ELASTIC AND INELASTIC LATERAL-TORSIONAL BUCKLING OF BOX BEAMS SUBJECTED TO PURE BENDING

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ELASTIC AND INELASTIC LATERAL-TORSIONAL BUCKLING OF BOX BEAMS SUBJECTED TO PURE BENDING

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The writer dedicates this dissertation to his wife, Fang Chu, whose love and patience have made this work possible. Finally, the writer extends his sincere thanks to Mrs. Claudine Taylor for her wonderful typing.

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LIST OF SYMBOLS

A	=	the area of a cross-section, also an arbitrary constant in the solution of a differential equation.
B	=	constant.
b_f	=	average width of a box section.
b_w	=	average depth of a box section.
$\vec{b}$	=	Unit Binormal Vector.
C_1, C_2, C_3	=	constants related to the torsion of a box section.
C_s	=	shear constant of a rectangular section.
E	=	Young's modulus of elasticity.
E_{st}	=	strain hardening modulus.
F	=	axial force.
f_y	=	yield stress of material.
G	=	shear modulus of elasticity.
H	=	shear force.
I	=	moment of inertia of a cross-section.
$\vec{i}$	=	unit vector along the X-axis.
$\vec{j}$	=	unit vector along the Y-axis.
K	=	curvature of a space curve, also curvature of a beam.
$\vec{k}$	=	unit vector along the Z-axis.
k	=	constant.
L	=	the span length of a beam.

LIST OF SYMBOLS (Continued)

L_{cr}	=	the laterally unsupported span length of a beam.
l	=	length of a curve, also the direction cosine between two axes.
M	=	bending moment, also torsional moment.
M_p	=	Plastic moment of a beam cross-section.
m	=	constant, also the direction cosine between two axes.
N	=	the normal direction at the boundary of a closed section.
n	=	constant, also the direction cosine between two axes.
o	=	the origin of a set of coordinate axes, also the shear center of a closed section.
p	=	constant.
$\vec{p}$	=	Unit Principal Normal Vector.
$\vec{R}$	=	projection of a vector on certain axis.
$\vec{r}$	=	directional vector of a point in space.
r_y	=	the radius of gyration of the full cross-section of a beam about the Y-axis.
S	=	the tangential direction at the boundary of a closed section.
s	=	arc length.
t	=	thickness.
t_f	=	thickness of the flanges of a box section.
t_w	=	thickness of the webs of a box section.
u	=	deformation in the X-direction.
$\vec{u}$	=	Unit Tangent Vector.
V	=	shear force.

LIST OF SYMBOLS (Continued)

v	=	deformation in the Y-direction.
w	=	deformation in the Z-direction, also warping displacement.
X-axis	=	the primary principal axis of a beam cross-section.
Y-axis	=	the secondary principal axis of a beam cross-section.
Z-axis	=	the axis of twisting. It is tangential to the centroidal axis of a beam.
α	=	constant
β	=	constant.
ϕ	=	rotational displacement of a box section, also the angular rotation of a beam axis.
ϵ	=	strain, also elastic strain.
Δ, δ	=	small increments of certain quantity.
θ	=	$d\phi/dz$, the twist per unit length of a beam.
φ, ψ, Ψ	=	functions introduced to help define the deformations of a box section under torsion.
ξ-axis	=	the primary principal axis of a deflected beam cross-section.
η-axis	=	the secondary principal axis of a deflected beam cross-section.
φ -axis	=	the axis of twisting. It is tangential to the centroidal axis of a deflected beam cross-section.
μ	=	constant
λ	=	constant.
τ	=	shear stress, also the twist of a space curve.
σ	=	tensile or compressive stresses
γ	=	shear deformation

SUMMARY

This dissertation examines the problem of lateral-torsional buckling under pure bending moments of simply supported, prismatic box beams with doubly symmetrical thin-walled rectangular cross-sections. This problem is considered both in the elastic range and in the inelastic range. In the elastic range, two solutions are obtained. The first elastic solution neglects the effect of deflections in the plane of the primary bending moment on the curvature of the beam. The second elastic solution, however, takes this effect into consideration. An approximate method is used to obtain the elastic solutions. In this method, one establishes the differential equation for non-uniform torsion of a box section based on the ordinary simple bending theory. The elastic solutions have been shown to be quite straight-forward. The formulae obtained can be easily applied in practical engineering works. In the inelastic range, an approximate lower bound solution is obtained based on the argument that, with proper modifications of the coefficients, the formulae for the elastic solution can be applicable in the inelastic range. These modified coefficients are computed on the basis that, at buckling, no previously yielded fibers will unload elastically, and that additional deformation is resisted by the unyielded elastic core of the cross-section. Due to the complications created by a partially yielded condition, no simple, explicit equation is given in the inelastic range. The method of solution is a numerical one. Two major conclusions were obtained in this paper:

i) It is quite unlikely for ordinary box beams with dimensions and laterally unsupported span length conceivable in engineering practices to fail in the mode of lateral-torsional buckling.

ii) Except for box beams with very small I_y/I_x ratios, the stability of box beams against lateral-torsional buckling does not diminish rapidly in the inelastic range. In many instances a box beam can become more stable against lateral-torsional buckling after the flanges of the box beam have been fully yielded.

CHAPTER I

INTRODUCTION

1. Some Practical Considerations of Box Beams

The problem considered in this dissertation is that of lateral-torsional buckling under pure bending moments of simply supported, prismatic box beams with doubly symmetrical thin-walled rectangular cross-sections. The buckling problem will be studied both in the elastic range and the inelastic range. It is believed that finding solutions to this problem may be very helpful in the practical application of box beams. One is familiar with the fact that a deep and narrow rectangular beam or wide-flange beam will buckle lateral torsionally under the action of pure bending moments acting about the primary principal axis of bending. Lateral bracing is often required to provide additional lateral rigidity in order that the beam can carry the primary bending moments. The cost of fabricating and installing lateral bracing is often a sizable proportion of the cost of a structure with respect to the weight of material and the cost of manufacturing and labor. In practice, laterally unsupported main structural members are sometimes desirable either when space required to put up the lateral support system is not available or when one tries to reduce the cost of manufacturing and erecting the lateral bracing systems. When the latter is the case, a box beam, due to its higher resistance against lateral-torsional buckling, may be more efficient and more economical than a wide-flange beam having approximately the same primary bending capacity.

An interesting comparison of a box beam and a wide-flange beam is made here using the result of tests on box beams carried out by Moran [1]* at Georgia Institute of Technology and the result of tests on wide-flange beams carried out by Lee and Galambos [7]. In the tests carried out by Moran, four laterally unsupported box beams of the same cross-section as shown in Figure 1a were tested for different span lengths under pure bending moments. It was found that in all four tests, in which the maximum unsupported span was 15 feet 7½ inches, the plastic moment of the section was attained and the beams sustained the plastic moment through large plastic rotations.** Local buckling was then observed in each of the four tests. In the experiments conducted by Lee and Galambos, five wide-flange beams (W10 x 25) were tested with different lengths between lateral supports. The purposes of their experiments were to determine the maximum permissible unsupported span lengths for wide-flange beams subjected to constant plastic moment and to study the post-buckling strength of wide-flange beams. It was found that the lateral stability of wide-flange beams is closely related to the ratio of unsupported span length (L_{cr}) divided by the radius of gyration about the secondary bending axis (r_y). Test result showed that for beams with $L_{cr}/r_y \leq 45$, failure was caused by local buckling of the compression flange and that these beams showed considerable post buckling strength, and in each case a plastic hinge of large rotation capacity was formed. It was also found that wide-flange beams with L_{cr}/r_y

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* Number in brackets refers to corresponding reference listed in the Bibliography section on Page 165.

** Of all the tests performed by Moran, the minimum ratio of rotation at failure to rotation at first yield (ϕ_f/ϕ_y) is equal to 4.0.

ratio larger than 45 failed by lateral-torsional buckling.

The wide-flange beam section shown in Figure 1b has approximately the same weight and plastic moment capacity as the box beam section shown in Figure 1a. If the wide-flange beam were required to support the plastic moment through large plastic rotations under the pure bending condition, the minimum spacing of lateral support can be computed by the test result obtained by Lee and Galambos. For the wide-flange beam shown in Figure 1b, $r_y = 0.845$ inch. Thus $L_{cr} = (45)(0.845) = 3$ feet 2 inches. The comparison between the box beam and the wide-flange beam in Figure 1 is summarized in Table 1. It can be seen that for a span length of 15 feet $7\frac{1}{2}$ inches the box beam can sustain the plastic moment with no lateral support necessary. On the other hand, lateral support is required at 3 feet 2 inches intervals if the wide-flange beam was to support the plastic moment. With this practical application in mind, it is thus worthwhile to carry out an investigation on the lateral-torsional buckling characteristics of box beams.

2. A Brief Historical Sketch of the Problem of Lateral-Torsional Buckling of Beams and the Analysis of Box Beams

Two possibilities arise when a beam buckles lateral-torsionally under the action of pure bending moments. In the first case, the stresses at every point of the beam are still within the elastic limit of the material. In the second case, part of the cross-section has been stressed above the elastic limit into the plastic range.

The problems of elastic and inelastic lateral-torsional buckling of solid rectangular beams and wide-flange beams subjected to pure bending have been

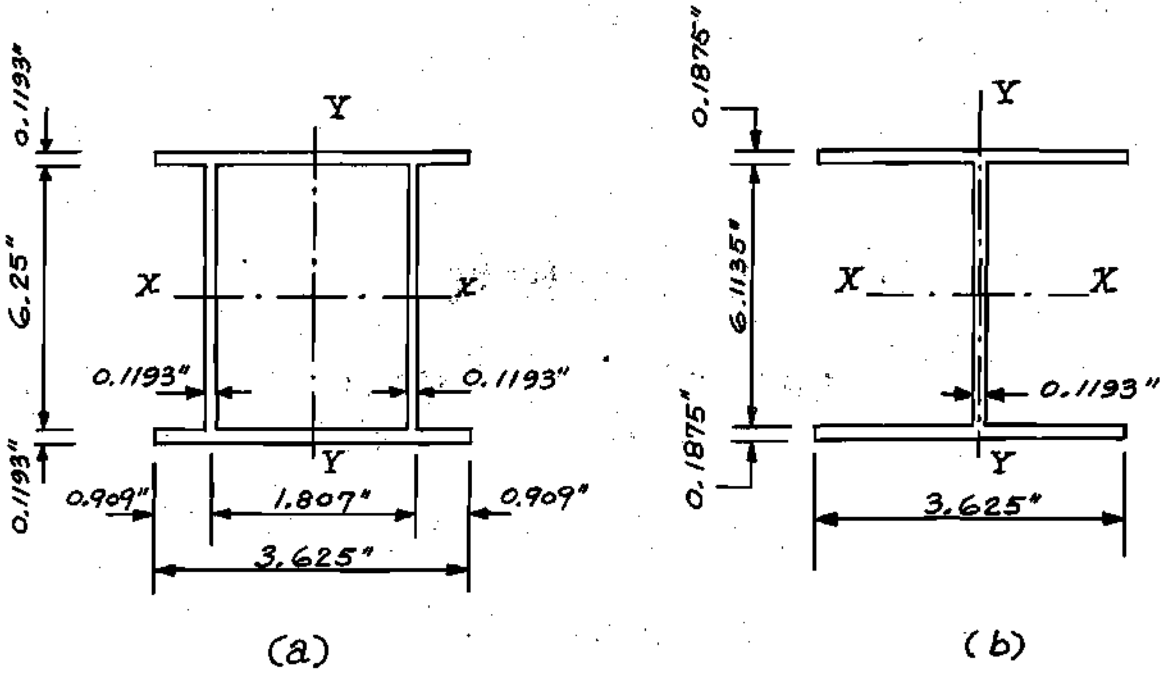


Figure 1. Configuration of the Box Beam Used in the Tests Carried Out by Moran [1] and a Wide-Flange Beam With Comparable Plastic Moment Capacity.

Table 1. Comparison Between the Box Beam and the Wide-Flange Beam as Shown in Figure 1

Type	Area (in^2)	M_p (k-in) f_y=39.7ksi	Maximum Unsupported Span Length Permitting Large Plastic Rotations	
			From Tests by Moran	From Tests by Lee and Galambos
Wide-flange Beam	2.088	214.0	-	$3^{ft}2^{in}$
Box Beam	2.390	207.0	$15^{ft}7\frac{1}{2}^{in}$	-

thoroughly studied by many investigators. Timoshenko [4] presented detailed solutions of elastic critical pure bending moments for both rectangular beams and wide-flange beams. Neal [5] derived an additional equation for elastic lateral-torsional buckling of rectangular beams taking into account the effect on the buckling characteristics of the curvature of the beam in the plane of the primary bending moment. He also derived a solution for the critical pure bending moment of a rectangular beam when the cross-section has partially yielded. His inelastic solution is valid for mild steel which has a pronounced upper yield stress. Wittrick [6] generalized Neal's inelastic solution for rectangular beams to include materials having a general type of stress-strain diagram.

Much theoretical and experimental work on the problems of inelastic lateral-torsional buckling of wide-flange beams has been done at Lehigh University during the last two decades or so. Their results and recommendations are summarized and adapted in the Commentary on Plastic Design in Steel [3], and in the AISC Specifications [2]. In the tests of wide-flange beams carried out by Lee and Galambos [7], it was found that under the pure bending condition, wide-flange beams with $L_{cr}/r_y \leq 45$ failed by local buckling of the compression flange. It was also found that for wide-flange beams with $L_{cr}/r_y > 45$, failure was caused by lateral-torsional buckling. Galambos [8] discussed the effect of residual stresses and obtained an inelastic solution for wide-flange beams under pure bending, assuming a known pattern of residual stresses throughout the cross-section. Lay and Galambos [9, 10] pointed out that the inelastic lateral-torsional buckling and inelastic local buckling of steel beams are functions of the strain-hardening modulus (Est) of the material. The lower the strain hardening stiffness, the lower is the resistance of the member against either lateral-torsional buckling or local buckling in the inelastic range. Strain-hardening is a property of increase in stress following the yielding stage. The importance of this property in the theories of plastic steel design was discussed by Hrennikoff [11]. He argued that the yielding property, although necessary, is not sufficient for the applicability of plastic steel design theories. The material must possess strain-hardening characteristics. Lay and Smith [12] used numerical examples to show that if a material does not possess strain-hardening characteristics, it is not possible for the members to form a mechanism. Hrennikoff [13] later ran an actual test to prove this argument.

There were also many other valuable papers. The above discussion was meant only to summarize studies made on the most important areas concerning the problem of lateral-torsional buckling of rectangular and wide-flange beams. Although the solutions for rectangular and wide-flange beams can not be directly applied to box beams, they nevertheless provide good references and valuable information. Not much theoretical work has been done on the problem of lateral torsional buckling of box beams. Most of the earlier studies of box beams were associated with the investigations of the stress distribution and deformations in an airplane wing. Reissner [14] has shown that the state of stress in a box beam under bending is necessarily different from that given by elementary beam theory. Consider the cantilevered box beam as shown in Figure 2a. The box beam is loaded in some manner on the upper flange. The downward loads are transmitted by the upper flange to the web plates at the sides (Figure 2b). The forces acting on the web plates are in equilibrium with vertical shear forces in these members. If the thickness of the web is small, the vertical shear stress can be considered as uniformly distributed across the thickness of the web. Due to the law of distribution of shear stresses, horizontal shear stresses are also acting on cross-sections of the web parallel to the Z-axis. Thus at the edges of the flange plates horizontal shear stresses are distributed as shown in Figure 2c. In the elementary theory, the state of stress in the flanges is given by a simple law such that the normal stresses will be constant across the flange plate, and shear stresses will vary linearly (Figure 2d). In reality, the state of stress in the flanges is not given by a simple law. The edge shear forces cause a strain in the plate which decreases

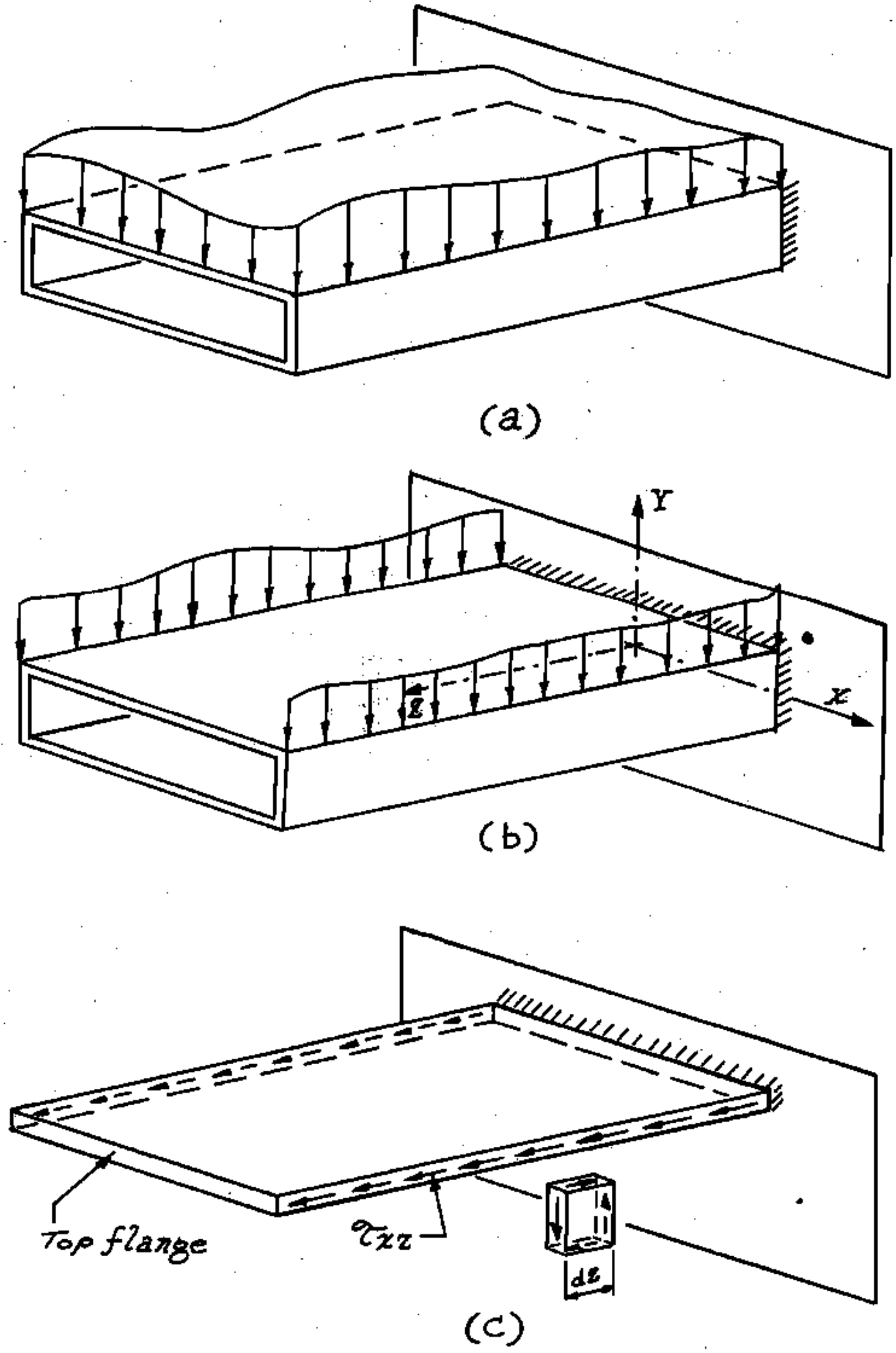
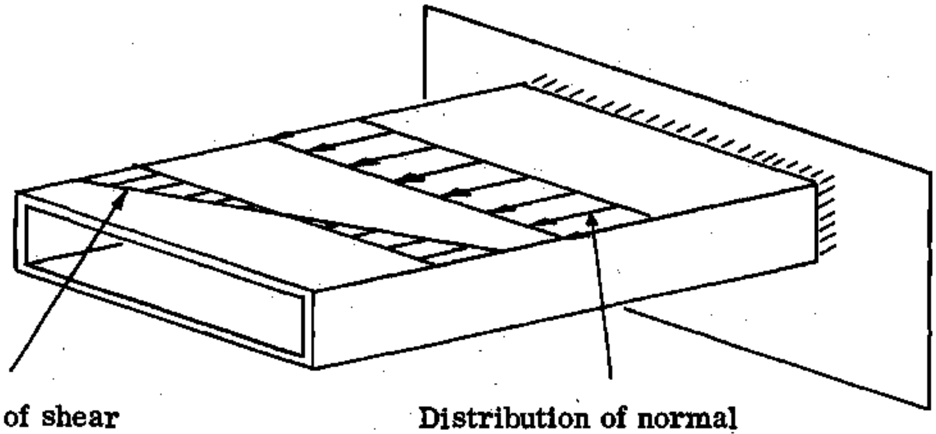


Figure 2. Stress Distribution in A Box Beam

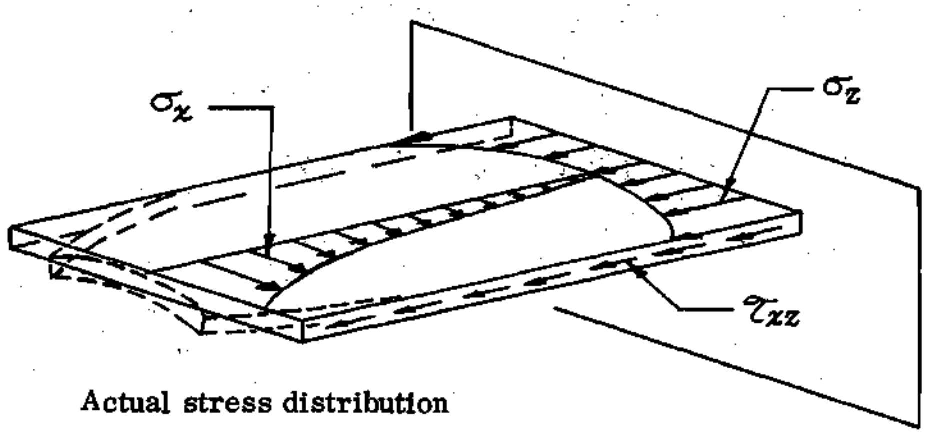
(due to Reissner [14]).



