g_i \frac{x^i}{i!} \quad .$$

Using the operators P_n of Example 3.1 restricted by (3.5), (3.6), and the operators $\tilde{P}_n$ of Example 3.2, define the approximate operators K_n and $\tilde{K}_n$, respectively. The integral equations equivalent to (1.1), (1.2) and (5.38), (5.39) are $(I - K)\ y = F$ and $(I - \tilde{K})\ y = \tilde{F}$, and the corresponding approximate equations are

$$(I - K_n)y_n = F \tag{5.41}$$

and

$$(I - \tilde{K}_n)\tilde{y}_n = \tilde{F} \quad . \tag{5.42}$$

Theorem 5.2 (with $q_1 = 0$ and s replaced by s + q) applies to equation (5.42). Hence there is some $N > 0$ such that for all $n \geq N$, there exists $r_n \leq \delta$ so that (5.42) has a unique solution $\tilde{y}_n \in C(y,\ r_n)$. (The hypothesis of Theorem 5.2 that f satisfies a Lipschitz condition is only necessary for the existence of the solution y of (1.1), (1.2). Here we know that y is also a solution of (5.38), (5.39)).

Also by Theorem 5.2, there is a constant C independent of n such that

$$||y - \tilde{y}_n||_{s+q-1} \le C||\tilde{K}_n y - \tilde{K}y||_{s+q-1}$$

$$= C||\int_o^{(\cdot)} \frac{(\cdot - t)^{s+q-1}}{(s + q - 1)!} [\tilde{P}_n(y^{(s+q)}) - y^{(s+q)}](t)\, dt||_{s+q-1}$$

$$= C||\int_o^{(\cdot)} \frac{(\cdot - t)^{s-1}}{(s - 1)!} (P_n(y^{(s)}) - y^{(s)})(t)\, dt||_{s+q-1}$$

$$\le C\ |\Delta_n|^{\tilde{p}-q+1} \quad ,$$

where we have also used Lemmas 3.6 and 3.4. Now using Lemma 3.6 and the fact that $\tilde{y}_n$ satisfies (5.39) we can show that

$$\tilde{y}_n = \tilde{F} + \tilde{K}_n\tilde{y}_n = F + K_n\tilde{y}_n \quad ,$$

that is, the solution $\tilde{y}_n$ of (5.42) is also a solution of (5.41). Conversely, each solution of (5.41) is a solution of (5.42). This equivalency establishes all the results of the theorem, except the last error estimate and the statement on Newton's method. To obtain this error estimate we start from (5.24), but with $a_k = 0$, $w \equiv 1$, which we write as

$$(y - y_n)^{(s+i)}(x) = [y^{(s)} - P_n(y^{(s)})]^{(i)}(x) + [P_n(f(\cdot; y) - f(\cdot; y_n))]^{(i)}(x),$$

$$x_{j-1} < x < x_j \quad , \quad q - 1 \le i \le \tilde{p} - 1 \quad . \qquad (5.43)$$

If $i = q - 1$, then by the first estimate of the theorem and Lemma 3.4,

$$\sup_{x_{j-1}<x<x_j} |(P_n(f(\cdot; y) - f(\cdot; y_n)))^{(q-1)}(x)| = O(|\Delta_n|^{\tilde{p}-q+1}) \ .$$

Applying now Lemma 4.5 to $(P_n(f(\cdot; y) - f(\cdot; y_n)))^{(q-1)}$ in each subinterval we have

$$\sup_{x_{j-1}<x<x_j} |(P_n(f(\cdot;\, y) - f(\cdot;\, y_n)))^{(q-1+u)}(x)| = O(|\Delta_n|^{\tilde{p}-q+1-u}) ,$$

$$o \le u \le \tilde{p} - q \quad . \tag{5.44}$$

Then from (5.43), (5.44) and Lemma 3.4, it follows that

$$||(y - y_n)^{(s+v)}||_{o,n} \le C\ |\Delta_n|^{\tilde{p}-v} \quad , \quad q \le v \le \tilde{p} - 1 \quad .$$

Finally, we prove the statement about Newton's method. By Theorem 5.2 we know that the iterates

$$\tilde{y}_{n,r+1} = \tilde{F} + \tilde{K}_n\tilde{y}_{n,r} - \tilde{K}'_n(\tilde{y}_{n,r})(\tilde{y}_{n,r} - \tilde{y}_{n,r+1}) \tag{5.45}$$

are defined in $C(y,\, r_n)$ and converge to $\tilde{y}_n (\equiv y_n)$.

However, by Lemma 3.6, and since $\tilde{y}_{n,r}$ and $\tilde{y}_{n,r+1}$ satisfy (5.39), we have

$$\tilde{y}_{n,r+1}(x) = \tilde{F}(x) + \int_o^x \frac{(x-t)^{s+q-1}}{(s+q-1)!}\ \tilde{P}_n[\tilde{T}\tilde{y}_{n,r} - \tilde{T}'(\tilde{y}_{n,r})(\tilde{y}_{n,r} - \tilde{y}_{n,r+1})](t)\ dt$$

$$= \tilde{F}(x) + \int_o^x \frac{(x-t)^{s+q-1}}{(x+q-1)!}\ \tilde{P}_n[(T\tilde{y}_{n,r})^{(q)} - (T'(\tilde{y}_{n,r})(\tilde{y}_{n,r} - \tilde{y}_{n,r+1}))^{(q)}](t)\,dt$$

$$= F(x) + \int_o^x \frac{(x-t)^{s-1}}{(s-1)!}\ P_n[T\tilde{y}_{n,r} - T'(\tilde{y}_{n,r})(\tilde{y}_{n,r} - \tilde{y}_{n,r+1})](t)\ dt$$

$$= F(x) + K_n\ \tilde{y}_{n,r} - K'_n(\tilde{y}_{n,r})(\tilde{y}_{n,r} - \tilde{y}_{n,r+1}) \quad , \tag{5.46}$$

where we have also used the fact that if u, v ε $C^{s+q-1}[o, b]$, then

$$(\tilde{T}'(u)v)(x) = (T'(u)v)^{(q)}(x) \tag{5.47}$$

Equation (5.47) is satisfied because

$$\frac{||\tilde{T}(u+h) - \tilde{T}(u) - (T'(u)h)^{(q)}||_o}{||h||_{s+q-1}} = \frac{||(T(u+h) - T(u) - T'(u)h)^{(q)}||_o}{||h||_{s+q-1}} \to o$$

$$\text{as } ||h||_{s+q-1} \to o \quad ,$$

which can be seen in the proof of Lemma 5.2, with some change in notation.

Equation (5.46) shows that $\tilde{y}_{n,r+1}$ is also a solution of (5.37) if $\tilde{y}_{n,r} = y_{n,r}$. Conversely, each solution $y_{n,r+1}$ of (5.37) is a solution of (5.45) if $\tilde{y}_{n,r} = y_{n,r}$. Hence both (5.37) and (5.45) have the same solution for each r, if both iterations start with the same initial function. This equivalency suffices to finish the proof of the theorem.

Even though Theorem 5.4 establishes the convergence of the method, the error estimates given there are not the best possible for the lower derivatives. As we did in the linear case, we can derive better estimates for the important case m = 0.

Corollary 5.4

Assume all the hypothesis of Theorem 5.4 but let m = 0. In addition, let $\{\Delta_n\}$ be quasi-uniform. Then if f ε $C^{\tilde{p}}(\eta)$ there is a constant C independent of n such that

$$||(y - y_n)^{(i)}||_o \leq C \; |\Delta_n|^{\tilde{p}} \quad , \quad o \leq i \leq s \quad ,$$

and

$$||(y - y_n)^{(s+i)}||_{o,n} \leq C\ |\Delta_n|^{\tilde{p}-i} , \quad 1 \leq i \leq \tilde{p} - 1 \quad .$$

However,

$$\max_{o \leq j \leq n} \{|(y_n - y)^{(s+i)}(x_j)|\} \leq C\ |\Delta_n|^{\tilde{p}} , \quad 1 \leq i \leq r_1 \quad .$$

Proof

Since

$$y_n(x) - y(x) = \int_o^x \frac{(x-t)^{s-1}}{(s-1)!} [P_n f(\cdot, y_n) - f(\cdot, y_n) + f(\cdot, y_n)$$

$$- f(\cdot, y)](t)\ dt \quad , \tag{5.48}$$

then by (2.1),

$$|y_n(x) - y(x)| \leq C_1\ ||P_n f(\cdot, y_n) - f(\cdot, y_n)||_{o,n} + C_2 \int_o^x |y_n(t) - y(t)|dt,$$

for some appropriate positive constants C_1, C_2. By Lemma 4.6,

$$|y_n(x) - y(x)| \leq C_1\ ||P_n f(\cdot, y_n) - f(\cdot, y_n)||_{o,n} \quad e^{C_2 b} , \ o \leq x \leq b. \tag{5.49}$$

By Lemma 3.4 there is a constant C independent of n such that

$$||(P_n f(\cdot, y_n) - f(\cdot, y_n))^{(i)}||_{o,n} \leq C\ |\Delta_n|^{\tilde{p}-i} ||(f(\cdot, y_n))^{(\tilde{p})}||_{o,n} \quad ,$$

$$o \leq i \leq \tilde{p} - 1 \quad . \tag{5.50}$$

But by Theorem (5.4), $||y_n^{(i)}||_{o,n}$ is uniformly bounded for each $i = 0, 1, \ldots, \tilde{p}$. Hence (5.49) and (5.50) imply

$$||y - y_n||_o \leq C\ |\Delta_n|^{\tilde{p}} \quad , \tag{5.51}$$

where C is independent of n.

