Analytic Inverse Kinematics for the Universal Robots UR-5/UR-10 Arms

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December 7, 2013

1 Introduction

The joints are sometimes referenced casually as shoulder pan (θ_1), shoulder lift (θ_2), elbow (θ_3), wrist 1 (θ_4), wrist 2 (θ_5), and wrist 3 (θ_6). For the remainder of the document, we use the short-hand $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, and for angle sums, $c_{ij} = \cos(\theta_i + \theta_j)$. The

2 Forward Kinematics

We first begin by giving the forward kinematics, describing the position of the end effector as a function of joint angles:

$$
{}_0B_6(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6) = {}_0B_1(\theta_1)_1B_2(\theta_2)_2B_3(\theta_3)_3B_4(\theta_4)_4B_5(\theta_5)_5B_6(\theta_6) = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1)
$$

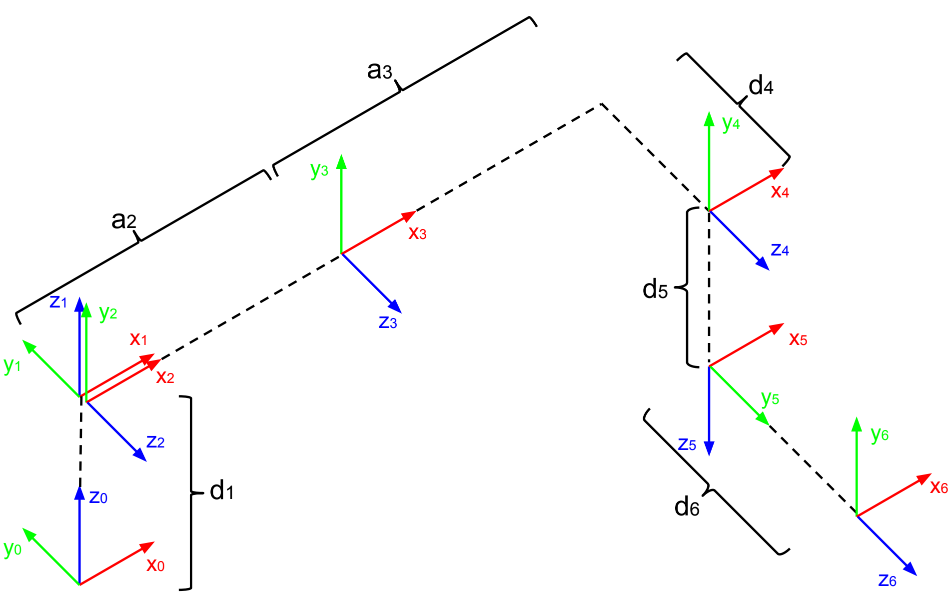

Figure 1: Coordinate frames for UR arm. Joints rotate around the z-axes and are pictured at $\theta_i = 0$ for $1 \le i \le 6$.

i	d_i	a_i	α_i
0	-	0	0
1	d_1	0	$\pi/2$
2	0	a_2	0
3	0	a_3	0
4	d_4	0	$\pi/2$
5	d_5	0	$-\pi/2$
6	d_6	-	-

Table 1: Denavit-Hartenberg parameters for the UR Arms

$$
\begin{aligned}
n_x =\ & c_6(s_1s_5 + ((c_1c_{234} - s_1s_{234})c_5)/2.0 + ((c_1c_{234} + s_1s_{234})c_5)/2.0) - (s_6((s_1c_{234} + c_1s_{234}) - (s_1c_{234} - \\
& c_1s_{234})))/2.0 \\
n_y =\ & c_6(((s_1c_{234} + c_1s_{234})c_5)/2.0 - c_1s_5 + ((s_1c_{234} - c_1s_{234})c_5)/2.0) + s_6((c_1c_{234} - s_1s_{234})/2.0 - (c_1c_{234} + \\
& s_1s_{234})/2.0) \\
n_z =\ & (s_{234}c_6 + c_{234}s_6)/2.0 + s_{234}c_5c_6 - (s_{234}c_6 - c_{234}s_6)/2.0
\end{aligned}
$$