Distribution of shear stresses σ_{xz} according to elementary theory

Distribution of normal stresses σ_z according to elementary theory

(d)



Actual stress distribution

(e)

Figure 2 (Continued). Stress Distribution in A Box Beam

(due to Reissner [14]).

from the edges toward the middle of the plate. Consequently the normal stresses decrease toward the middle of the plate (Figure 2e). And for reasons of equilibrium, there exist transverse normal stresses σ_x of the same order of magnitude as the longitudinal normal stresses σ_z.

Many investigators have made significant contributions in evaluating these stresses. Reissner [14, 15, 16, 17] solved the problem using both the theorem of least work which is the basic minimum principle for the stresses, and the theorem of minimum potential energy which is the basic minimum principle for the strain. However, his methods led to approximate results. Hildebrand [18] obtained exact solutions with a rigorous mathematical procedure. Many other contributions can be found in the literature concerning the stress distributions in airplane wing structures.

In studying the lateral-torsional characteristics of a box beam, an understanding of the behavior of a box section under torsion is of great importance. The assumption that plane cross-sections remain plane, which is a basis for the bending theory, is not valid for the case of torsion of a box section because warping of the cross section may occur under the action of the torsional moment. There are two cases of torsion of a bar, namely the uniform torsion and the non-uniform torsion conditions. It is well-known that the problem of uniform torsion of a prismatic bar leads to a simple solution. This solution, commonly known as the Saint Venant solution [19] for an open section and as the Bredt-Batho solution [20] for a hollow cylindrical section, gives a system of strains and stresses uniform along the longitudinal axis of the bar. Axial stresses vanish throughout the bar and shearing stresses due to uniform torsion produce warping of the cross-section which is constant along the longitudinal axis of the bar. In the case of non-uniform torsion, that is, when warping is prevented or limited in some manner or when the torque is not uniform along the length of the bar, the strains and stresses will vary from section to section, and there will be axial stresses in addition to shearing stresses. Warping will also vary along the longitudinal axis of the bar.

The nature of the warping displacements of a box beam under torsion can be visualized with Shanley's simplified model [21] as shown in Figure 3. The box in Figure 3a was constructed so that the two end sections were originally plane. Figure 3b shows the unfolded view of the plates from which the box is made. Note that the ends of the plates form straight lines before loading is applied. When a torsional moment M_z is applied at the end, each of the four plates constituting the box will have shear deformation. Assuming that the stress distribution is such that the amount of shear deformation is the same for each plate, then the edges of the unfolded box will remain as straight lines (Figure 3c). When these distorted plates are put back into a closed box the ends of the box beam will remain plane, provided the cross-section is not constrained. But if the cross-section is forced to remain rectangular, each plate must then rotate in its own plane. This will cause warping of the cross-section.

Another case of warping occurs when the shear strain distribution is not uniform over the four plates. Figure 3d is an example in which the wider plates AD and BC have a larger angle of rotation. If the right end of the box is forced to remain plane, the continuity of the plates will be violated. There will be gaps and overlaps as shown in Figure 3d. In order to regain the original continuity, each

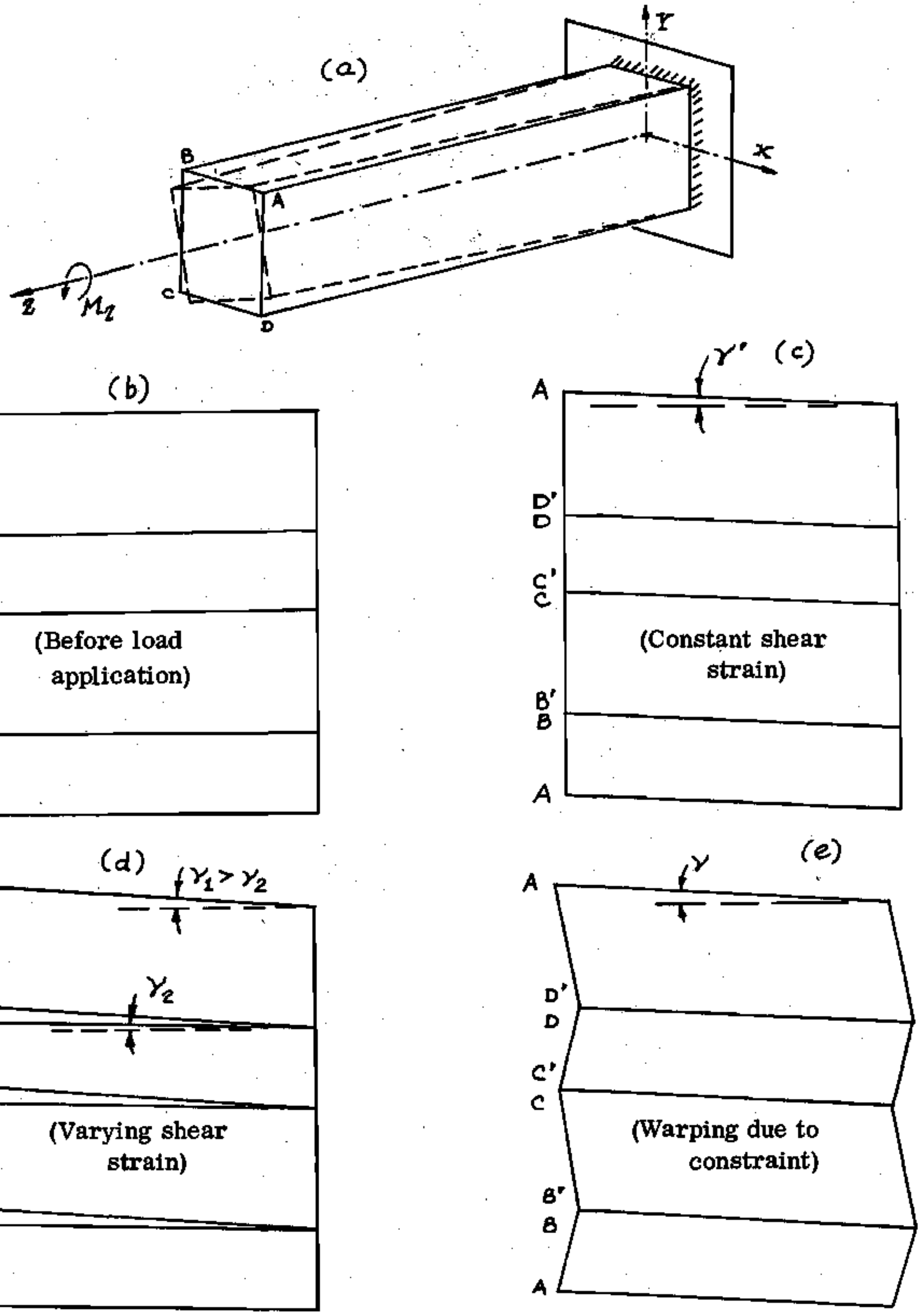
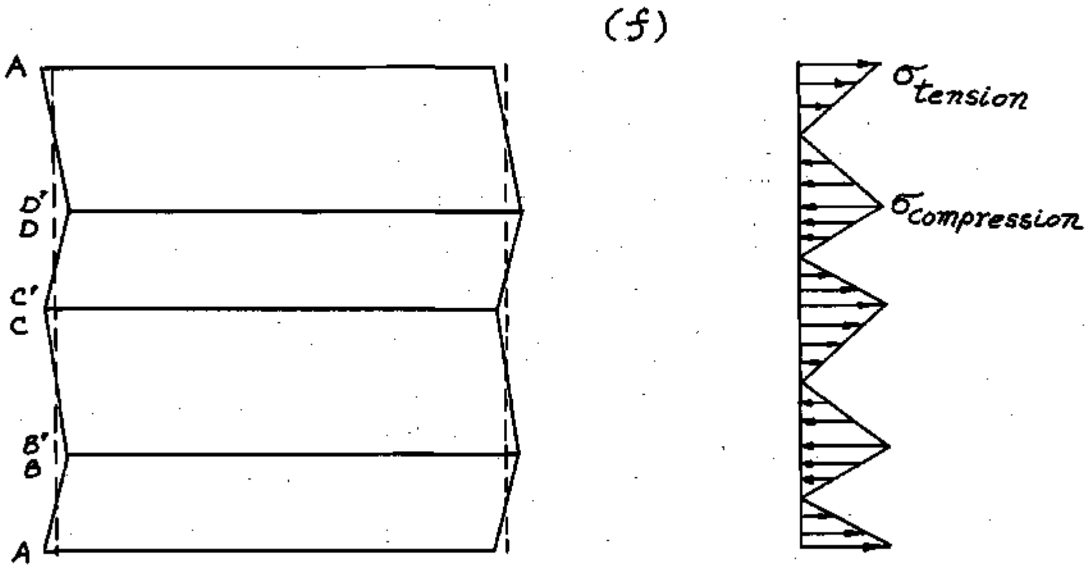


Figure 3. Shanley's Simplified Model [21] of A Box Beam Under Torsion.



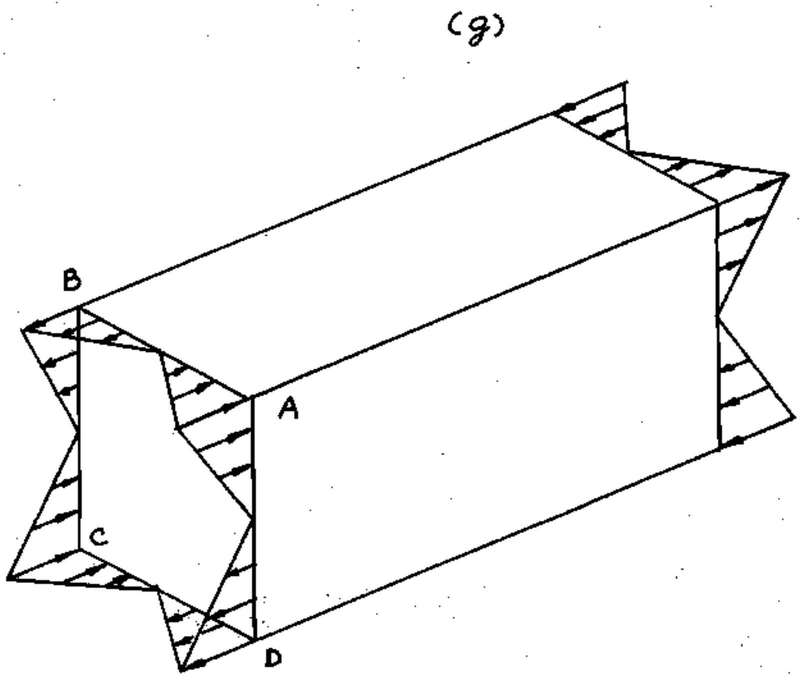


Figure 3 (Continued). Shanley's Simplified Model [21] of

A Box Beam Under Torsion.

plate must rotate. Plate AB must increase its angle of rotation and plate BC must decrease its angle of rotation, and so on. The final compatible configuration is shown in Figure 3e. The edges of the unfolded box are no longer straight lines. If these plates are put back into a box, the ends will not be plane surfaces. If the right end is forced to remain plane, axial stresses as shown in Figure 3f are produced. Some regions are under tensile stresses and some regions are under compressive stresses. Figure 3g shows the stress distribution when both ends are forced to remain plane. If the box beam is acted on by the torsional moment alone, the resultant of the axial stresses must be zero. However, as shown in Figure 3g, these stresses produce bending of each plate in its own plane. Thus part of the torsional moment is resisted by bending of the plates that form the box.

Williams [20] made thorough studies of the problem of non-uniform torsion of a rectangular box beam. His approach in obtaining a rigorous mathematical solution was first to solve for the warping displacements of a thin-walled rectangular box beam under uniform torsion using the "semi-inverse" method due to St. Venant, and then to solve for the stress distribution under non-uniform torsion by introducing proper stress functions to satisfy the differential equations of the problem and the corresponding boundary conditions. He later [22] obtained approximate solutions based on the simple bending theory. It was found that this approximate solution was more easily applicable in design and with negligible sacrifice of accuracy when compared to the rigorous method. This approximate method was further discussed by Payne [23] and McGuire [24]. It can be seen later that this approximate method becomes very useful in the development of this dissertation.

In a somewhat different manner, another approximate solution can be obtained if the variation of twist along the longitudinal axis of the bar is small. Then as a zero order approximation, the warping of the section can be calculated from the Saint-Venant theory corresponding to the local value of twist at any particular section. The axial rate of change of the warping so calculated gives an induced axial stress. This induced axial stress will then serve as the starting point of a first order approximation of the shear stresses due to variable warping--a correction to the Saint-Venant solution. In mathematical language, this method is an iteration process. Timoshenko, Goodier [19] and others used this method to solve the stability problems of an open section under torsion and bending, and under torsion and axial compression. Von Karman and Christensen [25] used this method to analyze open, closed, and multicell sections under non-uniform torsion. Smith [26] used a similar approach to analyze torsion of box beams with relatively thick walls.

The instability problems of a box beam were also studied to some extent. Lundquist [27] calculated critical stresses for local instability of symmetrical rectangular tubes. He utilized Timoshenko's solution for the critical stress of a rectangular plate under edge compression. Budiansky, Stein and Gilbert [28] gave theoretical solutions for the buckling of a long square tube in torsion and compression. It was found that an appreciable amount of torsion may be present without reducing the compression required for buckling. Falconer [29] discussed the effects of initial deviations from perfect flatness of the plating of a square box on its buckling behavior under torsion. He found that the development of buckling is greatly influenced by the magnitude of the initial deformations and that where there are initial deformations, however small, deviation from linearity of the relationship between rotation and torque will occur at a load well below the critical load.

Tests on the lateral-torsional buckling of box beams were carried out by Moran [1]. In his tests the box beams were subject to pure bending moments and were loaded well into the plastic range. No lateral buckling of the box beams was observed (cross-section of box beams is shown in Figure 1a). Failure was caused by local buckling of the compression flange. Leddick [30] also conducted tests on box beams at Georgia Institute of Technology. In his tests the box section consisted of two M8 x 6.5 beams welded flange to flange. Two beams and one portal frame were constructed with this box section. In the beam tests the maximum unsupported span length was equal to 13 feet 4 inches, a constant plastic moment was attained over a length of 6 feet 8 inches in the middle of the span. The beams sustained the plastic moment through considerable deflection and rotation* before failure occurred by a simultaneous action of local and lateral buckling near mid-span. The portal frame tested by Leddick had a span of 16 feet and a height of 8 feet 8 inches. The frame was left unbraced over its entire length--horizontal member and columns. Failure was initiated by local buckling at the top portions of columns. This was followed by lateral buckling in the horizontal member. The completely unbraced frame carried 96.2% of its predicted ultimate load through

*The minimum ratio of ϕ_f/ϕ_y for Leddick's beam tests was equal to 3.08.

considerable deflection, attaining a ratio of rotation at failure to rotation at first yield (ϕ_f/ϕ_y) of 3.45. Tests conducted by Moran [1] and Leddick [30] proved that a box beam has very high rigidity against lateral-torsional buckling.

Theoretical work on the problem of lateral torsional buckling of box beams is not yet available. It is thus the purpose of this dissertation to find analytical solutions for this problem.

2. Method of Solution

Two sets of solutions are intended herein, one to be applicable in the elastic range and the other to be applicable in the inelastic range. The elastic solutions are derived on the basis that the stability limit of the beam is defined as the point at which a slightly deflected equilibrium position of the beam becomes possible. In Chapter II the writer first derives the differential equation for non-uniform torsion of box sections under variable torque. This derivation follows the work of McGuire [24, 32] who derived the differential equation for non-uniform torsion of cantilevered box beams under constant torque. Using the differential equation for variable torque, the mathematical model for the first elastic solution is set up by neglecting the effect of deflections in the plane of the primary bending moment. The problem of finding the critical buckling moment is then reduced to solving a fourth order differential equation. In the later part of Chapter II this fourth order differential equation is solved and the critical buckling moment is obtained.

In Chapter III a second elastic solution is derived by considering the effect of deflections in the plane of the primary bending moment. In setting up the mathematical model, the writer utilizes Kirchhoff's general equilibrium equations for the bending and twisting of beams. The problem of finding the critical buckling moment is reduced to solving a fifth order differential equation. In the later part of Chapter III this fifth order differential equation is solved and the critical buckling moment is obtained.

The solutions in the inelastic range are derived using the argument that with proper modification of the stiffness coefficients the buckling equations for the elastic range can be applicable in the inelastic range. This argument has been recognized by practically all investigators in the analysis of lateral-torsional buckling of solid rectangular beams and wide-flange beams. Since the lateral stability problem of box beams is analogous to that of wide-flange beams, it is thus logical to utilize this argument in this dissertation. In Chapter IV formulae for the modified stiffness coefficients are derived and equations for determining the critical buckling moment in the inelastic range are obtained.

CHAPTER II

THE FIRST SET OF SOLUTIONS IN THE ELASTIC RANGE

1. Assumptions

A solution can be obtained for the critical pure bending moment in the elastic range, neglecting the effect of deflections in the plane of the primary bending moment (Y-Y plane in Figure 1). This solution is justifiable if the deflections in the plane of the primary bending moment are relatively small and the primary flexural rigidity (EI_x) is large compared to the secondary flexural rigidity (EI_y), the torsional rigidity (EC_1), and the warping constant (C_2). One shall consider the matter of an elastic body to be homogeneous and continuously distributed over its volume so that the smallest element cut from the body has the same specific physical properties as the body. We shall also consider the body to be isotropic, so that the elastic properties are the same in all directions. The box beam is assumed to be prismatic along its longitudinal axis, to be perfectly straight in its initial condition, and to have a doubly symmetrical cross-section. A linear relationship is assumed to exist between increments of stress and strain. The deviation from linearity of the distribution of bending stress in a box beam as explained in Article 2 of Chapter I (see Figure 2) is to be neglected. This is justifiable because this deviation from linearity is appreciable only when the width of the box beam is many times greater than its depth. But this is not generally the case for box beams used in civil engineering structures. The dimension of the width of such box beam is not usually many times greater than its depth. One can imagine that the shear forces built up at the edges of the flanges of a box beam are of the same order of magnitude as those built up at the flange-web junction of a wide-flange beam. Just as the effect of these shear forces on flange bending stress distribution is neglected for wide-flange beams, this effect shall also be neglected in the bending of box beams conceived in this paper.

It shall be assumed that the wall thickness of the box beam is thin, i.e. it is relatively small compared to other dimensions of the box cross-section. It shall also be assumed that premature failures such as local buckling do not occur prior to the initiation of lateral-torsional buckling of the box beam. In addition, we consider the box section to retain its rectangular shape upon buckling away from its original plane of stable equilibrium (Figure 4). Williams [20, 22] has discussed the justification of this assumption. He showed that in the case of a box beam subject to torsion the work done in deforming the shape-retaining diaphragms is small so that it is possible to assume they are rigid in their own planes and that there is an infinite number of them along the longitudinal axis of the box beam. This assumption was adopted by practically all investigators in analyzing a box beam under torsion. Finally, the possibility of the existence of residual stresses in the cross-section is to be neglected in the analysis.