Differentiating both sides of (4.48) up to s times and using (2.1), (5.50), (5.51), we obtain

$$||(y - y_n)^{(i)}||_o \leq C\ |\Delta_n|^{\tilde{p}}\ , \qquad 1 \leq i \leq s\ , \tag{5.52}$$

with C independent of n. To obtain the second estimates, define

$$R_2(f)(x) \equiv f(x, y_n) - f(x, y)\ .$$

By the Mean Value Theorem,

$$\sup_{x_{j-1}<x<x_j} |R_2(f)(x)| \leq C \sup_{x_{j-1}<x<x_j} |(y_n - y)(x)|$$

where C is a bound, independent of n, on the first partials of f in η. Now

$$(R_2 f)'(x) = R_2\ (\frac{\partial f}{\partial x})(x) + R(\frac{\partial f}{\partial y})(x)\ y'(x) + \frac{\partial f(x, y_n)}{\partial y}\ (y_n - y)'(x)\ ,$$

hence

$$\sup_{x_{j-1}<x<x_j} |(R_2 f)'(x)| \leq C \max_{o\leq v\leq 1} \sup_{x_{j-1}<x<x_j} |(y_n - y)^{(v)}(x)|\ ,$$

where C is a uniform bound on the first and second partials of f in η. In general, differentiating $R_2(f)$ up to i times and using a uniform bound on all the partial derivatives up to order i of f in η, we obtain

$$\sup_{x_{j-1}<x<x_j} |(R_2 f)^{(i)}(x)| \leq C \max_{o\leq v\leq i} \sup_{x_{j-1}<x<x_j} |(y_n - y)^{(i)}(x)|\ , \quad o \leq i \leq \tilde{p} - 1. \tag{5.53}$$

From (4.48),

$$(y_n - y)^{(s+i)}(x) = [P_n f(\cdot, y_n) - f(\cdot, y_n)]^{(i)}(x)$$

$$+ [f(\cdot, y_n) - f(\cdot, y)]^{(i)}(x) ,$$

$$x_{j-1} < x < x_j \quad , \quad o \leq i \leq \tilde{p} - 1 . \qquad (5.54)$$

From (5.50), (5.52) and (5.53) it follows that for some C independent of n,

$$||(y - y_n)^{(s+i)}||_{o,n} \leq C \, |\Delta_n|^{\tilde{p}-i} \quad , \quad o \leq i \leq \tilde{p} - 1 \; .$$

The last estimate of the corollary is a consequence of the first estimate, of

$$(y_n - y)^{(s+i)}(x_j) = (P_n f(\cdot, y_n))^{(i)}(x_j) - (f(\cdot, y))^{(i)}(x_j)$$

$$= (f(\cdot, y_n) - f(\cdot, y))^{(i)}(x_j) \quad , \quad o \leq i \leq r_1 \quad ,$$

and of

$$|(f(\cdot, y_n) - f(\cdot, y))^{(i)}(x_j)| \leq C \max_{o \leq v \leq i} \{|(y_n - y)^{(v)}(x_j)|\} \; .$$

<u>Higher Rates</u>

When the operators P_n of Example 3.1 are used, it may be possible to improve the rates of convergence for the lower derivatives, especially at the partition points, by choosing the points $\gamma_k \; \varepsilon [0, 1]$ so that

$$\int_o^1 \prod_{k=1}^{p} (t - \gamma_k)^{1+r_k} \, v(t) \, dt = 0 \qquad (5.55)$$

for every $v \in \Pi_r$, where r is some integer satisfying $r \leq \tilde{p} - 1$. Of course it is not always possible to find such real numbers $\gamma_k \; \varepsilon [0,1]$ for all $r \leq \tilde{p} - 1$, as the case $p = 2$, $r_1 = 1$, $r_2 = 1$ shows.

To obtain the higher rates we rely on a basic idea of deBoor and Swartz [16], modified to account for w and the fact that y_n is not necessarily a piecewise polynomial. For $a_k = 0$, and $w \equiv 1$, the next theorem contains the results of [16, Thm. 4.1], and of [59, Thm. 5] if $o \leq q \leq s - m - 1$, specialized to (1.1), (1.2).

Theorem 5.5

Assume all the hypothesis of Theorem 5.3, and choose the γ_k to satisfy (5.55) for some $r \leq \tilde{p} - 1$. Suppose in addition that $g_i^n = y^{(i)}(o)$, $o \leq i \leq s - 1$, and that $H \in C^{m_4}(\Omega)$, $w \in C^{m_5}[o, b]$, where $m_4 = \tilde{p} + r + 2$ and $m_5 = \max\{\tilde{p} + r + 1 + m - s,\ r + 1\}$.

Then the approximations y_n of Theorem 5.3 also satisfy for $n \geq N$,

$$\max_{o \leq j \leq n} |(y - y_n)^{(i)}(x_j)| \leq C\, |\Delta_n|^{\tilde{p}+r+1}, \qquad o \leq i \leq s - 1$$

and

$$||(y - y_n)^{(i)}||_o \leq C\, |\Delta_n|^{\tilde{p}+\min(r+1,s-i)}, \quad o \leq i \leq s - 1,$$

where C is independent of n.

Proof

Let N in Theorem 5.3 be sufficiently large so that the conclusions of Lemma 5.4 are valid for $n \geq N$, and let $R_n y$ be as in (5.26). Suppose

$$E(x, t) = u_t(x), \qquad o \leq t \leq x,$$

where $u_t(x)$ satisfies

$$(L - wT'(y))u_t = o, \quad u_t^{(i)}(t) = o, \quad o \leq i \leq s - 2, \quad u_t^{(s-1)}(t) = 1.$$

Then since $(y - R_n y)^{(i)}(o) = o$, $o \leq i \leq s - 1$, we have

$$(y - R_n y)(x) = \int_o^x E(x, t)(L - wT'(y))(y - R_n y)(t)\, dt$$

$$= \int_o^x E(x, t)\, w(t)\, v_n(t)\, dt \quad , \qquad o \leq x \leq b \quad ,$$

with

$$v_n(x) = \frac{1}{w(x)} (L - wT'(y))(y - R_n y)(x)$$

$$= (Ty - P_n Ty - T'(y)(y - R_n y) + P_n(T'(y)(y - R_n y)))(x) \quad , \qquad (5.56)$$

$$x_{j-1} < x < x_j \quad .$$

The last equality follows from (5.33) and $Ly = wTy$. Hence

$$(y - R_n y)^{(i)}(x) = \int_o^x E_i(x, t)\, w(t)\, v_n(t)\, dt, \quad o \leq x \leq b \ , \quad o \leq i \leq s-1 \quad (5.57)$$

where

$$E_i(x, t) = \frac{\partial^i E(x, t)}{\partial x^i} \quad .$$

Suppose $x \ \varepsilon [x_{v-1}, x_v)$ for some v, and define

$$F_e = \int_{x_{e-1}}^{x_e} E_i(x, t)\, w(t)\, v_n(t)\, dt \quad , \qquad 1 \leq e \leq v \quad .$$

For each $e = 1, 2, \ldots, v$, we can assume that $v_n \ \varepsilon \ C^{\tilde{p}+r+1}[x_{e-1}, x_e]$, using the limiting values of v_n in (5.56). Hence we see from (5.56) and the definition of P_n that

$$v_n^{(u)}(\gamma_{ke}) = 0, \quad 1 \leq k \leq p \ , \quad o \leq u \leq r_k \ , \quad 1 \leq e \leq v \ .$$

Therefore we can write

$$F_e = \int_{x_{e-1}}^{x_e} h_x(t) \prod_{k=1}^{p} (t - \gamma_{ke})^{1+r_k}\, dt, \quad 1 \leq e \leq v$$

where

$$h_x(t) = E_i(x,\ t)\ w(t)\ v_n[\underbrace{\gamma_{1e}, \ldots, \gamma_{1e}}_{1 + r_1 \text{ times}}, \ldots, \underbrace{\gamma_{pe}, \ldots, \gamma_{pe}}_{1 + r_p \text{ times}}, t] ,$$

and the last expression involving v_n is the $\tilde{p}$th divided difference of v_n on the points γ_{ke} with multiplicities $1 + r_k$.

If we write for $t \in [x_{e-1}, x_e]$,

$$h_x(t) = \sum_{u=o}^{r} h_x^{(u)}(x_{e-1}) \frac{(t - x_{e-1})^u}{u!} + h_x^{(r+1)}(\Psi_t)\frac{(t - x_{e-1})^{r+1}}{(r + 1)!} ,$$

$$x_{e-1} < \Psi_t < x_e ,$$

then

$$F_e = \int_{x_{e-1}}^{x_e} \prod_{k=1}^{p} (t - \gamma_{ke})^{1+r_k} \frac{(t - x_{e-1})^{r+1}}{(r + 1)!} h_x^{(r+1)}(\psi_t)\ dt ,$$

$$1 \leq e \leq v . \qquad (5.58)$$

By the properties of divided differences [26, p. 40] we have

$$h_x^{(r+1)}(t) = \sum_{u=o}^{r+1} \binom{r+1}{u} (E_i(x,\cdot)w)^{(r+1-u)}(t)\ v_n^{(\tilde{p}+u)}(z_{u,\ t}) \frac{u!}{(\tilde{p} + u)!} ,$$

$$x_{e-1} < t , \quad z_{u,\ t} < x_e .$$

Therefore

$$\sup_{x_{e-1}<t<x_e} |h_x^{(r+1)}(t)| \leq C \sup_{x_{e-1}<t<x_e} \sum_{u=o}^{r+1} |v_n^{(\tilde{p}+u)}(t)| ,$$

where C is independent of x and n, and we have used a uniform bound on

$$\sup_{o \le t \le x \le b} \left|(E_i(x,\cdot))^{(u)}(t)\right| \quad , \qquad o \le i \le s-1 \quad , \qquad o \le u \le r+1.$$

Now from (5.56) it follows by (5.27), (5.32) and Lemma 3.4 that for some C independent of e ,

$$\sup_{x_{e-1} < t < x_e} |v_n^{(i)}(t)| \le C \frac{|\Delta_n|^{\tilde{p}}}{(\Delta x_e)^{\tilde{p}}} \quad , \quad o \le i \le \tilde{p}+r+1 \ , \quad o \le e \le v \ . \quad (5.59)$$

This bound and (5.58) imply

$$|F_e| \le C\ |\Delta_n|^{\tilde{p}}\ (\Delta x_e)^{r+2} \quad , \qquad o \le e \le v-1 \ , \qquad (5.60)$$

where C is independent of n and x.

