

# New Tools and Results in Graph Structure Theory

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# New Tools and Results in Graph Structure Theory

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The desire to be right and the desire to have been right are two desires, and the sooner we separate them the better off we are. The desire to be right is the thirst for truth. On all accounts, both practical and theoretical, there is nothing but good to be said for it. The desire to have been right, on the other hand, is the pride that goeth before a fall. It stands in the way of our seeing we were wrong, and thus blocks the progress of our knowledge.

W. V. Quine and J. S. Ullian, *The Web of Belief*

*To my parents...*

*They have always put their children's interests ahead of their own, and have given me more love and support than I could have asked for, or deserved.*

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## SUMMARY

We first prove a “non-embeddable extensions” theorem for polyhedral graph embeddings. Let  $G$  be a “weakly 4-connected” planar graph. We describe a set of constructions that produce a finite list of non-planar graphs, each having a minor isomorphic to  $G$ , such that every non-planar weakly 4-connected graph  $H$  that has a minor isomorphic to  $G$  has a minor isomorphic to one of the graphs in the list. The theorem is more general and applies in particular to polyhedral embeddings in any surface.

We discuss an approach to proving Jorgensen’s conjecture, which states that if  $G$  is a 6-connected graph with no  $K_6$  minor, then it is apex, that is, it has a vertex  $v$  such that deleting  $v$  yields a planar graph. We relax the condition of 6-connectivity, and prove Jorgensen’s conjecture for a certain sub-class of these graphs.

We prove that every graph embedded in the Klein bottle with representativity at least 4 has a  $K_6$  minor. Also, we prove that every “locally 5-connected” triangulation of the torus, with one exception, has a  $K_6$  minor. (Local 5-connectivity is a natural notion of local connectivity for a surface embedding.) The above theorem uses a locally 5-connected version of the well-known splitter theorem for triangulations of any surface.

We conclude with a theoretically optimal algorithm for the following graph connectivity problem. A shredder in an undirected graph is a set of vertices whose removal results in at least three components. A 3-shredder is a shredder of size three. We present an algorithm that, given a 3-connected graph, finds its 3-shredders in time proportional to the number of vertices and edges, when implemented on a RAM (random access machine).

# CHAPTER I

## INTRODUCTION

This chapter is intended to be both a brochure and a map. In Section 1.1 (the brochure part), I have attempted to give a brief yet (hopefully) useful overview of the theme of this thesis. It is aimed primarily at the reader who is not already familiar with the particular neighborhood of discrete mathematics that my research inhabits, and it highlights some of the most important results in that theory. Section 1.2 (the map part) briefly describes the results in this thesis, along with the larger context that surrounded and motivated it. It is intended to serve as a navigational aid for the rest of the thesis.

### *1.1 The What and the Why*

Most of the research described in this thesis relates, directly or indirectly, to *Graph Structure Theory*. Before diving into it, a few words about what graphs are, and what they are good for, might be in order.

A *graph* can be defined abstractly as a pair of sets: a (finite, non-empty) set of *vertices* (or points), and a set of unordered pairs of vertices, called *edges* (or lines). An edge is said to join the two vertices specified in the pair, and the two vertices are said to be *adjacent*. (As far as a mental picture is concerned, it is almost always more fruitful to think of graphs as consisting of points, and lines joining those points.) Unless otherwise specified, the graphs that we consider are *simple graphs*, that is, they do not have two edges sharing both end-points, or an edge with coincident end-points.

It should not take much imagination to see why graphs are useful: they can model any sort of binary relationship among a set of objects. The objects themselves can be represented by vertices in a graph, and related pairs of objects give rise to edges. Thus the Internet is really a graph in disguise, and so is a family tree, or a map of the world. (For the latter example, think of the countries as the vertices, with edges joining pairs of countries

that share a border.)

### 1.1.1 Graph Structure Theory

Graph Structure Theory is the area of Graph Theory that concerns itself primarily with questions of the following two flavors:

1. When does some graph contain a certain other graph as a “substructure”?
2. How does having certain substructures affect other (and seemingly unrelated) properties of the graph?

Questions of the first kind are perhaps best motivated by an appeal to our inclination towards modularity. We like our problems, and solutions, to come in bite-sized pieces. The best way to understand a complex thing is usually to break it down into smaller components, and try to understand the components; graphs are no exception. The theorems in Chapters 3 and 5, for instance, illustrate problems of this nature.

As for questions of the second kind, there are two reasons to study them, one theoretical and one practical. From a historical standpoint, surprising connections between seemingly unrelated problems have always been of interest to mathematicians. On a more practical note, if some graph property has a structural characterization, it usually leads to an efficient algorithm to recognize it. For instance, the algorithm presented in Chapter 6 is motivated by such a structural characterization.

A good example of the algorithmic contribution of Graph Structure Theory is the use of “tree decompositions” in efficiently solving optimization problems that are intractable in general. Roughly speaking, a tree decomposition breaks a graph into pieces joined along a tree-like structure. If the pieces in this structure are of bounded size (“tree-width”), then certain traditionally hard problems can be solved for these graphs using a hierarchical (bottom-up) approach [40, 6]. Even when the tree-width is not bounded, these decompositions are useful in obtaining efficient approximation algorithms [18, 9].

Now that we agree that the above kinds of problems are worthwhile endeavors, the question is, what do we mean by a “substructure”? The substructure relation that has

proven most fruitful in this context, by far, is that of a *graph minor*. We say that a graph  $G$  is a *minor* of a graph  $H$  if the vertices of  $G$  are in one-to-one correspondence with disjoint connected subgraphs of  $H$ , in such a way that adjacencies (in  $G$ ) are preserved: that is, for every edge in  $G$ , there is an edge between the corresponding subgraphs of  $H$ . (Another way of thinking of a graph minor is the following:  $G$  is a minor of  $H$  if  $G$  can be obtained from  $H$  by “contracting” or deleting edges, or deleting vertices. Contracting an edge<sup>1</sup> means identifying its end-points  $u$  and  $v$  and joining the new vertex to any vertex that was adjacent to  $u$  or  $v$ , or both.)

The theory of graph minors is very rich, with theorems that have deep implications in graph theory and elsewhere. The most important result in this theory, and perhaps the deepest theorem in graph theory, is the following theorem of Robertson and Seymour:

**Theorem 1.1.1 (Graph Minor Theorem, formerly Wagner’s conjecture)** *Given any infinite sequence  $G_1, G_2, \dots$  of (finite) graphs, there exist integers  $i < j$  such that  $G_i$  is a minor of  $G_j$ .*

The original proof of the theorem appeared in a series of over twenty papers under the common title *Graph Minors*, most of which were published in the *Journal of Combinatorial Theory, Series B*. Simpler proofs have since been found for some parts of that proof. For a good overview, refer to [10].

Like most fundamental theorems in mathematics, the proof of the Graph Minor Theorem has spawned numerous techniques that are interesting in their own right. However, one particular implication of the theorem stands out for its far-reaching consequences in graph theory and the complexity of graph algorithms. Let  $\mathcal{F}$  be a family of graphs that is minor-closed (that is, if  $H \in \mathcal{F}$  and  $G$  is a minor of  $H$ , then  $G \in \mathcal{F}$ ). It is an easy exercise to show that  $\mathcal{F}$  can then be characterized in terms of a set of *forbidden* (or *excluded*) minors. In other words, a graph  $G$  is a member of  $\mathcal{F}$  if and only if none of the forbidden graphs is a minor of  $G$ . From the Graph Minor Theorem, we can conclude that in fact the set of forbidden minors can be chosen to be finite.

---

<sup>1</sup>There are minor variations of this definition, depending on the context in which graph minors are used.

Now, for a *fixed* graph  $X$  (not part of the input), there is an  $O(n^3)$ -time algorithm to test if an input graph  $G$  has a minor isomorphic to  $X$  (where  $n$  is the number of vertices). Thus it immediately follows that there exists an  $O(n^3)$ -time membership test for *any* minor-closed family. (However, the list of forbidden minors, and the algorithm, are only given implicitly.)

**Corollary 1.1.2** *Given any minor-closed family  $\mathcal{F}$  of graphs, there is a finite set of graphs such that a graph  $G$  is a member of  $\mathcal{F}$  if and only if no graph in this set is a minor of  $G$ .*

Among the more dramatic algorithmic consequences of the kind stated above is for the problem of deciding whether a graph is *knotless* (that is, embeddable in 3-space such that none of its cycles forms a non-trivial knot). Before the Graph Minor Theorem came along, it was not even known whether the problem was decidable, that is whether *any* algorithm, however slow, existed. To this day, no explicit algorithm for this problem is known. However, the family of knotless graphs is minor-closed, so the above corollary immediately implies the existence of a polynomial, indeed  $O(n^3)$ -time, algorithm.

Minor-closed graph families are quite common in graph theory. Among the most natural (and historically significant) of these is the family of graphs embeddable in a given surface. Thus another powerful corollary of the Graph Minor Theorem is the following:

**Corollary 1.1.3** *For any surface  $\Sigma$ , there is a finite list of forbidden minors, such that a graph is embeddable in  $\Sigma$  if and only if it has none of the forbidden minors.*

The prototypical theorem of the above kind, indeed one that is a precursor of Graph Structure Theory itself, is *Kuratowski's Theorem* [30], which states that a graph can be embedded in the plane (with no edge crossings) if and only if it does not have a minor isomorphic to either of two graphs,  $K_5$  and  $K_{3,3}$ . (The former is the complete graph on five vertices, while the latter is a graph consisting of six vertices, three in each of two classes, with all possible edges between the classes.) Thus the above corollary is a kind of “generalized Kuratowski theorem”. It had been conjectured in the 1930s by Erdős and König. It was verified for non-orientable surfaces in [4], and for orientable surfaces in [5].

Robertson and Seymour [44] gave a proof for general surfaces before their proof of the Graph Minor Theorem itself. Recently, Thomassen gave a simpler proof for the result in [56].

A major ingredient in the proof of the Graph Minor Theorem, and indeed a deep result in its own right, is the *Structure Theorem*, which gives for any integer  $n$  a structural description of the graphs that do not have a minor isomorphic to the complete graph  $K_n$ . The statement of the theorem is somewhat technical, and we will not state it precisely here. Very roughly speaking, however, it says that every graph not having a  $K_n$ -minor is a subgraph of a graph obtained by pasting together “nearly embeddable” graphs in a surface in which  $K_n$  itself is not embeddable, where the pasting is along complete subgraphs of a size bounded above by a number depending only on  $n$  (and not the size of the graph itself).

Yet another deep result in the Robertson-Seymour theory is a polynomial-time algorithm for the *Disjoint Rooted Paths* problem stated below:

**Instance:** A graph  $G$  and disjoint subsets  $S = \{s_1, \dots, s_k\}$  and  $T = \{t_1, \dots, t_k\}$  of “terminals” (vertices), where  $k$  is a *fixed* integer (not part of the input).

**Question:** Does  $G$  have disjoint paths joining  $s_i$  and  $t_i$ , for  $i = 1, \dots, k$ ?

Before its resolution, the above was a significant open problem in the theory of graph algorithms, and its complexity status was unknown. As with the Graph Minor Theorem, however, the significance of the algorithm lies as much in its repercussions as in the result itself.

The case  $k = 2$  is particularly nice, with the following beautiful structural characterization for the existence of the paths. A graph has two disjoint  $s_i$ - $t_i$  paths (modulo some natural connectivity requirements) if and only if it cannot be drawn in the plane with the vertices  $s_1, s_2, t_1, t_2$  on the infinite face, in that order. (It is trivial to check that the above condition is necessary.) The theorem was proved independently in [48], [49], and [54]. [49] gives an  $O(nm)$ -time algorithm that, given a graph with four terminals as above, either finds two paths as required, or produces a planar drawing demonstrating their infeasibility. (Here  $n$  and  $m$  are the numbers of vertices and edges respectively.) It should be noted that for the general  $k$ -paths problem above, no structural characterization is known for  $k \geq 3$ .

Graph coloring offers another tantalizing connection with Graph Structure Theory, with some deep questions yet to be answered. Consider the “chromatic number” of a graph, that is, the least number of colors that one can color the vertices with, such that no two adjacent vertices get the same color. When does a graph need, say, at least 10 colors? One natural possibility is when the graph has 10 mutually adjacent vertices. One can now reasonably ask whether containing such a “complete graph” (or clique) is the only reason for a high chromatic number. For the right notion of containment (that is, graph minors), this is in fact a beautiful and long-standing conjecture:

**Conjecture 1.1.4 (Hadwiger’s Conjecture)** *For every integer  $t$ , if a graph has chromatic number  $\geq t$ , then it has a minor isomorphic to the complete graph  $K_t$  on  $t$  vertices.*

Hadwiger’s conjecture is trivial for  $t \leq 3$  and easy for  $t = 4$ , but things go steeply uphill from there. For  $t = 5$ , it implies the four-color theorem, a deep theorem which had to wait over a hundred years for a proof. In fact, it is equivalent to the four-color theorem, as shown by Wagner’s structural characterization of graphs with no  $K_5$  minor [59]. A deep theorem of Robertson, Seymour and Thomas [45] establishes the case  $t = 6$ ; more precisely, it proves that Hadwiger’s conjecture for  $t = 6$  is also equivalent to the four-color theorem. The conjecture is open for all  $t \geq 7$ . (It is easy to show that the conjecture for  $t + 1$  implies it for  $t$ .)

Hadwiger’s conjecture has been shown to be true for “almost all” graphs. (More precisely: in  $\mathcal{G}(n, p)$ , a natural and well-studied probabilistic model for a random graph, the probability that a graph has a clique minor of size equal to its chromatic number has been shown to approach 1 as the number  $n$  of vertices tends to infinity.) Also, a deterministic, but approximate version of the conjecture is known: by a theorem independently proved in [29] and [53], a graph with chromatic number at least  $ct\sqrt{\log t}$  has a  $K_t$  minor. It should be noted that this theorem obtains the minor using merely the corresponding lower bound on the average degree. The main question that Hadwiger’s conjecture raises is whether the much stronger property of having a high chromatic number has a deeper connection to clique minors than the weaker property of having lots of edges.

## 1.2 *Research Summary*

### 1.2.1 **Non-embeddable Extensions of Embedded Minors**

A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends, where the paths are disjoint, except possibly for shared ends. A graph  $G$  is said to be a *topological minor* of a graph  $H$  if a subdivision of  $G$  is a subgraph of  $H$ . (A topological minor is a special case of a minor.)

Suppose a non-planar graph  $H$  has a minor isomorphic to a planar graph  $G$ . For various problems in Graph Structure Theory it is useful to know the minor-minimal graphs that (i) are a minor of  $H$ , (ii) have a minor isomorphic to  $G$ , and (iii) are non-planar. In other words, one wants to know what more does  $H$  contain on account of its non-planarity.

In the applications of such a result,  $G$  is explicitly known, whereas  $H$  is not, and the enlargement operations would furnish an explicit list of graphs such that (i)  $H$  has a minor isomorphic to one of the graphs on the list, and (ii) each graph on the list is a witness both to the fact that  $G$  is isomorphic to a minor of  $H$ , and that  $H$  is, in addition, non-planar. (The minor-minimality of the graphs in the list is required to avoid redundancy — if  $G_1, G_2$  are in the list with  $G_1$  being a minor of  $G_2$ , then we could throw away  $G_2$  from the list without affecting the validity of the previous statement.)

Under some mild connectivity assumptions, [41] finds these “minimal non-planar enlargements” of  $G$  when the containment relation is that of a topological minor. This theorem has been used in [11, 51], and will probably be useful elsewhere as well. However, in more complicated applications it is more efficient to work with minors, rather than topological minors.

Chapter 2 describes a theorem for finding non-planar extensions of planar minors. In fact, it turns out that such a tool can be formulated in terms of *disk systems*. (Chapter 2 gives a precise definition of a disk system. Roughly speaking, it is an abstract generalization of the set of facial cycles of a graph embedding.) In particular, the results of Chapter 2 apply to “polyhedral embeddings” in any surface, not just the plane.

[41] describes a related theorem for “non-apex extensions”. A graph is said to be *apex* if it has some vertex whose removal yields a planar graph. Thus, if an apex graph  $G$  is a minor

of a non-apex graph  $H$ , one would want to know the list of minimal non-apex extensions of  $G$ , such that  $H$  has a minor isomorphic to one of the graphs in the list.

Under connectivity assumptions similar to the ones in the above theorem, [41] gives a list of these non-apex extensions. A potential application of this result is described in Section 1.2.2.

### 1.2.2 Six-regular Graphs in $\mathcal{F}_6$

The proof in [45] establishing Hadwiger’s conjecture for  $t = 6$  considers a minor-minimal counterexample to the conjecture. In other words, it considers a graph that is minor-minimal subject to (i) having chromatic number  $\geq 6$ , and (ii) having no  $K_6$  minor. It then proves that such a graph must be apex. By the four-color theorem, it follows that the graph is 5-colorable, and thus not a counterexample after all. Mader [33] proved that a minor-minimal counterexample as above must be 6-connected. (A graph is  $k$ -connected if the removal of fewer than  $k$  vertices does not disconnect it.)

Jorgensen asked whether it is the 6-connectivity that is the crucial property here: more precisely, he made the following conjecture.

**Conjecture 1.2.1 (Jorgensen’s Conjecture, [24])** *Every 6-connected graph with no  $K_6$  minor is apex.*

From the discussion in the previous paragraph, it follows that Jorgensen’s conjecture immediately implies Hadwiger’s conjecture for  $t = 6$ . Understanding the structure of graphs with no  $K_t$  minor is an important problem since excluded minors play an important role with regard to several fundamental graph properties. Moreover, Jorgensen’s conjecture can be seen as a Kuratowski-like theorem for apex graphs — modulo 6-connectivity, graphs are apex if and only if they have no  $K_6$  minor. (The forward implication is trivial:  $K_6$  itself is non-apex since  $K_5$  is non-planar, and thus a graph with a  $K_6$  minor cannot be apex.)

The difficulty with trying to prove Jorgensen’s conjecture, however, is that the hypothesis of 6-connectivity (or  $k$ -connectivity, in general) does not behave well under taking minors. One possible approach to getting around this problem is thus to consider a different notion of connectivity. Mader considered one such notion in [32], where he defined a family

$\mathcal{F}_5$  of graphs. A graph is in  $\mathcal{F}_5$  if it has at least six vertices and has minimum degree five, *except* perhaps a set of mutually adjacent vertices of degree less than five. Thus being in  $\mathcal{F}_5$  is a weaker property than 5-connectivity. The intuition is that this weaker condition is more robust under taking minors, and that one can thus say what the minimal graphs in the family look like. Indeed, [32] explicitly lists all the minimal graphs in  $\mathcal{F}_5$  (there are four such graphs, two of which are  $K_6$  and the graph of the icosahedron).

It is thus reasonable to ask whether an analogous weakening of 6-connectivity might be useful for attacking Jorgensen’s conjecture. We thus define  $\mathcal{F}_6$  as the family of graphs with at least seven vertices and with minimum degree six, *except* perhaps a set of mutually adjacent vertices of degree less than six. A counterexample  $G$  to the conjecture must have a minor isomorphic to a minimal member of  $\mathcal{F}_6$ . If that minimal graph itself has a  $K_6$  minor, then we are done. If it is apex, however, then one could use the non-apex extensions theorem mentioned in Section 1.2.1 and verify whether applying them always leads to a  $K_6$  minor.

The list of minor-minimal graphs in  $\mathcal{F}_6$  is not known, and it seems that finding that list explicitly might involve a formidable effort. However, we verified that Jorgensen’s conjecture does hold for a certain subfamily of  $\mathcal{F}_6$ :

**Theorem 1.2.2** *Every minor-minimal graph in  $\mathcal{F}_6$  with no low-degree vertices (that is, with minimum degree 6) has a  $K_6$  minor.*

In particular, for this sub-class of  $\mathcal{F}_6$ , the implication in Jorgensen’s conjecture holds vacuously (in the sense that the antecedent is never true). A proof of the above theorem is given in Chapter 3.

### 1.2.3 A Splitter Theorem for Triangulations

If  $G$  and  $H$  are connected graphs such that  $G$  is a minor of  $H$ , then  $H$  can be obtained from  $G$  by repeatedly “splitting” vertices and adding edges. (Vertex splits, as explained in Chapter 4, are inverses of the usual edge contractions, for edges not contained in triangles.) Seymour’s well-known splitter theorem says that if  $G$  and  $H$  are both 3-connected, and  $G$

is not a wheel, then the above sequence of operations can be chosen such that every intermediate graph is also 3-connected. This theorem has been very useful in various contexts [52].

In Chapter 4, we prove a splitter theorem for “locally 5-connected” triangulations of a general surface. In other words, (i) the edge contractions and vertex splits are defined in the context of surface triangulations, (ii)  $G$  and  $H$  are both “locally 5-connected” (a natural definition of local connectivity for a surface embedding). The theorem, then, states that  $H$  can be obtained from  $G$  by applying a sequence of vertex splits such that every intermediate graph is also locally 5-connected. (Since  $G$  and  $H$  are both triangulations of the same surface, it is easy to see that edge additions are not needed.) One application of this splitter theorem is in the proof of Theorem 5.1.2.

#### 1.2.4 $K_6$ Minors in the Torus and the Klein Bottle

In Chapter 5, we prove two theorems about  $K_6$  minors in graphs embedded in the torus or the Klein bottle. The first of these states that “locally 5-connected” triangulations in the torus, with one particular exception, have a minor isomorphic to the complete graph  $K_6$ . The second theorem states that every graph embedded in the Klein bottle with representativity at least four has a minor isomorphic to  $K_6$ .

While these theorems might be of independent interest, our primary motivation for proving them was their usefulness for Theorem 1.2.2. Towards the end of the proof of that theorem, we encounter a minimal graph in  $\mathcal{F}_6$  that is a 6-regular triangulation of the torus or the Klein bottle. The theorems mentioned in the previous paragraph imply, in particular, that such a graph must indeed have a  $K_6$  minor.

#### 1.2.5 Finding 3-shredders Efficiently

A *shredder* in a graph is a set of vertices whose removal results in at least three connected components. (Thus a shredder is a special case of a vertex-cut.) A *3-shredder* is a shredder of size three. In Chapter 6, we present an algorithm that, given a 3-connected graph, finds its 3-shredders in time proportional to the number of vertices and edges, when implemented on a RAM (random access machine).

The motivation to study this problem came from the even directed cycle problem. Given a digraph  $D$ , the question is to decide whether  $D$  has a directed cycle of even length. This is equivalent [58] to several other problems of interest, for instance: Given a 0-1 square matrix  $A$ , can some of the 1's be changed to  $-1$ 's in such a way that the permanent of  $A$  equals the determinant of the modified matrix (Pólya's permanent problem, [38])? When does a bipartite graph have a "Pfaffian orientation" [26, 27])? When is a square matrix *sign-nonsingular* [7, 28]?

For the version above that is phrased in terms of Pfaffian orientations, [34, 35, 43] present an  $O(n^3)$  algorithm, based on a structural characterization of bipartite graphs that possess a Pfaffian orientation. The exact definition of a Pfaffian orientation is not important here; what is relevant is that the structural characterization is in terms of a "trismus" operation that pastes *three* smaller graphs along a cycle of 4 vertices. With careful implementation, the running time of that algorithm can be reduced to  $O(n^2)$ , but attempts at further improvements run into serious difficulty. In order to take advantage of the structure theorem of [43], one needs to be able, at the very least, to efficiently decide whether a 4-connected bipartite graph has a 4-shredder. It is not clear to us whether the bipartite-ness would help. Given that the corresponding problem for 3-shredders in 3-connected graphs was not known, we started with that as the first step. It should be noted that the 4-connected graphs in the above application have  $O(n)$  edges, so a linear ( $O(n + m)$ ) running time would indeed be an improvement over  $O(n^2)$ , and is theoretically optimal.

### 1.2.6 A Note on Terminology

Most of the notation used in this thesis is standard to Graph Theory.

Specific definitions are usually made just before their use, instead of being collected in a common glossary. General or common terminology, on the other hand, is explained here or at the beginning of the respective chapters.

Unless otherwise stated, graphs are finite, undirected and simple (that is, with no parallel edges or loops).

For a graph  $G$  and an edge  $e$  in  $G$ ,  $G/e$  denotes the graph obtained by *contracting* the

edge  $e$ , that is, identifying its end-points and removing any resulting parallel edges.  $G \setminus e$  denotes the graph obtained by deleting the edge  $e$ . (Similarly, if  $X$  is a subset of the vertex or edge sets, or is a subgraph of  $G$ , then  $G \setminus X$  denotes the graph obtained by deleting the vertices and/or edges in  $X$ .)

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends, where the paths are disjoint, except possibly for shared ends. The replacement paths are called *segments*, and their ends are called *branch-vertices*. A graph is a *topological minor* of another if a subdivision of the first is a subgraph of the second.

*Paths* and *cycles* have no “repeated” vertices. For  $Z \subseteq V(G)$ ,  $G[Z]$  denotes the subgraph *induced* by  $Z$ , that is, the subgraph consisting of  $Z$  and all edges with both ends in  $Z$ . A subgraph of  $G$  is said to be *induced* if it is induced by its vertex set.

A *separation* of a graph  $G$  is a pair  $(A, B)$  of subsets of  $V(G)$  such that  $A \cup B = V(G)$ , and there is no edge between  $A - B$  and  $B - A$ . The separation is called *non-trivial* or *proper* if both  $A$  and  $B$  are proper subsets of  $V(G)$ . The *order* of  $(A, B)$  is  $|A \cap B|$ . A separation of order  $k$  (respectively, of order  $\leq k$ ) is called a  $k$ -separation (respectively, a  $\leq k$ -separation).

## CHAPTER II

# NON-EMBEDDABLE EXTENSIONS OF EMBEDDED MINORS

### 2.1 *Introduction*

Let a non-planar graph  $H$  have a subgraph isomorphic to a subdivision of a planar graph  $G$ . For various problems in Graph Structure Theory it is useful to know the minimal subgraphs of  $H$  that are isomorphic to a subdivision of  $G$  and are non-planar. In other words, one wants to know what more does  $H$  contain on account of its non-planarity. In [41] it is shown that under some mild connectivity assumptions these “minimal non-planar enlargements” of  $G$  are quite nice. In the applications of the result,  $G$  is explicitly known, whereas  $H$  is not, and the enlargement operations would furnish an explicit list of graphs such that (i)  $H$  has a subgraph isomorphic to a subdivision of one of the graphs on the list, and (ii) each graph on the list is a witness both to the fact that  $G$  is a topological minor of  $H$ , and that  $H$  is, in addition, non-planar. (The minimality of the graphs in the list is required to avoid redundancy.) Before we state that result, we need a few definitions.

A graph  $G$  is *weakly 4-connected* if  $G$  is 3-connected, has at least five vertices, and for every separation  $(A, B)$  of  $G$  of order at most three, one of the graphs  $G[A]$ ,  $G[B]$  has at most four edges.

A cycle  $C$  in a graph  $G$  is called *peripheral* if it is induced and  $G \setminus V(C)$  is connected. It is well-known [57, 60] that the peripheral cycles in a 3-connected planar graph are precisely the cycles that bound faces in some (or, equivalently, every) planar embedding of  $G$ .

Let  $S$  be a subgraph of a graph  $H$ . An  $S$ -*path* in  $H$  is a path with both ends in  $S$ , and otherwise disjoint from  $S$ . Let  $C$  be a cycle in  $S$ , and let  $P_1$  and  $P_2$  be two disjoint  $S$ -paths in  $H$  with ends  $u_1, v_1$  and  $u_2, v_2$ , respectively, such that  $u_1, u_2, v_1, v_2$  belong to  $V(C)$  and occur on  $C$  in the order listed. In those circumstances we say that the pair  $P_1, P_2$  is an

$S$ -cross in  $H$ . We also say that it is an  $S$ -cross on  $C$ . We say that  $u_1, v_1, u_2, v_2$  are the feet of the cross. We say that the cross  $P_1, P_2$  is free if

(F1) for  $i = 1, 2$  no segment of  $S$  includes both ends of  $P_i$ , and

(F2) no two segments of  $S$  that share a vertex include all the feet of the cross.

The following was proved in [41].

**Theorem 2.1.1** *Let  $G$  be a weakly 4-connected planar graph, and let  $H$  be a weakly 4-connected non-planar graph such that a subdivision of  $G$  is isomorphic to a subgraph of  $H$ . Then there exists a subgraph  $S$  of  $H$  isomorphic to a subdivision of  $G$  such that one of the following conditions holds:*

1. *there exists an  $S$ -path in  $H$  such that its ends belong to no common peripheral cycle in  $S$ , or*
2. *there exists a free  $S$ -cross in  $H$  on some peripheral cycle of  $S$ .*

This theorem has been used in [11, 51], and will probably be useful elsewhere as well. However, in more complicated applications it is more efficient to work with minors, rather than topological minors. We sketch one such application in Section 2.8. For any fixed graph  $G$ , there exists a finite and explicitly constructible set  $\{G_1, G_2, \dots, G_t\}$  of graphs such that a graph  $H$  has a minor isomorphic to  $G$  if and only if it has a topological minor isomorphic to one of the graphs  $G_i$ . Thus one can apply Theorem 2.1.1  $t$  times to deduce the desired conclusion about  $G$ , but it would be nicer to have a more direct route to the result that involves less potential duplication. Furthermore, if the outcome is allowed to be a minor of  $H$  rather than a topological minor, then the outcomes (i) and (ii) above can be strengthened to require that the ends of the paths involved are branch-vertices of  $S$ , as we shall see.

It turns out that Theorem 2.1.1 is not exclusively about face boundaries of planar graphs, but that an appropriate generalization holds under more general circumstances. Thus rather than working with peripheral cycles in planar graphs we will introduce an appropriate set of axioms for a set of cycles of a general graph. We do so now in order to avoid having to restate our definitions later when we present the more general form of our results.

A *segment* in a graph  $G$  is a maximal path such that its internal vertices all have degree in  $G$  exactly two. If a graph  $G$  has no vertices of degree two, then the segments of a subdivision of  $G$  defined earlier coincide with the notion just defined. Since we will not consider subdivisions of graphs with vertices of degree two there is no danger of confusion. A *cycle double cover* in a graph  $G$  is a set  $\mathcal{D}$  of distinct cycles of  $G$ , called *disks*, such that

(D1) each edge of  $G$  belongs to precisely two members of  $\mathcal{D}$ .

A cycle double cover  $\mathcal{D}$  is called a *disk system* in  $G$  if

- (D2) for every vertex  $v$  of  $G$ , the edges incident with  $v$  can be arranged in a cyclic order such that for every pair of consecutive edges in this order, there is precisely one disk in  $\mathcal{D}$  containing that pair of edges, and
- (D3) the intersection of any two distinct disks in  $\mathcal{D}$  either has at most one vertex or is a segment.

A cycle double cover satisfying (D3) is called a *weak disk system*. It is easy to see that if a connected graph has a disk system, then it is a subdivision of a 3-connected graph. Also, note that in a 3-connected graph, Axiom (D3) is equivalent to the requirement that every two distinct disks intersect in a complete subgraph on at most two vertices. The peripheral cycles of a 3-connected planar graph form a disk system. More generally, if  $G$  is a subdivision of a 3-connected graph embedded in a surface  $\Sigma$  in such a way that every homotopically nontrivial closed curve intersects the graph at least three times (a “polyhedral embedding”), then the face boundaries of this embedding form a disk system in  $G$ . Conversely, it can be shown that a disk system in a graph is the set of face boundaries of a polyhedral embedding of the graph in some surface. Weak disk systems correspond to face boundaries of embeddings into pseudo-surfaces (surfaces with “pinched” points).

Let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ . Two vertices or edges of  $G$  are said to be *confluent* if there is a disk containing both of them. If  $\mathcal{D}$  is a cycle double cover in a graph  $G$  and  $S$  is a subdivision of  $G$ , then  $\mathcal{D}$  induces a cycle double cover  $\mathcal{D}'$  in  $S$  in the obvious way, and vice versa. We say that  $\mathcal{D}'$  is the *disk system induced in  $S$  by  $\mathcal{D}$* .

Let  $v$  be a vertex of a graph  $G$  with degree at least 4. Partition the set of its neighbors into two disjoint sets  $N_1$  and  $N_2$ , with at least two vertices in each set. Let  $G'$  be obtained from  $G$  by replacing the vertex  $v$  with two adjacent vertices  $v_1, v_2$ , with  $v_i$  adjacent to the vertices in  $N_i$  for  $i = 1, 2$ . The graph  $G'$  is said to be obtained from  $G$  by *splitting* the vertex  $v$ . It is easy to see that if  $G$  is 3-connected, then so is  $G'$ . The vertices  $v_1$  and  $v_2$  are called the *new vertices* of  $G'$  and the edge  $v_1v_2$  of  $G'$  is called the *new edge* of  $G'$ .

Suppose a graph  $G$  has a cycle double cover  $\mathcal{D}$ . The above splitting operation on a vertex  $v$  of  $G$  is said to be a *conforming split* (with respect to  $\mathcal{D}$ ) if

- (S1) among the disks that use the vertex  $v$ , there are exactly two, say  $D_1$  and  $D_2$ , that use one vertex each from  $N_1$  and  $N_2$ , and
- (S2)  $D_1$  and  $D_2$  intersect precisely in the vertex  $v$ .

The split is then said to be *along*  $D_1$  (and along  $D_2$ ). A split that is not conforming as above is said to be a *non-conforming split*.

Let  $G, G'$  be as in the above paragraph. If  $G$  is a 3-connected planar graph, then  $G'$  is planar if and only if the split is conforming with respect to the disk system of peripheral cycles of  $G$ . More generally, to each cycle  $C$  of  $G$  there corresponds a unique cycle  $C'$  of  $G'$ , and so to  $\mathcal{D}$  there corresponds a uniquely defined set of cycles  $\mathcal{D}'$  of  $G'$ . If  $\mathcal{D}$  is a weak disk system, then so is  $\mathcal{D}'$ , and if  $\mathcal{D}$  is a disk system, then so is  $\mathcal{D}'$ . We call  $\mathcal{D}'$  the (weak) disk system *induced* in  $G'$  by  $\mathcal{D}$ . This is the purpose of conditions (S1) and (S2). If  $\mathcal{D}$  is a disk system, then an equivalent way to define a conforming split of a vertex  $v$  is to say that both  $N_1$  and  $N_2$  form contiguous intervals in the cyclic order induced on the neighborhood of  $v$  by  $\mathcal{D}$ . Similarly, an equivalent condition for a split to be non-conforming with respect to a disk system is the existence of vertices  $a, c \in N_1$  and  $b, d \in N_2$  such that  $a, b, c$  and  $d$  appear in the cyclic order listed around  $v$  (as given by  $\mathcal{D}$  in (D2)). The reason we use the definition above is that it applies more generally to weak disk systems.

A graph  $G'$  obtained from a graph  $G$  by repeatedly splitting vertices of degree at least four is said to be an *expansion* of  $G$ . In particular, each graph is an expansion of itself. Each split leading to an expansion of  $G$  has exactly one new edge; the set of these edges

are the *new edges of the expansion of  $G$* . The new edges form a forest in  $G'$ . If  $G$  has a cycle double cover  $\mathcal{D}$ , the expansion is called a *conforming* expansion if each of the splits involved in it is conforming (with respect to  $\mathcal{D}$ ). If at least one of the splits involved is not conforming, then the expansion is called *non-conforming*. From the above discussion, it is clear that a disk system in  $G$  induces a unique disk system in a conforming expansion.

We now describe seven enlargement operations. Let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ , and let  $G^+$  be the graph obtained from  $G$  by applying one of the operation described below.

1. (non-conforming jump)  $G^+$  is obtained from  $G$  by adding an edge  $uv$  where  $u$  and  $v$  are non-confluent vertices of  $G$ .
2. (cross) Let  $a, b, c, d$  be vertices appearing on a disk of  $G$  in that cyclic order. Add the edges  $ac$  and  $bd$  to obtain  $G^+$ .
3. (non-conforming split)  $G^+$  is obtained from  $G$  by performing a non-conforming split of a vertex of  $G$ .
4. (split + non-conforming jump) Let  $u, v$  be non-adjacent vertices on some disk  $C \in \mathcal{D}$ . Perform a conforming split of  $v$  into  $v_1, v_2$  such that  $u$  and  $v_2$  are non-confluent vertices. (In particular, the split is not along  $C$ .) Now add the edge  $uv_2$  to obtain  $G^+$ .
5. (double split + non-conforming jump) Let  $u, v$  be adjacent vertices and  $C_1, C_2$  be the two disks containing the edge  $uv$ . Make a conforming split of  $u$  into  $u_1, u_2$  along  $C_1$  and a conforming split of  $v$  into  $v_1, v_2$  along  $C_2$  such that both splits are conforming and  $u_1$  and  $v_1$  are adjacent in the resulting graph. Now add the edge  $u_2v_2$  to obtain  $G^+$ .
6. (split + cross) Let  $u, v, w$  be vertices on a disk  $C$  such that  $u$  is not adjacent to  $v$  or  $w$ . Perform a conforming split of  $u$  into  $u_1, u_2$ , along  $C$ , with  $u_1, u_2, v, w$  in that cyclic order on the new disk corresponding to  $C$ . Now add the edges  $u_1v$  and  $u_2w$  to obtain  $G^+$ .

7. (double split + cross) Let  $u, v$  be non-adjacent vertices on a disk  $C$ . Perform conforming splits of  $u$  and  $v$ , into  $u_1, u_2$  and  $v_1, v_2$ , respectively such that both splits are along  $C$ . Let  $u_1, u_2, v_1, v_2$  appear in that cyclic order on the new disk corresponding to  $C$ . Now add the edges  $u_1v_1$  and  $u_2v_2$  to obtain  $G^+$ .

If  $G^+$  is obtained as in paragraph  $i$  above, then we say that  $G^+$  is an  $i$ -enlargement of  $G$  with respect to  $\mathcal{D}$ . When the disk system  $\mathcal{D}$  is implied by context, we may simply refer to an  $i$ -enlargement of  $G$ . We are now ready to state a preliminary form of our main result, a counterpart of Theorem 2.1.1, with minors instead of topological minors.

**Theorem 2.1.2** *Let  $G$  be a weakly 4-connected planar graph, let  $H$  be a weakly 4-connected non-planar graph such that  $G$  is isomorphic to a minor of  $H$ , and let  $\mathcal{D}$  be the disk system in  $G$  consisting of all peripheral cycles. Then there exists an integer  $i \in \{1, 2, \dots, 7\}$  such that  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$  with respect to  $\mathcal{D}$ .*

Theorem 2.1.1 is definitely easier to state than Theorem 2.1.2. So what are the advantages of the latter result? First, in the applications one is usually concerned with minors rather than topological minors, and so Theorem 2.1.2 gives a more direct route to the desired results. Second, while the number of types of outcome is larger in Theorem 2.1.2, in most cases the actual number of cases needed to examine will be smaller. (Notice that, for instance, in Theorem 2.1.1 one must examine all  $S$ -paths between non-confluent ends, whereas in Theorem 2.1.2 one is only concerned with those between non-confluent branch-vertices.)

Third, while a graph listed as an outcome of Theorem 2.1.1 may fail to be weakly 4-connected (and may do so in a substantial way), an  $i$ -enlargement of a weakly 4-connected graph is again weakly 4-connected. This has two advantages. In the applications we are often seeking to prove that weakly 4-connected graphs, with a minor isomorphic to some weakly 4-connected graph embeddable in a surface  $\Sigma$ , that themselves do not embed into  $\Sigma$  have a minor isomorphic to a member of a specified list  $\mathcal{L}$  of graphs. In order to get a meaningful result we would like each member of  $\mathcal{L}$  to satisfy the same connectivity requirement imposed on the input graphs.

From a more practical viewpoint, the advantage of maintaining the same connectivity in the outcome graph is that the theorem can then be applied repeatedly. That will become important when we consider a generalization to arbitrary surfaces (that is, in the context of theorems 2.2.1 and 2.7.5). While a weakly 4-connected graph  $G$  has at most one planar embedding, it may have several embeddings in a non-planar surface  $\Sigma$ . Now one application of the generalization of Theorem 2.1.2 will dispose of one embedding into  $\Sigma$ , but some other embedding might extend naturally to those outcome graphs. So it may be necessary to apply the theorem in turn to those outcome graphs in place of  $G$ . It will be important that the outcomes of (the generalization of) Theorem 2.1.2 satisfy the same requirement as the input graph. We can then apply such a theorem repeatedly till we get a list of graphs that no longer embed in  $\Sigma$  — in other words, we would have obtained the non-embeddable extensions of  $G$ . This will be illustrated in Section 2.8.

## 2.2 *Main Theorem*

Our main theorem applies to arbitrary disks systems, at the expense of having to add two additional outcomes, the following. As before, let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ , and let  $G^+$  be obtained by one of the operations below.

8. (non-separating triad) Let  $x_1, x_2, x_3$  be three vertices of  $G$  such that (i) they are pairwise confluent, but not all contained in any single disk, and (ii)  $\{x_1, x_2, x_3\}$  is independent, and does not separate  $G$ . To obtain  $G^+$ , add a new vertex to  $G$  adjacent to  $x_1, x_2$  and  $x_3$ .
9. (non-conforming T-edge) Let a vertex  $u$  and an edge  $xy$  be such that (i)  $u$  is not confluent with the edge  $xy$ , but is confluent with both  $x$  and  $y$ , (ii)  $u$  is not adjacent to either  $x$  or  $y$ , and (iii)  $\{u, x, y\}$  does not separate  $G$ . Subdivide the edge  $xy$  and join  $u$  to the new vertex, to obtain  $G^+$ .

As before, if  $G^+$  is obtained as in paragraph  $i$  above, then we say that  $G^+$  is an  *$i$ -enlargement of  $G$  with respect to  $\mathcal{D}$* .

We also need to define an appropriate analogue of being non-planar in the context of cycle double covers. That is the objective of this paragraph and the next. Let  $S$  be a subgraph of a graph  $H$ . An  $S$ -bridge of  $H$  is a subgraph  $B$  of  $H$  such that either  $B$  consists of a unique edge of  $E(H) - E(S)$  and its ends, where the ends belong to  $S$ , or  $B$  consists of a component  $J$  of  $H \setminus V(S)$  together with all edges from  $V(J)$  to  $V(S)$  and all their ends. For an  $S$ -bridge  $B$ , the vertices of  $B \cap S$  are called the *attachments* of  $B$ . Let  $\mathcal{D}$  be a cycle double cover in  $S$ . We say that  $\mathcal{D}$  is *locally planar* in  $H$  if the following conditions are satisfied:

- (i) for every  $S$ -bridge  $B$  of  $H$  there exists a disk  $C_B \in \mathcal{D}$  such that all the attachments of  $B$  lie on  $C_B$ , and
- (ii) for every disk  $C \in \mathcal{D}$  the subgraph  $\bigcup B \cup C$  of  $H$  has a planar drawing with  $C$  bounding the unbounded face, where the big union is taken over all  $S$ -bridges  $B$  of  $H$  with  $C_B = C$ .

Let  $G$  have a weak disk system  $\mathcal{D}$  and  $H$  have a minor isomorphic to  $G$ . It is easy to see that there is an expansion  $G'$  of  $G$ , such that  $G'$  is a topological minor of  $H$ . We say that  $\mathcal{D}$  has a *locally planar extension* into  $H$  if:

- (i) there exists a *conforming* expansion  $G'$  of  $G$  such that a subdivision of  $G'$  is isomorphic to a subgraph  $S$  of  $H$ , and
- (ii) the weak disk system  $\mathcal{D}'$  induced in  $S$  by  $\mathcal{D}$  is locally planar in  $H$ .

We are now ready to state the main result. A graph is a *prism* if its complement is a cycle of length six.

**Theorem 2.2.1** *Let  $G$  and  $H$  be weakly 4-connected graphs such that  $H$  has a minor isomorphic to  $G$  and  $G$  is not a prism. Let  $G$  have a disk system  $\mathcal{D}$  that has no locally planar extension into  $H$ . Then  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$ , for some  $i \in \{1, \dots, 9\}$ .*

Let us deduce Theorem 2.1.2 from Theorem 2.2.1.

**Proof of Theorem 2.1.2, assuming Theorem 2.2.1.** Let  $i \in \{8, 9\}$ . By Theorem 2.2.1 it suffices to show that a weakly 4-connected planar graph  $G$  has no  $i$ -enlargement with respect to the disk system  $\mathcal{D}$  consisting of all peripheral cycles of  $G$ . Indeed, suppose for a contradiction that such an  $i$ -enlargement exists, and let  $u, x, y$  be the three vertices of  $G$  as in the definition of  $i$ -enlargement. Since every pair of vertices among  $u, x, y$  are confluent, it follows that  $G \setminus \{u, x, y\}$  is disconnected, a contradiction.  $\square$

### 2.3 Outline of Proof

The purpose of this section is to outline the proof of the main theorem. In fact, we prove a more general result, stated as Theorem 2.7.5, that applies to weak disk systems. However, this more general theorem requires three additional outcomes, and we introduce them now. Let  $J$  be a  $K_4$  subgraph of a graph  $G$ , and assume that each of the four peripheral cycles of  $J$  are disks of  $G$ . In those circumstances we say that  $J$  is a *detached  $K_4$  subgraph* of  $G$ . Again, let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ , and let  $G^+$  be obtained from  $G$  by one of the operations below.

10. (non-conforming double split) Let  $v$  be a vertex of  $G$  of degree at least 6. Partition the set of edges incident with  $v$  into three disjoint sets  $E_1, E_2, E_3$ , each of size at least 2, such that of the disks that use the vertex  $v$ , exactly two, say  $D_1$  and  $D_2$ , use edges of two distinct sets  $E_i$ . Moreover, let  $D_1$  and  $D_2$  intersect precisely in the vertex  $v$  and let them use no edge of  $E_3$ . Then  $G^+$  is obtained from  $G$  by replacing  $v$  with three vertices  $v_1, v_2, v_3$ , where  $v_i$  is incident with edges in  $E_i$  (for  $i = 1, 2, 3$ ) and  $G^+[\{v_1, v_2, v_3\}]$  has edge-set  $\{v_1v_3, v_2v_3\}$ .
11. (simplex on detached  $K_4$ ) The graph  $G^+$  is obtained from  $G$  by adding a new vertex adjacent to all four vertices of a detached  $K_4$  subgraph of  $G$ .
12. (enlargement of a prism) Let  $G$  be a prism, and let  $G^+$  be obtained from  $G$  by selecting two edges of  $G$  that do not belong to a common peripheral cycle but both belong to a triangle, subdividing them, and joining the two new vertices by an edge.

Similarly as before, if  $G^+$  is obtained as in paragraph  $i$  above, then we say that  $G^+$  is an  $i$ -enlargement of  $G$  with respect to  $\mathcal{D}$ . Thus if  $G$  is not a prism, then it has no 12-enlargement, and if  $G$  is a prism, then its 12-enlargement is unique, up to isomorphism. The unique 12-enlargement of the prism is known as  $V_8$ .

Our main tool for the proof of Theorem 2.2.1 will be its counterpart for subdivisions, proved in [41]. Before we can state it we need one more definition. Let  $S$  be a subgraph of a graph  $H$ , and let  $\mathcal{D}$  be a cycle double cover in  $S$ . Let  $x \in V(H) - V(S)$  and let  $x_1, x_2, x_3$  be distinct vertices of  $S$  such that every two of them are confluent, but no disk of  $S$  contains all three. Let  $L_1, L_2, L_3$  be three paths such that (i) they share a common end  $x$ , (ii) they share no internal vertex among themselves or with  $S$ , and (iii) the other end of  $L_i$  is  $x_i$ , for  $i = 1, 2, 3$ . The paths  $L_1, L_2, L_3$  are then said to form an  $S$ -triad. The vertices  $x_1, x_2, x_3$  are called the *feet* of the triad. We are now ready to state the result from [41].

**Theorem 2.3.1** ([41]) *Let  $G$  be a graph with no vertices of degree two, let  $H$  be a weakly 4-connected graph, let  $\mathcal{D}$  be a weak disk system in  $G$ , and let a subdivision of  $G$  be isomorphic to a subgraph of  $H$ . Then there exists a subgraph  $S$  of  $H$  isomorphic to a subdivision of  $G$  such that, letting  $\mathcal{D}'$  denote the weak disk system induced in  $S$  by  $\mathcal{D}$ , one of the following conditions holds:*

1. *there exists an  $S$ -path in  $H$  such that its ends are not confluent in  $S$ , or*
2. *there exists a free  $S$ -cross in  $H$  on some disk of  $S$ , or*
3. *the graph  $H$  has an  $S$ -triad, or*
4. *the graph  $S$  has a detached  $K_4$  subgraph  $J$  such that the attachments of some  $S$ -bridge of  $H$  are precisely the branch-vertices of  $J$ , or*
5. *the weak disk system  $\mathcal{D}'$  is locally planar in  $H$ .*

Now let  $G$ ,  $\mathcal{D}$  and  $H$  be as in Theorem 2.2.1. It is easy to see that there exists an expansion  $G'$  of  $G$  such that a subdivision of  $G'$  is isomorphic to a subgraph  $S$  of  $H$ . (If  $G$  itself is a topological minor of  $H$ , then  $G' = G$ .) In Lemma 2.4.4 we prove that if  $G'$  is a

nonconforming expansion, then there exists a 3-enlargement or a 10-enlargement of  $G$  that is isomorphic to a minor of  $H$ . Thus from now on we may assume that  $G'$  is a conforming expansion of  $G$ . By Lemma 2.3.1 applied to  $S$  and  $H$  we deduce that one of the outcomes of that lemma holds. Notice that those outcomes correspond to 1-enlargement, 2-enlargement, 8-enlargement and 11-enlargement, respectively, except that in the enlargements the vertices in question are required to be branch-vertices of  $S$ , whereas in Lemma 2.3.1 they are allowed to be interior vertices of segments. We deal with this in Section 2.5 by showing that each of the outcomes mentioned leads to a suitable enlargement of  $G'$ . To be precise, at this point we settle for what we call weak 8- and weak 9-enlargements, and in Section 2.6 show that these weak enlargements can be replaced by ordinary enlargements, possibly of a different expansion of  $G$  and of a different kind. Finally, in Section 2.7 we complete the proof of Theorem 2.2.1 by showing that the expansion  $G'$  can be chosen to be equal to  $G$ .

## 2.4 Preliminaries

Let  $G'$  be an expansion of a graph  $G$ . Then every vertex  $v$  of  $G$  corresponds to a connected subgraph  $T_v$  of  $G'$ . We call  $V(T_v)$  the *branch-set corresponding to  $v$* .

**Lemma 2.4.1** *Let  $G'$  be an expansion of a graph  $G$ , let  $u, v \in V(G)$  be distinct, and let  $T_u, T_v$  be the corresponding subgraphs of  $G'$ . Then  $T_u$  and  $T_v$  are induced subtrees of  $G'$ . If  $u$  is adjacent to  $v$  then exactly one edge of  $G'$  has one end in  $V(T_u)$  and the other in  $V(T_v)$ , and if  $u$  is not adjacent to  $v$ , then no such edge exists.*

An expansion of a weakly 4-connected graph may fail to be weakly 4-connected, but only in a limited way. The next definition and lemma make that precise. Let  $(A, B)$  be a nontrivial separation of order three in a graph  $G$ . We say that  $(A, B)$  is *degenerate* if the vertices in  $A \cap B$  can be numbered  $v_1, v_2, v_3$  such that either

- (1)  $|A - B| = 1$  and  $A \cap B$  is an independent set, or
- (2) there exists a triangle  $u_1u_2u_3$  in  $G[A]$  such that for  $i = 1, 2, 3$  the vertices  $u_i$  and  $v_i$  are either adjacent or equal,  $A \subseteq \{u_1, u_2, u_3, v_1, v_2, v_3\}$ , and each edge of  $G[A]$  is of the form  $u_i v_i$  for  $1 \leq i \leq 3$  or  $u_i u_j$  for  $1 \leq i < j \leq 3$ .

The following two lemmas are routine, and we omit the straightforward proof.

**Lemma 2.4.2** *Let  $G$  be an expansion of a weakly 4-connected graph. Then  $G$  is 3-connected, and if it is not a prism, then for every nontrivial separation  $(A, B)$  of  $G$  of order three, exactly one of  $(A, B)$ ,  $(B, A)$  is degenerate.*

**Lemma 2.4.3** *Let  $G'$  be expansion of a weakly 4-connected graph  $G$ , let  $(A, B)$  be a degenerate separation of  $G$  of order three satisfying condition (2) of the definition of degenerate separation, and let  $u_1, u_2, u_3, v_1, v_2, v_3$  be as in that condition. Then for at least two integers  $i \in \{1, 2, 3\}$  either  $u_i = v_i$  or  $u_i v_i$  is a new edge of  $G'$ .*

We now show that a non-conforming expansion of  $G$  must have a minor isomorphic to a 3-enlargement or a 10-enlargement of  $G$ .

**Lemma 2.4.4** *Let  $G'$  be a non-conforming expansion of a graph  $G$ . Then  $G'$  has a minor isomorphic to a 3-enlargement or 10-enlargement of  $G$ .*

*Proof:* We may assume that for every new edge  $e$  of  $G'$  the graph  $G'/e$  is a conforming expansion of  $G$ . We shall refer to this as the *minimality* of  $G'$ . We will prove that  $G'$  is either a 3-enlargement or a 10-enlargement of  $G$ .