2. The Differential Equation for Non-Uniform Torsion of A Thin-Walled Box Section

In Appendix A the warping displacement due to uniform torsion of a box section as shown in Figure 5 is derived. It is assumed in the derivation that the

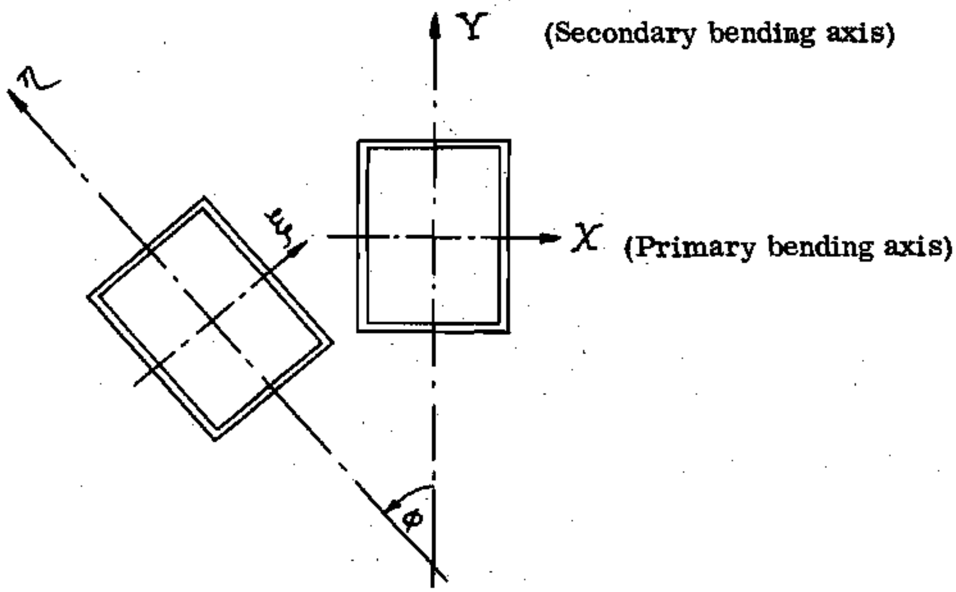
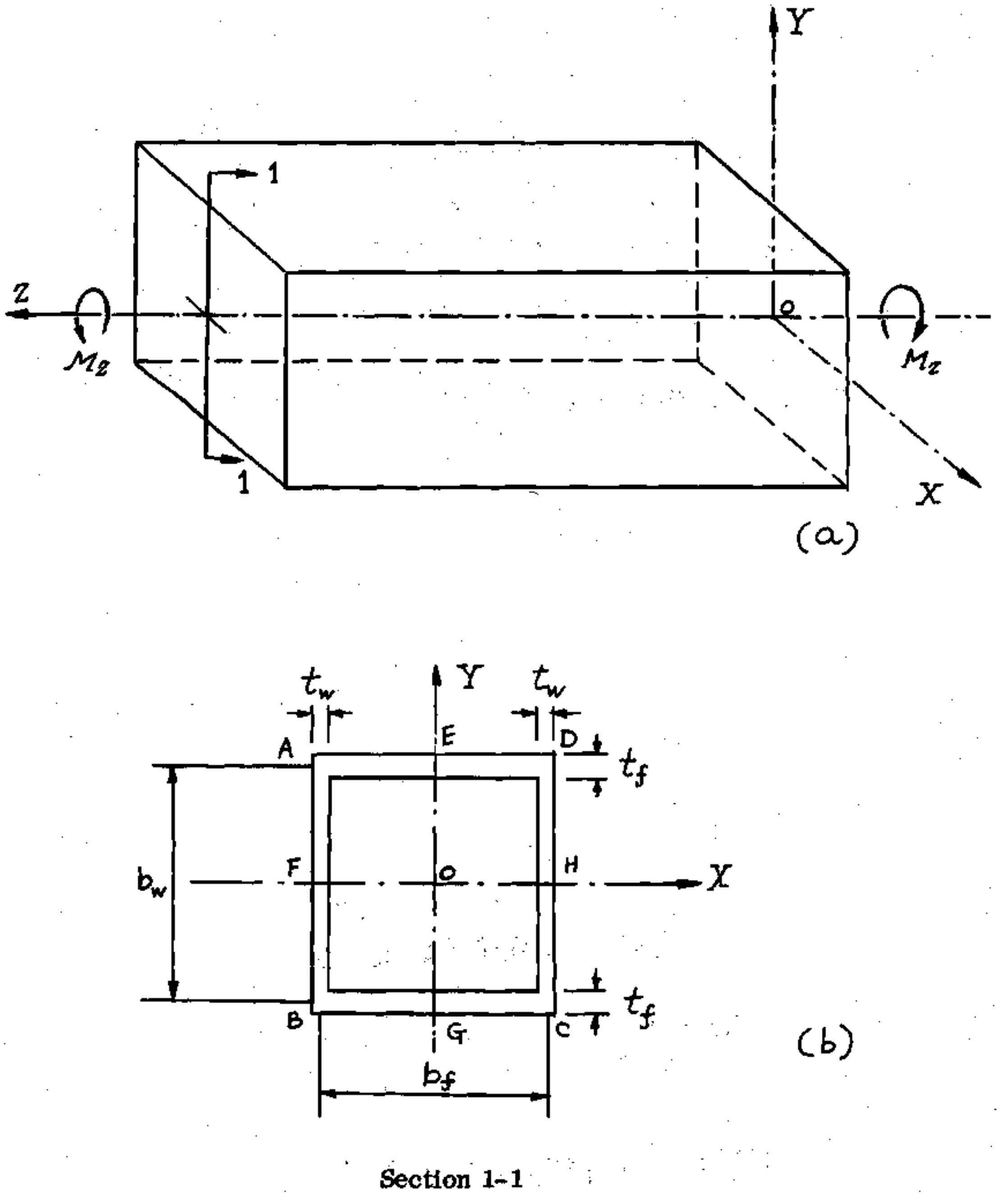


Figure 4. A Box Beam Retains its Rectangular Shape Upon Buckling Away from its Original Plane of Stable Equilibrium.

torque (M_z) is applied at the ends of the beam and that the cross-sections of the beam are free to warp. Under such conditions warping is the same for all cross-sections and takes place without any axial strain in the longitudinal fibers. The case of non-uniform torsion occurs if any cross-section is not free to warp or if the torque varies along the length of the beam. The amount of warping will then vary along the beam. The difference in warping displacements between two adjacent sections produces axial strains. The longitudinal fibers of the beam will be subjected to tensile or compressive axial stresses depending on whether the fiber is extended or compressed. Since the beam is acted on by torques alone,



Section 1-1

Figure 5. Coordinate System for the Analytical Model of A Box Beam.

the resultant of the warping stresses at any cross-section must equal to zero. In addition, the angle of twist per unit length (θ) will no longer be constant but will vary along the axis of the beam. Obtaining a solution of non-uniform torsion of a box section using a rigorous mathematical procedure can be quite involved and the results may not be easily applied to every day engineering practice. Fortunately, an approximate solution is possible based on the ordinary theory of simple bending. It has been demonstrated by Williams [22] and Payne [23] that solution based on the ordinary bending theory leads to results that are much more easily applicable to engineering practice. Williams [22] has shown that the accuracy of the approximate solution is comparable to that of the rigorous solution. In an unpublished note [32], McGuire solved the problem of cantilevered box beams under the action of a constant twisting moment using the approximate method. In this article the differential equation for non-uniform torsion of a box section based on the simple bending theory is derived, following the work of McGuire. A variable twisting moment is considered here in contrast to the constant twisting moment considered by McGuire.

Consider now a case of non-uniform torsion as shown in Figure 6, in which the box beam is fixed at one end and a torque M_z is applied at the other end. Since warping of the kind shown in Figure 30 of Appendix A cannot occur at the fixed end, the original plane cross-section at that end is forced to remain plane. But this requires that the fixed base exert forces F in each plate acting in the directions as indicated in Figure 6b (for the purpose of clarity, only the bottom and right-side plates are illustrated, similar forces are also acting on the other two plates but in

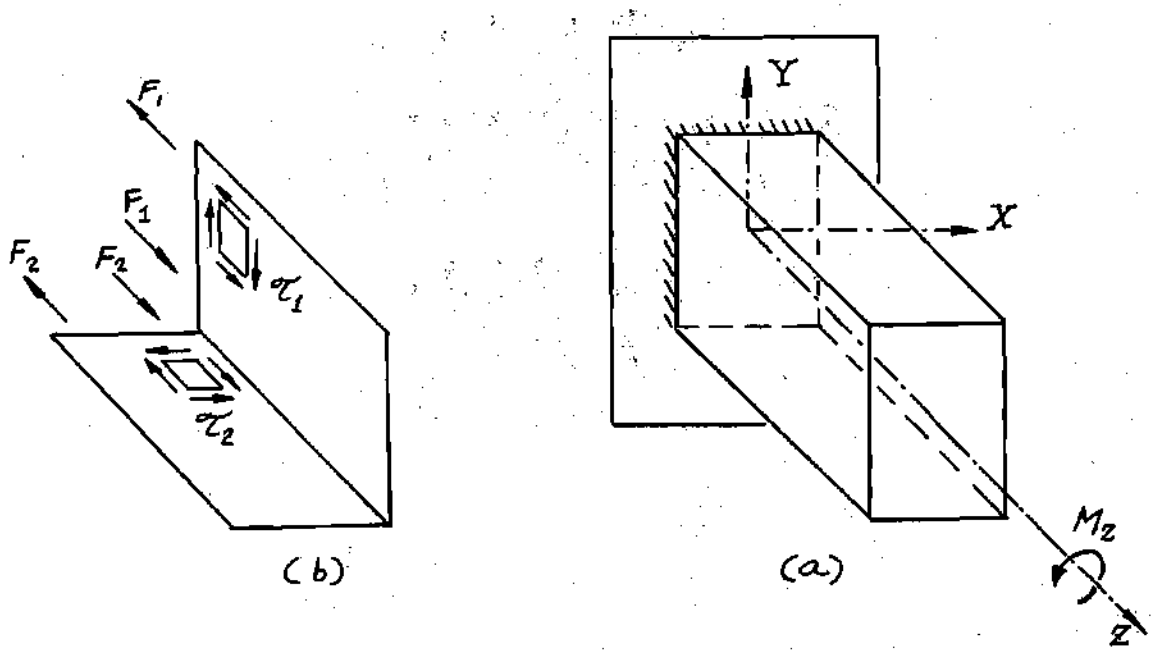


Figure 6. Internal Forces Developed in A Box Beam Under Torsion.

opposite directions). These F forces produce bending of each plate in its own plane. The magnitudes of the bending moments diminish as the distance from the fixed end increases. Shear stresses (τ_1 and τ_2) are produced by the effect of differential bending and are acting in the directions shown. These shear stresses are solely due to the effect of non-uniform torsion and are in addition to the shear stresses caused by uniform torsion of the cross section. There is, therefore, a change in the manner in which the total torque M_z is resisted. Part of M_z is balanced by shear stresses due to uniform torsion and part balanced by resistance to bending of the plates that make up the box beam. It is important to note that since there is no externally applied axial force, the resultant of the F forces must be zero.

The portion of torque (denoted by M_{z1}) that is balanced by uniform torsion can be expressed by Equation (2-1). This equation was derived by Bredt [19] and has since been used as a standard equation* for uniform torsion of a general thin-walled closed section.

$$M_{z1} = \left(\frac{4A^2G}{\oint \frac{ds}{t}} \right) \cdot \frac{d\phi}{dz} \qquad \text{(2-1)}^*$$

Here A is the mean of the areas enclosed by the outer and the inner boundaries of the box section, t is the thickness of the wall, and s is the arc length which is positive when increasing in the counterclockwise direction. Equation (2-1) can be further rearranged into a more convenient form by denoting

$$C_1 = \frac{4A^2G}{\oint \frac{ds}{t}} \qquad \text{(2-2)}$$

Hence,

$$M_{z1} = C_1 \frac{d\phi}{dz} \qquad \text{(2-3)}$$

The coefficient C_1 for a box beam as shown in Figure 5 can be evaluated as follows.

*For the derivation of Equation (2-1), one is referred to the work of Timoshenko [19].

$$C_1 = \frac{4 \cdot (b_w \cdot b_f)^2 \cdot G}{\frac{b_w}{t_w} + \frac{b_f}{t_f} + \frac{b_w}{t_w} + \frac{b_f}{t_f}}$$

$$C_1 = \frac{2 \cdot (b_w^2 \cdot b_f^2 \cdot t_w \cdot t_f) \cdot G}{(b_w t_f + b_f t_w)} \quad \text{(2-4)}$$

Next consider a general case of non-uniform torsion of a box section as shown in Figure 7. The directions of shear forces are assigned based on the discussions related to Figure 30 of Appendix A and Figure 6 (i.e. assuming $b_w t_f > b_f t_w$).

Hence

$$M_z = H \cdot b_w + V \cdot b_f \quad \text{(2-5)}$$

$$M_{z1} = H_1 \cdot b_w + V_1 \cdot b_f \quad \text{(2-6)}$$

$$M_{z2} = H_2 \cdot b_w - V_2 \cdot b_f \quad \text{(2-7)}$$

and

$$M_z = M_{z1} + M_{z2} \quad \text{(2-8)}$$

Knowing that both bending and shear stresses exist and vary along the longitudinal axis, one can imagine that the curvature of each plate of the box beam consists of

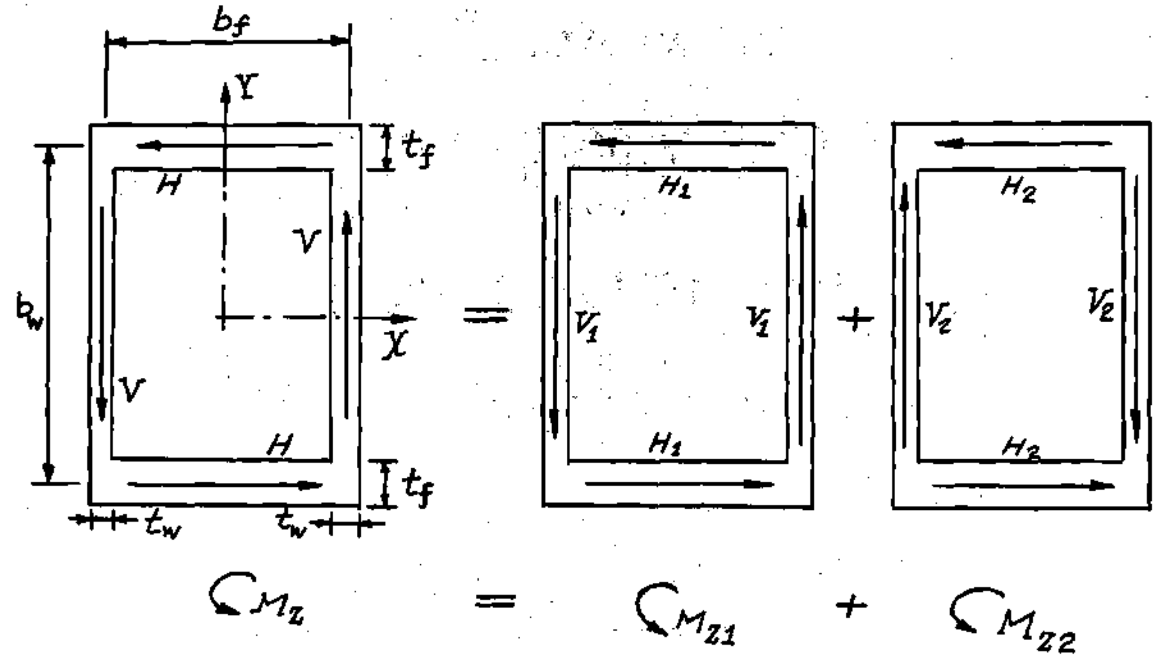


Figure 7. Shear Forces in A Box Beam Due to Non-Uniform Torsion.

two portions, one portion due to simple bending and the other due to shear. Consider first the effect of the curvature due to simple bending alone. In Figure 8b is shown an elementary strip of the bottom flange. The bending moments are shown in their positive directions. By taking moment about point n and neglecting second order terms, we have

$$\cancel{M_f} + dM_f - \cancel{M_f} - H_2 \cdot dz + d\,\cancel{H_2 \cdot dz} = 0$$

$$H_2 = \frac{dM_f}{dz} \qquad \text{(2-9)}$$

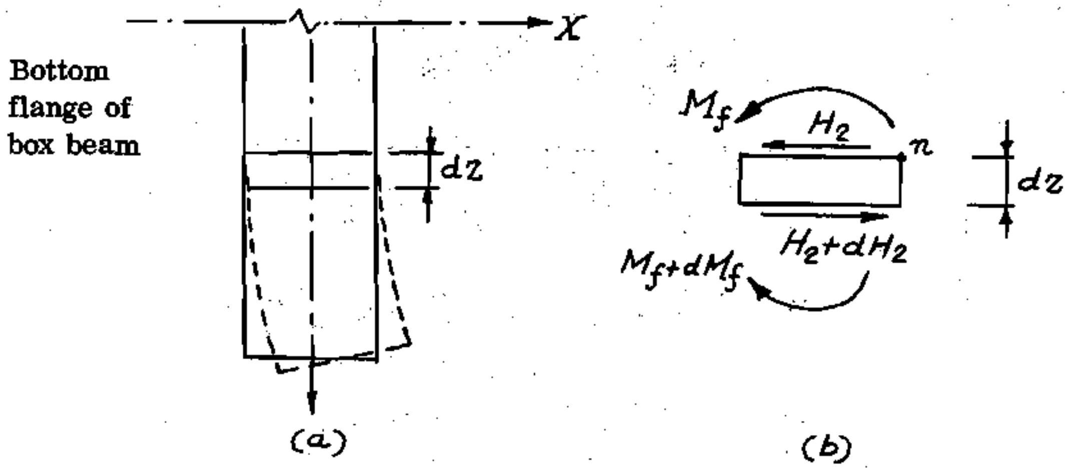


Figure 8. Free Body Diagrams of Bottom Flange of A Box Beam

Under the Action of Simple Bending Alone.

Similarly, from the elementary strip of the right side web shown in Figure 9b, we have

$$\cancel{M_W} + dM_W - \cancel{M_W} - V_2 \cdot dz + d\cancel{V_2 \cdot} dz = 0$$

$$V_2 = \frac{dM_W}{dz} \qquad \text{(2-10)}$$

It is necessary to point out that there is an important difference between the character of the displacement due to simple bending and the displacement due to shear. The displacement of a beam cross-section due to simple bending has the

Right side web of box beam

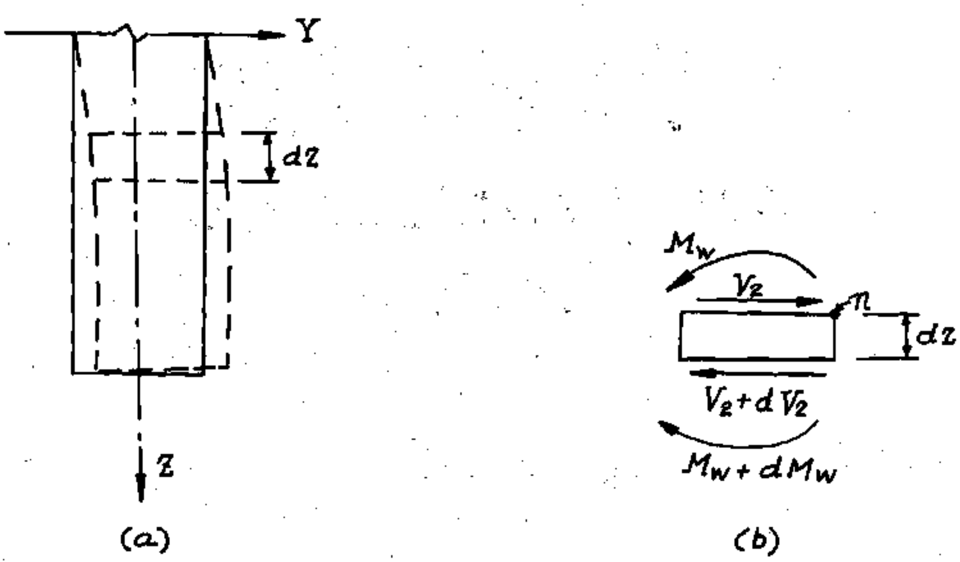


Figure 9. Free Body Diagrams of Right-Side Web of A Box Beam Under the Action of Simple Bending Alone.

effect that all cross-sections of a beam perpendicular to the neutral axis before displacement remain perpendicular after displacement. On the other hand, the displacement due to shear has the effect that cross-sections parallel before displacement remain parallel after displacement, so that if the end cross-section of a horizontal beam is fixed in a vertical position all cross-sections originally vertical remain vertical as long as the displacement is caused by shear alone. It is thus clear that displacements at the longitudinal edges of a beam in the direction of its axis can be produced only by actions of simple bending. For displacements perpendicular to the beam axis, both the actions of bending and shear must be taken into consideration.

The displacements of a box section under non-uniform torsion have to meet certain compatibility conditions. The first of these conditions calls for identical axial displacement at the common edges of the four plates. Consider for example the axial displacement at point C of Figure 5b. Knowing that this axial displacement can only be caused by actions of bending alone, we have

$$\left\{\begin{matrix}\text{Axial displacement of} \\ \text{point C in plate BC.}\end{matrix}\right\} = \left\{\begin{matrix}\text{Axial displacement of} \\ \text{of point C in plate DC.}\end{matrix}\right\},$$

$$\frac{d^2 x_b}{dz^2}\cdot(\Delta z)\cdot\left(\frac{b_f}{2}\right) = -\frac{d^2 y_b}{dz^2}\cdot(\Delta z)\cdot\left(\frac{b_w}{2}\right) \qquad \text{(2-11)}$$

where x_b and y_b are deflections due to bending only. The negative sign on the right side of the equation is due to the negative curvature in the right-side web. The relationship established by Equation (2-11) actually dictates that the curvatures of the bottom flange and the right-side web must be of opposite signs* in order to meet the compatibility requirement at the common edge (Figure 10). Furthermore,

$$\left\{\begin{aligned} \frac{d^2 x_b}{dz^2} &= -\frac{M_f}{EI_f} \\ \frac{d^2 y_b}{dz^2} &= -\frac{M_w}{EI_w} \end{aligned}\right. \qquad \text{(2-12)}$$

*Similar condition applies to the other two plates.

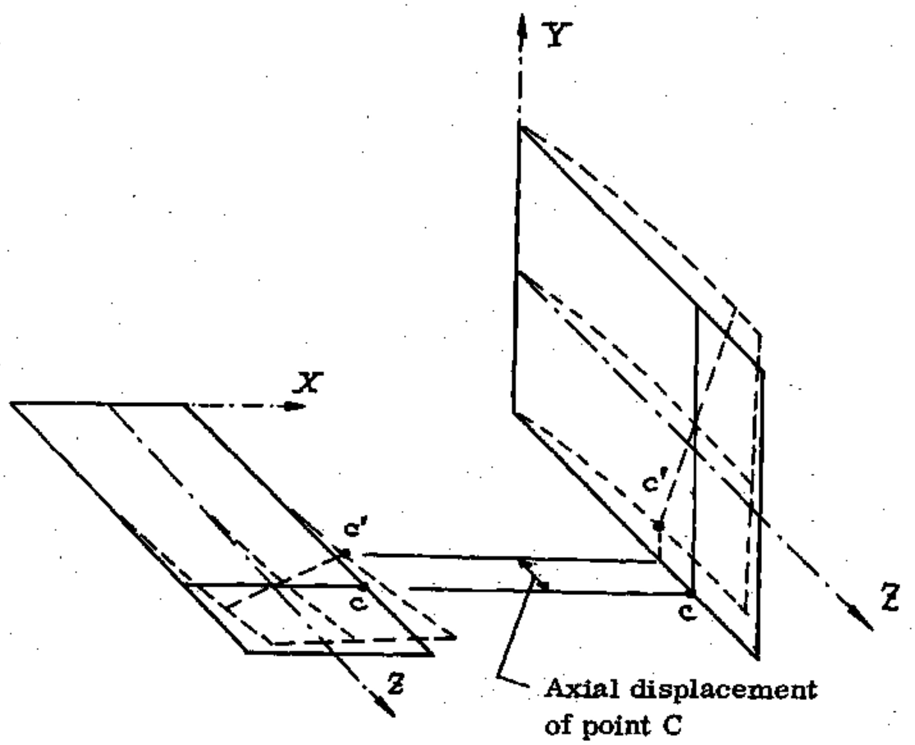


Figure 10. Compatibility Condition at the Common Edge Between Two Plates of A Box Beam.

where $I_f = \frac{t_f b_f^3}{12}$ and $I_w = \frac{t_w b_w^3}{12}$, we have

$$-\frac{M_f}{E I_f} \cdot \left(\frac{b_f}{2}\right) = \frac{M_w}{E I_w} \cdot \left(\frac{b_w}{2}\right) \qquad \text{(2-13)}$$

$$M_f = -\frac{b_w I_f}{b_f I_w} \cdot M_w \qquad \text{(2-14)}$$

Substituting Equation (2-14) into Equation (2-9), we obtain

$$H_2 = \frac{dM_f}{dz} = \frac{d}{dz}\left(-\frac{b_w I_f}{b_f I_w}\right)\cdot M_w$$

$$H_2 = -\frac{b_w I_f}{b_f I_w}\cdot\left(\frac{dM_w}{dz}\right) \qquad \text{(2-15)}$$

Substituting Equations (2-10) and (2-15) into Equation (2-7), we have

$$M_{z2} = H_2 b_w - V_2 b_f = -\frac{b_w^2 I_f}{b_f I_w}\cdot\frac{dM_w}{dz} + b_f\cdot\frac{dM_w}{dz}$$

$$M_{z2} = -\left(\frac{b_w^2\cdot I_f - b_f^2\cdot I_w}{b_f I_w}\right)\cdot\frac{dM_w}{dz} \qquad \text{(2-16)}$$

$$M_{z2} = -\left(\frac{b_w^2\cdot\frac{t_f b_f^3}{12} - b_f^2\cdot\frac{t_w b_w^3}{12}}{b_f\cdot\frac{t_w b_w^3}{12}}\right)\cdot\frac{dM_w}{dz}$$

$$M_{z2} = \frac{b_f}{t_w}(b_w t_w - b_f t_f)\cdot\frac{dM_w}{dz}$$

It is seen from the above equation that there exists a special case of non-uniform torsion in which the resisting moment M_{z2} due to warping is reduced to zero when $b_w t_w = b_f t_f$. There is warping of the cross-section. But the amount of warping is such that the resisting moments $H_2 \cdot b_w$ and $V_2 \cdot b_f$ cancel each other (see Figure 7).