$$
\begin{aligned}
o_x =\ & -(c_6((s_1c_{234} + c_1s_{234}) - (s_1c_{234} - c_1s_{234})))/2.0 - s_6(s_1s_5 + ((c_1c_{234} - s_1s_{234})c_5)/2.0 + ((c_1c_{234} + \\
& s_1s_{234})c_5)/2.0) \\
o_y =\ & c_6((c_1c_{234} - s_1s_{234})/2.0 - (c_1c_{234} + s_1s_{234})/2.0) - s_6(((s_1c_{234} + c_1s_{234})c_5)/2.0 - c_1s_5 + ((s_1c_{234} - \\
& c_1s_{234})c_5)/2.0) \\
o_z =\ & (c_{234}c_6 + s_{234}s_6)/2.0 + (c_{234}c_6 - s_{234}s_6)/2.0 - s_{234}c_5s_6
\end{aligned}
$$

$$
\begin{aligned}
a_x =\ & c_5s_1 - ((c_1c_{234} - s_1s_{234})s_5)/2.0 - ((c_1c_{234} + s_1s_{234})s_5)/2.0 \\
a_y =\ & -c_1c_5 - ((s_1c_{234} + c_1s_{234})s_5)/2.0 + ((c_1s_{234} - s_1c_{234})s_5)/2.0 \\
a_z =\ & (c_{234}c_5 - s_{234}s_5)/2.0 - (c_{234}c_5 + s_{234}s_5)/2.0
\end{aligned}
$$

$$
\begin{aligned}
p_x =\ & -(d_5(s_1c_{234} - c_1s_{234}))/2.0 + (d_5(s_1c_{234} + c_1s_{234}))/2.0 + d_4s_1 - (d_6(c_1c_{234} - s_1s_{234})s_5)/2.0 - (d_6(c_1c_{234} + \\
& s_1s_{234})s_5)/2.0 + a_2c_1c_2 + d_6c_5s_1 + a_3c_1c_2c_3 - a_3c_1s_2s_3) \\
p_y =\ & -(d_5(c_1c_{234} - s_1s_{234}))/2.0 + (d_5(c_1c_{234} + s_1s_{234}))/2.0 - d_4c_1 - (d_6(s_1c_{234} + c_1s_{234})s_5)/2.0 - (d_6(s_1c_{234} - \\
& c_1s_{234})s_5)/2.0 - d_6c_1c_5 + a_2c_2s_1 + a_3c_2c_3s_1 - a_3s_1s_2s_3) \\
p_z =\ & d_1 + (d_6(c_{234}c_5 - s_{234}s_5))/2.0 + a_3(s_2c_3 + c_2s_3) + a_2s_2 - (d_6(c_{234}c_5 + s_{234}s_5))/2.0 - d_5c_{234}
\end{aligned}
$$

3 IK Solution for UR 6-DOF Arm

The analytic inverse kinematics problem is to find the set of joint configurations $Q = \{q_i\}$ where $q_i = (\theta_1^i, \ldots, \theta_6^i) \in [0, 2\pi)^6$ that satisfies

$$
{}_0B_6(\theta_1^i, \theta_2^i, \theta_3^i, \theta_4^i, \theta_5^i, \theta_6^i) = ({}_0B_6^d) = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}
$$

where ${}_0B_6^d$ describes the desired position and orientation of the final link.