Write (5.57) as

$$(y - R_n y)^{(i)}(x) = \sum_{e=1}^{v-1} F_e + \int_{x_{v-1}}^{x} h_x(t) \prod_{k=1}^{p} (t - \gamma_{kv})^{1+r_k}\, dt \ . \qquad (5.61)$$

If $x = x_{v-1}$, then by (5.60) ,

$$(y - R_n y)^{(i)}(x_{v-1}) = O(|\Delta_n|^{\tilde{p}+r+1}) \quad , \qquad o \le i \le s-1 \ . \qquad (5.62)$$

If $x_{v-1}<x<x_v$ then since $E_i(x, t)$ has an $(s-1-i)$ th-fold zero at $x = t$, (5.59), (5.61) and the definition of $h_x(t)$ imply

$$(y - R_n y)^{(i)}(x) = O(|\Delta_n|^{\tilde{p}+r+1}) + O(|\Delta_n|^{\tilde{p}+s-i})$$

$$= O(|\Delta_n|^{\tilde{p}+\min(r+1,s-i)}) \quad , \quad o \le i \le s-1 \ . \qquad (5.63)$$

The conclusions of the theorem are now obtained from (5.28), (5.62) and (5.63), since $r+1 \le \tilde{p}$.

If $r_k = 0$, $1 \leq k \leq p$, then (5.55) is satisfied by the Gaussian points if $r = p - 1$, by the Radau points if $r = p - 2$, and by the Lobatto points if $r = p - 3$. By Theorem 5.5, the corresponding methods give error rates of $O(|\Delta_n|^{2p})$, $O(|\Delta_n|^{2p-1})$, and $O(|\Delta_n|^{2p-2})$ respectively, at the partition points. Several results concerning equation (5.55) can be found in [13], especially if all $r_k = 0$. Turán [55] studied (5.55) with all r_k equal to a fixed even positive integer and showed it is satisfied for $r = p - 1$ by the zeroes of an appropriate polynomial, which have been called multiple Gaussian points. Stroud and Stancu [53] tabulated these points for $1 \leq p \leq 7$ and each $r_k = 2, 4$. If each $r_k = 2$, for example, and $s - m \geq 3$, Theorem 5.5 gives error rates of $O(|\Delta_n|^{4p})$ at the partition points. Kastlunger and Wanner [35] have used the multiple Gaussian points to obtain general implicit Runge-Kutta methods for first order problems. As we shall see, their methods are computationally equivalent to our methods if $a_k = 0$, $w \equiv 1$, $s = 1$, but our theory does not apply, since for first-order problems we can use derivatives only if (3.5) and (3.6) are satisfied.

Stancu and Stroud [52] showed that one can fix n_1 of the p points γ_k, assign them arbitrary multiplicities, and find the remaining $p - n_1$ points of multiplicity one to satisfy (5.55) for $r = p - n_1 - 1$. They have fixed one, two, or three points at the ends and/or the middle of the interval with multiplicities ranging from one to six, and have computed the points of multiplicity one for several values of p up to eight. Wittenbrink [59] has looked at methods for nth order equations which use such points, with the restriction that only the endpoints can have multiplicities bigger than one. Our theory allows a more general choice of points and multiplicities for any nth order problem (see Corollary 5.4), but in general the error bounds given do not take advantage of the choice of points. However such points can be used to advantage if the highest chosen multiplicity does not exceed s - 1 - m. For example, if s = 2, m = 0, we can fix the endpoints with multiplicity 2 (q = 1), and choose the remaining p - 2 points as in [52]. By Theorem (5.5) the error rate at the partition points is $O(|\Delta_n|^{2p})$.

CHAPTER VI

EXTENSIONS OF METHODS CONSIDERED IN THE LITERATURE

The idea of constructing global approximations more general than piecewise polynomials can be useful if the differential equation exhibits characteristics which can be exploited, e.g., if f has slow-varying partial derivatives, since once can use larger values of $|\Delta_n|$ and obtain smaller errors. This advantage may disappear as $|\Delta_n| \to o$. Ixaru [31] approximates the coefficients and inhomogeneous term of second-order linear differential equations by step functions, then solves the resulting differential equation with constant coefficients exactly at each step, obtaining a global approximation. The method is of low-order, but it is explicit and can be an improvement over standard methods for certain equations. Pruess [42] obtained methods of arbitrarily high-order by replacing the coefficients and inhomogeneous term of an n^{th} order linear differential equation by piecewise polynomial functions, then using Taylor series techniques to solve the resulting problem exactly. Cooper [11] made use of a weight function to deal with difficult behavior in his study of discrete methods for n^{th} order nonlinear problems, but did not consider the global approximations.

For the remainder of this chapter we will assume that $a_k = o$, $o \leq k \leq m$, $w \equiv 1$, and $g_i^n = y^{(i)}(o)$, $o \leq i \leq s - 1$. We will show how particular choices of P_n reduce our methods to methods already considered by several authors. Many of these methods were not originally analyzed using functional analysis techniques, so our framework gives a unifying approach which makes possible generalizations in several directions. Our theorems are used to rediscover, strengthen, or produce new error bounds for some existing methods.

Collocation With Differentiation

Let P_n be the operators of Example 3.1. From corollary 5.3 we see that the approximate function y_n is a piecewise polynomial of degree $\leq \tilde{p} + s - 1$ at least in $C^{s-1}[o, b]$, which satisfies

$$y_n^{(s+v)}(\gamma_{kj}) = (f(\cdot; y_n))^{(v)}(\gamma_{kj}) , \quad 1 \leq k \leq p , \quad o \leq v \leq r_k , \quad 1 \leq j \leq n .$$

If $r_k = o$, $1 \leq k \leq p$, this is just the method of collocation with piecewise polynomials [16], [43], [59], which include some methods of [8], [9], [20], [27], [28], [41], and [57]. Wittenbrink [59] has considered $\gamma_1 = o$, $\gamma_p = 1$, $r_1 = r_p$, r_p arbitrary, but $r_i = o$, $2 \leq i \leq p - 1$, and has called the methods collocation with differentiation. We will use this term more generally.

We have studied two cases, depending on the value of q:

(a) $0 \leq q \leq s - m - 1$, but with an arbitrary partition (3.1) and $o \leq r_i \leq q$, $1 \leq i \leq p$, with at least one r_i equal to q. This includes collocation.

(b) $q \geq 1$, but the partition D_o given by (3.5) and the r_i satisfying (3.6). This includes the collocation with differentiation of [59].

We have given a fairly complete analysis of case (a), which includes obtaining error rates (5.23) for the higher derivatives, not given in [16], [43], [59], and also generalizing the higher-order methods based on special points γ_k of [16], [59]. Newton's method is not considered in [59].

For case (b) we have established the convergence of the method and have obtained some error bounds for the general problem. If m = o, however, better bounds are given in Corollary (5.4), including again bounds for the higher derivatives. But we have not obtained the bounds of [59] for his special choice of points γ_k.

Wright [61] pointed out that for first-order problems, collocation is equivalent to a subclass of the implicit Runge-Kutta methods. Here we point out that collocation with differentiation for first-order problems is a subclass of the implicit Runge-Kutta methods with multiple nodes of Kastlunger and Wanner [34]. To see this, write (5.19), as

$$y_n(x) = y_n(x_{j-1}) + \sum_{k=1}^{p} \sum_{i=o}^{r_k} c_{ki}(x)(\Delta x_j)^{i+1}(f(\cdot,\ y_n))^{(i)}(\gamma_{kj}) \quad ,$$

$$x_{j-1} \leq x \leq x_j \quad , \qquad (6.1)$$

where

$$c_{ki}(x) = \int_o^{t(x)} \ell_{k,i}(u)\ du, \qquad t(x) = \frac{x - x_{j-1}}{\Delta x_j} \ .$$

The function $\ell_{k,i}$ is defined in Example 3.1. Equation (6.1) is in the form of Equation 15 in [34], which is used to define the Runge-Kutta methods.

The Hermite methods of Loscalzo [39] for first-order problems, which produce a global approximation, are contained in case (b) with $p = 2,\ \gamma_1 = o,\ \gamma_2 = 1,\ r_1 = r_2 = q$. Loscalzo gives a fairly complete historical account of the Hermite formulae on which his methods are based. Some or all of these formulae have been rediscovered by different means and used to construct discrete methods for the numerical solution of first-order problems, but the global aspect was first considered in [38]. Loscalzo gave error bounds for the ith derivative, $o \leq i \leq 2q + 2$. Later, Varga [56] gave improved bounds for $1 \leq i \leq q + 1$. The estimates of Corollary 5.4 improve the bounds of Varga for $1 \leq i \leq q + 1$, and those of Loscalzo for $q + 2 \leq i \leq 2q + 2$. Computationally, these methods are best handled in the form (6.1), from which a single nonlinear equation in the unknown $y_n(x_j)$ is obtained at each step. By extrapolating y_n from one subinterval to the next, Loscalzo also constructed predictor-corrector methods.

Hung [29] uses a Hermite method for quintic splines ($q = 1$, $p = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $r_1 = r_2 = 1$) for the second-order equation. If $m = o$, his method falls into case (a); the error bounds given by Theorem 5.3 are the same as in [29]. If $m = 1$, the method falls into case (b), and Corollary 5.4 gives bounds in agreement with [29]. The computations are best handled through equation (5.19) which can be written as

$$y_n(x) = y_n(x_{j-1}) + y_n'(x_{j-1})(x - x_{j-1})$$

$$+ \sum_{k=1}^{2} \sum_{i=o}^{1} (\Delta x_j)^{i+1} (f(\cdot\,; y_n))^{(i)} (\gamma_{kj})\, h_{ki}(x) \;, \quad x_{j-1} < x < x_j \qquad (6.2)$$

where

$$h_{ki}(x) = \int_o^{t(x)} (x - t)\, \ell_{k,i}(u)\, du \;, \qquad t(x) = \frac{x - x_{j-1}}{\Delta x_j} \;.$$

From (6.2) one obtains two nonlinear equations in the two unknowns

$y_n(x_j)$, $y_n'(x_j)$.