With $M_{z2} = 0$, Equation (2-8) is reduced to Equation (2-3). That is, the differential equation for the case of uniform torsion now becomes the governing differential equation.

Next consider the effect of the curvature due to shear alone. Consider the bottom flange as shown in Figure 11.

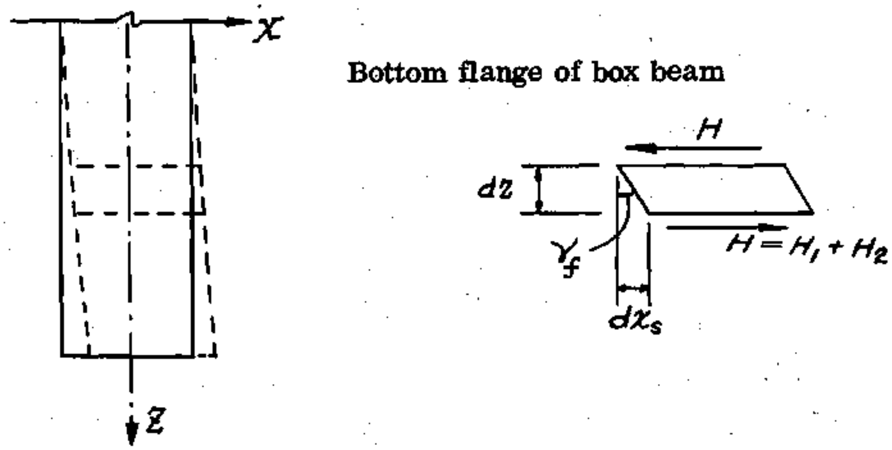


Figure 11. Free Body Diagrams of Bottom Flange of A Box Beam Under the Action of Shear Alone.

$$\gamma_f = \frac{C_s \cdot H}{b_f t_f G} \qquad (2\text{-}17)$$

where C_s is a constant [33, 34] whose magnitude depends on the shape of the cross-section. For a rectangular cross-section $C_s = 1.2$. Since $\gamma_f = \frac{dx_s}{dz}$, where x_s denotes the deflection due to shear alone, we have

$$\frac{dx_s}{dz} = \frac{C_s \cdot H}{b_f t_f G} \qquad (2\text{-}18)$$

Similarly considering the right-side web as shown in Figure 12, we have

$$\gamma_w = \frac{dy_s}{dz} = \frac{C_s \cdot V}{b_w t_w G} \qquad \text{(2-19)}$$

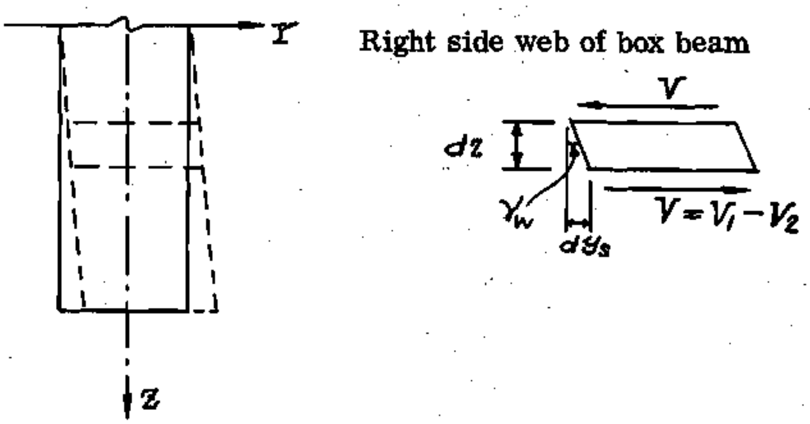


Figure 12. Free Body Diagrams of the Right-Side Web of A Box Beam Under the Action of Shear Alone.

From Equations (2-18) and (2-19), we obtain the curvature of the plates due to the action of shear alone:

$$\begin{cases} \dfrac{d^2\chi_s}{dz^2} = \dfrac{C_s}{b_f t_f G} \cdot \dfrac{dH}{dz} \\[2ex] \dfrac{d^2 y_s}{dz^2} = \dfrac{C_s}{b_w t_w G} \cdot \dfrac{dV}{dz} \end{cases} \qquad \text{(2-20)}$$

The total curvature of the plates can be obtained by adding the Equations in (2-12) and (2-20) respectively. Thus

$$\begin{cases} \dfrac{d^2x}{dz^2} = \dfrac{d^2x_b}{dz^2} + \dfrac{d^2x_s}{dz^2} = -\dfrac{M_f}{EI_f} + \dfrac{C_s}{b_f t_f G}\cdot\dfrac{dH}{dz} \\ \\ \dfrac{d^2y}{dz^2} = \dfrac{d^2y_b}{dz^2} + \dfrac{d^2y_s}{dz^2} = -\dfrac{M_w}{EI_w} + \dfrac{C_s}{b_w t_w G}\cdot\dfrac{dV}{dz} \end{cases} \qquad \text{(2-21)}$$

The second compatibility condition of the box section under non-uniform torsion is expressed in terms of the total deflections in the X and Y directions. This condition is based on an earlier assumption that the rectangular shape of the box remains undistorted under load, so that the angle of twist ϕ at any cross-section can be correctly defined by either $\phi = 2x/b_w$ or $\phi = 2y/b_f$ (Figure 13), considering small displacements. Thus

$$\begin{cases} x = \dfrac{b_w\cdot\phi}{2} \\ \\ y = \dfrac{b_f\cdot\phi}{2} \end{cases} \qquad \text{(2-22)}$$

$$\begin{cases} \dfrac{dx}{dz} = \dfrac{b_w}{2}\cdot\dfrac{d\phi}{dz} \\ \\ \dfrac{dy}{dz} = \dfrac{b_f}{2}\cdot\dfrac{d\phi}{dz} \end{cases} \qquad \text{(2-23)}$$

$$\left\{\begin{aligned} \frac{d^2 x}{dz^2} &= \frac{b_w}{2} \cdot \frac{d^2\phi}{dz^2} \\ \\ \frac{d^2 y}{dz^2} &= \frac{b_f}{2} \cdot \frac{d^2\phi}{dz^2} \end{aligned}\right. \qquad \text{(2-24)}$$

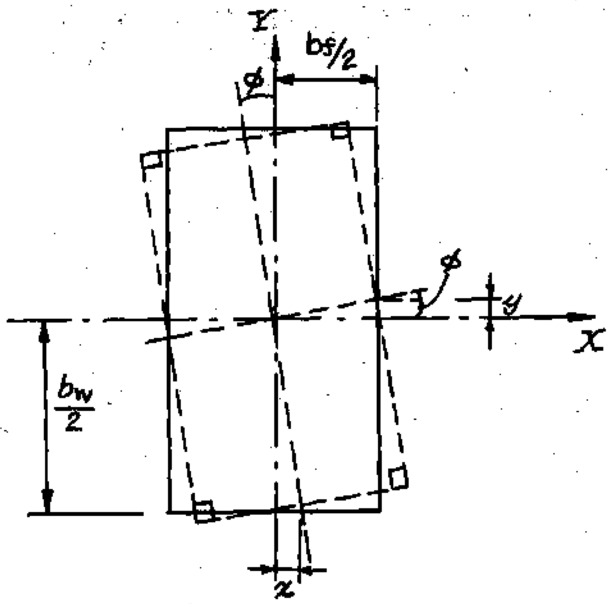


Figure 13. Rotational Displacements of A Box Beam Under Torsion.

Also since $\phi = \phi$

we have $$x = \frac{b_w}{b_f} \cdot y \qquad \text{(2-25)}$$

$$\frac{dx}{dz} = \frac{b_w}{b_f} \cdot \frac{dy}{dz} \qquad \text{(2-26)}$$

$$\frac{d^2x}{dZ^2} = \frac{b_w}{b_f} \cdot \frac{d^2y}{dZ^2} \qquad \text{(2-27)}$$

Substituting Equation (2-24) into Equation (2-21), we have

$$\left\{\begin{array}{l} \dfrac{d^2\phi}{dZ^2} = -\dfrac{2}{b_w E I_f} \cdot M_f + \dfrac{2C_s}{b_w b_f t_f G} \cdot \dfrac{dH}{dZ} \\[2ex] \dfrac{d^2\phi}{dZ^2} = -\dfrac{2}{b_f E I_w} M_w + \dfrac{2C_s}{b_w b_f t_w G} \cdot \dfrac{dV}{dZ} \end{array}\right. \qquad \text{(2-28)}$$

From Equation (2-5), we have

$$M_Z = H \cdot b_w + V \cdot b_f$$

$$H = \frac{M_Z}{b_w} - \frac{b_f}{b_w} \cdot V$$

Since M_z varies along Z-axis,

$$\frac{dH}{dZ} = \frac{1}{b_w} \cdot \frac{dM_Z}{dZ} - \frac{b_f}{b_w} \cdot \frac{dV}{dZ} \qquad \text{(2-29)}$$

Substituting Equations (2-14) and (2-29) into the first of Equations (2-28), we have

$$\frac{d^2\phi}{dZ^2} = \left(-\frac{2}{b_w E I_f}\right)\left(-\frac{b_w I_f}{b_f I_w}\right)\cdot M_w$$

$$+\left(-\frac{2C_s}{b_w b_f t_f G}\right)\left(\frac{1}{b_w}\right)\cdot\frac{dM_z}{dZ} - \left(\frac{2C_s}{b_w b_f t_f G}\right)\left(\frac{b_f}{b_w}\right)\cdot\frac{dV}{dZ}$$

$$\frac{d^2\phi}{dZ^2} = \frac{2}{b_f E I_w}\cdot M_w - \frac{2\cdot C_s}{b_w^2 t_f G}\cdot\frac{dV}{dZ} + \frac{2C_s}{b_w^2 b_f t_f G}\cdot\frac{dM_z}{dZ}$$

Multiplying both sides by $b_w t_f$,

$$b_w t_f \frac{d^2\phi}{dZ^2} = \frac{2 b_w t_f}{b_f E I_w}\cdot M_w - \frac{2C_s}{b_w G}\cdot\frac{dV}{dZ} + \frac{2C_s}{b_w b_f G}\cdot\frac{dM_z}{dZ} \qquad \text{(2-30)}$$

Also multiplying both sides of the second of Equations (2-28) by $b_f t_w$,

$$b_f t_w \frac{d^2\phi}{dZ^2} = -\frac{2 b_f t_w}{b_f E I_w}\cdot M_w + \frac{2C_s}{b_w G}\cdot\frac{dV}{dZ} \qquad \text{(2-31)}$$

Adding Equations (2-30) and (2-31), we have

$$(b_w t_f + b_f t_w)\frac{d^2\phi}{dZ^2} = \frac{2}{b_f E I_w}(b_w t_f - b_f t_w)\cdot M_w + \frac{2C_s}{b_w b_f G}\cdot\frac{dM_z}{dZ} \qquad \text{(2-32)}$$

In the discussion at the end of Appendix A it has been shown that if $b_w t_f = b_f t_w$ there is no warping of the cross-section under torsion. If the section does not warp, the resisting moment M_{z2} due to warping is equal to zero. Then the problem reduces to that of a case of uniform torsion and the governing differential equation is given by Equation (2-3). Thus there are two conditions for which the problem of non-uniform torsion of a box section is reduced to the case of uniform torsion. The first condition is $b_w^2 I_f = b_f^2 I_w$ (i.e. $b_w t_w = b_f t_f$), and the second condition is $b_w t_f = b_f t_w$. In the following discussions of this Article, these two conditions for which $M_{z2} = 0$ are to be excluded. They will be discussed specifically in Article 3 of Chapter IV and in Appendix C.

Assuming now that $b_w t_f - b_f t_w \neq 0$ and dividing both sides of Equation (2-32) by $(b_w t_f - b_f t_w)$ and rearranging, we have

$$M_w = \frac{b_f E I_w}{2} \left(\frac{b_w t_f + b_f t_w}{b_w t_f - b_f t_w} \right) \frac{d^2\phi}{dz^2} - \frac{C_s E I_w}{b_w G} \cdot \frac{1}{(b_w t_f - b_f t_w)} \cdot \frac{dM_z}{dz}$$

$$\frac{dM_w}{dz} = \frac{b_f E I_w}{2} \cdot \left(\frac{b_w t_f + b_f t_w}{b_w t_f - b_f t_w} \right) \frac{d^3\phi}{dz^3} - \frac{C_s E I_w}{b_w G \cdot (b_w t_f - b_f t_w)} \cdot \frac{d^2 M_z}{dz^2}$$

Substituting into Equation (2-16), we have

$$M_{z2} = - \frac{b_w^2 I_f - b_f^2 I_w}{b_f I_w} \cdot \frac{b_f E I_w}{2} \cdot \left(\frac{b_w t_f + b_f t_w}{b_w t_f - b_f t_w} \right) \cdot \frac{d^3\phi}{dz^3}$$

$$+\frac{b_w^2 I_f - b_f^2 I_w}{b_f I_w} \cdot \frac{C_s E I_w}{b_w G(b_w t_f - b_f t_w)} \cdot \frac{d^2 M_z}{d z^2}$$

$$M_{zz} = -\left(\frac{b_w^2 I_f - b_f^2 I_w}{2}\right)\left(\frac{b_w t_f + b_f t_w}{b_w t_f - b_f t_w}\right) \cdot E \cdot \frac{d^3 \phi}{d z^3}$$

$$+\frac{(b_w^2 I_f - b_f^2 I_w)}{b_w b_f (b_w t_f - b_f t_w)}\left(\frac{C_s E}{G}\right)\frac{d^2 M_z}{d Z^2}$$

Denote

$$C_2 = \left(\frac{b_w^2 I_f - b_f^2 I_w}{2}\right)\left(\frac{b_w t_f + b_f t_w}{b_w t_f - b_f t_w}\right) \cdot E$$

$$= \frac{(b_f t_f - b_w t_w)}{(b_w t_f - b_f t_w)} \cdot \left\{\frac{1}{24}(b_w^2 b_f^2)(b_w t_f + b_f t_w) \cdot E\right\} \qquad (2\text{-}33)$$

and

$$C_3 = \frac{b_w^2 I_f - b_f^2 I_w}{(b_w b_f)(b_w t_f - b_f t_w)} \cdot \left(\frac{C_s E}{G}\right)$$

$$= \frac{(b_f t_f - b_w t_w)}{(b_w t_f - b_f t_w)} \cdot \left\{\frac{1}{12}(b_w b_f)\left(\frac{C_s E}{G}\right)\right\} \qquad (2\text{-}34)$$

Note that in both Equations (2-33) and (2-34), we have $(b_w t_f - b_f t_w) \neq 0$ and $(b_w^2 I_f - b_f^2 I_w) \neq 0$. Thus

$$M_{Z2} = -C_2 \frac{d^3\phi}{dZ^3} + C_3 \frac{d^2 M_Z}{dZ^2} \qquad (2\text{-}35)$$

By substituting Equations (2-3) and (2-35) into (2-8), one obtains the differential equation for non-uniform torsion of a box section.

$$M_Z = M_{Z1} + M_{Z2}$$

$$M_Z = C_1 \frac{d\phi}{dZ} - C_2 \frac{d^3\phi}{dZ^3} + C_3 \frac{d^2 M_Z}{dZ^2} \qquad (2\text{-}36)$$

where C_1, C_2 and C_3 are defined by Equations (2-2), (2-33) and (2-34) respectively, and $C_2 \neq 0$, $C_3 \neq 0$.

3. An Elastic Solution Neglecting the Effect of Deflections in the Plane of the Primary-Bending Moment

The differential equations for this solution can be set up using the elastic buckling concept as defined in the classical theory of elastic stability.* This concept states that the critical pure bending moment is determined by the criterion

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*This method has been thoroughly discussed by Timoshenko [4], Bleich [34] and many others.

that the stability limit of the beam is reached when a deflected configuration infinitesimally near to the equilibrium form in the vertical plane is possible, indicating bifurcation of the equilibrium position. This means that as long as the pure bending moment on the box beam is below the critical value, the beam will be stable. As the pure bending moment increases in magnitude, a condition is reached at which a slightly deflected and twisted form of equilibrium infinitesimally away from the vertical plane of bending becomes possible. The plane configuration of the beam is thus unstable, and the lowest moment at which this critical condition occurs represents the critical pure bending moment.

In Figure 14 the box beam is subjected to pure bending moments acting in the Y-Z plane, which is the plane of maximum flexural rigidity. In deriving the differential equations one shall use the fixed coordinate axes X, Y, Z. In addition, the origin of the coordinate axes ξ, η, ζ is taken at the shear center of a cross-section (which is coincident with the centroid for a box section). The axes ξ and η are respectively the major and minor principal axes of the cross-section and ζ-axis is in the direction of the tangent to the deflected axis of the box beam after buckling. The deflection of the beam is defined by the components u and v of the displacement of the centroid (or the shear center) of the cross-section in the X and Y directions respectively, and by the angle of rotation ϕ of the cross-section.

For small relative deflections, the curvatures of the deflected axis in the XZ and YZ planes can be taken as $\frac{d^2u}{dz^2}$ and $\frac{d^2v}{dz^2}$ respectively. For small angle of twist ϕ, one can also assume that the curvatures in the $\xi\zeta$ and $\eta\zeta$ planes be taken

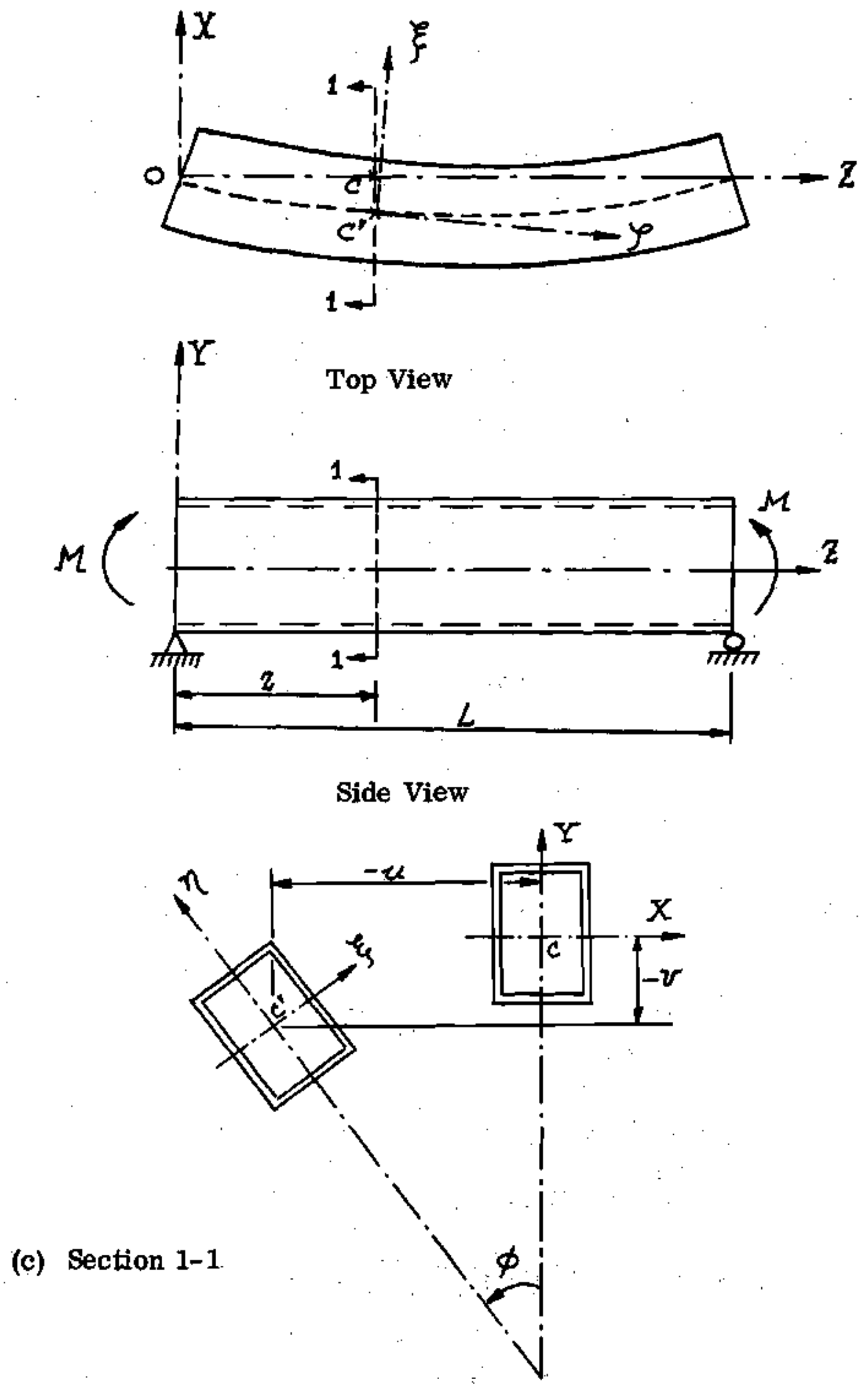


Figure 14. Lateral-Torsional Buckling of A Box Beam

Under Pure Bending Moments.

as $\frac{d^2u}{dz^2}$ and $\frac{d^2v}{dz^2}$ respectively. Thus the moment-curvature relationships for bending of the box beam about the ξ amd η axes can be written as

$$EI_\xi \frac{d^2v}{dz^2} = M_\xi \tag{2-37}$$

$$EI_\eta \frac{d^2u}{dz^2} = M_\eta \tag{2-38}$$

The positive directions of M_ξ and M_η are as shown in Figure 15.