We begin by finding θ_1 using the position of the 5th joint. Analyzing the tranformation from frame 1 to frame 5 using equations (1) and (2), we can form the equality

$$
[{}_1B_5]_{LHS} = [{}_1B_5]_{RHS} \tag{3}
$$

$$
[({}_0B_1)^{-1}({}_0B_6^d)({}_5B_6)^{-1}]_{LHS} = [({}_1B_2)({}_2B_3)({}_3B_4)({}_4B_5)]_{RHS} \tag{4}
$$

We can then see that the y-coordinate of the position of this frame is

$$
[({}_1p_{15})_y]_{LHS} = (p_x - d_6z_x)(-s_1) + (p_y - d_6z_y)(c_1) = \begin{bmatrix} -s_1 & c_1 & 0 \end{bmatrix} \begin{bmatrix} p_x - d_6z_x \\ p_y - d_6z_y \\ p_z - d_6z_z \end{bmatrix} = ({}_0y_1)^T({}_0p_{05}) \tag{5}
$$

$$
[({}_1p_{15})_y]_{RHS} = -d_4 \tag{6}
$$

The equation

$$
(p_x - d_6z_x)(-s_1) + (p_y - d_6z_y)(c_1) = -d_4 \tag{7}
$$

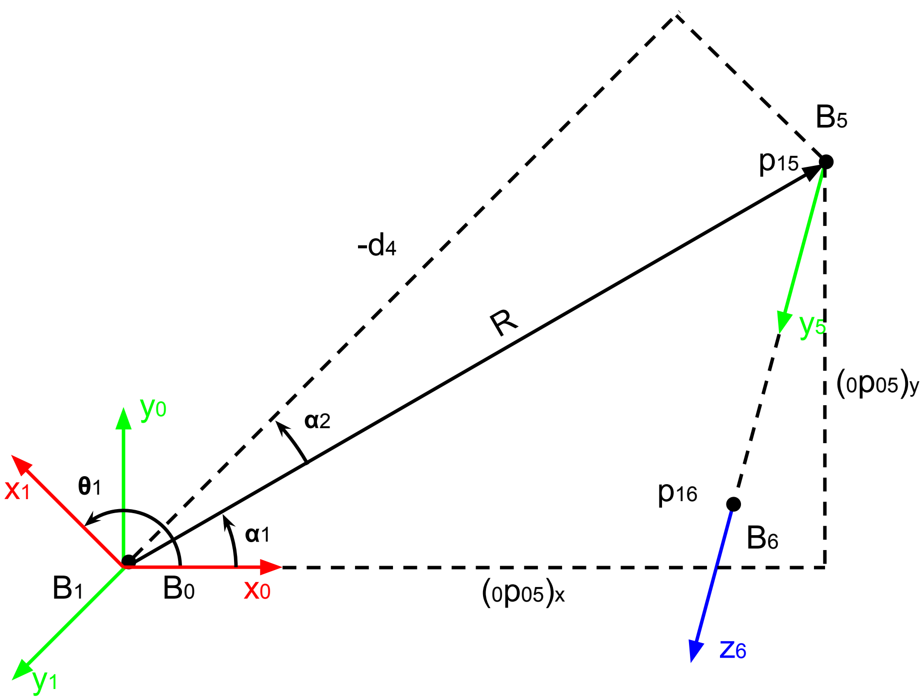

Figure 2: Illustration of the geometry of finding θ_1. This is essentially an overhead view of the robot, looking down on the x-y plane.

is known as a phase-shift equation whose solution can be found as

$$\tan\alpha_1 = \frac{({}_0p_{05})_y}{({}_0p_{05})_x} \tag{8}$$

$$\cos\alpha_2 = \frac{d_4}{R} = \frac{d_4}{\sqrt{({}_0p_{05})_x^2 + ({}_0p_{05})_y^2}} \tag{9}$$

$$\theta_1 = \alpha_1 + \alpha_2 + \pi/2 = \text{atan2}(({}_0p_{05})_y, ({}_0p_{05})_x) \pm \cos^{-1}\frac{d_4}{R} + \pi/2 \tag{10}$$

We can see that there are generally 2 solutions for θ_1, which correspond to configurations where the shoulder is “left” or “right”. Using the function atan2 is essential for insuring correct signs and behavior when $({}_0p_{05})_x = 0$. By looking at the figure 2, it is easy to see that physically, no configuration is possible which makes $\sqrt{({}_0p_{05})_x^2 + ({}_0p_{05})_y^2} \leq |d_4| < 0$. Thus, both α_1 and α_2 always exist if an inverse solution exists.