The first family of Ehle's [19] first class of discrete L-acceptable methods for first-order problems, which include Enright's [21] formula for k = 1 (or Liniger and Willoughby's [37] formula for a = b = 1/3) falls into case (b) for p = 2, $\gamma_1 = o$, $\gamma_2 = 1$, $r_1 = q - 1$, $r_2 = q$, since case (b) then gives $y_n(x_1)$ as the first subdiagonal Padé approximant to $e^{\lambda x_1}$ when the method is applied to $y' = \lambda y$, y(o) = 1. (See Chapter VII). Corollary 5.4 gives error bounds for the global approximation, which was not considered in [19]. Again, this method is used best with equation (6.1), from which one obtains a nonlinear equation in the unknown $y_n(x_j)$. Computationally, the second family of Ehle's [19] first class of discrete L-acceptable methods corresponds to collocation with differentiation with p = 2, $\gamma_1 = o$, $\gamma_2 = 1$, $r_1 = q - 2$, $r_2 = q$, $q \geq 2$, but it is not covered by case (a) or (b).

Block Implicit Methods

From equation (5.19) we can obtain a nonlinear system of equations whose unknowns are the values $y_n^{(v)}(\gamma_{kj})$ appearing on the right-hand side. We may be interested in such values only and not in the global approximant $y_n(x)$; this is often the case if the γ_k are equally spaced, e. g.,

$$\gamma_k = \frac{k - 1}{p - 1} \quad , \quad 1 \leq k \leq p \quad .$$

If P_n is an in Example 3.1 with each $r_k = q = o$, and if p is odd, the higher estimates of Theorem 5.5 hold with $\tilde{p} = p$, r = o, since (5.55) is satisfied for r = o (see for example [30, pg. 309]). In general, for any p, and s = 1, this choice of P_n corresponds to the Newton-Cotes block implicit discrete methods of Watts and Shampine [57], some of whose global aspects were investigated by Callender [9] and Williams and deHoog [58]. Their global error bounds are contained in Theorems 5.3 and 5.5. If $s = 2$, we obtain the spline methods of Micula [41]. If p is odd, Theorems 5.3 and 5.5 contain his error bounds and improve the bound for the first derivative.

If P_n is derived from the operator Q_3 of Example 3.3 with $\ell = p$, one obtains the natural g-spline block implicit methods of Andria, Byrne, and Hill [2] for first-order problems. Equation (5.15) is written as

$$y_n(x) = y(x_{j-1}) + \sum_{i=o}^{p} f(\gamma_{kj}, y_n(\gamma_{kj})) \int_{x_{j-1}}^{x} T_i\left(\frac{t - x_{j-1}}{\Delta x_j}\right) dt \quad ,$$

from which one obtains a nonlinear system in the p unknowns $y_n(\gamma_{kj})$, $1 \leq k \leq p$. From Corollary 5.1 and Lemma 3.8 we can obtain results similar to those of Corollary 5.3, including error bounds for derivations up to order ℓ. Global approximations were not considered in [2].

CHAPTER VII

STABILITY OF THE NUMERICAL METHODS

First-Order Problems

Discrete implicit methods began to be studied extensively after the more common explicit methods run into stability difficulties when they were applied to certain types of equations called stiff equations. A constant coefficient system $y' = Ay$ is stiff if all the eigenvalues of A are in the left half plane $R_e z < o$ and if the real parts differ greatly in magnitude. Implicit methods were constructed which allow a larger step size to be used in the numerical integration without running into stability problems. The usual difficulty encountered by the explicit methods in integrating stiff equations numerically is that the more rapidly exponentially decreasing terms impose technical restrictions on using relatively large step lengths, even when such terms become insignificant.

The first methods proposed to handle stiff problems efficiently had the property of A-stability [12]. A numerical method is A-stable if all its solutions Y(ih) tend to zero as $i \to \infty$ when the method is applied with arbitrary fixed positive step size h to any differential equation of the form

$$y' = \lambda y, \quad y(o) = 1 \quad , \tag{7.1}$$

where λ is a complex constant with negative real part. This definition basically says that decreasing solutions should be approximated by decreasing functions. Dahlquist studied multistep methods which use only values of y and y'; he showed that the explicit methods were not A-stable and that an implicit A-stable multistep method could not be of order larger than 2.

An A-stable method does not guarantee that the numerical solution will exhibit the rapidly decaying behavior of the solution if $|\lambda h| >> 1$, and sometimes the numerical solution may be slowly damped and rapidly oscillating, which is frequently undesirable. For example, the trapezoidal rule gives

$$Y(ih) = \left(\frac{1 + \frac{\lambda h}{2}}{1 - \frac{\lambda h}{2}} \right)^i , \qquad i = 0, 1, 2, \ldots \quad ,$$

so the method is A-stable, but as $Re(\lambda h) \to -\infty$, with fixed h, the approximate solution exhibits this bad behavior. To guarantee that Y(ih) tends to zero rapidly when $|\lambda h| >> 1$, a subclass of A-stable methods was introduced. For a one-step method, the solution may be written as

$$Y(ih) = (P(\lambda h))^i \quad , \qquad i = 0, 1, 2, 3, \ldots \quad .$$

The method is called strongly A-stable if $|P(\lambda h)| < 1$ for $Re(\lambda h) < o$, and $|P(\lambda h)| \to o$ as $Re(\lambda h) \to -\infty$ (Chipman [10]).

The general methods we have studied are trivially strongly A-stable if we choose

$$a_o = \frac{\partial f(o, y(o))}{\partial y}$$

A large number of A-stable and strongly A-stable methods have been studied in the past few years. We present here a unified treatment of many of these, from the point of view of collocation with differentiation.

Let P_n be the operators of Example 3.1. Then if $a_k = o$, $o \leq k \leq m$, $w \equiv 1$, $\Delta x_j = h$, $g_o^n = 1$, the methods applied to (7.1) produce an approximation y_n which is a piecewise polynomial of degree $\tilde{p}$ in $C[o, b]$, and which satisfies $y_n(o) = 1$ and

$$y_n^{(1+u)}(\gamma_{kj}) = \lambda y_n^{(u)}(\gamma_{kj}) \quad , \quad 1 \leq k \leq p, \quad o \leq u \leq r_k, \quad 1 \leq j \leq n .$$

Then in particular, $y_n' - \lambda y_n$ is a polynomial of degree $\tilde{p}$ in $[o, h]$ with a zero of multiplicity $1 + r_k$ at γ_{k1}, i.e.,

$$y_n'(x) - \lambda y_n(x) = \tau \prod_{k=1}^{p} (x - h\gamma_k)^{1+r_k} \quad , \quad o \leq x \leq h \quad , \tag{7.2}$$

where τ is some constant. The differential equation (7.2) can be solved exactly to obtain

$$y_n(h) = e^{\lambda h} + \tau\, e^{\lambda h}\, h^{\tilde{p}+1} \int_o^1 e^{-\lambda u h} \prod_{k=1}^{p} (u - \gamma_k)^{1+r_k} \, du \quad . \tag{7.3}$$

This approach is due to Wright [61] who has also shown that

$$y_n(h) = \frac{\sum_{r=o}^{\tilde{p}} r!\; v_r(\lambda h)^{\tilde{p}-r}}{\sum_{r=o}^{\tilde{p}} r!\; u_r(\lambda h)^{\tilde{p}-r}} \tag{7.4}$$

where u_r, v_r are defined by

$$\prod_{k=1}^{p} (x - \gamma_k)^{1+r_k} = \sum_{k=o}^{\tilde{p}} u_r x^r \quad , \tag{7.5}$$

$$\prod_{k=1}^{p} (x + 1 - \gamma_k)^{1+r_k} = \sum_{k=o}^{\tilde{p}} v_r x^r \quad . \tag{7.6}$$

Since $y_n(ih) = (y_n(h))^i$, one needs to investigate the behavior of $y_n(h)$ for fixed h and any λ with $\mathrm{Re}\,\lambda < o$. Axelsson [5], Ehle [18], Watts and Shampine [57], Wright [61] and others have several results on the A-stability of the methods for the case of Lagrange interpolation $(r_k = o, o \leq k \leq m)$. Here we are interested in the connection between the expression $y_n(h)$ given by (7.4) and the Padé approximants to $e^{\lambda h}$, since a great deal is known about the latter. Our treatment is based on equations (7.3) - (7.6), which was the approach taken by Wright for the case $r_k = o$. First we summarize his results. Expanding $e^{-\lambda uh}$ in a Taylor series, then (7.3) implies

(i) $y_n(h) - e^{\lambda h} = O(h^{2p+1})$, if the γ_k are the Gaussian points;

(ii) $y_n(h) - e^{\lambda h} = O(h^{2p})$, if the γ_k are the Radau points (with right-endpoint fixed) ;

(iii) $y_n(h) - e^{\lambda h} = O(h^{2p-1})$, if the γ_k are the Lobatto points.

But from (7.4) - (7.6) we see that $y_n(h)$ is a rational function with numerator and denominator of degree p, p for case (i), of degree p - 1, p for case (ii) and of degree p - 1, p - 1 for case (iii). Hence Wright concluded that $y_n(h)$ is the (p, p) diagonal Padé approximant to $e^{\lambda h}$ for case (i), the (p - 1, p) subdiagonal Padé approximant for case (ii), and the (p - 1, p - 1) diagonal Padé approximant for case (iii). Birkoff and Varga [7] had shown that the diagonal Padé approximants $P_{n,n}$ satisfy $|P_{n,n}(\lambda h)| < 1$ for $\mathrm{Re}(\lambda h) < o$, and later Ehle [19] proved that the first and second subdiagonal approximants $P_{n-1,n}$, $P_{n-2,n}$ have the same property. Therefore, cases (i) and (iii) give A-stable methods, and case (ii) gives strongly A-stable methods.