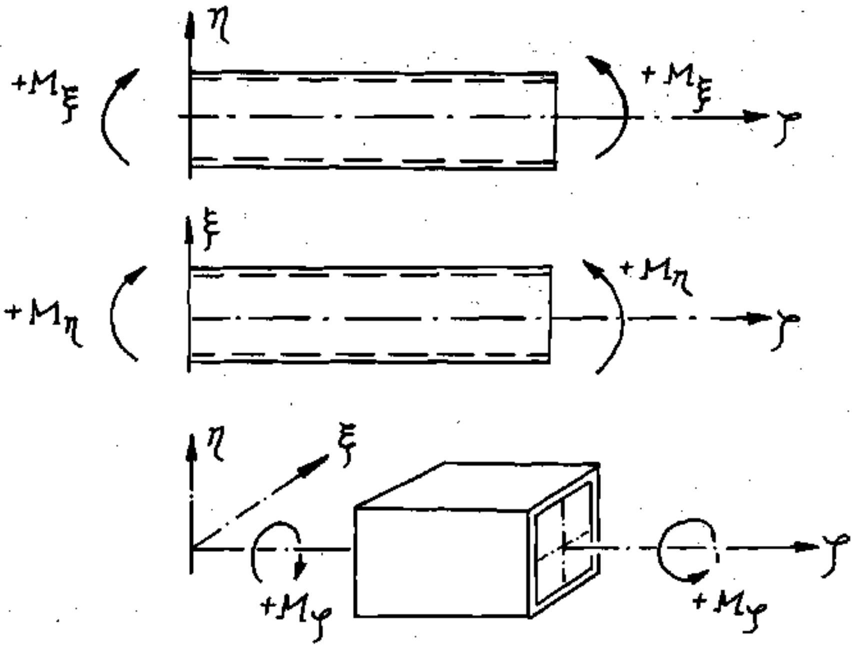


Figure 15. Sign Convention of Moments.

The differential equation for non-uniform torsion of a box section is given by Equation (2-36). Applying this equation to the buckled box beam, we have

$$C_1 \frac{d\phi}{dZ} - C_2 \frac{d^3\phi}{dZ^3} - M_\zeta + C_3 \frac{d^2 M_\zeta}{dZ^2} = 0 \qquad \text{(2-39)}$$

where C_1, C_2 and C_3 are constants defined by Equations (2-2), (2-33) and (2-34) respectively, and $C_2 \neq 0$, $C_3 \neq 0$.

The bending and twisting moments at any cross-section of the buckled box beam (Figure 15) can be found by taking components of the applied moment M about the ξ, η and ζ axes. To do this one shall need the expressions of the cosines of the angles between the coordinate axes X, Y, Z and ξ, η, ζ. These expressions are given in Table 2 (see also Figure 16). They are valid with the assumption that the deflections u, v, ϕ are small.

Thus using the first column of Table 2 and using the sign convention of moments according to Figure 15, one can compute the components of the applied pure bending moment M taken at any cross-section as follows.

$$\begin{cases} M_\xi = M \cdot \cos(\xi X) = M \\ M_\eta = -M \cdot \cos(\eta X) = \phi \cdot M \\ M_\zeta = -M \cdot \cos(\zeta X) = -\frac{du}{dZ} \cdot M \end{cases}$$

Substituting these values into Equations (2-37), (2-38) and (2-39), one obtains

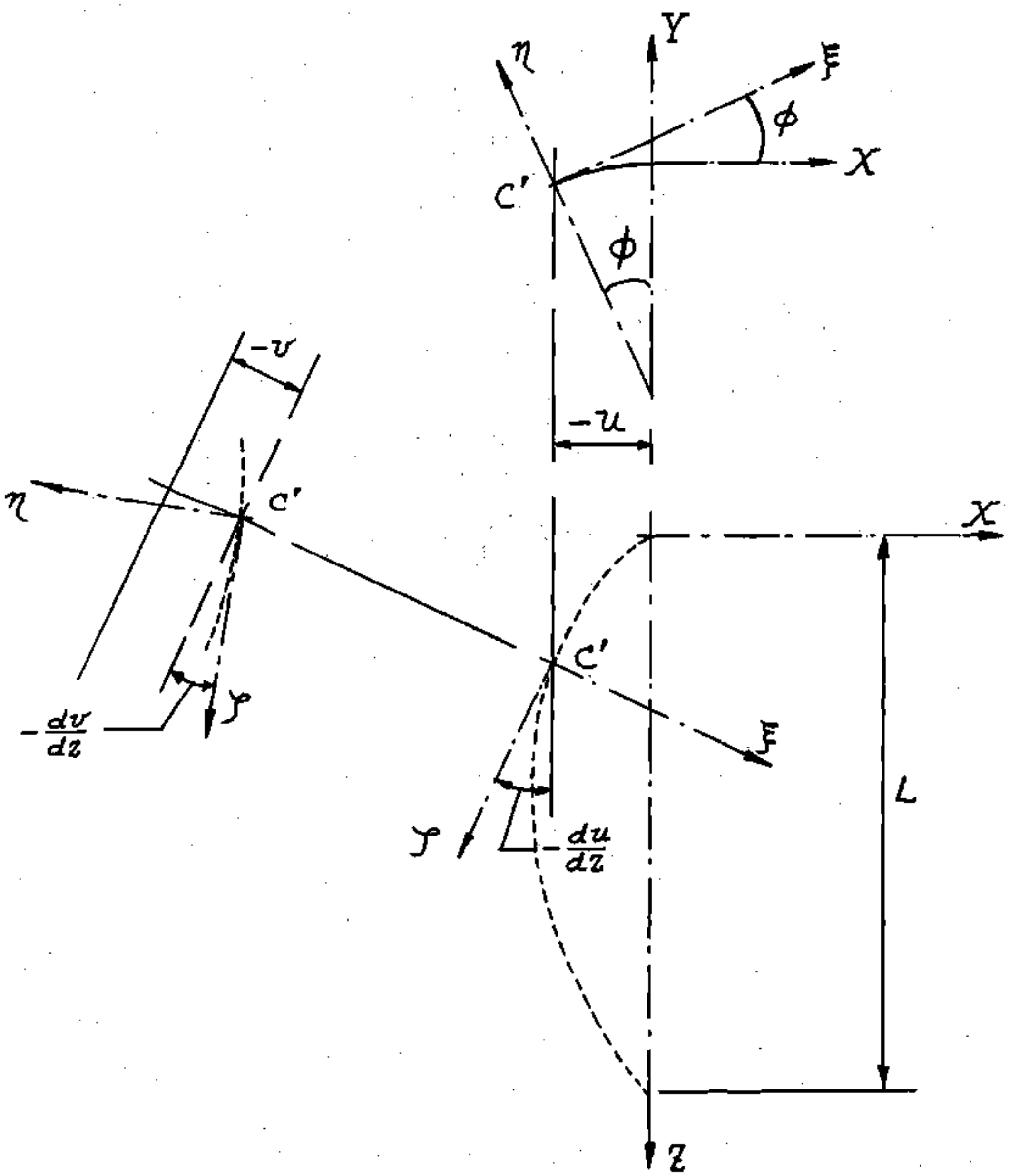


Figure 16. Displacements of A Box Beam After Lateral-Torsional Buckling Has Occurred.

Table 2. Cosines of Small Angles Between

Axes ξ, η, ζ and Axes X, Y, Z.

	x	y	z
ξ	1	ϕ	$(-)\frac{du}{dz}$
η	$-\phi$	1	$(-)\frac{dv}{dz}$
ζ	$\frac{du}{dz}$	$\frac{dv}{dz}$	1

$$EI_{\xi}\frac{d^2v}{dz^2} - M = 0 \qquad (2\text{-}40)$$

$$EI_{\eta}\frac{d^2u}{dz^2} - \phi M = 0 \qquad (2\text{-}41)$$

$$C_1\frac{d\phi}{dz} - C_2\frac{d^3\phi}{dz^3} + M\frac{du}{dz} - MC_3\frac{d^3u}{dz^3} = 0 \qquad (2\text{-}42)$$

Equations (2-40), (2-41) and (2-42) are the governing differential equations for the problem of lateral-torsional buckling of a box beam. With proper given boundary conditions these differential equations can readily be solved and the critical pure bending moment can be determined. By differentiating Equation (2-41) twice by z, we have

$$EI_\eta \frac{d^3u}{dz^3} - M \cdot \frac{d\phi}{dz} = 0$$

$$EI_\eta \frac{d^4u}{dz^4} - M \cdot \frac{d^2\phi}{dz^2} = 0$$

$$\frac{d^4u}{dz^4} = \frac{M}{EI_\eta} \cdot \frac{d^2\phi}{dz^2} \qquad \text{(2-43)}$$

Differentiating Equation (2-42) by z,

$$C_1 \cdot \frac{d^2\phi}{dz^2} - C_2 \cdot \frac{d^4\phi}{dz^4} + M \cdot \frac{d^2u}{dz^2} - M \cdot C_3 \cdot \frac{d^4u}{dz^4} = 0 \qquad \text{(2-44)}$$

Substituting equations (2-41) and (2-43) into Equation (2-44), we have

$$C_1 \frac{d^2\phi}{dz^2} - C_2 \cdot \frac{d^4\phi}{dz^4} + \frac{M^2}{EI_\eta} \cdot \phi - \frac{C_3 \cdot M^2}{EI_\eta} \cdot \frac{d^2\phi}{dz^2} = 0$$

$$C_2 \frac{d^4\phi}{dz^4} - \left(C_1 - \frac{C_3 \cdot M^2}{EI_\eta}\right) \frac{d^2\phi}{dz^2} - \frac{M^2}{EI_\eta} \cdot \phi = 0$$

since $C_2 \neq 0$, we have

$$\frac{d^4\phi}{dz^4} - \frac{1}{C_2}\left(C_1 - C_3 \cdot \frac{M^2}{EI_\eta}\right) \cdot \frac{d^2\phi}{dz^2} - \frac{1}{C_2} \cdot \frac{M^2}{EI_\eta} \cdot \phi = 0 \qquad \text{(2-45)}$$

There are three possible solutions to Equation (2-45), depending on the signs of the coefficients. From Equations (2-33) and (2-34), it can be shown that C_2 and C_3 always have the same sign. The first two solutions can be obtained by assuming C_1 and C_3 are both negative constants. Let

$$B_2 = |C_2| \;, \quad B_3 = |C_3| \tag{2-46}$$

we have $C_2 = -B_2 \;, \quad C_3 = -B_3$

Equation (2-45) becomes

$$\frac{d^4\phi}{dZ^4} + \frac{1}{B_2}\cdot\left(C_1 + B_3\cdot\frac{M^2}{EI_\eta}\right)\cdot\frac{d^2\phi}{dZ^2} + \frac{1}{B_2}\cdot\frac{M^2}{EI_\eta}\cdot\phi = 0$$

Let

$$\left\{\begin{aligned} \alpha_1 &= \frac{1}{2B_2}\cdot\left(C_1 + B_3\cdot\frac{M^2}{EI_\eta}\right) > 0 \\ \beta_1 &= \frac{1}{B_2}\cdot\frac{M^2}{EI_\eta} > 0 \end{aligned}\right. \tag{2-47}$$

we have

$$\frac{d^4\phi}{dZ^4} + 2\alpha_1\frac{d^2\phi}{dZ^2} + \beta_1\phi = 0$$

The auxiliary equation of this differential equation is

$$p^4 + 2\alpha_1 p^2 + \beta_1 = 0 \tag{2-48}$$

The first solution can be derived by assuming

$$\alpha_1^2 \geqslant \beta_1$$

Thus $$\alpha_1^2 - \beta_1 \geqslant 0$$

And the roots of the auxiliary equation are given as follows.

$$\begin{cases} p_{1,2}^2 = -\alpha_1 + \sqrt{\alpha_1^2 - \beta_1} = -(\alpha_1 - \sqrt{\alpha_1^2 - \beta_1}) \\ p_{3,4}^2 = -\alpha_1 - \sqrt{\alpha_1^2 - \beta_1} = -(\alpha_1 + \sqrt{\alpha_1^2 - \beta_1}) \end{cases}$$

$$\begin{cases} p_1 = i\sqrt{\alpha_1 - \sqrt{\alpha_1^2 - \beta_1}} \\ p_2 = -i\sqrt{\alpha_1 - \sqrt{\alpha_1^2 - \beta_1}} \\ p_3 = i\sqrt{\alpha_1 + \sqrt{\alpha_1^2 - \beta_1}} \\ p_4 = -i\sqrt{\alpha_1 + \sqrt{\alpha_1^2 - \beta_1}} \end{cases}$$

Let

$$\begin{cases} m = \sqrt{\alpha_1 - \sqrt{\alpha_1^2 - \beta_1}} \\ n = \sqrt{\alpha_1 + \sqrt{\alpha_1^2 - \beta_1}} \end{cases}$$

we have

$$\phi = A_1 e^{imz} + A_2 e^{-imz} + A_3 e^{inz} + A_4 e^{inz}$$

$$\phi = A_1 \cos(mz) + A_1 i \cdot \sin(mz) + A_2 \cos(mz) - A_2 i \cdot \sin(mz)$$

$$+ A_3 \cos(nz) + A_3 i \cdot \sin(nz) + A_4 \cos(nz) - A_4 i \cdot \sin(nz)$$

$$\phi = (A_1 + A_2)\cos(mz) + (A_1 - A_2) i \cdot \sin(mz)$$

$$+ (A_3 + A_4)\cos(nz) + (A_3 - A_4) i \cdot \sin(nz)$$

Substituting a new constant A_1 for $(A_1 + A_2)$ and A_2 for $(A_1 - A_2)i$ and so on, we have

$$\phi = A_1 \cos(mz) + A_2 \sin(mz) + A_3 \cos(nz) + A_4 \sin(nz) \tag{2-49}$$

Consider the case of a simply supported box beam in which the ends of the box beam cannot rotate about the Z-axis but are free to warp (A schematic picture of this end condition is shown in Figure 17). We have, since it cannot rotate at the ends, the first two boundary conditions, namely,

$$\begin{cases} \phi = 0 \quad at \quad z = 0 \\ \phi = 0 \quad at \quad z = L \end{cases} \tag{2-50}$$

If the ends are free to warp, there will be no bending moments due to warping and thus the curvature due to bending is zero. Since only small deflection is considered,

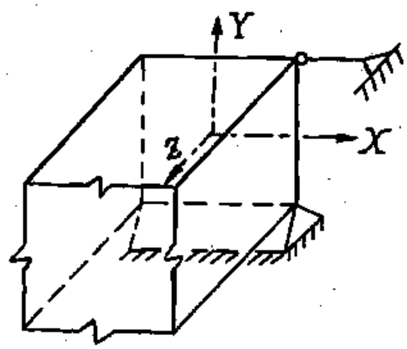


Figure 17. A Schematic Picture of A Particular End Condition.

the curvature due to shear at the ends is very small and can be neglected. Thus

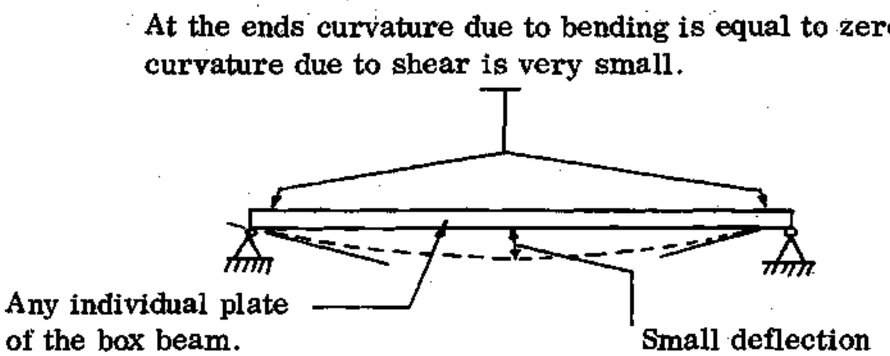


Figure 18. Deflection and Curvature of an Individual Plate of A Box Beam.

from Equations (2-21) and (2-24), one arrives at the last two boundary conditions, which are approximately true for all practical purposes, if only small deflection is considered.

$$\begin{cases} \dfrac{d^2\phi}{dz^2} = 0, \quad \text{at} \quad z = 0 \\ \\ \dfrac{d^2\phi}{dz^2} = 0, \quad \text{at} \quad z = L \end{cases} \tag{2-51}$$

Applying the first two boundary conditions to Equation (2-49), we have

$$0 = A_1 + A_3 \quad , \qquad \therefore \quad A_1 = -A_3$$

and

$$0 = A_1 \cos(mL) + A_2 \sin(mL) + A_3 \cos(nL) + A_4 \sin(nL)$$

Also, by differentiating Equation (2-49) twice, we have

$$\frac{d^2\phi}{dz^2} = -A_1 m^2 \cos(mz) - A_2 m^2 \sin(mz) - A_3 n^2 \cos(nz) - A_4 n^2 \sin(nz)$$

Using the last two boundary conditions, we have

$$\begin{cases} 0 = -A_1 m^2 - A_3 n^2 \\ \\ 0 = -A_2 m^2 \sin(mL) - A_4 n^2 \sin(nL) \end{cases}$$

Since $\quad m^2 - n^2 = \alpha_1 - \sqrt{\alpha_1^2 - \beta_1} - \alpha_1 - \sqrt{\alpha_1^2 - \beta_1}$

$$= -2\sqrt{\alpha_1^2 - \beta_1} \quad \neq 0$$

the first of these equations gives

$$A_1 - A_3 = 0$$

There are left two constants to be solved from the following homogeneous equations

$$\begin{cases} A_2 \sin(mL) + A_4 \sin(nL) = 0 \\ \\ -A_2 m^2 \sin(mL) - A_4 n^2 \sin(nL) = 0 \end{cases}$$

For a non-trivial solution the determinant of these equations must equal zero.

Thus

$$-(\sin mL)(n^2 \sin nL) + (\sin nL)(m^2 \sin mL) = 0$$

$$(\sin mL)(\sin nL)(m^2 - n^2) = 0$$

Since $m^2 - n^2 \neq 0$,

Let $\quad$ Sin mL = 0.

The smallest root of this equation is

$$m = \frac{\pi}{L}$$

$$\sqrt{\alpha_1 - \sqrt{\alpha_1^2 - \beta_1}} = \frac{\pi}{L}$$

$$\alpha_1 - \sqrt{\alpha_1^2 - \beta_1} = \frac{\pi^2}{L^2}$$

$$\alpha_1^2 - \beta_1 = \alpha_1^2 - 2\alpha_1 \frac{\pi^2}{L^2} + \frac{\pi^4}{L^4}$$

$$\beta_1 = -\frac{\pi^4}{L^4} + 2\alpha_1 \frac{\pi^2}{L^2} \qquad \text{(2-52)}$$

$$\frac{M^2}{B_2 E I_\eta} = -\frac{\pi^4}{L^4} + \frac{1}{B_2}\left(C_1 + B_3 \frac{M^2}{E I_\eta}\right)\cdot\frac{\pi^2}{L^2}$$

$$\frac{M^2}{E I_\eta}\left(\frac{1}{B_2} - \frac{B_3}{B_2}\cdot\frac{\pi^2}{L^2}\right) = \frac{\pi^2}{L^2}\left(\frac{C_1}{B_2} - \frac{\pi^2}{L^2}\right)$$

$$M^2 = (E I_\eta)\cdot\frac{\pi^2}{L^2}\cdot\frac{\left(\frac{C_1}{B_2} - \frac{\pi^2}{L^2}\right)}{\left(\frac{1}{B_2} - \frac{B_3}{B_2}\cdot\frac{\pi^2}{L^2}\right)}$$

$$M_1 = \left(\sqrt{\frac{\pi^2 E I_\eta}{L^2}\cdot C_1}\right)\sqrt{\frac{\left(\frac{C_1}{B_2} - \frac{\pi^2}{L^2}\right)}{C_1\cdot\left(\frac{1}{B_2} - \frac{B_3}{B_2}\cdot\frac{\pi^2}{L^2}\right)}}$$

Let

$$k_1 = \sqrt{\frac{\frac{\pi^2}{L^2} - \frac{C_1}{B_2}}{\frac{C_1}{B_2}\left(B_3\cdot\frac{\pi^2}{L^2} - 1\right)}} \qquad \text{(2-53)}$$

We have

$$M_1 = (-k_1)\sqrt{\frac{\pi^2 E I_z}{L^2}} \cdot C_1 \qquad \text{(2-54)}$$

Next, let Sin(nL)= 0. The smallest root of this equation gives

$$n = \frac{\pi}{L}$$

$$\sqrt{\alpha_1 + \sqrt{\alpha_1^2 - \beta_1}} = \frac{\pi}{L}$$

$$\alpha_1 + \sqrt{\alpha_1^2 - \beta_1} = \frac{\pi^2}{L^2}$$

$$\alpha_1^2 - \beta_1 = \frac{\pi^4}{L^4} - 2\alpha_1 \frac{\pi^2}{L^2} + \alpha_1^2$$

$$\beta_1 = -\frac{\pi^4}{L^4} + 2\alpha_1 \frac{\pi^2}{L^2}$$

Comparing this equation with Equation (2-52), one finds that it gives the same solution of M as Equation (2-54).