Given a particular θ_1, we can solve for θ_5. Using the tranformation from frame 1 to frame 6, we can form the equality

$$[({}_0B_1)^{-1}({}_0B_6^d)]_{LHS} = [({}_1B_2)({}_2B_3)({}_3B_4)({}_4B_5)({}_5B_6)]_{RHS} \tag{11}$$

We can then see that the y-coordinate of the position of this frame is

$$[({}_1p_{16})_y]_{LHS} = p_x(-s_1) + p_y(c_1) = \begin{bmatrix} -s_1 & c_1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = ({}_0y_1)^T({}_0p_{06}) \tag{12}$$

$$[({}_1p_{16})_y]_{RHS} = -d_4 - c_5 d_6 \tag{13}$$

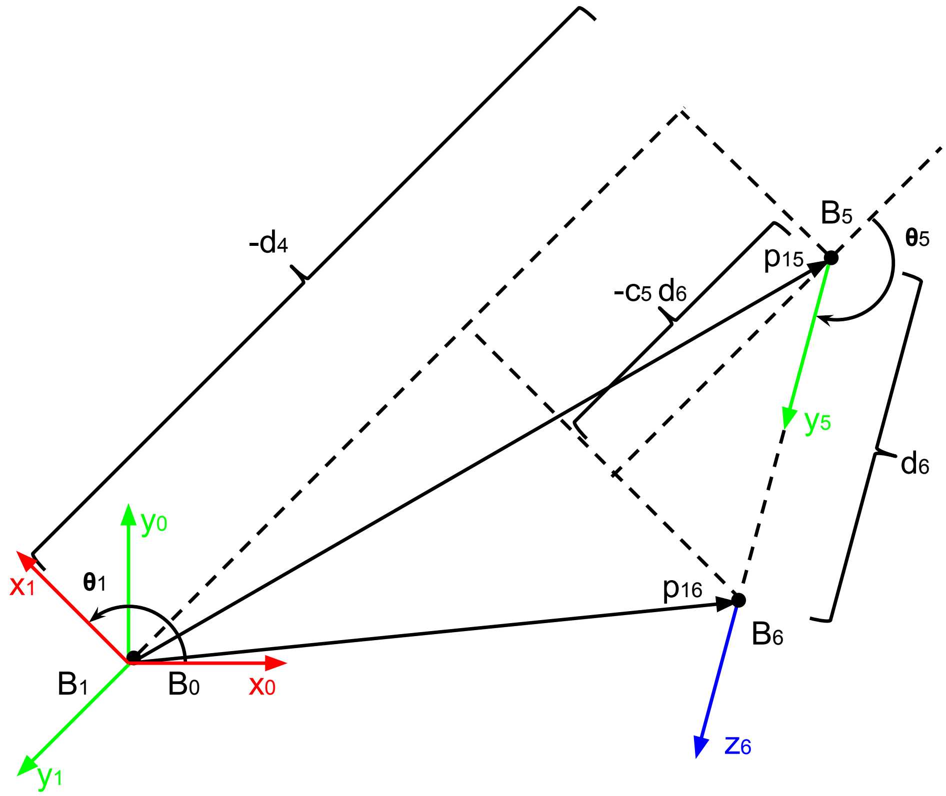

Figure 3: Illustration of the geometry of finding θ_5. This is an overhead view of the robot, looking down on the x-y plane.

$$\theta_5 = \pm \cos^{-1} \frac{p_x s_1 - p_y c_1 - d_4}{d_6} \tag{14}$$

Again, we find that there are 2 solutions for θ_5, which correspond to configurations where the wrist is "in/down" or "out/up". This occurs due to the fact that the joint sum θ_{234} can allow the ${}_1B_5$ to achieve orientations where ${}_1y_5$ is pointing in the same direction, but that ${}_1z_5$ is pointing in the opposite direction. This flip can then be reversed very simply by the 6th joint. This joint has a solution so long as the argument of $\cos^{-1}$ has magnitude not greater than 1, or $|({}_1p_{16})_y - d_4| \leq |d_6|$.