We now investigate the expression $y_n(h)$ if some $r_k \geq 1$. If p = 2,

$\gamma_1 = o$ and $\gamma_2 = 1$, then $\tilde{p} = 2 + r_1 + r_2$, and by (7.3),

$$y_n(h) - e^{\lambda h} = 0(h^{3+r_1+r_2})$$

Moreover, (7.4) - (7.6) show that $y_n(h)$ is a rational function with numerator and denominator of degree $1 + r_1$ and $1 + r_2$, respectively. We conclude that $y_n(h)$ is the $(1 + r_1, 1 + r_2)$ Padé approximant of $e^{\lambda h}$. Hence if $r_1 = r_2$, the methods are A-stable; this case is equivalent to the Hermite methods of Loscalzo [39], who had shown the A-stability, but it was Varga [56] who pointed out the connection with the Padé approximants. If $0 \leq r_1 = r_2 - 1$ or $0 \leq r_1 = r_2 - 2$, the methods are strongly A-stable. These two cases are equivalent to the first class of L-acceptable methods of Ehle [19], who constructed formulae directly to yield the first and second subdiagonal approximants.

A recent paper by Saff and Varga [46] gives general results concerning the location of the poles of the Padé approximants to e^z. For example, the first four subdiagonals of the Padé table are analytic for $Re(z) \leq o$; also, proceeding sufficiently far enough along any subdiagonal of the table, all entries will be analytic for $Re(z) \leq o$. If it can be established that any such entry is bounded by one on the imaginary axis, then by the maximum modulus theorem, it is bounded by one on the entire left-half plane. The values of r_1 and r_2 which correspond to such entry will produce a method of collocation with differentiation which is strongly A-stable.

Finally, we consider the fixed multiple nodes and simple Gaussian nodes of Stancu and Stroud [52]. Let $p > 2$, $\gamma_1 = 1$, $\gamma_p = 2$, r_1 and r_p be arbitrary nonnegative integers, and $r_i = o$, $2 \leq i \leq p - 1$. Then there exist p - 2 points $\gamma_k \in [o, 1]$, $2 \leq k \leq p - 2$, such that (5.55) is satisfied for $r = p - 3$. From (7.3) we see that

$$y_n(h) - e^{\lambda h} = 0(h^{2p+r_1+r_p - 1}).$$

But (7.4) - (7.6) show that $y_n(h)$ is a rational function with numerator and denominator of degree $p + r_1 - 1$ and $p + r_p - 1$ respectively. Therefore, $y_n(h)$ is the $(p + r_1 - 1, p + r_p - 1)$ Padé approximant to $e^{\lambda h}$. If $r_1 = r_p$ the methods are A-stable; if $r_1 = r_p - 1$ or $r_1 = r_p - 2$, the methods are strongly A-stable.

Second-Order Problems

The initial value problem

$$y''(x) = f(x, y(x)) \quad , \quad o \leq x \leq b \quad , \quad y(o) = y_o, \; y'(o) = y'_o \tag{7.7}$$

arises frequently in practice, e.g., in mechanics and astronomy. Various special methods have been developed to solve it numerically, which apply to (7.7) directly; see for example [14], [15], [22], [23], [33], [50], and [60].

The stability of one-step methods has been treated in several papers using the fundamental analysis of Rutishauser [44], [45]. The methods are applied with fixed step h to the initial value problem

$$y''(x) = \alpha y \;, \quad y(o) = y_o \;, \quad y'(o) = y'_o \;, \tag{7.8}$$

where α is a real number, to obtain the numerical solution

$$\begin{bmatrix} Y(ih) \\ Y'(ih) \end{bmatrix} = A^i \begin{bmatrix} y_o \\ y'_o \end{bmatrix} \quad , \qquad i = 0, 1, 2, \ldots \tag{7.9}$$

where $A = [a_{ij}]$ is a matrix independent of i. The case $\alpha = \lambda^2$ is used to compute the order of the method as follows: the exact solution y satisfies

$$\begin{bmatrix} y(ih) \\ y'(ih) \end{bmatrix} = e^{ihB} \begin{bmatrix} y_o \\ y'_o \end{bmatrix} \quad , \quad \text{where } B = \begin{bmatrix} 0 & 1 \\ \lambda^2 & 0 \end{bmatrix} \quad . \tag{7.10}$$

Then the order of the one-step method is an integer $r > o$ for which

$$\lim_{h \to o} \frac{e^{hB} - A}{h^{r+1}}$$

exists and is different from zero for at least one value of λ. Hence, the order of the method is actually a measure of how well it approximates the solutions of the family of problems (7.8) with $\alpha > o$. Since our theorems give the order of convergence independently of any particular family of problems, we will not pursue the above approach further, but instead concentrate on the case $\alpha = -\lambda^2 < 0$. We introduce the following definition.

<u>Definition 7.1</u>

Suppose a one-step method applied with a fixed step size h to any problem of the form (7.8), where $\alpha = -\lambda^2$ is a real number, gives approximations in the form (7.9). If the eigenvalues of A do not exceed one in modulus for all $\lambda h > o$, the method is said to be B-stable.

We expect a B-stable method to give approximations which stay close to the oscillatory exact solution of (7.8) as $i \to \infty$, when using a moderate fixed step size h, even if λ is very large. Several authors, e. g., [15], [23], [33], [50], have shown that their methods produce a matrix A whose eigenvalues do not exceed one in modulus only if λh is small. A typical bound is $o < \lambda^2 h^2 < 9.6$. The method of Cowell and

Numerov [24] is B-stable, but it is not self-starting and is a discrete method. Any of our methods with

$$a_o = \frac{\partial f(o,\ y_o)}{\partial y} \quad , \quad a_1 = 0$$

is trivially B-stable since it solves (7.8) exactly.

We will next derive a useful expression for the matrix A for all methods of collocation with differentiation with piecewise polynomials. For a particular choice of collocation points, one can examine the eigenvalues of A directly to determine if the method is B-stable, and conversely, the entries in A can suggest appropriate choices of collocation points.

If P_n is the operator of Example 3.1, then our methods, with all $a_k = o$, $w \equiv 1$, $\Delta x_j = h$, $g_o^n = y_o$, $g_1^n = y_o'$, applied to (7.8) with $\alpha = -\lambda^2$, produce an approximation y_n which is a piecewise polynomial of degree $\tilde{p} + 1$ in $C'[o,\ b]$, and which satisfies $y_n(o) = y_o$, $y_n'(o) = y_o'$ and

$$y_n^{(2+v)}(\gamma_{kj}) = -\lambda^2 y_n(\gamma_{kj}), \quad 1 \leq k \leq p, \quad o \leq v \leq r_k \ , \quad 1 \leq j \leq n \ .$$

In particular, $y_n'' + \lambda^2 y_n$ is a polynomial of degree $\tilde{p} + 1$ in $[o,\ h]$ with a zero of multiplicity $1 + r_k$ at γ_{k1}, i.e.,

$$y_n''(x) + \lambda^2 y_n(x) = \tau \prod_{k=1}^{p+1} (x - h\gamma_k)^{1+r_k} \quad , \quad o \leq x \leq h \qquad (7.11)$$

where τ and γ_{p+1} are some constants.

Equation (7.11) has the solution

$$\begin{bmatrix} y_n(x) \\ y_n'(x) \end{bmatrix} = e^{Ax} \left\{ \begin{bmatrix} y_o \\ y_o' \end{bmatrix} + \tau \int_o^x e^{-At} \prod_{k=1}^{p+1} (t - h\ \gamma_k)^{1+r_k} \, dt \right\}, \quad o \leq x \leq h \quad ,$$

where

$$e^{Ax} = \begin{bmatrix} \cos(\lambda x) & \frac{1}{\lambda}\sin(\lambda x) \\ -\lambda\sin(\lambda x) & \cos(\lambda x) \end{bmatrix} .$$

Setting x = h and combining terms we have

$$y_n(h) = y_o \cos(\lambda h) + \frac{y_o'}{\lambda}\sin(\lambda h) + \frac{\tau}{\lambda} h^{\tilde{p}+2} \int_o^1 \sin(\lambda h - \lambda hu) \prod_{k=1}^{p+1} (u - \gamma_k)^{1+r_k} du \tag{7.12}$$

and

$$y_n'(h) = -y_o \lambda\sin(\lambda h) + y_o' \cos(\lambda h) + \tau h^{\tilde{p}+2} \int_o^1 \cos(\lambda h - \lambda hu) \prod_{k=1}^{p+1} (u - \gamma_k)^{1+r_k} du . \tag{7.13}$$

Integration by parts gives

$$\int_o^1 \sin(\lambda h - \lambda hu)\, u^r \, du = \begin{cases} \displaystyle\sum_{j=o}^{\frac{r}{2}} \frac{(-1)^j}{(\lambda h)^{2j+1}} \frac{r!}{(r-2j)!} + (-1)^{1+\frac{r}{2}} \frac{r!}{(\lambda h)^{r+1}} \cos(\lambda h) , & r = 0, 2, 4, \ldots \\ \displaystyle\sum_{j=o}^{\frac{r-1}{2}} \frac{(-1)^j}{(\lambda h)^{2j+1}} \frac{r!}{(r-2j)!} + (-1)^{\frac{r+1}{2}} \frac{r!}{(\lambda h)^{r+1}} \sin(\lambda h) , & r = 1, 3, 5, \ldots \end{cases} \tag{7.14}$$

If we define u_r and v_r by

$$\prod_{k=1}^{p+1} (x - \gamma_k)^{1+r_k} = \sum_{r=o}^{\tilde{p}+1} u_r x^r \tag{7.15}$$

and

$$\prod_{k=1}^{p+1} (x + 1 - \gamma_k)^{1+r_k} = \sum_{r=o}^{\tilde{p}+1} v_r x^r \quad , \tag{7.16}$$

then by induction

$$v_k = \frac{1}{k!} \sum_{r=k}^{\tilde{p}+1} \frac{r!}{(r-k)!} u_r \quad , \qquad o \leq k \leq \tilde{p} + 1 \quad . \tag{7.17}$$

Equation (7.14) implies that

$$\sum_{r=o}^{\tilde{p}+1} u_r \int_o^1 \sin(\lambda h - \lambda h t)\, t^r \, dt = \sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} \frac{(-1)^j}{(\lambda h)^{2j+1}} \left(\sum_{r=2j}^{\tilde{p}+1} \frac{r!}{(r-2j)!} u_r \right)$$

$$+ \sum_{r=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^{r+1} \frac{(2r)!}{(\lambda h)^{2r+1}} u_{2r} \cos(\lambda h)$$

$$+ \sum_{r=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^{r+1} \frac{(2r+1)!}{(\lambda h)^{2r+2}} u_{2r+1} \sin(\lambda h) \; , \tag{7.18}$$

where [i] = greatest integer contained in i.