The next solution of M may be derived by assuming

$$\alpha_1^2 - \beta_1 \leqslant 0$$

Starting with Equation (2-48),

$$p^4 + 2\alpha_1 p^2 + \beta_1 = 0$$

$$p^4 + 2\sqrt{\beta_1}\, p^2 + \beta_1 - 2\sqrt{\beta_1}\, p^2 + 2\alpha_1 p^2 = 0$$

Since $\alpha_1^2 - \beta_1 < 0$, $\alpha_1 > 0$ and $\beta_1 > 0$, we have

$$\sqrt{\beta_1} - \alpha_1 > 0$$

Hence

$$(p^2 + \sqrt{\beta_1})^2 - \left[p \cdot \sqrt{2(\sqrt{\beta_1} - \alpha_1)}\,\right]^2 = 0$$

$$\left[p^2 + p \cdot \sqrt{2(\sqrt{\beta_1} - \alpha_1)} + \sqrt{\beta_1}\right]\left[p^2 - p \cdot \sqrt{2(\sqrt{\beta_1} - \alpha_1)} + \sqrt{\beta_1}\right] = 0$$

Let
$$\begin{cases} m = \sqrt{\dfrac{\sqrt{\beta_1} - \alpha_1}{2}} \\ n = \sqrt{\beta_1} \end{cases}$$

$$p_{1,2} = \frac{-2m \pm \sqrt{4m^2 - 4n}}{2} = -m \pm \sqrt{m^2 - n}$$

$$p_{3,4} = \frac{2m \pm \sqrt{4m^2 - 4n}}{2} = m \pm \sqrt{m^2 - n}$$

Since

$$\sqrt{m^2 - n} = \sqrt{\frac{\sqrt{\beta_1} - \alpha_1}{2} - \sqrt{\beta_1}} = i \cdot \sqrt{\frac{\sqrt{\beta_1} + \alpha_1}{2}} \,,$$

one reassigns $n = \sqrt{\frac{\sqrt{\beta_1} + \alpha_1}{2}}$, and the four roots of the auxiliary equation become

$$\begin{cases} p_1 = -m + in \\ p_2 = -m - in \\ p_3 = m + in \\ p_4 = m - in \end{cases}$$

Hence the general solution of Equation (2-45) is

$$\phi = A_1 e^{(-m+in)z} + A_2 e^{(-m-in)z} + A_3 e^{(m+in)z} + A_4 e^{(m-in)z}$$

$$= (A_1 e^{-mz} + A_3 e^{mz}) e^{inz} + (A_2 e^{-mz} + A_4 e^{mz}) e^{-inz}$$

$$= (A_1 e^{-mz} + A_3 e^{mz})(\cos nz + i \sin nz) + (A_2 e^{-mz} + A_4 e^{mz})(\cos nz - i \sin nz)$$

$$= [(A_1 + A_2) e^{-mz} + (A_3 + A_4) e^{mz}] \cos nz + [i(A_1 - A_2) e^{-mz} + i(A_3 - A_4) e^{mz}] \sin nz$$

Substituting new constants A_1 for $(A_1 + A_2)$, A_2 for $(A_3 + A_4)$, A_3 for $i \cdot (A_1 - A_2)$ and A_4 for $i(A_3 - A_4)$, we have

$$\phi = (A_1 e^{-mz} + A_2 e^{mz}) \cos nz + (A_3 e^{-mz} + A_4 e^{mz}) \sin nz$$

$$\frac{d\phi}{dz} = (-mA_1e^{-mz} + mA_2e^{mz})\cos nz - (nA_1e^{-mz} + nA_2e^{mz})\sin nz$$

$$+ (-mA_3e^{-mz} + mA_4e^{mz})\sin nz + (nA_3e^{-mz} + nA_4e^{mz})\cos nz$$

$$= [(-mA_1 + nA_3)e^{-mz} + (mA_2 + nA_4)e^{mz}]\cos nz$$

$$+ [(-nA_1 - mA_3)e^{-mz} + (-nA_2 + mA_4)e^{mz}]\sin nz$$

$$\frac{d^2\phi}{dz^2} = [(m^2A_1 - mnA_3)e^{-mz} + (m^2A_2 + mnA_4)e^{mz}]\cos nz$$

$$+ [(mnA_1 - n^2A_3)e^{-mz} + (-mnA_2 - n^2A_4)e^{mz}]\sin nz$$

$$+ [(mnA_1 + m^2A_3)e^{-mz} + (-mnA_2 + m^2A_4)e^{mz}]\sin nz$$

$$+ [(-n^2A_1 - mnA_3)e^{-mz} + (-n^2A_2 + mnA_4)e^{mz}]\cos nz$$

$$\frac{d^2\phi}{dz^2} = \{[(m^2-n^2)A_1 - 2mnA_3]e^{-mz} + [(m^2-n^2)A_2 + 2mnA_4]e^{mz}\}\cos nz$$

$$+ \{[2mnA_1 + (m^2-n^2)A_3]e^{-mz} + [-2mnA_2 + (m^2-n^2)A_4]e^{mz}\}\sin nz$$

Using the boundary condition $\begin{cases}\phi = 0 \\ z = 0\end{cases}$, **we have**

$$A_1 = -A_2$$

Using the boundary condition $\begin{cases} \frac{d^2\phi}{dz^2} = 0 \\ z = 0 \end{cases}$, we have

$$0 = [(m^2 - n^2)A_1 - 2mnA_3] + [(m^2 - n^2)A_2 + 2mnA_4]$$

$$A_3 = A_4$$

Eliminating A_2 and A_4 from the expressions of ϕ and $\frac{d^2\phi}{dz^2}$.

$$\phi = A_1(e^{-mz} - e^{mz})\cos nz + A_3(e^{-mz} + e^{mz})\sin nz$$

$$\frac{d^2\phi}{dz^2} = \left\{[(m^2 - n^2)A_1 - 2mnA_3]e^{-mz} + [-(m^2 - n^2)A_1 + 2mnA_3]e^{mz}\right\}\cos nz$$

$$+\left\{[(2mnA_1 + (m^2 - n^2)A_3]e^{-mz} + [2mnA_1 + (m^2 - n^2)A_3]e^{mz}\right\}\sin nz$$

Using the boundary condition $\begin{cases} \phi = 0 \\ z = L \end{cases}$, we have

$$A_1(e^{-mL} - e^{mL})\cos nL + A_3(e^{-mL} + e^{mL})\sin nL = 0 \qquad \text{(2-55)}$$

Using the boundary condition $\begin{cases} \frac{d^2\phi}{dz^2} = 0 \\ z = L \end{cases}$, have

$$\left\{[(m^2 - n^2)A_1 - 2mnA_3]e^{-mL} + [-(m^2 - n^2)A_1 + 2mnA_3]e^{-mL}\right\}\cos nL$$

$$+\left\{[2mnA_1 + (m^2 - n^2)A_3]e^{-mL} + [2mnA_1 + (m^2 - n^2)A_3]e^{mL}\right\}\sin nL = 0$$

$$\{[(m^2-n^2)A_1-2mnA_3]e^{-mL}+[-(m^2-n^2)A_1+2mnA_3]e^{mL}\}\cos nL$$

$$+\{[2mnA_1+(m^2-n^2)A_3]e^{-mL}+[2mnA_1+(m^2-n^2)A_3]e^{mL}\}\sin nL=0$$

$$\{[(m^2-n^2)e^{-mL}-(m^2-n^2)e^{mL}]\cos nL+[2mne^{-mL}+2mne^{mL}]\sin nL\}A_1$$

$$+\{[-2mne^{-mL}+2mne^{mL}]\cos nL+[(m^2-n^2)e^{-mL}+(m^2-n^2)e^{mL}]\sin nL\}A_3=0$$

$$\{(e^{-mL}-e^{mL})(m^2-n^2)(\cos nL)+(e^{-mL}+e^{mL})(2mn)(\sin nL)\}A_1$$

$$+\{(e^{-mL}-e^{mL})(-2mn)(\cos nL)+(e^{-mL}+e^{mL})(m^2-n^2)(\sin nL)\}A_3=0 \quad \text{—— (2-56)}$$

For a non-trivial solution of M, the determinant of Equations (2-55) and (2-56) must equal zero. Hence

$$\begin{vmatrix} (e^{-mL}-e^{mL})\cos nL & (e^{-mL}+e^{mL})\sin nL \\ \begin{bmatrix}(e^{-mL}-e^{mL})(m^2-n^2)\cos nL \\ +(e^{-mL}+e^{mL})(2mn)\sin nL\end{bmatrix} & \begin{bmatrix}(e^{-mL}-e^{mL})(-2mn)\cos nL \\ +(e^{-mL}+e^{mL})(m^2-n^2)\sin nL\end{bmatrix} \end{vmatrix}=0$$

$$-(e^{-mL}-e^{mL})^2(2mn)(\cos nL)^2+(e^{-2mL}-e^{2mL})(m^2-n^2)(\cos nL)(\sin nL)$$

$$-(e^{-2mL}-e^{2mL})(m^2-n^2)(\cos nL)(\sin nL)-(e^{-mL}+e^{mL})^2(2mn)(\sin nL)^2=0$$

$$2mn[(e^{-2mL}-2+e^{2mL})(\cos nL)^2+(e^{-2m}+2+e^{2mL})(\sin nL)^2]=0$$

$$(2mn)\left[e^{-2mL}+e^{2mL}+2(\sin^2 nL-\cos^2 nL)\right]=0 \qquad \text{(2-57)}$$

If $2mn = 0$, then either $m = 0$, or $n = 0$ will give a solution of M. In either case, one arrives at the condition

$$\beta_1 = \alpha_1^2 .$$

But this condition has already been covered by the solution previously derived: Equations (2-53) and (2-54). Thus it gives no new solution of M. Next the second term of Equation (2-57) is set to zero.

$$e^{-2mL}+e^{2mL}+2(\sin^2 nL-\cos^2 nL)=0 \qquad \text{(2-58)}$$

This is a transcendental equation. The first two terms of this equation represent a hyperbolic function (Figure 19a). The smallest value of the hyperbolic function $e^{-2mL}+e^{2mL}$ is equal to 2.

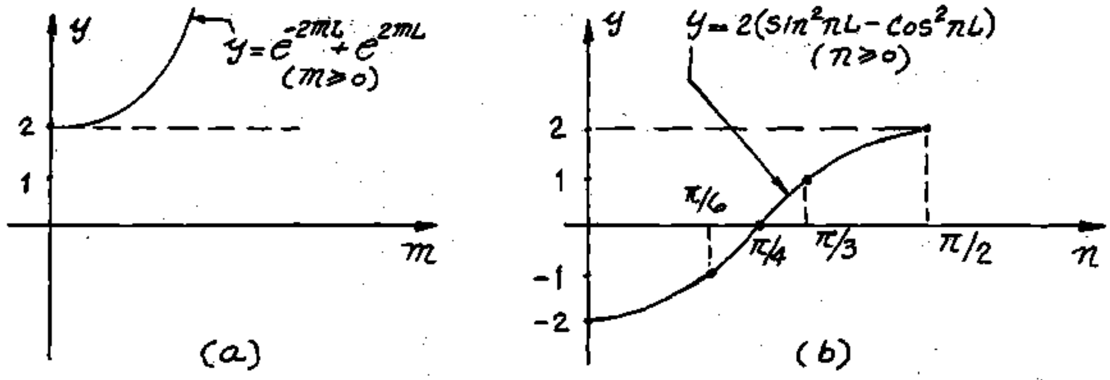


Figure 19. Graphical Solution of Equation (2-58).

The third term of this equation is a trigonometric function illustrated in Figure 19b. The smallest algebraic value of this function is -2. It can be shown that the only possible root of the transcendental equation is such that m and n are both equal to zero simultaneously. This leads to the condition of $\beta_1 = \alpha_1^2$. Again, the solution of M for this condition is given by Equations (2-53) and (2-54). Thus it can be concluded that there is no solution of the critical moment under the assumption $\alpha_1^2 - \beta_1 < 0$. This implies that the box beam is in a state of stable equilibrium.

Another solution of M can be obtained by assuming C_2 and C_3 to be both positive. Then Equation (2-45) can be rewritten as

$$\frac{d^4\phi}{dz^4} - \frac{1}{B_2}\left(C_1 - B_3 \cdot \frac{M^2}{E I_\eta}\right) \cdot \frac{d^2\phi}{dz^2} - \frac{1}{B_2} \cdot \frac{M^2}{E I_\eta} \cdot \phi = 0$$

where B_2 and B_3 are defined by Equations (2-46). Let

$$\begin{cases} \alpha_2 = \dfrac{1}{2B_2}\left(C_1 - B_3 \dfrac{M^2}{E I_\eta}\right) \\ \\ \beta_2 = \dfrac{1}{B_2} \cdot \dfrac{M^2}{E I_\eta} > 0 \end{cases} \qquad \text{(2-59)}$$

we have

$$\frac{d^4\phi}{dz^4} - 2\alpha_2 \frac{d^2\phi}{dz^2} - \beta_2 \cdot \phi = 0 \qquad \text{(2-60)}$$

The auxiliary equation of the differential equation (2-60) is

$$p^4 - 2\alpha_2 p^2 - \beta_2 = 0 \qquad \text{(2-61)}$$

Observing that

$$\begin{cases} \alpha_2^2 + \beta_2 > 0 \\ \sqrt{\alpha_2^2 + \beta_2} + \alpha_2 > 0 \\ \sqrt{\alpha_2^2 + \beta_2} - \alpha_2 > 0 \end{cases} \qquad \text{(2-62)}$$

Equations (2-62) are true regardless of the sign of α_2. Thus the roots of Equation (2-61) can be written as

$$\begin{cases} p_{1,2}^2 = \sqrt{\alpha_2^2 + \beta_2} + \alpha_2 \\ p_{3,4}^2 = -\sqrt{\alpha_2^2 + \beta_2} + \alpha_2 = -\left(\sqrt{\alpha_2^2 + \beta_2} - \alpha_2\right) \end{cases}$$

$$\begin{cases} p_1 = +\sqrt{\sqrt{\alpha_2^2 + \beta_2} + \alpha_2} \\ p_2 = -\sqrt{\sqrt{\alpha_2^2 + \beta_2} + \alpha_2} \\ p_3 = +i \cdot \sqrt{\sqrt{\alpha_2^2 + \beta_2} - \alpha_2} \\ p_4 = -i \cdot \sqrt{\sqrt{\alpha_2^2 + \beta_2} - \alpha_2} \end{cases}$$

Let

$$\begin{cases} m = \sqrt{\sqrt{\alpha_2^2 + \beta_2} + \alpha_2} > 0 \\ n = \sqrt{\sqrt{\alpha_2^2 + \beta_2} - \alpha_2} > 0 \end{cases}$$

Hence the general solution of the differential equation (2-60) becomes

$$\phi = A_1 e^{mz} + A_2 e^{-mz} + A_3 e^{inz} + A_4 e^{-inz}$$

$$= A_1 e^{mz} + A_2 e^{-mz} + A_3(\cos nz + i\sin nz) + A_4(\cos nz - i\cdot\sin nz)$$

$$= A_1 e^{mz} + A_2 e^{-mz} + (A_3 + A_4)\cos nz + (A_3 - A_4) i\cdot\sin nz$$

Substituting a new constant A_3 for $A_3 + A_4$ and A_4 for $(A_3 - A_4)$ i, we have

$$\phi = A_1 e^{mz} + A_2 e^{-mz} + A_3 \cos nz + A_4 \sin nz \text{ ——— } (2\text{-}63)$$

Differentiating Equation (2-63) twice, we have

$$\frac{d^2\phi}{dz^2} = A_1 m^2 e^{mz} + A_2 m^2 e^{-mz} - A_3 n^2 \cos nz - A_4 n^2 \sin nz \text{ ——— } (2\text{-}64)$$

Applying the boundary condition $\begin{cases}\phi = 0 \\ z = 0\end{cases}$ to Equation (2-63), we have,

$$0 = A_1 + A_2 + A_3 \text{ ——— } (2\text{-}65)$$

Applying the boundary condition $\begin{cases} \dfrac{d^2\phi}{dz^2} = 0 \\ z = 0 \end{cases}$ to Equation (2-65), we have,

$$0 = A_1 m^2 + A_2 m^2 - A_3 n^2 \qquad (2\text{-}66)$$

Adding Equations (2-65) and (2-66), we obtain

$$0 = (m^2 + n^2)A_1 + (m^2 + n^2)A_2$$

$m^2 + n^2 \neq 0$, hence

$$A_1 = -A_2$$

Substituting into Equation (2-65), we have

$$A_3 = 0$$

Thus $\quad \phi = A_1 e^{mz} - A_1 e^{-mz} + A_4 \sin nz$

$$\phi = A_1(e^{mz} - e^{-mz}) + A_4 \sin nz \qquad (2\text{-}67)$$

$$\frac{d^2\phi}{dz^2} = A_1 m^2 (e^{mz} - e^{-mz}) - A_4 n^2 \sin nz \qquad (2\text{-}68)$$

Substituting the boundary conditions $\begin{cases} \phi = 0 \\ z = L \end{cases}$ and $\begin{cases} \dfrac{d^2\phi}{dz^2} = 0 \\ z = L \end{cases}$ into Equations (2-67) and (2-68) respectively, we have

$$\begin{cases} A_1(e^{mL}-e^{-mL}) + A_4 \sin nL = 0 \\ A_1 m^2(e^{mL}-e^{-mL}) - A_4 n^2 \sin nL = 0 \end{cases}$$

For a non-trivial solution, the determinant of the above homogeneous equations must equal to zero. Hence,

$$\begin{vmatrix} (e^{mL}-e^{-mL}) & \sin nL \\ m^2(e^{mL}-e^{-mL}) & -n^2 \sin nL \end{vmatrix} = 0$$

$$-n^2(e^{mL}-e^{-mL})\sin nL - m^2(e^{mL}-e^{-mL})\sin nL = 0$$

$$(m^2+n^2)(e^{mL}-e^{-mL})\sin nL = 0$$

$$(2\sqrt{\alpha_2^2+\beta_2})(e^{mL}-e^{-mL})\sin nL = 0$$

$(2\sqrt{\alpha_2^2+\beta_2})(e^{mL}-e^{-mL})$ does not always equal zero, hence

$$\sin nL = 0$$

The smallest root of this equation gives

$$n = \frac{\pi}{L}$$

$$\sqrt{\sqrt{\alpha_2^2+\beta_2}-\alpha_2}=\frac{\pi}{L}$$

$$\sqrt{\alpha_2^2+\beta_2}-\alpha_2=\frac{\pi^2}{L^2}$$

$$\sqrt{\alpha_2^2+\beta_2}=\alpha_2+\frac{\pi^2}{L^2}$$

$$\alpha_2^2+\beta_2=\alpha_2^2+2\alpha_2\frac{\pi^2}{L^2}+\frac{\pi^4}{L^4}$$

$$\beta_2=2\alpha_2\frac{\pi^2}{L^2}+\frac{\pi^4}{L^4} \qquad \text{(2-69)}$$

$$\frac{M^2}{B_2EI_\eta}=\frac{1}{B_2}\left(C_1-B_3\frac{M^2}{EI_\eta}\right)\frac{\pi^2}{L^2}+\frac{\pi^4}{L^4}$$

$$=\frac{C_1}{B_2}\cdot\frac{\pi^2}{L^2}-\frac{B_3}{B_2}\cdot\frac{M^2}{EI_\eta}\cdot\frac{\pi^2}{L^2}+\frac{\pi^4}{L^4}$$

$$\frac{M^2}{EI_\eta}\left\{\frac{1}{B_2}+\frac{B_3}{B_2}\cdot\frac{\pi^2}{L^2}\right\}=\frac{\pi^2}{L^2}\left\{\frac{C_1}{B_2}+\frac{\pi^2}{L^2}\right\}$$

$$M^2=\frac{\pi^2EI_\eta}{L^2}\cdot\frac{\left\{\frac{C_1}{B_2}+\frac{\pi^2}{L^2}\right\}}{\left\{\frac{1}{B_2}+\frac{B_3}{B_2}\cdot\frac{\pi^2}{L^2}\right\}}$$

Let

$$k_2 = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{B_2}}{\frac{C_1}{B_2}\left(B_3 \cdot \frac{\pi^2}{L^2} + 1\right)}} \tag{2-70}$$

then,

$$M_2 = k_2 \cdot \sqrt{\frac{\pi^2 E I_\eta}{L^2} \cdot C_1} \tag{2-71}$$

Equations (2-70) and (2-71) are applicable when C_2 and C_3 are both positive. Equations (2-54) and (2-71) can be combined into one equation. Let

$$k = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{C_2}}{\frac{C_1}{C_2}\left(C_3 \cdot \frac{\pi^2}{L^2} + 1\right)}} \tag{2-72}$$

we have

$$M = k \cdot \sqrt{\frac{\pi^2 E I_\eta}{L^2} \cdot C_1} \tag{2-73}$$

Thus Equations (2-72) and (2-73) define the first solution of the elastic buckling moment M. It is to be noted that this elastic buckling moment does not depend on the primary flexural rigidity (EI_ξ) of the box beam. This is a consequence of the assumption that the deflections in the vertical plane are small and that the primary flexural rigidity (EI_ξ) is large when compared with the secondary ridigity (EI_η) and the uniform torsion constant (C_1). If the magnitude of EI_η is not too small when compared with EI_ξ, then a second solution is necessary to include the effect of bending in the Y-Z plane. This solution is to be discussed in Chapter III.