To solve for the 6th joint, we look at the ${}_6y_1$ coordinate axis:

$$[{}_6y_1]_{LHS} = \begin{bmatrix} -x_x s_1 + x_y c_1 \\ -y_x s_1 + y_y c_1 \\ -z_x s_1 + z_y c_1 \end{bmatrix} \tag{15}$$

$$[{}_6y_1]_{RHS} = \begin{bmatrix} -c_6 s_5 \\ s_6 s_5 \\ -c_5 \end{bmatrix} \tag{16}$$

As figure 4 shows, this equality forms a spherical coordinate expression for the vector $-{}_6y_1$ where θ_6 is the azimuthal angle and θ_5 is the polar angle. The x and y coordinates of this vector form a system which can be easily solved as

$$\theta_6 = \text{atan2}\left(\frac{-y_x s_1 + y_y c_1}{s_5}, \frac{-(-x_x s_1 + x_y c_1)}{s_5}\right) \tag{17}$$

The demoninators of each argument can be replaced by $\text{sign}(s_5)$. This solution is undefined in two circumstances, when both of the numerators are 0 or $s_5 = 0$. Inspection of equations (15) and (16) shows that

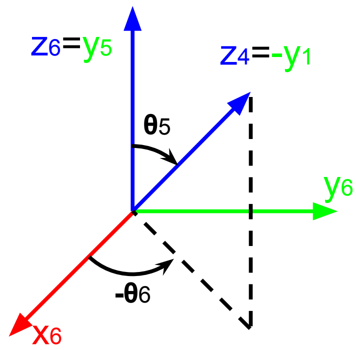

Figure 4: Overhead view of the kinematics problem.

these conditions actually imply each other. When $s_5 = 0$, we know $c_5 = \pm 1$, which indicates that the joints 2, 3, 4, and 6 are all parallel and the solution is underdetermined. When this occurs, a desired θ_6 can be supplied to fully determine the system.

The final 3 joints can be found easily, understanding that they together form a classical 3R planar arm. Since we have the other 3 joints, we solve for the location of the base and end effector of the 3R arm, and use those equations to solve. The solution has two possible configurations, where the arm is elbow "up" or "down". No solutions exist when the distance to the 4th joint exceeds the sum $|a_2 + a_3|$ or is less than the difference $|a_2 - a_3|$. If $a_2 = a_3$, a displacement singularity exists when $\theta_3 = \pi$, making θ_2 arbitrary.

4 IK Extension for 7th Axis

Suppose the arm is mounted to a linear actuator aligned with the x-axis such that

$$ {}_{-1}B_6(a_{-1}, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) = ({}_{-1}B_0)({}_0B_6) \tag{18} $$

$$ ({}_{-1}B_0) = \begin{bmatrix} 1 & 0 & 0 & a_{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19} $$

where a_{-1} is the controllable linear translation. Since the system is now underdetermined, we must add new constraints to solve the system. One potential method of constraining is to arbitrarily set θ_1 to a desired value, and then solve for the other 5 joint angles and a_{-1}.

We can find the new arm tranformation

$$ {}_0B_6^{new} = ({}_{-1}B_0)^{-1}({}_{-1}B_6) \tag{20} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & -a_{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x - a_{-1} \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21} $$

so $p_x^{new} = p_x - a_{-1}$. If we substitute p_x^{new} into the equation (7), we can solve for

$$ a_{-1} = p_x + \frac{-d_4 - (p_y - d_6 z_y)(c_1)}{s_1} - d_6 z_x \tag{22} $$

Choose θ_1 such that $s_1 \neq 0$, and we can find the unique solution for a_{-1}. By producing p_x^{new}, we can solve for ${}_0B_6^{new}$ using the remainder of the IK algorithm, starting with joint 5.