Let  $\widehat{G}$  be an expansion of  $G$  such that  $G'$  is obtained from  $\widehat{G}$  by splitting a vertex  $v$  into  $v_1$  and  $v_2$ . By the minimality of  $G'$  this split is non-conforming, and  $\widehat{G}$  is a conforming expansion of  $G$ . If  $G = \widehat{G}$ , then  $G'$  is a 3-enlargement of  $G$ , and so we may assume that  $G \neq \widehat{G}$ . Let  $e$  be a new edge of  $\widehat{G}$ . If  $e$  is not incident with  $v$ , then  $G'/e$  is a non-conforming expansion of  $\widehat{G}/e$ , contrary to the minimality of  $G'$ . Now let us consider  $e$  as an edge of  $G'$ . From the symmetry between  $v_1$  and  $v_2$  we may assume that  $e$  is incident with  $v_2$  in  $G'$ ; let  $v_3$  be its other end. The split of the vertex  $v$  of the graph  $\widehat{G}$  into  $v_1$  and  $v_2$  violates (S1) or (S2). But it does not violate (S1), for otherwise the same violation occurs in the analogous split of  $\widehat{G}/e$ , contrary to the minimality of  $G'$ . Thus the split of the vertex  $v$  of the graph  $\widehat{G}$  into  $v_1$  and  $v_2$  satisfies (S1); let  $D_1$  and  $D_2$  be the corresponding disks. It follows that the disks violate (S2), but they do not do so for the corresponding split in  $\widehat{G}/e$ . It follows

that  $e \in E(D_1) \cap E(D_2)$ . Since  $G'/e$  is a conforming expansion of  $G$  it follows that  $G'$  is a 10-enlargement of  $\widehat{G}/e$ . But  $\widehat{G}/e = G$  by the minimality of  $G$ , as desired.  $\square$

The following lemma will be useful.

**Lemma 2.4.5** *Let  $G'$  be a conforming expansion of a graph  $G$  with respect to a weak disk system  $\mathcal{D}$ , and let  $\mathcal{D}'$  be the weak disk system induced in  $G'$  by  $\mathcal{D}$ . Let  $qr$  be a new edge of  $G'$ , and let the vertex  $p \in V(G') - \{q, r\}$  share distinct disks  $D_q, D_r$  of  $G'$  with  $q$  and  $r$ , respectively, such that  $D_r$  does not contain  $q$ . Then  $p$  is adjacent to  $r$  and the disks  $D_q, D_r$  both contain the edge  $pr$ .*

*Proof:* The disks of  $G/qr$  that correspond to  $D_q$  and  $D_r$  share  $p$  and the new vertex of  $G/qr$ , say  $w$ . By (D3)  $p$  is adjacent to  $w$  in  $G/qr$  and the edge  $pw$  belongs to both those disks. By Lemma 2.4.1 the vertex  $p$  is adjacent to exactly one of  $q, r$ . But  $q \notin V(D_r)$ , and hence  $p$  is adjacent to  $r$  and  $D_q, D_r$  both contain the edge  $pr$ , as desired.  $\square$

We end this section with a lemma about fixing separations in weakly 4-connected graphs, a special case of a lemma from [23]. First some additional notation: when a graph  $G$  is a minor of a graph  $H$ , we say that an *embedding*  $\eta$  of  $G$  into  $H$  is a mapping with domain  $V(G) \cup E(G)$  as follows.  $\eta$  maps vertices  $v \in G$  to connected subgraphs  $\eta(v)$  of  $H$ , with distinct vertices being mapped to disjoint vertex-disjoint subgraphs. Further,  $\eta$  maps edges  $uv$  of  $G$  to paths  $\eta(uv)$  in  $H$  with one end in  $\eta(u)$  and the other in  $\eta(v)$ , and otherwise disjoint from  $\eta(w)$  for any vertex  $w$  of  $G$ . Also, for edges  $e \neq e'$  of  $G$ , if  $\eta(e)$  and  $\eta(e')$  share a vertex, then it must be an end of both the paths.

**Lemma 2.4.6** *Let  $G_1$  be a graph isomorphic to a minor of a weakly 4-connected graph  $H$ . Let  $P = \{p_1, p_2\}$ ,  $Q = \{q_1, q_2, q_3\}$  and  $R$  be such that  $(P, Q, R)$  is a partition of  $V(G_1)$ , and  $G_1$  has all possible edges between  $P$  and  $Q$ , and no edge with both ends in  $Q$ . Further, suppose  $R$  has at least two vertices, and that  $(P \cup Q, Q \cup R)$  is a (non-trivial) 3-separation of  $G_1$ . Then  $H$  has a minor isomorphic to a graph  $G_1^+$  that is obtained from  $G_1$  by*

1. adding an edge between  $p_i$  and  $r$  for some  $i \in \{1, 2\}$  and  $r \in R$ , or

2. *splitting*  $q_j$  for some  $j \in \{1, 2, 3\}$  into vertices  $q_j^1$  and  $q_j^2$  such that  $q_j^1$  is adjacent to  $p_1$  and  $q_j^2$  is adjacent to  $p_2$

*Proof:* Call an embedding  $\eta$  of  $G_1$  into  $H$  *minimal* if for every embedding  $\eta'$  of  $G$  into  $H$ ,

$$\sum_{j=1}^3 |E(\eta(q_j))| \leq \sum_{j=1}^3 |E(\eta'(q_j))|$$

In particular, if  $\eta$  is minimal,  $\eta(q_j)$  is a tree for every  $j$ . Further, we say that a vertex  $q_j$  is *good* for  $\eta$  if the paths  $\eta(p_1q_j)$  and  $\eta(p_2q_j)$  are vertex-disjoint (in other words, their ends in  $\eta(q_j)$  are distinct).

Consider a minimal embedding  $\eta$  of  $G_1$  into  $H$ . Suppose there exists a  $q_j$  that is good for  $\eta$ . For  $i = 1, 2$ , let  $p'_i$  be the endpoint of  $\eta(p_iq_j)$  in  $\eta(q_j)$ . Let  $e$  be an edge in the unique path between  $p'_1$  and  $p'_2$  in  $\eta(q_j)$ , and let  $T_1, T_2$  be the two subtrees obtained by deleting  $e$  from  $\eta(q_j)$ , such that  $p'_i \in T_i$  for  $i = 1, 2$ . For  $i = 1, 2$ , define  $N_i$  to be the set of neighbors  $r \in R$  of  $q_j$  in  $G$  that  $\eta(rq_j)$  has an endpoint in  $T_i$ . Now  $N_1, N_2$  are non-empty by the minimality of  $\eta$ . (If, say,  $N_1$  were empty, then we could replace  $\eta(q_j)$  by  $T_2$  and modify  $\eta(p_1q_j)$  accordingly to get a better embedding  $\eta'$ , a contradiction.) It is easy to see that conclusion 2 of the lemma is satisfied, with the neighborhoods of  $q_j^1$  and  $q_j^2$  being  $N_1 \cup \{p_1\}$  and  $N_2 \cup \{p_2\}$ , respectively.

Hence we may assume that there is no minimal embedding of  $G$  into  $H$  with a vertex in  $Q$  being good for it. Let  $\eta$  be an embedding of  $G$  into  $H$ . For  $j = 1, 2, 3$ , there exist vertices  $t_j$  such that both  $\eta(p_1q_j)$  and  $\eta(p_2q_j)$  have  $t_j$  as an end. Define  $J_1$  as the union of  $\eta(p_i)$ ,  $i = 1, 2$  and of  $\eta(e)$  for all edges  $e$  with at least one end in  $P$ . Define  $J_2$  as the union of  $\eta(v)$  for  $v \in Q \cup R$  and of  $\eta(e)$  for every edge  $e$  of  $G$  with both ends in  $Q \cup R$ . Now  $V(J_1) \cap V(J_2) = \{t_1, t_2, t_3\}$ . Since  $H$  is weakly 4-connected, there is a path in  $H$  with ends  $a \in V(J_1) \setminus V(J_2)$  and  $b \in V(J_2) \setminus V(J_1)$ , and otherwise disjoint from  $J_1 \cup J_2$ . If  $b$  belongs to  $\eta(q_j) \setminus t_j$  for some  $j$ , then we can modify  $\eta$  to get a minimal embedding where  $q_j$  is a good vertex, which is a contradiction. Thus  $b$  belongs to  $\eta(r)$  for some  $r \in R$  or  $b$  is an internal vertex of  $\eta(e)$  for an edge  $e$  of  $G$  that has an end in  $R$  (recall that  $Q$  is an independent set). In either case, it is easy to see that conclusion 1 holds.  $\square$

## 2.5 The Enlargements of an Expansion of $G$

Let  $G$  and  $H$  be as in Theorem 2.2.1. In order to apply Theorem 2.3.1 we select an expansion  $G'$  of  $G$  such that a subdivision of  $G'$  is isomorphic to a subgraph of  $H$ . By Lemma 2.4.4 we may assume that  $G'$  is a conforming expansion. In this section we prove three lemmas, one corresponding to each of the first three outcomes of Theorem 2.3.1. The lemmas together almost imply that the conclusion of Theorem 2.2.1 holds for  $G'$ . The reason for the word almost is that for convenience we allow a weaker form of 8-enlargements and 9-enlargements.

The weaker form of 9-enlargements is defined as follows. Let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ , and let  $u, x, y \in V(G)$ , where  $x$  and  $y$  are adjacent and  $u$  is not confluent with the edge  $xy$ . Let  $G^+$  be obtained from  $G$  by subdividing the edge  $xy$  and adding an edge joining the new vertex to  $u$ . We say that  $G^+$  is a weak 9-enlargement of  $G$ . Later, in Lemma 2.6.3, we show how to move from a weak 9-enlargement to a 9-enlargement or another useful outcome. Our first lemma deals with the first outcome of Theorem 2.3.1.

**Lemma 2.5.1** *Let  $G, H$  be graphs such that  $G$  is connected, has at least five vertices and no vertices of degree two. Let  $\mathcal{D}$  be a weak disk system in  $G$ , let  $S$  be a subgraph of  $H$  isomorphic to a subdivision of  $G$ , and let  $P$  be an  $S$ -path in  $H$  such that its ends are not confluent in the weak disk system  $\mathcal{D}'$  induced in  $S$  by  $\mathcal{D}$ . Then  $H$  has a minor isomorphic to a 1-enlargement, 3-enlargement or a weak 9-enlargement of  $G$ .*

*Proof:* Let  $s, t$  be the ends of  $P$ . If both  $s$  and  $t$  are branch-vertices in  $S$ , then  $H$  has a minor isomorphic to a 1-enlargement of  $G$ , and we are done. If one of  $s$  and  $t$  is a branch-vertex and the other is an internal vertex of a segment of  $S$ , then  $H$  has a minor isomorphic to a weak 9-enlargement of  $G$ , as desired.

Thus we may assume that  $s$  and  $t$  are internal vertices of two different segments  $Q_1$  and  $Q_2$  of  $S$ , respectively. Let  $Q_1$  correspond to an edge  $uv \in E(G)$ , and let  $Q_2$  correspond to an edge  $xy \in E(G)$ . Now, if  $u$  is not confluent with the edge  $xy$ , then  $H$  has a minor isomorphic to a weak 9-enlargement of  $G$ , and the lemma holds. Thus, we may assume that  $u$  shares a disk  $D_1$  with the edge  $xy$ . By symmetry, we get a disk  $D_2$  shared by  $v$  and the edge  $xy$ , and disks  $D_3, D_4$  that the edge  $uv$  shares with vertices  $x$  and  $y$  respectively. (The

disks  $D_i$  may not be pairwise distinct.)

The disks  $D_1$  and  $D_3$ , however, must be distinct, since the vertices  $s, t$  are not confluent. Notice, however, that they share the vertices  $u$  and  $x$ . It follows that  $u, v, x, y$  are pairwise distinct, for if  $v = y$ , say, then  $u, v = y, x$  all belong to  $V(D_1 \cap D_3)$ , and hence  $D_1 = D_3$  by (D3), a contradiction. By (D3) this implies that  $u$  is adjacent to  $x$  in  $G$  and the intersection of  $D_1$  and  $D_3$  is precisely the edge  $ux$ . In other words, the vertices  $u$  and  $x$  must be adjacent in  $G$ , and  $D_1, D_3$  are precisely the two disks containing the edge  $ux$ . By a similar argument, it follows that  $u$  and  $y$  are adjacent, and  $D_1, D_4$  are precisely the two disks containing the edge  $uy$ . Thus  $u$  is adjacent with each of  $v, x, y$  in  $G$ , and the edges  $uv, ux, uy$  are pairwise confluent.

By symmetry, we get similar conclusions about the vertices  $v, x, y$ . Thus  $G[u, v, x, y]$  is a detached  $K_4$  subgraph of  $G$ . Since  $G$  has at least five vertices and is connected, we may assume, without loss of generality, that  $u$  has a neighbor in  $G$  outside of the set  $\{v, x, y\}$ . Let  $N$  be the set of all such neighbors of  $u$ . But then delete the edges of the segment of  $S$  corresponding to the edge  $ux$  and contract the edges of the subpath of  $Q_2$  between  $t$  and the end corresponding to  $x$ . It follows that  $H$  has a minor isomorphic to a graph obtained from  $G$  by splitting  $u$  corresponding to the partition  $\{\{v, x\}, N \cup \{y\}\}$  of its neighbors. This split is non-conforming since the disks  $D_1$  and  $D_4$  violate condition (S2) in the definition of a conforming split. Hence  $H$  has a minor isomorphic to a 3-enlargement of  $G$ .  $\square$

**Lemma 2.5.2** *Let  $G, H$  be graphs such that  $H$  is weakly 4-connected, and  $G$  is connected, has at least 5 vertices and has no vertices of degree two. Let  $\mathcal{D}$  be a weak disk system in  $G$ , let  $S$  be a subgraph of  $H$  isomorphic to a subdivision of  $G$ , such that  $\mathcal{D}$  induces the weak disk system  $\mathcal{D}'$  in  $S$ . Further, let there exist a free  $S$ -cross on some disk of  $S$ . Then  $H$  has a minor isomorphic to a 2-enlargement or a 3-enlargement or a weak 9-enlargement of  $G$ .*

*Proof:* Let the free cross consist of paths  $P_1, P_2$ , in a disk  $C'$  of  $S$ , that corresponds to a disk  $C$  of  $G$ . We shall call the paths  $P_1, P_2$  the *legs* of the cross. Recall that the ends of  $P_1, P_2$  are called the *feet* of the cross.

If  $C$  has at least four vertices, then we claim that  $H$  has a minor isomorphic to a 2-enlargement of  $G$ . We define an auxiliary bipartite graph  $B$ , with the vertex set being the set of feet of the cross and the set of branch-vertices of  $S$  that belong to  $C'$ . A foot  $f$  and a branch-vertex  $b$  are adjacent if one of the subpaths of  $C'$  with ends  $f$  and  $b$  includes no feet or branch-vertices in its interior. Since the cross is free, it follows from Hall's bipartite matching theorem that  $B$  has a complete matching from the set of feet to the set of branch-vertices (in other words, one that matches each of the feet). By contracting the edges of the paths that correspond to this matching, we deduce that  $H$  has a minor isomorphic to a 2-enlargement of  $G$ , as desired.

Hence we may assume that  $C$  is in fact a triangle on vertices  $u_1, u_2$  and  $u_3$ , say. For  $i = 1, 2, 3$ , if  $u_i$  has degree 3 in  $G$ , then define  $v_i$  to be its third neighbor (that is, the neighbor not in  $C$ ). Otherwise, define  $v_i = u_i$ . Let  $u'_1, u'_2, u'_3, v'_1, v'_2, v'_3$  be the corresponding vertices of  $S$ . Let  $Q_i$  denote the segment of  $S$  corresponding to the edge  $u_i v_i$  if  $u_i \neq v_i$  and let  $Q_i$  be the null graph otherwise, and let  $A = V(C' \cup P_1 \cup P_2)$  and  $B = (V(S) - V(C' \cup Q_1 \cup Q_2 \cup Q_3)) \cup \{v'_1, v'_2, v'_3\}$ . There exist three vertex-disjoint paths in  $H$  linking  $\{u'_1, u'_2, u'_3\}$  to  $\{v'_1, v'_2, v'_3\}$ . Since  $H$  is weakly 4-connected, it follows that there is no 3-cut in  $H$  separating  $A$  from  $B$ . Hence, by a variant of Menger's theorem,  $H$  contains four vertex-disjoint paths  $L_1, \dots, L_4$  linking  $\{v'_1, v'_2, v'_3, y\}$  to  $\{u'_1, u'_2, u'_3, x\}$  (not necessarily in that order), where  $x \in A$  and  $y \in B$ . We assume the numbering of the paths is such that for  $i = 1, 2$  and  $3$ ,  $L_i$  has end  $v'_i \in B$ . (The remaining path  $L_4$  then has end  $y \in B$ .) Let  $w$  be a nearest node of  $y$  that is not in  $\{v_1, v_2, v_3\}$ . (Note that such a nearest node exists by Lemma 2.4.1, and Lemma 2.4.3.)

We may assume that  $x \in V(C')$ . If not, then we may contract edges suitably in  $P_1$  or  $P_2$  such that the vertex corresponding to  $x$ , after the contraction, lies on  $C'$ . (Note that this contraction does not affect the graph  $S$ , neither does it destroy the cross.)

Relabel the vertices  $u'_1, u'_2, u'_3, x$  as  $a, b, c, d$ , in the order in which they appear on  $C'$  (in some orientation), such that  $L_4$  joins  $d$  to  $y$ . (Note that  $d$  need not be the same as  $x$ .) Let  $(d, a, b)$  denote the interior vertices of the subpath of  $C'$  with ends  $d$  and  $b$  that includes  $a$  in its interior, and let  $(d, c, b)$  be defined analogously.

We claim that there is a leg of the cross with feet  $f, g$  such that  $f \in (d, a, b)$  and  $g \in (d, c, b)$ . Since the cross is free, there exists a leg with foot in  $(d, a, b)$ . We may assume the other foot of this leg does not belong to  $(d, c, b)$ , but then the other leg of the cross satisfies the claim.

Choose a leg as above such that there is no foot between  $f$  and  $a$ , and no foot between  $g$  and  $c$ . (Such a choice must be possible, due to the freeness of the cross.) Let the other leg of the cross have feet  $h, i$ , such that  $b$  and  $h$  are joined by a subpath of the cycle  $C'$  that is disjoint from  $\{f, g\}$ . By contracting disjoint subpaths of  $C'$  with ends  $(a, f)$ ,  $(c, g)$ , and  $(b, h)$  respectively, it follows that  $H$  has a minor isomorphic to the graph  $G'$  obtained from  $G$  by adding a new vertex  $z$  adjacent to  $u_1, u_2, u_3$  and  $w$ .

If  $w$  is not confluent with the edge  $u_1u_2$  then  $G' \setminus u_1u_2 \setminus zu_3$  is isomorphic to a weak 9-enlargement of  $G$ , and we are done. Thus we may assume that  $w$  is confluent with the edge  $u_1u_2$ , and by symmetry, with the edges  $u_2u_3$  and  $u_1u_3$  as well. It follows similarly as in the proof of Lemma 2.5.1 that  $G[u_1, u_2, u_3, w]$  is a detached  $K_4$  subgraph of  $G$ . Since  $G$  is connected, and  $|V(G)| \geq 5$ , we may assume, without loss of generality, that  $u_1$  has a neighbor in  $G$  outside that set. It follows that a graph obtained from  $G$  by a non-conforming split of  $u_1$  is isomorphic to a minor of  $H$ .  $\square$

We now define the weaker form of 8-enlargements. Let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ , and let  $x_1, x_2, x_3$  be vertices of  $G$  such that no disks contains all three. Let  $G^+$  be obtained from  $G$  by adding a vertex with neighborhood  $\{x_1, x_2, x_3\}$ . We say that  $G^+$  is a weak 8-enlargement of  $G$ . Our third lemma deals with the third outcome of Theorem 2.3.1.

**Lemma 2.5.3** *Let  $G, H$  be graphs such that  $G$  is connected, has at least five vertices and no vertices of degree two. Let  $\mathcal{D}$  be a weak disk system in  $G$ , let  $S$  be a subgraph of  $H$  isomorphic to a subdivision of  $G$ , and let there exist an  $S$ -triad in  $H$ . Then  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$  for  $i = 1$  or  $3$ , or a weak  $i$ -enlargement of  $G$  for  $i = 8$  or  $9$ .*

*Proof:* We proceed by induction of  $|E(H)|$ . Let the  $S$ -triad be  $L_1, L_2, L_3$ , and let its feet be  $x_1, x_2, x_3$ . If each  $x_i$  is a branch-vertex of  $S$ , then  $S \cup L_1 \cup L_2 \cup L_3$  gives rise to a

minor of  $H$  isomorphic to a weak 8-enlargement, as desired. We may therefore assume that  $x_3$  is an internal vertex of a segment  $Q_3$  of  $S$  with ends  $u_3$  and  $v_3$ . Let  $f$  be an edge of  $Q_3$ . By induction applied to  $G$ ,  $H/f$ , and  $S/f$ , we may assume that  $f$  is incident with  $x_3$  and one end of  $Q_3$ , say  $u_3$ , and that there exists a disk  $D_1$  in  $G$  containing  $x_1, x_2, u_3$ . Similarly, we may assume that there exists a disk  $D_2$  in  $G$  containing  $x_1, x_2, v_3$ . Then  $D_1 \neq D_2$ , because otherwise  $D_1 = D_2$  includes the segment  $Q_3$  by (D3), and hence each of  $x_1, x_2, x_3$ , a contradiction. Since  $x_1$  and  $x_2$  belong to  $D_1 \cap D_2$ , it follows from (D3) that  $x_1$  and  $x_2$  belong to a common segment  $Q$  of  $S$ .

Let  $S'$  be obtained from  $S$  by replacing  $Q[x_1, x_2]$  by  $L_1 \cup L_2$ . Applying Lemma 2.5.1 to  $G, H, S'$  and the  $S'$ -path  $L_3$ , the lemma now follows.  $\square$

Using Theorem 2.3.1 we can summarize Lemmas 2.5.1–2.5.3 as follows.

**Lemma 2.5.4** *Let  $G, H$  be weakly 4-connected graphs, let  $G$  have a weak disk system  $\mathcal{D}$  with no locally planar extension into  $H$ , and let a subdivision of  $G$  be isomorphic to a subgraph of  $H$ . Then  $H$  has a minor isomorphic to*

- (i) *an  $i$ -enlargement of  $G$  for some  $i \in \{1, 2, 3, 11\}$ , or*
- (ii) *a weak  $i$ -enlargement of  $G$  for some  $i \in \{8, 9\}$ .*

*Proof:* Let  $G, H$  and  $\mathcal{D}$  be as stated. By Theorem 2.3.1 we deduce that there exists a subgraph  $S$  of  $H$  isomorphic to a subdivision of  $G$  such that the induced weak disk system in  $S$  satisfies one of the outcomes (1)–(4) of Theorem 2.3.1. If it satisfies (1)–(3), then (i) or (ii) of this lemma hold by Lemmas 2.5.1–2.5.3. If (4) of Theorem 2.3.1 holds, then  $H$  has a minor isomorphic to a 11-enlargement of  $G$ , as desired.  $\square$

## **2.6 From Weak Enlargements to Enlargements**

The purpose of this section is to replace weak enlargements by enlargements in Lemma 2.5.4(ii). We start with a special case of weak 9-enlargements.

**Lemma 2.6.1** *Let  $G$  be a graph with a cycle double cover  $\mathcal{D}$ , let  $G^+$  be a weak 9-enlargement of  $G$ , and let  $u, x, y$  be as in the definition of weak 9-enlargement. If  $G \setminus \{u, x, y\}$  is connected, then  $G^+$  has a minor isomorphic to an  $i$ -enlargement of  $G$  for some  $i \in \{1, 3, 9\}$ .*

*Proof:* Let  $z$  be the new vertex of  $G^+$  that resulted from the subdivision of the edge  $xy$ . If  $u$  and  $x$  are not confluent, then contracting the edge  $xz$  of  $G^+$  produces a 1-enlargement of  $G$ , and so the lemma holds. Thus we may assume that  $u$  and  $x$  are confluent, and, by symmetry, we may assume that  $u$  and  $y$  are confluent. If  $u$  is not adjacent to  $x$  or  $y$ , then  $G^+$  is a 9-enlargement of  $G$ , and the lemma holds. Thus, from the symmetry, we may assume that  $u$  is adjacent to  $x$ . The edges  $xy$  and  $xu$  are not confluent, for otherwise  $u$  is confluent with the edge  $xy$ , contrary to what a weak 9-enlargement stipulates. But then deleting the edge  $xu$  from  $G^+$  yields a graph isomorphic to a 3-enlargement of  $G$  — more specifically, a graph obtained by a non-conforming split of the vertex  $x$ .  $\square$

**Lemma 2.6.2** *Let  $G$  be an expansion of a weakly 4-connected graph, let  $H$  be a weakly 4-connected graph, let  $\mathcal{D}$  be a weak disk system in  $G$ , let  $v$  be a vertex of  $G$  of degree three and let  $u, x, y$  be the neighbors of  $v$ . Let  $G^+$  be the graph obtained from  $G$  by adding a new vertex  $z$  adjacent to  $u, x, y$  and deleting all edges with both ends in  $\{u, x, y\}$ . If  $H$  has a minor isomorphic to  $G^+$ , then  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$  for some  $i \in \{1, 2, 3, 4, 9, 12\}$ .*

*Proof:* Since  $G$  is an expansion of a weakly 4-connected graph, Lemma 2.4.2 implies that at most one edge of  $G$  has both ends in  $\{u, x, y\}$ . Thus we may assume that  $u$  is not adjacent to  $x$  or  $y$ . Since  $v$  has degree three, it follows from (D1) and (D3) that the triangle  $vxy$  is a disk in  $G$ . We can apply Lemma 2.4.6 to  $G^+ = G_1$  and  $H$ , with  $P = \{v, z\}$ ,  $Q = \{u, x, y\}$  and  $R = V(G^+) - (P \cup Q)$ . From the lemma, using the symmetry between  $x$  and  $y$ , and the symmetry among  $x, y$  and  $u$  if  $x$  is not adjacent to  $y$ , we get the following three cases:

**Case 1:**  $H$  has a minor isomorphic to a graph  $G^{++}$  that is obtained from  $G^+$  by adding an edge between a vertex  $p \in P$  and a vertex  $r \in R$ . Note that the vertices  $v$  and  $z$  are symmetric for the application of Lemma 2.4.6. Hence we may assume that  $p = v$ . Now if

$r$  is not confluent with  $v$  in  $G$ , then  $G^{++}$  above has a minor isomorphic to a 1-enlargement of  $G$ . Thus we may assume that  $r$  is confluent with  $v$  in  $G$ . Furthermore, we may assume, without loss of generality, that the disk  $D_3$  shared by  $r$  and  $v$  contains the edges  $vu$  and  $vy$ . (Note that  $v$  has degree 3 in  $G$ .) On the disk  $D_3$ , the vertices  $u, v, y$  and  $r$  occur in that cyclic order. Now in  $G^{++}$ , contracting the edge  $yz$  gives a cross in the disk  $D_3$  with arms  $uy$  and  $rv$ . In other words,  $G^{++}$ , and hence  $H$ , has a minor isomorphic to a 2-enlargement of  $G$ , as desired.

**Case 2:** The vertices  $x$  and  $y$  are adjacent in  $G$  and  $H$  has a minor isomorphic to a graph  $G^{++}$  that is obtained from  $G^+$  by splitting the vertex  $x$  into  $x_1$  and  $x_2$ , with  $x_1$  adjacent to  $v$  and  $x_2$  adjacent to  $z$ . Let  $N_i$  be the neighbors of  $x_i$  in  $G^{++}$  other than  $v, z, x_1, x_2$ . The neighborhood of  $x$  in  $G$  is thus  $N_1 \cup N_2 \cup \{v, y\}$ . In  $G$ , let  $D_4$  be the disk that contains the edge  $xy$ , other than the triangle  $vxy$ . The disk  $D_4$  must contain a vertex in either  $N_1$  or  $N_2$ , and from the symmetry between  $v$  and  $z$  we may assume that it contains a vertex in  $N_1$ . Then, in  $G^{++}$ , delete the edge  $uz$  and contract the edge  $x_2z$ . This gives a graph that is a 3-enlargement of  $G$  (non-conforming split of  $x$ , with the disks  $vxy$  and  $D_4$  violating condition (S2) in the definition of a conforming split), as desired.

**Case 3:**  $H$  has a minor isomorphic to a graph  $G^{++}$  that is obtained from  $G^+$  by splitting the vertex  $u$  into  $u_1$  and  $u_2$ , with  $u_1$  adjacent to  $v$  and  $u_2$  adjacent to  $z$ . Let  $N_i$  be the set of neighbors of  $u_i$  other than  $v, z, u_1, u_2$ . Thus in  $G$ , the neighborhood of  $u$  is  $N_1 \cup N_2 \cup \{v\}$ .

Let  $D_1$  be the disk in  $G$  shared by the edges  $xv$  and  $vu$ , and  $D_2$  be the disk in  $G$  shared by the edges  $yv$  and  $vu$ . The disks  $D_1$  and  $D_2$  both contain exactly one vertex each from  $N_1 \cup N_2$ . Let us assume first that  $|N_2| \geq 2$ . Contract the edge  $xz$  in  $G^{++}$ , and if  $x$  is not adjacent to  $y$  in  $G$ , then delete also the resulting edge  $xy$  to obtain a graph  $G_1$ , and let  $G_2$  be the graph obtained from  $G_1$  by further deleting the edge  $u_2x$ . Now  $G_2$  is isomorphic to a graph obtained from  $G$  by splitting the vertex  $u$  into  $u_1$  and  $u_2$ . If this split is non-conforming, then  $G_2$  is a 3-enlargement of  $G$ , and we are done. Otherwise, the split is not along  $D_1$  or  $D_2$ , and from the symmetry we may assume it is not along  $D_1$ . Thus  $G_1$  is a 4-enlargement of  $G$ . (Note that in  $G$ ,  $u$  and  $x$  are non-adjacent, and hence non-consecutive on  $D_1$ .) This completes the case when  $|N_2| \geq 2$ .

From the symmetry we may therefore assume that  $|N_1| = |N_2| = 1$ . Thus the degree of  $u$  in  $G$  is three. For  $i = 1, 2$  let  $N_i = \{n_i\}$ . We may assume that the edge  $un_i$  belongs to the disk  $D_i$ . It follows that the vertex  $x$  and edge  $un_2$  are not confluent in  $G$ , for if some disk  $D$  contained both of them, then the intersection  $D \cap D_1$  would violate (D3), because  $u$  is not adjacent to  $x$ . The graph  $G_1$  from the previous paragraph is a weak 9-enlargement of  $G$ , and so by Lemma 2.6.1 we may assume that  $G \setminus \{x, u, n_2\}$  is disconnected. Since  $u$  has degree three, the weak 4-connectivity of  $G$  implies that  $n_1$  has degree three and its neighbors are  $x, u, n_2$ . Since  $G \setminus \{n_2, y\}$  is connected, we deduce that  $G$  is isomorphic to the prism, and  $G^{++}$  is isomorphic to a 12-enlargement of  $G$ , as desired.  $\square$

Now we are ready to eliminate weak 9-enlargements.

**Lemma 2.6.3** *Let  $G$  be an expansion of a weakly 4-connected graph, let  $\mathcal{D}$  be a weak disk system in  $G$ , and let  $G^+$  be a weak 9-enlargement of  $G$  such that  $G^+$  is isomorphic to a minor of a weakly 4-connected graph  $H$ . Then  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$  for some  $i \in \{1, 2, 3, 4, 9, 12\}$ .*

*Proof:* Let  $u, x, y$  be as in the definition of weak 9-enlargement. By Lemma 2.6.1 we may assume that  $G \setminus \{u, x, y\}$  is disconnected. Since  $x$  is adjacent to  $y$  and  $G$  is an expansion of a weakly 4-connected graph, Lemma 2.4.2 implies that the neighborhood of some vertex  $v$  of  $G$  is precisely the set  $\{u, x, y\}$ . Thus  $G^+$  is as described in Lemma 2.6.2, and the conclusion follows from that lemma.  $\square$

We now turn to weak 8-enlargements. In order to save effort we prove a weaker analogue of Lemma 2.6.3, the following.

**Lemma 2.6.4** *Let  $G_1$  be an expansion of a weakly 4-connected graph  $G$ , let  $\mathcal{D}$  be a weak disk system in  $G$ , and let  $G^+$  be a weak 8-enlargement of  $G_1$  such that  $G^+$  is isomorphic to a minor of a weakly 4-connected graph  $H$ . Then there exists an expansion  $G_2$  of  $G$  obtained from  $G_1$  by contracting a possibly empty set of new edges such that  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G_2$  for some  $i \in \{1, 2, 3, 4, 8, 9, 12\}$ .*

*Proof:* We proceed by induction on  $|E(G_1)|$ . Let  $G^+$  be obtained from  $G_1$  by adding a vertex joined to  $v_1, v_2, v_3$ . If some edge of  $G_1$  has both ends in the set  $\{v_1, v_2, v_3\}$ , then by deleting that edge we obtain a graph isomorphic to a weak 9-enlargement of  $G_1$ , and the lemma follows from Lemma 2.6.3. Thus we may assume that  $\{v_1, v_2, v_3\}$  is an independent set in  $G_1$ . We may also assume that every pair of vertices in  $\{v_1, v_2, v_3\}$  is confluent, for otherwise  $G^+$  has a minor isomorphic to a 1-enlargement of  $G$ , and the lemma holds. Thus we may assume that  $G \setminus \{v_1, v_2, v_3\}$  is disconnected, for otherwise  $G^+$  is an 8-enlargement of  $G_1$ .

Let  $(A, B)$  be a non-trivial separation of  $G$  with  $A \cap B = \{v_1, v_2, v_3\}$ . By Lemma 2.4.2 we may assume that  $(A, B)$  is degenerate. If  $|A - B| = 1$ , then the lemma follows from Lemma 2.6.2. Thus we may assume that  $|A - B| \geq 2$ . Let  $v_1, v_2, v_3, u_1, u_2, u_3$  be as in the definition of degenerate. Since  $\{v_1, v_2, v_3\}$  is independent, we may assume from the symmetry that  $u_1 \neq v_1$  and  $u_2 \neq v_2$ . Now one of  $u_1v_1, u_2v_2$  is a new edge of  $G_1$ , and so we may assume the former is. Thus  $G^+ / u_1v_1$  is a weak 8-enlargement of  $G_1 / u_1v_1$ , and hence the lemma follows by the induction hypothesis applied to the graph  $G_1 / u_1v_1$ .  $\square$

The lemmas of this section allow us to upgrade Lemma 2.5.4 to the following.

**Lemma 2.6.5** *Let  $G, H$  be weakly 4-connected graphs, let  $G$  have a weak disk system  $\mathcal{D}$  with no locally planar extension into  $H$ , and let  $G'$  be a conforming expansion of  $G$  such that a subdivision of  $G'$  is isomorphic to a subgraph of  $H$ . Then there exists a conforming expansion  $G''$  of  $G$  obtained from  $G'$  by contracting a possibly empty set of new edges such that, letting  $\mathcal{D}''$  denote the weak disk system induced in  $G''$  by  $\mathcal{D}$ , the graph  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G''$  with respect to  $\mathcal{D}''$  for some  $i \in \{1, 2, 3, 4, 8, 9, 11, 12\}$ .*

*Proof:* By Lemma 2.5.4 we may assume that a weak 8-enlargement or a weak 9-enlargement of  $G'$  is isomorphic to a minor of  $H$ . By Lemmas 2.6.3 and 2.6.4 there exists a required conforming expansion  $G''$  of  $G$  such that  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G''$  for some  $i \in \{1, 2, 3, 4, 8, 9, 12\}$ .  $\square$

## 2.7 Proof of the Main Theorem

Lemma 2.6.5 gives an  $i$ -enlargement of an expansion  $G''$  of  $G$ . Our final objective is to show that we can choose  $G'' = G$ . We break the proof into several lemmas depending on the value of  $i$ .

**Lemma 2.7.1** *Let  $G$  and  $H$  be weakly 4-connected graphs, and let  $\mathcal{D}$  be a weak disk system in  $G$  with no locally planar extension into  $H$ . Let  $G'$  be a conforming expansion of  $G$  such that  $H$  has a minor isomorphic to a 1-enlargement of  $G'$ . Then  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$  for some  $i \in \{1, 3, 4, 5\}$ .*

*Proof:* We may assume that  $G'$  is as stated in the lemma, and subject to that, it is minor-minimal. By hypothesis,  $H$  has a minor isomorphic to  $G^+$ , a graph obtained from  $G'$  by adding an edge between two vertices  $x$  and  $y$  that are not confluent. Let  $e$  be a new edge of  $G'$ . By the minimality of  $G'$ , it follows that

- (i) one end of  $e$  must be in  $\{x, y\}$ , and
- (ii) the other end of  $e$  must be confluent with the vertex in  $\{x, y\}$  other than the one above.

Recall that branch-sets of an expansion were defined at the beginning of Section 2.4. Thus all branch sets that are disjoint from  $\{x, y\}$  are singleton sets. Let  $T_p$  and  $T_q$  be the branch sets corresponding to vertices  $p, q \in V(G)$  such that they contain  $x$  and  $y$  respectively ( $p$  and  $q$  may be identical). We claim that the degree of  $x$  in the branch set containing it is at most one (that is,  $x$  is a leaf of the tree  $G'[T_p]$ ). Suppose not; hence  $x$  has (at least) two neighbors  $x_1$  and  $x_2$  in  $T_p$ . By (ii) above,  $y$  shares disks  $D_1$  and  $D_2$  of  $G'$  with  $x_1$  and  $x_2$  respectively. Then  $x \notin V(D_1 \cup D_2)$ , for  $x, y$  are not confluent. It follows that  $D_1 \neq D_2$ , for otherwise  $D_1$  is not a cycle in  $G'/x_1x/x_2x$ , and yet  $D_1$  corresponds to a disk in  $G$ . Also,  $y$  is not adjacent to both  $x_1$  and  $x_2$ , by Lemma 2.4.1. But then contracting edges  $xx_1$  and  $xx_2$  violates Axiom (D3) in  $G$ . This proves the claim. Thus  $x$ , and by symmetry  $y$ , are leaf vertices in  $G'[T_p]$  and  $G'[T_q]$  respectively.

If  $p = q$ , then it follows that  $T_p = T_q$  must be a path of length 2, with a middle vertex  $z$ . Let  $D'_1, D'_2$  be the two disks in  $G'$  that include the edge  $xz$ , and let  $D'_3, D'_4$  be the two disks that include the edge  $yz$ . Note that, since  $x$  and  $y$  are not confluent in  $G'$ , all four disks above are distinct. Let  $D_1, D_2, D_3, D_4$  be the corresponding disks in  $G$ . Let  $N_1, N_2$  be the partition of the set of neighbors of  $p$  in  $G$ , corresponding to the partition  $\{x, y\}, \{z\}$  of  $V(T_p)$ . Clearly,  $N_1$  has at least two vertices, but so does  $N_2$ , by Axiom (D3) applied to  $\tilde{D}_1, \tilde{D}_2$ . In  $G^+$  (which has the edge  $xy$ ), contract the edge  $xy$ . This gives a graph  $G^{++}$  that can be obtained from  $G$  by splitting  $p$  with respect to the partition  $N_1, N_2$  of its neighbors. This split is non-conforming, since the disks  $D_1, \dots, D_4$  violate condition (S1) in the definition of a conforming split. Thus  $G^{++}$  is a 3-enlargement of  $G$ , as desired.

If  $p \neq q$ , then from (i) and (ii) above,  $T_p$  is either  $\{x\}$  or  $\{x, x_1\}$ . By symmetry,  $T_q$  is either  $\{y\}$  or  $\{y, y_1\}$ . If  $T_p$  and  $T_q$  are both singletons, then clearly  $G' = G$  and we are done.

Suppose exactly one of the two branch sets, say  $T_q$ , is a singleton, and  $T_p$  consists of  $\{x, x_1\}$ , where  $x_1$  shares a disk  $D$  with  $y$  in  $G'$ . If  $x_1$  and  $y$  are not adjacent, then  $G^+$  is a 4-enlargement of  $G$ , and we are done. Thus we may assume that  $x_1$  and  $y$  are adjacent, and hence by Axiom (D3), they are consecutive in  $D$ . Let  $D_1, D_2$  be the two disks in  $G'$  containing the edge  $xx_1$ . They are both distinct from  $D$ , since  $x$  and  $y$  are not confluent in  $G'$ . By Axiom (D3) applied to  $D_1$  and  $D_2$ , the vertex  $x_1$  has at least two neighbors in  $G'$  other than  $x$  and  $y$ . Now in  $G^+$  (which contains the edge  $xy$ ), delete the edge  $x_1y$ . This gives a graph  $\tilde{G}$  obtained from  $G$  by splitting  $p$  in the same way as in  $G'$ , except that  $y$  is adjacent to  $x$  rather than  $x_1$ . Further, it is a non-conforming split, as the disks  $D, D_1$  and  $D_2$  violate condition (S1) in the definition of a conforming split. Thus  $\tilde{G}$ , which is isomorphic to a minor of  $H$ , is a 3-enlargement of  $G$ , and we are done.

Finally, suppose  $T_p = \{x, x_1\}$  and  $T_q = \{y, y_1\}$ , where  $x$  shares a disk  $D'_1$  with  $y_1$  and  $y$  shares a disk  $D'_2$  with  $x_1$ . Let  $D_1, D_2$  be the corresponding disks in  $G$ . Since  $x$  and  $y$  are not confluent in  $G'$ ,  $D'_1$  does not contain  $y$  and  $D'_2$  does not contain  $x$ . (In particular,  $D'_1$  and  $D'_2$  are distinct.) Apply Lemma 2.4.5 to  $\hat{G} = G'/xx_1$ , with the vertices  $p, y, y_1$  in that graph corresponding to  $p, q, r$  in the lemma. Thus the (conforming) split of the vertex  $q$  in  $G$  that produces  $\hat{G}$  is along  $D_2$ , and  $D_2$  is one of the disks containing the edge  $pq$  in  $G$ .

Also, since  $x$  and  $y$  are not confluent in  $G'$ , the (conforming) split of  $p$  in  $\widehat{G}$  that produces  $G'$  must be along  $D_1$ , and  $D_1$  is the other disk in  $G$  containing  $pq$ . It now follows that  $G^+$  is a 5-enlargement of  $G$ . This finishes the proof of the lemma.  $\square$

**Lemma 2.7.2** *Let  $G$  and  $H$  be weakly 4-connected graphs, and let  $\mathcal{D}$  be a weak disk system in  $G$  with no locally planar extension into  $H$ . Let  $G'$  be a conforming expansion of  $G$  such that  $H$  has a minor isomorphic to a 2-enlargement of  $G'$ . If  $G' \neq G$ , then there exists a conforming expansion  $G''$  of  $G$  obtained from  $G'$  by contracting at least one new edge such that  $H$  has a minor isomorphic to an  $i$ -enlargement or a weak 9-enlargement of  $G''$  for some  $i \in \{2, 6, 7\}$ .*

*Proof:* We may assume that  $G'$  is as stated in the lemma, and subject to that, it is minor-minimal. By hypothesis, there are vertices  $u, v, x, y$  appearing on a disk  $C'$  in  $G'$ , in that cyclic order, such that  $H$  has a minor isomorphic to a graph obtained from  $G'$  by adding the edges  $ux$  and  $vy$ . Let  $C$  be the cycle in  $G$  corresponding to  $C'$ . The minimality of  $G'$  implies that every new edge of  $G'$  has both ends in  $\{u, v, x, y\}$ , and hence it belongs to  $C'$  by (D3). We may therefore assume that  $uv$  is a new edge of  $G'$ . We claim that if  $v$  is adjacent to  $x$ , then the lemma holds. To prove this claim suppose that  $v$  and  $x$  are adjacent in  $G'$ , and let  $G_1 = G^+ \setminus vx$ . If  $v$  has degree three in  $G'$ , then  $G_1$  is isomorphic to a weak 9-enlargement of  $G'/uv$  (the new edge is  $yv$ ; notice that  $y$  is not confluent with the edge of  $G'/uv$  that is being subdivided by (D3)), and hence the lemma holds. Thus we may assume that  $v$  has degree at least four in  $G'$ . In that case  $G_1$  is isomorphic to a 4-enlargement of  $G'/uv$ , for a graph isomorphic to  $G_1$  can be obtained by a conforming split of the new vertex of  $G'/uv$ , not along  $C'$ , and joining one of the new vertices to  $y$ . This proves our claim, and hence we may assume that  $v$  is not adjacent to  $x$ . By symmetry we may also assume that  $u$  is not adjacent to  $y$ .

If  $uv$  is the only new edge of  $G'$ , then  $G'$  is a 6-enlargement of  $G$ , and the lemma holds. Thus we may assume that  $G'$  has another new edge, and so that edge must be  $xy$  and there are no other new edges. It follows that  $G'$  is a 7-enlargement of  $G$ , and so the lemma holds.  $\square$

**Lemma 2.7.3** *Let  $G$  and  $H$  be graphs, let  $\mathcal{D}$  be a weak disk system in  $G$ , and let  $G'$  be a conforming expansion of  $G$  such that  $H$  has a minor isomorphic to a 9-enlargement  $G^+$  of  $G'$ . If  $G' \neq G$ , then there exists a conforming expansion  $G''$  obtained from  $G'$  by contracting at least one new edge such that  $H$  has a minor isomorphic to a 3-enlargement or a weak 9-enlargement of  $G''$ .*

*Proof:* Let  $u, x, y \in V(G')$  be such that  $G^+$  is obtained from  $G'$  by subdividing the edge  $xy$  and joining the new vertex to  $u$ , and let  $f$  be a new edge of  $G'$ . Then  $f \neq xy$ , for otherwise Lemma 2.4.5 implies that  $u$  is confluent with the edge  $xy$ , a contradiction. We may assume that  $f$  is incident with  $u$ , and that contracting  $f$  makes the new vertex confluent with the edge  $xy$ , for otherwise  $G^+/f$  is a weak 9-enlargement of  $G'/f$ , and the lemma holds. Hence the other end  $v$  of  $f$  must share a disk  $D_1$  with the edge  $xy$ . Since  $u$  is not confluent with  $xy$ ,  $D_1$  does not contain  $u$ . Let  $D_2$  and  $D_3$  be disks shared by  $u$  and  $x$ , and by  $u$  and  $y$ , respectively. These three disks are pairwise distinct, since  $u$  is not confluent with the edge  $xy$  in  $G'$ . Now apply Lemma 2.4.5 with  $x$  as the vertex  $p$ , and  $u, v$  as the vertices  $q, r$  respectively. It follows that  $v$  and  $x$  are adjacent, and that  $D_1$  and  $D_2$  are the two disks containing the edge  $vx$ . Apply Lemma 2.4.5 again, this time with  $y$  in place of  $x$ . It follows that the edges  $vu, vx$  and  $vy$  are covered twice each by the three disks  $D_1, D_2$  and  $D_3$ . In particular,  $D_1$  is a triangle.

If  $f' \neq f$  is a new edge of  $G'$ , then by what we have shown about  $f$  it follows that  $f'$  is incident with  $u$  and its other end belongs to a disk  $D'_1$  that contains the edge  $xy$ . Since  $D_1$  is a triangle consisting of  $x, y$  and an end of  $f$ , we see that  $D'_1 \neq D_1$ . But the disks that correspond to  $D_1$  and  $D'_1$  in  $G'/f/f'$  have three vertices in common, contrary to (D3). Thus  $f$  is the only new edge of  $G'$ , and hence  $G = G'/f$ . Let  $p$  be the new vertex of  $G = G'/f$ .

Since  $G^+$  is a 9-enlargement of  $G'$ , the graph  $G' \setminus \{u, x, y\}$  is connected, and hence  $v$  has a neighbor outside  $\{u, x, y\}$ . (In fact, it must then have at least three neighbors outside  $\{u, x, y\}$ .) Let  $z$  be the new vertex of  $G^+$  created by subdividing the edge  $xy$ . The graph  $G^+ \setminus vx/xz$  is isomorphic to a graph obtained from  $G$  by splitting  $p$  into two vertices. This split is non-conforming, since the two disks in  $G$  that contain  $py$  violate condition (S2) in the definition of a conforming split. Thus  $H$  has a minor isomorphic to a 3-enlargement of

$G$ . This finishes the proof of the lemma.  $\square$

**Lemma 2.7.4** *Let  $G$  and  $H$  be graphs, let  $\mathcal{D}$  be a weak disk system in  $G$ , and let  $G'$  be a conforming expansion of  $G$  such that  $H$  has a minor isomorphic to a 8-enlargement  $G^+$  of  $G'$ . If  $G' \neq G$ , then there exists a conforming expansion  $G''$  obtained from  $G'$  by contracting at least one new edge such that  $H$  has a minor isomorphic to a 3-enlargement, 10-enlargement, or a weak 9-enlargement of  $G''$ .*

*Proof:* Let  $G^+$  be obtained from  $G'$  by adding a vertex adjacent to  $x_1, x_2, x_3$ , and let  $f$  be a new edge of  $G'$ . We may assume that upon contracting  $f$  the vertices that correspond to  $x_1, x_2, x_3$  belong to a common disk, for otherwise  $G^+/f$  is a weak 8-enlargement of  $G'/f$ , and the lemma holds. Thus  $f$  is incident with at least one of  $x_1, x_2, x_3$ , say  $x_1$ , and there exists a disk  $\mathcal{D}$  in  $G'$  that includes  $y, x_2, x_3$ , where  $y$  is the other end of  $f$ .

Apply Lemma 2.4.5 twice, once with  $x_2$  as the vertex  $p$ , and next with  $x_3$  as the vertex  $p$ . In both applications, let  $x_1$  and  $y$  be the vertices  $q$  and  $r$  respectively. It follows that  $y$  is adjacent to  $x_2$  and  $x_3$ , and that  $yx_1 \in E(D_2 \cap D_3)$ ,  $yx_2 \in E(D \cap D_3)$  and  $yx_3 \in E(D \cap D_2)$ . Since  $G^+$  is a 8-enlargement of  $G'$  the graph  $G' \setminus \{x_1, x_2, x_3\}$  is connected, and hence  $y$  has degree at least four. Let  $N$  be the neighbors of  $y$  in  $G'$  other than  $x_1, x_2, x_3$ . Let  $G'$  be obtained from  $G$  by splitting  $x_1$  in such a way that the neighborhood of one of the new vertices is  $N$ . Then  $G'$  is isomorphic to a minor of  $G^+$ , and it is a 3-enlargement of  $G'$ . Thus the lemma follows from Lemma 2.4.4.  $\square$

We are finally ready to state and prove the generalization of Theorem 2.2.1 mentioned earlier, and then deduce Theorem 2.2.1.

**Theorem 2.7.5** *Let  $G$  and  $H$  be weakly 4-connected graphs such that  $H$  has a minor isomorphic to  $G$ . Let  $G$  have a weak disk system  $\mathcal{D}$  that has no locally planar extension into  $H$ . Then  $H$  has a minor isomorphic to an  $i$ -enlargement of  $G$ , for some  $i \in \{1, 2, \dots, 12\}$ .*

*Proof:* There exists an expansion of  $G$  whose subdivision is isomorphic to a subgraph of  $H$ . If this expansion is not conforming, then the theorem holds by Lemma 2.4.4, and so we may assume that the expansion is conforming. By Lemma 2.6.5 there exists a conforming

expansion  $G'$  of  $G$  such that  $H$  has a minor isomorphic to an  $i$ -enlargement  $G^+$  of  $G'$  for some  $i \in \{1, 2, \dots, 12\}$ . We may choose  $G'$  and  $G^+$  such that  $|E(G')|$  is minimum. If  $i \in \{1, 4, 5, 6, 7\}$ , then  $G^+$  is isomorphic to a 1-enlargement of a conforming expansion of  $G'$ , and the theorem holds by Lemma 2.7.1. If  $i \in \{3, 10\}$ , then the theorem holds by Lemma 2.4.4. If  $i = 12$ , then the minimality of  $G'$  implies that  $G = G'$ , and if  $i \in \{2, 8, 9, 11\}$ , then the same conclusion follows from Lemmas 2.7.2, 2.7.4, 2.7.3 and 2.4.1, respectively, using Lemmas 2.6.3 and 2.6.4. Thus the theorem holds.  $\square$

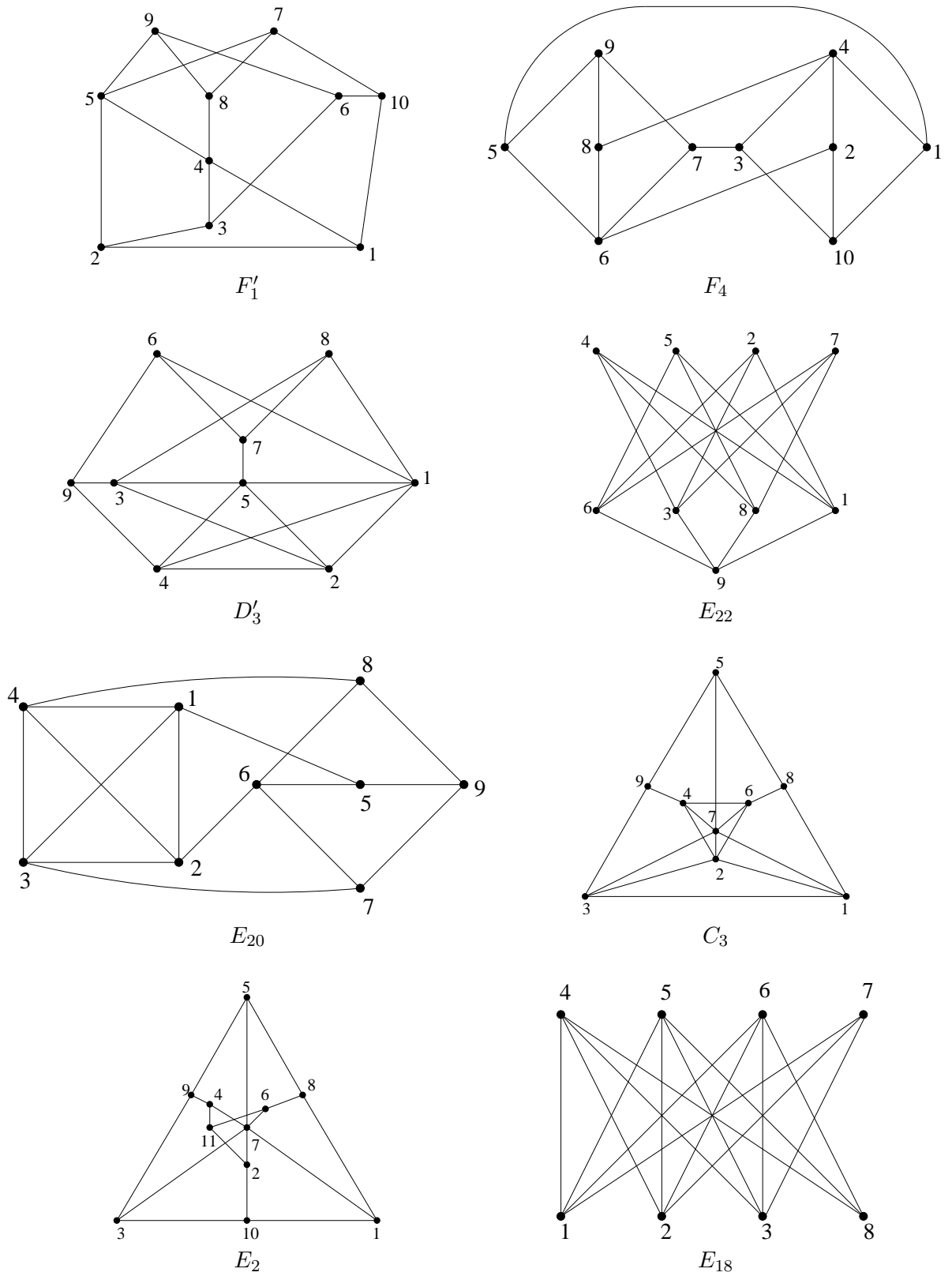
**Proof of Theorem 2.2.1.** Let  $G, \mathcal{D}$  and  $H$  be as in Theorem 2.2.1. By Theorem 2.7.5 it suffices to show that  $G$  has no  $i$ -enlargement for  $i \in \{10, 11, 12\}$ . This is clear for  $i = 12$ , and follows easily from (D2) when  $i = 10$  or  $i = 11$ , as desired.  $\square$

## 2.8 An Application

In this section, we illustrate an application of Theorem 2.2.1. The projective plane is the non-orientable surface obtained from a closed disk by identifying all pairs of antipodal points on its boundary. Archdeacon [2, 3] proved that a graph  $H$  does not embed in the projective plane if and only if it has a minor isomorphic to some graph in an explicitly constructed list of 35 graphs. One might hope that if we assume that  $H$  is sufficiently connected, then the list may be shortened. Mohar and Thomas (work in progress) developed a strategy for a proof, but it will be a lengthy project with several intermediate steps. Here we complete one such step: under the assumptions that  $H$  is weakly 4-connected and has a minor isomorphic to the Petersen graph, Theorem 2.8.1 below gives a list of eight forbidden minors, each of which are weakly 4-connected.