As an example of how to apply Equations (2-72) and (2-73), consider the box section (Figure 1) used in the tests conducted by Moran [1] at Georgia Institute of Technology. First, we shall evaluate the constants C_1, C_2 and C_3. Taking $C_s = 1.2$, $G/E = 1/3$ and $E = 30,000$ K/in^2, we have

$$C_1 = \frac{2(b_w b_f)^2 (t_w t_f) G}{(b_w b_f + b_f t_w)}$$

$$= \frac{(2)[(6.25)(1.807)]^2 (0.1193)(0.1193) G}{(6.25)(0.1193) + (1.807)(0.1193)}$$

$$= \frac{(2)[127.5](0.01425) G}{(0.746) + (0.216)}$$

$$= (3.77) G \qquad (K\text{-}in^2)$$

$$C_2 = \frac{(b_f t_f - b_w t_w)}{(b_w t_f - b_f t_w)} \left\{ \frac{1}{24} (b_w b_f)^2 (b_w t_f + b_f t_w) E \right\}$$

$$= \frac{(1.807)(0.1193) - (6.25)(0.1193)}{(6.25)(0.1193) - (1.807)(0.1193)} \left\{ \frac{1}{24} (6.25 \times 1.807)^2 (0.746 + 0.216) E \right\}$$

$$= -5.12\, E \qquad (K\text{-}in^4)$$

$$C_3 = \frac{(b_f t_f - b_w t_w)}{(b_w t_f - b_f t_w)} \left\{ \frac{1}{12}(b_w b_f) \frac{C_s E}{G} \right\}$$

$$= -\frac{1}{12}(11.3)(1.2)(3)$$

$$= -3.39 \quad (in^2)$$

Substituting the values of C_1, C_2 and C_3 into Equations (2-72) and (2-73), we have

$$k = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{C_2}}{\frac{C_1}{C_2}\left(C_3 \cdot \frac{\pi^2}{L^2} + 1\right)}}$$

$$\frac{\pi^2}{L^2} = \frac{9.89}{(187.5)^2} = \frac{9.89}{35{,}200} = 0.00028 \qquad \left(\frac{1}{in^2}\right)$$

$$\frac{C_1}{C_2} = \frac{3.77G}{-5.12E} = -0.246 \qquad \left(\frac{1}{in^2}\right)$$

$$k = \sqrt{\frac{0.00028 - 0.246}{(-0.246)[(-3.39)(0.00028) + 1]}} = \sqrt{\frac{0.2457}{0.2457}}$$

$$k = 1.0$$

$$M = (1.0)\sqrt{\frac{\pi^2 E I_\eta}{L^2} \cdot C_1} \qquad \text{(2-74)}$$

The value under the radical sign in Equation (2-74) represents the lateral-torsional buckling moment of the box beam under uniform torsion (i.e. without the benefits of warping resistance). Equation (2-74) shows that for the box beam shown in Figure 1, the benefit of warping resistance is negligible. The resistance due to warping is anticipated not to be large. Since in Figure 7 it can be shown that the warping resistance of a box section consists of two couples acting in the opposite directions. One couple is adding to and the other couple is cancelling out the resistance against torsion. But the main reason for the low value of k is because this first elastic solution is not exactly accurate. In this particular example EI_{η} (2.08E in^4) is not too small when compared to EI_{ξ} (13.94E in^4). A more accurate solution is necessary,therefore, to include the effect of bending in the plane of the primary bending moment (YZ plane). The value of the critical moment in kip-inches is evaluated as follows:

$$\frac{\pi^2 E I_{\eta}}{L^2} \cdot C_1 = \sqrt{\frac{(9.89)(30{,}000)(2.17)}{35{,}000}(3.77)(10{,}000)} = 830 \quad (K\text{-}in)$$

$$M = (1.0)(830) = 830 \quad (K\text{-}in)$$

Since the elastic lateral-torsional buckling moment M is much greater than the plastic moment (M_p = 207 K-in for f_y = 36 Ksi) of the box section considered. This indicates that the box beam will not buckle in the elastic range. This conclusion had been proven by the tests conducted by Moran [1] at Georgia Institute of Technology.

CHAPTER III

THE SECOND SET OF SOLUTIONS IN THE ELASTIC RANGE

1. An Elastic Solution Considering the Effect of Deflections in the Plane of the Primary Bending Moment

In Appendix B the derivation of Kirchhoff's general equilibrium equations for a bent and twisted beam in space has been shown. These equations are given in the following.

$$
\left\{
\begin{aligned}
&\frac{dV_x}{ds} - V_y \tau_z + F_z K_{zx} + \overline{V_x} = 0 \\
&\frac{dV_y}{ds} + V_x \tau_z - F_z K_{zy} + \overline{V_y} = 0 \\
&\frac{dF_z}{ds} + V_y K_{zy} - V_x K_{zx} + \overline{F_z} = 0 \\
&\frac{dM_x}{ds} - M_y \tau_z + M_z K_{zx} - V_y + \overline{M}_x = 0 \\
&\frac{dM_y}{ds} - M_z K_{zy} + M_x \tau_z + V_x + \overline{M_y} = 0 \\
&\frac{dM_z}{ds} - M_x K_{zx} + M_y K_{zy} + \overline{M_z} = 0
\end{aligned}
\right.
$$

In the above equations there are nine unknown quantities namely V_x, V_y,

F_z, M_x, M_y, M_z, K_{zx}, K_{zy} and τ_z. The first six quantities are the resultant internal forces and moments acting on the face of a cross-section. They are defined by Equation (B-11) of Appendix B. K_{zx} and K_{zy} are the components of curvature of the deformed centroidal axis on ZX plane and ZY plane respectively. τ_z is the twist per unit length of the deformed centroidal axis. $\bar{V}_x$, $\bar{V}_y$, $\bar{F}_z$ are the components of the resultant external force per unit length of the centroidal axis, and $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ are the components of the resultant external moment per unit length of the centroidal axis. If there are provided three additional equations correlating these nine unknown quantities, one then has sufficient number of equations to determine all unknown quantities. In the situation where uniform torsion of the beam exists, one can get a solution by assuming

$$\begin{cases} M_x = E I_x K_{zy} \\ M_y = E I_y K_{zx} \\ M_z = C_1 \tau_z \end{cases} \qquad (3\text{-}1)$$

Since only small deflections are considered, the difference between ds and dz* can be neglected. Hence $\tau_z = \frac{d\phi}{ds} = \frac{d\phi}{dz}$. In the situation where non-uniform torsion of the beam exists, the third of Equations (3-1) is replaced by Equation (2-36):

---

*dz differs from ds only by the square of a small quantity since

$$ds - dz = dz \cdot \sqrt{1 + \left(\frac{dx}{dz}\right)^2} - dz \approx \frac{1}{2}\left(\frac{dx}{dz}\right)^2 \cdot dz$$

$$M_z = C_1 \frac{d\phi}{dz} - C_2 \frac{d^3\phi}{dz^3} + C_3 \frac{d^2 M_z}{dz^2}$$

When a beam is subjected to the action of pure bending moments applied at its ends, the potential energy of the beam continues to build up as the moments increase gradually in magnitude. When the critical value of the moments is reached the potential energy of the beam is at a relative maximum. The equilibrium position of the beam becomes unstable, the beam will tend to assume a new deflected equilibrium position for which the potential energy of the beam is less than that of the relative maximum value. One asks now, when will the beam under the action of pure bending moments buckle out of its vertical equilibrium position? That is, when will such a variation of the condition from $K_{zx} = 0$, $\frac{d\phi}{dz} = 0$ to a condition where K_{zx} and $\frac{d\phi}{dz}$ do not vanish give a possible equilibrium form?

Observing that for pure bending, we have

$$F_z = 0,$$

$$\bar{V}_x = 0,$$

$$\bar{V}_y = 0,$$

$$\bar{F}_z = 0,$$

$$\bar{M}_x = 0,$$

$$\bar{M}_y = 0,$$

$$\bar{M}_z = 0,$$

$$M_x = M$$

So that the variation gives us the following equations:

$$\left\{\begin{aligned}
&\frac{dV_x}{dz} - V_y\frac{d\phi}{dz} = 0\\
&\frac{dV_y}{dz} + V_x\frac{d\phi}{dz} = 0\\
&V_y K_{zy} - V_x K_{zx} = 0\\
&\frac{dM_x}{dz} - M_y\frac{d\phi}{dz} + M_z K_{zx} - V_y = 0\\
&\frac{dM_y}{dz} - M_z K_{zx} + M_x\frac{d\phi}{dz} + V_x = 0\\
&\frac{dM_z}{dz} - M_x K_{zx} + M_y K_{zy} = 0\\
&M_x = EI_x K_{zy}\\
&M_y = EI_y K_{zx}\\
&M_z = C_1\frac{d\phi}{dz} - C_2\frac{d^3\phi}{dz^3} + C_3\frac{d^2M_z}{dz^2}
\end{aligned}\right. \tag{3-2}$$

Eliminating $\frac{d\phi}{dz}$ from the first and second equations,

$$\begin{cases} V_x \dfrac{dV_x}{dz} - V_x V_y \dfrac{d\phi}{dz} = 0 \\ \\ V_y \dfrac{dV_y}{dz} + V_x V_y \dfrac{d\phi}{dz} = 0 \end{cases}$$

$$\frac{1}{2}\frac{d}{dz}(V_x^2) + \frac{1}{2}\frac{d}{dz}(V_y^2) = 0$$

$$\frac{d}{dz}(V_x^2 + V_y^2) = 0$$

Thus $V_x^2 + V_y^2$ = Constant for z = 0 to z = L. Since $V_x^2 + V_y^2 = 0$ at the ends of the beam, it follows that $V_x = V_y = 0$ for z = 0 to z = L. The first three equations of (3-2) are thus identically satisfied. Assuming small deflections, one can write (as discussed in Article 3 of Chapter II),

$$M_x = EI_x K_{zy} = M$$

Using this equation plus the fifth through the ninth equation of (3-2), one obtains the following set of three simultaneous differential equations:

$$\frac{dK_{zx}}{dz} = \frac{M}{(EI_x)(EI_y)} M_z - \frac{M}{EI_y} \cdot \frac{d\phi}{dz} \qquad \text{(3-3)}$$

$$\frac{dM_z}{dz} = M\left(1 - \frac{EI_y}{EI_x}\right) K_{zx} \qquad \text{(3-4)}$$

$$M_z = C_1 \frac{d\phi}{dz} - C_2 \frac{d^3\phi}{dz^3} + C_3 \frac{d^2 M_z}{dz^2} \qquad (3\text{-}5)$$

These are the governing differential equations for the second set of solutions of lateral-torsional buckling of box beams.

Differentiating Equation (3-4) with respect to z, we have

$$\frac{d^2 M_z}{dz^2} = M\left(1 - \frac{EI_y}{EI_x}\right)\frac{dK_{zx}}{dz} \qquad (3\text{-}6)$$

Substituting Equation (3-3) into (3-6),

$$\frac{d^2 M_z}{dz^2} = M\left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M}{EI_x \cdot EI_y}\right)M_z - M\left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M}{EI_y}\right)\frac{d\phi}{dz}$$

$$\frac{d^2 M_z}{dz^2} = \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M^2}{EI_x \cdot EI_y}\right)M_z - \left(1 - \frac{EI_y}{EI_x}\right)\frac{M^2}{EI_x} \cdot \frac{d\phi}{dz} \qquad (3\text{-}7)$$

Substituting Equation (3-7) into (3-5),

$$M_z = C_1 \frac{d\phi}{dz} - C_2 \frac{d^3\phi}{dz^3} + \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{C_3}{EI_x}\right)\left(\frac{M^2}{EI_y}\right)M_z - \left(1 - \frac{EI_y}{EI_x}\right)(C_3)\left(\frac{M^2}{EI_y}\right)\frac{d\phi}{dz}$$

$$\left[1 - \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{C_3}{EI_x}\right)\left(\frac{M^2}{EI_y}\right)\right]M_z = \left[C_1 - \left(1 - \frac{EI_y}{EI_x}\right)(C_3)\left(\frac{M^2}{EI_y}\right)\right]\frac{d\phi}{dz} - C_2 \frac{d^3\phi}{dz^3}$$

The factor $\left[1-\left(1-\frac{E\,I_y}{E\,I_x}\right)\left(\frac{C_3}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)\right]$ does not always equal to zero. Hence,

$$M_z = \frac{\left[C_1-\left(1-\frac{E\,I_y}{E\,I_x}\right)(C_3)\left(\frac{M^2}{E\,I_y}\right)\right]}{\left[1-\left(1-\frac{E\,I_y}{E\,I_x}\right)\left(\frac{C_3}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d\phi}{dz} - \frac{C_2}{\left[1-\left(1-\frac{E\,I_y}{E\,I_x}\right)\left(\frac{C_3}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d^3\phi}{dz^3}$$

Let

$$\begin{cases} A = \dfrac{E\,I_y}{E\,I_x} \\[2ex] B = \dfrac{C_1}{E\,I_x} \\[2ex] C = \dfrac{C_2}{E\,I_x} \\[2ex] D = \dfrac{C_3}{E\,I_x} \end{cases} \qquad \text{———} \quad (3\text{-}8)$$

Note that while the constants A and B are unitless, the unit of C is equal to in^2 and the unit of D is equal to 1/Kips.

Thus,

$$M_z = \frac{\left[C_1-(1-A)(C_3)\left(\frac{M^2}{E\,I_y}\right)\right]}{\left[1-(1-A)(D)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d\phi}{dz} - \frac{C_2}{\left[1-(1-A)(D)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d^3\phi}{dz^3} \qquad \text{———} \quad (3\text{-}9)$$

Substituting Equation (3-9) into (3-7)

$$\frac{\left[C_1-(1-A)(C_3)\left(\frac{M^2}{E\,I_y}\right)\right]}{\left[1-(1-A)(D)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d^3\phi}{dz^3}-\frac{C_2}{\left[1-(1-A)(D)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d^5\phi}{dz^5}=$$

$$(1-A)\left(\frac{1}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)\frac{\left[C_1-(1-A)(C_3)\left(\frac{M^2}{E\,I_y}\right)\right]}{\left[1-(1-A)(D)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d\phi}{dz}-(1-A)\left(\frac{1}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)\frac{C_2}{\left[1-(1-A)(D)\left(\frac{M^2}{E\,I_y}\right)\right]}\cdot\frac{d^3\phi}{dz^3}$$

$$-(1-A)\left(\frac{M^2}{E\,I_y}\right)\frac{d\phi}{dz}$$

$$C_2\frac{d^5\phi}{dz^5}-\left[C_1-(1-A)(C_3)\left(\frac{M^2}{E\,I_y}\right)+(1-A)\left(\frac{C_2}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)\right]\cdot\frac{d^3\phi}{dz^3}$$

$$+\left\{(1-A)\left(\frac{C_1}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)-(1-A)^2\left(\frac{C_3}{E\,I_x}\right)\left(\frac{M^2}{E\,I_y}\right)-(1-A)\left(\frac{M^2}{E\,I_y}\right)\right.$$

$$\left.+(1-A)^2(D)\left(\frac{M^2}{E\,I_y}\right)\right\}\cdot\frac{d\phi}{dz}=0$$

$$C_2\frac{d^5\phi}{dz^5}-\left[C_1-(1-A)(C_3-C)\left(\frac{M^2}{E\,I_y}\right)\right]\frac{d^3\phi}{dz^3}-\left[(1-A)(1-B)\left(\frac{M^2}{E\,I_y}\right)\right]\cdot\frac{d\phi}{dz}=0$$

Since $C_2 \neq 0$, we have

$$\frac{d^5\phi}{dZ^5} - \frac{1}{C_2}\left[C_1 - (1-A)(C_3-C)\left(\frac{M^2}{EI_y}\right)\right]\frac{d^3\phi}{dZ^3} - \left[(1-A)(1-B)\left(\frac{M^2}{EI_y}\right)\right]\left(\frac{1}{C_2}\right)\frac{d\phi}{dZ} = 0 \quad \text{(3-10)}$$

The auxiliary equation of the above differential equation is

$$p^5 - \frac{1}{C_2}\left[C_1 - (1-A)(C_3-C)\left(\frac{M^2}{EI_y}\right)\right]p^3 - \frac{1}{C_2}\left[(1-A)(1-B)\left(\frac{M^2}{EI_y}\right)\right]p = 0$$

$$p\left\{p^4 - \frac{1}{C_2}\left[C_1 - (1-A)(C_3-C)\left(\frac{M^2}{EI_y}\right)\right]p^2 - \frac{1}{C_2}\left[(1-A)(1-B)\left(\frac{M^2}{EI_y}\right)\right]\right\} = 0$$

Assuming that p_1, p_2, p_3 and p_4 are the roots of the second factor of the above equation, then the general solution of Equation (3-10) can be written in the form:

$$\phi = A_1 e^{p_1 Z} + A_2 e^{p_2 Z} + A_3 e^{p_3 Z} + A_4 e^{p_4 Z} + A_5$$

It can be easily seen that the constant A_5 represents a rigid body rotation of the beam. Since we do not consider support rotations or settlements, there can be no rigid body rotation of the beam. Hence $A_5 = 0$, and it follows that the solution of the differential equation

$$\frac{d^4\phi}{dZ^4} - \frac{1}{C_2}\left[C_1 - (1-A)(C_3-C)\left(\frac{M^2}{EI_y}\right)\right]\frac{d^2\phi}{dZ^2} - \frac{1}{C_2}\left[(1-A)(1-B)\left(\frac{M^2}{EI_y}\right)\right]\phi = 0 \quad \text{(3-11)}$$

will be fully sufficient to define the deflected shape of the beam.

It is to be noted that in the case where EI_x of the box beam is very large and that the ratios EI_y/EI_x and C_1/EI_x are small enough to be neglected when compared to unity; and $1/EI_x$ is small enough to be neglected when compared to C_3/C_2, one can write

$$\begin{cases} (1-A) = 1 \\ (1-B) = 1 \\ \left(\frac{C_3}{C_2} - \frac{C}{C_2}\right) = \frac{C_3}{C_2} \end{cases} \qquad \text{(3-12)}$$

Thus Equation (3-11) is reduced to the following form:

$$\frac{d^4\phi}{dz^4} - \frac{1}{C_2}\left[C_1 - C_3 \cdot \frac{M^2}{EI_y}\right]\frac{d^2\phi}{dz^2} - \frac{1}{C_2} \cdot \frac{M^2}{EI_y} \cdot \phi = 0$$

This is exactly the same equation as Equation (2-45), in which the effect of the curvature in the plane of the primary bending moment is not considered.

In most practical cases, EI_x is relatively large and the ratios $\frac{EI_y}{EI_x}$, $\frac{C_1}{EI_x}$, $\frac{1}{EI_x}$ are rather small so that one can make the following assumptions

$$\begin{cases} (1-A) > 0 \\ (1-B) > 0 \\ \frac{C_3}{C_2} - \frac{C}{C_2} = \frac{C_3}{C_2} - \frac{1}{EI_x} > 0 \end{cases} \qquad \text{(3-13)}$$

With these assumptions, one can solve Equation (3-11) following the same procedure as discussed in Article 3 of Chapter II.

Let

$$B_2 = \left| C_2 \right| , \quad B_3 = \left| C_3 \right| \qquad \text{(3-14)}$$

and let C_2 and C_3 be both negative quantities. Hence $C_2 = -B_2$, $C_3 = -B_3$

Equation (3-11) becomes

$$\frac{d^4\phi}{dZ^4} + \frac{1}{B_2}\left[C_1 + (1-A)(B_3 - |c|)\frac{M}{EI_y}\right]\frac{d^2\phi}{dZ^2} + \frac{1}{B_2}\left[(1-A)(1-B)\frac{M^2}{EI_y}\right]\phi = 0$$

Let

$$\begin{cases} \alpha_1' = \dfrac{1}{2B_2}\left[C_1 + (1-A)(B_3 - |c|)\cdot\dfrac{M^2}{EI_y}\right] > 0 \\ \\ \beta_1' = \dfrac{1}{B_2}\left[(1-A)(1-B)\cdot\dfrac{M^2}{EI_y}\right] \end{cases} \qquad \text{(3-15)}$$

we have

$$\frac{d^4\phi}{dZ^4} + 2\alpha_1'\frac{d^2\phi}{dZ^2} + \beta_1'\phi = 0 \qquad \text{(3-16)}$$

The first solution can be obtained by assuming

$$\alpha_1'^2 \geqslant \beta_1'$$

$$\alpha_1'^2 - \beta_1' \geqslant 0$$

Solution for this case leads to Equation (2-52):

$$\beta_1' = -\frac{\pi^4}{L^4} + 2\alpha_1' \frac{\pi^2}{L^2}$$

from which,

$$\frac{1}{B_2}(1-A)(1-B)\frac{M^2}{E I_y} = -\frac{\pi^4}{L^4} + (2)\frac{1}{2B_2}\left[C_1 + (1-A)(B_3-|c|)\frac{M^2}{E I_y}\right]\frac{\pi^2}{L^2}$$

$$\frac{M^2}{E I_y}\left\{\frac{1}{B_2}(1-A)(1-B) - \frac{(B_3-|c|)}{B_2}(1-A)\frac{\pi^2}{L^2}\right\} = \frac{\pi^2}{L^2}\left\{\frac{C_1}{B_2} - \frac{\pi^2}{L^2}\right\}$$

$$M^2 = \left[\frac{\pi^2 E I_y}{L^2}\right] \cdot \frac{\frac{C_1}{B_2} - \frac{\pi^2}{L^2}}{(1-A)\left\{\frac{(1-B)}{B_2} - \frac{(B_3-|c|)}{B_2}\cdot\frac{\pi^2}{L^2}\right\}}$$

Let

$$k_1' = \sqrt{\frac{\frac{\pi^2}{L^2} - \frac{C_1}{B_2}}{\frac{C_1}{B_2}(1-A)\left[(B_3-|c|)\frac{\pi^2}{L^2} - (1-B)\right]}} \qquad (3\text{-}17)$$

we have

$$M_1' = (k_1')\sqrt{\frac{\pi^2 E I_y}{L^2}\cdot C_1} \qquad (3\text{-}18)$$

As it has been discussed in Article 3 of Chapter II, the condition $\alpha_1^2 - \beta_1' < 0$ does not lead to a solution of the critical buckling moment. This is an indication that the box beam is at a state of stable equilibrium.