Using (7.15), (7.17), and (7.18) in (7.12) we obtain

$$y_n(h) = y_o \cos(\lambda h) + \frac{y'_o}{\lambda} \sin(\lambda h) + \frac{\tau}{\lambda} h^{\tilde{p}+2} \sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} \frac{(-1)^j}{(\lambda h)^{2j+1}} (2j)!\, v_{2j}$$

$$+ \cos(\lambda h) \frac{\tau}{\lambda} h^{\tilde{p}+2} \sum_{r=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^{r+1} \frac{(2r)!}{(\lambda h)^{2r+1}} u_{2r}$$

$$+ \sin(\lambda h) \frac{\tau}{\lambda} h^{\tilde{p}+2} \sum_{r=0}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^{r+1} \frac{(2r+1)!}{(\lambda h)^{2r+2}} u_{2r+1} \; . \tag{7.19}$$

Since $y_n(h)$ is a polynomial and (7.19) is satisfied for all h, we must have

$$y_o + \frac{\tau}{\lambda} h^{\tilde{p}+2} \sum_{r=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^{r+1} \frac{(2r)!}{(\lambda h)^{2r+1}} u_{2r} = 0 \quad , \tag{7.20}$$

and

$$\frac{y_o'}{\lambda} + \frac{\tau}{\lambda} h^{\tilde{p}+2} \sum_{r=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^{r+1} \frac{(2r+1)!}{(\lambda h)^{2r+2}} u_{2r+1} = 0 \quad . \tag{7.21}$$

Therefore (7.19) becomes

$$y_n(h) = \frac{\tau}{\lambda} h^{\tilde{p}+2} \sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} \frac{(-1)^j}{(\lambda h)^{2j+1}} (2j)! v_{2j} = \frac{\tau}{\lambda^{\tilde{p}+3}} \sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^j (\lambda h)^{\tilde{p}+1-2j} (2j)! v_{2j} \; . \tag{7.22}$$

In a similar manner,

$$y_n'(h) = \frac{\tau}{\lambda^{\tilde{p}+2}} \sum_{j=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^j (\lambda h)^{\tilde{p}-2j} (2j+1)! \; v_{2j+1} \; . \tag{7.23}$$

To obtain the matrix A, we find u_r and v_r in terms of γ_{p+1} from (7.15) and (7.16). Then letting $y_o = 1$, $y_o' = o$, equation (7.21) gives γ_{p+1} and equation (7.20) gives τ. Substituting these values of γ_{p+1} and τ in (7.22) and (7.23), we get the first column of A. Similarly, letting $y_o = o$, $y_o' = 1$, equation (7.20) gives γ_{p+1} and equation (7.21) gives τ. Substituting these new values of γ_{p+1} and τ in (7.22) and (7.23) we get the second column of A.

Hence $A = [-\alpha_{ij}]$, with

$$\alpha_{11} = \frac{\sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^j (\lambda h)^{\tilde{p}+1-2j} \; (2j)! \; v_{2j}(1, o)}{\sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^{j+1} (\lambda h)^{\tilde{p}+1-2j} \; (2j)! \; u_{2j}(1, o)}$$

$$\alpha_{12} = \frac{\frac{1}{\lambda} \sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^j (\lambda h)^{\tilde{p}+1-2j} \; (2j)! \; v_{2j}(o, 1)}{\sum_{j=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^{j+1} (\lambda h)^{\tilde{p}-2j} \; (2j+1)! \; u_{2j+1}(o, 1)} ,$$

$$\alpha_{21} = \frac{\lambda \sum_{j=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^j (\lambda h)^{\tilde{p}-2j} \; (2j+1)! \; v_{2j+1}(1, o)}{\sum_{j=o}^{\left[\frac{\tilde{p}+1}{2}\right]} (-1)^{j+1} (\lambda h)^{\tilde{p}+1-2j} \; (2j)! \; u_{2j}(1, o)}$$

$$\alpha_{22} = \frac{\sum_{j=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^j (\lambda h)^{\tilde{p}-2j} \; (2j+1)! \; v_{2j+1}(o, 1)}{\sum_{j=o}^{\left[\frac{\tilde{p}}{2}\right]} (-1)^{j+1} (\lambda h)^{\tilde{p}-2j} \; (2j+1)! \; u_{2j+1}(o, 1)} .$$

Here $v_{2j}(1, o)$, for example, represents the values v_{2j} obtained when $y_o = 1$, $y_o' = o$.

To illustrate the computations of this section, and to obtain a B-stable method of order three, let $p = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $r_1 = 0$, $r_2 = 1$. From equations (7.15) and (7.16) we find

$$u_o = 0,\ u_1 = -\gamma_3,\ u_2 = 2\gamma_3 + 1,\ u_3 = -\gamma_3 - 2,\ u_4 = 1 \quad ,$$

$$v_o = 0,\ v_1 = 0,\ v_2 = 1 - \gamma_3,\ v_3 = 2 - \gamma_3,\ v_4 = 1 \quad .$$

If $y_o = 1$, $y'_o = 0$, equation (7.21) gives

$$\gamma_3 = \frac{12}{(\lambda h)^2 - 6}$$

So if we let $\beta = (\lambda h)^2$ and $\delta = 1 + \frac{\beta}{12} + \frac{\beta^2}{72}$, we have

$$\alpha_{11} = \frac{1}{\delta}\,(1 - \frac{5}{12}\,\beta + \frac{\beta^2}{72}) \quad , \qquad \alpha_{21} = \frac{1}{\delta h}\left(\frac{\beta^2}{12} - \beta\right) \quad .$$

Now with $y_o = o$, $y'_o = 1$, equation (7.20) gives

$$\gamma_3 = \frac{12 - \beta}{2\beta} \quad ,$$

and so

$$\alpha_{12} = \frac{h}{\delta}\,(1 - \frac{\beta}{12}) \quad , \qquad \alpha_{22} = \frac{1}{\delta}\,(1 - \frac{5}{12}\,\beta\,) \quad .$$

The characteristic roots of A are

$$\ell = \frac{1}{2\delta}(2 - \frac{5}{6}\,\beta + \frac{\beta^2}{72} \pm (\frac{\beta^4}{5184} - \frac{\beta^3}{36} + \frac{2}{3}\,\beta^2 - 4\beta)^{\frac{1}{2}}) \quad .$$

The characteristic roots are complex for (approximately)

$$o < \beta < 9.54 \quad \text{and} \quad 18.78 < \beta < 115.67 \quad .$$

One verifies algebraically that $|\ell| \leq 1$ for $\lambda h > 0$.

CHAPTER VIII

NUMERICAL EXAMPLES

This chapter contains several examples which illustrate the rates of convergence and the improvement caused by choosing some $a_k \neq 0$ or $w \not\equiv 1$. In addition, we present some details of the formulae used to obtain the numerical solutions. All calculations were carried out on a CDC 6400 in single precision (which is approximately fourteen decimal digits).

Example 8.1

The first example is the problem

$$y'(x) = y^2(x) \ , \qquad o \le x \le 1, \qquad y(o) = -1 \ ,$$

which is used to illustrate the conclusions of Corollary 5.4. The numerical approximations y_n are obtained using the operators P_n of Example 3.1 with $p = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $r_1 = 1$, $r_2 = 1$. We choose $a_o = 0$, $w \equiv 1$ and $\Delta x_j = h$, $1 \le j \le n$. Equation (5.19) gives

$$y_n(x) = y_n(x_{j-1}) + hy_n^2(x_{j-1})(\frac{u^4}{2} - u^3 + u) + hy_n^2(x_j)(u^3 - \frac{u^4}{2})$$

$$+ 2h^2y_n^3(x_{j-1})(\frac{u^2}{2} - \frac{2}{3}u^3 + \frac{u^4}{4}) + 2h^2y_n^3(x_j)(\frac{u^4}{4} - \frac{u^3}{3}) \ ,$$

$$x_{j-1} \le x \le x_j \ , \tag{8.1}$$

where $u = \dfrac{x - x_{j-1}}{h}$.

Setting $x = x_j$ in the above equation, we obtain the nonlinear equation

$$y_n(x_j) = y_n(x_{j-1}) + \frac{h}{2}(y_n^2(x_{j-1}) + y_n^2(x_j)) + \frac{h^2}{6}(y_n^3(x_{j-1}) - y_n^3(x_j)) \quad ,$$

which is solved easily by Newton's method at each step. The value $y_n(x_j)$ then determines $y_n(x)$ uniquely in $[x_{j-1}, x_j]$ through (8.1). As we have mentioned before, this is a method studied by Loscalzo [39], but Corollory 5.4 gives improved rates of convergence for the ith derivative, $1 \leq i \leq 4$. Table 1 contains the quantities

$$\ell_n^{(i)}(h) = ||(y_n - y)^{(i)}||_o \quad , \qquad i = 0, 1, 2,$$

$$\ell_n^{(i)}(h) = ||(y_n - y)^{(i)}||_{o,n} \quad , \qquad i = 3, 4,$$

and

$$\ell_{n,\Delta}^{(2)}(h) = \max_{1 \leq j \leq n} |(y_n - y)^{(2)}(x_j)| \quad .$$

The column next to each column of errors represents the computed orders of convergence

$$\frac{\log (\ell_n(h_1)/\ell_n(h_2))}{\log (h_1/h_2)} \quad .$$

For convenience we denote 1.52×10^{-3} by 1.52(-3) .