Figure 2.1 shows these eight graphs (with a vertex-labeling for each of them). All of these graphs, with the exception of  $F'_1$  and  $D'_3$ , appear in the list of 35 forbidden minors for the projective plane.  $F'_1$  and  $D'_3$ , however, are obtained from two graphs in that list ( $F_1$  and  $D_3$ , respectively) by splitting exactly one vertex. (The reason we list  $F'_1, D'_3$  instead of  $F_1, D_3$  is that the latter two graphs are not weakly 4-connected.)

**Theorem 2.8.1** *Let  $H$  be a weakly 4-connected graph that has a minor isomorphic to the*

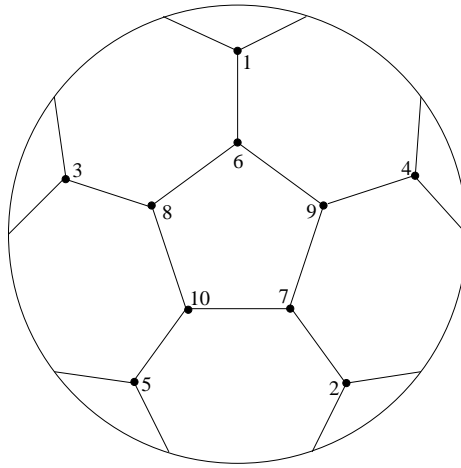


**Figure 2.1:** The eight graphs of Theorem 2.8.1

*Petersen graph. Then  $H$  does not embed in the projective plane if and only if it has a minor isomorphic to one of the eight graphs  $F'_1, F_4, D'_3, E_{22}, E_{20}, C_3, E_2,$  or  $E_{18}$  shown in Figure 2.1.*

Before we derive Theorem 2.8.1 from Theorem 2.2.1, we describe some notation that will be convenient in the proof.

Let  $P_{10}$  denote a labeling of the Petersen graph as shown in Figure 2.2. In fact, Figure 2.2 shows an embedding of  $P_{10}$  in the projective plane. The disk system  $\mathcal{D}$  associated with this embedding consists of the 5-cycles 6-9-7-10-8, 1-5-10-7-2, 4-3-8-10-5, 2-1-6-8-3, 5-4-9-6-1, and 3-2-7-9-4.



**Figure 2.2:** One of the two projective-planar embeddings of the Petersen graph

$P_{10}$  has exactly one other embedding in the projective plane. This embedding is distinct from the above embedding, but is isomorphic to it. (An isomorphism of embeddings is an isomorphism  $\tau$  of the underlying graphs such that a cycle  $C$  is facial in one embedding if and only if  $\tau(C)$  is facial in the other.) The disk system  $\mathcal{D}'$  associated with the second embedding consists of the 5-cycles 1-2-3-4-5, 6-9-4-3-8, 7-10-5-4-9, 8-6-1-5-10, 9-7-2-1-6, and 10-8-3-2-7.

We now describe notation that will let us denote specific enlargements of a (labeled) graph as given by Theorem 2.2.1. Recall the operations 1–9 and the definition of a split, as described in Sections 2.1 and 2.2.

Let  $G$  be a graph whose vertices are labeled  $1, \dots, n$ . For vertices  $u, v$ , the graph

$G+(u, v)$  denotes the graph obtained from  $G$  by adding an edge joining  $u$  and  $v$  (if none existed before). Also, the graph  $G*v(N_1)$  denotes the graph obtained by splitting the vertex  $v$ , where  $N_1$  is as in the definition of a split. We follow the convention that the vertex  $v_1$  retains the same label as  $v$ , while  $v_2$  is assigned the label  $n + 1$ .

Since operations 1–7 are defined in terms of vertex splits and edge additions, the above notation lets us specify  $i$ -enlargements for  $i = 1, \dots, 7$ . An 8-enlargement of  $G$  is specified as  $G+(x_1, x_2, x_3)$ , where the vertices  $x_i$  are as in the definition of operation 8. The new vertex  $x$  gets the label  $n + 1$ .

Finally, a 9-enlargement of  $G$  is specified as  $G+(u, x-y)$ , where  $u, x, y$  are as in the definition of operation 9. The new vertex obtained by subdividing the edge  $xy$  gets the label  $n + 1$ .

### 2.8.1 Proof of Theorem 2.8.1.

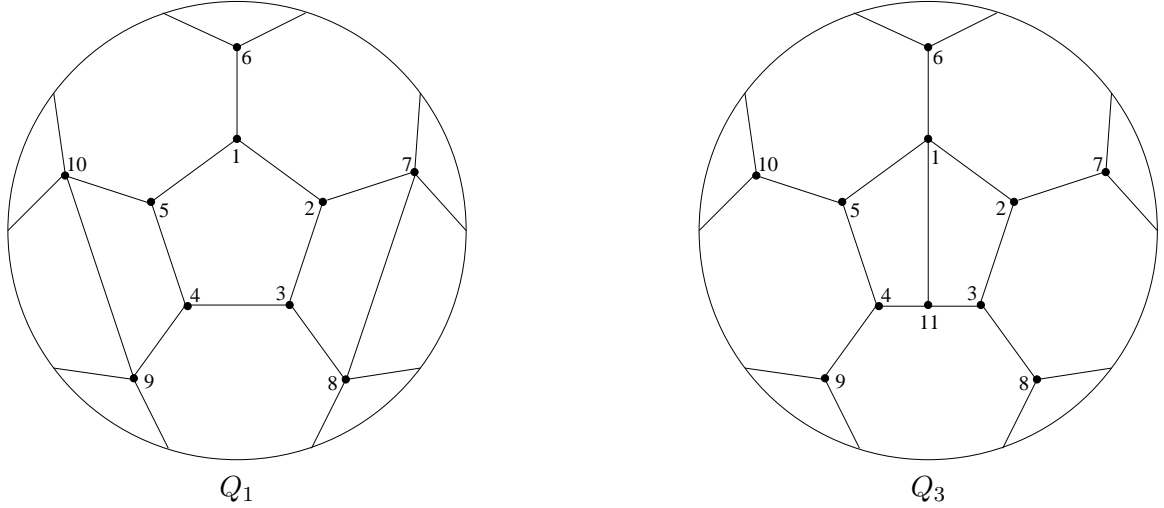
For the backward implication of Theorem 2.8.1, recall that each of the eight graphs specified is either isomorphic to one of the 35 forbidden minors of [3] or is obtained from one of them by splitting a vertex. In particular, none of these eight graphs embed in the projective plane, and so  $H$  does not embed either.

For the forward implication,  $H$ , by hypothesis, does not embed in the projective plane, and has a minor isomorphic to  $P_{10}$ . Clearly, the disk system  $\mathcal{D}$  of  $P_{10}$  has no locally planar extension to  $H$ . Applying Theorem 2.2.1 to  $P_{10}, \mathcal{D}$  and  $H$ , it is easy to check that  $H$  has a minor isomorphic to one of three enlargements, up to isomorphism:

1. a 2-enlargement  $Q_1 = P_{10} + (7, 8) + (9, 10)$
2. an 8-enlargement  $Q_2 = P_{10} + (2, 4, 6)$
3. a 9-enlargement  $Q_3 = P_{10} + (1, 3 - 4)$

$Q_2$  has a minor isomorphic to  $E_{18}$ , as witnessed by the branch sets  $\{1, 5\}, \{3, 8\}, \{7, 9\}, \{2\}, \{4\}, \{6\}, \{10\}$ , and  $\{11\}$ . (The order of the branch sets follows that of the corresponding vertex labels in  $E_{18}$ , as shown in Figure 2.1.)

Thus we may assume that  $H$  has a minor isomorphic to  $Q_1$  or  $Q_3$ . The disk system  $\mathcal{D}'$  of  $P_{10}$  extends in a natural way to disk systems  $\mathcal{D}_1, \mathcal{D}_3$  in the enlargements  $Q_1, Q_3$ . Thus  $Q_1, Q_3$  each embed (*uniquely*) in the projective plane. The embeddings are shown in Figure 2.3.



**Figure 2.3:** The graphs  $Q_1$  and  $Q_3$

We now apply Theorem 2.2.1 to  $Q_1, \mathcal{D}_1, H$  and  $Q_3, \mathcal{D}_3, H$  and deduce Theorem 2.8.1. This involves a fair amount of case-checking, which is summarized in Tables 2.1 and 2.2. Each row in the tables lists an enlargement of  $Q_1$  or  $Q_3$ , along with one of the eight graphs from the list that is a minor of the enlargement. The branch sets in the rightmost column follow the order of the vertex labels of the corresponding graph in the preceding column. For clarity, singleton sets are not enclosed in braces.

Tables 2.1 and 2.2 respectively list all possible enlargements of  $Q_1$  and  $Q_3$  up to isomorphism, with the exception of 8-enlargements and 9-enlargements of  $Q_1$ , and 8-enlargements of  $Q_3$ . Every 8-enlargement of  $Q_1$  with respect to  $\mathcal{D}_1$  has a subgraph isomorphic to  $Q_2$ , and thus has a minor isomorphic to  $E_{18}$ . Every 8-enlargement of  $Q_3$  with respect to  $\mathcal{D}_3$  either has a minor isomorphic to  $Q_2$ , or is isomorphic to the 8-enlargement listed in Table 2.2. Finally, every 9-enlargement of  $Q_1$  with respect to  $\mathcal{D}_1$  is either isomorphic to the 9-enlargement listed in Table 2.1 or is isomorphic to a 2-enlargement of  $Q_3$  with respect to  $\mathcal{D}_3$  (and is thus listed in Table 2.2 instead). This finishes the proof of Theorem 2.8.1.  $\square$

**Table 2.1:** Applying Theorem 2.2.1 to  $Q_1$

Type	Enlargement	Minor	Branch sets of the minor
1	$Q_1+(2, 10)$	$D'_3$	$\{2, 3\}, 7, 9, 8, 10, 1, 5, 4, 6$
	$Q_1+(3, 10)$	$F'_1$	$8, 7, 2, 3, 10, 1, 9, 4, 5, 6$
2	$Q_1+(2, 8)+(3, 7)$	$E_{20}$	$2, 7, 3, 8, \{1, 6\}, 9, 4, 10, 5$
	$Q_1+(2, 4)+(3, 5)$	$E_{22}$	$2, 3, 5, 4, 7, 8, 10, 9, \{1, 6\}$
	$Q_1+(1, 4)+(3, 5)$	$F_4$	$2, 4, 5, 1, 7, 9, 10, 6, 8, 3$
	$Q_1+(1, 4)+(2, 5)$	$F_4$	$1, 3, 5, 2, 6, 8, 10, 7, 9, 4$
	$Q_1+(3, 9)+(4, 8)$	$C_3$	$3, 4, 1, 10, 7, 9, 2, 5, \{6, 8\}$
	$Q_1+(3, 9)+(4, 6)$	$C_3$	$3, 4, 1, 10, 7, 9, 2, 5, \{6, 8\}$
	$Q_1+(4, 6)+(8, 9)$	$E_{20}$	$8, 7, 10, 9, 3, \{1, 2\}, 5, 6, 4$
	$Q_1+(2, 9)+(6, 7)$	$D'_3$	$\{1, 2\}, 7, 10, 6, 9, 3, 4, 5, 8$
	$Q_1+(1, 9)+(6, 7)$	$F'_1$	$1, 5, 4, 9, 10, 3, 7, 6, 8, 2$
	$Q_1+(1, 9)+(2, 6)$	$F_4$	$1, 10, 4, 9, 6, 8, 3, 7, 2, 5$
	$Q_1+(1, 7)+(2, 9)$	$D'_3$	$9, 7, 8, 2, \{1, 6\}, 4, 5, 10, 3$
	$Q_1+(1, 7)+(2, 6)$	$C_3$	$1, 2, 4, 8, 10, 7, 5, 3, \{6, 9\}$
3	$Q_1*7(2, 10)$	$F'_1$	$\{1, 6\}, 5, 4, 9, 10, 3, 7, 11, 8, 2$
	$Q_1*8(3, 10)$	$F'_1$	$2, 7, 11, \{1, 6\}, 9, 8, 4, 5, 10, 3$
4	$Q_1*7(2, 9)+(1, 11)$	$F'_1$	$2, 7, 11, \{1, 6\}, 9, 8, 4, 5, 10, 3$
	$Q_1*7(2, 9)+(6, 11)$	$F'_1$	$3, 4, 5, \{8, 10\}, 9, 1, 7, 11, 6, 2$
	$Q_1*7(2, 8)+(3, 11)$	$F'_1$	$8, 7, 2, 3, 11, 1, 9, 4, \{5, 10\}, 6$
	$Q_1*8(3, 7)+(2, 11)$	$F'_1$	$\{5, 10\}, 1, 6, 11, 2, 9, 3, 8, 7, 4$
	$Q_1*8(3, 7)+(1, 8)$	$F'_1$	$\{5, 10\}, 11, 6, 1, 8, 9, 3, 2, 7, 4$
	$Q_1*8(3, 7)+(5, 8)$	$F'_1$	$\{1, 2\}, 3, 4, 5, 8, 9, 11, 10, 7, 6$
	$Q_1*8(3, 6)+(4, 11)$	$F'_1$	$8, 3, \{2, 7\}, 11, 4, 1, 9, 10, 5, 6$
	$Q_1*8(3, 6)+(9, 11)$	$E_{20}$	$7, 10, 9, 11, 2, \{1, 5, 6\}, 4, 8, 3$
5	$Q_1*7(8, 10)*8(3, 7)+(11, 12)$	$F'_1$	$\{1, 2\}, 6, 9, 11, 12, \{3, 4\}, 10, 7, 8, 5$
	$Q_1*7(2, 8)*8(7, 10)+(11, 12)$	$F'_1$	$\{3, 4\}, 9, 6, 12, 11, \{1, 2\}, 10, 8, 7, 5$
	$Q_1*7(2, 9)*9(4, 6)+(9, 11)$	$F'_1$	$\{3, 4\}, 8, 6, 9, 11, \{1, 2\}, 10, 12, 7, 5$
	$Q_1*7(2, 8)*9(4, 10)+(7, 9)$	$F'_1$	$8, \{3, 4\}, 5, 10, 9, \{1, 2\}, 12, 11, 7, 6$
	$Q_1*7(2, 9)*10(9, 11)+(7, 12)$	$F'_1$	$\{1, 6\}, 2, 3, \{8, 11\}, 7, 4, 12, 10, 9, 5$
	$Q_1*7(2, 8)*10(5, 9)+(7, 10)$	$F'_1$	$3, \{1, 2\}, 6, 8, 7, 9, 10, 12, 11, \{4, 5\}$
6	$Q_1*7(2, 8)+(1, 11)+(6, 7)$	$F'_1$	$2, 7, 6, 1, 11, 8, \{4, 9\}, 5, 10, 3$
	$Q_1*8(3, 7)+(4, 11)+(8, 9)$	$F'_1$	$\{1, 6\}, 5, 10, 11, 4, 7, 3, 8, 9, 2$
	$Q_1*8(3, 6)+(1, 11)+(8, 10)$	$F'_1$	$2, 7, 11, \{1, 6\}, 9, 8, 4, 5, 10, 3$
7	$Q_1*8(3, 7)*9(4, 10)+(8, 12)+(9, 11)$	$F'_1$	$\{2, 7\}, 10, 5, \{1, 6\}, 11, 4, 8, 12, 9, 3$
9	$Q_1+(1, 7-8)$	$F'_1$	$2, 7, 11, \{1, 6\}, 9, 8, 4, 5, 10, 3$

**Table 2.2:** Applying Theorem 2.2.1 to  $Q_3$

Type	Enlargement	Minor	Branch sets of the minor
1	$Q_3+(2, 4)$	$F_1'$	1, 11, 3, 2, {4, 5}, 8, 9, 7, 10, 6
	$Q_3+(2, 5)$	$F_1'$	1, 11, 4, 5, {2, 3}, 9, 8, 10, 7, 6
2	$Q_3+(1, 7)+(2, 6)$	$D_3'$	{3, 8, 11}, 2, 7, 6, 1, 4, 5, 10, 9
	$Q_3+(1, 9)+(2, 6)$	$F_1'$	9, 4, 5, 1, {3, 11}, 10, 2, 6, 8, 7
	$Q_3+(1, 9)+(6, 7)$	$F_4$	5, 11, 9, 1, 10, {3, 8}, 6, 2, 7, 4
	$Q_3+(2, 9)+(6, 7)$	$E_{18}$	1, {3, 8}, {4, 9}, 2, {5, 10}, 6, 11, 7
	$Q_3+(1, 7)+(2, 9)$	$D_3'$	1, 2, {3, 8}, 7, {6, 9}, 5, 4, 11, 10
	$Q_3+(2, 8)+(3, 7)$	$F_4$	11, 9, 5, {1, 6}, 3, 7, 10, 2, 8, 4
	$Q_3+(2, 10)+(3, 7)$	$E_{22}$	1, 2, 3, 11, 5, 10, 7, {4, 9}, {6, 8}
	$Q_3+(2, 10)+(7, 8)$	$F_4$	3, 7, 10, 8, 11, {4, 9}, 5, 6, 1, 2
	$Q_3+(3, 10)+(7, 8)$	$F_4$	5, 11, 6, 1, 10, 3, 8, 2, 7, {4, 9}
	$Q_3+(2, 8)+(3, 10)$	$F_1'$	2, 8, 6, 1, {3, 11}, 9, 10, 5, 4, 7
	$Q_3+(3, 9)+(4, 8)$	$E_{22}$	{1, 6}, 2, 3, 11, 5, {7, 10}, 8, 4, 9
	$Q_3+(3, 9)+(8, 11)$	$F_1'$	11, 4, 5, {1, 6}, 9, 10, 3, 2, 7, 8
	$Q_3+(3, 4)+(8, 11)$	$D_3'$	{4, 5}, 11, 8, 1, {2, 3}, 9, 7, 10, 6
	$Q_3+(6, 11)+(8, 9)$	$F_1'$	10, 7, 2, {3, 8}, 9, 1, 4, 11, 6, 5
	$Q_3+(3, 9)+(6, 11)$	$F_1'$	10, 7, 2, {3, 8}, 9, 1, 4, 11, 6, 5
	$Q_3+(3, 4)+(6, 11)$	$D_3'$	{1, 2}, 11, 4, 6, {3, 8}, 7, 10, 5, 9
	$Q_3+(4, 6)+(8, 9)$	$F_4$	5, 11, 6, 1, 10, {3, 8}, 9, 2, 7, 4
	$Q_3+(3, 9)+(4, 6)$	$F_4$	5, 11, 6, 1, 10, {3, 8}, 9, 2, 7, 4
	$Q_3+(3, 6)+(8, 11)$	$E_{20}$	2, 1, 3, 11, 7, {6, 9}, 8, {4, 5}, 10
$Q_3+(1, 3)+(2, 11)$	$F_1'$	2, 3, 6, 1, 11, 9, {8, 10}, 5, 4, 7	
3	$Q_3*1(2, 5)$	$F_1'$	7, 10, 5, {1, 2}, {3, 8}, 4, 6, 12, 11, 9
4	$Q_3*1(5, 6)+(10, 12)$	$F_1'$	12, 1, 5, 10, {6, 8}, 4, {2, 3}, 7, 9, 11
	$Q_3*1(5, 6)+(8, 12)$	$F_1'$	9, {1, 6}, 5, {4, 11}, 12, 10, 2, 3, 8, 7
	$Q_3*1(5, 6)+(1, 3)$	$F_1'$	10, 8, 6, {1, 5}, 3, {4, 9}, 2, 12, 11, 7
6	$Q_3*1(5, 6)+(1, 7)+(9, 12)$	$F_1'$	8, 10, 5, {1, 6}, 7, 4, 2, 12, 9, {3, 11}
8	$Q_3+(2, 9, 11)$	$F_4$	1, 12, 3, 2, {4, 5}, 9, {6, 8}, 7, 10, 11
9	$Q_3+(8, 1-11)$	$F_1'$	1, 12, 11, {2, 3}, {6, 8}, 4, 10, 7, 9, 5
	$Q_3+(8, 1-2)$	$F_1'$	9, 4, 5, {1, 6}, {3, 11}, 10, 2, 12, 8, 7
	$Q_3+(10, 1-2)$	$F_4$	11, 5, 9, {1, 6}, {3, 8}, 10, 7, 12, 2, 4
	$Q_3+(6, 2-3)$	$F_1'$	2, 12, 6, 1, {3, 8, 11}, 9, 10, 5, 4, 7
	$Q_3+(9, 2-3)$	$F_1'$	5, 4, 11, {1, 6}, 9, {3, 8}, 7, 2, 12, 10
	$Q_3+(3, 1-6)$	$F_4$	2, 11, 12, 1, 7, {4, 9}, {6, 8}, 5, 10, 3
	$Q_3+(1, 3-8)$	$F_4$	2, 11, 12, 1, 7, {4, 6, 9}, 8, 5, 10, 3
	$Q_3+(2, 6-8)$	$F_4$	3, 7, 12, {8, 10}, 11, {4, 9}, 6, 5, 1, 2
	$Q_3+(7, 6-8)$	$F_4$	11, 9, 5, {1, 6}, {2, 3}, 7, 10, 12, 8, 4
	$Q_3+(3, 7-9)$	$F_4$	5, 11, 9, {1, 6}, 10, {3, 8}, 12, 2, 7, 4
	$Q_3+(8, 7-9)$	$F_4$	5, 11, 9, {1, 6}, 10, {3, 8}, 12, 2, 7, 4
	$Q_3+(1, 7-10)$	$E_2$	2, 9, 12, 11, 5, 8, {1, 6}, 3, 7, 4, 10
$Q_3+(6, 7-10)$	$E_2$	2, 9, 12, 11, 5, 8, {1, 6}, 3, 7, 4, 10	

## CHAPTER III

### SIX-REGULAR GRAPHS IN $\mathcal{F}_6$

#### 3.1 Preliminaries

We begin with some basic notation and observations that will be used in the proof of this chapter's main theorem. Refer to Section 1.2.2 for a discussion of the theorem and its motivation, in the context of Jorgensen's conjecture (Conjecture 1.2.1).

Let  $v$  be a vertex in a graph  $G$ , and  $X$  be a set of vertices in  $G$ , that may or may not contain  $v$ . We define  $N_X(v)$  as the set of vertices in  $X$  that are adjacent to  $v$  in  $G$ , and  $\deg_X(v)$  as the size of this set.

A vertex of  $G$  with degree less than 6 is called a *low-degree* vertex.

$\mathcal{F}_6$  is defined to be the family of graphs  $G$  such that  $G$  has at least 7 vertices, and the set of low-degree vertices in  $G$  forms a clique.

We are now ready to state the main theorem of this chapter:

**Theorem 3.1.1** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ . Then  $G$  has a minor isomorphic to  $K_6$ .*

We now make four easy observations that will be used repeatedly in the proof of Theorem 3.1.1. In the following discussion, we shall denote  $X$  to be a clique in  $G$  of maximum size. For a vertex  $x \in X$ , let  $N'(x)$  be the set of neighbors of  $x$  in  $V(G) \setminus X$ . Let  $Y$  be the union of  $N'(x)$  over all vertices  $x \in X$ . For a vertex  $y \in Y$ ,  $\deg_X(y) \geq 1$ .

For the following observations, let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ . Also, assume that  $G$  has at least eight vertices. (Clearly, if it has seven vertices, then it must be isomorphic to  $K_7$  and thus Theorem 3.1.1 holds trivially for it.) Let  $X$  and  $Y$  be defined as above.

**Observation 1** *Any two adjacent vertices  $u, v$  in  $G$  must have at least two common neighbors that are mutually non-adjacent. If  $u$  or  $v$  (or both) are in the clique  $X$ , then at least*

one of those common neighbors must be outside  $X$ .

**Observation 2** *Let the clique  $X$  be of size 4, with  $x \in X$  and  $y_1 \in Y$  such that  $x$  is the unique neighbor of  $y_1$  in  $X$ . Let  $y_2$  and  $y_3$  be the other two neighbors of  $x$  outside  $X$ . Then  $y_1, y_2, y_3$  span precisely the two edges  $y_1y_2$  and  $y_1y_3$ .*

*Proof:* This follows by applying Observation 1 to  $x, y_1$ . □

**Observation 3** *Let the clique  $X$  be of size 4. No vertex  $x \in X$  is adjacent to two vertices  $y_1, y_2 \in Y$  such that  $\deg_X(y_1) = \deg_X(y_2) = 1$ .*

*Proof:* This follows immediately from Observation 2.

Recall that a *separation* of a graph  $G$  is a pair  $(A, B)$  of subsets of  $V(G)$  such that  $A \cup B = V(G)$ , and there is no edge between  $A - B$  and  $B - A$ . The separation is called *proper* or *non-trivial* if both  $A - B$  and  $B - A$  are non-empty; otherwise, it is called *trivial*. The *order* of the separation is defined as  $|A \cap B|$ . A separation of order  $\leq k$  is called a  $\leq k$ -*separation*.

**Observation 4**  *$G$  does not have a separation  $(A, B)$  such that (i) both  $A - B$  and  $B - A$  are non-empty, and (ii)  $G[A]$  has a clique minor rooted at the vertices of  $A \cap B$  (that is, a clique minor in which each vertex of  $A \cap B$  is contained in a distinct branch set).*

*Proof:* This follows by noting that adding to  $G[B]$  all possible edges spanning  $A \cap B$  yields a graph in which the low-degree vertices form a clique. Further, the graph has at least 7 vertices, and is thus in  $\mathcal{F}_6$ . Thus from the minor-minimality of  $G$  it follows that a separation satisfying conditions (i) and (ii) cannot exist in  $G$ . □

The definition of  $\mathcal{F}_6$  is motivated by a desire to relax 6-connectivity. (Refer to Section 1.2.2 for a relevant discussion.) However, on the face of it, the definition of  $\mathcal{F}_6$  does not say anything directly about separations (and hence about connectivity). Observation 4 above realizes the connectivity implications of that definition.

For  $Z \subseteq V(G)$ ,  $G[Z]$  denotes the subgraph *induced* by  $Z$ , that is, the subgraph consisting of  $Z$  and all edges with both ends in  $Z$ . A subgraph of  $G$  is said to be *induced* if it is induced by its vertex set.

Let  $S$  be a subgraph of a graph  $G$ . An  $S$ -bridge of  $G$  is a subgraph  $\mathcal{B}$  of  $G$  such that either  $\mathcal{B}$  consists of a unique edge of  $E(G) - E(S)$  and its ends, where the ends belong to  $S$ , or  $\mathcal{B}$  consists of a component  $J$  of  $G \setminus V(S)$  together with all edges from  $V(J)$  to  $V(S)$  and all their ends. For an  $S$ -bridge  $\mathcal{B}$ , the vertices of  $\mathcal{B} \cap S$  are called the *attachments* of  $\mathcal{B}$ .

For  $Z \subseteq V(G)$ , a  $Z$ -bridge of  $G$  refers to an  $S$ -bridge as above, where  $S$  is the subgraph  $G[Z]$  induced by  $Z$ .

When we say that  $G$  has a *rooted subgraph isomorphic to a graph  $H$*  shown in some figure, we mean that  $G$  has a subgraph isomorphic to  $H$ , with the convention that solid vertices in the figure are not incident to any more edges in  $G$  than the ones shown. We use vertex labels in the figures to also denote the corresponding vertices in  $G$  (relative to some fixed copy of  $H$  in  $G$ ).

A *society*  $(G, \Omega)$  consists of a graph  $G$ , and a cyclic order  $\Omega$  on a subset  $V(\Omega)$  of  $V(G)$ . The society  $(G, \Omega)$  is said to have a *cross* if  $G$  has two vertex-disjoint paths with ends  $(s_1, t_1)$  and  $(s_2, t_2)$  respectively, such that  $s_1, s_2, t_1, t_2$  appear in  $\Omega$ , in the given cyclic order. The vertices  $s_1, s_2, t_1, t_2$  are called the *feet* of the cross.

**Theorem 3.1.2** ([48, 49, 54]) *Let  $(G, \Omega)$  be a society such that  $G$  has no  $\leq 3$ -separation  $(A, B)$  with  $V(\Omega) \subseteq A$ . Then exactly one of the following outcomes hold:*

1.  $G$  can be drawn in the plane with the vertices in  $V(\Omega)$  on the boundary of the infinite face, in the cyclic order given by  $\Omega$
2.  $(G, \Omega)$  has a cross

### 3.1.1 Outline of the Main Proof

If  $G$  is not six-regular, then deleting an edge incident to a vertex of degree  $\geq 7$  creates at most one low-degree vertex, and hence the resulting graph is also in  $\mathcal{F}_6$ , contradicting the minimality of  $G$ . Hence it follows that  $G$  is *6-regular*. Also, we may assume that the size of a maximum clique is  $\geq 3$  by Observation 1, and  $\leq 5$  (otherwise Theorem 3.1.1 holds trivially). The proof of the theorem is split into cases depending on the clique number (that is, size of the largest clique) of  $G$ . The main lemmas 3.1.3, 3.2.1 and 3.3.2 handle these

cases, with subcases being handled by other lemmas of their own. The subsidiary lemmas usually follow the main lemma that they serve.

**Lemma 3.1.3** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and let  $G$  have a maximum clique  $X$  of size 5. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* Let  $x_1, \dots, x_5$  be the vertices of  $X$ , and let  $y_1, y_2$  be the two neighbors of  $x_1$  outside  $X$ . Applying Observation 1 to each pair of the form  $x_1, x_i$  for  $i = 2, \dots, 5$ , it follows that each vertex in  $X$  is adjacent to  $y_1$  or  $y_2$  (or both). Further, applying Observation 1 to  $x_1$  and  $y_1$ , it follows that  $y_1$  and  $y_2$  are adjacent. But then contracting  $y_1 y_2$  in  $G$  gives a  $K_6$  minor.  $\square$

### 3.2 When $G$ has Clique Number 4

**Lemma 3.2.1** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and let  $G$  have a maximum clique  $X$  of size 4. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* Let  $x_1, \dots, x_4$  be the vertices of  $X$ . Since  $G$  has no clique of size 5, every vertex in  $Y$  is incident with at most 3 vertices in  $X$ . Each vertex in  $X$  has exactly three neighbors outside  $X$ , and hence there are 12  $X$ - $Y$  edges. Thus there are at most four vertices  $y$  such that  $\deg_X(y) = 3$ .

If there are exactly four such vertices, then we claim that  $G$  must have a  $K_6$  minor. Indeed,  $G$  cannot have a vertex outside  $X \cup Y$ , otherwise Observation 4 applied to the separation  $(X \cup Y, V(G) - X)$  yields a contradiction. Thus  $V(G) = X \cup Y$ . Since  $G$  is six-regular, it follows that it is isomorphic to the graph obtained from  $K_8$  by removing a perfect matching. Thus it has a  $K_6$  minor.

Thus the number of vertices  $y$  with  $\deg_X(y) = 3$  is three, two, one or zero. By Lemmas 3.2.3, 3.2.4, 3.2.7 and 3.2.10, it follows that  $G$  has a minor isomorphic to  $K_6$ .  $\square$

The rest of this section is devoted to the subsidiary lemmas referenced in the above proof.

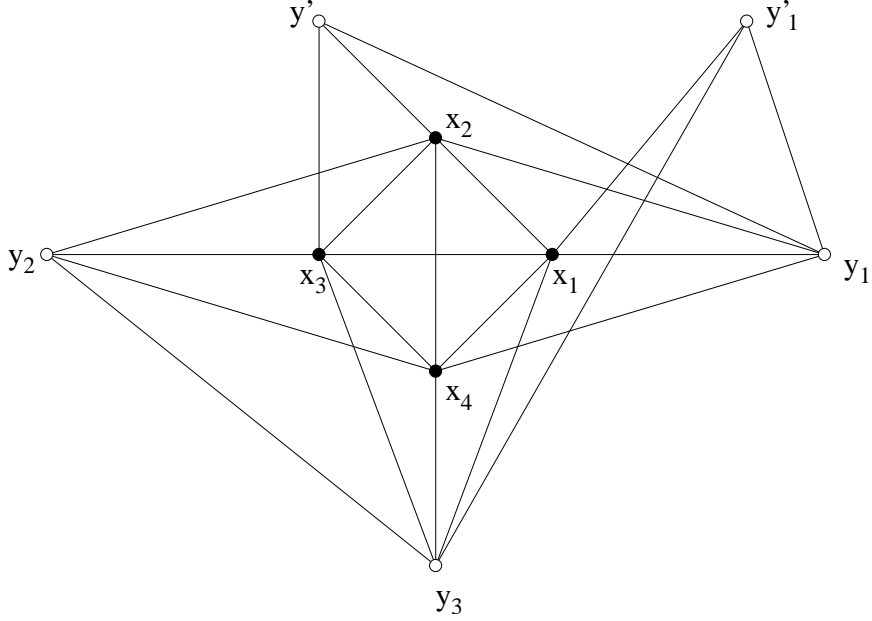
**Lemma 3.2.2** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and let  $G$  have a largest clique  $X$  of size 4. No two vertices  $y_1, y_2$  share the same set of three neighbors in  $X$ .*

Suppose  $y_1, y_2 \in Y$  are both adjacent to  $x_1, x_2$  and  $x_3$ . Since  $G$  has no clique of size 5,  $y_1$  and  $y_2$  are non-adjacent. For  $i = 1, 2, 3$ , let  $y'_i$  be the unique neighbor of  $x_i$  in  $Y$ , other than  $y_1$  and  $y_2$ . (Note that the vertices  $y'_i$  may not all be distinct.) Applying Observation 1 in turn to  $y_1, x_i$  for  $i = 1, 2, 3$ , we conclude that  $y_1$  must be adjacent to each  $y'_i$ . By the same argument,  $y_2$  must be adjacent to each  $y'_i$ , for  $i = 1, 2, 3$ . Further, applying Observation 3 to  $x_4$ , it follows that  $x_4$  must be adjacent to at least two *distinct* vertices  $y'_i$  and  $y'_j$ . But then contracting the edges  $y_1y'_i$  and  $y_2y'_j$  gives a  $K_6$  minor in  $G$ .  $\square$

**Lemma 3.2.3** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ . Let  $G$  have a largest clique  $X$  of size 4, such that there are exactly three vertices  $y \in Y$  with  $\deg_X(y) = 3$ . Then  $G$  has a minor isomorphic to  $K_6$ .*

Let  $y_1, y_2, y_3$  be the three vertices with the given property. Without loss of generality, we may assume:  $y_1$  is adjacent to  $x_4, x_1, x_2$ ;  $y_2$  is adjacent to  $x_4, x_2, x_3$ ;  $y_3$  is adjacent to  $x_4, x_1, x_3$ . For  $i = 1, 2, 3$ , let  $y'_i$  be the unique neighbor of  $x_i$  in  $Y$ , other than  $y_1, y_2, y_3$ . (Note that the vertices  $y'_i$  may not all be distinct.) In fact, we claim that they cannot all be distinct. Applying Observation 2 to  $y'_1$  and  $y'_3$ , we conclude that  $y_3$  can be adjacent neither to  $y_1$  nor  $y_2$ . But then applying Observation 1 to  $x_4y_3$  yields a contradiction. Thus the vertices  $y'_i$  are not all distinct. (They cannot all be the same either, because of the hypothesis of the lemma.)

Now suppose that  $y'_2 = y'_3 = y'$ , and that  $y'_1$  is distinct from that vertex. Applying Observation 2 to  $x_1, y'_1$ , we conclude that  $y'_1$  is adjacent to both  $y_1$  and  $y_3$ , and that  $y_1, y_3$  are non-adjacent. Applying Observation 1 in turn to  $x_4y_3$  and  $x_2y'$ , it follows respectively that  $y_3$  is adjacent to  $y_2$  and  $y'$  is adjacent to  $y_1$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_1$  shown in Figure 3.1. But then contracting the edges  $x_2y_2$ ,  $x_3y'$  and  $y_1y'_1$  yields a  $K_6$  minor in  $G$ .



**Figure 3.1:** Graph  $H_1$

Finally, suppose that  $y'_1 = y'_3 = y'$ , and that  $y'_2$  is distinct from that vertex. Applying Observation 1 in turn to  $x_1y'$  and  $x_3y'$ , it follows that  $y'$  is adjacent to  $y_1$  and  $y_2$  respectively. Again, applying Observation 1 to  $x_4y_3$ , it follows that  $y_3$  must be adjacent to at least one vertex among  $y_1$  and  $y_2$ , say  $y_1$ . But then contracting the edges  $y_1y_3$  and  $y_2y'$  yields a  $K_6$  minor in  $G$ .  $\square$

**Lemma 3.2.4** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ . Let  $G$  have a largest clique  $X$  of size 4, such that there are exactly two vertices  $y \in Y$  with  $\deg_X(y) = 3$ . Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By Lemma 3.2.2, without loss of generality, we may assume that there exist  $y_1, y_2 \in Y$ , with  $y_1$  adjacent to  $x_1, x_2, x_3$  and  $y_2$  adjacent to  $x_1, x_4, x_3$ . Applying Observation 1 to  $x_2, x_4$ , we conclude that there exists  $y_3$  adjacent to  $x_2, x_4$  (and thus distinct from  $y_1, y_2$ ).

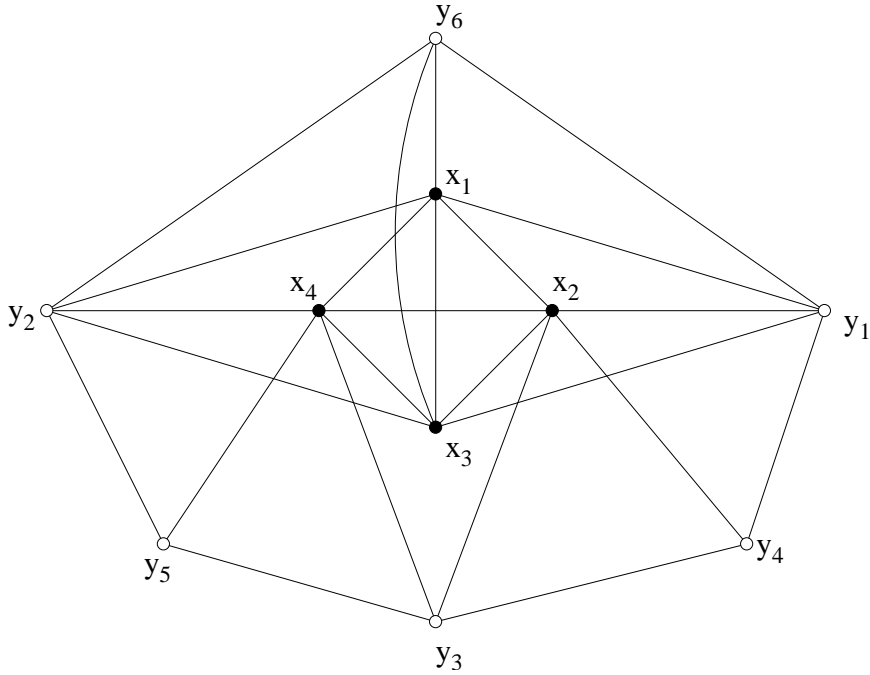
We claim that there cannot exist another vertex  $y_4$  that is adjacent to both  $x_2$  and  $x_4$ . Suppose the contrary. Applying Observation 1 in turn to  $y_3x_2$  and  $y_3x_4$ , it follows that  $y_3$  is respectively adjacent to  $y_1$  and  $y_2$ . By a similar argument,  $y_4$  is also adjacent to  $y_1$  and  $y_2$ . But then  $G$  has a minor isomorphic to  $K_6$ . This proves the claim.

Thus there exist vertices  $y_4$  and  $y_5$ , distinct from each other and from  $y_1, y_2, y_3$ , and respectively adjacent to  $x_2$  and  $x_4$ . We now split the argument into cases:

**Case 1:**  $\deg_X(y_4) = \deg_X(y_5) = 1$ .

Applying Observation 2 to  $x_2y_4$ , it follows that  $y_4$  is adjacent to  $y_1, y_3$  (and that  $y_1$  and  $y_3$  are non-adjacent). Similarly,  $y_5$  is adjacent to  $y_2$  and  $y_3$ .

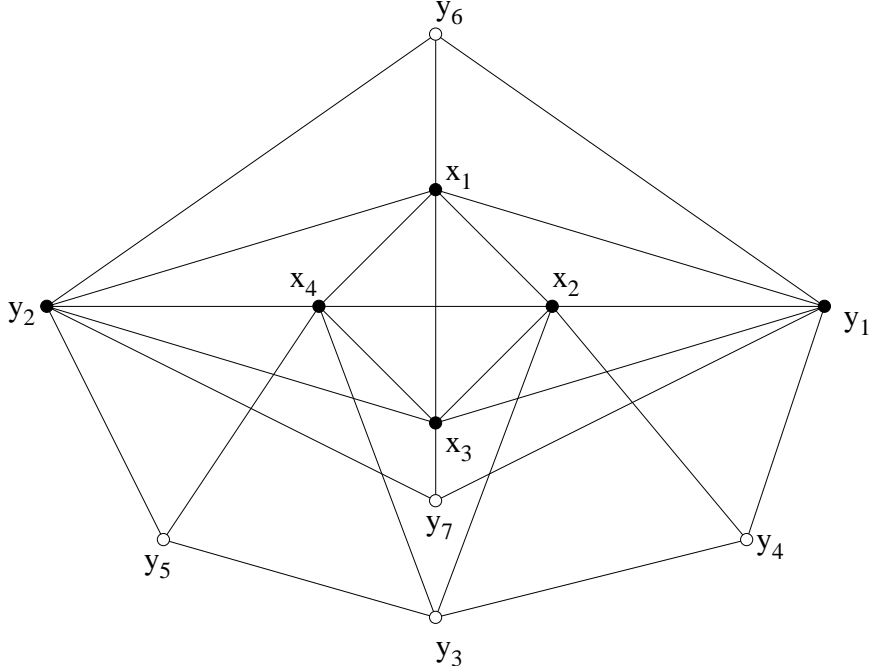
Now, suppose that  $x_1$  and  $x_3$  have a common neighbor  $y_6$  distinct from  $y_1, y_2$ , so that  $Y$  consists of the six distinct vertices  $y_1, \dots, y_6$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_2$  shown in Figure 3.2. By Lemma 3.2.5, we are done.



**Figure 3.2:** Graph  $H_2$

Thus  $x_1$  and  $x_3$  have neighbors  $y_6$  and  $y_7$  respectively, that are distinct from each other and from  $y_1, \dots, y_5$ . Applying Observation 2 in turn to  $x_1y_6$  and  $x_3y_7$ , it follows that  $G$  has a rooted subgraph isomorphic to the graph  $H_3$  shown in Figure 3.3. By Lemma 3.2.6, we are done. This finishes Case 1.

**Case 2:** Among  $y_4$  and  $y_5$ , one vertex has exactly two neighbors in  $X$  and the other vertex has exactly one neighbor in  $X$ . Without loss of generality, we may assume that  $y_4$  is adjacent to  $x_2$  and  $x_3$ , whereas  $y_5$ , as before, is adjacent to  $x_4$  (and has no other neighbor in  $X$ ). This means that there is a vertex  $y_6$  whose unique neighbor in  $X$  is  $x_1$ ,

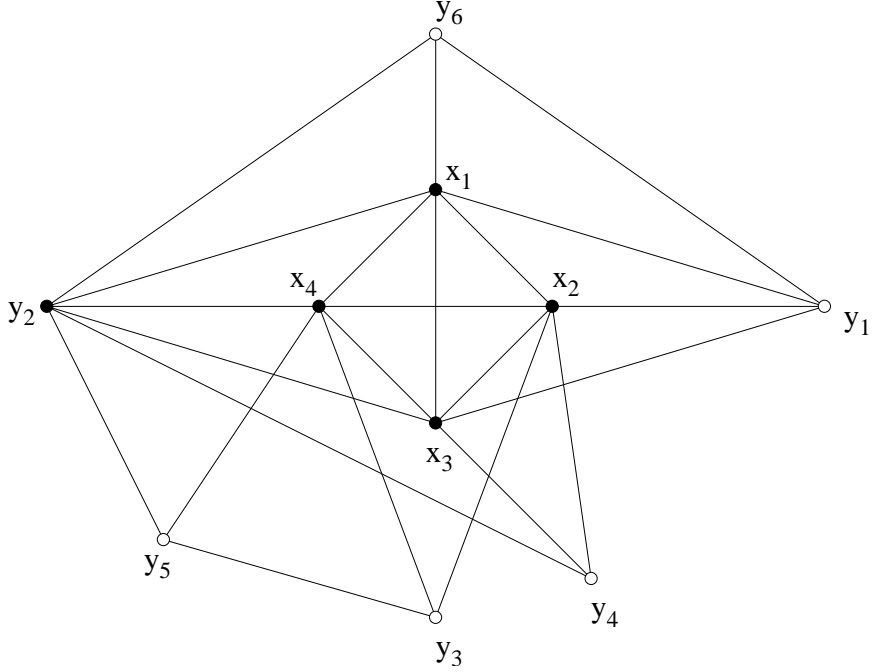


**Figure 3.3:** Graph  $H_3$

and  $Y = \{y_1, \dots, y_6\}$ . Applying Observation 2 to  $x_4y_5$ , it follows that  $y_5$  is adjacent to  $y_2$  and  $y_3$ . Applying Observation 2 to  $x_1y_6$ , it follows that  $y_6$  is adjacent to  $y_1$  and  $y_2$ . Finally, applying Observation 1 to  $x_3y_4$ , we conclude that  $y_4$  must be adjacent to  $y_2$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_4$  shown in Figure 3.4. We may assume that the edge  $y_1y_3$  does not exist (otherwise contracting the edges  $y_1y_3$  and  $y_2y_4$  yields a  $K_6$  minor). Since  $y_3$  has degree 6 in  $G$ , there must be a vertex in  $V(G) - X - Y$ . Now the vertices  $y_1, y_3, y_4, y_5, y_6$  separate  $X \cup \{y_2\}$  from the rest of  $G$ , and the resulting separation violates Observation 4. This finishes Case 2.

**Case 3:** Both  $y_4$  and  $y_5$  have exactly two neighbors in  $X$ . Without loss of generality, we may assume that  $y_4$  is adjacent to  $x_2, x_3$  and  $y_5$  is adjacent to  $x_4, x_1$ . Applying Observation 1 in turn to  $x_1y_5$  and  $x_3y_4$ , it follows that  $y_5$  is adjacent to  $y_1$  and  $y_4$  is adjacent to  $y_2$ . Further, applying Observation 1 in turn to  $x_2y_4$  and  $x_4y_5$ , it follows that both  $y_4$  and  $y_5$  are adjacent to  $y_3$ . But then contracting the edges  $y_1y_5$  and  $y_2y_4$  yields a minor isomorphic to  $K_6$ . This finishes Case 3, and the proof of the lemma.  $\square$

**Lemma 3.2.5** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such*



**Figure 3.4:** Graph  $H_4$

that it has a rooted subgraph isomorphic to the graph  $H_2$  shown in Figure 3.2. Then  $G$  has a minor isomorphic to  $K_6$ .

*Proof:* By Lemma 3.1.3, we may assume that  $G$  has no clique of size 5. Applying Observation 1 in turn to  $y_1y_6$  and  $y_1y_4$ , it follows that one (or both) of the following outcomes happen:

1.  $y_6$  is adjacent to  $y_4$
2.  $y_1, y_4, y_6$  have a common neighbor  $z_1$  outside  $X \cup Y$

By symmetry, we also get one (or both) of the following outcomes:

- A.  $y_6$  is adjacent to  $y_5$
- B.  $y_2, y_5, y_6$  have a common neighbor  $z_2$  outside  $X \cup Y$

First, suppose either outcome 1 or outcome A (or both) happen. Without loss of generality, we may assume that outcome 1 happens. If  $y_1$  is adjacent to  $y_3$  or  $y_5$ , then  $G$  has a minor isomorphic to  $K_6$ . Also,  $y_1$  is not adjacent to  $y_2$  (by applying Observation 1 to  $x_1y_6$ , for

instance). Thus  $y_1$  has a neighbor outside  $X \cup Y$ . Let  $\mathcal{B}$  be the  $(X \cup Y)$ -bridge of  $G$  that contains that vertex. Suppose  $\mathcal{B}$  does not attach at  $y_3$  or  $y_5$ . Since  $y_6$  is either adjacent to  $y_5$  or has a common neighbor  $z_2$  with it,  $\mathcal{B}$  cannot attach at  $y_6$  either. But then the vertices  $y_1, y_2, y_4$  separate  $\mathcal{B}$  from the rest of  $G$ , and the resulting separation violates Observation 4. Thus  $\mathcal{B}$  must attach at  $y_3$  or  $y_5$  (or both). But then  $G$  has a minor isomorphic to  $K_6$  (with branch sets given by  $\mathcal{B} \cup \{y_1, y_3, y_5\}, \{y_2, y_6, y_4\}, \{x_1\}, \dots, \{x_4\}$ ).

Thus we may assume that both outcome 2 and outcome B happen, and that  $y_6$  is adjacent to neither  $y_4$  nor  $y_5$ . Suppose that  $z_1 = z_2 = z$ . If  $y_6$  is adjacent to  $y_3$ , then  $G$  has a  $K_6$  minor. Since  $y_6$  has degree 6 in  $G$ , it has a neighbor outside  $X \cup Y \cup \{z\}$ , and let  $\mathcal{B}$  be the  $(X \cup Y)$ -bridge of  $G$  that contains that neighbor. If  $\mathcal{B}$  attaches at  $y_3$ , then  $G$  has a  $K_6$  minor as before. Thus the vertices  $y_4, y_5, y_6, z$  separate  $\mathcal{B}$  from the rest of  $G$ , and the resulting separation violates Observation 4.

Hence  $z_1, z_2$  must be distinct. Applying Observation 1 to  $y_6 z_1$ , it follows that  $z_1$  is adjacent to  $z_2$ . Let  $P = X \cup \{y_1, y_2, y_6\}$ . The five vertices  $z_1, z_2, y_5, y_3, y_4$  separate  $P$  from the rest of  $G$ , and the graph induced by those five vertices is a 5-cycle, in that order. We claim that  $G \setminus P$  is not a planar graph. Indeed, if the number of vertices in  $G \setminus P$  is  $n$ , the number of edges is  $\frac{1}{2}(6(n-5) + 20) = 3n - 5$ . Let  $G_0 = G \setminus P$  and  $\Omega = z_1, z_2, y_5, y_3, y_4$  (in that cyclic order).

We claim that  $G_0$  has no  $\leq 3$ -separation  $(A, B)$  with  $V(\Omega) \subseteq A$ . Suppose it does; choose  $(A, B)$  as above such that its order  $k$  is minimal. Then by Menger's theorem, there exist  $k$  vertex disjoint paths from  $V(\Omega)$  to  $A \cap B$ . It now follows easily that the separation  $(A \cup P, B)$  in  $G$  violates Observation 4. This proves our claim.

Thus by Theorem 3.1.2, the society  $(G_0, \Omega)$  has a cross. By symmetry, there are three possibilities to consider for the cross, and it is easy to check that  $G$  has a  $K_6$  minor in each of those cases. □

**Lemma 3.2.6** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it has a rooted subgraph isomorphic to the graph  $H_3$  shown in Figure 3.3. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By Lemma 3.1.3, we may assume that  $G$  has no clique of size 5. Applying Observation 1 to  $y_1y_4$ , it follows that  $y_4$  is adjacent to either  $y_6$  or  $y_7$  (or both). By symmetry,  $y_5$  is also adjacent to either  $y_6$  or  $y_7$  (or both). If there are two disjoint edges between  $\{y_4, y_5\}$  and  $\{y_6, y_7\}$ , then  $G$  has a  $K_6$  minor. If not, we may assume, without loss of generality, that  $y_6$  is adjacent to both  $y_4$  and  $y_5$ , and  $y_7$  is adjacent to neither. Further, applying Observation 1 to  $y_1y_7$ , it follows that  $y_7$  is adjacent to  $y_6$ . Since  $y_4$  has degree 6, it must have a neighbor outside  $X \cup Y$ . But then  $\{y_3, y_4, y_5, y_7\}$  separates  $X \cup \{y_1, y_2, y_6\}$  from the rest of  $G$ , and the resulting separation violates Observation 4.  $\square$

**Lemma 3.2.7** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ . Let  $G$  have a largest clique  $X$  of size 4, such that there is exactly one vertex  $y_{123} \in Y$  with  $\deg_X(y_{123}) = 3$ . Then  $G$  has a minor isomorphic to  $K_6$ .*

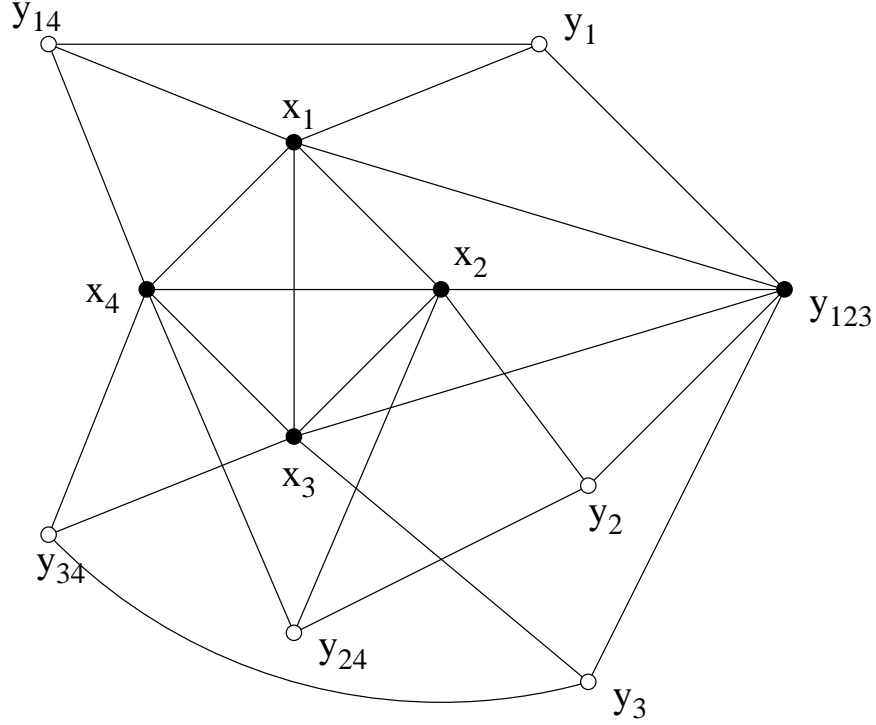
*Proof:* Let  $y_{123}$  be adjacent to  $x_1, x_2, x_3$ . By the hypothesis of the lemma, and by Observation 1 applied in turn to  $x_ix_4$  for  $i = 1, 2, 3$ , it follows that there are vertices  $y_{i4}$  adjacent to precisely  $x_i, x_4$  (and no other vertex in  $X$ ). Now for  $i = 1, 2, 3$ ,  $x_i$  has a unique neighbor  $y_i$  distinct from  $y_{123}, y_{14}, y_{24}, y_{34}$ . (Note that the vertices  $y_i$  may not all be distinct.)

First, suppose that the vertices  $y_i$  are all distinct. Applying Observation 2 in turn to  $x_iy_i$ , for  $i = 1, 2, 3$ , it follows that  $y_i$  is adjacent to both  $y_{123}$  and  $y_{i4}$  (and that  $y_{123}, y_{i4}$  are non-adjacent). Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_5$  shown in Figure 3.5. By Lemma 3.2.8, we are done.

Hence we may assume, without loss of generality, that  $y_1 = y_2 = y_{12}$ , and  $y_3$  is distinct from that vertex. Applying Observation 1 in turn to  $x_1y_{12}$  and  $x_2y_{12}$ , it follows that  $y_{12}$  is adjacent to  $y_{14}$  and  $y_{24}$ . Applying Observation 2 to  $x_3y_3$ , it follows that  $y_3$  is adjacent to  $y_{123}$  and  $y_{34}$  (and that the latter two vertices are non-adjacent).

Suppose  $y_{123}$  is not adjacent to  $y_{12}$ . Then apply Observation 1 in turn to  $x_1y_{123}, x_2y_{123}, x_4y_{34}$ ; it follows respectively that  $y_{123}$  is adjacent to both  $y_{14}$  and  $y_{24}$ , and  $y_{34}$  is adjacent to  $y_{14}$  or  $y_{24}$  (or both). But then  $G$  has a  $K_6$  minor. Hence we may assume that  $y_{123}$  is adjacent to  $y_{12}$ .

Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_6$  shown in Figure 3.6. By



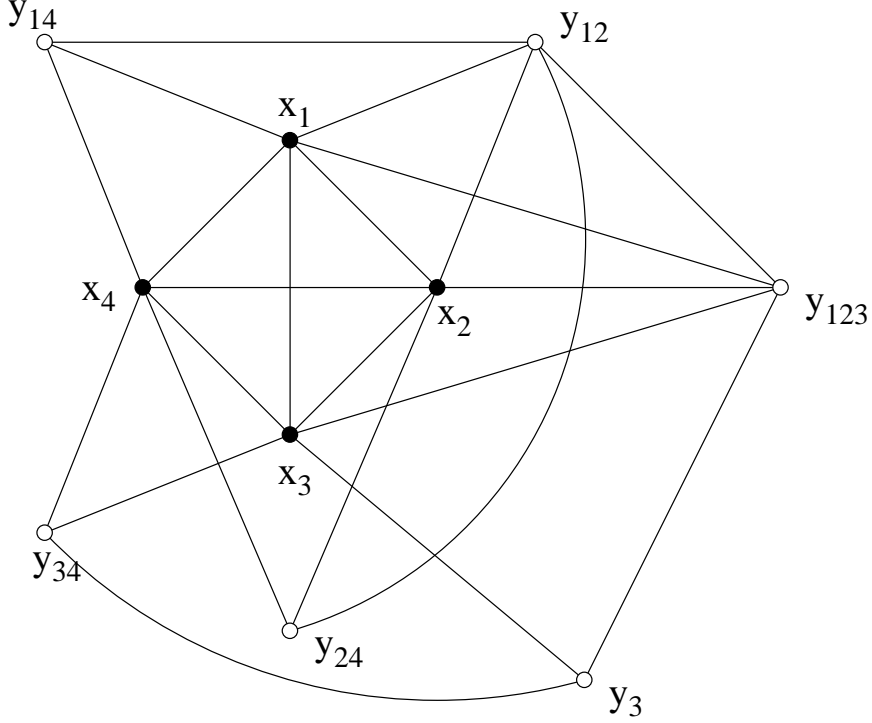
**Figure 3.5:** Graph  $H_5$

Lemma 3.2.9, we are done. □

**Lemma 3.2.8** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it has a rooted subgraph isomorphic to the graph  $H_5$  shown in Figure 3.5. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By Lemma 3.1.3, we may assume that  $G$  has no clique of size 5. Let  $P = \{y_i : i = 1, 2, 3\}$ ,  $Q = \{y_{i4} : i = 1, 2, 3\}$ ,  $X = \{x_1, \dots, x_4\}$  and  $Y = \{y_{123}\} \cup P \cup Q$ . For  $i = 1, 2, 3$ , apply Observation 1 to each of  $y_i y_{123}$ , and to each of  $y_{i4} x_4$ . It then follows that the graphs induced by  $P$  and  $Q$  are both connected.

First, suppose that  $V(G) = X \cup Y$ . Since  $G$  is 6-regular,  $G[P \cup Q]$  is 4-regular. Hence each vertex in  $P$  is adjacent to at least two vertices in  $Q$ , and vice-versa. Since  $G[P]$  is connected, we may assume, without loss of generality, that  $y_2$  is adjacent to  $y_3$ . Now  $y_1$  is adjacent to at least one vertex among  $y_{24}$  or  $y_{34}$ , say  $y_{34}$ . Further,  $y_{14}$  is adjacent to at least one vertex among  $y_2, y_3$ . But then  $G$  has a  $K_6$  minor (with branch sets given by  $\{y_{123}, y_1, y_{34}\}, \{y_{24}, y_2, y_3, y_{14}\}, \{x_1\}, \dots, \{x_4\}$ ).



**Figure 3.6:** Graph  $H_6$

Thus we may assume that  $G$  has a vertex outside  $X \cup Y$ . Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$ . We claim that  $\mathcal{B}$  has at least 5 attachments in  $P \cup Q$ . Suppose the contrary, and let  $B_0$  be the set of its attachments. Consider the separation  $(G \setminus (V(\mathcal{B}) - B_0), V(\mathcal{B}))$ . If there exists a matching between  $B_0$  and  $X$  that saturates  $B_0$ , then contracting it violates Observation 4. Hence, we may assume, without loss of generality, that  $B_0 = \{y_1, y_2, y_{14}, y_{24}\}$ . Since  $G[P]$  is connected,  $y_3$  is adjacent to at least one vertex among  $y_1$  and  $y_2$ , say  $y_1$ . But then contracting the sets  $\{x_4, y_{24}\}, \{y_1, y_3, y_{34}\}, \{y_2, x_2, x_1\}$  still violates Observation 4. This proves the claim (that  $|B_0| \geq 5$ ).

Now since  $G[P]$  is connected, without loss of generality,  $y_2$  is adjacent to both  $y_1$  and  $y_3$ . Since  $G[Q]$  is also connected, we thus get the following two cases:

Suppose  $y_{24}$  is adjacent to both  $y_{14}$  and  $y_{34}$ . Since  $B_0$  has at least 5 vertices, it contains either  $\{y_1, y_{34}\}$  or  $\{y_3, y_{14}\}$  (or both). In any case, it is easy to check that  $G$  has a  $K_6$  minor.

Alternatively, suppose  $y_{14}$  is adjacent to both  $y_{24}$  and  $y_{34}$ . If  $\mathcal{B}$  does not attach at  $y_1$ , then it attaches at all other vertices in  $P \cup Q$ , and  $G$  has a  $K_6$  minor. If  $\mathcal{B}$  attaches at  $y_1$ ,

then it also attaches at  $y_{24}$  or  $y_{34}$  (or both). In either case,  $G$  has a  $K_6$  minor. This finishes the proof of the lemma.  $\square$

**Lemma 3.2.9** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it has a rooted subgraph isomorphic to the graph  $H_6$  shown in Figure 3.6. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By Lemma 3.1.3, we may assume that  $G$  has no clique of size 5. Suppose that  $y_{123}$  is adjacent to  $y_{24}$ . Applying Observation 1 to  $y_{123}y_{24}$ , it follows that  $y_{24}$  is adjacent to  $y_3$ . Since  $\deg(y_{24}) = 6$ ,  $y_{24}$  is not adjacent to at least one vertex among  $y_{14}$  and  $y_{34}$ , say the former. Applying Observation 1 to  $x_4y_{14}$ , it follows that  $y_{14}$  is adjacent to  $y_{34}$ . But then  $G$  has a  $K_6$  minor. Hence we may assume that  $y_{123}$  is not adjacent to  $y_{24}$ . By a similar argument, we may also assume that  $y_{123}$  is not adjacent to  $y_{14}$ .

From Observation 2 applied to  $x_3y_3$ ,  $y_{123}$  is not adjacent to  $y_{34}$ . Now this means that  $y_{123}$  has a neighbor outside  $X \cup Y$ . Applying Observation 1 to  $y_{123}y_{12}$ , we get one of the following two cases:

**Case 1:** The vertex  $y_{12}$  is adjacent to  $y_3$ . But then consider the non-trivial separation  $(X \cup Y, V(G) \setminus (X \cup \{y_{12}\}))$ . Contracting the edges  $y_{12}x_1, y_{34}x_4, y_{24}x_2, y_{12}y_3, y_{123}x_3$ , it follows that the separation violates Observation 4, a contradiction.

**Case 2:** The vertices  $y_{123}$  and  $y_{12}$  have a common neighbor  $z_1$  outside  $X \cup Y$ . Applying Observation 1 to  $y_{123}y_3$ , it follows that  $y_3$  is adjacent to  $z_1$ . Applying Observation 1 to  $y_{12}y_{14}$ , it follows that  $y_{14}$  is adjacent to either  $z_1$  or  $y_{24}$  (or both). By symmetry,  $y_{24}$  is adjacent to either  $z_1$  or  $y_{14}$  (or both). Thus, if  $y_{14}, y_{24}$  are non-adjacent, then  $z_1$  must be adjacent to both of them. Further, applying Observation 1 to  $x_4y_{34}$ , it follows that  $y_{34}$  is adjacent to at least one vertex among  $y_{14}, y_{24}$ , and then  $G$  has a  $K_6$  minor. Hence we may assume that  $y_{14}, y_{24}$  are adjacent.

Now if  $z_1$  has no neighbor outside  $X \cup Y$ , then it must be adjacent to  $y_{14}$  and  $y_{24}$ , and again,  $G$  has a  $K_6$  minor. Thus  $z_1$  must have a neighbor outside  $X \cup Y$ . Let  $\mathcal{B}$  be an  $(X \cup Y \cup \{z_1\})$ -bridge of  $G$  containing such a neighbor.

If  $\mathcal{B}$  attaches at both  $y_{14}$  and  $y_{24}$ , then  $G$  has a  $K_6$  minor as before. Hence, without loss of generality, we may assume that it does not attach at  $y_{14}$ . But then  $\{z_1, y_3, y_{24}, y_{34}\}$  is a 4-cut separating  $\mathcal{B}$  from the rest of  $G$ , and the resulting separation violates Observation 4. This finishes the proof of the lemma.  $\square$

**Lemma 3.2.10** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ . Let  $G$  have a largest clique  $X$  of size 4, such that there is no vertex  $y \in Y$  with  $\deg_X(y) = 3$ . Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By the hypothesis of the lemma, and by Observation 1 applied to each edge in  $X$ , it follows that  $Y$  consists precisely of the six vertices  $y_{ij}$  for  $1 \leq i < j \leq 4$ , where  $y_{ij}$  is a common neighbor of  $x_i, x_j$ . Applying Observation 1 to  $x_1y_{12}$  and  $x_2y_{12}$ , it follows that the degree of  $y_{12}$  in  $G[Y]$  is at least 2. By symmetry, the minimum degree in  $G[Y]$  is at least two. Also, the maximum degree in  $G[Y]$  is clearly at most 4.

First, suppose that no vertex in  $y_{ij} \in Y$  is such that it shares two common neighbors in  $Y$  with each of  $x_i$  and  $x_j$ . (In particular, every vertex in  $Y$  has a neighbor outside  $X \cup Y$ .) By Observation 1 applied to  $x_1y_{1i}$  for  $i = 2, 3, 4$ ,  $y_{12}, y_{13}, y_{14}$  span a connected subgraph. Hence, without loss of generality, we may assume that  $y_{12}$  is adjacent to  $y_{13}$  and  $y_{14}$ . Further,  $G[y_{12}, y_{23}, y_{24}]$  consists of exactly two edges:  $y_{12}y_{23}, y_{23}y_{24}$  or  $y_{12}y_{24}, y_{24}y_{23}$ . By symmetry, we may assume the first outcome. Now since  $G[y_{13}, y_{23}, y_{34}]$  is connected,  $y_{13}$  must be adjacent to  $y_{34}$ . Finally,  $y_{24}$  is adjacent to either  $y_{14}$  or  $y_{34}$  (or both). If the former is true, then the branch sets  $\{y_{14}, y_{24}, y_{23}\}, \{y_{12}, y_{13}, y_{34}\}, \{x_1\}, \dots, \{x_4\}$  witness a  $K_6$  minor in  $G$ . If the latter is true, then the branch sets  $\{y_{13}, y_{34}, y_{24}\}, \{y_{23}, y_{12}, y_{14}\}, \{x_1\}, \dots, \{x_4\}$  witness a  $K_6$  minor in  $G$ .

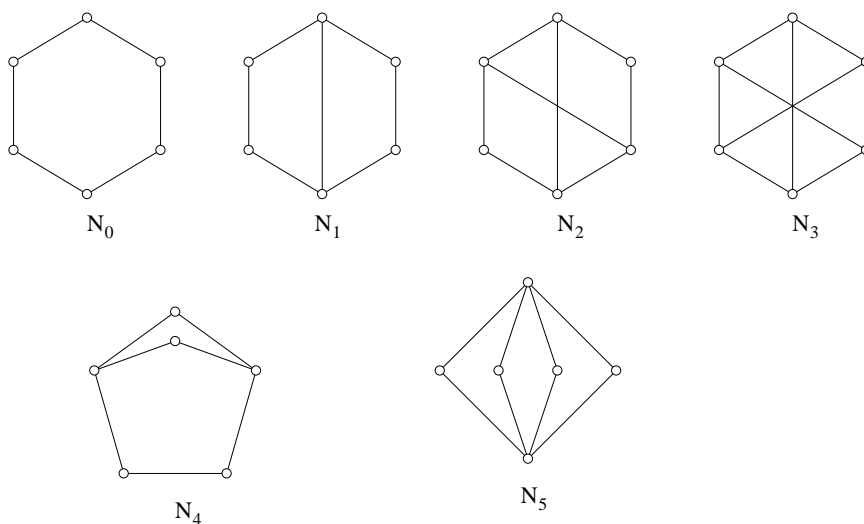
Thus we may assume, without loss of generality, that  $y_{12}$  shares two common neighbors in  $Y$  with each of  $x_1$  and  $x_2$ , that is, it is adjacent to  $y_{13}, y_{14}, y_{23}, y_{24}$ . If there is no vertex outside  $X \cup Y$ , then  $G[Y]$  must be 4-regular, and thus has all possible edges except those of the form  $y_{ij}y_{kl}$  with distinct  $i, j, k, l$ . But then  $\{y_{12}, y_{13}, y_{34}\}, \{y_{14}, y_{24}, y_{23}\}$ , and the four vertices in  $X$  form the branch sets of a  $K_6$  minor. Thus we may assume that  $(X \cup Y, V(G) \setminus (X \cup \{y_{12}\}))$  is a non-trivial separation. But contracting the edges  $x_1y_{14}, x_2y_{23}, x_3y_{34}, x_4y_{24}, y_{12}y_{13}$  makes

it violate Observation 4. This finishes the proof of the lemma.  $\square$

### 3.3 When $G$ has Clique Number 3

First, we analyze the possible isomorphism classes of  $G[N(x)]$ , where  $N(x)$  is the set of neighbors of a vertex  $x$ .

**Lemma 3.3.1** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and let the size of the largest clique in  $G$  be 3. Then for any vertex  $x$ , the set  $N(x)$  of its neighbors induces a graph isomorphic to one of the graphs  $N_0, \dots, N_5$  shown in Figure 3.7*



**Figure 3.7:** Possible neighborhoods of a vertex

*Proof:* Let  $N$  be the graph induced in  $G$  by  $N(x)$ . By Observation 1, the minimum degree in  $N$  is two, and hence it has a cycle.  $N$  has no triangles, since  $G$  has no clique of size 4. Let  $C$  be a longest cycle in  $N$ . If  $C$  has length 6, then the only other edges in  $N$  can be the main diagonals of  $C$ . Thus  $N$  is isomorphic to one of  $N_0, \dots, N_3$ . If  $C$  has length 5, then the vertex outside  $C$  is adjacent to an independent set in  $C$ . Thus that vertex is adjacent to exactly two vertices in  $C$ , and we get the graph  $N_4$ . Finally, suppose  $C$  has length 4. The two vertices outside  $C$  cannot be adjacent, otherwise  $N$  would contain a cycle of length greater than 4. It now follows that  $N$  must in fact be isomorphic to  $N_5$  (which is the complete bipartite graph  $K_{2,4}$ ).  $\square$

For a graph  $N$ , we say that  $G$  is  $N$ -free if no vertex has a neighborhood that induces a graph isomorphic to  $N$ .

For the following lemma, refer to Chapters 4 and 5 for the definitions of internal 5-connectivity, graph embedding and representativity (of an embedding).

**Lemma 3.3.2** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and let the size of the largest clique in  $G$  be 3. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By Lemma 3.3.1, the graphs induced by vertex neighborhoods are isomorphic to one of the graphs  $N_0, \dots, N_5$  shown in Figure 3.7. Suppose there exists a vertex  $x$  such that  $N(x)$  is isomorphic to  $N_5$ . Then by Lemma 3.3.3, we are done. Thus  $G$  is  $N_5$ -free.

Now applying Lemma 3.3.4, it follows that  $G$  is  $N_i$ -free for  $i = 1, 2, 3, 5$ . Finally, applying Lemma 3.3.11, we conclude that all neighborhoods in  $G$  are induced 6-cycles. It now follows that  $G$  has an embedding in a surface  $\Sigma$  such that each face (region of  $\Sigma - G$ ) is homeomorphic to the open disk and is bounded by a triangle in  $G$ . By Euler's formula, it now follows that the Euler genus of  $\Sigma$  is two. Thus  $G$  is a triangulation of either the torus or the Klein bottle.

First, let  $\Sigma$  be the torus. We claim that  $G$  is internally 5-connected. Suppose not, and let  $G$  have a separation  $(A, B)$  of order  $\leq 4$ . Then it follows that one of  $G[A], G[B]$  (say,  $G[A]$ ) has a planar drawing with the vertices of  $A \cap B$  on the exterior face. A simple application of Euler's formula now implies that  $A - B$  is empty (and the order of  $(A, B)$  is  $\leq 3$ ). This proves our claim. By Corollary 5.1.3,  $G$  has a  $K_6$  minor.

Finally, let  $\Sigma$  be the Klein bottle. Since  $G$  is a triangulation of  $\Sigma$ , with the neighborhoods of each vertex being an induced 6-cycle, it easily follows that the representativity of the embedding must be  $\geq 4$ . From Theorem 5.2.3, it follows that  $G$  has a  $K_6$  minor.  $\square$

The rest of this section is devoted to the subsidiary lemmas referenced in the above proof.

**Lemma 3.3.3** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and let the size of the largest clique in  $G$  be 3. Let  $x_1 \in V(G)$  be such that the set  $N(x_1)$  of its*

neighbors induces a graph isomorphic to the graph  $N_5$  shown in Figure 3.7. Then  $G$  has a minor isomorphic to  $K_6$ .

*Proof:* First, notice that among  $N_0, \dots, N_5$ , all but  $N_5$  have maximum degree  $\leq 3$ .

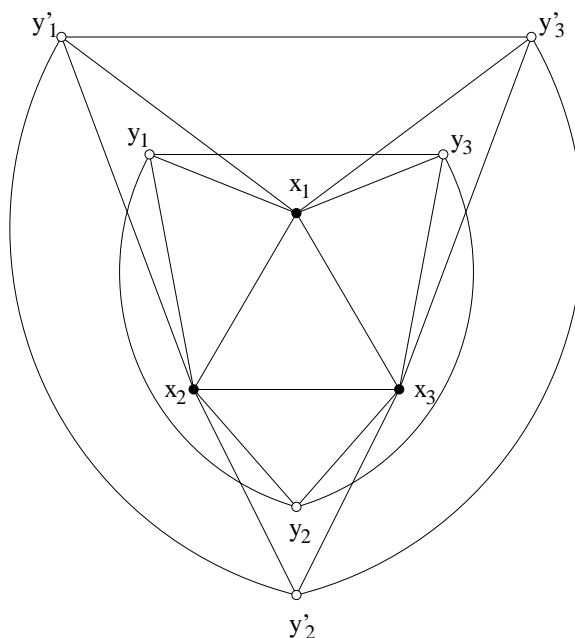
Let  $N(x_1) = \{x_2, x_3, y_1, \dots, y_4\}$  such that  $x_2$  and  $y_1$  are the two vertices of degree 4 in  $G[N(x_1)]$ . Since  $x_1$  has degree 4 in  $N(x_2)$ ,  $N(x_2)$  must also be isomorphic to  $N_5$ . Now, the other vertex of degree 4 in  $N(x_2)$  cannot be  $y_1$ , since  $G$  has no clique of size 4. Hence there must be a neighbor  $y_5$  of  $x_2$  that is adjacent to  $x_3, y_2, y_3, y_4$ . Now  $x_2$  has degree 4 in  $N(y_5)$ , and hence  $N(y_5)$  is also isomorphic to  $N_5$ . The other vertex of degree 4 in  $N(y_5)$  (besides  $x_2$ ) cannot be  $x_1$ . We claim that it cannot be  $y_1$  either. Suppose the contrary. Then,  $N(x_3)$  has the 4-cycle  $y_1x_1x_2x_5y_1$ . However, since  $N(x_3)$  cannot contain  $y_2, y_3$  or  $y_4$ , this 4-cycle is a connected component of  $G[N(x_3)]$ , which is a contradiction. This proves the claim.

Thus  $y_5$  has a neighbor  $y_6$  (distinct from  $y_1$ ) such that it is adjacent to  $x_3, y_2, y_3, y_4$ . But now  $y_5$  has degree 4 in  $N(y_6)$  (which is thus isomorphic to  $N_5$ ). The other vertex  $y_7$  of degree 4 in  $N(y_6)$  cannot be identical to  $x_2$  since  $G$  has no clique of size 4. It cannot be identical to  $x_1$  since  $G$  is 6-regular. By an argument similar to the one used for the above claim,  $y_7$  is not identical to  $y_1$  either. Thus  $G[N(x_3)]$  contains a path  $y_1x_1x_2y_5y_6y_7$  of length 6. It then follows that  $G[N(x_3)]$  contains a 6-cycle (and that  $y_7$  is adjacent to  $y_1$ ). But then,  $G$  has a  $K_6$  minor with branch sets given by  $\{x_1\}, \{x_2\}, \{x_3\}, \{y_2, y_5\}, \{y_3, y_6\}, \{y_4, y_1, y_7\}$ .  $\square$

**Lemma 3.3.4** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_5$ -free, and  $x_1 \in V(G)$  be such that the set  $N(x_1)$  of its neighbors induces a graph isomorphic to one of the graphs  $N_1, N_2, N_3$  shown in Figure 3.7. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* The neighborhood of  $x_1$  consists of a 6-cycle with at least one main diagonal. Let the main diagonal be the edge  $x_2x_3$  and the 6-cycle be  $y_1x_2y'_1y'_3x_3y_3y_1$ . (Note that the other two diagonal edges,  $y_1y'_3$  and  $y_3y'_1$ , may or may not exist.) By Observation 1 and 6-regularity respectively,  $x_2$  and  $x_3$  share at least one, and at most two, common neighbors besides  $x_1$ .

Suppose that  $x_2$  and  $x_3$  share two common neighbors  $y_2$  and  $y'_2$ . Notice that in  $G[N(x_2)]$ ,  $x_1$  and  $x_3$  are adjacent vertices of degree exactly three. Applying Lemma 3.3.1 to  $x_2$ , it follows that there must be a perfect matching between  $\{y_1, y'_1\}$  and  $\{y_2, y'_2\}$ . Using a similar argument for  $x_3$ , it follows that there must be a perfect matching between  $\{y_3, y'_3\}$  and  $\{y_2, y'_2\}$ . Without loss of generality, we may assume the edges  $y_1y_2$  and  $y'_1y'_2$ . Now if the second perfect matching between  $\{y_3, y'_3\}$  and  $\{y_2, y'_2\}$  has the edges  $y_2y'_3$  and  $y'_2y_3$ , then  $G$  has a  $K_6$  minor: contract the edges  $y_1y_3, y_2y'_3, y'_1y'_2$ . Thus the perfect matching has the edges  $y_2y_3$  and  $y'_2y'_3$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_7$  shown in Figure 3.8.



**Figure 3.8:** Graph  $H_7$

If there are two disjoint edges between  $\{y_1, y_2, y_3\}$  and  $\{y_4, y_5, y_6\}$ , then there is a 6-cycle passing through those 6 vertices, and contracting three independent edges in that cycle yields a  $K_6$  minor in  $G$ . Hence, without loss of generality, we may assume that the only edges between  $\{y_1, y_2, y_3\}$  and  $\{y_4, y_5, y_6\}$  are, possibly,  $y_1y'_2$  and  $y_1y'_3$ . (Note that  $y_1$  cannot be adjacent to  $y'_1$  since  $G$  has no clique of size 4.) But now, contract the edges  $x_iy_i$  for  $i = 1, 2, 3$  in  $G$  to get a graph  $G'$ . The only vertex in  $G'$  with degree less than 6 is possibly  $y'_3$ , and  $G'$  has at least 7 vertices because  $V(G) \neq X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$

and  $Y = \{y_1, y_2, y_3, y'_1, y'_2, y'_3\}$ . Thus  $G' \in \mathcal{F}_6$ , contradicting the minimality of  $G$ .

Thus  $x_2$  and  $x_3$  share exactly one common neighbor besides  $x_1$ , and let  $y_3$  be that common neighbor. Let  $y_4$  and  $y_5$  be the sixth neighbors, respectively, of  $x_2$  and  $x_3$ . Notice that in  $G[N(x_2)]$ ,  $x_1$  has degree exactly 3. Applying Lemma 3.3.1 to  $x_2$ , we get one of the following outcomes, by symmetry:

1.  $G$  contains the edges  $y_1y_3, y_3y_4, y'_1y_4$  ( $G[N(x_2)]$  is isomorphic to the graph  $N_4$ )
2.  $G$  contains the edges  $y_1y_3, y_1y_4, y'_1y_4$  ( $G[N(x_2)]$  is isomorphic to one of the graphs  $N_1, N_2, N_3$ )
3.  $G$  contains the edges  $y_3y_4, y_1y_4, y'_1y_4$  ( $G[N(x_2)]$  is isomorphic to the graph  $N_4$ )

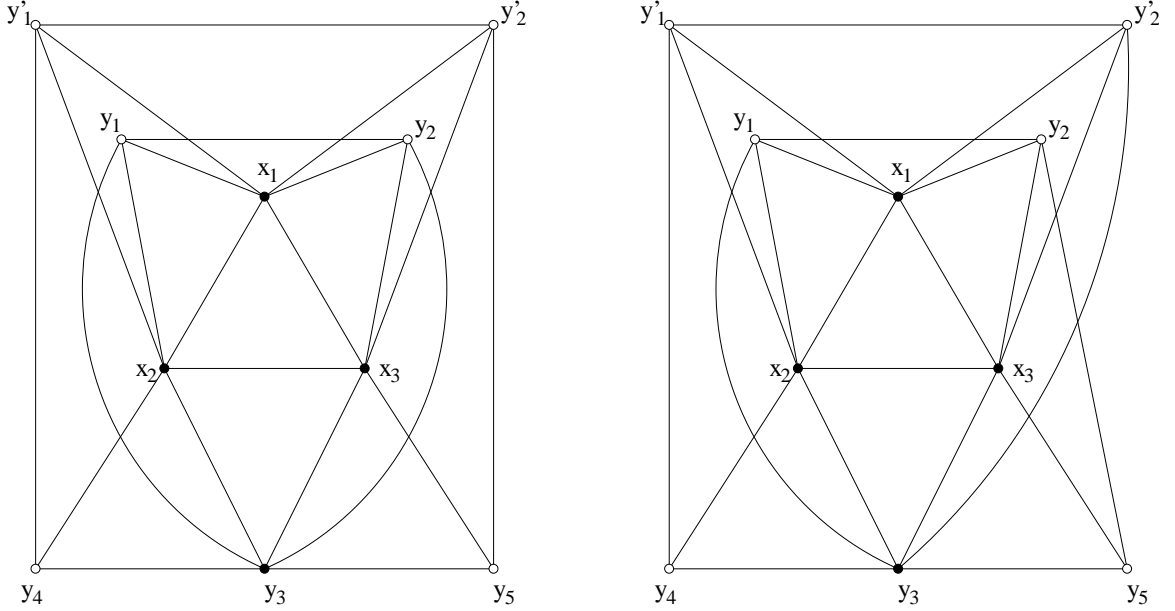
Applying Lemma 3.3.1 to  $x_3$ , we additionally get one of the following outcomes:

- A.  $G$  contains the edges  $y_2y_3, y_3y_5, y'_2y_5$  or the edges  $y'_2y_3, y_3y_5, y_2y_5$  ( $G[N(x_3)]$  is isomorphic to the graph  $N_4$ )
- B.  $G$  contains the edges  $y_2y_3, y_2y_5, y'_2y_5$  or the edges  $y'_2y_3, y_2y_5, y'_2y_5$  ( $G[N(x_3)]$  is isomorphic to one of the graphs  $N_1, N_2, N_3$ )
- C.  $G$  contains the edges  $y_3y_5, y_2y_5, y'_2y_5$  ( $G[N(x_3)]$  is isomorphic to the graph  $N_4$ )

Note that for  $x_2$ , in outcomes 1 and 2, there are two realizations each that are symmetric. However, for  $x_3$ , in the corresponding outcomes A and B, that symmetry is broken, and we need to consider both realizations. Note also that in outcomes 1 and 3 (respectively, A and C), there are no more edges in  $G[N(x_2)]$  (respectively,  $G[N(x_3)]$ ) than the ones indicated.

By symmetry, the combinations of two outcomes that can happen above are: 1 and A, 1 and B, 1 and C, 2 and B, 2 and C, and 3 and C. In these cases,  $G$  has a rooted subgraph isomorphic, respectively, to the graphs  $H_8$  or  $H'_8, H_9$  or  $H'_9, H_{10}, H_{11}$  or  $H'_{11}, H_{13}, H_{14}$  shown in Figures 3.9, 3.10, 3.11, 3.12, 3.14, 3.15 respectively. By applying Lemmas 3.3.5, 3.3.6, 3.3.7, 3.3.8, 3.3.9 and 3.3.10 in turn, it follows that  $G$  has a  $K_6$  minor. This finishes the proof of the lemma.  $\square$

**Lemma 3.3.5** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it is  $N_5$ -free and has a rooted subgraph isomorphic to one of the graphs  $H_8$  or  $H'_8$  shown in Figure 3.9. Then  $G$  has a minor isomorphic to  $K_6$ .*

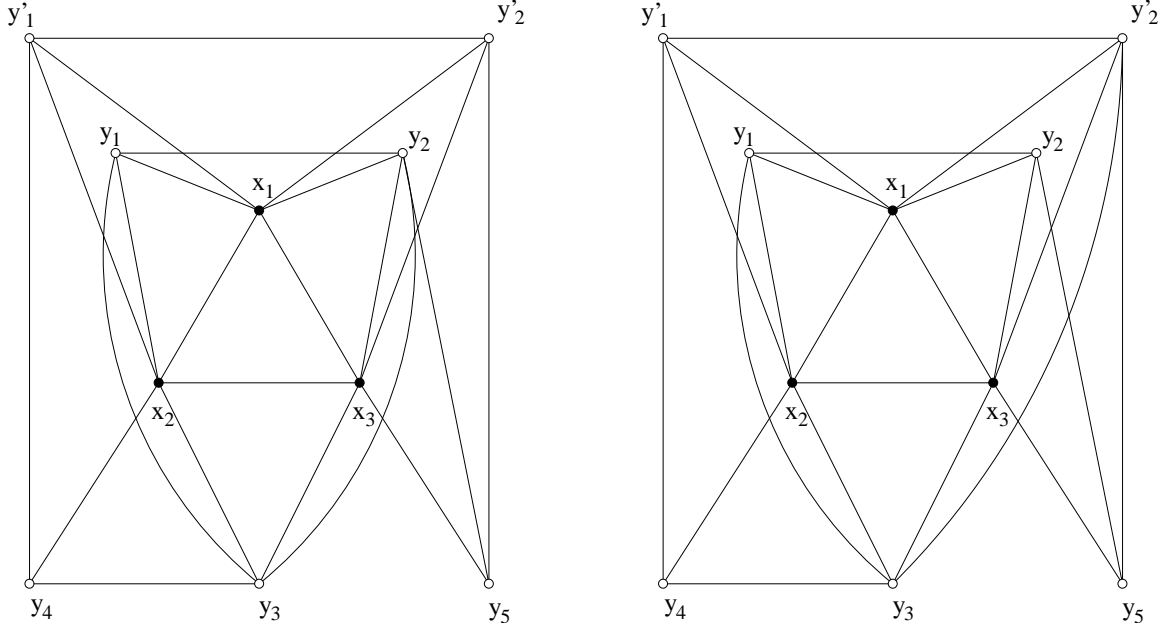


**Figure 3.9:** Graphs  $H_8$  and  $H'_8$

*Proof:* By Lemma 3.2.1, we may assume that  $G$  has no clique of size 4. Suppose that  $G$  has a rooted subgraph isomorphic to  $H_8$ . If either of the edges  $y_1y_5$  or  $y_2y_4$  exist, then  $G$  has a  $K_6$  minor. Now  $y_4$  must have a neighbor outside  $X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y'_1, y_2, y'_2, y_3, y_4, y_5\}$ ). Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. If  $\mathcal{B}$  attaches at  $y_2$  as well, or at  $y_1$  and  $y_5$ , then  $G$  has a  $K_6$  minor as before. Hence the possible attachments of  $\mathcal{B}$  are restricted to  $y'_1, y'_2, y_4$  and at most one of  $y_1, y_5$ . In any case, the resulting separation in  $G$  violates Observation 4.

Now suppose  $G$  has a rooted subgraph isomorphic to  $H'_8$ . Note that if either of the edges  $y'_1y_5$  or  $y_2y_4$  exist, then  $G$  has a  $K_6$  minor. The rest of the proof then proceeds as for graph  $H_8$  above, and this finishes the proof of the lemma.  $\square$

**Lemma 3.3.6** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it is  $N_5$ -free and has a rooted subgraph isomorphic to one of the graphs  $H_9$  or  $H'_9$  shown in Figure 3.10. Then  $G$  has a minor isomorphic to  $K_6$ .*



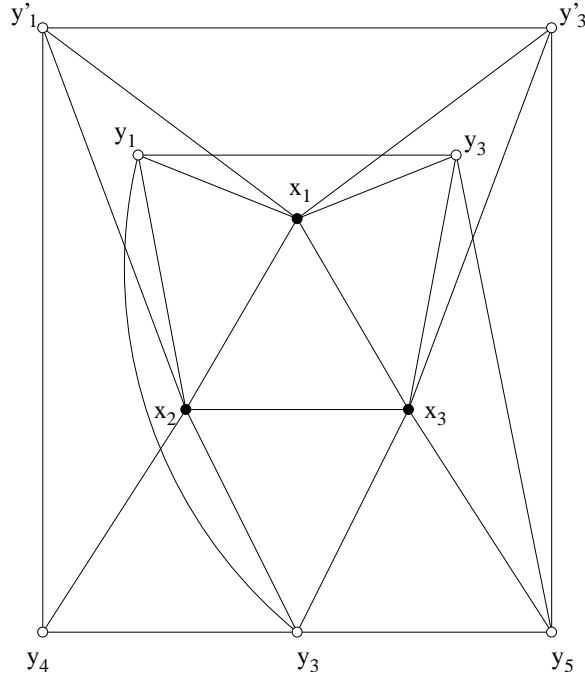
**Figure 3.10:** Graphs  $H_9$  and  $H'_9$

*Proof:* By Lemma 3.2.1, we may assume that  $G$  has no clique of size 4. Suppose first that  $G$  has a rooted subgraph isomorphic to  $H_9$ . If the edge  $y_1y_5$  exists, then  $G$  has a  $K_6$  minor. Otherwise, it must have a neighbor outside  $X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y'_1, y_2, y'_2, y_3, y_4, y_5\}$ ). Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. Now  $\mathcal{B}$  does not attach at  $y_5$ , otherwise  $G$  has a  $K_6$  minor as before. If  $\mathcal{B}$  attaches at  $y_2$ , then  $y_2$  has its sixth neighbor in  $\mathcal{B}$ . From Lemma 3.3.1 applied to  $y_2$ , and the fact that  $G$  is  $N_5$ -free, it then follows that this neighbor must also be adjacent to  $y_5$ , but then  $G$  has a  $K_6$  minor. Thus  $\mathcal{B}$  cannot attach at  $y_2$ . The possible attachments of  $\mathcal{B}$  are now restricted to  $y_1, y'_1, y_4$  and at most one vertex among  $y'_2, y_3$ . (If  $\mathcal{B}$  attaches at both  $y'_2$  and  $y_3$ , then  $G$  has a  $K_6$  minor witnessed by the branch sets  $\{x_1, y'_1, y_4\}, \{x_2, x_3\}, \{y_2, y_5\}, \{y'_2\} \cup V(\mathcal{B}), \{y_1\}, \{y_3\}$ .) In either case, the resulting separation in  $G$  violates Observation 4.

Thus we may assume that  $G$  has a rooted subgraph isomorphic to  $H'_9$ . If  $y_4$  is adjacent to either  $y_2$  or  $y_5$ , then  $G$  has a  $K_6$  minor. Since  $G$  has no clique of size 4,  $y_4$  is not adjacent to  $y_1$  either. Thus  $y_4$  has a neighbor outside  $X \cup Y$ . Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. Clearly,  $\mathcal{B}$  does not attach at  $y_2$  or  $y_5$ , otherwise  $G$  has a  $K_6$  minor as above.

We claim that  $\mathcal{B}$  cannot attach at  $y'_2$ . Suppose it does. Then  $y'_2$  has its sixth neighbor  $z$  in  $\mathcal{B}$ . Also, note that  $y'_1$  is not adjacent to  $y_3$  since  $G$  has no clique of size 4. Now from Lemma 3.3.1 applied to  $y'_2$ , it follows that  $G$  either has the edge  $y'_1y_5$  or the edges  $y'_1z, zy_5$ . In either case,  $G$  has a  $K_6$  minor. This proves the claim. But then the possible attachments of  $\mathcal{B}$  are restricted to  $y_1, y'_1, y_3, y_4$ , and the resulting separation in  $G$  violates Observation 4. This proves the lemma.  $\square$

**Lemma 3.3.7** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it is  $N_5$ -free and has a rooted subgraph isomorphic to the graph  $H_{10}$  shown in Figure 3.11. Then  $G$  has a minor isomorphic to  $K_6$ .*

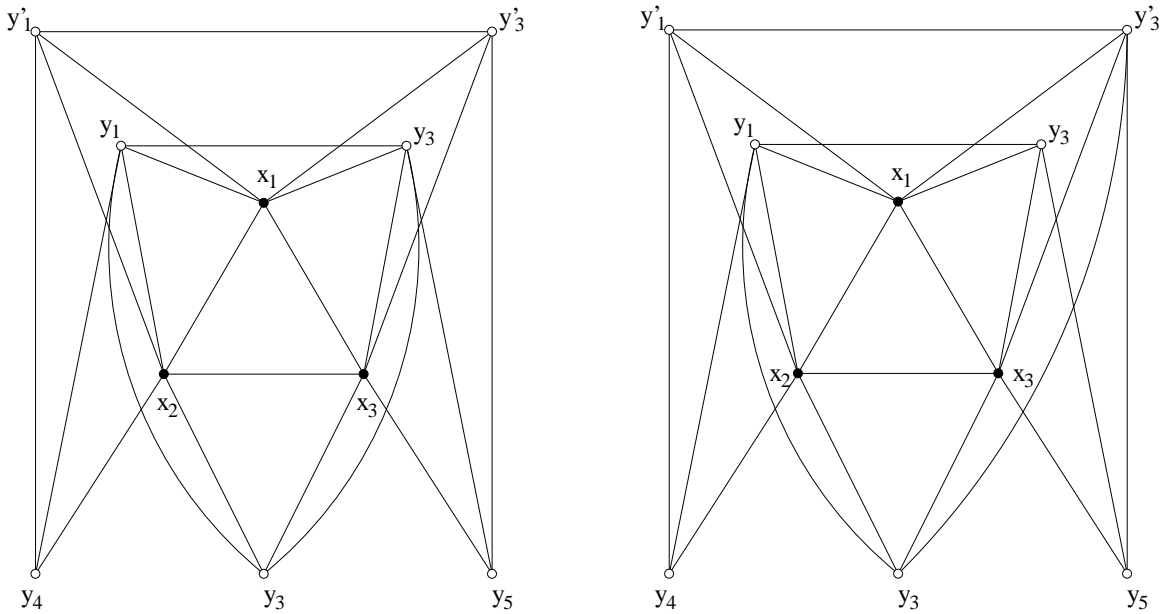


**Figure 3.11:** Graph  $H_{10}$

*Proof:* By Lemma 3.2.1, we may assume that  $G$  has no clique of size 4. If  $y_2$  is adjacent to  $y_4$ , then  $G$  has a  $K_6$  minor. Also, if  $y_2$  is adjacent to  $y'_1$ , then  $G$  has a  $K_6$  minor, witnessed by the branch sets  $\{x_1\}, \{x_2, x_3\}, \{y_1, y_3, y_4\}, \{y'_1\}, \{y_2\}, \{y'_2, y_5\}$ . Now  $y_2$  has a neighbor outside  $X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y'_1, y_2, y'_2, y_3, y_4, y_5\}$ ). Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. If  $\mathcal{B}$  attaches at  $y_4$  or  $y'_1$ , then  $G$  has a  $K_6$  minor as before. Thus the possible attachments of  $\mathcal{B}$  are now restricted to  $\{y_1, y_2, y'_2, y_3, y_5\}$ .

But then the attachments of  $\mathcal{B}$  separate it from the rest of  $G$ , and that separation violates Observation 4. (To get the  $K_5$  needed for the hypothesis of Observation 4, choose the branch sets  $\{y_1, x_2, x_3\}, \{y_2, x_1\}, \{y_5\}, \{y_3, y_4, y'_1\}, \{y'_2\}$ .) This contradiction finishes the proof of the lemma.  $\square$

**Lemma 3.3.8** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it is  $N_5$ -free and has a rooted subgraph isomorphic to one of the graphs  $H_{11}$  or  $H'_{11}$  shown in Figure 3.12. Then  $G$  has a minor isomorphic to  $K_6$ .*



**Figure 3.12:** Graphs  $H_{11}$  and  $H'_{11}$

*Proof:* By Lemma 3.2.1, we may assume that  $G$  has no clique of size 4. First, suppose that  $G$  has a rooted subgraph isomorphic to  $H_{11}$ . Applying Lemma 3.3.1 to  $y_1$ , it follows that exactly one of the following outcomes occur:

1. the vertex  $y'_2$ , or
2. the vertex  $y_5$ , or
3. a vertex  $z$  outside  $X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y'_1, y_2, y'_2, y_3, y_4, y_5\}$ )

By symmetry, applying Lemma 3.3.1 to  $y_2$  yields similar results. Now if  $y_1$  is adjacent to  $y'_2$  (outcome 1 above), then  $y'_2$  must be adjacent to  $y_4$  (from Lemma 3.3.1). Looking at the

five neighbors of  $y'_2$  uncovered so far, and applying Lemma 3.3.1 to it, we conclude that  $y_5$  is adjacent to either  $y'_1$  or  $y_4$ . In either case,  $G$  has a  $K_6$  minor.

Now suppose  $y_1$  is adjacent to  $y_5$ . From Lemma 3.3.1 applied to  $y_1$ , the edge  $y_4y_5$  exists. Also, we may assume that  $y_2$  is not adjacent to  $y_4$  or  $y'_1$ , otherwise  $G$  has a  $K_6$  minor. Thus  $y_2$  has its sixth neighbor  $z$  outside  $X \cup Y$ . Applying Lemma 3.3.1 to  $y_2$ , it follows that  $z$  is adjacent to both  $y_3$  and  $y_5$ . But then, applying Lemma 3.3.1 to  $y_3$  now yields a contradiction.

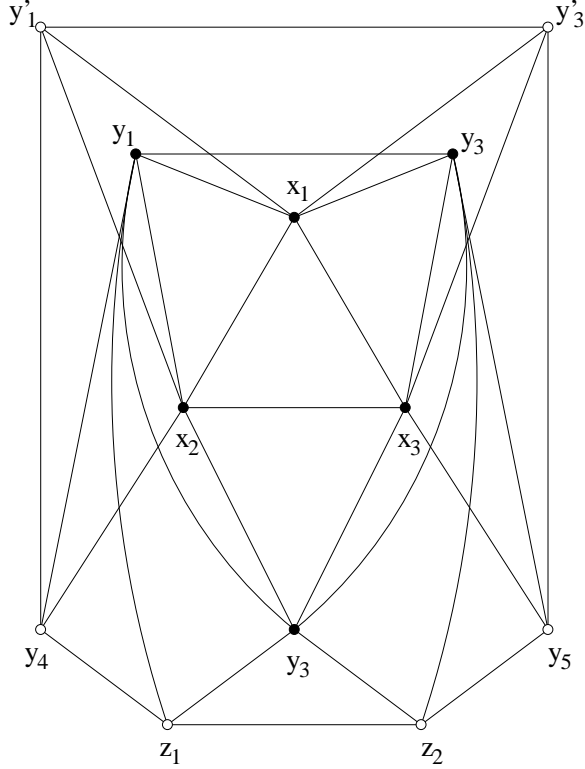
Thus the only possibility left is that outcome 1 above occurs for  $y_1$ , and its symmetrical version occurs for  $y_2$ . Thus they both have neighbors  $z_1, z_2$  respectively outside  $X \cup Y$  (where  $z_1, z_2$  may be identical). If  $z_1 = z_2 = z$ , then from Lemma 3.3.1 applied in turn to  $y_1$  and  $y_2$ , it follows that  $z$  is adjacent to  $y_4$  and  $y_5$ . Note that  $y_3$  cannot be adjacent to  $y_4, y_5$  or  $z$  since  $G$  has no clique of size 4. Applying Lemma 3.3.1 to  $y_3$ , it now follows that the two additional neighbors of  $y_3$  must be  $y'_1$  and  $y'_2$  (and that  $G[N(y_3)]$  is isomorphic to  $N_1$ ). But then  $G$  has a  $K_6$  minor.

Thus  $z_1, z_2$  are distinct vertices. Now applying Lemma 3.3.1 to  $y_1$  and  $y_2$ , we get:  $z_1$  is adjacent to  $y_3$  and  $y_4$ , and  $z_2$  is adjacent to  $y_3$  and  $y_5$ . Further, applying Lemma 3.3.1 to  $y_3$ , it follows that  $z_1, z_2$  are adjacent. Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{12}$  shown in Figure 3.13.

Let  $P = X \cup \{y_1, y_2, y_3\}$ . Consider the graph  $G_0 = G \setminus P$ , with  $\Omega = y'_1, y_4, z_1, z_2, y_5, y'_2$  (in that cyclic order). Since  $G_0$  has  $\frac{1}{2}(6(n-6) + 24) = 3n - 6$  edges (where  $n$  is the number of its vertices), it cannot have a planar drawing with  $V(\Omega)$  on the boundary of the infinite face.

We claim that  $G_0$  has no  $\leq 3$ -separation  $(A, B)$  with  $V(\Omega) \subseteq A$ . Suppose it does; choose  $(A, B)$  as above such that its order  $k$  is minimal. Then by Menger's theorem, there exist  $k$  vertex disjoint paths from  $V(\Omega)$  to  $A \cap B$ . It now follows easily that the separation  $(A \cup P, B)$  in  $G$  violates Observation 4. This proves our claim.

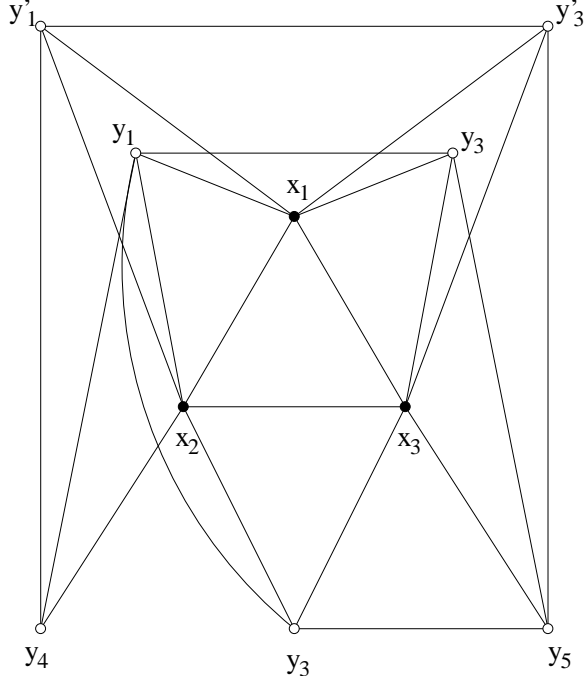
Thus by Theorem 3.1.2 the society  $(G_0, \Omega)$  has a cross. It is now easy to check that  $G$  has a  $K_6$  minor: let  $V(\Omega)$  and the cross form four branch sets of the  $K_6$ , and let the other two branch sets be  $\{x_1, y_1, y_2\}$  and  $\{x_2, x_3, y_3\}$ .