All the computed rates of convergence in Table 1 agree with the rates given in Corollary 5.4. Earlier, Loscalzo [39] had given rates of $O(h^4)$, $O(h^2)$, $O(h)$, $O(1)$, $O(1)$ for ℓ_n, $\ell_n^{(1)}$, $\ell_n^{(2)}$, $\ell_n^{(3)}$, $\ell_n^{(4)}$, respectively. Later, Varga [56] gave improved rates of $O(h^3)$, $O(h^2)$

Table 1. Illustration of Error Rates of Corollary 5.4

h	ℓ_n		$\ell_n^{(1)}$		$\ell_n^{(2)}$		$\ell_n^{(3)}$		$\ell_n^{(4)}$		$\ell_{n,\Delta}^{(2)}$	
1/4	7.18(-5)		5.60(-4)		7.66(-3)		3.55(-1)		1.01(1)		2.62(-4)	
1/8	4.53(-6)	3.99	5.05(-5)	3.47	1.32(-3)	2.54	1.16(-1)	1.61	6.05(0)	0.74	1.63(-5)	4.01
1/16	2.80(-7)	4.02	3.85(-6)	3.71	1.96(-4)	2.75	3.34(-2)	1.80	3.35(0)	0.85	1.03(-6)	3.98
1/32	1.74(-8)	4.01	2.67(-7)	3.85	2.68(-5)	2.87	9.02(-3)	1.89	1.77(0)	0.92	6.45(-8)	4.00
1/64	1.08(-9)	4.01	1.76(-8)	3.92	3.50(-6)	2.94	2.35(-3)	1.94	9.11(-1)	0.96	4.04(-9)	4.00
1/128	6.74(-11)	4.00	1.13(-9)	3.96	4.48(-7)	2.97	5.98(-4)	1.97	4.62(-1)	0.98	2.52(-10)	4.00
1/256	4.09(-12)	4.04	7.18(-11)	3.98	5.66(-8)	2.98	1.51(-4)	1.99	2.33(-1)	0.99	1.55(-11)	4.02

for $\ell_n^{(1)}$, $\ell_n^{(2)}$, respectively.

Example 8.2

Our second example

$$y'(x) = \sqrt{x}\, y(x) \quad , \qquad o \leq x \leq 1 \; , \qquad y(o) = 1 \quad ,$$

shows how the rates of convergence can be improved by choosing $w \not\equiv 1$. The numerical approximations y_n are obtained using the operators P_n of Example 3.1 with $p = 2$, $\gamma_1 = 0$, $\gamma_2 = 1$, $r_1 = 0$, $r_2 = 0$. In addition, $a_o = 0$, $w(x) = \sqrt{x}$ and $\Delta x_j = h$.

Equation (5.19) gives

$$y_n(x) = y_n(x_{j-1}) + \frac{2}{5h}\,[(-y_n(x_{j-1}) + y_n(x_j))(x^{5/2} - x_{j-1}^{5/2})]$$

$$+ \frac{2}{3h}\,[(x_j y_n(x_{j-1}) - x_{j-1} y_n(x_j))(x^{3/2} - x_{j-1}^{3/2})] \;, \quad x_{j-1} \leq x \leq x_j \quad .$$

Setting $x = x_j$, we easily find $y_n(x_j)$, which determines $y_n(x)$ completely in $[x_{j-1}, x_j]$.

Table 2 has the results given by this method and also by the method of collocating at the endpoints of each subinterval with piecewise quadratic polynomials. The notation is as in Example 8.1.

Table 2 shows that not only are ℓ_n and $\ell_n^{(1)}$ smaller for the new method, but that the new method converges faster. The reason is that the error decreases with $||P_n\, \tilde{H}(\cdot;\, y) - \tilde{H}(\cdot;\, y)||_{o,n}$, and $\tilde{H}(x;\, y) = \sqrt{x}\; y(x)$ for collocation, while $\tilde{H}(x;\, y) = y(x)$ for the new method. Since $y \in C'[o,\, 1]$, Theorem 4.3 gives $\ell_n = O(\omega(\sqrt{x}\; y(x),\, h))$ for collocation and $\ell_n = O(h)$ for the new method. A small modification in the proof of Theorem 4.3 gives these same rates for $\ell_n^{(1)}$.

Table 2. Use of a Weight Function

	Collocation				New Method			
h	ℓ_n		$\ell_n^{(1)}$		ℓ_n		$\ell_n^{(1)}$	
1/2	5.22(-2)		1.51(-1)		3.79(-2)		8.04(-2)	
1/4	2.59(-2)	1.01	1.19(-1)	0.34	9.26(-3)	2.03	2.46(-2)	1.71
1/8	1.18(-2)	1.13	8.62(-2)	0.47	2.30(-3)	2.01	6.95(-3)	1.82
1/16	4.80(-3)	1.30	6.16(-2)	0.48	5.72(-4)	2.01	1.86(-3)	1.90
1/32	1.86(-3)	1.37	4.37(-2)	0.50	1.43(-4)	2.00	4.82(-4)	1.95

Example 8.3

Before we consider a specific example we will derive some general formulas for first-order problems. Suppose

$$y'(x) = f(x, y), \quad o \le x \le b \quad , \quad y(o) = y_o \qquad (8.2)$$

Then if $a_o = -k$, $w \equiv 1$, equation (2.7) becomes

$$y_n(x) = y_n(x_{j-1})\, e^{-k(x-x_{j-1})} + \int_{x_{j-1}}^{x} e^{-k(x-t)}\, P_n(f(\cdot, y_n) + k\, y_n)(t)\, dt,$$

$$x_{j-1} \le x \le x_j \quad . \qquad (8.3)$$

Let P_n be as in Example 3.1 with $p = 3$, $\gamma_1 = 0$, $\gamma_2 = 0.5$, $\gamma_3 = 1$, $r_1 = r_2 = r_3 = 0$. Also let $\Delta x_j = h$. Then equation (8.3) with $x = \gamma_{2j}$ and $x = \gamma_{3j}$ gives

$$y_n(\gamma_{ij}) = y_n(x_{j-1})\, e^{-k\gamma_i h} + h[A_i H(\gamma_{1j}, y_n(\gamma_{1j})) + B_i\, H(\gamma_{2j}, y_n(\gamma_{2j}))$$

$$+ C_i\, H(\gamma_{3j}, y_n(\gamma_{3j}))] \ , \quad 2 \le i \le 3 \ , \ (8.4)$$

where

$$H(x, y_n(x)) = f(x, y_n(x)) + k\, y_n(x) \quad ,$$

$$\gamma_{ij} = x_{j-1} + \gamma_i h , \qquad 1 \le i \le 3 \quad ,$$

$$A_2 = (r - 4 + e^{r/2} (r^2 - 3r + 4))/r^3 \quad ,$$

$$B_2 = -4(r^2/4 - 2 + e^{r/2} (-r + 2))/r^3 \; ,$$

$$C_2 = (-r - 4 + e^{r/2} (-r + 4))/r^3 \quad ,$$

$$A_3 = (-r - 4 + e^{r} (r^2 - 3r + 4))/r^3 \quad ,$$

$$B_3 = -4(-r - 2 + e^{r}(-r + 2))/r^3 \quad ,$$

$$C_3 = (-r^2 - 3r - 4 + e^{r}(-r + 4))/r^3 \quad ,$$

and

$$r = -kh \quad .$$

Once k is chosen, equations (8.4) can be interpreted as representing a block implicit method of size two to approximate the solution of (8.2). Since A_i, B_i and C_i, $2 \le i \le 3$, are computed only once (for each value of k), this method requires very little more work that the method of collocation obtained using the same operators P_n as above (but $a_o = 0$) which gives

$$y_n(\gamma_{2j}) = y_n(x_{j-1}) + h[\frac{5}{24} f(\gamma_{1j}, y_n(\gamma_{1j})) + \frac{1}{3} f(\gamma_{2j}, y_n(\gamma_{2j})) - \frac{1}{24} f(\gamma_{3j}, y_n(\gamma_{3j}))] \quad , \tag{8.5}$$

and

$$y_n(\gamma_{3j}) = y_n(x_{j-1}) + h[\frac{1}{6} f(\gamma_{1j}, y_n(\gamma_{1j})) + \frac{2}{3} f(\gamma_{2j}, y_n(\gamma_{2j})) + \frac{1}{6} f(\gamma_{3j}, y_n(\gamma_{3j}))] \quad . \tag{8.6}$$

Notice that these equations are obtained from (8.4) as $k \to o$ or as $h \to o$.

We will apply equations (8.4) and (8.5), (8.6) to approximate the solution of

$$y'(x) = (-5 + x)y(x) \ , \quad o \le x \le 4 \ , \quad y(o) = 1 \ .$$

We have let k = 4 for the interval [o, 2] and k = 2 for the interval [2, 4]. The errors $\ell_n^{(i)}$ in Tables 3 and 4 are defined by

$$\ell_n^{(i)} = \max_{1 \le j \le n} |(y - y_n)^{(i)}(x_j)| \ , \quad i = 0, 1,$$

$$\ell_n^{(i)} = \max_{1 \le j \le n} |(y - y_n)^{(i)}(\bar{x}_j)| \ , \quad i = 2, 3 \ .$$

Table 3. Collocation With Piecewise Cubic Polynomials

h	ℓ_n		$\ell_n^{(1)}$		$\ell_n^{(2)}$		$\ell_n^{(3)}$	
1/4	1.46(-3)		6.96(-3)		2.06(0)		3.97(1)	
1/8	8.56(-5)	4.09	4.07(-4)	4.10	7.13(-1)	1.53	3.06(1)	0.38
1/16	5.47(-6)	3.97	2.63(-5)	3.95	2.12(-1)	1.75	1.92(1)	0.67
1/32	3.41(-7)	4.00	1.64(-6)	4.00	5.78(-2)	1.87	1.08(1)	0.83

Table 4. Collocation With Modified Functions

h	ℓ_n		$\ell_n^{(1)}$		$\ell_n^{(2)}$		$\ell_n^{(3)}$	
1/4	2.86(-4)		1.36(-3)		4.66(-1)		7.98(0)	
1/8	1.72(-5)	4.06	8.37(-5)	4.02	1.80(-1)	1.37	7.35(0)	0.12
1/16	1.12(-6)	3.94	5.41(-6)	3.95	5.64(-2)	1.67	5.00(0)	0.56
1/32	7.01(-8)	4.00	3.38(-7)	4.00	1.58(-2)	1.84	2.92(0)	0.78