**Figure 3.13:** Graph  $H_{12}$

Finally, suppose  $G$  has a rooted subgraph isomorphic to the graph  $H'_{11}$ . Note that if  $y_5$  is adjacent to either  $y'_1$  or  $y_4$ , then  $G$  has a  $K_6$  minor. Thus  $y_5$  has a neighbor outside  $X \cup Y$ . Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. Clearly,  $\mathcal{B}$  does not attach at  $y'_1$  or  $y_4$ , otherwise  $G$  has a  $K_6$  minor as before. If  $\mathcal{B}$  attaches at  $y_1$ , then  $y_1$  has its sixth neighbor  $z$  in  $\mathcal{B}$ . Note that  $z$  is not adjacent to  $y_4$ , since  $\mathcal{B}$  does not attach at  $y_4$ . Applying Lemma 3.3.1 to  $y_1$ , it now follows that  $G$  has the edge  $y_2y_4$ . But then  $G$  has a  $K_6$  minor. Thus we may assume that  $\mathcal{B}$  does not attach at  $y_1$ . The possible attachments of  $\mathcal{B}$  are now restricted to  $y_2, y'_2, y_3, y_5$ , and the resulting separation in  $G$  violates Observation 4. This finishes the proof of the lemma.  $\square$

**Lemma 3.3.9** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it is  $N_5$ -free and has a rooted subgraph isomorphic to the graph  $H_{13}$  shown in Figure 3.14. Then  $G$  has a minor isomorphic to  $K_6$ .*



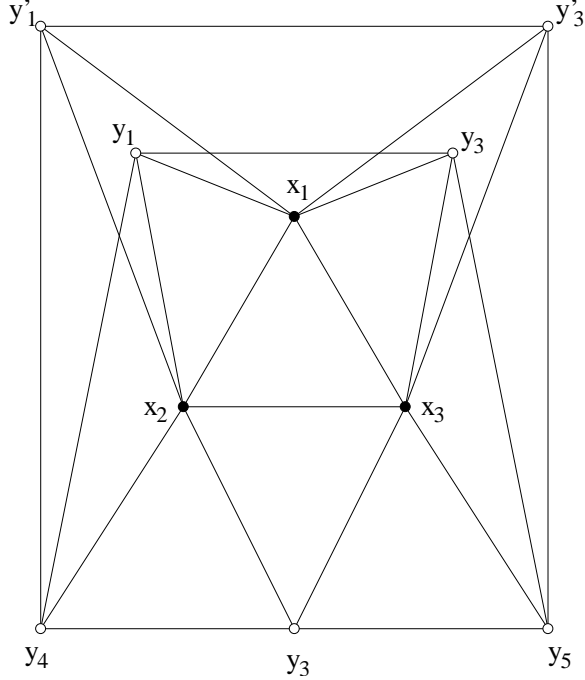
**Figure 3.14:** Graph  $H_{13}$

*Proof:* By Lemma 3.2.1, we may assume that  $G$  has no clique of size 4. If  $y_2$  is adjacent to either  $y'_1$  or  $y_4$ , then  $G$  has a  $K_6$  minor. Thus  $y_2$  has a neighbor outside  $X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y'_1, y_2, y'_2, y_3, y_4, y_5\}$ ). Let  $\mathcal{B}$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. Clearly,  $\mathcal{B}$  does not attach at  $y'_1$  or  $y_4$ , otherwise  $G$  has a  $K_6$  minor as before.

We claim that  $\mathcal{B}$  does not attach at  $y_1$ . Suppose it does; then  $y_1$  has a sixth neighbor  $z$  outside  $X \cup Y$ . Note that  $z$  is not adjacent to  $y_4$  since  $\mathcal{B}$  does not attach at  $y_4$ . Also,  $y_4$  is not adjacent to  $y_3$  since  $G$  has no clique of size 4. But then, applying Lemma 3.3.1 to  $y_1, y_4$  is adjacent to  $y_2$ , and  $G$  has a  $K_6$  minor. This proves the claim.

Now the possible attachments of  $\mathcal{B}$  are restricted to  $y_2, y'_2, y_3, y_5$ , and the resulting separation in  $G$  violates Observation 4. This proves the lemma.  $\square$

**Lemma 3.3.10** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , such that it is  $N_5$ -free and has a rooted subgraph isomorphic to the graph  $H_{14}$  shown in Figure 3.15. Then  $G$  has a minor isomorphic to  $K_6$ .*



**Figure 3.15:** Graph  $H_{14}$

*Proof:* By Lemma 3.2.1, we may assume that  $G$  has no clique of size 4. If either of the edges  $y_1y_2'$  or  $y_2y_1'$  exist, then  $G$  has a  $K_6$  minor. Similarly, we may assume there is no  $(X \cup Y)$ -bridge of  $G$  that attaches at  $y_1, y_2'$  or at  $y_2, y_1'$ .

Now  $y_1$  has a neighbor outside  $X \cup Y$  (where  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_1', y_2, y_2', y_3, y_4, y_5\}$ ). Let  $\mathcal{B}_1$  be an  $(X \cup Y)$ -bridge of  $G$  containing such a neighbor. If  $\mathcal{B}_1$  does not attach at  $y_5$ , then its attachments are restricted either to  $y_1, y_2, y_3, y_4$  or  $y_1, y_1', y_3, y_4$ . In either case, the resulting separation in  $G$  violates Observation 4. Thus  $\mathcal{B}_1$  must attach at  $y_5$ .

Using a similar argument as in the previous paragraph, we get an  $(X \cup Y)$ -bridge  $\mathcal{B}_2$  of  $G$  that attaches at  $y_2$  and  $y_4$ . If  $\mathcal{B}_1 \neq \mathcal{B}_2$ , then  $G$  has a  $K_6$  minor. Thus  $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$ . If  $\mathcal{B}$  does not attach at  $y_3$ , then its only attachments are  $y_1, y_2, y_4, y_5$ , and the resulting separation in  $G$  violates Observation 4. Thus  $\mathcal{B}$  attaches precisely at  $y_1, \dots, y_5$ .

Interchanging the roles of  $y_1, y_2$  with  $y_1', y_2'$ , and using the above argument, we further get an  $(X \cup Y)$ -bridge  $\mathcal{B}'$  that attaches precisely at  $y_1', y_2', y_3, y_4, y_5$ . By 6-regularity,  $y_4$  and  $y_5$  are each incident to exactly one edge of  $\mathcal{B}$  and  $\mathcal{B}'$  respectively. But now, it follows that the number of edges in  $\mathcal{B}$  that are incident to exactly one attachment is five, an odd number.

This contradicts the 6-regularity of  $G$ , and finishes the proof of the lemma.  $\square$

We now turn towards eliminating  $N_4$  from the list of graphs induced by the neighborhood of a vertex in  $G$ . For the subsequent discussion,  $G$  is a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , with no clique of size 4, and which is  $N_i$ -free for  $i = 1, 2, 3, 5$ .

For an edge  $uv$  in  $G$ , the vertices  $u$  and  $v$  share either two or three common neighbors. By “a common neighbor of the edge  $e$ ”, we mean a common neighbor of its endpoints. We shall call an edge *thick* if it has three common neighbors. The following observations are trivial.

**Observation 5** *If  $uv$  is a thick edge, then  $G[N(u)]$  and  $G[N(v)]$  are both isomorphic to  $N_4$ . Also,  $v$  is one of the two vertices of degree three in  $G[N(u)]$  (and vice versa).*

**Observation 6** *The thick edges in  $G$  induce a 2-regular subgraph (unless there are no thick edges at all).*

**Observation 7** *No triangle in  $G$  has more than one thick edge. In particular, the graph induced by the thick edges consists of cycles of length at least 4.*

**Observation 8** *If  $N(x)$  contains a 4-cycle, for some vertex  $x$ , then the two thick edges incident with  $x$  join it to two alternate vertices on the 4-cycle.*

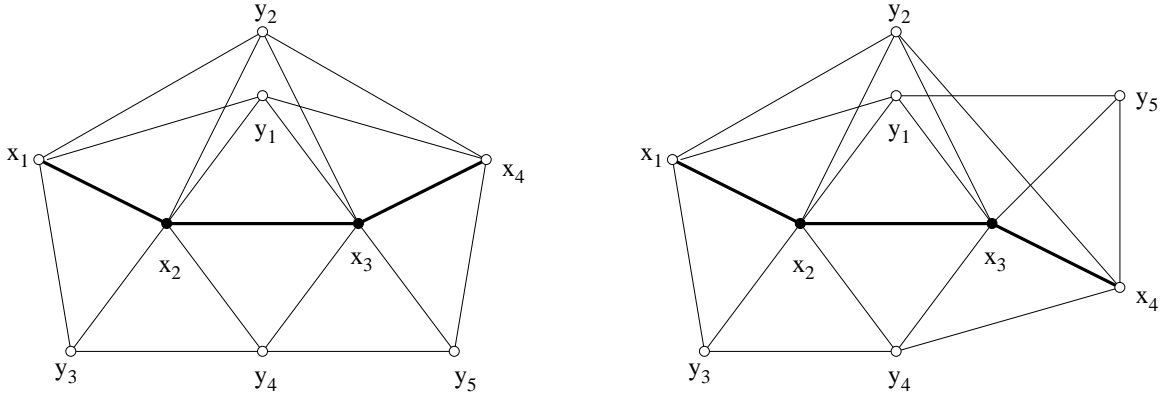
On inspecting  $N_4$ , it follows that two adjacent thick edges must share exactly two of their common neighbors. (In other words, if  $x_1x_2$  and  $x_2x_3$  are both thick edges, then there are exactly two vertices  $y$  that are incident to  $x_1, x_2, x_3$ .) Now let  $e$  be a thick edge and  $e', e''$  be the two thick edges incident to it. It now follows that the number of common neighbors that  $e, e', e''$  share is either one or two. If they share two common neighbors, then we call  $e$  a *regular* edge. Otherwise (that is, if they share exactly one common neighbor), we call  $e$  an *irregular* edge.

**Lemma 3.3.11** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and  $x \in V(G)$  be such that the set  $N(x)$  of its neighbors induces a graph isomorphic to the graph  $N_4$  shown in Figure 3.7. Then  $G$  has a minor isomorphic to  $K_6$ .*

*Proof:* By the hypothesis of the lemma,  $G$  has a thick edge. (In fact, since the thick edges form a 2-regular subgraph,  $G$  has at least 3 thick edges.)

Consider two adjacent thick edges. If they are both regular, then by Lemma 3.3.13,  $G$  has a  $K_6$  minor. If exactly one of them is regular, then we are done by Lemma 3.3.14. Finally, if they are both irregular, then we are done by Lemma 3.3.18.  $\square$

**Lemma 3.3.12** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ . Let  $x_1x_2, x_2x_3, x_3x_4$  be three thick edges (with  $x_1 \neq x_4$ ). Thus the graph induced by  $P = \{x_2, x_3\} \cup N(x_2) \cup N(x_3)$  is isomorphic to one of those shown in Figure 3.16 (depending on whether  $x_2x_3$  is a regular edge.) Then  $x_4$  does not share any neighbor with  $x_2$  other than the ones it shares with  $x_3$  (that is, other than  $y_1, y_2$ ). By symmetry,  $x_1, x_3$  also have no common neighbors other than  $y_1, y_2$ .*



**Figure 3.16:** Neighborhood of a thick edge

*Proof:* Clearly,  $y_1$  and  $y_2$  have no more edges in  $G[P]$  than those indicated, since  $G$  has no cliques of size 4.

First, suppose  $x_2x_3$  is irregular, that is,  $G[P]$  is the graph on the right in Figure 3.16. If the conclusion of the lemma were false, then  $x_4$  must be adjacent to either  $x_1$  or  $y_3$ . But then  $N(y_1)$  or  $N(y_4)$  (respectively) has a 4-cycle. By Observation 8, one of the edges  $y_1x_2, y_1x_3$  (respectively,  $y_4x_2, y_4x_3$ ) must be thick, which contradicts Observation 6.

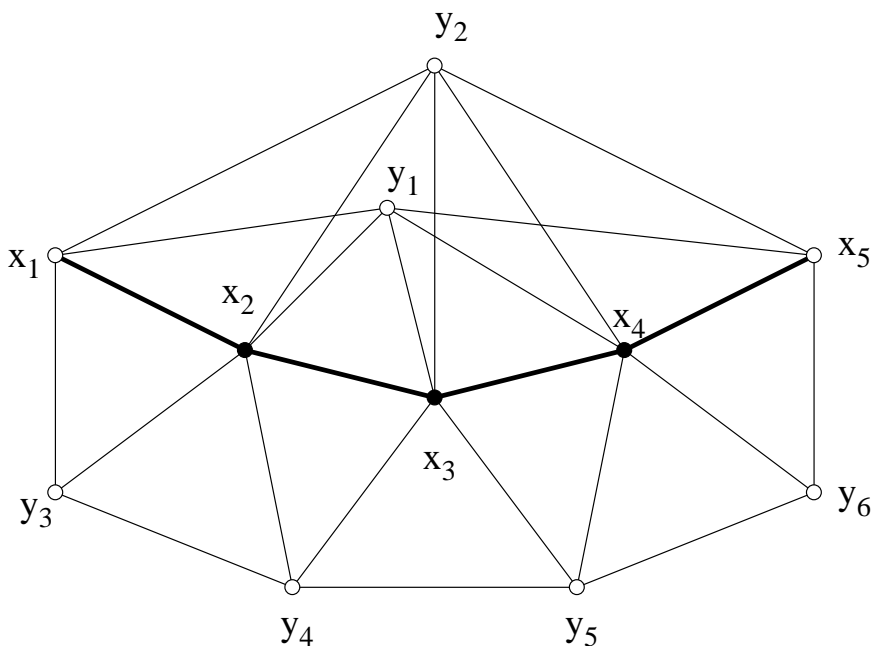
Thus, we may assume that  $x_2x_3$  is regular, that is,  $G[P]$  is the graph on the left in Figure 3.16. Suppose that the conclusion of the lemma is false, and that  $x_4$  is adjacent to

$x_1$ . Since  $G[N(y_1)]$  contains a 4-cycle. This leads to a similar contradiction as above.

Now suppose  $x_4$  is adjacent to  $y_3$ . Since  $y_3$  is not adjacent to  $y_1, y_2$  or  $x_3$ , it follows that  $x_4y_3$  has exactly two common neighbors. Thus the edge  $x_4y_3$  cannot be thick, and  $y_3$  is adjacent to  $y_5$ . But then  $G[N(y_4)]$  contains a 4-cycle, and neither  $y_4x_2$  nor  $y_4x_3$  is a thick edge, which is a contradiction as above.  $\square$

**Lemma 3.3.13** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and let  $G$  have two adjacent thick edges that are both regular. Then  $G$  has a  $K_6$  minor.*

*Proof:* From Lemma 3.3.12, it follows that  $G$  has a rooted subgraph isomorphic to the graph shown in Figure 3.17. (The edges  $x_2x_3, x_3x_4$  are the two adjacent regular edges.) The sixth



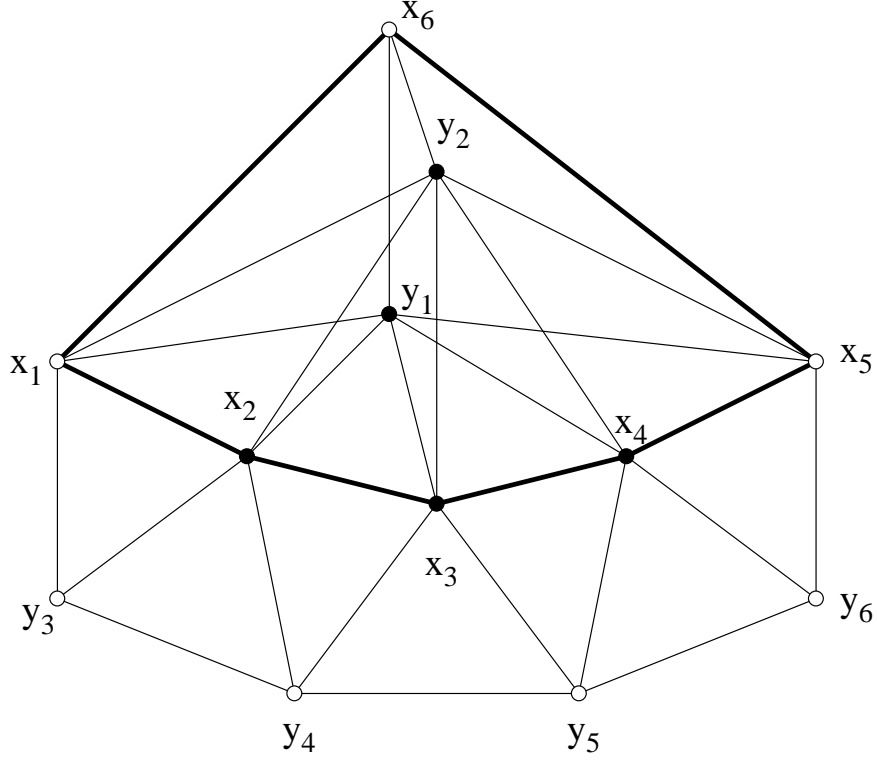
**Figure 3.17:** Graph with two adjacent regular edges

neighbor of  $y_1$  cannot be in  $\{y_2, \dots, y_6\}$ . Also,  $N(y_1)$  must be isomorphic to  $N_0$  (a 6-cycle). Thus  $y_1$  has a neighbor  $z_1$  adjacent to  $x_1$  and  $x_5$ . Similarly,  $y_2$  has a neighbor  $z_2$  adjacent to  $x_1$  and  $x_5$ .

If  $z_1, z_2$  are distinct, then one of the edges  $x_5z_1, x_5z_2$  must be thick. (Note that by Observation 7, none of the edges  $x_5y_1, x_5y_2, x_5y_6$  can be thick.) We may assume, without

loss of generality, that  $x_5z_1$  is thick. Since  $N(x_5)$  is isomorphic to  $N_4$ , it follows that  $z_1, z_2$  are adjacent. But then  $N(z_2)$  has a 4-cycle  $x_1y_2x_5z_1x_1$ . But neither of the edges  $z_2y_2, z_2x_5$  are thick, which contradicts Observation 8. Thus  $z_1 = z_2$ , and call this vertex  $x_6$ .

Now  $N(x_6)$  has a 4-cycle  $x_1y_1x_5y_2$ , and by Observation 8, it follows that the edges  $x_6x_1, x_6x_5$  are thick. Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{15}$  shown in Figure 3.18.

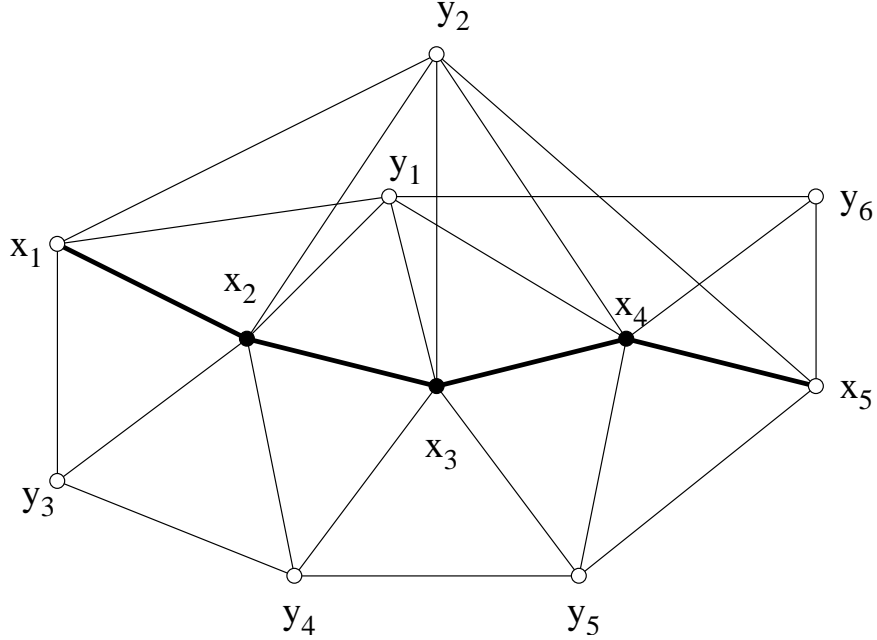


**Figure 3.18:** Graph  $H_{15}$

By examining  $N(x_1), N(x_5)$  respectively, it follows that  $x_6$  cannot be adjacent to  $y_3$  or  $y_6$ . Now  $N(x_6)$  has two more vertices other than the ones shown, and  $x_1, x_5$  must each be adjacent to one of them. Neither of those vertices can be  $y_4$  or  $y_5$ , since by Lemma 3.3.12, the edges  $x_1y_5, x_5y_4$  cannot exist. Thus  $x_6$  has neighbors  $y_7, y_8$  distinct from all the vertices in  $H_{15}$ , such that  $x_1$  is adjacent to  $y_7$  and  $x_5$  is adjacent to  $y_8$ . Also, by examining  $N(x_1), N(x_5)$  and  $N(x_6)$  in turn, we get the edges  $y_7y_3, y_7y_8$  and  $y_8y_6$ . But then, contracting a perfect matching between  $\{x_1, \dots, x_6\}$  and  $\{y_3, \dots, y_8\}$  yields a graph in  $\mathcal{F}_6$ , a contradiction. This proves the lemma.  $\square$

**Lemma 3.3.14** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and let  $G$  have two adjacent thick edges  $x_2x_3, x_3x_4$ , such that  $x_2x_3$  is regular and  $x_3x_4$  is irregular. Then  $G$  has a  $K_6$  minor.*

*Proof:* By Lemma 3.3.12, without loss of generality,  $G$  has a rooted subgraph isomorphic to the graph  $H_{16}$  shown in Figure 3.19.



**Figure 3.19:** Graph  $H_{16}$

First, suppose  $x_4x_5$  is regular. Let the second thick edge incident with  $x_5$  join it to a vertex  $x_6$ . We claim that  $x_6$  cannot be any vertex in  $H_{16}$ . (By Lemma 3.3.12, it suffices to show that it is not  $x_1$  or  $y_3$ .) Since  $N(x_5)$  is isomorphic to  $N_4$ , it follows that  $x_6$  is adjacent to at least one vertex among  $y_2, y_5$ . This means that  $x_6 \neq y_3$ . (Note that  $x_3$  cannot be adjacent to  $y_5$ , otherwise  $N(y_4)$  violates Observation 8.) If  $x_6 = x_1$ , on the other hand,  $N(y_2)$  has a 5-cycle (and hence is isomorphic to  $N_5$ ). But none of the edges  $y_2x_i$  for  $i = 2, 3, 4$  can be thick, which is a contradiction. This proves the claim that  $x_6$  is outside  $V(H_{16})$ .

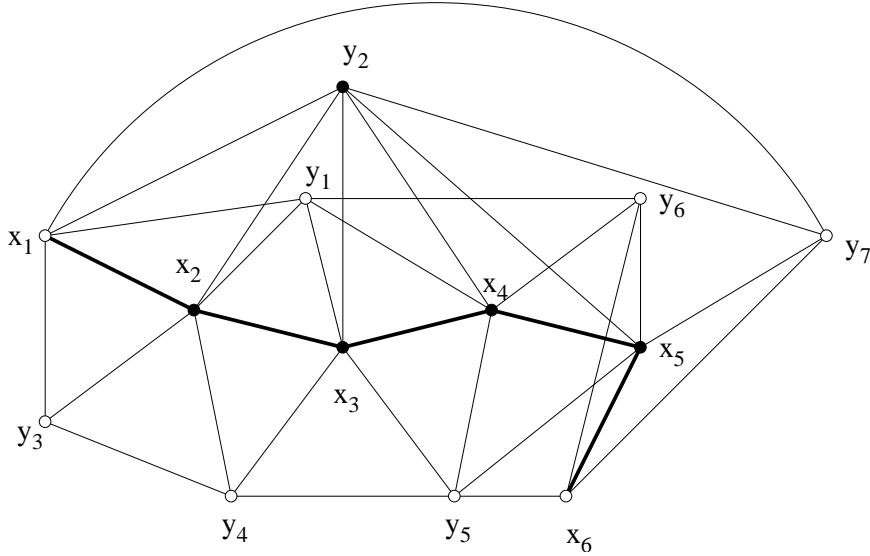
Now since  $x_4x_5$  is regular,  $x_6$  shares three neighbors with  $x_5$ , comprised of: (i) the vertex  $y_6$ , (ii) exactly one vertex among  $y_2, y_5$  and (iii) a vertex  $y_7$ . It follows by an argument similar to the above claim that  $y_7$  cannot be in  $V(H_{16}) \cup \{x_6\}$ .

Suppose now that  $x_6$  is adjacent to  $y_6, y_5, y_7$ . From  $N(x_5)$ , we get the edges  $x_5y_7, y_2y_7$ . Now  $N(y_2)$  cannot be isomorphic to  $N_4$ . Hence we get the edge  $x_1y_7$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{17}$  shown in Figure 3.20, and by Lemma 3.3.15, we are done.

On the other hand, if  $x_6$  is adjacent to  $y_6, y_2, y_7$ , then  $N(y_2)$  must be a 6-cycle, and we get the edge  $x_6x_1$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{18}$  shown in Figure 3.21, and by Lemma 3.3.16, we are done.

Finally, suppose  $x_4x_5$  is irregular. Then  $G$  has a rooted subgraph isomorphic to the graph  $H_{19}$  shown in Figure 3.22, and by Lemma 3.3.17, we are done.  $\square$

**Lemma 3.3.15** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and let  $G$  have a rooted subgraph isomorphic to the graph  $H_{17}$  shown in Figure 3.20. Then  $G$  has a  $K_6$  minor.*



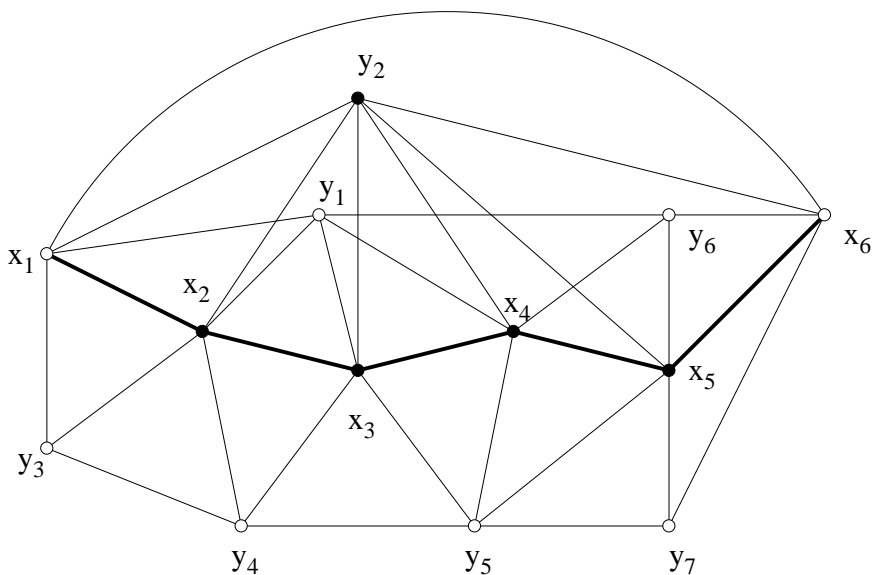
**Figure 3.20:** Graph  $H_{17}$

*Proof:*  $N(y_1)$  cannot be isomorphic to  $N_4$ . Let  $y_8$  be the sixth neighbor of  $y_1$ . Thus  $y_8$  is adjacent to  $x_1$  and  $y_6$ . Now  $y_8$  is either  $x_6$  or  $y_7$  or a vertex outside  $V(H_{17})$ .

If  $y_8 = x_6$ , then we get the edge  $x_1x_6$ . Now  $N(x_1)$  is isomorphic to  $N_4$  and the second thick edge incident to it (that is, other than  $x_1x_2$ ) is either  $x_1x_6$  or  $x_1y_7$ . But then we respectively get the edges  $x_6y_3$  or  $y_7y_3$ . In either case,  $G$  has a  $K_6$  minor.

If  $y_8 = y_7$ , then from  $N(y_1)$  we get the edge  $y_6y_7$ . But  $G$  has no clique of size 4, so this is a contradiction. Thus  $y_8$  is outside  $V(H_{17})$  (and is adjacent to  $x_1$  and  $y_6$ ). Again, consider  $N(x_1)$ , which is isomorphic to  $N_4$ . The second thick edge incident to it (that is, other than  $x_1x_2$ ) is either  $x_1x_7$  or  $x_1y_8$ . From  $N(y_1)$  we get the edges  $y_7y_3$  or  $y_8y_3$  respectively in the first and second cases. In either case,  $G$  has a  $K_6$  minor, which proves the lemma.  $\square$

**Lemma 3.3.16** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and let  $G$  have a rooted subgraph isomorphic to the graph  $H_{18}$  shown in Figure 3.21. Then  $G$  has a  $K_6$  minor.*



**Figure 3.21:** Graph  $H_{18}$

*Proof:* The vertex  $y_1$  has a neighbor  $y_8$  other than the five shown in Figure 3.21. By Lemma 3.3.12,  $y_8$  is either  $x_6$  or  $y_7$  or is outside  $V(H_{18})$ .

If  $y_8 = x_6$ , then consider  $N(x_6)$ , which is isomorphic to  $N_4$ . Now the edge  $x_6y_1$  is not thick, as  $N(y_1)$  induces a 6-cycle. Thus  $x_6x_1$  must be thick. But then  $x_1x_2, x_2x_3$  are adjacent regular edges. By Lemma 3.3.13, we are done.

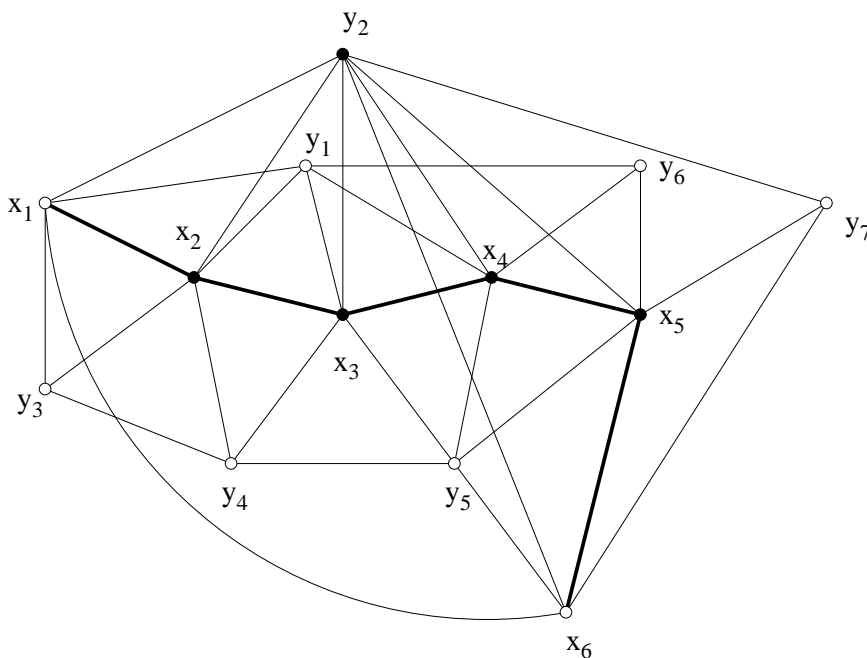
If  $y_8 = y_7$ , then from  $N(y_1)$  we get the edges  $y_7x_1, y_7y_6$ . But then  $G$  has a  $K_6$  minor.

Finally, suppose that  $y_8$  is outside  $V(H_{18})$ . From  $N(y_1)$  we get the edges  $y_8x_1, y_8y_6$ . Consider  $N(x_1)$ , which is isomorphic to  $N_4$ . Exactly one of the edges  $x_1x_6, x_1y_8$  is thick.

If  $x_1x_6$  is thick, then from  $N(x_1)$ , we get the edges  $x_6y_3, x_6y_8$ . But then  $x_6$  has degree at least seven, a contradiction.

Thus  $x_1y_8$  is thick. Then from  $N(x_1)$ , we get the edges  $y_8x_6, y_8y_3$ . But then  $G$  has a  $K_6$  minor, which proves the lemma.  $\square$

**Lemma 3.3.17** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and let  $G$  have a rooted subgraph isomorphic to the graph  $H_{19}$  shown in Figure 3.22. Then  $G$  has a  $K_6$  minor.*



**Figure 3.22:** Graph  $H_{19}$

*Proof:*  $N(y_5)$  cannot be  $N_4$ , and hence  $y_5$  a neighbor  $y_8$  that is adjacent to  $y_4, x_6$ . By Lemma 3.3.12,  $y_8$  must be outside  $V(H_{19})$ .

Now  $N(x_6)$  is isomorphic to  $N_4$ , and exactly one of the edges  $x_6x_1, x_6y_8$  is thick. If  $x_6x_1$  is thick, then from  $N(x_6)$  we get the edges  $x_1y_7, x_1y_8$ . But then  $x_1$  has degree at least 7, a contradiction.

Thus the edge  $x_6y_8$  is thick. But then, from  $N(x_6)$  we get the edge  $y_8y_7$ , and  $G$  has a  $K_6$  minor. This proves the lemma.  $\square$

**Lemma 3.3.18** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and such that all thick edges in  $G$  are irregular. Then  $G$  has a  $K_6$  minor.*

*Proof:*  $G$  has a rooted subgraph isomorphic to the graph on the right of Figure 3.16. By Lemma 3.3.12,  $x_4$  has neighbors  $x_5$  and  $y_6$  besides the ones shown in the figure, where  $x_4x_5$  is a thick edge. Since the edge  $x_3x_4$  is also irregular by the hypothesis of the lemma, it follows that  $x_5$  is adjacent to: (i)  $y_5$  and  $y_6$ , and (ii) exactly one vertex among  $y_4, y_2$ .

First, suppose  $x_5$  is adjacent to  $y_5, y_6, y_4$ . Now  $N(y_4)$  must induce a 6-cycle, and the sixth neighbor of  $y_4$  is outside  $\{x_1, \dots, x_5, y_1, \dots, y_6\}$ . This neighbor  $y_7$  is adjacent to  $y_3$  and  $x_5$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{20}$  shown in Figure 3.23, and by Lemma 3.3.19, we are done.

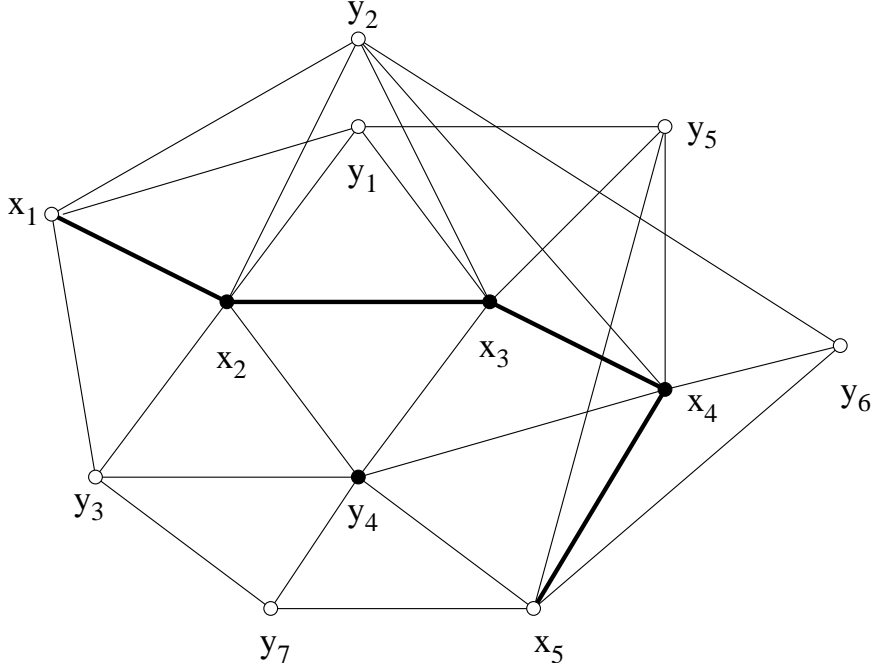
Now suppose that  $x_5$  is adjacent to  $y_5, y_6, y_2$ . From  $N(x_4)$  we also get the edge  $y_4y_6$ . Now  $N(y_4)$  must induce a 6-cycle, and thus  $y_4$  must have a neighbor  $y_7$  outside  $\{x_1, \dots, x_5, y_1, \dots, y_6\}$ , such that  $y_7$  is adjacent to  $y_3$  and  $y_6$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{22}$  shown in Figure 3.25, and by Lemma 3.3.20, we are done. This proves the lemma.  $\square$

**Lemma 3.3.19** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and such that all thick edges in  $G$  are irregular. Let  $G$  have a rooted subgraph isomorphic to the graph  $H_{20}$  shown in Figure 3.23. Then  $G$  has a  $K_6$  minor.*

*Proof:*  $N(y_2)$  must induce a 6-cycle. We claim that the sixth neighbor of  $y_2$ , other than the five shown in the figure, must be outside  $V(H_{20})$ . Suppose not; then that vertex must be  $y_7$ , since  $G$  has no clique of size 4.

From  $N(y_2)$  we now get the edges  $y_7y_6, y_7x_1$ , and it follows that  $N(y_7)$  also induces a 6-cycle. In particular, the edge  $y_7x_5$  is not thick. But  $N(x_5)$  contains a 4-cycle  $x_4y_6y_7y_4$ , and by Observation 8  $x_5y_7$  must be thick, which is a contradiction. This proves the claim.

Thus  $y_2$  has a neighbor  $y_8$  outside  $V(H_{20})$ , such that  $y_8$  is adjacent to  $x_1$  and  $y_6$ . Now if  $x_5, x_1$  are adjacent, then they can share at most one neighbor, by six-regularity. This is a



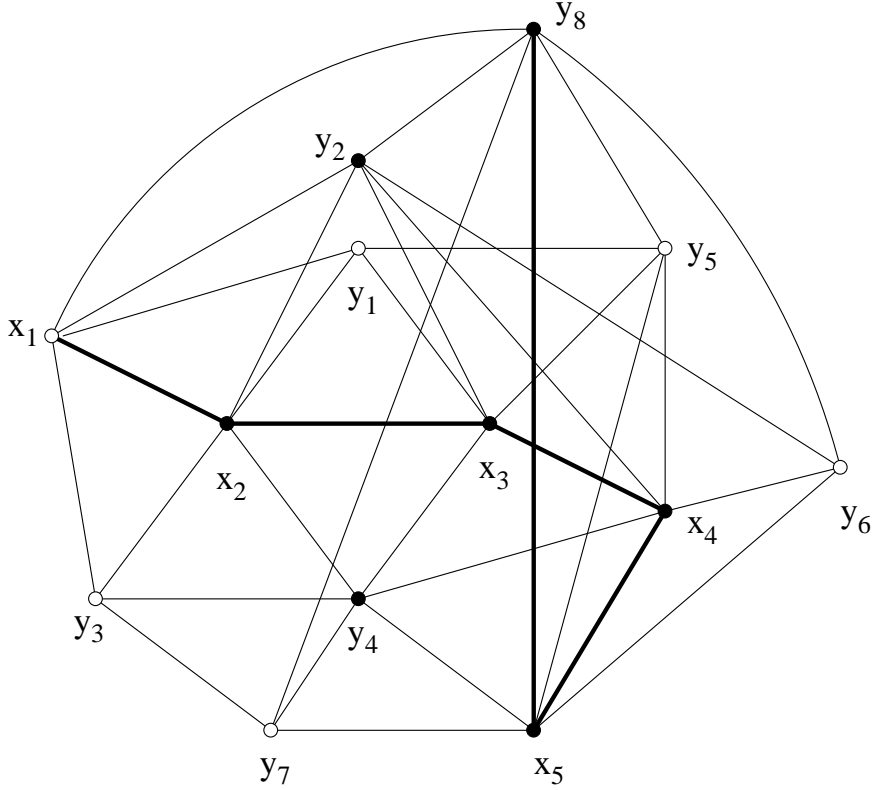
**Figure 3.23:** Graph  $H_{20}$

contradiction, so  $x_5$  cannot be adjacent to  $x_1$ . Thus, by Lemma 3.3.12, the sixth neighbor  $z$  of  $x_5$  (other than those shown in Figure 3.23) is either  $y_8$  or is a vertex outside  $V(H_{20})$ .

Suppose  $z = y_8$ . Then by examining  $N(x_5)$  (and noting that  $x_4x_5$  is regular), we get the edges  $y_8y_5, y_8y_7$ . In particular, the edge  $x_5y_8$  must be thick. Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{21}$  shown in Figure 3.24. Since  $N(y_2)$  induces a 6-cycle, the edge  $y_8y_2$  is not thick. Hence  $y_8x_1$  must be a thick edge, and from  $N(y_8)$  we deduce that  $x_1$  must be adjacent to  $y_7$  and (exactly) one vertex among  $y_5, y_6$ . This means  $x_1$  has degree at least 7, a contradiction.

Finally, suppose the vertex  $z$  is outside  $V(H_{20})$ . From  $N(x_5)$  we deduce that  $z$  must be adjacent to  $y_7$  and  $y_8$ . But then  $G$  has a  $K_6$  minor, which proves the lemma.  $\square$

**Lemma 3.3.20** *Let  $G$  be a minor-minimal graph in  $\mathcal{F}_6$ , with minimum degree  $\geq 6$ , and with no clique of size 4. Let  $G$  be  $N_i$ -free for  $i = 1, 2, 3, 5$ , and such that all thick edges in  $G$  are irregular. Let  $G$  have a rooted subgraph isomorphic to the graph  $H_{22}$  shown in Figure 3.25. Then  $G$  has a  $K_6$  minor.*

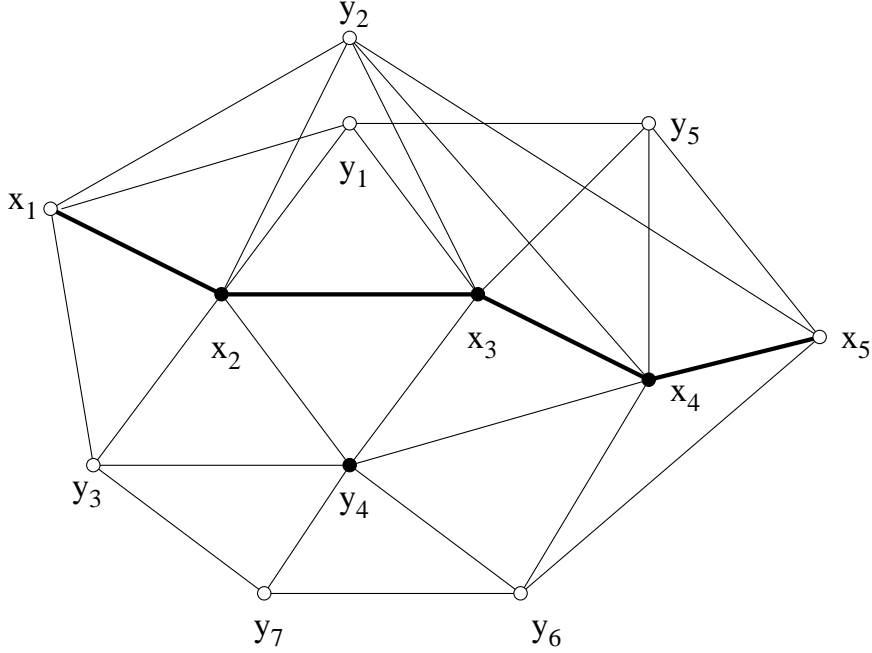


**Figure 3.24:** Graph  $H_{21}$

*Proof:*  $N(y_2)$  must induce a 6-cycle. We claim that the sixth neighbor of  $y_2$ , other than the five shown in the figure, must be outside  $V(H_{22})$ . Suppose not; then that vertex must be  $y_7$ , which must then be adjacent to  $x_1$  and  $x_5$ . But then  $N(y_6)$  contains a 4-cycle that violates Observation 8. This proves the claim.

Thus  $y_2$  has a neighbor  $y_8$  outside  $V(H_{22})$ , such that  $y_8$  is adjacent to  $x_1$  and  $x_5$ . Thus  $G$  has a rooted subgraph isomorphic to the graph  $H_{23}$  shown in Figure 3.26.

Note that the graph  $H_{23}$  has left-right symmetry as shown in the figure. Now  $N(x_1)$  induces a graph isomorphic to  $N_4$ . Let  $z_1$  be the sixth neighbor of  $x_1$  (other than the five shown in the figure). From  $N(x_1)$ , and the fact that  $x_1x_2$  is irregular, we deduce the edges  $z_1x_1, z_1y_8, z_1y_1$ . We claim that  $z_1$  must be outside  $V(H_{23})$ . Suppose not; then by Lemma 3.3.12,  $z_1$  must be identical to  $x_5, y_6$  or  $y_7$ . However,  $z_1 \neq x_5$  since  $G$  has no clique of size 4. Also,  $z_1 \neq y_7$ , otherwise  $N(y_3)$  has a 4-cycle that violates Observation 8. Finally,  $z_1 \neq y_6$ , otherwise  $G$  has a  $K_6$  minor. This proves our claim that  $z_1$  is outside  $V(H_{23})$ .



**Figure 3.25:** Graph  $H_{22}$

By symmetry,  $x_5$  also has a neighbor  $z_2$  outside  $V(H_{23})$ , such that  $z_2$  is adjacent to  $x_5$ ,  $y_8$  and  $y_5$ . (Note that  $z_1, z_2$  may be identical.) First, suppose that  $z_1, z_2$  are distinct. If the edge  $y_8x_1$  is thick, then from  $N(x_1)$  it follows that  $y_8$  is adjacent to  $y_3$ . Similarly, if  $y_8x_5$  is thick, then  $y_8$  must be adjacent to  $y_6$ . Since  $\deg(y_8) = 6$ , it follows that at least one of the edges  $y_8x_1, y_8x_5$  (say, the former) is *not* thick. From  $N(x_1)$ , we now deduce the edge  $z_1y_3$ . But then  $G$  has a  $K_6$  minor.

Thus, we may assume that  $z_1 = z_2 = z$ . Now  $N(y_8)$  has a 4-cycle  $x_1y_2x_5zx_1$ . Since  $N(y_2)$  induces a 6-cycle, the edge  $y_8y_2$  is not thick. Thus by Observation 8, both the edges  $y_8x_1$  and  $y_8x_5$  must be thick. From  $N(x_1), N(x_5)$  respectively, we now deduce the edges  $y_8y_3, y_8y_6$ . But then, from  $N(y_8)$ , it follows that  $y_3, y_6$  are adjacent. This contradicts the fact that  $G$  has a no clique of size 4, and thus proves the lemma.  $\square$

### 3.4 Proof of Main Theorem

We now prove Theorem 3.1.1. Let  $G$  be as in the statement of the theorem. If  $G$  is not six-regular, then deleting an edge incident to a vertex of degree  $\geq 7$  creates at most one low-degree vertex, and hence the resulting graph is also in  $\mathcal{F}_6$ , contradicting the minimality

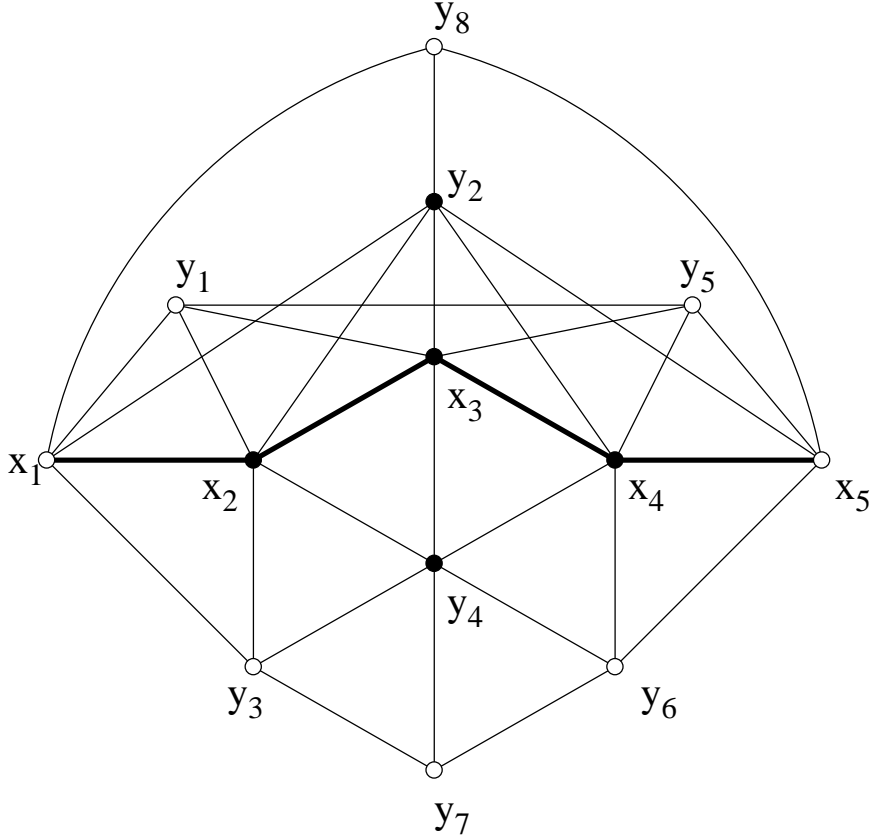


Figure 3.26: Graph  $H_{23}$

of  $G$ . Hence it follows that  $G$  is  $6$ -regular. Also, we may assume that the size of a maximum clique is  $\geq 3$  by Observation 1, and  $\leq 5$  (otherwise Theorem 3.1.1 holds trivially). The theorem now follows by applying lemmas 3.1.3, 3.2.1 and 3.3.2.  $\square$

### 3.5 Jorgensen's Conjecture in Surfaces of Non-negative Euler Characteristic

In the proof of Lemma 3.3.2, the cases when  $G$  is a triangulation of the torus or the Klein bottle also follow respectively from some recent work of Fijavž [13, 14], which is based on results by Altshuler [1], Negami [37, 36], and Thomassen [55]. More precisely, [13] proves that there are precisely 4 minor-minimal 6-regular graphs in the torus ( $K_7$ ,  $K_8 - 4K_2$ ,  $K_9 - C_9$ , and  $K_9 - 3K_3$ ). It is easy to see that each of the 4 graphs has a  $K_6$  minor. Similarly, [14] proves that there are precisely 3 (explicitly given) minor-minimal 6-regular graphs in the Klein bottle, and it is easy to check that each of them has a  $K_6$  minor.

By Euler's formula, a 6-connected graph in the torus or the Klein bottle must be 6-regular (and is hence a triangulation). From the above discussion, it now follows that Jorgensen's conjecture (Conjecture 1.2.1) is true for graphs embedded in the torus or the Klein bottle.

Also, by Euler's formula, no graph embedded in the projective plane is 6-connected; thus Jorgensen's conjecture holds vacuously for projective-planar graphs. More interestingly, however, the corresponding statement for 5-connected graphs is also true: Fijavž and Mohar [15] proved that a 5-connected graph in the projective plane has a  $K_6$  minor if and only if it has representativity at least 3. It is easy to see that a graph embedded in the projective plane with representativity  $\leq 2$  is apex, and hence a 5-connected graph in the projective plane that has no  $K_6$  minor must be apex.

## CHAPTER IV

# A SPLITTER THEOREM FOR SURFACE TRIANGULATIONS

### 4.1 *Introduction*

We begin with some basic notation. For a motivation of the splitter theorem proved in this chapter, refer to Section 1.2.3. Graphs in this chapter are allowed to have parallel edges or loops.

By a *graph embedding*  $G$ , we mean a graph embedded in a surface (compact 2-dimensional manifold)  $\Sigma$ , such that every face (that is, region of  $\Sigma - G$ ) is homeomorphic to an open disk. (Such an embedding is sometimes called a *cellular* embedding.)

Given a graph  $H$  embedded in a surface  $\Sigma$ , performing a sequence of contractions and deletions of edges yields a graph embedding  $G$  in  $\Sigma$  that is called a *surface minor* of  $H$ . We follow the usual convention for edge contractions in graph embeddings — when contracting an edge  $e$  that lies in a triangular face, one of the resulting parallel edges created by the contraction is deleted. (Note that parallel edges that are created by non-facial triangles are *not* deleted by the operation.)

A closed curve in a surface  $\Sigma$  is called *contractible* if it is homotopic to a point. Otherwise, it is called *non-contractible* or *essential*. Thus a simple closed curve is contractible if and only if it bounds an open disk. Given a graph embedding  $G$  and a closed curve  $C$  in  $\Sigma$ , the  $G$ -*degree* of  $C$  (or simply degree, when the embedding is understood) is defined to be  $|C \cap G|$ .

Let  $G$  be a graph embedded in a surface  $\Sigma$ , and  $C$  be a contractible simple closed curve in  $\Sigma$  that does not meet the interior of any edge of  $G$ . (In other words,  $C \cap G \subseteq V(G)$ .) Let  $R_1$  and  $R_2$  be the closures of the two regions of  $\Sigma - C$ , such that  $R_1$  is a closed disk. (Then  $R_2$  is not a closed disk, unless  $\Sigma$  is the sphere  $S^2$ .) Let  $A, B$  be the intersections of  $V(G)$

with  $R_1, R_2$  respectively. The resulting separation  $(A, B)$  of  $G$  is called a *local separation*, and  $C$  is called a *boundary curve* for that separation.

If  $(A, B)$  is a local separation in  $H$ , then fixing a boundary curve for the separation and then performing an edge contraction or deletion yields a corresponding local separation in the new graph. Thus if  $G$  is obtained from  $H$  by a sequence of edge contractions and deletions, then a local separation in  $H$  uniquely defines a local separation in  $G$ .

We now define the notion of connectivity for our splitter theorem. An embedding  $G$  in a surface  $\Sigma$  is *locally 5-connected* if

1.  $G$  has at least six vertices,
2. for every local  $\leq 3$ -separation  $(A, B)$ ,  $A - B$  is empty (in particular,  $(A, B)$  is trivial),  
and
3. for every local 4-separation  $(A, B)$ ,  $A - B$  has at most one vertex.

Local 5-connectivity is weaker than 5-connectivity, in that it allows vertices of degree 4, and vertex cuts (of any size) that lie on non-contractible closed curves in  $\Sigma$ .

An embedding  $G$  in a surface  $\Sigma$  is called a *triangulation* if every face (that is, region of  $\Sigma \setminus G$ ) is bounded by a triangle of  $G$ . Thus a triangulation cannot have loops. It can have parallel edges, however. (A pair of parallel edges would always trace a non-contractible curve.)

By “triangulation”, we sometimes refer to the underlying graph, if no ambiguity arises. Thus, we say that a triangulation has a  $K_6$  minor when the underlying graph has a minor isomorphic to  $K_6$ .

A triangulation is called *simple* if the underlying graph is simple, that is, has no loops or parallel edges.

For a vertex  $v$  in an embedding, the *star of  $v$*  denotes the set of edges incident with  $v$ . Given an embedding  $G$  in a surface  $\Sigma$ , fixing an orientation (say, clockwise) at every vertex  $v$  yields a cyclic permutation of the star of  $v$ . (This is sometimes referred to as the *rotation* at  $v$ , and the collection of rotations is called a *rotation scheme* for  $G$ .) A subset of the star of  $v$  that is contiguous with respect to the rotation at  $v$  is said to be a *fan around  $v$* .

We now define splits, which are the inverses of edge contractions performed on triangulations. (There are more general variants of this operation which are sometimes also called splits (see Chapter 2, for instance) , but the following definition is relevant particularly to triangulations.) Let  $E_1, E_2$  be two fans around  $v$  such that they intersect in precisely two edges, and their union is the star of  $v$ . Now replace  $v$  by two adjacent vertices  $v_1, v_2$ , with  $v_i$  incident with the edges in  $E_i$  for  $i = 1, 2$ . The resulting triangulation is said to be obtained from the original one by *splitting* the vertex  $v$ . Conversely, *contracting* the edge  $v_1v_2$  in the new triangulation (and deleting the resulting parallel edges) gives back the original triangulation.

A triangulation  $H$  obtained by applying a sequence of splits to a triangulation  $G$  is called an *expansion* of  $G$ . For a vertex  $v$  in  $G$ , the set of vertices in  $H$  resulting from  $v$  is called the *branch set* of  $v$  in  $H$ . (Note that the branch sets are determined uniquely by a sequence of splits that leads from  $G$  to  $H$ . However, such a sequence itself may not be uniquely determined.)

Now we define a stronger version of the split operation that, for simple triangulations, turns out to be precisely the kind that preserves local 5-connectivity. A split as defined in the paragraph above is called a *strong split* if the following conditions are satisfied: (i)  $E_1, E_2$  each have at least 3 edges, and at least one of them has  $\geq 4$  edges (in particular,  $v$  has degree  $\geq 5$ ) and (ii) for  $i = 1, 2$ , if  $E_i$  has precisely three edges  $e_1, e_2, e_3$ , in that order according to the rotation at  $v$ , then the end-point  $x_2$  of  $e_2$  (other than  $v$ ) must have degree  $\geq 5$  in  $G$ .

We now prove that strong splits are precisely those that preserve local 5-connectivity, as long as the vertex  $v$  being split is not incident with any parallel edges. (This technical assumption about  $v$  can be eliminated by suitably defining local 5-connectivity and strong splits, but we choose not to do so for simplicity.)

**Lemma 4.1.1** *Let  $G$  be a locally 5-connected triangulation and  $v$  a vertex of  $G$  that is not incident with any parallel edges. Let  $G'$  be the triangulation obtained by splitting  $v$ . Then  $G'$  is locally 5-connected if and only if the split is strong.*

*Proof:* The forward implication is trivial to check: if conditions (i) or (ii) in the definition of a strong split are violated, then  $G'$  violates conditions (ii) or (iii) respectively of the definition of local 5-connectivity. (Notice that if  $v$  has degree 4 but is incident with two parallel edges tracing a non-contractible curve, then a split may preserve local 5-connectivity without being strong.)

For the converse implication, suppose  $v_1, v_2$  are the vertices created by a strong split of  $v$ , such that the resulting triangulation  $G'$  is not locally 5-connected. Condition (i) in the definition of a strong split ensures that  $G'$  does not violate condition (ii) of local 5-connectivity. Thus it has a local 4-separation  $(A', B')$  with  $|A' - B'| \geq 2$ .

We claim that  $B' - A'$  must be disjoint from  $\{v_1, v_2\}$ . Suppose not; then contracting the edge  $v_1v_2$  back yields a local 4-separation  $(A, B)$  in  $G$  with  $|A - B| \geq 2$ . This contradicts the local 5-connectivity of  $G$ , which proves our claim.

Suppose  $A' - B'$  contains both  $v_1$  and  $v_2$ . Since  $G$  is locally 5-connected, the resulting local 4-separation  $(A, B)$  must be such that  $A - B = \{v\}$  (and  $A' - B' = \{v_1, v_2\}$ ). It follows that condition (i) of a strong split must have been violated, which is a contradiction.

Now if  $A' - B'$  contains exactly one of  $v_1, v_2$ , it is easy to check that one of the conditions in the definition of a strong split must be violated. Thus we may assume that  $A' \cap B'$  contains both  $v_1$  and  $v_2$ . Since  $v$  is not incident with any parallel edges,  $v_1, v_2$  must appear consecutively along any boundary curve for  $(A', B')$ . But now, contracting the edge  $v_1v_2$  back gives a local  $\leq 3$ -separation  $(A, B)$  with  $|A - B| \geq 2$ . Again, this contradicts the local 5-connectivity of  $G$ , and thus proves the lemma.  $\square$

## 4.2 Preliminaries

First, we make the following easy observations for triangulations:

**Observation 9** *If  $G$  and  $H$  are triangulations of a surface  $\Sigma$  such that  $G$  is a surface minor of  $H$ , then  $H$  is an expansion of  $G$ , that is,  $H$  can be obtained from  $G$  by a sequence of splits alone. Every intermediate embedding in this sequence is also a triangulation.*

*Proof:*  $G$  can be obtained from  $H$  by a sequence of edge contractions and deletions. It is easy to see that the contractions and deletions can be done in any order without affecting

the final triangulation  $G$  in the sequence. We choose to perform all the contractions before doing any edge deletions. Let  $G'$  be the graph obtained from  $H$  by performing all the contractions. Clearly,  $G'$  is also a triangulation, and has the same number of vertices as  $G$ . Since  $G$  is a subgraph of  $G'$ , it follows that  $G = G'$ . In other words,  $H$  is an expansion of  $G$ . The conclusion now follows easily.  $\square$

**Observation 10** *If  $(A, B)$  is a local separation in a triangulation  $G$ , then there is a cycle  $C$  in  $G$  with  $V(C) = A \cap B$ , such that the simple closed curve traced by  $C$  is homotopic to a boundary curve for  $(A, B)$ .*

*Proof:* Let  $\tilde{C}$  be a boundary curve for  $(A, B)$ . Thus  $\tilde{C} \cap G = A \cap B$ . Since  $G$  is a triangulation, it follows that  $\tilde{C}$  is homotopic to a cycle  $C$  in  $G$  with the required properties.  $\square$

If the cycle  $C$  is as in Observation 10 above, we say that  $C$  *induces* the local separation  $(A, B)$ . Thus the above Observation states that every local separation in a triangulation is induced by some cycle.

**Observation 11** *Let  $(A, B)$  is a local  $\leq 5$ -separation in a locally 5-connected triangulation  $H$ . Then  $A - B$  induces a connected subgraph of  $H$ . Further, if  $|A - B| \geq 2$ , then every vertex of  $A \cap B$  is adjacent to at least one vertex of  $A - B$ .*

*Proof:* By the local 5-connectivity of  $H$ , and the fact that  $H[A]$  has a planar drawing with  $A \cap B$  on the exterior face, it follows that  $H[A - B]$  is connected. If  $|A - B| \geq 2$ , then the second conclusion follows trivially from local 5-connectivity.  $\square$

For the splitter theorem, we need the following crucial but easy lemmas that roughly mean the following: “If a triangulation  $H$  has a surface minor isomorphic to a locally 5-connected triangulation  $G$ , and  $(A, B)$  is a local  $\leq 3$ -separation of  $H$ , then  $A - B$  contributes no vertex in the minor. If  $(A, B)$  is a  $\leq 4$ -separation, then  $A - B$  contributes at most one vertex in the minor.”

**Lemma 4.2.1** *Let  $G, H$  be triangulations of a surface  $\Sigma$  such that  $G$  is a surface minor of  $H$ . Let  $G$  be locally 5-connected, and let  $H$  have a local  $\leq 3$ -separation  $(A, B)$ . Then  $G$  is a surface minor of the triangulation obtained from  $H$  by deleting  $A - B$ .*

*Proof:* By Observation 9,  $H$  can be obtained from  $G$  by a sequence of splits. Thus vertices in  $G$  correspond to branch sets in  $H$ . It follows from the local 5-connectivity of  $G$  that no branch set is contained in  $A - B$ . Since  $G$  is a triangulation, the conclusion of the lemma easily follows.  $\square$

**Lemma 4.2.2** *Let  $G, H$  be triangulations of a surface  $\Sigma$  such that  $G$  is a surface minor of  $H$ . Let  $G$  be locally 5-connected, and let  $H$  have a local  $\leq 4$ -separation  $(A, B)$ . Then  $G$  is a surface minor of the graph obtained from  $H$  by replacing  $A - B$  with a single vertex joined to every vertex in  $A \cap B$ .*

*Proof:* By Observation 9,  $H$  can be obtained from  $G$  by a sequence of splits. Thus vertices in  $G$  correspond to branch sets in  $H$ . It follows from the local 5-connectivity of  $G$  that  $A - B$  contains at most one branch set. Since  $H[A]$  has a planar drawing with  $A \cap B$  on the infinite face, and since  $G$  is a triangulation, the conclusion of the lemma easily follows.  $\square$

For an edge  $e$ , we define  $\deg(e)$  to be the minimum of  $\deg(x), \deg(y)$ , where  $x$  and  $y$  are the end-points of  $e$ . Clearly, in a locally 5-connected triangulation,  $\deg(e) \geq 4$  for every edge  $e$ .

The proof of the splitter theorem proceeds by considering two cases, depending on whether contracting an edge  $e$  in  $H$  creates a violating separation of order 3 or 4. The latter case uses the following technical lemma.

**Lemma 4.2.3** *Let  $G$  and  $H$  be locally 5-connected triangulations of a surface  $\Sigma$  such that  $G$  is a surface minor of  $H$ . Let an edge  $e$  of  $H$  be such that  $H/e$  has a surface minor isomorphic to  $G$ . Let  $e$  be contained in a 5-cycle  $C$  that induces a local separation  $(A, B)$  with  $|A - B| \geq 2$ . Further, suppose that the 5-cycle  $C$  is chosen such that  $|A - B|$  is maximum, and let  $H_0$  be the triangulation obtained by replacing  $A - B$  by a single vertex  $x$  joined to every vertex in  $A \cap B$ . Then one of the following outcomes must hold:*

1.  $H_0$  is locally 5-connected
2. there is an edge  $e'$  of  $H$  such that  $H/e'$  has a surface minor isomorphic to  $G$ , and  $\deg(e') < \deg(e)$

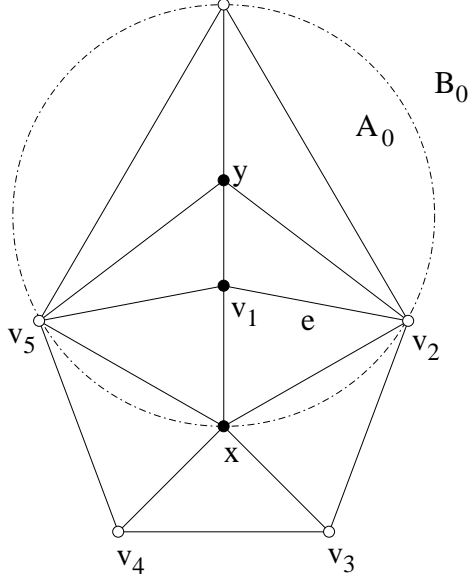
*Proof:* Let  $(A', B')$  be the local 4-separation in  $H/e$ , corresponding to the 5-separation  $(A, B)$  in  $H$ . By Lemma 4.2.2 applied to  $G, H/e$ ,  $G$  is a surface minor of the triangulation obtained from  $H/e$  by replacing  $A' - B'$  by a single vertex joined to every vertex of  $A' \cap B'$ . It is easy to see that the above triangulation is isomorphic to  $H_0/e$ . Thus  $G$  is a surface minor of  $H_0$ . In particular,  $H_0$  has at least six vertices. We may assume that  $H_0$  is not locally 5-connected, otherwise the outcome 1 of the lemma holds. Thus  $H_0$  has a violating local  $\leq 4$ -separation  $(A_0, B_0)$ .

By the local 5-connectivity of  $H$ ,  $A_0 \cap B_0$  must contain  $x$ . Also,  $A_0 - B_0$  and  $B_0 - A_0$  must each contain at least one neighbor of  $x$ . (Those two neighbors, being on opposite sides of a separation, must be non-adjacent vertices of  $C$ .) Thus  $A_0 \cap B_0$  must contain at least two of the neighbors of  $x$ . Now if  $x$  has exactly one neighbor  $v$  in  $B_0 - A_0$ , then  $((A_0 - \{x\}) \cup (A - B) \cup \{v\}, B_0 - \{x\})$  is a local separation that contradicts the local 5-connectivity of  $H$ . Thus it follows that  $x$  has exactly two neighbors  $v_3, v_4$  in  $B_0 - A_0$ , exactly one neighbor  $v_1$  in  $A_0 - B_0$ , and exactly two neighbors  $v_2, v_5$  in  $A_0 \cap B_0$  (such that  $V(C)$  consists of  $v_1, \dots, v_5$ , in that order). But then, consider the local separations  $(X_1, Y_1)$  and  $(X_2, Y_2)$  in  $H$ , where

$$\begin{aligned} X_1 &= (A_0 - \{x\}) \cup (A - B) \cup \{v_3, v_4\} & Y_1 &= B_0 - \{x\} \\ X_2 &= A_0 - \{x\} & Y_2 &= (B_0 - \{x\}) \cup (A - B) \cup \{v_1\} \end{aligned}$$

Now  $|X_1 \cap Y_1| = |A_0 \cap B_0| + 1$ . Thus it follows that  $(X_1, Y_1)$  has order 5,  $(A_0, B_0)$  has order 4, and  $|A_0 - B_0| \geq 2$  (since it violates local 5-connectivity, by assumption). But then  $(X_2, Y_2)$  is also a local 4-separation (in  $H$ ), which means that  $A_0 - B_0$  has exactly two vertices, where  $y$  is the vertex of  $A_0 - B_0$  other than  $v_1$ . (The schematic in Figure 4.1 shows the separation  $(A_0, B_0)$  in  $H_0$ , where the dotted circle is a boundary curve for the separation. We follow the convention that solid vertices are not incident with any more edges than the ones indicated.) Applying Lemma 4.2.2 to  $G, H_0$  and the separation  $(A_0, B_0)$ , it follows that  $G$  is a surface minor of  $H_0/yv_1$  (which in turn is a surface minor of  $H/yv_1$ ).

Now  $e$  must be either  $v_1v_2$  or  $v_1v_5$ . (If it is any of the other three edges in  $C$ , then the cycle  $y, v_2, \dots, v_5$  contradicts the choice of  $C$ .) By symmetry, we may assume that  $e$



**Figure 4.1:** The local separation  $(A_0, B_0)$  in  $H_0$

is the edge  $v_1v_2$ . Clearly,  $v_2$  has degree  $\geq 5$  in  $H_0$ , and hence in  $H$ . In fact,  $v_1$  also must have degree  $\geq 5$  in  $H$ , otherwise the separation  $(A_0, B_0)$  in  $H_0$  will correspond to a local 4-separation in  $H$  that violates its local 5-connectivity.

Thus  $\deg(e) \geq 5$  (in  $H$ ). Outcome 2 of the lemma now follows with  $e'$  being the edge  $yv_1$ .  $\square$

### 4.3 The Main Theorem

**Theorem 4.3.1** *Let  $G$  and  $H$  be locally 5-connected triangulations of a surface  $\Sigma$  such that  $G$  is a surface minor of  $H$ . Then  $H$  can be obtained from  $G$  by a sequence of vertex splits, such that every intermediate triangulation in the sequence is also locally 5-connected.*

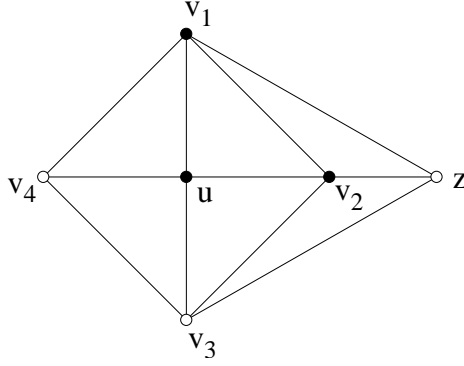
*Proof:* By Observation 9,  $G$  can be obtained from  $H$  by a sequence of edge contractions. It suffices to show that the contractions can be chosen in such a way that every intermediate embedding stays locally 5-connected. We use induction on  $|H| - |G|$  (that is, the number of contractions). For the base case, where  $|H| - |G| = 1$ , there is nothing to prove. Since  $|G| \geq 6$  by local 5-connectivity, we may thus assume that  $|H| \geq 8$ .

Now there is an edge  $e$  in  $H$  such that  $G$  is a surface minor of  $H/e$ . Choose such an edge  $e$  with  $\deg(e)$  minimum. If  $H/e$  is locally 5-connected, we are done by induction. Since  $H/e$

has at least seven vertices, we may thus assume that  $H/e$  has a violating local separation  $(A', B')$ . Let  $(A, B)$  be the corresponding local separation in  $H$ .

First, suppose that  $(A, B)$  has order 5. Thus  $(A', B')$  has order 4, and  $|A - B| = |A' - B'| \geq 2$ . By Observation 10, it follows that  $e$  is in a 5-cycle  $C$  that induces  $(A, B)$ . Choose  $C$  such that  $|A - B|$  is maximum. Let  $H_0$  be the triangulation obtained from  $H$  by replacing  $A - B$  with a single vertex  $x$  joined to every vertex in  $C$ . By Lemma 4.2.2 applied to  $G, H/e$ , and the separation  $(A', B')$ , it follows that  $G$  is a surface minor of  $H_0/e$ . (In particular,  $G$  is a surface minor of  $H_0$ , and  $|G| < |H_0|$ .) Now apply Lemma 4.2.3 to  $G, H$ , and the edge  $e$ . If outcome 1 of the lemma holds, we can apply induction to  $G, H_0$  and  $H_0, H$ . Thus we may assume that outcome 2 holds. But then the edge  $e'$  contradicts the choice of  $e$ .

Thus  $(A, B)$  has order 4 (and  $(A', B')$  has order 3). By the local 5-connectivity of  $H$ , it follows that  $A - B$  has exactly one vertex  $u$  of degree 4, joined to every vertex in  $A \cap B$ . Label the vertices in  $A \cap B$  as  $v_1, \dots, v_4$ , in that cyclic order according to the rotation at  $u$ , such that  $\deg(v_1) \leq \deg(v_i)$  for  $i = 1, \dots, 4$ . Now, instead of contracting  $e$  in  $H$ , we contract the edge  $e' = uv_1$ . If  $H/e'$  is locally 5-connected, then we are done by induction. Also, since  $\deg(e') = 4$ , it follows from the argument in the previous paragraph that  $H/e'$  has no violating local 4-separation. Thus we may assume that it has a local 3-separation  $(A'', B'')$  with  $A'' - B''$  non-empty. By the local 5-connectivity of  $H$ , it follows that  $A'' - B''$  has a unique vertex, and that vertex must be one of  $v_2, v_4$ , say  $v_2$ . Let  $z$  be the fourth neighbor of  $v_2$  (other than  $v_1, u, v_3$ ). Figure 4.2 shows the neighborhoods of  $u$  and  $v_2$  in  $H$ . (Again, we follow the convention that solid vertices are not incident with any more edges than the ones indicated.) Clearly,  $z$  is distinct from  $v_1, v_3$  (but may be identical to  $v_4$ ). If  $z \neq v_4$ , then the triangle  $zv_3v_4$  yields a local 3-separation that contradicts the local 5-connectivity of  $H$ . Thus  $z = v_4$ . (In particular, the triangle  $v_2uv_4$  traces a non-contractible curve). Notice that  $\deg(v_2) = 4$  (in  $H$ ), so by the numbering of  $v_1, \dots, v_4$ , we conclude that  $\deg(v_1) = 4$ , and that the star of  $v_1$  consists of the edges  $v_1u, v_1v_2$ , and two parallel edges incident with  $v_4$ . Moreover, the rotation at  $v_1$  must be such that the two parallel  $v_1v_4$  edges must be shared by some face, which contradicts the fact that  $H$  is a triangulation. This finishes the proof



**Figure 4.2:** The neighborhoods of  $u$  and  $v_2$  in  $H$

of the theorem. □

From Lemma 4.1.1, we immediately obtain the following corollary:

**Corollary 4.3.2** *Let  $G$  and  $H$  be simple locally 5-connected triangulations of a surface  $\Sigma$  such that  $G$  is a surface minor of  $H$ . Then  $H$  can be obtained from  $G$  by a sequence of strong splits. (Equivalently, every intermediate triangulation in the sequence is also locally 5-connected.)*

□

## CHAPTER V

### $K_6$ MINORS IN THE TORUS AND THE KLEIN BOTTLE

We use the notation from Chapter 4, except that graphs in this chapter are restricted to being simple, that is, they do not have parallel edges or loops.

#### 5.1 $K_6$ Minors in the Torus

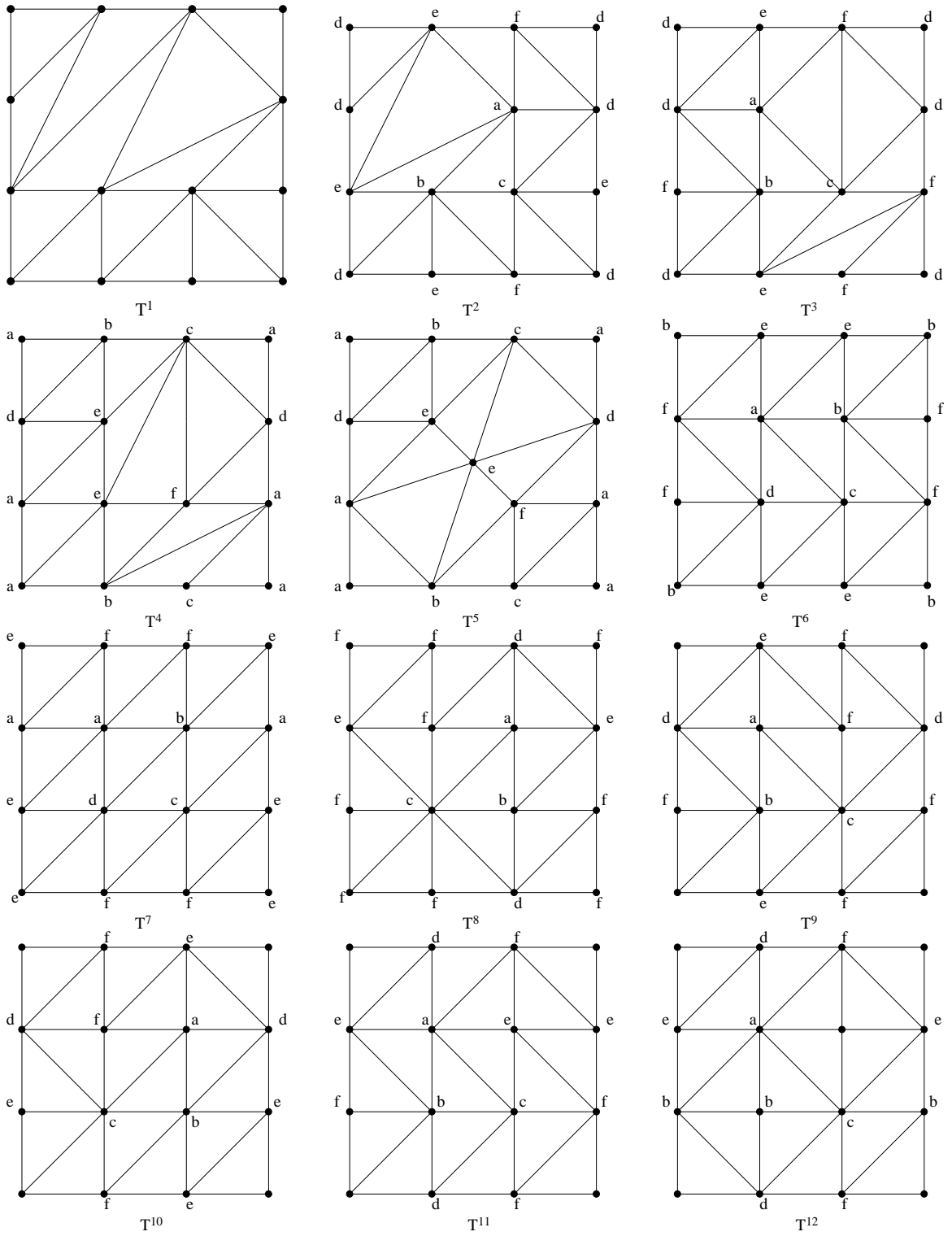
An edge in a (simple) triangulation is called *contractible* if it can be contracted (in the sense of a “surface contraction” defined in Chapter 4) without creating a loop or a parallel edge. Thus an edge is contractible if and only if it is not contained in any non-facial triangle.

A triangulation of the torus is called *irreducible* if it has no contractible edges. [31] gives a list of the irreducible triangulations of the torus. (This list was originally found by Duke and Grünbaum [12], but they did not publish their result.) There are twenty one triangulations in the list, up to isomorphism; they are shown in Figures 5.1, 5.2, and 5.3. (The figures use the usual unfolding of the torus into a rectangle. The torus is obtained by pasting opposite pairs of this rectangle.)

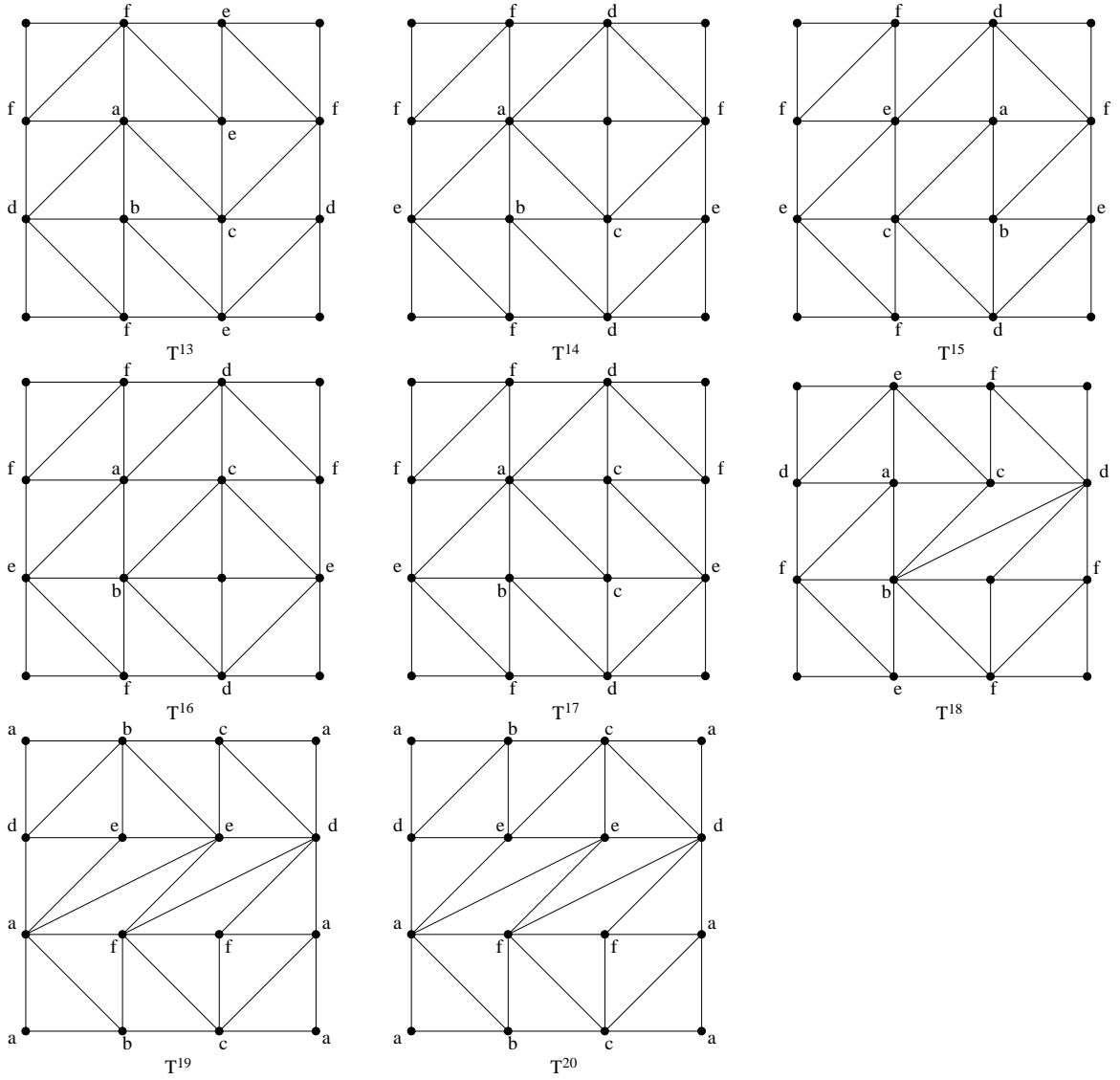
**Theorem 5.1.1** ([12, 31]) *There are precisely 21 irreducible triangulations  $T^1, \dots, T^{21}$  of the torus, up to isomorphism.*

**Theorem 5.1.2** *Every locally 5-connected (simple) triangulation of the torus, with the exception of  $T^{21}$  shown in Figure 5.3, has a minor isomorphic to  $K_6$ .*

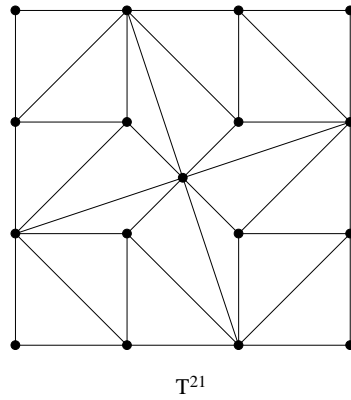
*Proof:* Let  $H$  be a locally 5-connected triangulation of the torus not isomorphic to  $T^{21}$ . By Theorem 5.1.1,  $H$  has a surface minor isomorphic to one of the irreducible triangulations  $T^1, \dots, T^{21}$ . Figures 5.1 and 5.2 show the triangulations  $T^1, \dots, T^{20}$ .  $T^1$  is the complete graph  $K_7$ , and the labeling on the remaining 19 triangulations demonstrates that they all have a  $K_6$  minor. (The labels  $a$  through  $f$  represent the branch sets of the six vertices of



**Figure 5.1:** Triangulations  $T^1, \dots, T^{12}$  of the torus



**Figure 5.2:** Triangulations  $T^{13}, \dots, T^{20}$  of the torus



**Figure 5.3:** Triangulation  $T^{21}$  of the torus

$K_6$ .) Thus if  $H$  has a surface minor isomorphic to any of these twenty triangulations, then  $H$  also has a  $K_6$  minor.

Thus we may assume that  $H$  has a surface minor isomorphic to  $T^{21}$  (but is not itself isomorphic to  $T^{21}$ ). By Corollary 4.3.2, it suffices to show that

1.  $T^{21}$  is locally 5-connected, and
2. Any triangulation obtained from  $T^{21}$  by applying a strong split has a  $K_6$  minor

The vertices of  $T^{21}$  can be labeled  $x_1, \dots, x_5, y_1, \dots, y_5$  such that for  $i = 1, \dots, 5$ ,  $x_i$  is adjacent to all other vertices except  $y_i$ . Let  $X = \{x_1, \dots, x_5\}$  and  $Y = \{y_1, \dots, y_5\}$ .

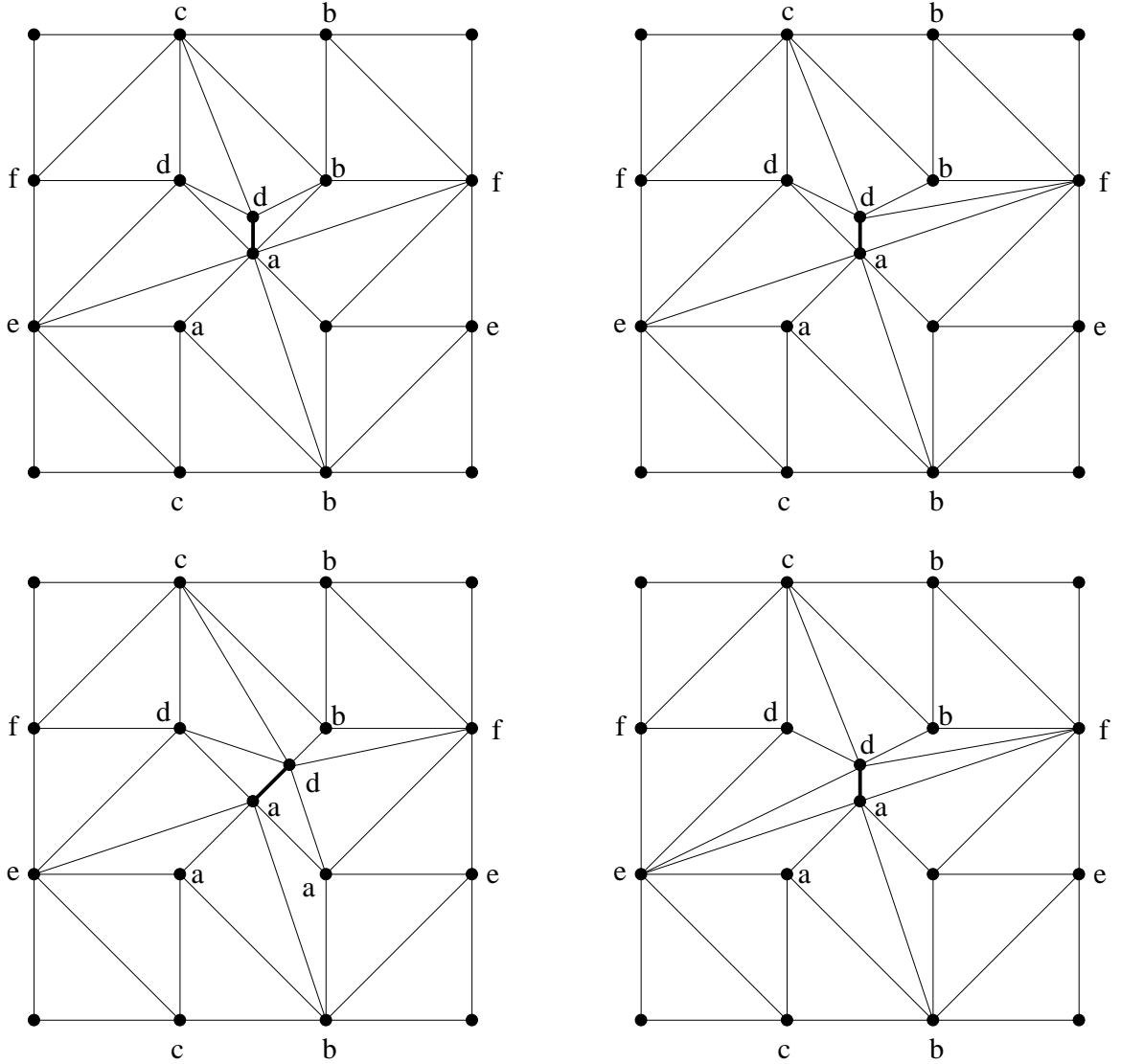
$T^{21}$  is 4-connected and simple, and hence has no local 3-separation  $(A, B)$  with  $A - B$  non-empty. Suppose there is a local 4-separation  $(A, B)$  with  $|A - B| \geq 2$ . Clearly,  $A - B$  is disjoint from  $X$ , and hence contains at least two vertices in  $Y$ . But then  $A \cap B$  must contain  $X$ , a contradiction. Thus  $T^{21}$  is locally 5-connected.

The vertices in  $X$  are symmetric among themselves (and so are the vertices of  $Y$ ). More precisely, for all  $i, j$ , there is an automorphism of  $T^{21}$  that maps  $x_i$  to  $x_j$  and respects the embedding (that is, maps faces to faces). Furthermore, the embedding is such that, for any  $x_i$ , its four neighbors in  $X$  and  $Y$  respectively appear alternately in the rotation at  $x_i$ .

We claim that, up to symmetry, there are exactly four ways to apply a strong split to  $T^{21}$ . By condition (i) of a strong split, none of the vertices in  $Y$  can be split, as they have degree 4 each. By symmetry, it suffices to consider any one vertex in  $X$ , say  $x_1$ . Let  $N_1$  and  $N_2$  be the two fans around  $x_1$ , as defined by the split, such that  $|N_1| \leq |N_2|$ . Now  $|N_1| + |N_2| = \deg(x_1) + 2 = 10$ , and by the condition (i) of a strong split, we have  $|N_1|, |N_2| \geq 3$ . Thus  $(|N_1|, |N_2|)$  is either  $(3, 7)$ ,  $(4, 6)$  or  $(5, 5)$ . It is easy to check that, up to symmetry, cases 1 and 2 lead to one type of split each, and case 3 leads to two types of splits.

Figure 5.4 shows, up to isomorphism, the four triangulations that can be obtained from  $T^{21}$  by applying a strong split, along with labellings that demonstrate a  $K_6$  minor in each of them. This proves the theorem. □

A graph  $G$  is said to be *internally 5-connected* if (i)  $G$  is 4-connected and has at least



**Figure 5.4:** Strong splits in  $T^{21}$ , up to isomorphism

six vertices, and (ii) for every separation  $(A, B)$  of  $G$  of order at most four, at least one of  $|A - B|, |B - A|$  is  $\leq 1$ . For a simple graph embedding  $G$ , it is easy to check that internal 5-connectivity implies local 5-connectivity. Thus we have the following:

**Corollary 5.1.3** *Every internally 5-connected (simple) triangulation of the torus, with the exception of  $T^{21}$  shown in Figure 5.3, has a minor isomorphic to  $K_6$ .*

□

## 5.2 $K_6$ Minors in the Klein Bottle

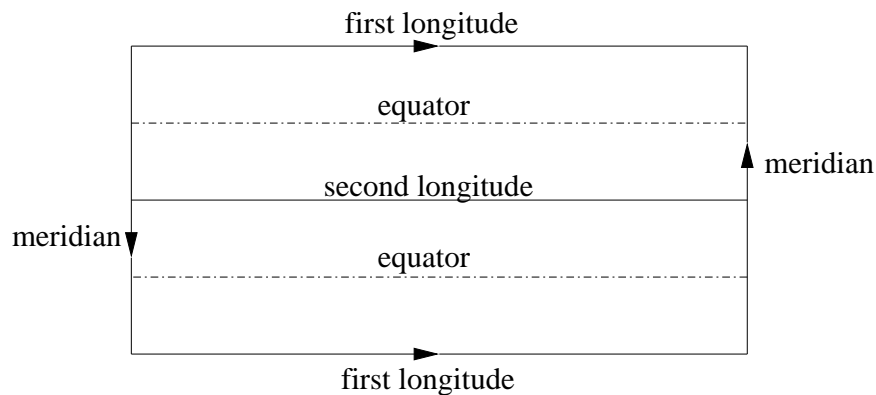
We now prove that every graph embedded in the Klein bottle with representativity at least 4 has a minor isomorphic to  $K_6$ .

The *representativity* of an embedding is the minimum  $k$  such that there is a non-contractible curve in the surface meeting the graph in at most  $k$  points. This invariant of the embedding is also called *face-width*. It is a measure of how locally planar the embedding is. (If representativity is  $> k$ , then any subgraph of diameter  $< k$  must be planar.)

A simple closed curve  $C$  in a surface  $\Sigma$  is called *separating* if  $\Sigma \setminus C$  consists of at least two regions, and is called *non-separating* otherwise.

A simple closed curve is called *orientation-preserving* if traversing it once preserves the sense of orientation (that is, clockwise or anti-clockwise); it is called *orientation-reversing* otherwise.

The Klein bottle can be obtained from identifying the pairs of opposite sides of a rectangle as shown in Figure 5.5. There are three kinds of non-contractible simple closed curves



**Figure 5.5:** Non-contractible simple closed curves in the Klein bottle

in the Klein bottle, up to homeomorphism:

1. Orientation-preserving, non-separating curves called *meridians*: Any two meridians are homotopic to each other.
2. Orientation-preserving, separating curves called *equators*: Any two equators are also homotopic to each other. (The union of the two dotted arcs in Figure 5.5 forms one such equator.)

3. Orientation-reversing, non-separating curves called *longitudes*: Any two longitudes are homeomorphic, but not necessarily homotopic to each other. Every maximal family of disjoint longitudes has size two. (The curves “first longitude” and “second longitude” in Figure 5.5 comprise one such family. Any other longitude in the Klein bottle is homotopic to exactly one of them.) Furthermore, if a simple closed curve crosses a meridian exactly once, then it must be a longitude.

The proof of Theorem 5.2.3 relies on reducing the problem to a graph in the projective plane. To this end, we need the following lemma, a variant of which was shown in [46].

Given a graph  $G$  embedded in the Klein bottle, a cycle in  $G$  is called *equatorial* (respectively, *longitudinal*) if the simple closed curve that it traces is an equator (respectively, a longitude) in the Klein bottle.

**Lemma 5.2.1** *Let  $G$  be graph embedded in the Klein bottle with representativity at least 4. Then  $G$  satisfies at least one of the following outcomes:*

1.  $G$  has two vertex-disjoint equatorial cycles  $E_1, E_2$
2.  $G$  has two longitudinal cycles  $L_1, L_2$  and an equatorial cycle  $E$  such that the three cycles are pairwise vertex-disjoint

*Proof:* Choose a meridian  $M$  in the Klein bottle such that  $M$  meets  $G$  in as few points as possible. Without loss of generality, we may assume that the points of intersection are all vertices of  $G$ . Let  $v_1, \dots, v_k$  be those vertices. By hypothesis,  $k \geq 4$ . Cutting open the Klein bottle along  $M$  yields an embedding  $G'$  in a cylinder, such that the vertices on the two boundary components are, respectively,  $x_1, \dots, x_k$  and  $y_k, \dots, y_1$  (where the vertices appear in those cyclic orders along the respective components, for some fixed orientation around the axis of the cylinder).  $G$  can be obtained from  $G'$  by identifying  $x_i, y_i$  into the vertex  $v_i$ , for  $i = 1, \dots, k$ .

We claim that in  $G'$  there are  $k$  vertex-disjoint paths  $P_1, \dots, P_k$  such that  $P_i$  has ends  $x_i$  and  $y_{\sigma(i)}$ , for  $i = 1, \dots, k$ . Indeed, by Menger’s theorem, if these paths do not exist, then there exists a meridian  $M'$  in the Klein bottle that meets  $G$  in fewer than  $k$  points, contrary to the choice of  $M$ . This proves the claim.

Since the paths  $P_i$  are pairwise disjoint, it follows that  $\sigma(1), \dots, \sigma(k)$  is simply a cyclic shift of  $k, \dots, 1$ . It is easy to check that  $\sigma(\sigma(i)) = i$ , for  $i = 1, \dots, k$ . If  $\sigma(i) = i$ , then the path  $P_i = P_{\sigma(i)}$ , in  $G$ , forms a cycle that crosses  $M$  exactly once, and is hence longitudinal. On the other hand, if  $\sigma(i) \neq i$ , then  $P_i \cup P_{\sigma(i)}$ , in  $G$ , forms a cycle that crosses  $M$  twice, and furthermore traces a separating curve in the Klein bottle. Thus that cycle must be equatorial. Since  $k \geq 4$ , the conclusion of the lemma now follows easily.  $\square$

We now reduce the problem on the Klein bottle to one on the projective plane:

**Lemma 5.2.2** *Let  $G$  be a graph embedded in the Klein bottle with representativity at least 4. Then  $G$  has a minor isomorphic to a graph  $G'$  embedded in the projective plane with representativity at least 4.*

*Proof:* By Lemma 5.2.1,  $G$  satisfies one of the outcomes in that lemma.

First, suppose  $G$  satisfies outcome 1, and has equatorial cycles  $E_1, E_2$  as specified. The curve traced by  $E_1$  separates the Klein bottle into two regions. Let  $R_1, R_2$  be the closures of those two regions. (Both  $R_1$  and  $R_2$  are homeomorphic to a Möbius band.) Since  $E_2$  is vertex-disjoint from  $E_1$ ,  $V(E_2)$  is contained in  $V(G) \cap R_i$  for some  $i = 1, 2$  (say,  $i = 1$ ). Let  $G'$  be the graph determined by the vertices and edges contained in  $R_1$ . Indeed, by attaching a disk to  $R_1$  along  $E_1$ , we get an embedding  $G'$  in the projective plane (with  $E_1$  being one of the facial cycles). We claim that  $G'$  has representativity at least 4. Suppose not, and let  $C$  be a non-contractible simple closed curve in the projective plane that meeting  $G'$  in at most 3 points. It follows that  $C$  must meet the interior of the face bounded by  $E_1$  (otherwise, the corresponding curve in the Klein bottle violates the representativity of  $G$ .) But then  $C$  would have to meet each of  $E_1, E_2$  in two distinct points, which is a contradiction. This proves our claim.

Now suppose that  $G$  satisfies outcome 2 of Lemma 5.2.1 and has cycles  $L_1, L_2$  and  $E$  as specified. Contract the cycle  $L_2$  to a single vertex  $v$ . It is an elementary topological argument to show that the resulting graph has an embedding in the projective plane. Let  $G'$  be the embedding so obtained. We claim that  $G'$  has representativity at least 4. Suppose not, and let  $C$  be a non-contractible simple closed curve in the projective plane that meets

$G'$  in at most 3 points. Since  $G$  has representativity 4,  $C$  must contain the vertex  $v$ . Also, it meets  $L_1$  in exactly one vertex. (Every two non-contractible curves in the projective plane cross at exactly one point.) In doing so,  $C$  would have to meet  $E$  in at least two points. Thus  $C$  meets  $G'$  in at least 4 points, a contradiction. This proves the lemma.  $\square$

We are now ready to prove the theorem:

**Theorem 5.2.3** *Every graph  $G$  embedded in the Klein bottle with representativity at least 4 has a minor isomorphic to  $K_6$ .*

*Proof:* By Lemma 5.2.2,  $G$  has a minor isomorphic to a graph  $G'$  embedded in the projective plane with representativity at least 4. By Lemma 5.2.5,  $G'$  has a minor isomorphic to  $K_6$ , and the theorem follows.  $\square$

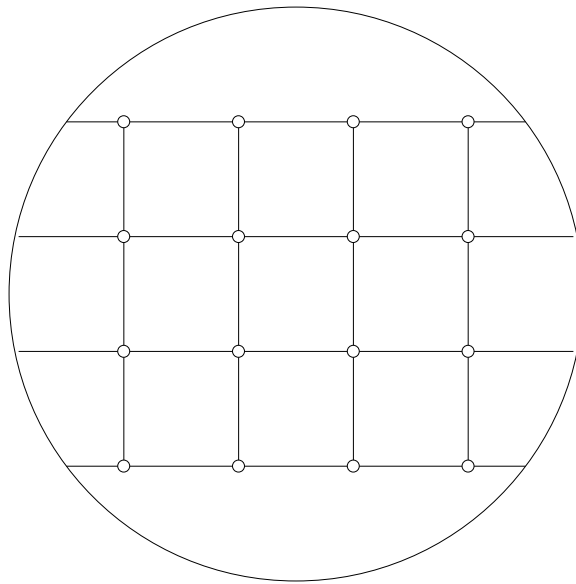
To prove Lemma 5.2.5, we need the following two well-known operations on graph embeddings: (i) a  $Y\Delta$ -operation deletes a vertex of degree 3, and adds a (facial) triangle passing through its 3 neighbors, while (ii) a  $\Delta Y$ -operation deletes a facial triangle and adds a new vertex joined to the 3 vertices.

Randby (see [47]) proved the following theorem:

**Theorem 5.2.4** *Every graph embedded in the projective plane with representativity at least 4 has a minor isomorphic to one of the graphs obtained from the  $4 \times 4$ -grid in the projective plane, shown in Figure 5.6, by a sequence of  $Y\Delta$  and  $\Delta Y$ -operations.*

**Lemma 5.2.5** *Every graph  $G'$  embedded in the projective plane with representativity at least 4 has a minor isomorphic to  $K_6$ .*

*Proof:* By Theorem 5.2.4,  $G'$  has a minor isomorphic to one of the graphs obtained from the  $4 \times 4$ -grid by a sequence of  $Y\Delta$  and  $\Delta Y$ -operations. There are 270 such graphs, and each of them has a  $K_6$  minor. The generation of the list, and the verification of a  $K_6$  minor in each graph therein, was done with the help of computer programs. It now follows that  $G'$  too must have a  $K_6$  minor.  $\square$



**Figure 5.6:** The  $4 \times 4$ -grid in the projective plane

## CHAPTER VI

### FINDING 3-SHREDDERS EFFICIENTLY

#### 6.1 Introduction

Connectivity is an important invariant of graphs. Efficient algorithms for determining the connectivity properties of a graph are both theoretically interesting and practically useful. For general  $k$ , the fastest algorithm to test for  $k$ -vertex connectivity runs in time  $O((n + \min(k^{5/2}, kn^{3/4}))m)$  [16]. The problem of counting the number of  $k$ -vertex cuts in a  $k$ -connected graph, for general  $k$ , however, is #P-complete [8].

For  $k \leq 3$ , linear time algorithms are known for testing  $k$ -vertex-connectivity and for finding the vertex cuts of size  $k - 1$ . The cases  $k = 1, 2$  are easily solved using depth-first search. For  $k = 3$ , see [22] and [20]. (The classical 3-connectivity algorithm of Hopcroft and Tarjan [22] finds, in  $O(n + m)$  time, a decomposition into 3-connected components. One could then read off all the 2-cuts in the graph by looking at the decomposition.) For  $k = 4$ , [25] gives an almost-linear time algorithm for testing 4-connectivity and maintaining a decomposition into 4-connected components online, under certain update operations.

In this chapter, we consider those vertex cuts that not only disconnect the graph, but do so into at least three components. More precisely, a *shredder* in a graph  $G$  is a vertex cut  $S$  such that  $G \setminus S$  has at least three components. A shredder having  $k$  vertices is called a *k-shredder*. For instance, if  $G$  is a tree, the 1-shredders of  $G$  correspond precisely to the vertices of degree at least three. One application of shredders is in node connectivity augmentation; (see [8]), for instance. [8] also presents an algorithm to find, for general  $k$ , the set of  $k$ -shredders of a  $k$ -vertex connected graph on  $n$  vertices that runs in time  $O(k^2n^2 + k^3n^{1.5})$ .

This chapter presents an algorithm to find the set of 3-shredders of a 3-vertex connected graph in time proportional to the number of vertices and edges in the graph. (The connectivity assumption is without loss of generality, because of the tri-connectivity algorithm

in [22].) The best known current bound for this problem is  $O(n^2)$ , which follows from the general  $k$ -shredders algorithm in [8].

For a discussion of the motivation of this problem in the context of Graph Structure Theory, refer to Section 1.2.5.

## 6.2 Notation

Given a simple undirected graph  $G$  with  $n$  vertices and  $m$  edges, we can test for 3-connectivity in linear ( $O(n + m)$ ) time using the algorithm in [22] (see also [20].) The 3-shredders algorithm will then proceed in several steps. In the first step, we generate a certain set of triples of vertices that includes all the 3-shredders. In subsequent steps, we eliminate those triples that are not 3-shredders. The basic strategy for these steps is depth-first search (dfs). We first find a depth-first spanning tree  $T$ , starting at an arbitrary vertex, which will henceforth be called the root. The edges of  $T$  (tree edges) will be directed from parent to child, and the remaining edges (back edges) will be directed from descendant to ancestor. We denote tree edges by  $u \rightarrow v$  and back edges by  $u \leftrightarrow v$ . The adjacency list  $\text{Adj}(u)$  denotes the set of all edges with tail  $u$ .  $\text{Adj}^R(u)$  will denote the set of *back edges* with head  $u$ . (We consider the tree  $T$  to be “growing downwards”, with the root being the top vertex, and the children of a vertex  $u$  being listed from left to right according to their order in  $\text{Adj}(u)$ .) The generation procedure and the subsequent steps will have the general format given in Table 6.1.

It is important that during the generation procedure and the subsequent steps, the edges in an adjacency list are processed in a specific order. Before we describe what the order is, we need to define the quantities HIGH1, HIGH2 and HIGH3, which are defined for all edges of  $G$ , and for all vertices except for the root of  $T$ . The value of HIGH1, HIGH2 or HIGH3 is either a vertex of  $G$ , or as a special case, infinity. (Vertices are later going to be identified with their post-order numbers with respect to a certain depth-first traversal of  $G$ ; the  $\infty$  notation is intended to be consistent with that numbering.) It should be noted that the concepts of HIGH1 and HIGH2 are more or less the same as “LOWPT1” and “LOWPT2”, introduced in [22], except when they are defined to be infinity.

**Table 6.1:** General Format for the Dfs-based Steps

```
A: (statement to be inserted here)
dfs_step(root);

procedure dfs_step( $u$ )
begin
  for  $e \in \text{Adj}(u)$  do begin
    forward_visit( $e$ )
    if  $e$  is a tree edge  $u \rightarrow v$  then begin
      dfs_step( $v$ )
      backward_visit( $e$ )
    end
  end
  B: (statement to be inserted here)
end
```

For  $v \in V$ , let  $D(v)$  be the set of descendants of  $v$  in the depth-first spanning tree (including  $v$  itself), and let  $ND(v) := |D(v)|$ . (With a slight abuse of notation, we sometimes refer to  $D(v)$  as a “subtree” rather than the vertex set of a subtree.) If  $e$  is a back edge  $u \hookrightarrow v$ , we define  $\text{HIGH1}(e) = v$  and  $\text{HIGH2}(e) = \text{HIGH3}(e) = u$ . Now let  $e$  be a tree edge  $u \rightarrow v$ . We call a vertex  $a$  an *attachment* of the subtree  $D(v)$  if it is a proper ancestor of  $u$  and  $v' \hookrightarrow a$  for some  $v' \in D(v)$ .  $\text{HIGH1}(e)$  is defined as the attachment of  $D(v)$  that is highest in the tree, i.e. closest to the root. If no such attachment exists, we define  $\text{HIGH1}(e) = \infty$ .  $\text{HIGH2}(e)$  is defined as the second highest attachment of  $D(v)$  ( $\infty$  if no such attachment exists). Finally,  $\text{HIGH3}(e)$  is defined as the third highest attachment of  $D(v)$  ( $\infty$  if no such attachment exists.) Note that since  $G$  is 3-connected,  $\text{HIGH1}(u \rightarrow v) \neq \infty$  unless  $u$  is the root and  $v$  is its (unique) child. Similarly,  $\text{HIGH2}(u \rightarrow v) \neq \infty$  unless  $u$  is the root or its child, and  $v$  is its (unique) child.

For a vertex  $v$  that is not the root, we denote by  $\text{HIGH1}(v)$  the value of  $\text{HIGH1}$  for the (unique) tree edge  $u \rightarrow v$ . The quantities  $\text{HIGH1}$ ,  $\text{HIGH2}$  and  $\text{HIGH3}$  can be easily computed in a bottom-up fashion by a dfs.