Notice that the errors are always smaller for the new method even though the rates of convergence are about the same. The rates are in agreement with Theorems 4.3 and 5.5, with the exception of $\ell_n^{(1)}$ which is higher. This is because for example,

$$(y_n - y)'(x_j) = P_n((-5+(\cdot))y_n)(x_j) - (-5+x_j)y(x_j)$$

$$= (-5 + x_j)(y_n(x_j) - y(x_j)) \quad .$$

Example 8.4

Consider the general second-order problem

$$y''(x) = f(x, y(x)) \ , \quad o \le x \le b \ , \quad y(o) = y_o, \quad y'(o) = y'_o \ ,$$

and suppose we choose $a_o = -k^2 \neq 0$, $w \equiv 1$. Then equation (2.7) becomes

$$y_n(x) = y_n(x_{j-1})\cos(k(x - x_{j-1})) + \frac{y_n'(x_{j-1})}{k} \sin(k(x - x_{j-1}))$$

$$+ \int_{x_{j-1}}^{x} \frac{\sin(k(x-t))}{k} P_n(f(\cdot, y_n))(t)\,dt \ , \quad x_{j-1} \le x \le x_j . \qquad (8.7)$$

To illustrate the higher rates given by Theorem 5.5, we let P_n be as in Example 3.1 with $p = 2$, $\gamma_1 = (1 - (1/3)^{1/2})/2$, $\gamma_2 = 1 - \gamma_1$, $r_1 = 0$, $r_2 = 0$. Since γ_1 and γ_2 are the Gaussian points translated to $[o, 1]$, equation (5.55) is satisfied for $r = 1$.

We choose $\Delta x_j = h$ and

$$k = \left[\frac{1}{2}(g(1 + \gamma_1 h) + g(1 + \gamma_2 h))\right]^{1/2} \ ,$$

where

$$g(x) = (3 + 4x)/16x^2$$

to approximate the solution of the problem

$$y''(x) = -(3 + 4x)/16x^2 , \quad 1 \leq x \leq 6 , \quad y(1) = 1 , \quad y'(1) = 0 , \tag{8.8}$$

which has the solution $y(x) = x^{1/4} (\cos(x^{1/2} - 1) - \frac{1}{2} \sin(x^{1/2} - 1))$.

Equation (8.7) becomes

$$y_n(x) = y_n(x_{j-1}) \cos u + \frac{y_n'(x_{j-1})}{k} \sin u - \frac{f(\gamma_{1j}, y_n(\gamma_{1j}))}{(\gamma_2 - \gamma_1) kh} \left(\frac{1}{k}(x - \gamma_{2j}) \right.$$

$$\left. - \frac{1}{k}(x_{j-1} - \gamma_{2j}) \cos u - \frac{1}{k^2} \sin u\right) + \frac{f(\gamma_{2j}, y_n(\gamma_{2j}))}{(\gamma_2 - \gamma_1) kh} \left(\frac{1}{k}(x - \gamma_{1j}) \right.$$

$$\left. - \frac{1}{k}(x_{j-1} - \gamma_{1j}) \cos u - \frac{1}{k^2} \sin u\right) , \tag{8.9}$$

where $u = k(x - x_{j-1})$ and $\gamma_{ij} = x_{j-1} + \gamma_i h$, $i = 1, 2$. Letting $x = \gamma_{ij}$, $i = 0, 1$, in the above equation we obtain a system of two equations in the two unknowns $y_n(\gamma_{ij})$, $i = 0, 1$. These two values $y_n(\gamma_{ij})$ determine $y_n(x)$ uniquely in $[x_{j-1}, x_j]$.

Table 5 shows the agreement of the computed rates and the rates given by Theorem 5.5. Here

$$\ell_n^{(i)} = ||(y - y_n)^{(i)}||_o , \quad i = 0, 1 ,$$

$$\ell_n^{(i)} = ||(y - y_n)^{(i)}||_{o,n} , \quad i = 2 ,$$

and

$$\ell_{n,\Delta}^{(i)} = \max_{o \leq j \leq n} |(y - y_n)^{(i)}(x_j)| , \quad i = 0, 1 .$$

Table 5. Higher Rates of Theorem 5.5

h	$\ell_{n,\Delta}$		$\ell^{(1)}_{n,\Delta}$		ℓ_n		$\ell^{(1)}_n$		$\ell^{(2)}_n$	
1/2	1.5(-4)		3.5(-5)		1.6(-4)		9.6(-4)		2.1(-2)	
1/4	1.0(-5)	3.91	2.4(-6)	3.87	1.3(-5)	3.62	1.5(-4)	2.68	6.5(-3)	1.69
1/8	6.5(-7)	3.94	1.6(-7)	3.91	9.0(-7)	3.85	2.2(-5)	2.77	1.8(-3)	1.85
1/16	4.1(-8)	3.99	9.9(-9)	4.01	6.0(-8)	3.91	2.9(-6)	2.92	4.9(-4)	1.88
1/32	2.6(-9)	3.98	6.0(-10)	4.04	3.9(-9)	3.94	3.8(-7)	2.93	1.3(-4)	1.91

Example 8.5

We will use the same method of Example 8.4 to illustrate the great improvement which is possible if some characteristics of the differential equation are taken into account.

The problem is Bessel's equation

$$y''(x) = (-100 - 1/4x^2)y(x), \qquad 1 \le x \le 6 \quad , \tag{8.10}$$

with the initial values chosen so that the solution is

$$y(x) = x^{1/2} J_0(10x) \quad .$$

It is known [22] that a differential equation of the form

$$y''(x) = -k^2(1 + p(x))y(x) \quad , \qquad a \le x \quad , \tag{8.11}$$

where k is a positive constant and p(x) satisfies

$$\int_a^\infty |p(x)|\, dx < \infty \quad ,$$

has solutions of the form

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx) + o(1)$$

for large x. Hence we expect particularly favorable results when our method is applied to (8.10) with k = 10. Gautschi [22] has studied trigonometric multistep methods which can be advantageously applied to (8.11). In particular, we list in Table 6 the results obtained when his Störmer interpolation method of trigonometric order two (with $T = \pi/5$) is applied to (8.10). Also shown are the results of collocation at the Gaussian points γ_1, γ_2 [16], of the Runge-Kutta method, of Numerov's

Table 6. Application to a Special Second-Order Linear Problem

	New			Collocation			Gautschi	R. - K.	Numerov	Lobatto
x	ε	$\varepsilon^{(1)}$	$\varepsilon^{(2)}$	ε	$\varepsilon^{(1)}$	$\varepsilon^{(2)}$	ε	ε	ε	ε
2	4.7(-10)	1.9(-8)	4.7(-5)	3.8(-7)	1.1(-5)	8.1(-2)	1.1(-7)	6.4(-6)	3.0(-6)	3.5(-7)
3	1.8(-9)	1.7(-8)	1.3(-5)	1.6(-6)	1.5(-5)	5.4(-2)	4.7(-7)	4.7(-5)	1.4(-5)	1.1(-6)
4	2.6(-9)	4.1(-9)	6.1(-7)	2.9(-6)	6.2(-6)	1.9(-2)	8.5(-7)	9.8(-5)	2.5(-5)	1.3(-6)
5	2.5(-9)	1.2(-8)	4.1(-6)	3.3(-6)	1.5(-5)	3.6(-2)	9.7(-7)	1.3(-4)	2.9(-5)	8.6(-7)
6	1.5(-9)	2.5(-8)	5.1(-6)	2.2(-6)	4.0(-5)	7.2(-2)	6.3(-7)	1.0(-4)	1.9(-5)	3.5(-7)

method [24] and of a method based on Lobatto quadrature of Jain and Sharma [33]. We have let h = 0.02, and made use of the results presented in [14] for the Runge-Kutta, Numerov's, and Gautschi's method. The results on the Lobatto method are from [33]. Here

$$\varepsilon^{(i)}(x) = |(y - y_n)^{(i)}(x)| \quad , \qquad i = 0, 1,$$

and

$$\varepsilon^{(2)}(x) = |(y - y_n)^{(i)}(x^-)| \quad .$$

<u>Example 8.6</u>

The final example is an application of the method of example 8.4 to the problem

$$y''(x) = -y(x)(100 + 3y^2(x)) \ , \quad o \le x \le 2 \ , \quad y(o) = 1 \ , \ y'(o) = 0.$$

We choose $k = \left(\frac{\partial f}{\partial x}(o,\ y(o))\right)^{\frac{1}{2}} = (109)^{\frac{1}{2}}$.

Table 7 compares the new method to collocation at the Gaussian points γ_1, γ_2. In this table,

$$\varepsilon_n^{(i)} = \max \{ |(y - y_n)^{(i)}(\tfrac{r}{4})| \ : \ 1 \le r \le 8, \ \ r \text{ an integer} \} .$$

The values of the exact solution where found using the same method with h = 1/256.

Table 7. Application to a Special Second-Order Nonlinear Problem

	New						Collocation					
h	ε_n		$\varepsilon_n^{(1)}$		$\varepsilon_n^{(2)}$		ε_n		$\varepsilon_n^{(1)}$		$\varepsilon_n^{(2)}$	
1/4	1.9(-2)		2.2(-1)		5.2(0)		9.2(-2)		1.4(0)		4.6(1)	
1/8	1.4(-3)	3.76	1.3(-2)	4.08	1.4(0)	1.89	1.1(-2)	3.06	7.9(-2)	4.15	1.3(1)	1.82
1/16	8.0(-5)	4.13	5.9(-4)	4.46	3.2(-1)	2.13	6.6(-4)	4.06	5.3(-3)	3.90	3.5(0)	1.89
1/32	5.0(-6)	4.00	3.5(-5)	4.08	7.2(-2)	2.15	5.0(-5)	3.72	3.4(-4)	3.96	9.1(-1)	1.94
1/64	3.1(-7)	4.01	2.2(-6)	3.99	1.6(-2)	2.17	3.2(-6)	3.97	2.1(-5)	4.02	2.3(-1)	1.98

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