We are now ready to describe the order on  $\text{Adj}(u)$ . An edge  $e$  will precede an edge  $f$  in  $\text{Adj}(u)$  if either  $\text{HIGH1}(e)$  is higher in the tree than  $\text{HIGH1}(f)$ , or  $\text{HIGH1}(e) = \text{HIGH1}(f)$

and  $\text{HIGH2}(e)$  is higher than  $\text{HIGH2}(f)$ , or  $\text{HIGH1}(e) = \text{HIGH1}(f)$  and  $\text{HIGH2}(e) = \text{HIGH2}(f)$  and  $\text{HIGH3}(e)$  is higher than  $\text{HIGH3}(f)$ . ( $\infty$  is considered higher in the tree than any vertex. Ties are broken arbitrarily in the above order.) The adjacency lists can be sorted in  $O(n + m)$  time using a slight modification of radix sort with  $n + 1$  buckets, and future depth-first searches will use this ordering to process the edges in an adjacency list. The ordering of the adjacency lists here is somewhat similar to that described in [22], but differs in two respects. Firstly, the ordering here is lexicographic with respect to three quantities instead of two, as is the case in [22]. Secondly, the  $\infty$  part of the definition of  $\text{HIGH1}$ ,  $\text{HIGH2}$  etc. ensures the following: Among all edges of  $\text{Adj}(u)$  having some fixed value of  $\text{HIGH1}$ , back edges will appear in the front of the ordering. Similarly, among all tree edges in  $\text{Adj}(u)$  having some fixed values for  $\text{HIGH1}$  and  $\text{HIGH2}$ , tree edges with  $\text{HIGH3}$  being  $\infty$  (that is, those for which the corresponding subtree has only two attachments) appear in the front of the ordering. This fact turns out to be useful for the 3-shredders algorithm, whereas it is immaterial in the case of the 3-connectivity algorithm in [22]. In fact, with respect to the ordering in [22], the above-mentioned edges appear at the end of the respective sublists of  $\text{Adj}(u)$ .

The vertices are then numbered 1 through  $n$  in the order in which they are *last* examined by a dfs (using the new ordering on the adjacency lists.) Henceforth, we will identify the vertices with their post-order number as given above, and refer to a vertex and its integer label interchangeably. For instance, a range of integers can be construed as a subset of  $V(G)$  (if it falls between 1 and  $n$ .) The quantities  $\text{HIGH1}$ ,  $\text{HIGH2}$  and  $\text{HIGH3}$  may also be treated as integers with respect to the above numbering. Note that the numbering *respects height on the tree* i.e. if  $u$  is an ancestor of  $v$  ( $u \rightarrow^* v$ ), then  $u \geq v$ . (We use “ $\rightarrow^*$ ” to denote a path of 0 or more tree edges.)

The first edge in an adjacency list is called a *leftmost edge*. We call  $v$  a *leftmost vertex* if it is not the root and the (unique) tree edge  $u \rightarrow v$  is leftmost. Otherwise, we call  $v$  non-leftmost. A path consisting of leftmost edges is called a leftmost path. If  $u \rightarrow^* v$  is a leftmost path,  $v$  is called a leftmost descendant of  $u$ .

We need to define two more quantities,  $\text{LOW1}$  and  $\text{RCH}$  (for “reach”), as follows. Let

$e = (u \rightarrow v)$  be a tree edge. We define  $\text{LOW1}(e)$  to be the *lowest* attachment of  $D(v)$  distinct from  $u$ . (By lowest, we mean farthest from the root.) If no such attachment exists, define  $\text{LOW1}$  to be 0. Note that if the graph is 2-connected, every edge  $u \rightarrow v$  has a non-zero  $\text{LOW1}$  value unless  $u$  is the root and  $v$  its (unique) child. Section 6.4.1 describes how to compute  $\text{LOW1}$  for all tree edges using a Union-Find procedure.

Further, we define, for every vertex  $u$ ,  $\text{RCH}(u) = \min\{w \mid w \hookrightarrow u\}$ , where the minimum is  $\infty$  if the set is empty. It is easy to compute  $\text{RCH}(u)$  for all vertices  $u$  in a bottom-up fashion in a dfs.

Next we need a few observations about the possible arrangement of the vertices of a 3-shredder in  $G$ , with respect to the dfs tree. We say that two vertices are comparable under the ancestor relation if one of them is an ancestor of the other (in the dfs tree.) It can be seen that if three vertices  $a_1, a_2$  and  $a_3$  are not mutually comparable under the ancestor relation i.e. they are not all on a (directed) path of tree edges, then  $G \setminus \{a_1, a_2, a_3\}$  has at most two components. In other words, we have the following:

**Lemma 6.2.1** *When  $G$  is 3-connected, a 3-shredder in  $G$  is always of the form  $(a_1, a_2, a_3)$  with  $a_1 \rightarrow^* a_2 \rightarrow^* a_3$ .*

*Proof:* Suppose not. Note that, since  $G$  is 3-connected,  $\{a_1, a_2, a_3\}$  is a minimal vertex-cut, hence each of the three vertices is adjacent to some vertex from each of the components of  $G \setminus \{a_1, a_2, a_3\}$ . Now we may assume, without loss of generality, that one of the following three situations occurs:

No two of the three vertices are comparable. In this case, define  $V_0$  to be  $V(G) \setminus (\cup_{i=1}^3 D(a_i))$  and  $V_i$  to be  $D(a_i) \setminus \{a_i\}$ , for  $i = 1, \dots, 3$ . Clearly,  $V_0$  is non-empty, and spans a connected subgraph of  $G \setminus \{a_1, a_2, a_3\}$ . For  $i = 1, 2, 3$ ,  $V_i$  consists of the vertex sets of subtrees rooted at the children of  $a_i$ . Since  $a_i$  is not a cut-vertex, each of these subtrees has an attachment in  $V_0$ . But then, this means that  $G \setminus \{a_1, a_2, a_3\}$  is connected, which is a contradiction.

Now suppose  $a_1 \rightarrow^* a_2$  but  $a_3$  is not comparable to  $a_1$  or  $a_2$ . Since  $a_3$  is not a cut-vertex, for every child  $w$  of  $a_3$ ,  $D(w)$  has an attachment that is a proper ancestor of  $a_3$ . Since every

neighbor of  $a_3$  is either a descendant or an ancestor of  $a_3$ , it follows that  $a_3$  is adjacent to only one component of  $G \setminus \{a_1, a_2, a_3\}$ , which is a contradiction.

Finally, suppose  $a_1$  is an ancestor of both  $a_2$  and  $a_3$ , but  $a_2$  and  $a_3$  are mutually incomparable. Let  $a_1^2$  be the child of  $a_1$  that is an ancestor of  $a_2$ . Define  $V_0$  to be  $V(G) \setminus D(a_1)$  and  $V_1^2$  to be  $D(a_1^2) \setminus D(a_2)$ . Since  $\{a_1, a_2\}$  is not a vertex-cut, for every child  $w$  of  $a_2$ ,  $D(w)$  has an attachment in either  $V_0$  or  $V_1^2$ . It follows that  $a_2$  is adjacent to at most two components of  $G \setminus \{a_1, a_2, a_3\}$ , which is a contradiction.  $\square$

Henceforth, we shall consider a potential 3-shredder as an ordered triple of vertices, where the ordering refers to that along the tree path. Now let  $(a, b, c)$  be *any* triple of distinct vertices with  $a \rightarrow p \rightarrow^* b \rightarrow q \rightarrow^* c$ . With respect to the above triple  $(a, b, c)$ , we define the following sets of vertices (refer to Figure 6.1):

- $\mathcal{A} = A \cup A' \cup A''$ , where
  - $A = V(G) \setminus (D(p) \cup \{a\})$
  - $A'$  consists of all subtrees of the form  $D(v)$  where  $v$  is a child of  $b$  different from  $q$  and with  $\text{HIGH1}(v) > a$
  - $A''$  consists of all subtrees of the form  $D(v)$  where  $v$  is a child of  $c$  with  $\text{HIGH1}(v) > a$
- $\mathcal{B} = B \cup B' \cup B''$ , where
  - $B$  consists of  $D(p) \setminus D(b)$
  - $B'$  consists of all subtrees of the form  $D(v)$  where  $v$  is a child of  $b$  different from  $q$  with  $\text{HIGH1}(v) \leq a$
  - $B''$  consists of all subtrees of the form  $D(v)$  where  $v$  is a child of  $c$  with  $(b < \text{HIGH1}(v) < a)$  OR  $(\text{HIGH1}(v) = a$  AND  $b < \text{HIGH2}(v) < a)$
- $\mathcal{C} = C \cup C'$ , where
  - $C$  consists of  $D(q) \setminus D(c)$
  - $C'$  consists of all subtrees of the form  $D(v)$  where  $v$  is a child of  $c$  with  $(\text{HIGH1}(v) \leq b)$  OR  $(\text{HIGH1}(v) = a$  AND  $c < \text{HIGH2}(v) < b)$  OR  $(\text{HIGH1}(v) = a$  AND  $\text{HIGH2}(v) = b$  AND  $c < \text{HIGH3}(v) < b)$
- $\mathcal{D}$ , which consists of subtrees of the form  $D(v)$  where  $v$  is a child of  $c$  with  $\text{HIGH1}(v) =$

$a$ ,  $\text{HIGH2}(v) = b$  and  $\text{HIGH3}(v) = \infty$ . These subtrees, if any, are clearly components of  $G \setminus \{a, b, c\}$  by themselves, and we will refer to these as the *singular* components of the triple  $(a, b, c)$

**Lemma 6.2.2** *The sets  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ , and the vertex sets of the singular components (defined above) all span connected subgraphs of  $G \setminus \{a, b, c\}$ . Further, these sets partition  $V(G) \setminus \{a, b, c\}$ .*

*Proof:* Denote  $G \setminus \{a, b, c\}$  by  $G'$ . Since  $a$  is not a cut-vertex,  $\mathcal{A}$  spans a connected subgraph of  $G'$ . It now follows by definition that  $\mathcal{A}$  also spans a connected subgraph of  $G'$ .

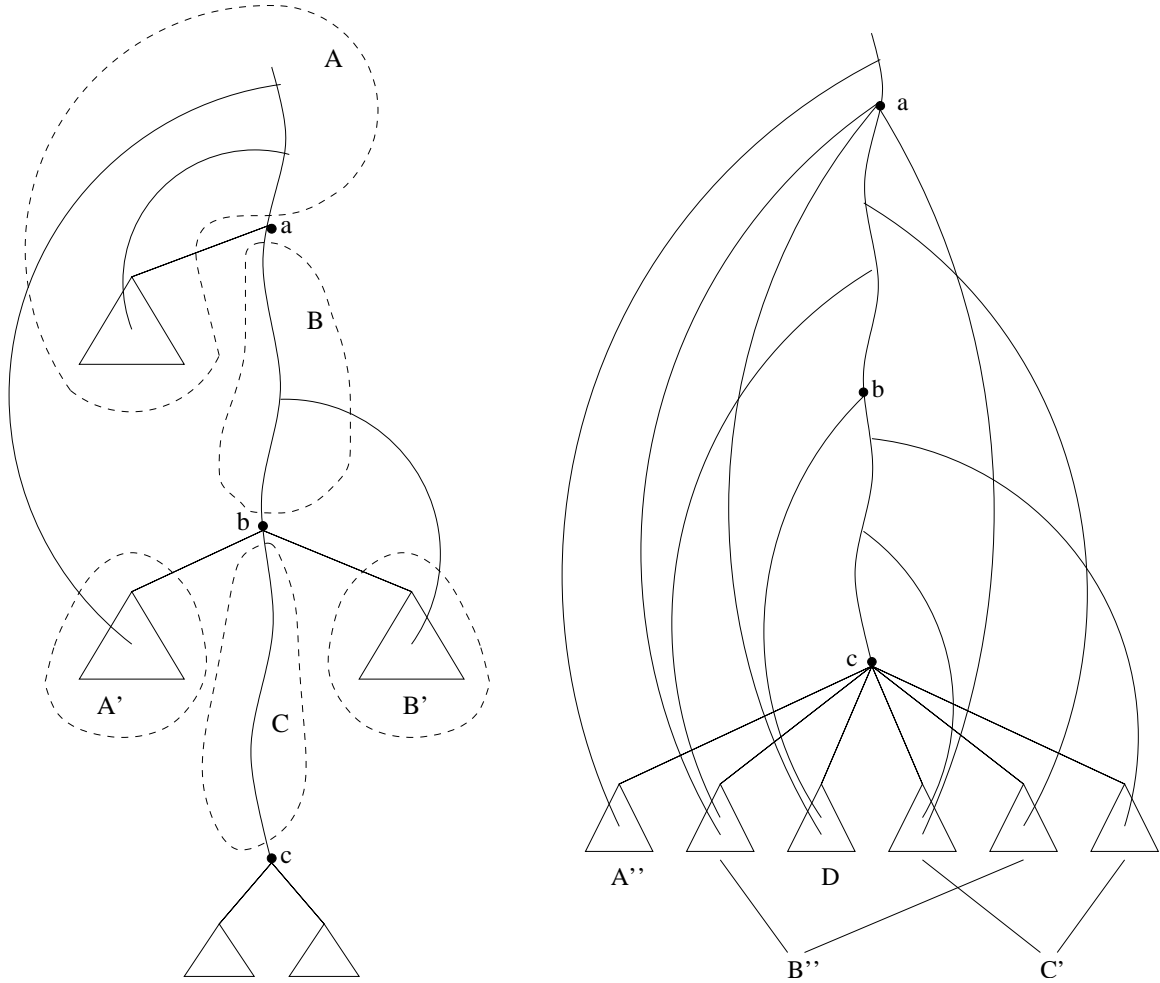
Clearly,  $\mathcal{B}$  spans a connected subgraph of  $G'$ . It follows by definition, that  $B'$  (respectively,  $B''$ ) consists of the vertex sets of those subtrees rooted at children of  $b$  (respectively,  $c$ ) that have an attachment in  $\mathcal{B}$  but are not included in  $\mathcal{A}$  (respectively,  $\mathcal{A}''$ ). It follows that  $\mathcal{B}$  spans a connected subgraph of  $G'$ , and is disjoint from  $\mathcal{A}$ . A similar argument shows that  $\mathcal{C}$  spans a connected subgraph of  $G'$ , and is disjoint from  $\mathcal{A}$  and  $\mathcal{B}$ . The lemma now follows.  $\square$

A triple as above (respectively, a shredder) that has a singular component is called a *singular* triple (or shredder). Conversely, a triple (shredder) that does not have any singular components is called *non-singular*. Further, a singular triple (shredder) is called *degenerate* if  $\text{HIGH1}(q) \leq a$ , and *non-degenerate* otherwise. Note that for a non-singular triple  $(a, b, c)$ , the maximum number of components of  $G \setminus \{a, b, c\}$  is three (which is achieved only when  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$  all span distinct components of  $G \setminus \{a, b, c\}$ .) Hence, if the triple is to be a 3-shredder, then there must not be any edge between the vertex sets  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ . By “ $X$ - $Y$  edge”, for  $X, Y \subseteq V(G)$ , we mean an edge with one end in  $X$  and the other end in  $Y$ , disregarding the direction that we are associating with the edges.

The above decomposition of  $V(G) \setminus \{a, b, c\}$  divides the set of proper descendants of  $c$  into  $\mathcal{A}''$ ,  $\mathcal{B}''$ ,  $\mathcal{C}''$  and  $\mathcal{D}$ . The ordering of  $\text{Adj}(c)$  (in particular, the ordering of  $c$ 's children in the list) implies that the subtrees in  $\mathcal{A}''$  occur before all the other subtrees. We define the “corner” vertex  $\alpha$  for the triple  $(a, b, c)$  as the “lower left corner” of the first subtree not in  $\mathcal{A}''$ , i.e.  $\alpha$  is the lowest numbered vertex, among the descendants of  $c$ , that is not in  $\mathcal{A}''$ .

More precisely, if there exists a child  $v$  of  $c$  with  $\text{HIGH1}(v) \leq a$ , let  $v_0$  be the first<sup>1</sup> such  $v$  and  $\alpha = v_0 - \text{ND}(v_0) + 1$ , otherwise  $\alpha = c$ . It follows that  $A'' = [c - \text{ND}(c) + 1, \alpha)$ .

We define  $\text{corner}(e)$  for an edge  $e \in \text{Adj}(u)$  as follows: let  $e' = u \rightarrow v'$  be the first *tree* edge to follow  $e$  in  $\text{Adj}(u)$  (if  $e$  is itself a tree edge,  $e' = e$ ). We set  $\text{corner}(e) = v' - \text{ND}(v') + 1$  (If no such  $v'$  exists, we set  $\text{corner}(e)$  to  $u$ .) The idea here is that, in the generation step, when we explore an edge  $e \in \text{Adj}(u)$ , we generate part of a triple with  $c = u$  and  $a = \text{HIGH1}(e)$ . The value of  $\text{corner}(e)$ , as defined above, gives us the right value of  $\alpha$  for



**Figure 6.1:** Potential Components of  $G \setminus \{a, b, c\}$  (triangles denote subtrees)

this triple. (Note that the definition of  $\alpha$  for a triple  $(a, b, c)$  does not involve  $b$ .)

We will use the following basic lemmas about any triple  $(a, b, c)$  of vertices with  $a \rightarrow$

<sup>1</sup>“first” refers to the usual ordering of the adjacency lists

$p \rightarrow^* b \rightarrow q \rightarrow^* c$ .

**Lemma 6.2.3** *If the path  $p \rightarrow^* b$  is not leftmost, then there is an  $A$ - $B$  edge. In particular, for a non-singular shredder  $(a, b, c)$ , the path  $p \rightarrow^* b$  is leftmost.*

*Proof:* Suppose the path  $p \rightarrow^* b$  is not leftmost, and let  $s \rightarrow t$  be the first non-leftmost edge in it. Now  $\text{HIGH1}(p) > a$  since  $a$  is not a cut-vertex, and since  $s$  is a leftmost descendant of  $p$ ,  $\text{HIGH1}(s) > a$ . (In general, if  $y$  is a leftmost descendant of  $x$ , it is easy to see that  $\text{HIGH1}(y) = \text{HIGH1}(x)$ , assuming  $\text{HIGH1}(x)$  is defined i.e  $x$  is not the root.) Let  $e$  be the first edge in  $\text{Adj}(s)$ . It follows that  $\text{HIGH1}(e) > a$ , which means there is an  $A$ - $B$  edge. The latter inference about non-singular shredders follows from the fact that for a non-singular triple  $(a, b, c)$  to be a 3-shredder,  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  must be vertex sets of distinct components of  $G \setminus \{a, b, c\}$ .  $\square$

**Lemma 6.2.4** *Either  $\text{HIGH1}(c) > a$ , or there is an  $\mathcal{A}$ - $\mathcal{B}$  or  $\mathcal{A}$ - $\mathcal{C}$  edge. In particular, for a non-singular shredder  $(a, b, c)$ ,  $\text{HIGH1}(c) > a$ .*

*Proof:* If  $\text{HIGH1}(c) \leq a$ , then  $c$  cannot be adjacent to any vertex in  $\mathcal{A}$ , and hence  $\mathcal{A}$  is not the vertex set of a component of  $G \setminus \{a, b, c\}$  by itself. It follows that there must be an  $\mathcal{A}$ - $\mathcal{B}$  or  $\mathcal{A}$ - $\mathcal{C}$  edge.  $\square$

**Lemma 6.2.5** *If the path  $q \rightarrow^* c$  is not leftmost, then either  $\text{HIGH1}(q) \leq a$  or there is an  $\mathcal{A}$ - $\mathcal{C}$  edge. In particular, for a non-singular shredder  $(a, b, c)$ , the path  $q \rightarrow^* c$  is leftmost.*

*Proof:* The proof of the first statement is similar to the proof of Lemma 6.2.3. For the latter inference, note that for a non-singular shredder  $(a, b, c)$ , it follows, from Lemma 6.2.4, that  $\text{HIGH1}(c)$  (and hence  $\text{HIGH1}(q)$ ) must be greater than  $a$ , and hence the path  $q \rightarrow^* c$  is leftmost.  $\square$

For the following lemmas, let  $(a, b, c)$  be a non-singular shredder.

**Lemma 6.2.6**  *$B''$  is non-empty or there is a back edge  $c \leftrightarrow v$  with  $b < v < a$ .*

*Proof:* This follows from the fact that  $c$  must be adjacent to a vertex in  $\mathcal{B}$ .  $\square$

**Lemma 6.2.7** *One (or both) of the following conditions must hold:*

- (i)  $a = \text{HIGH1}(e)$  for some edge  $e$  in  $\text{Adj}(c)$
- (ii)  $\exists u$  on the path  $q \rightarrow^* c$ ,  $u \neq c$ , s.t.  $a = \text{HIGH1}(e)$  for some non-leftmost edge  $e$  in  $\text{Adj}(u)$ .

*Proof:* This follows from the fact that  $a$  must be adjacent to a vertex in  $\mathcal{C}$ . □

**Lemma 6.2.8** *One (or both) of the following conditions must hold:*

- (i)  $\exists e \neq (b \rightarrow q)$  in  $\text{Adj}(b)$  s.t.  $\text{HIGH1}(e) > a$ . (In particular, if  $b \rightarrow q$  is not leftmost, then this condition is automatically satisfied.)
- (ii)  $\exists v \in A''$  with  $v \leftrightarrow b$  (i.e.  $b$  “sees” a back edge from a vertex in  $A''$ ). In particular, if  $b \rightarrow q$  is a leftmost edge, i.e. if  $b \rightarrow^* c$  is a leftmost path, then this condition is equivalent to saying  $\text{RCH}(b) < \alpha$ .

*Proof:* This follows from the fact that  $b$  must be adjacent to a vertex in  $\mathcal{A}$ . □

Finally, we need the following lemma about degenerate shredders.

**Lemma 6.2.9** *If  $(a, b, c)$  is a (singular and) degenerate shredder, then it has no  $\mathcal{A}$ - $\mathcal{C}$  edges, and it must have an  $\mathcal{A}$ - $\mathcal{B}$  edge.*

*Proof:* By definition, a degenerate shredder has  $\text{HIGH1}(c) \leq \text{HIGH1}(q) \leq a$ . It is easy to see then that there cannot be any  $\mathcal{A}$ - $\mathcal{C}$  edges, and that  $c$  cannot be adjacent to any vertex in  $\mathcal{A}$ . It follows that  $\mathcal{A}$  cannot be the vertex set of a component of  $G \setminus \{a, b, c\}$  by itself, and hence there must be an  $\mathcal{A}$ - $\mathcal{B}$  edge. □

### 6.3 The Generation Step

As mentioned before, the generation step follows the general format given in Table 6.1. The pseudo-code for replacing the lines `forward_visit( $e$ )` and `backward_visit( $e$ )` is given in tables 6.2 and 6.3 respectively.

A singular shredder  $(a, b, c)$ , by definition, has a singular component. Hence there is an edge  $e = (c \rightarrow v)$  in  $\text{Adj}(c)$  such that  $\text{HIGH1}(e) = a$ ,  $\text{HIGH2}(e) = b$  and  $\text{HIGH3}(e) = \infty$ .

Hence we can generate the triple when we are about to explore  $e$ . In order to find out whether the triple is degenerate or not, we need to know what  $q$  is, i.e. we need to know which child of  $b$  is currently active (in the dfs). We keep track of this information in the array `active_child`, which is updated whenever a recursive call is made. The rest of the section gives an informal description of how the generation step finds non-singular shredders, before giving proofs of correctness and the time bound for the generation step.

Let  $(a, b, c)$  be a non-singular shredder. By Lemma 6.2.7,  $a = \text{HIGH1}(e)$ , where  $e$  is as in that lemma. We will generate “candidate pairs”  $(a, c)$  and find the corresponding vertex  $b$  later. The candidate pairs will be stored in a data structure that we call `PSTACK`. This will be a stack of “blocks”, separated by end-markers, similar to the stack in a recursion. The individual blocks will be ordered lists of candidate pairs. Before each call `dfs_step(v)` for a *non-leftmost vertex*  $v$ , an end-marker is inserted on top of `PSTACK`, signifying that a fresh block is now on top of `PSTACK`. After the exit from `dfs_step(v)`, the topmost block (and the end-marker) are removed from the `PSTACK`. The first step of generating a triple  $(a, b, c)$  begins when some edge  $e$  in  $\text{Adj}(c)$  is explored, at which point  $\text{HIGH1}(e)$  will be our guess for  $a$ , which might be revised later. In addition, the value of  $\alpha$  for this “candidate pair” is set to  $\text{corner}(e)$ . This candidate pair  $(a, c)$  is added to the beginning of the current (topmost) block of `PSTACK`, and will be removed from `PSTACK` either to be moved to a list of triples (when the vertex  $b$  is detected), or to be discarded. As the dfs backs up over the tree path  $a \rightarrow^* c$ , we expect to recognize a vertex on the path as the right “ $b$ ” for the pair. There are three situations in which we realize that we have come across  $b$ , corresponding to the conditions in Lemma 6.2.8:

1. Whenever we back up over a non-leftmost edge  $u \rightarrow v$ , we mark  $u$  as the vertex  $b$  for all the candidate pairs in the current block of `PSTACK` (and move them to a list of triples). In particular, the topmost block of `PSTACK` (and the end-marker below it) are removed.
2. Whenever we explore a non-leftmost edge  $e$  in  $\text{Adj}(u)$  and see a candidate pair  $(a, c)$  in the current block of `PSTACK` with  $a < \text{HIGH1}(e)$ , we create the triple  $(a, u, c)$ .

3. Whenever we back up over a leftmost edge  $u \rightarrow v$ , and see a candidate pair  $(a, c)$  in the current block of PSTACK with  $\text{RCH}(u) < \alpha$ , we create the triple  $(a, u, c)$  and remove the pair  $(a, c)$  from PSTACK.

In situation 2 above, in addition to generating the corresponding triple, we also *revise* the value of  $a$  in the pair to  $\text{HIGH1}(e)$  (the revised pair stays on PSTACK). This corresponds to Lemma 6.2.7(ii). A pair that has already been revised once, however, is discarded. This is essential for keeping the overall time taken for the PSTACK operations linear.

**Table 6.2:** Generation step: pseudo-code for  $\text{forward\_visit}(e)$

```

1  if  $e = (u \rightarrow v)$  AND  $\text{HIGH3}(v) = \infty$  then   comment generating a singular triple
2      generate the triple  $(a=\text{HIGH1}(v), b=\text{HIGH2}(v), c=u)$  and mark it non-degenerate or
      degenerate depending on whether  $\text{HIGH1}(q) > a$  or not (where  $q = \text{active\_child}(b)$ )
3  if  $e$  is non-leftmost then begin
4      let  $(a, c)$  be the first candidate pair in the current block of PSTACK (set it to null
      if the end-marker is encountered instead)
5      while  $(a, c)$  is not null AND  $a < \text{HIGH1}(e)$  do begin
6          create the triple  $(a, b=u, c)$ 
7          if the pair  $(a, c)$  is unrevised then
8              set  $a = \text{HIGH1}(e)$  and mark the pair  $(a, c)$  as revised
9          else discard the pair  $(a, c)$  from PSTACK
10         set  $(a, c)$  to the next pair in the current block (set it to null if the end-marker
            is encountered instead)
11     end
12     add the pair  $(a=\text{HIGH1}(e), c = u)$  to the beginning of the current block; set  $\alpha =$ 
        corner( $e$ )
13     if  $e$  is a tree edge then add an end-marker on top of PSTACK
14 end
15 if  $e$  is a tree edge then set  $\text{active\_child}(u) = v$ 

```

If  $u$  is the current vertex in the dfs, and  $(a, c)$  is a candidate pair in the current block, then  $c$  would be a leftmost descendant of  $u$ , and  $a$  would be an ancestor of  $u$ . Thus the pairs in the current block consist of vertices that are all on a tree path containing  $u$ . Moreover, the pairs  $(a_1, c_1), (a_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, c_2, c_1, u, a_1, a_2, \dots$  appear on this path in the order listed, and that  $\dots \leq \alpha_2 \leq \alpha_1$ . This is essential for efficiently updating the pairs on PSTACK.

**Theorem 6.3.1** *The generation step runs in time  $O(n + m)$  and therefore the number of*

**Table 6.3:** Generation step: pseudo-code for backward\_visit( $e$ )

```

1 if  $e = (u \rightarrow v)$  is non-leftmost then   comment backing up over a non-leftmost edge
2     move all pairs in the current block to a list of triples, setting  $b = u$ ; remove the
   end-marker
3 else   comment backing up over a leftmost edge
4     while the pair  $(a, c)$  in the beginning of the current block has  $\alpha > \text{RCH}(u)$  do
5         remove the pair from PSTACK and create the triple  $(a, b = u, c)$ 

```

*triples generated is also  $O(n + m)$ .*

*Proof:* Since the generation step has the format given in Table 6.1, we only need to verify that the **while** loops in the pseudo-code for forward\_visit and backward\_visit (line 5 of Table 6.2 and line 4 of Table 6.3 respectively) and line 2 of Table 6.3 take  $O(n + m)$  time overall. The total number of distinct candidate pairs processed on PSTACK is at most  $2m$ , since each edge leads to the generation of at most one candidate pair, and this pair may be revised only once. Since the time taken by the **while** loops and line 2 of Table 6.3 is at most the number of distinct candidate pairs plus the number of edges, it follows that the overall time taken is  $O(n + m)$ .  $\square$

Before the next lemma, we need a definition. For a vertex  $u$ , the (maximal) tree path joining the root to the unique leaf that is a leftmost descendant of  $u$  is called the *canonical path* containing  $u$ .

**Lemma 6.3.2** *During the generation step, the following condition holds immediately before the **while** loops in forward\_visit( $e$ ) and backward\_visit( $e$ ) (line 5 of Table 6.2 and line 4 of Table 6.3 respectively). If  $u$  is the current vertex in the search, then the pairs in the current block of PSTACK consist of vertices that are all on the canonical path containing  $u$ . Moreover, the pairs  $(a_1, c_1), (a_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, a_2, a_1, u, c_1, c_2, \dots$  appear on this path in the order listed (possibly with repetition), and  $\dots \leq \alpha_2 \leq \alpha_1$ .*

*Proof:* We shall proceed by induction, proving that all the operations that change the current vertex, or the current block of PSTACK (or both) preserve the properties stated in the lemma. Suppose that at some time instant in the generation step, the pairs in the current

block satisfy the assertion of the lemma. Let these pairs (in order) be  $(a_2, c_2), (a_3, c_3), \dots$ , and let  $u$  be the current vertex. Suppose the search is exploring a non-leftmost edge  $e$  in  $\text{Adj}(u)$  with  $\text{HIGH1}(e) = a_1$ . Since the revision process (the **while** loop on line 5, Table 6.2) either discards pairs with  $a < a_1$  or revises them (by setting  $a = a_1$ ), it is clear that it preserves the required property of PSTACK. If the generation step then adds the pair  $(a_1, c_1 = u)$  while exploring an edge in  $\text{Adj}(u)$ , it is easy to see that  $\alpha_2 \leq \alpha_1$ , and that  $a_1 \leq a_2$  because of the revision process. Thus the assertion of the lemma still holds after this pair is added. Now suppose the recursive call  $\text{dfs\_step}(v)$  (that makes  $v$  the current vertex) is made after the tree edge  $u \rightarrow v$  is explored (i.e. after  $\text{forward\_visit}(u \rightarrow v)$ .) If  $u \rightarrow v$  is non-leftmost, then a fresh block would have been started before the recursive call, so the lemma holds trivially immediately after the recursive call. If  $u \rightarrow v$  is leftmost, then no candidate pair with  $c = u$  is yet on PSTACK, since pairs are generated only while exploring non-leftmost edges. Hence the lemma still holds immediately after the recursive call. While backing up over a non-leftmost edge  $u \rightarrow v$ , the current block is removed, restoring PSTACK to its state just before the recursive call  $\text{dfs\_step}(v)$ , and hence the lemma still holds by induction. Finally, suppose that the search is backing up over a leftmost edge  $u \rightarrow v$ , with  $(a, c)$  being the first pair in the current block. It suffices to verify that  $a$  is a proper ancestor of  $u$  (unless  $a = u$  is the root) to prove that the lemma still holds after the search backs up over  $u \rightarrow v$ . Since the pair  $(a, c)$  has survived on PSTACK till the search has backed up over to the vertex  $u$ , we claim by induction that there is no edge between  $A \cup A''$  and  $D(v) \setminus (A'' \cup \{c\})$ , where  $A$  is defined for the pair  $(a, c)$  analogous to the definition for triples, and  $A'' = [c - ND(c) + 1, \alpha)$ . If there is such an edge, say between  $A''$  and  $D(v) \setminus (A'' \cup \{c\})$ , it would be detected in the **while** loop in Table 6.3. If there is an edge between  $A$  and  $D(v) \setminus (A'' \cup \{c\})$ , then either there is a non-leftmost edge in the path  $v \rightarrow^* c$  (which will be detected by line 2 in Table 6.3), or  $\exists v'$  on the path  $v \rightarrow^* c$ ,  $v' \neq c$ , such that  $\text{HIGH1}(e) > a$  for some non-leftmost edge  $e$  in  $\text{Adj}(v')$  (which will be detected in the **while** loop in Table 6.2.) This proves the claim and implies that  $a$  must be a proper ancestor of  $u$  (unless  $a = u$  is the root), otherwise  $A \cup A''$  and  $D(v) \setminus (A'' \cup \{c\})$  would be (non-empty) vertex sets of components of  $G \setminus \{a, c\}$ . This proves the lemma.  $\square$

**Theorem 6.3.3** *The generation step finds a (multi-)set of triples that includes the set of 3-shredders of  $G$ . Furthermore, the non-singular triples generated have no  $\mathcal{A}$ - $\mathcal{C}$  edges, the path  $q \rightarrow^* c$  is leftmost for all such triples, and they satisfy the properties given in Lemma 6.2.7.*

*Proof:* First, let  $(a, b, c)$  be a singular shredder and  $D(v)$  be the vertex set of a singular component of  $(a, b, c)$ . Then the triple  $(a, b, c)$  will be generated when the edge  $c \rightarrow v$  is explored.

Consider a non-singular shredder  $(a, b, c)$ . Let  $e_1$  be the first edge in  $\text{Adj}(c)$  with  $b < \text{HIGH1}(e_1) \leq a$ , and let  $\text{HIGH1}(e_1) = a_1$ . (We know such an edge exists because of Lemma 6.2.6.) Now  $\text{HIGH1}(e_1) \leq a < \text{HIGH1}(c)$  by Lemma 6.2.4; in particular,  $e_1$  is non-leftmost. It follows that the candidate pair  $(a_1, c)$  is generated when  $e_1$  is explored (line 12 of Table 6.2), with the correct value of  $\alpha$ . Furthermore, if  $a_1 \neq a$ , then by Lemma 6.2.7, there is a vertex  $u$  on the path  $q \rightarrow^* c$ ,  $u \neq c$ , such that  $a = \text{HIGH1}(e_2)$  for some non-leftmost edge  $e_2$  in  $\text{Adj}(u)$ . Subject to the above condition, choose  $u$  to be closest to  $c$  and subject to this, choose  $e_2$  to be the earliest in  $\text{Adj}(u)$ . Since the shredder  $(a, b, c)$  does not have any  $\mathcal{A}$ - $\mathcal{C}$  or  $\mathcal{B}$ - $\mathcal{C}$  edges, that candidate pair  $(a_1, c)$  remains unrevised till the edge  $e_2$  is explored, at which point the pair will be revised with  $a_1$  being changed to  $a$ . (The fact that the revision process reaches the pair follows from Lemma 6.3.2.) Again, since  $(a, b, c)$  has no  $\mathcal{A}$ - $\mathcal{C}$  edges, it follows that the pair  $(a, c)$  then stays on  $\text{PSTACK}$  until the search backs up to the vertex  $b$ . By Lemma 6.2.8, the vertex  $b$  is detected by one of the two **while** loops in tables 6.2 and 6.3, or line 2 in Table 6.3. (Again, the fact that  $b$  is detected as above follows from Lemma 6.3.2.) Thus the triple  $(a, b, c)$  will be generated.

Also, if a triple  $(a, b, c)$  is generated from a candidate pair  $(a, c)$ , the fact that the candidate pair survived on  $\text{PSTACK}$  till the search backed up over the edge  $b \rightarrow q$  means that there is no edge between  $A \cup A''$  and  $D(q) \setminus (A'' \cup \{c\})$ , where  $A'' = [c - ND(c) + 1, \alpha]$ . (This is simply the claim proved at the end of the proof of Lemma 6.3.2.) In particular, the non-singular triples generated have no  $\mathcal{A}$ - $\mathcal{C}$  edges. Similarly, it follows that the path  $q \rightarrow^* c$  is leftmost for all the non-singular triples generated. To see that Lemma 6.2.7 holds for the non-singular triples, note that if a triple is generated from an unrevised candidate pair, then Lemma 6.2.7(i) is satisfied, and if a triple  $(a, b, c)$  is generated from a candidate

pair  $(a, c)$  that had been revised at a vertex  $u$ , then  $b$  must be a proper ancestor of  $u$ , i.e. Lemma 6.2.7(ii) is satisfied.  $\square$

The next section describes some intermediate computation that needs to be done before we move on to the detection steps.

## 6.4 Intermediate Computation

Before the detection steps described in Section 6.5, we eliminate multiple copies of non-singular and singular triples generated by the generation step. From the proof of Theorem 6.3.3, it can be seen that for non-singular triples, among multiple copies of a triple, the one with the lowest value of  $\alpha$  should be retained. This can be efficiently done by simultaneously sorting (using radix sort) for all vertices  $u$ , the lists of triples with  $c = u$  in lexicographic order of  $(a, b, \alpha)$ , and then scanning the lists for multiple entries. Similarly, if a triple  $(a, b, c)$  is generated both as a non-singular and a singular triple, then the non-singular triple is discarded. Furthermore, non-singular triples that do not satisfy the property  $\text{HIGH1}(c) > a$  (as in Lemma 6.2.4) are discarded, and triples with  $\mathcal{A}, \mathcal{B}$  or  $\mathcal{C}$  empty (i.e.  $a = \text{root}$  or  $a$  is the parent of  $b$  or  $b$  is the parent of  $c$ , respectively) are discarded.

### 6.4.1 Computing LOW1

This section describes how to compute LOW1 for all the tree edges in linear time, using a Union-Find procedure. Let  $z \hookrightarrow u$  be a back edge such that no proper descendant of  $u$  is the head of any back edge, and let  $u \rightarrow v \rightarrow w \rightarrow^* z$  be the tree path from  $u$  to  $z$ . It is easy to see that for all tree edges  $e$  along the path  $v \rightarrow^* z$ ,  $\text{LOW1}(e) = u$ . If we now discard the back edge  $z \hookrightarrow u$  and consider another back edge as above, we can compute, in the same way, LOW1 for other tree edges for which the quantity has not yet been computed. Proceeding in this fashion, we can compute LOW1 eventually for all tree edges. However, in order to not examine edges repeatedly (and hence spend too much time), we need to contract a tree edge once its LOW1 value is computed. It can be seen that in the situation above, *contracting* any edge on the path  $v \rightarrow^* z$  keeps the LOW1 values of the remaining tree edges the same. (By contracting a tree edge  $x \rightarrow y$ , we mean removing the edge and the vertex  $y$ , and replacing the end  $y$  in any other (tree and back) edges with the vertex

$x$ .) In order to pick the back edges in the manner described above, all the back edges are collected in a list, sorting the edges  $z \leftrightarrow u$  in increasing order of  $u$ . The back edges are then picked from this list, in order, computing LOW1 for the relevant tree edges as above and contracting them. Note that contracting those edges does not change the head of any other back edge, and hence does not affect the ordering of the list of back edges.

The edge contraction is implemented symbolically using Union-Find. A Union-Find algorithm implements a data structure for a partition of a ground set (the set of vertices in this case). Each set has a *representative* that is used to refer to the set. The Union-Find algorithm supports two operations:

$\text{union}(x,y)$ , which forms the union of the sets containing  $x$  and  $y$  (destroying the original sets), and

$\text{find-set}(x)$ , which returns the representative of the set containing  $x$ .

As the contractions are carried out, the vertex set is dynamically represented as a partition of the original vertex set. Initially all vertices are in singleton sets. In general, the vertex represented by  $x$  is the representative of the set containing  $x$ .

Contraction of an edge  $x \rightarrow y$  is done by merging the sets containing  $x$  and  $y$ , setting the representative of the merged set to that of the former set. (If the Union-Find algorithm does not implement the latter requirement, we can implement an additional function on the vertex set that maps the representative assigned by the algorithm to the one desired.) Computing the LOW1 values as above requires  $O(m)$  operations (union and find-set) to be performed on a ground set of size  $n$ . Note also that the unions all correspond to edges in the dfs tree (i.e. they are all of the form  $\text{union}(x,y)$  where  $x$  and  $y$  are adjacent in the dfs tree.) Hence this is a “graphical Union-Find” where the graph is actually a tree. This special case of Union-Find can be solved in time  $O(m + n)$  on a random-access machine (see [17] or [19].) The classical algorithm for Union-Find (based on “weighted-union of trees with finds executing path compression”; see [50]) performs the above Union-Find in time  $O((m + n)\alpha(m + n))$ , where  $\alpha$  is the functional inverse of an Ackermann-like function.

### 6.4.2 The “Corner” Vertices for a Triple

The vertex  $\alpha$  for a triple  $(a, b, c)$  was defined in such a way that the interval  $[c - ND(c) + 1, \alpha)$  is precisely  $A''$ . Similarly we define, for a triple  $(a, b, c)$ , vertices  $\beta, \gamma, \delta$  and  $\epsilon$  for demarcating  $B'', C'$  and  $\mathcal{D}$ , as follows:

$\beta$ : If there exists a child  $v$  of  $c$  with  $(\text{HIGH1}(v) = a \text{ AND } \text{HIGH2}(v) \leq b)$  OR  $\text{HIGH1}(v) < a$ , let  $v_0$  be the first such  $v$  and  $\beta = v_0 - ND(v_0) + 1$

otherwise  $\beta = c$

$\gamma$ : If there exists a child  $v$  of  $c$  such that  $(\text{HIGH1}(v) = a \text{ AND } \text{HIGH2}(v) < b)$  OR  $(\text{HIGH1}(v) = a \text{ AND } \text{HIGH2}(v) = b \text{ AND } \text{HIGH3}(v) < b)$  OR  $\text{HIGH1}(v) < a$ , let  $v_0$  be the first such  $v$  and  $\gamma = v_0 - ND(v_0) + 1$

otherwise  $\gamma = c$

$\delta$ : If there exists a child  $v$  of  $c$  with  $\text{HIGH1}(v) < a$ , let  $v_0$  be the first such  $v$  and  $\delta = v_0 - ND(v_0) + 1$

otherwise  $\delta = c$

$\epsilon$ : If there exists a child  $v$  of  $c$  with  $\text{HIGH1}(v) \leq b$ , let  $v_0$  be the first such  $v$  and  $\epsilon = v_0 - ND(v_0) + 1$

otherwise  $\epsilon = c$

Note that  $A'' = [c - ND(c) + 1, \alpha)$ ,  $B'' = ([\alpha, \beta) \cup [\delta, \epsilon))$ ,  $C'' = ([\gamma, \delta) \cup [\epsilon, c))$ , and  $\mathcal{D} = [\beta, \gamma)$ .

### 6.4.3 Computing the Corner Vertices for the Triples

For the following discussion,  $\text{list}(u)$ , for all vertices  $u$ , is the list of triples  $(a, b, c)$  with  $c = u$ .

**computing  $\epsilon$**

sort  $\text{list}(u)$  in decreasing order of  $b \forall u \in V(G)$

**for  $u \in V(G)$  do begin**

**for tree edge  $e \in \text{Adj}(u)$  do**

mark off triples from  $\text{list}(u)$  with  $\epsilon = \text{corner}(e)$  (and remove them from the list) until a triple with  $b < \text{HIGH1}(e)$  is encountered

set  $\epsilon = u$  for all triples left over in  $\text{list}(u)$

**end**

**computing  $\delta$**

sort  $\text{list}(u)$  in decreasing order of  $a \forall u \in V(G)$

**for  $u \in V(G)$  do begin**

**for tree edge  $e \in \text{Adj}(u)$  do**

mark off triples from  $\text{list}(u)$  with  $\delta = \text{corner}(e)$  (and remove them from the list) until a triple with  $a \leq \text{HIGH1}(e)$  is encountered

set  $\delta = u$  for all triples left over in  $\text{list}(u)$

**end**

**computing  $\gamma$**

sort  $\text{list}(u)$  in decreasing lexicographic order of  $(a, b) \forall u \in V(G)$

**for  $u \in V(G)$  do begin**

**for tree edge  $e \in \text{Adj}(u)$  do**

**if  $\text{HIGH3}(v) = \infty$  then**

mark off triples from  $\text{list}(u)$  with  $\gamma = \text{corner}(e)$  (and remove them from the list) until a triple with  $(a = \text{HIGH1}(e) \text{ AND } b \leq \text{HIGH2}(e)) \text{ OR } a < \text{HIGH1}(e)$  is encountered

**else**

mark off triples from  $\text{list}(u)$  with  $\gamma = \text{corner}(e)$  (and remove them from the list) until a triple with  $(a = \text{HIGH1}(e) \text{ AND } b < \text{HIGH2}(e)) \text{ OR } a < \text{HIGH1}(e)$  is encountered

set  $\gamma = u$  for all triples left over in  $\text{list}(u)$

**end**

**computing  $\beta$**

sort  $\text{list}(u)$  in decreasing lexicographic order of  $(a, b) \forall u \in V(G)$

**for  $u \in V(G)$  do begin**

**for tree edge  $e \in \text{Adj}(u)$  do**

mark off triples from  $\text{list}(u)$  with  $\beta = \text{corner}(e)$  (and remove them from the list) until a triple with  $(a = \text{HIGH1}(e) \text{ AND } b < \text{HIGH2}(e)) \text{ OR } a < \text{HIGH1}(e)$  is encountered  
 set  $\beta = u$  for all triples left over in  $\text{list}(u)$

**end**

**computing**  $\alpha$  (for singular triples)

similar to computing  $\delta$ ; simply replace the condition  $a \leq \text{HIGH1}(e)$  with  $a < \text{HIGH1}(e)$ .

We need to define (and compute for all the triples) an additional corner vertex to demarcate  $A'$  and  $B'$ , but it can be done in a similar fashion as discussed above, and is hence omitted. Finally, in preparation for the detection steps carried out in Sections 6.5.2 and 6.5.3, we need to mark those triples in which  $c$  is not adjacent to any vertex in  $\mathcal{B}$ . The above happens for a triple if and only if  $B''$  is empty (i.e.  $\alpha = \beta$  and  $\delta = \epsilon$ ) and there is no back-edge  $c \hookrightarrow v$  with  $b < v < a$ . Checking the first condition for all triples is trivial. The second condition is also easily checked by looking at the first child  $v$  of  $c$  with  $\text{HIGH1}(v) \leq b$  (this information can be computed for all triples along with  $\epsilon$ ) and then examining whether the edge  $e$  preceding  $c \rightarrow v$  in  $\text{Adj}(c)$  has  $\text{HIGH1}(e) < a$  or not.

## 6.5 Weeding out the Non-shredders

The generation procedure gives us a set of triples that includes the set of 3-shredders of the graph. It remains to weed out those triples that are not 3-shredders, by recognizing those that have edges between the vertex sets of the (potential) components  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ . A non-singular triple is discarded as soon as such a “bad” edge is found. A singular triple, on the other hand, could have these bad edges as long as it has enough singular components to make it a 3-shredder. We detect these edges in several dfs-like steps, each step dealing with a certain type or types of these bad edges. The general idea behind all the steps is the same (except for the one described in Section 6.5.6). The types of bad edges handled by a detection step would all have either their head or tail in one of the sets  $B$  or  $C$ ; so the detection of the bad edges for a given triple is carried out while the search is inside  $B$  or  $C$ , more precisely, as the search backs up over the tree paths  $p \rightarrow^* b$  or  $q \rightarrow^* c$  respectively.

Furthermore, the other end of each of the bad edges would be in a set of vertices that we shall call the *forbidden set*. For instance, if we are dealing with  $\mathcal{A}\text{-}\mathcal{C}$  edges, then the bad edges would all have one end in  $C$  and the other end in  $A \cup A''$ . Thus the detection will be done while backing up over the path  $q \rightarrow^* c$ , and the forbidden set in this case will be  $A \cup A''$ .

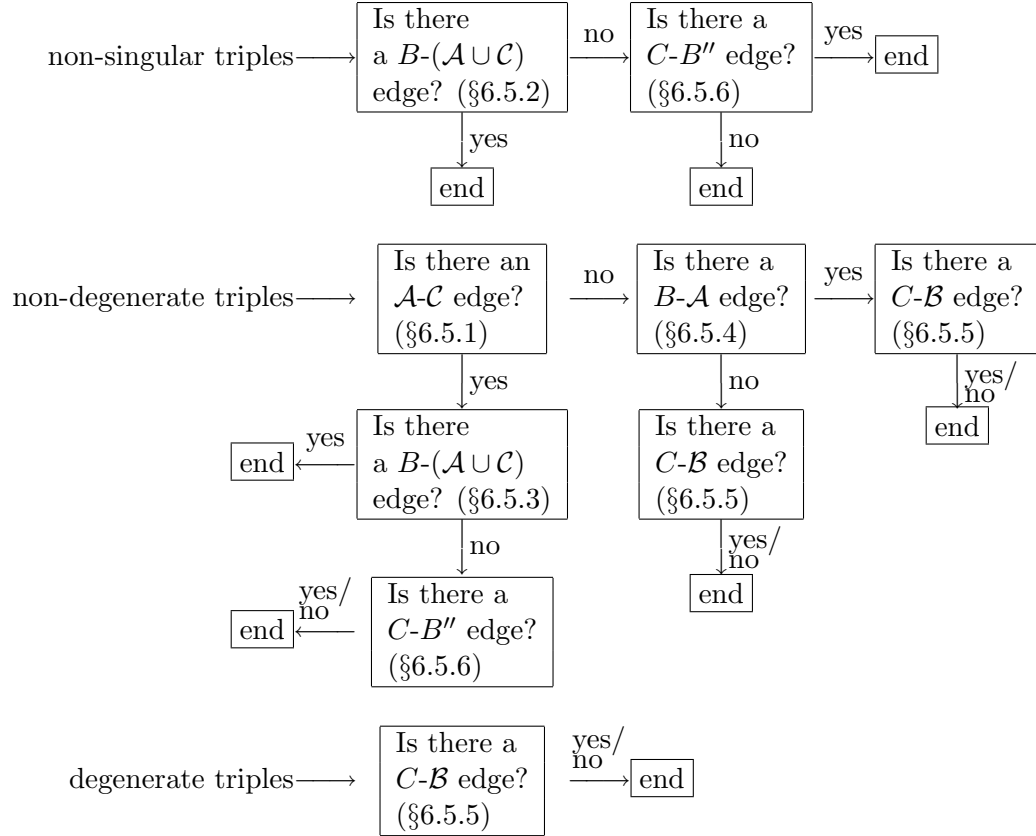
The triples are stored and processed on a stack similar to the one used in the generation step. The current block of the stack would contain the triples  $(a, b, c)$  for which the current vertex  $u$  is in  $p \rightarrow^* b$  or  $q \rightarrow^* c$ , as the case may be. We examine the back edges with head  $u$ , and for each such edge  $e$ , mark off (and remove) those triples in the current block for which the other end of  $e$  is in the forbidden set. Also, while exploring a non-leftmost tree edge  $e$  in  $\text{Adj}(u)$ , we mark off and remove those triples with  $\text{HIGH1}(e)$  in the forbidden set. In order to do this step efficiently, we need to keep the forbidden sets of the triples in the current block of the stack monotone i.e. if  $(a_1, b_1, c_1)$  appears before  $(a_2, b_2, c_2)$  in the current block, then the forbidden set of  $(a_1, b_1, c_1)$  contains the forbidden set of  $(a_2, b_2, c_2)$ .

The general format of the detection steps will be as given in Table 6.1, with statement A being replaced by the subroutine *sort\_lists* and statement B being replaced by the subroutine *load\_triples(u)*. The subroutine *sort\_lists* sorts  $\text{list}(u)$  for all vertices  $u$ , where  $\text{list}(u)$  is the list of triples with  $c = u$  (or  $b = u$  as the case may be). The subroutine *load\_triples(u)* loads  $\text{list}(u)$  at the beginning of the current block of the stack while maintaining the property mentioned in the previous paragraph. The flowchart in Figure 6.2 describes the order in which the detection steps are applied to the triples to eventually determine the set of 3-shredders of the graph. The following subsections describe the individual steps in detail.

### 6.5.1 Detecting $\mathcal{A}\text{-}\mathcal{C}$ Edges

From Theorem 6.3.3, the non-singular triples generated do not have any  $\mathcal{A}\text{-}\mathcal{C}$  edges. Furthermore, degenerate triples do not have any  $\mathcal{A}\text{-}\mathcal{C}$  edges, as  $\text{HIGH1}(q) \leq a$ . Hence this step is only required for singular, non-degenerate triples.

The forbidden set for a triple  $(a, b, c)$  is  $A \cup A''$  and the detection is carried out on the



**Figure 6.2:** Flowchart describing the order of application of the detection steps

path  $q \rightarrow^* c$ . For all vertices  $u$ ,  $\text{list}(u)$  is the list of non-degenerate triples with  $c = u$ . The subroutine `sort_lists` sorts  $\text{list}(u) \forall u \in V$  in increasing order of  $a$ . Note that this is consistent with the monotonicity required for the forbidden sets i.e. the forbidden set  $(A \cup A'')$  of  $(a_1, b_1, u)$  contains the forbidden set of  $(a_2, b_2, u)$  if  $a_1 \leq a_2$ . A computational remark is in order here. The above sorting can be done in linear time with a radix sort with  $n + 1$  buckets, similar to the sorting of the adjacency lists described in Section 6.2. The pseudo-code for replacing `forward_visit(e)` and `backward_visit(e)` is given in tables 6.4 and 6.5 respectively. Table 6.6 gives the pseudo-code for the subroutine `load_triples(u)`. (Recall that  $\text{RCH}(u)$ , for a vertex  $u$ , is defined in Section 6.2.)

**Lemma 6.5.1** *During the detection step, the following condition holds immediately before the **while** loops (line 2 of Table 6.4 and line 7 of Table 6.5). If  $u$  is the current vertex in the search, then the triples  $(a, b, c)$  in the current block of `STACK` are such that  $a, b$  and  $c$*

**Table 6.4:** Detecting  $\mathcal{A}\text{-}\mathcal{C}$  edges: pseudo-code for `forward_visit(e)`

```
1 if  $e$  is non-leftmost then begin
2     while the triple  $(a, b, c)$  in the beginning of the current block of STACK has  $a <$ 
        HIGH1( $e$ ) do
3         mark the triple as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge and remove it from STACK
4     if  $e$  is a tree edge then add an end-marker on top of STACK
5 end
```

**Table 6.5:** Detecting  $\mathcal{A}\text{-}\mathcal{C}$  edges: pseudo-code for `backward_visit(e)`

```
1 remove the triples with  $q = v$  from the current block of STACK
2 if  $e = (u \rightarrow v)$  is non-leftmost then begin comment backing up over a
        non-leftmost edge
3     mark all triples  $(a, b, c)$  in the current block of STACK as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge
4     remove the block (and the end-marker) from STACK
5 end
6 else comment backing up over a leftmost edge
7     while the triple  $(a, b, c)$  in the beginning of the current block has  $\alpha > \text{RCH}(u)$ 
        do
8         mark the triple as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge and remove it from STACK
```

are all on the canonical path containing  $u$ , with  $a$  and  $b$  being proper ancestors of  $u$  and  $c$  a (leftmost) descendant of  $u$ . Moreover, the triples  $(a_1, b_1, c_1), (a_2, b_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, a_2, a_1, u, c_1, c_2, \dots$  appear on this path in the order listed (possibly with repetition), and that  $\dots \leq \alpha_2 \leq \alpha_1$ . It follows that  $(A_1 \cup A_1'') \supseteq (A_2 \cup A_2'') \dots$  and so on.

*Proof:* We shall use an inductive argument similar to the one used in the proof of Lemma 6.3.2. It is clear that `forward_visit(e)` and the recursive call preserve the property asserted in the lemma. After backing up a non-leftmost edge  $u \rightarrow v$ , the property still holds by induction because STACK is restored to its state just before the recursive call `dfs_step(v)`. After backing up a leftmost edge, the property still holds because of line 1 in Table 6.5.

Suppose now that during the search, we are at the end of `dfs_step(u)` and want to load `list(u)` on STACK. We need to verify that `load_triples(u)` preserves the required property of STACK. The procedure `load_triples(u)` looks at the triple  $(a_1, b_1, c_1)$  at the end of `list(u)`

**Table 6.6:** Detecting  $\mathcal{A}\text{-}\mathcal{C}$  edges: pseudo-code for `load_triples( $u$ )`

```

1  while list( $u$ ) is non-empty do begin
2      let  $(a_1, b_1, c_1)$  be at the end of list( $u$ )
3      if the current block of STACK is empty then
4          remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current block
          of STACK
5      else begin
6          let  $(a_2, b_2, c_2)$  be at the beginning of the current block of STACK
7          if  $c_1 > c_2$  AND  $a_1 > a_2$  then
8              mark  $(a_2, b_2, c_2)$  as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge and remove it from STACK
9          else remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current
          block of STACK
10     end
11 end

```

and compares it with the triple  $(a_2, b_2, c_2)$  at the beginning of the current block of STACK. It either moves  $(a_1, b_1, c_1)$  to STACK if it finds that doing so maintains the property stated in the lemma, or discards one of the triples, marking it as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge. (This is continued until all triples in `list( $u$ )` have either been loaded onto STACK or marked as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge.) If  $c_1 = c_2$  (this happens if  $(a_2, b_2, c_2)$  is originally from `list( $u$ )`), then it follows that  $(a_1, b_1, c_1)$  can be moved from `list( $u$ )` to STACK while maintaining the property given in the lemma. If  $c_1 \neq c_2$ , then  $c_2$  is a proper leftmost descendant of  $c_1$ . Also, since  $(a_2, b_2, c_2)$  is still on STACK and the search is at the end of `dfs_step( $c_1$ )`, it follows that  $b_2$  is a proper ancestor of  $c_1$ . Hence  $c_1 \in \mathcal{C}_2$ . Suppose  $a_1 > a_2$ , and let  $s$  be the leftmost child of  $c_1$ .  $D(s)$  cannot be a singular component of  $(a_1, b_1, c_1)$ , since it has two distinct attachments  $a_2$  and  $b_2$  other than  $a_1$ . But then a singular component of  $(a_1, b_1, c_1)$  gives rise to an  $\mathcal{A}\text{-}\mathcal{C}$  edge for  $(a_2, b_2, c_2)$ , which is hence removed from STACK and marked as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge. If  $a_1 \leq a_2$ , it follows that  $(a_1, b_1, c_1)$  can be moved from `list( $u$ )` to STACK while maintaining the property given in the lemma.  $\square$

**Theorem 6.5.2** *The detection step correctly marks, in  $O(n + m)$  time, those of its input triples that have an  $\mathcal{A}\text{-}\mathcal{C}$  edge.*

*Proof:* Suppose a triple  $(a, b, c)$  is marked by the search as having an  $\mathcal{A}\text{-}\mathcal{C}$  edge. We need to verify that it indeed has one. If the triple is marked by the subroutine `load_triples`, then

it can be easily seen from the proof of the previous lemma that it has an  $\mathcal{A}\mathcal{C}$  edge. If the triple is marked by line 3 in Table 6.4, the edge  $e = u \rightarrow v$  is such that the subtree  $D(v) \subseteq C$  has an attachment higher than  $a$ , that is, in  $A$ . Hence the triple has an  $\mathcal{A}\mathcal{C}$  edge. If the triple is marked by line 3 in Table 6.5, the edge  $e = u \rightarrow v$  is non-leftmost, hence by Lemma 6.2.5 it follows that the triple has an  $\mathcal{A}\mathcal{C}$  edge (since  $\text{HIGH1}(q) > a$ , by definition, for non-degenerate triples.) Finally, if the triple is marked by line 8 in Table 6.5, the vertex  $u \in C$  has a back edge coming in from a vertex in  $A''$ , hence the triple has an  $\mathcal{A}\mathcal{C}$  edge. Conversely, suppose a triple  $(a, b, c)$  has an  $\mathcal{A}\mathcal{C}$  edge. Then it has either a  $C\text{-}A''$  edge, or a  $C\text{-}A$  edge (or both.) From the previous lemma, it then follows that in the first case, such an edge would be detected by the **while** loop on line 7 in Table 6.5. In the second case, the edge would be detected by the **while** loop in Table 6.4 or line 2 in Table 6.5. Hence a triple with an  $\mathcal{A}\mathcal{C}$  edge will be marked accordingly.

Finally, for the time bound, note that the subroutine `load_triples` takes  $O(1)$  time per triple, and hence loading `list(u)` for all vertices  $u$  takes  $O(n+m)$  time overall (as there are  $O(n+m)$  triples.) Also, line 1 in Table 6.5 can be efficiently executed by maintaining, for every vertex  $v$ , the list of triples with  $q = v$  (where the triple  $(a, b, c)$  is such that  $a \rightarrow p \rightarrow^* b \rightarrow q \rightarrow^* c$ .) The time taken for executing line 1 would then be proportional to the number of triples, and hence  $O(n + m)$ .  $\square$

### 6.5.2 Detecting $B\text{-}(\mathcal{A} \cup \mathcal{C})$ Edges in Non-singular Triples

This step is only executed for non-singular triples. A triple is discarded as soon as such an edge is found. Before the detection step itself, triples  $(a, b, c)$  such that  $c$  is not adjacent to any vertex in  $\mathcal{B}$  are discarded. (Note that for such triples,  $\mathcal{B}$  cannot be the vertex set of a component of  $G \setminus \{a, b, c\}$  by itself, and hence there is a  $\mathcal{B}\text{-}(\mathcal{A} \cup \mathcal{C})$  edge. In fact, since  $c$  is not adjacent to any vertex in  $\mathcal{B}$ , such an edge must be a  $B\text{-}(\mathcal{A} \cup \mathcal{C})$  edge.)

The forbidden set for a triple is  $\mathcal{A} \cup \mathcal{C}$  and the detection is carried out on the path  $p \rightarrow^* b$ . For all vertices  $u$ , `list(u)` is the list of non-singular triples with  $b = u$ . The subroutine `sort_lists` sorts `list(u)`  $\forall u \in V$  such that  $(a_1, u, c_1)$  precedes  $(a_2, u, c_2)$  in the list iff  $a_1 < a_2$ , or  $a_1 = a_2$  and  $c_1 \geq c_2$ .

The pseudo-code for replacing  $\text{forward\_visit}(e)$  and  $\text{backward\_visit}(e)$  is given in tables 6.7 and 6.8 respectively. Table 6.9 gives the pseudo-code for the subroutine  $\text{load\_triples}(u)$ .

**Table 6.7:** Detecting  $B-(\mathcal{A} \cup \mathcal{C})$  edges in non-singular triples: pseudo-code for  $\text{forward\_visit}(e)$

```

1  if  $e$  is non-leftmost then begin
2      while the triple  $(a, b, c)$  in the beginning of the current block of STACK has  $a <$ 
         $\text{HIGH1}(e)$  do
3          discard the triple
4      if  $e$  is a tree edge then add an end-marker on top of STACK
5  end

```

**Table 6.8:** Detecting  $B-(\mathcal{A} \cup \mathcal{C})$  edges in non-singular triples: pseudo-code for  $\text{backward\_visit}(e)$

```

1  while the triple  $(a, b, c)$  at the beginning of the current block of STACK has  $a = u$  do
2      remove the triple from STACK
3  if  $e = (u \rightarrow v)$  is non-leftmost then comment backing up over a
        non-leftmost edge
4      discard all triples  $(a, b, c)$  in the current block of STACK; remove the block (and the
        end-marker) from STACK
5  else comment backing up over a leftmost edge
6      for edge  $w \hookrightarrow u$  in  $\text{Adj}^R(u)$  do
7          while the triple  $(a, b, c)$  at the beginning of the current block has  $w \in$ 
             $(\mathcal{A} \cup \mathcal{C})$  do
8              discard the triple

```

**Lemma 6.5.3** *During the detection step, the following condition holds immediately before the **while** loops on line 2 of Table 6.7 and line 7 of Table 6.8. If  $u$  is the current vertex in the search, then the triples  $(a, b, c)$  in the current block of STACK are such that  $a$  and  $b$  are on the canonical path containing  $u$ , with  $a$  being a proper ancestor of  $u$  and  $b$  a (leftmost) descendant of  $u$ . Moreover, the triples  $(a_1, b_1, c_1), (a_2, b_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, a_2, a_1, u, b_1, b_2, \dots$  appear on this path in the order listed (possibly with repetition, and  $\dots (\mathcal{A}_2 \cup \mathcal{C}_2) \supseteq (\mathcal{A}_1 \cup \mathcal{C}_1)$ ).*

**Table 6.9:** Detecting  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edges in non-singular triples: pseudo-code for `load_triples( $u$ )`

```

1  while list( $u$ ) is non-empty do begin
2      let  $(a_1, b_1, c_1)$  be at the end of list( $u$ )
3      if the current block of STACK is empty then
4          remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current block
          of STACK
5      else begin
6          let  $(a_2, b_2, c_2)$  be at the beginning of the current block of STACK
7          if  $b_1 = b_2$  then
8              if  $q_1 \neq q_2$  then
9                  if  $\text{HIGH1}(q_1) > a_2$  then
10                     discard  $(a_2, b_2, c_2)$ 
11                 else remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the
                    current block of STACK
12                 else if  $c_1 = c_2$  then
13                     remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the cur-
                    rent block of STACK
14                 else comment  $c_1 > c_2$ 
15                     discard  $(a_2, b_2, c_2)$ 
16                 else if  $a_1 > a_2$  then
17                     if  $q_1$  is non-leftmost then
18                         discard  $(a_2, b_2, c_2)$ 
19                     else if the second edge  $e$  in  $\text{Adj}(b_1)$  has  $\text{HIGH1}(e) > a_1$  then
20                         discard  $(a_2, b_2, c_2)$ 
21                     else if  $c_1 > b_2$  then
22                         discard  $(a_2, b_2, c_2)$ 
23                     else if  $c_1 = b_2$  then
24                         if  $\text{HIGH1}(q_2) > a_1$  then
25                             discard  $(a_1, b_1, c_1)$ 
26                         else discard  $(a_2, b_2, c_2)$ 
27                     else if  $c_2 \geq c_1$  then
28                         discard  $(a_2, b_2, c_2)$ 
29                     else if  $\text{RCH}(b_1) < \alpha_2$  then
30                         discard  $(a_2, b_2, c_2)$ 
31                     else discard  $(a_1, b_1, c_1)$ 
32                 else if  $c_1 \geq b_2$  OR  $c_1 = c_2$  then
33                     remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current
                    block of STACK
34                 else discard  $(a_2, b_2, c_2)$ 
35          end
36 end

```

*Proof:* We shall use an inductive argument similar to the one used in the proof of Lemma 6.3.2. It is clear that `forward_visit(e)` and the recursive call preserve the property asserted in the lemma. After backing up a non-leftmost edge  $u \rightarrow v$ , the property still holds by induction because `STACK` is restored to its state just before the recursive call `dfs_step(v)`. After backing up a leftmost edge, the property still holds because of the **while** loop on line 1 in Table 6.8.

Suppose now that during the search, we are at the end of `dfs_step(u)` and want to load `list(u)` on `STACK`. The following is an outline of a proof that `load_triples(u)` preserves the required property of `STACK`.

line 10: Since  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ , this implies a  $B_2$ - $A'_2$  edge.

line 11: In this case,  $A_2 \subseteq A_1$  because of the ordering of `list(u)`, and  $(A'_2 \cup D(q_2)) \subseteq A'_1$ .

line 13: In this case,  $A_2 \subseteq A_1$ ,  $A'_2 \subseteq A'_1$ ,  $A''_2 \subseteq A''_1$ ,  $C_2 = C_1$  and  $C'_2 \subseteq (A''_1 \cup C'_1)$ .

line 15: Note, in this case, that  $c_1 < c_2$  cannot happen. (If  $a_1 = a_2$ , this follows from the ordering of `list(u)`. If  $a_1 < a_2$ , then it follows because otherwise,  $(a_1, b_1, c_1)$  would never have been generated.) Since  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ , this implies a  $B_2$ - $C_2$  edge.

lines 16–34: If  $b_1 \neq b_2$ , note that  $b_2$  is a proper leftmost descendant of  $b_1$  and that  $a_2$  is a proper ancestor of  $b_1$ .

line 18:  $c_1 \in B_2$ , hence  $\text{HIGH1}(c_1) > a_1$  implies a  $B_2$ - $A_2$  edge.

line 20: there is a  $B_2$ - $A_2$  edge.

line 22: Lemma 6.2.7 for  $(a_1, b_1, c_1)$  implies a  $B_2$ - $A_2$  edge.

line 25: Lemma 6.2.7 for  $(a_2, b_2, c_2)$  implies a  $B_1$ - $A'_1$  edge.

line 26: In this case, since  $\text{RCH}(b_1) < \alpha_1$ , it follows that there is a  $B_2$ - $A''_2$  edge.

line 28: Note that, in this case,  $c_1$  and  $c_2$  may be incomparable under the ancestor relation (if  $q_2$  is non-leftmost) or  $c_2$  is an ancestor of  $c_1$ . In the first case,  $\text{RCH}(b_1) < \alpha_1$  implies a  $B_2$ - $A'_2$  edge. In the second case, it implies a  $B_2$ - $A''_2$  edge.

line 30: In this case, there is a  $B_2$ - $A''_2$  edge.

line 31: Since  $\{a_1, c_2\}$  cannot be a vertex cut, it follows that there is a vertex  $v$  in the interior of the path  $a_1 \rightarrow^* c_2$  such that either  $\text{RCH}(v) < \alpha_2$  or there is a non-leftmost edge  $e$  in  $\text{Adj}(v)$  with  $\text{HIGH1}(e) > a_1$ . Now  $v$  cannot be in the interior of the paths  $b_2 \rightarrow^* c_2$  or

$b_1 \rightarrow^* c_1$  from Theorem 6.3.3 applied to  $(a_2, b_2, c_2)$  and  $(a_1, b_1, c_1)$  respectively. Also, it is clear that  $v \neq b_1$ . Hence  $v$  must be in the interior of the path  $a_1 \rightarrow^* b_1$ , and that implies a  $B_1$ - $\mathcal{A}_1$  edge.

line 33: Clearly,  $A_2 \subseteq A_1$ . If  $q_1$  is non-leftmost, then  $D(b_2) \subseteq A'_1$ . Otherwise, if  $c_1$  is an ancestor of  $b_2$ , then  $(A'_2 \cup D(q_2)) \subseteq A''_1$ . Otherwise, if  $c_1 = c_2$ , then  $(A'_2 \cup C_2) \subseteq C_1$ ,  $A''_2 \subseteq (A''_1 \cup C'_1)$  and  $C'_2 \subseteq (A''_1 \cup C'_1)$ .

line 34: If  $q_2$  is non-leftmost, the fact that  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$  implies a  $B_2$ - $A'_2$  edge. Otherwise, if  $c_2$  is a proper ancestor of  $c_1$ , it implies a  $B_2$ - $A''_2$  edge. Otherwise,  $c_1$  is a proper ancestor of  $c_2$  and the fact that  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$  implies a  $B_2$ - $C_2$  edge.  $\square$

**Theorem 6.5.4** *The detection step correctly discards, in  $O(n+m)$  time, those of its input triples that have a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge.*

*Proof:* Suppose a triple  $(a, b, c)$  is discarded by the search for having a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge. We need to verify that it indeed has one. If the triple is discarded by the subroutine `load_triples`, then it can be easily seen from the proof of the previous lemma that it has a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge. If the triple is discarded by line 3 in Table 6.7, the edge  $e = u \rightarrow v$  is such that the subtree  $D(v) \subseteq B$  has an attachment higher than  $a$ , that is, in  $A$ . Hence the triple has a  $B$ - $A$  edge. If the triple is marked by line 4 in Table 6.8, the edge  $e = u \rightarrow v$  is non-leftmost, hence by Lemma 6.2.3 it follows that the triple has a  $B$ - $A$  edge. Finally, if the triple is marked by line 8 in Table 6.8, the vertex  $u \in B$  has a back edge coming in from a vertex in  $\mathcal{A} \cup \mathcal{C}$ , hence the triple has a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge.

Conversely, suppose a triple  $(a, b, c)$  has a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge. Then it has either a  $B$ - $(A' \cup A'' \cup C \cup C')$  edge, or a  $B$ - $A$  edge (or both.) From the previous lemma, it then follows that in the first case, such a triple would be discarded by the **while** loop on line 7 in Table 6.8. In the second case, it would be discarded by the **while** loop in Table 6.7 or line 4 in Table 6.8. Hence a triple with a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge will be discarded.

Finally, for the time bound, note that the subroutine `load_triples` takes  $O(1)$  time per triple, and hence loading `list(u)` for all vertices  $u$  takes  $O(n+m)$  time overall (as there are  $O(n+m)$

triples.) □

### 6.5.3 Detecting $B-(\mathcal{A} \cup \mathcal{C})$ Edges in Singular Triples

This step is for non-degenerate triples with an  $\mathcal{A}$ - $\mathcal{C}$  edge. Before the detection step, the triples  $(a, b, c)$  such that  $c$  is not adjacent to any vertex in  $\mathcal{B}$  are marked as having a  $B-(\mathcal{A} \cup \mathcal{C})$  edge, and hence not examined during the detection step. (Refer to the corresponding argument in Section 6.5.2.)

The forbidden set for a triple is  $\mathcal{A} \cup \mathcal{C}$  and the detection is carried out along the path  $p \rightarrow^* b$ . The subroutine `sort_lists` sorts  $\text{list}(u) \forall u \in V$  such that  $(a_1, u, c_1)$  precedes  $(a_2, u, c_2)$  in the list iff  $a_1 < a_2$ , or  $a_1 = a_2$  and  $c_1 \leq c_2$ . The pseudo-code for `forward_visit(e)` and `backward_visit(e)` is similar to that in tables 6.7 and 6.8 respectively. The difference is that instead of discarding a triple, we only mark it as having a  $B-(\mathcal{A} \cup \mathcal{C})$  edge and remove it from `STACK`. The pseudo-code for `load_triples(u)` is given in Table 6.10.

**Lemma 6.5.5** *During the detection step, the following condition holds immediately before the **while** loops on line 2 of Table 6.7 and line 7 of Table 6.8. If  $u$  is the current vertex in the search, then the triples  $(a, b, c)$  in the current block of `STACK` are such that  $a$  and  $b$  are on the canonical path containing  $u$ , with  $a$  being a proper ancestor of  $u$  and  $b$  a (leftmost) descendant of  $u$ . Moreover, the triples  $(a_1, b_1, c_1), (a_2, b_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, a_2, a_1, u, b_1, b_2, \dots$  appear on this path in the order listed (possibly with repetition), and  $\dots (\mathcal{A}_2 \cup \mathcal{C}_2) \supseteq (\mathcal{A}_1 \cup \mathcal{C}_1)$ .*

*Proof:* We shall use an inductive argument similar to the one used in the proof of Lemma 6.3.2. It is clear that `forward_visit(e)` and the recursive call preserve the property asserted in the lemma. After backing up a non-leftmost edge  $u \rightarrow v$ , the property still holds by induction because `STACK` is restored to its state just before the recursive call `dfs_step(v)`. After backing up a leftmost edge, the property still holds because of the **while** loop on line 1 in Table 6.8.

Suppose now that during the search, we are at the end of `dfs_step(u)` and want to load `list(u)` on `STACK`. The following is an outline of a proof that `load_triples(u)` preserves the required property of `STACK`.

line 10: Since  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ , this implies a  $B_2$ - $A'_2$  edge.

line 11: Clearly,  $A_2 \subseteq A_1$  and  $A'_2 \subseteq A'_1$ . Since  $\text{HIGH1}(q_2) > a_2 \geq a_1$ ,  $D(q_2) \subseteq A'_1$ .

line 14: Clearly,  $(a_1, b_1, c_1)$  can be moved from  $\text{list}(u)$  to  $\text{STACK}$ .

line 15: Since  $(\mathcal{A}_2 \cup \mathcal{C}_2) \not\subseteq (\mathcal{A}_1 \cup \mathcal{C}_1)$ ,  $c_1 \in A''_2$ . But since  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ , this implies a  $B_2$ - $A_2$  edge.

line 17: Clearly, since  $a_2 \geq a_1$ ,  $A_2 \subseteq A_1$ ,  $A'_2 \subseteq A'_1$ ,  $A''_2 \subseteq A''_1$  and  $C'_2 \subseteq (A''_1 \cup C'_1)$

line 18: If  $c_1$  and  $c_2$  are incomparable under the ancestor relation, then  $c_1 \in C_2$  and since  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ , it follows that there is a  $B_2$ - $C_2$  edge. On the other hand, if  $c_1$  is a proper ancestor of  $c_2$ , then  $c_1 \in C_2$  and  $a_1 < a_2$  (because of the ordering of  $\text{list}(u)$ .) Let  $s$  be the child of  $c_1$  with  $c_2 \in D(s)$ . Clearly,  $D(s) \subseteq A''_1$ , and since  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ , it follows that there is a  $B_2$ - $C_2$  edge.

lines 19–37: If  $b_1 \neq b_2$ , note that  $b_2$  is a proper leftmost descendant of  $b_1$  and that  $a_2$  is a proper ancestor of  $b_1$ .

line 21: If  $q_1$  is non-leftmost, then clearly there is a  $B_2$ - $A_2$  edge. If  $c_1$  is a proper ancestor of  $b_2$ , let  $s$  be the child of  $c_1$  with  $b_2 \in D(s)$ . Since a singular component of  $(a_2, b_2, c_2)$  has an attachment at  $a_2$ ,  $D(s)$  is not a singular component of  $(a_1, b_1, c_1)$  and it follows that there is a  $B_2$ - $A_2$  edge. Finally, if  $c_1 = b_2$  or  $c_1$  is a descendant of  $b_2$  not contained in  $D(q_2)$ , a similar argument shows that there is a  $B_2$ - $A'_2$  edge.

line 23: It can be seen that  $c_2$  is not contained in any singular component of  $(a_1, b_1, c_1)$ . It follows that there is a  $B_2$ - $C_2$  edge.

line 24: It can be seen that  $c_1$  is not contained in any singular component of  $(a_2, b_2, c_2)$ . It follows that there is a  $\mathcal{B}_1$ - $\mathcal{C}_1$  edge.

line 26: If  $q_1$  is non-leftmost, then  $D(b_2) \subseteq A'_1$ . If  $c_1$  is an ancestor of  $b_2$ , then  $(A'_2 \cup D(q_2)) \subseteq A''_1$ . (Note that  $\text{HIGH1}(q_2) > a_2 \geq a_1$  since the triples in this step are non-degenerate.)

line 29: Clearly,  $(a_1, b_1, c_1)$  can be moved from  $\text{list}(u)$  to  $\text{STACK}$ .

line 30: This happens only when  $c_1 \in A'_2$ , in which case, there is a  $B_2$ - $A'_2$  edge (as  $c_1$  is adjacent to a vertex in  $\mathcal{B}_1$ .)

line 33: Clearly,  $(a_1, b_1, c_1)$  can be moved from  $\text{list}(u)$  to  $\text{STACK}$ .

line 34: In this case,  $c_1 \in A''_2$ , and there is a  $B_2$ - $A''_2$  edge (as  $c_1$  is adjacent to a vertex in

**Table 6.10:** Detecting  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edges in singular triples: pseudo-code for `load_triples( $u$ )`

```
1  while list( $u$ ) is non-empty do begin
2      let  $(a_1, b_1, c_1)$  be at the end of list( $u$ )
3      if the current block of STACK is empty then
4          remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current block
          of STACK
5      else begin
6          let  $(a_2, b_2, c_2)$  be at the beginning of the current block of STACK
7          if  $b_1 = b_2$  then
8              if  $q_1 \neq q_2$  then
9                  if  $\text{HIGH1}(q_1) > a_2$  then
10                     mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
11                 else remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the
                    current block of STACK
12             else if  $c_2$  is a proper ancestor of  $c_1$  then
13                 if  $(\mathcal{A}_2 \cup \mathcal{C}_2) \subseteq (\mathcal{A}_1 \cup \mathcal{C}_1)$  then
14                     remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the
                    current block of STACK
15                 else mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
16             else if  $c_1 = c_2$  then
17                 remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the cur-
                    rent block of STACK
18             else mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{C}$  edge and remove it from STACK
19         else if  $a_1 > a_2$  then
20             if  $c_1 \geq b_2$  OR  $c_1 \notin D(q_2)$  then
21                 mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
22             else if  $c_1 > c_2$  then
23                 mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{C}$  edge and remove it from STACK
24             else mark  $(a_1, b_1, c_1)$  as having a  $\mathcal{B}$ - $\mathcal{C}$  edge and remove it from list( $u$ )
25         else if  $c_1 \geq b_2$  then
26             remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current
                    block of STACK
27         else if  $c_1 \notin D(q_2)$  then
28             if  $(\mathcal{A}_2 \cup \mathcal{C}_2) \subseteq (\mathcal{A}_1 \cup \mathcal{C}_1)$  then
29                 remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the cur-
                    rent block of STACK
30             else mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
31         else if  $c_2$  is a proper ancestor of  $c_1$  then
32             if  $(\mathcal{A}_2 \cup \mathcal{C}_2) \subseteq (\mathcal{A}_1 \cup \mathcal{C}_1)$  then
33                 remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the cur-
                    rent block of STACK
34             else mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
35         else if  $c_1 = c_2$  then
36             remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current
                    block of STACK
37         else mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{C}$  edge and remove it from STACK
38     end
39 end
```

$\mathcal{B}_1$ .)

line 36: In this case,  $(A'_2 \cup C_2) \subseteq C_1$ ,  $A''_2 \subseteq A''_1$  and  $C'_2 \subseteq (A''_1 \cup C'_1)$

line 37: It can be seen that  $c_2$  is not in any singular component of  $(a_1, b_1, c_1)$ , hence there is a  $B_2$ - $C_2$  edge.  $\square$

**Theorem 6.5.6** *The detection step correctly detects, in  $O(n + m)$  time, those of its input triples that have a  $B$ - $(\mathcal{A} \cup \mathcal{C})$  edge.*

*Proof:* The proof of this is nearly identical to the proof of Theorem 6.5.4 and is hence omitted.

#### 6.5.4 Detecting $B$ - $\mathcal{A}$ Edges

This step is for non-degenerate triples with no  $\mathcal{A}$ - $\mathcal{C}$  edges. The forbidden set for a triple is  $\mathcal{A}$  and the detection is carried out on the path  $p \rightarrow^* b$ . The subroutine `sort_lists` sorts `list(u)`  $\forall u \in V$  such that  $(a_1, u, c_1)$  precedes  $(a_2, u, c_2)$  in the list iff  $a_1 < a_2$ , or  $a_1 = a_2$  and  $c_1 \geq c_2$ .

The pseudo-code for `forward_visit(e)` and `backward_visit(e)` is similar to that in tables 6.7 and 6.8 respectively. The difference is that instead of discarding a triple, we only mark it as having a  $B$ - $\mathcal{A}$  edge and remove it from `STACK`. Also, the forbidden set in this case is  $\mathcal{A}$  instead of  $\mathcal{A} \cup \mathcal{C}$  (see line 7 in Table 6.8.) The pseudo-code for `load_triples(u)` is given in Table 6.11.

**Lemma 6.5.7** *During the detection step, the following condition holds immediately before the **while** loops on line 2 of Table 6.7 and line 7 of Table 6.8. If  $u$  is the current vertex in the search, then the triples  $(a, b, c)$  in the current block of `STACK` are such that  $a$  and  $b$  are on the canonical path containing  $u$ , with  $a$  being a proper ancestor of  $u$  and  $b$  a (leftmost) descendant of  $u$ . Moreover, the triples  $(a_1, b_1, c_1), (a_2, b_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, a_2, a_1, u, b_1, b_2, \dots$  appear on this path in the order listed (possibly with repetition), and  $\dots \mathcal{A}_2 \supseteq \mathcal{A}_1$ .*

*Proof:* We shall use an inductive argument similar to the one used in the proof of Lemma 6.3.2. It is clear that `forward_visit(e)` and the recursive call preserve the property asserted in the

lemma. After backing up a non-leftmost edge  $u \rightarrow v$ , the property still holds by induction because STACK is restored to its state just before the recursive call  $\text{dfs\_step}(v)$ . After backing up a leftmost edge, the property still holds because of the **while** loop on line 1 in Table 6.8.

Suppose now that during the search, we are at the end of  $\text{dfs\_step}(u)$  and want to load  $\text{list}(u)$  on STACK. The following is an outline of a proof that  $\text{load\_triples}(u)$  preserves the required property of STACK. Note that for non-degenerate triples with no  $\mathcal{A}$ - $\mathcal{C}$  edges, an argument similar to the one in Lemma 6.2.5 shows that the path  $q \rightarrow^* c$  is leftmost.

line 11: Clearly, there is a  $B_2$ - $A'_2$  edge.

line 12: Note that this happens only when  $a_1 = a_2$  and  $\text{LOW1}(q_1) = a_1$ . Since  $c_1$  is not adjacent to any vertex in  $\mathcal{B}_1$ , there is a  $\mathcal{B}_1$ - $(\mathcal{A}_1 \cup \mathcal{C}_1)$  edge. In fact, since  $\text{LOW1}(q_1) = a_1$ , there cannot be any  $\mathcal{B}_1$ - $\mathcal{C}_1$  edge, hence there must be a  $B_1$ - $\mathcal{A}_1$  edge.

line 13: Clearly,  $A_2 \subseteq A_1$  and  $(A'_2 \cup D(q_2)) \subseteq A'_1$ .

line 14: Suppose  $c_2 > c_1$ . This means that  $a_2 > a_1$  (because of the order of  $\text{list}(u)$ ). It follows that  $c_1$  is not in any singular component of  $(a_2, b_2, c_2)$ , which implies an  $\mathcal{A}_1$ - $\mathcal{C}_1$  edge, a contradiction. Hence  $c_1 \geq c_2$ . It can now be seen that  $A'_2 \subseteq A'_1$  and  $A''_2 \subseteq A''_1$ .

lines 15–22: If  $b_1 \neq b_2$ , note that  $b_2$  is a proper leftmost descendant of  $b_1$  and that  $a_2$  is a proper ancestor of  $b_1$ .

line 17: If  $q_1$  is non-leftmost, clearly there is a  $B_2$ - $A_2$  edge. If  $c_1$  is a proper ancestor of  $b_2$ , it can be seen that  $b_2$  is not contained in a singular component of  $(a_1, b_1, c_1)$ , hence there is again a  $B_2$ - $A_2$  edge. If  $c_1 = b_2$ ,  $D(q_2)$  is not a singular component of  $(a_1, b_1, c_1)$ . It follows that there is a  $B_2$ - $A'_2$  edge.

line 19: First note that  $c_1 > c_2$  is not possible. (In that case,  $c_1$  would be a proper ancestor of  $c_2$ . Since  $c_2$  is not contained in a singular component of  $(a_1, b_1, c_1)$ , this would imply an  $\mathcal{A}_2$ - $\mathcal{C}_2$  edge, a contradiction.) Hence  $c_2 \geq c_1$ . If  $c_2$  is an ancestor of  $c_1$ , it follows that a singular component of  $(a_1, b_1, c_1)$  implies a  $B_2$ - $A''_2$  edge. If not (i.e. if  $q_2$  is non-leftmost) then a singular component of  $(a_1, b_1, c_1)$  implies a  $B_2$ - $A'_2$  edge.

line 21: If  $q_1$  is non-leftmost, then  $D(b_2) \subseteq A'_1$ . If  $c_1$  is an ancestor of  $b_2$ , then  $(A'_2 \cup D(q_2)) \subseteq A''_1$ . (Note that  $\text{HIGH1}(q_2) > a_2 \geq a_1$  since the triples in this step are non-degenerate.)

**Table 6.11:** Detecting  $B$ - $\mathcal{A}$  edges: pseudo-code for  $\text{load\_triples}(u)$

```

1  while  $\text{list}(u)$  is non-empty do begin
2      let  $(a_1, b_1, c_1)$  be at the end of  $\text{list}(u)$ 
3      if the current block of STACK is empty then
4          remove  $(a_1, b_1, c_1)$  from  $\text{list}(u)$  and add it at the beginning of the current block
          of STACK
5      else begin
6          let  $(a_2, b_2, c_2)$  be at the beginning of the current block of STACK
7          if  $b_1 = b_2$  then
8              if  $q_1 \neq q_2$  then
9                  if  $\text{HIGH1}(q_1) > a_2$  then
10                     if  $\text{LOW1}(q_1) < a_2$  then
11                         mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from
                          STACK
12                     else mark  $(a_1, b_1, c_1)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from
                           $\text{list}(u)$ 
13                     else remove  $(a_1, b_1, c_1)$  from  $\text{list}(u)$  and add it at the beginning of the
                          current block of STACK
14                     else remove  $(a_1, b_1, c_1)$  from  $\text{list}(u)$  and add it at the beginning of the
                          current block of STACK
15                 else if  $a_1 > a_2$  then
16                     if  $c_1 \geq b_2$  then
17                         mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
18                     else comment  $c_2 \geq c_1$ 
19                         mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
20                 else if  $c_1 \geq b_2$  then
21                     remove  $(a_1, b_1, c_1)$  from  $\text{list}(u)$  and add it at the beginning of the current
                          block of STACK
22                 else mark  $(a_2, b_2, c_2)$  as having a  $B$ - $\mathcal{A}$  edge and remove it from STACK
23             end
24 end

```

line 22: If  $c_1$  is an ancestor of  $c_2$ , then  $b_2$  cannot be adjacent to any vertex in  $\mathcal{A}_2$  (otherwise there would be an  $\mathcal{A}_1$ - $\mathcal{C}_1$  edge.) But then this means there is an  $\mathcal{A}_2$ - $(\mathcal{B}_2 \cup \mathcal{C}_2)$  edge, and since there cannot be a  $\mathcal{A}_2$ - $\mathcal{C}_2$  edge, there is in fact a  $B_2$ - $\mathcal{A}_2$  edge. On the other hand, if  $c_2 > c_1$ , either  $c_2$  is a proper ancestor of  $c_1$ , or  $q_2$  is non-leftmost. In the first case,  $c_1 \in A_2''$  (since  $(a_2, b_2, c_2)$  is non-degenerate) and there is a  $B_2$ - $A_2''$  edge. Similarly, in the latter case,  $c_1 \in A_2'$  and there is a  $B_2$ - $A_2'$  edge.  $\square$

**Theorem 6.5.8** *The detection step correctly detects, in  $O(n + m)$  time, those of its input triples that have a  $B$ - $\mathcal{A}$  edge.*

*Proof:* The proof of this is nearly identical to the proof of Theorem 6.5.4 and is hence omitted. □

### 6.5.5 Detecting $C$ - $\mathcal{B}$ Edges

This step is for non-degenerate triples with no  $\mathcal{A}$ - $\mathcal{C}$  edges, and also for degenerate triples, which, as observed before, do not have any  $\mathcal{A}$ - $\mathcal{C}$  edges either. The forbidden set for a triple is  $\mathcal{A} \cup \mathcal{B}$  and the detection is carried out on the path  $q \rightarrow^* c$ . ( $\mathcal{A}$  is included in the forbidden set only for convenience; it does not affect the detection step as it is only used for triples with no  $\mathcal{A}$ - $\mathcal{C}$  edges.) The subroutine `sort_lists` sorts `list(u)`  $\forall u \in V$  such that  $(a_1, b_1, u)$  precedes  $(a_2, b_2, u)$  in the list iff  $b_1 < b_2$ , or  $b_1 = b_2$  and  $a_1 \leq a_2$ .

The pseudo-code for replacing `forward_visit(e)` and `backward_visit(e)` is given in tables 6.12 and 6.13 respectively. Table 6.14 gives the pseudo-code for `load_triples(u)`.

**Table 6.12:** Detecting  $C$ - $\mathcal{B}$  edges: pseudo-code for `forward_visit(e)`

```

1  if  $e$  is non-leftmost then begin
2      while the triple  $(a, b, c)$  in the beginning of the current block of STACK has
           $\text{HIGH1}(e) > b$  and  $\text{HIGH1}(e) \neq a$  do
3          mark the triple as having a  $C$ - $\mathcal{B}$  edge and remove it from STACK
4      if  $e$  is a tree edge then add an end-marker on top of STACK
5  end

```

**Table 6.13:** Detecting  $C$ - $\mathcal{B}$  edges: pseudo-code for `backward_visit(e)`

```

1  while the triple  $(a, b, c)$  at the beginning of the current block of STACK has  $b = u$  do
2      remove the triple from STACK
3  if  $e = (u \rightarrow v)$  is non-leftmost then begin   comment backing up over a
                                                         non-leftmost edge
4      mark all triples in the current block of STACK as having a  $C$ - $\mathcal{B}$  edge
5      remove the block (and the end-marker) from STACK
6  end
7  else   comment backing up over a leftmost edge
8      for edge  $w \hookrightarrow u$  in  $\text{Adj}^R(u)$  do
9          while the triple  $(a, b, c)$  at the beginning of the current block has  $w \in$ 
               $(\mathcal{A} \cup \mathcal{B})$  do
10             mark the triple as having a  $C$ - $\mathcal{B}$  edge

```

**Table 6.14:** Detecting  $C$ - $\mathcal{B}$  edges: pseudo-code for `load_triples(u)`

```

1  while list( $u$ ) is non-empty do begin
2      let  $(a_1, b_1, c_1)$  be at the end of list( $u$ )
3      if the current block of STACK is empty then
4          remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current block
          of STACK
5      else begin
6          let  $(a_2, b_2, c_2)$  be at the beginning of the current block of STACK
7          if  $c_1 = c_2$  then
8              if  $b_1 = b_2$  then
9                  mark  $(a_2, b_2, c_2)$  as having a  $C$ - $\mathcal{B}$  edge and remove it from STACK
10             else if  $a_1 \leq b_2$  OR  $a_1 = a_2$  then
11                 remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the cur-
                 rent block of STACK
12             else mark  $(a_2, b_2, c_2)$  as having a  $C$ - $\mathcal{B}$  edge and remove it from STACK
13         else if  $b_1 > b_2$  then
14             mark  $(a_2, b_2, c_2)$  as having a  $C$ - $\mathcal{B}$  edge
15         else if  $b_1 < b_2$  then
16             if  $a_1 \leq b_2$  OR  $a_1 = a_2$  then
17                 remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the cur-
                 rent block of STACK
18             else mark  $(a_2, b_2, c_2)$  as having a  $C$ - $\mathcal{B}$  edge and remove it from STACK
19         else if  $a_1 = a_2$  then
20             remove  $(a_1, b_1, c_1)$  from list( $u$ ) and add it at the beginning of the current
             block of STACK
21         else mark  $(a_2, b_2, c_2)$  as having a  $C$ - $\mathcal{B}$  edge
22     end
23 end

```

**Lemma 6.5.9** *During the detection step, the following condition holds immediately before the **while** loops on line 2 of Table 6.12 and line 9 of Table 6.13. If  $u$  is the current vertex in the search, then the triples  $(a, b, c)$  in the current block of STACK are such that  $a, b$  and  $c$  are all on the canonical path containing  $u$ , with  $a$  and  $b$  being proper ancestors of  $u$  and  $c$  a (leftmost) descendant of  $u$ . Moreover, the triples  $(a_1, b_1, c_1), (a_2, b_2, c_2), \dots$  in the block (in order) will be such that the vertices  $\dots, b_2, b_1, u, c_1, c_2, \dots$  appear on this path in the order listed (possibly with repetition), and  $\dots (\mathcal{A}_2 \cup \mathcal{B}_2) \supseteq (\mathcal{A}_1 \cup \mathcal{B}_1)$ .*

*Proof:* We shall use an inductive argument similar to the one used in the proof of Lemma 6.3.2. It is clear that `forward_visit(e)` and the recursive call preserve the property asserted in the lemma. After backing up a non-leftmost edge  $u \rightarrow v$ , the property still holds by induction

because STACK is restored to its state just before the recursive call  $\text{dfs\_step}(v)$ . After backing up a leftmost edge, the property still holds because of the **while** loop on line 1 in Table 6.13.

Suppose now that during the search, we are at the end of  $\text{dfs\_step}(u)$  and want to load  $\text{list}(u)$  on STACK. The following is an outline of a proof that  $\text{load\_triples}(u)$  preserves the required property of STACK.

line 9: Since  $a_2 > a_1$ , and there is no  $\mathcal{A}_1\text{-}\mathcal{C}_1$  edge,  $a_2$  cannot be adjacent to any vertex in  $\mathcal{C}_2$ . It follows that there must be a  $\mathcal{C}_2\text{-}\mathcal{B}_2$  edge.

line 11: If  $a_1 \leq b_2$ , then  $(A_2 \cup B_2 \cup A'_2 \cup B'_2) \subseteq A_1$  and  $(A''_2 \cup B''_2) \subseteq A''_1$ . If  $a_1 = a_2$ , then  $A_2 = A_1$ ,  $(B_2 \cup A'_2 \cup B'_2) \subseteq B_1$ ,  $A''_2 = A''_1$  and  $B''_2 \subseteq B''_1$ .

line 12: Since there is no  $\mathcal{A}_2\text{-}\mathcal{C}_2$  edge, by considering a singular component of  $(a_1, b_1, c_1)$  it follows that  $b_2 < a_1 < a_2$ , and hence there is a  $\mathcal{C}_2\text{-}\mathcal{B}_2''$  edge.

lines 13–21: If  $c_1 \neq c_2$ , note that  $c_2$  is a proper leftmost descendant of  $c_1$  and that  $b_2$  is a proper ancestor of  $c_1$ .

line 14: Note that  $c_2$  is not contained in any singular component of  $(a_1, b_1, c_1)$ . Now a singular component of  $(a_1, b_1, c_1)$  must have an attachment other than  $b_2$  and  $a_2$ , and since there is no  $\mathcal{A}_2\text{-}\mathcal{C}_2$  edge, it follows that there must be a  $\mathcal{C}_2\text{-}\mathcal{B}_2$  edge.

line 17: If  $a_1 \leq b_2$ , then  $(A_2 \cup B_2 \cup A'_2 \cup B'_2) \subseteq A_1$  and  $(A''_2 \cup B''_2) \subseteq D(c_2) \subseteq A''_1$ . (Note that this case cannot happen for degenerate triples.) If  $a_1 = a_2$ , then  $A_2 = A_1$  and  $(B_2 \cup A'_2 \cup B'_2) \subseteq B_1$ . Furthermore,  $(A''_2 \cup B''_2) \subseteq A''_1$  in the case of non-degenerate triples, whereas  $(A''_2 \cup B''_2) \subseteq B''_1$  in the case of degenerate triples.

line 18: The argument is similar to that for line 12.

line 20: In this case,  $A_2, B_2, A'_2$  and  $B'_2$  are respectively identical to  $A_1, B_1, A'_1$  and  $B'_1$ . In the case of non-degenerate triples,  $(A''_2 \cup B''_2) \subseteq A''_1$ . In the case of degenerate triples,  $(A''_2 \cup B''_2) \subseteq B''_1$ .

line 21: In this case,  $a_1 < a_2$ , since there is no  $\mathcal{A}_2\text{-}\mathcal{C}_2$  edge. Now a singular component of  $(a_1, b_1, c_1)$  gives a  $\mathcal{C}_2\text{-}\mathcal{B}_2$  edge. □

**Theorem 6.5.10** *The detection step correctly marks, in  $O(n + m)$  time, those of its input triples that have a  $\mathcal{C}\text{-}\mathcal{B}$  edge.*

*Proof:* Suppose a triple  $(a, b, c)$  is marked by the search as having a  $C\text{-}\mathcal{B}$  edge. We need to verify that it indeed has one. If the triple is marked by the subroutine `load_triples`, then it can be easily seen from the proof of the previous lemma that it has a  $C\text{-}\mathcal{B}$  edge. If the triple is marked by line 3 in Table 6.12, the edge  $e = u \rightarrow v$  is such that the subtree  $D(v) \subseteq C$  has an attachment distinct from  $a$  and higher than  $b$ , that is, in  $B$  (since the input triples for this step have no  $\mathcal{A}\text{-}\mathcal{C}$  edges.) Hence the triple has a  $C\text{-}\mathcal{B}$  edge. If the triple is marked by line 4 in Table 6.13, the edge  $e = u \rightarrow v$  is non-leftmost, hence by Lemma 6.2.5 it follows that the triple has  $\text{HIGH1}(q) \leq a$  (i.e. the input triples are degenerate.) It follows from the ordering of  $\text{Adj}(u)$  that the first edge  $e_0$  in  $\text{Adj}(u)$  (or any edge before  $e$ ) must have  $\text{HIGH1}(e_0) = a$  and hence be a tree edge. Now if  $\text{HIGH2}(e_0) \leq b$ , it would mean  $\text{HIGH2}(e) \leq b$ , and hence  $c$  cannot be adjacent to any vertex in  $\mathcal{B}$ . Since the triple is degenerate, this would mean that it has a  $C\text{-}\mathcal{B}$  edge. On the other hand, if  $\text{HIGH2}(e_0) > b$ , and  $e_0 = u \rightarrow v_0$ , say, then the subtree  $D(v_0) \subseteq C$  has an attachment in  $B$ , so the triple has a  $C\text{-}\mathcal{B}$  edge. Finally, if the triple is marked by line 10 in Table 6.13, the vertex  $u \in C$  has a back edge coming in from a vertex in  $B''$ , hence the triple has an  $C\text{-}B''$  edge.

Conversely, suppose a triple  $(a, b, c)$  has a  $C\text{-}\mathcal{B}$  edge. Then it has either a  $C\text{-}B''$  edge, or a  $C\text{-}\mathcal{B}$  edge (or both.) From the previous lemma, it then follows that in the first case, such an edge would be detected by the **while** loop on line 9 in Table 6.13. In the second case, the edge would be detected by the **while** loop in Table 6.12 or line 4 in Table 6.13. Hence a triple with a  $C\text{-}\mathcal{B}$  edge will be marked accordingly.

Finally, for the time bound, note that the subroutine `load_triples` takes  $O(1)$  time per triple, and hence loading `list(u)` for all vertices  $u$  takes  $O(n+m)$  time overall (as there are  $O(n+m)$  triples.) □

### 6.5.6 Detecting $C\text{-}B''$ Edges

This step is carried out for non-singular triples that do not have any  $B\text{-}(\mathcal{A} \cup \mathcal{C})$  edges, and for non-degenerate triples that have an  $\mathcal{A}\text{-}\mathcal{C}$  edge but no  $\mathcal{B}\text{-}(\mathcal{A} \cup \mathcal{C})$  edges. For all vertices  $u$ , `list(u)` is the list of triples, as above, with  $c = u$ , sorted such that  $(a_1, b_1, u)$  precedes  $(a_2, b_2, u)$  in the list iff  $a_1 > a_2$ , or  $a_1 = a_2$  and  $b_1 \leq b_2$ .

For each vertex  $u$ , we first divide up  $\text{list}(u)$  into ordered *clusters* and the tree edges in  $\text{Adj}(u)$  into *sublists* corresponding to the clusters as follows. We scan  $\text{list}(u)$  in order and form clusters corresponding to (contiguous) subsequences of triples with non-decreasing values of  $b$ . It can be seen that the triples  $(a_1, b_1, u), (a_2, b_2, u), \dots, (a_k, b_k, u)$  in a cluster (in order) are such that the vertices  $a_1, \dots, a_k$  and  $b_1, \dots, b_k$  appear in the order  $a_1, a_2, \dots, a_k, b_k, \dots, b_2, b_1$  (possibly with repetition) on the path from the root to  $u$ . We then scan  $\text{Adj}(u)$  in its usual order and make sublists of the *tree* edges in it, each sublist corresponding to a cluster in such a way that an edge  $e$  appears in the sublist corresponding to a cluster as above if  $b_1 < \text{HIGH1}(e) \leq a_1$ . Note that an edge can appear in at most one such sublist because the clusters do not “overlap” i.e. if we have a cluster  $(a_1, b_1, u), (a_2, b_2, u), \dots$  as above, and a cluster  $(a'_1, b'_1, u), (a'_2, b'_2, u), \dots$ , then  $a_1, a'_1, b_1$  and  $b'_1$  cannot appear in that order (without repetition) on the path from the root to  $u$ . For non-singular triples, this is not possible because then Lemma 6.2.8 would imply that  $(a'_1, b'_1, u)$  has a  $B$ - $\mathcal{A}$  edge. Similarly, for non-degenerate triples, this is not possible because a singular component of  $(a_1, b_1, u)$  would then give rise to a  $B$ - $\mathcal{A}$  edge for  $(a'_1, b'_1, u)$ .

We then sort all the sublists of  $\text{Adj}(u)$  for all vertices  $u$  in increasing order of  $LOW1$ . Note that for this sorting to be done in linear time overall, we use bucket sort to sort all the sublists (for all vertices  $u$ ) *simultaneously*. The pseudo-code for detecting  $C$ - $B''$  edges is given in Table 6.15.

**Table 6.15:** Detecting  $C$ - $B''$  edges

```

1  for each vertex  $u$  do
2      for each cluster of  $\text{list}(u)$  do
3          let  $e$  be the first edge in the sublist of  $\text{Adj}(u)$  corresponding to the cluster and
            $(a, b, u)$  be the first triple in the cluster
4          while the end of the cluster or the end of the sublist is reached do
5              if  $LOW1(e) \geq b$  then
6                  set  $(a, b, u)$  to the next triple in the cluster
7              else if  $b < \text{HIGH1}(e) < a$  OR
                    $(\text{HIGH1}(e) = a \text{ AND } b < \text{HIGH2}(e) < a)$  then
8                  mark the triple as having a  $C$ - $B''$  edge and set  $(a, b, u)$  to the next
                   triple in the cluster
9              else set  $e$  to the next edge in the sublist

```

**Theorem 6.5.11** *The procedure in Table 6.15 detects, in  $O(m + n)$  time, the triples that have a  $C$ - $B''$  edge.*

*Proof:* For a particular cluster and its corresponding sublist, the **while** loop takes as many steps as the number of triples in the cluster plus the number of edges in the sublist (plus a constant.) Since the clusters and sublists are disjoint, this implies that for a vertex  $u$ , the time taken by the inner **for** loop is linear in the size of  $\text{list}(u)$  plus the degree of  $u$ . Since there are  $O(m + n)$  triples, the linear time bound follows.

Consider a triple  $(a, b, u)$  in  $\text{list}(u)$ . If it is marked by the procedure as having a  $C$ - $B''$  edge, then it follows from the condition in line 7 of Table 6.15 that the triple indeed has a  $C$ - $B''$  edge. On the other hand, suppose the triple has such an edge. We need to show that it is marked so by the procedure. Let  $e = (u \rightarrow v)$  be the first edge in the sublist corresponding to the cluster containing the triple such that  $D(v) \in B''$  and  $D(v)$  has an attachment in  $C$ . Then either  $b < \text{HIGH1}(e) < a$ , or  $\text{HIGH1}(e) = a$  and  $b < \text{HIGH2}(e) < a$ . Since the sublist is ordered in increasing order of  $LOW1$ , the triple is still being considered (or waiting to be considered) when the edge  $e$  is being examined by the procedure. Furthermore, the triple is then marked before the procedure moves to the next edge in the sublist. This is because, for a triple occurring before  $(a, b, u)$  in the cluster, one of the conditions on lines 5 or 7 must hold, and hence the last option in the **if** statement (line 9) cannot happen. In other words, before the procedure moves to the next edge (after  $e$ ) in the sublist, the triple  $(a, b, u)$  will be considered, and hence will be marked as having a  $C$ - $B''$  edge. This proves the correctness of the procedure, and hence proves the theorem.  $\square$

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