Another solution for M can be obtained by assuming C_2 and C_3 to be both positive and recognizing the validity of Equations (3-13). Now that

$$C_2 = B_2, \; C_3 = B_3.$$

Equation (3-11) can be rewritten as

$$\frac{d^4\phi}{dz^4} - \frac{1}{B_2}\left[C_1 - (1-A)(B_3-C)\frac{M^2}{EI_y}\right]\frac{d^2\phi}{dz^2} - \frac{1}{B_2}\left[(1-A)(1-B)\frac{M^2}{EI_y}\right]\phi = 0$$

Let

$$\begin{cases} \alpha_2' = \dfrac{1}{2B_2}\left[C_1 - (1-A)(B_3-C)\dfrac{M^2}{EI_y}\right] \\ \\ \beta_2' = \dfrac{1}{B_2}\left[(1-A)(1-B)\dfrac{M^2}{EI_y}\right] > 0 \end{cases} \qquad \text{(3-19)}$$

We have

$$\frac{d^4\phi}{dz^4} - 2\alpha_2'\frac{d^2\phi}{dz^2} - \beta_2'\phi = 0 \qquad \text{(3-20)}$$

The solution of Equation (3-20) leads to Equation (2-69):

$$\beta_2' = (2\alpha_2')\frac{\pi^2}{L^2} + \frac{\pi^4}{L^4}$$

$$\frac{1}{B_2}\left[(1-A)(1-B)\frac{M^2}{EI_y}\right] = \left\{\frac{1}{B_2}\left[C_1 - (1-A)(B_3-C)\frac{M^2}{EI_y}\right]\right\}\frac{\pi^2}{L^2} + \frac{\pi^4}{L^4}$$

$$= \frac{C_1}{B_2}\cdot\frac{\pi^2}{L^2} - \frac{(1-A)(B_3-C)}{B_3}\cdot\frac{M^2}{EI_y}\cdot\frac{\pi^2}{L^2} + \frac{\pi^4}{L^4}$$

$$\frac{M^2}{EI_y}\left\{\frac{(1-A)(1-B)}{B_2} + \frac{(1-A)(B_3-C)}{B_2}\cdot\frac{\pi^2}{L^2}\right\} = \frac{\pi^2}{L^2}\left\{\frac{C_1}{B_2} + \frac{\pi^2}{L^2}\right\}$$

$$M^2 = \left(\frac{\pi^2 E I_y}{L^2}\right)\cdot\frac{\frac{\pi^2}{L^2} + \frac{C_1}{B_2}}{\frac{(1-A)}{B_2}\left[(B_3-C)\frac{\pi^2}{L^2} + (1-B)\right]}$$

Let

$$k'_2 = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{B_2}}{\frac{C_1}{B_2}(1-A)\left[(B_3-C)\frac{\pi^2}{L^2} + (1-B)\right]}} \qquad \text{(3-21)}$$

we have

$$M'_2 = (k'_2)\sqrt{\frac{\pi^2 E I_y}{L^2}\cdot C_1} \qquad \text{(3-22)}$$

Equations (3-21) and (3-22) are applicable when C_2 and C_3 are both positive. Again, Equations (3-18) and (3-22) can be combined into one equation by letting

$$k' = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{C_2}}{\frac{C_1}{C_2}(1-A)\left[(C_3-C)\frac{\pi^2}{L^2} + (1-B)\right]}} \qquad \text{(3-23)}$$

Hence,

$$M' = (k')\sqrt{\frac{\pi^2 E I_y}{L^2} \cdot C_1} \qquad \text{(3-24)}$$

Equations (3-23) and (3-24) define the second solution of the elastic buckling load M', in which the effect of the bending curvature in the ZY pl ane is included. Note that if one observes the validity of Equations (3-12), then Equations (3-23) and (3-24) will be reduced to the form of Equations (2-72) and (2-73), which are the governing equations of the solution in which the effect of the bending curvature in the ZY plane is not considered.

It is to be noted also that if the ratio EI_y/EI_x is approaching to unity, the value of k' and therefore the critical buckling moment M' approach infinity. This means that for a box beam in which $EI_y = EI_x$, lateral torsional buckling will not occur under the action of pure bending moments.

Applying Equations (3-23) and (3-24) to the box beam shown in Figure 1, we have

$$I_x = 13.94 \ (in^4) \ , \quad I_y = 2.08 \ (in^4)$$

$$C_1 = 3.77G \ (K\text{-}in^2) \ , \quad C_2 = -5.12E \ (K\text{-}in^4)$$

$$C_3 = -3.39 \ (in^2) \ , \quad G = E/3 \ , \quad \pi^2/L^2 = 0.00028 \left(\frac{1}{in^2}\right)$$

$$A = \frac{EI_y}{EI_x} = \frac{2.08}{13.94} = 0.149$$

$$B = \frac{C_1}{EI_x} = \frac{3.77G}{13.94E} = (0.27)(\tfrac{1}{3}) = 0.09$$

$$C = \frac{C_2}{EI_x} = \frac{-5.12E}{13.94E} = -0.367 \ (in^2)$$

$$\frac{C_1}{C_2} = \frac{3.77G}{-5.12E} = -(0.738)(\tfrac{1}{3}) = -0.246 \ (\tfrac{1}{in^2})$$

$$k' = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{C_2}}{\frac{C_1}{C_2}(1-A)\left[(C_3 - C)\frac{\pi^2}{L^2} + (1-B)\right]}}$$

$$= \sqrt{\frac{(0.00028 - 0.246)}{(-0.246)(0.851)[-(3.023)(0.00028) + 0.91]}}$$

$$= \sqrt{\frac{-0.24572}{(-0.210)(0.909)}} = \sqrt{\frac{-0.24572}{-0.191}} = 1.135$$

$$M' = (1.135)\sqrt{\frac{\pi^2 E I_y}{L^2} \cdot C_1} = (1.135)(830) = 944 \ (K\text{-}in)$$

Thus a more accurate solution shows that when both the existence of warping resistence and the effect of bending in the Y-Z plane are considered, the critical buckling moment is 13.5% more than that for the case in which the box beam is under uniform torsion. In this particular example, the more accurate critical moment is 944 (K-in) compared to 830 (K-in) as obtained at the end of Article 3 of Chapter II.

CHAPTER IV

LATERAL-TORSIONAL BUCKLING OF BOX BEAMS IN THE INELASTIC RANGE

1. Introduction to the Inelastic Solution

Except for the fact that stress and strain have exceeded the elastic limit in part of the cross-section, the assumptions cited for the elastic solutions are applicable for the inelastic solution. In addition, it shall be assumed that the stress-strain diagram of the material is as shown in Figure 20. This is the typical idealized stress-strain curve adopted in the theoretical works of plastic design of structural steels [3]. The stress-strain curve is assumed to be the same in tension and compression. If the material is loaded beyond the elastic limit and the stress is reversed slightly, the material will respond elastically with a modulus equal to the Young's modulus (E). Although the occurrence of strain hardening is a possibility, it has been shown [8] that it is reasonable to assume that the stresses may nowhere exceed the yield stress f_y. The strain distribution is assumed to be linear across a vertical cross-section of the box beam.

In the classical theory of elastic stability it is assumed that, when the critical load is reached, buckling occurs while the load remains constant. In the inelastic range, however, it is difficult to give a clear cut definition to the critical load. When yielding has occurred under the action of the primary bending

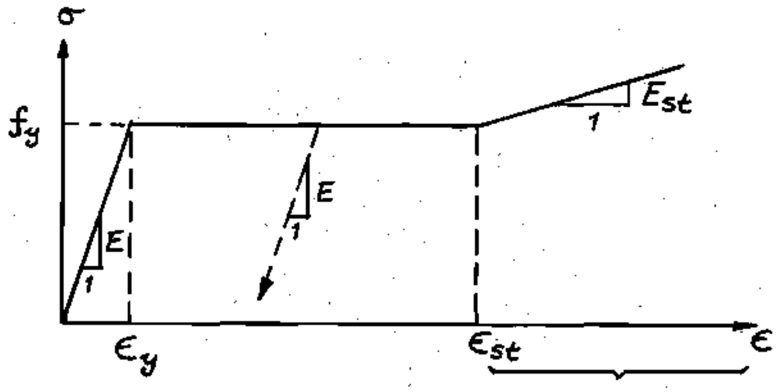
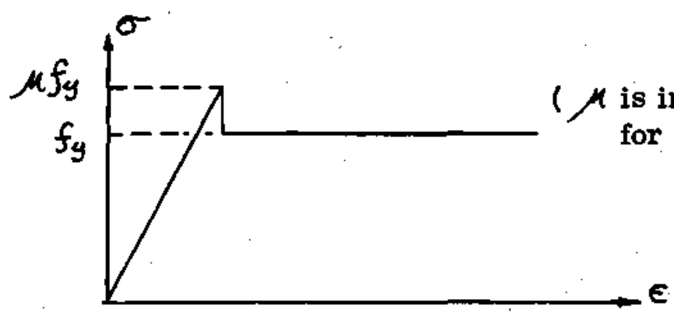


Figure 20. Idealized Stress-Strain Curve.

moment before lateral deflection takes place, one has to determine exactly what is meant by the critical bending moment. In principle, the problem of inelastic buckling of beams is analogous to the problem of inelastic buckling of perfectly straight columns. In the column problem, if it is assumed that the column remains straight until a certain critical load is reached and then buckling occurs under constant load, the critical load is given by the reduced--modulus formula of Von Kármán [39]. Shanley [40] has shown, however, that this is not the smallest load at which the column can assume a deflected position. He stipulates that lateral deflections can occur with increasing load at a value given by the tangent modulus formula of Engesser. He concludes that the true buckling load lies somewhere between the tangent modulus load and the reduced modulus load.

In the problem of lateral-torsional buckling of beams under pure bending, by analogy, one can define a lower-bound critical bending moment at which lateral deflection and twist can occur with increasing bending moment and define an upper-bound critical bending moment at which, if no previous lateral deflection has occurred, the beam can buckle lateral-torsionally under constant bending moment. The true buckling moment will lie somewhere between the lower and upper critical moments.

Neal [5] and Wittrick [6] have shown that for a solid rectangular beam made of annealed mild steel whose stress-strain curve is as shown in Figure 21, the difference between the lower and upper critical pure bending moments is very small, being of the order of 0.5%. Experiments conducted by Neal [5] have verified that the true buckling moments fall very close to the theoretical upper-bound critical moment. Wittrick later demonstrated that Neal's conclusion can hold true only for the particular material which has the kind of stress-strain diagram in which the material exhibits a high ratio of upper to lower yield stress. He has further shown that for material which does not exhibit a pronounced upper yield stress (such as aluminum alloy), the difference between the lower-bound and upper-bound critical moments is quite appreciable (being of the order of 5% for aluminum alloy over a large part of the range considered). It is felt that this is also true



(μ is in the order of 1.4 for annealed mild steel.)

Figure 21. Stress-Strain Curve of Annealed Mild Steel.

for other more complicated sections than the simple rectangular section. Moreover, the lower-bound critical moment gives a slightly conservative estimate of the true buckling moment and thus bears greater practical significance than the upper-bound critical moment. For these reasons, this paper is restricted to the search of a lower-bound solution for the critical buckling moment.

2. An Approximate Lower-Bound Solution of Inelastic Lateral-Torsional Buckling of Box Beams

It can be seen by the works of Wittrick [6] that a more exact lower bound solution (considering unloading of some yielded portion) is rather involved and tedious, and that the resulting accuracy does not warrant the amount of labor. The idea of Figure 22 was first proposed by Galambos [8]. In these figures are shown schematically the load--deflection curves for box beams failing by lateral buckling in the inelastic range. Figure 22a shows two possible deflection configurations into which a box beam under pure bending may deform. In the first possible configuration, the only deformation is the vertical deflection v, as the moment is increased from zero to M_{cr}. If no lateral buckling were to occur, the curve would follow the broken line portion until it would approach the full plastic moment of the cross-section. During the entire course of this curve the beam's vertical axis would remain vertical and parallel to the plane of the primary bending moment. The second possible configuration represents the buckled form of the box beam. The corresponding deformations are the vertical deflection v, the lateral deflection u, and the rotation ϕ. Bifurcation of the equilibrium position occurs at the critical

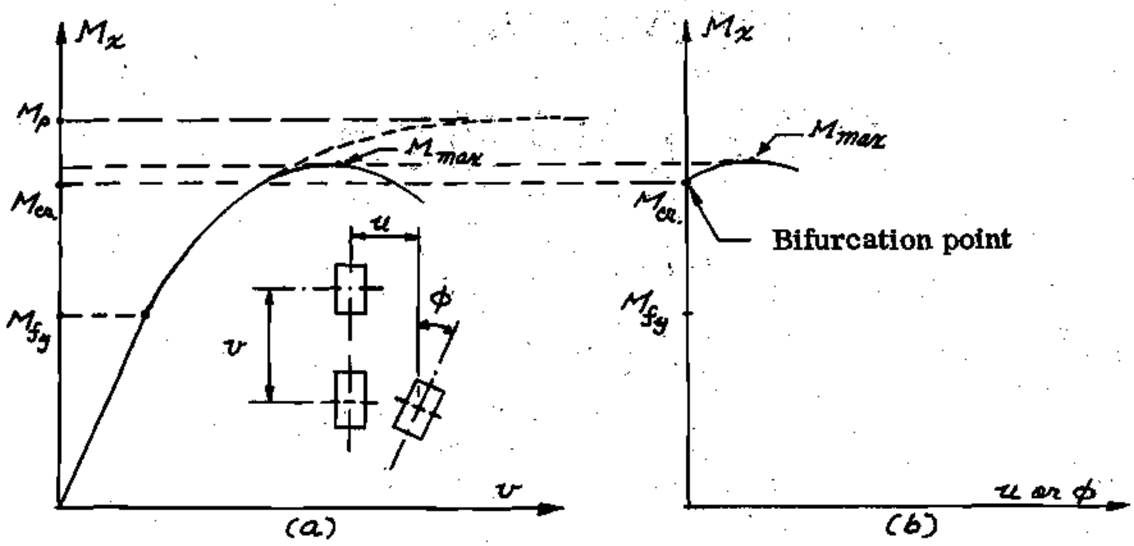


Figure 22. Schematic Moment-Deflection Curves

(Due to Galambos [8]).

moment M_{cr}, in which M_{cr} is above the elastic limit moment M_{fy}. The cross-section moves from its undeflected position to an infinitesimally close buckled position, and the deflection curve deviates from its original course because of lateral buckling. The beam will still be able to sustain a certain amount of increase in moment until the point M_{max} is reached (Figure 22b), after which initiation of unloading indicates failure. In solving the problem of inelastic lateral buckling of wide-flange beams, Galambos [8] has shown that M_{cr} can be used as a lower bound to the maximum moment. This lower bound moment is computed on the basis that, at buckling, no previously yielded fibers will unload elastically, and that additional deformation is resisted by the unyielded elastic core of the cross-section. The critical moment M_{cr} corresponds to the tangent modulus load of axially loaded columns failing in the inelastic range [35]. By an analogy that the tangent modulus load is taken as the critical load for axially loaded column, the moment M_{cr} causing initiation of lateral buckling of a box beam under pure bending is taken as the critical moment at which the moment capacity of the box beam is reached.

It has been demonstrated [5, 6, 8] that with proper modification of the stiffness coefficients (EI_x, EI_y, C_1, C_2 and C_3) the buckling equations for the elastic range can be applicable in the inelastic range. For an approximate lower bound M_{cr}, these coefficients will be modified according to the assumptions cited in the previous paragraph.

The Buckling Equations

For a box beam under pure bending, one uses the more accurate elastic solution which is represented by Equations (3-23) and (3-24), namely,

$$M_{cr} = \sqrt{\frac{\frac{\pi^2}{L^2} + \frac{C_1}{C_2}}{\frac{C_1}{C_2}(1-A)\left[(C_3-C)\frac{\pi^2}{L^2} + (1-B)\right]}} \cdot \sqrt{\frac{\pi^2 E I_y}{L^2} \cdot C_1}$$

By letting

$$\begin{cases} S_x = EI_x, \\ S_y = EI_y, \end{cases}$$

the above equation can be rearranged in the following manner:

$$M_{cr}^2 = \frac{\left(\frac{\pi}{r_y}\right)^2\left(\frac{r_y}{L}\right)^2 + \frac{C_1}{C_2}}{\frac{C_1}{C_2}(1-A)\left[(C_3-C)\left(\frac{\pi}{r_y}\right)^2\left(\frac{r_y}{L}\right)^2 + (1-B)\right]} \cdot \left(\frac{\pi}{r_y}\right)^2\left(\frac{r_y}{L}\right)^2 S_y \cdot C_1$$

$$M_{cr}^2 = \frac{\left[\left(\frac{\pi}{r_y}\right)^2 + \left(\frac{C_1}{C_2}\right)\left(\frac{L}{r_y}\right)^2\right]\left[\left(\frac{\pi}{r_y}\right)^2 S_y \cdot C_1\right]}{\frac{C_1}{C_2}(1-A)\left[(1-B)\left(\frac{L}{r_y}\right)^4 + (C_3-C)\left(\frac{\pi}{r_y}\right)^2\left(\frac{L}{r_y}\right)^2\right]} \qquad \text{(4-1)}$$

in which r_y is the radius of gyration of the full cross-section, taken about the secondary bending axis. Equation (4-1) expresses the elastic critical bending moment interns of the slenderness ratio L/r_y of the box beam. This equation can be further expanded as follows.

$$M_{cr}^2 \cdot \left(\frac{C_1}{C_2}\right)(1-A)(1-B)\left(\frac{L}{r_y}\right)^4 + \left[M_{cr}^2\left(\frac{C_1}{C_2}\right)(1-A)(C_3-C) - \left(\frac{C_1}{C_2}\right) \cdot S_y \cdot C_1\right]\left(\frac{\pi}{r_y}\right)^2\left(\frac{L}{r_y}\right)^2 - S_y \cdot C_1 \cdot \left(\frac{\pi}{r_y}\right)^4 = 0$$

Assuming that $(1-A) \neq 0$ and $(1-B) \neq 0$, we have

$$\left(\frac{L}{r_y}\right)^4 + \left[\frac{(C_3-C)}{(1-B)} - \frac{S_y \cdot C_1}{M_{cr}^2(1-A)(1-B)}\right]\left(\frac{\pi}{r_y}\right)^2\left(\frac{L}{r_y}\right)^2 - \frac{S_y \cdot C_2}{M_{cr}^2(1-A)(1-B)} \cdot \left(\frac{\pi}{r_y}\right)^4 = 0$$

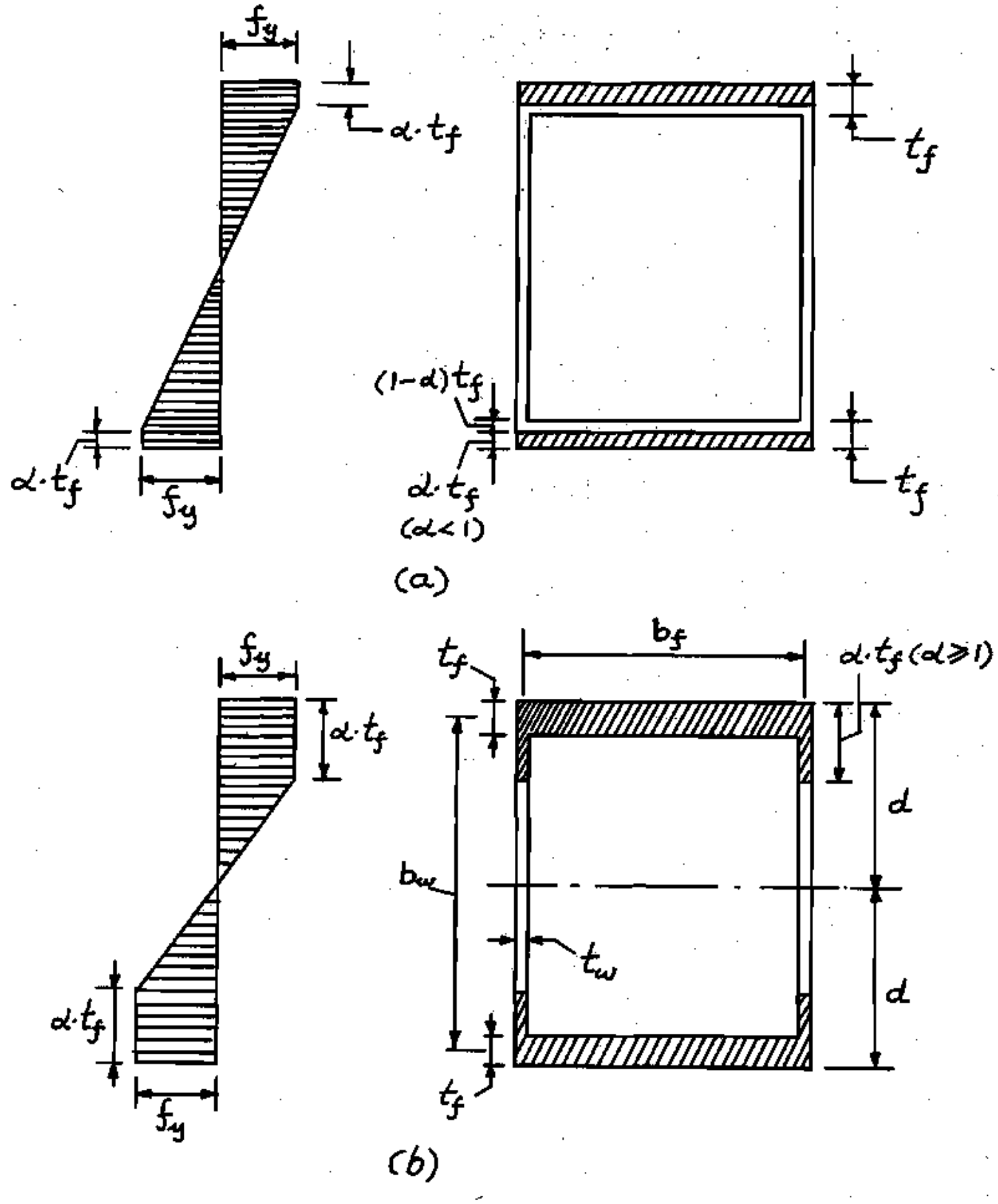


Figure 23. Partially Yielded Cross-Sections of A Box Beam.

Let

$$\begin{cases} p_1 = \left[\dfrac{S_y \cdot C_1}{M_{cr}^2(1-A)(1-B)} - \dfrac{(C_3 - C)}{(1-B)}\right]\left(\dfrac{\pi}{r_y}\right)^2 \\ \\ p_2 = \dfrac{S_y \cdot C_2}{M_{cr}^2(1-A)(1-B)}\left(\dfrac{\pi}{r_y}\right)^4 \end{cases} \qquad \text{(4-2)}$$

we have

$$\left(\frac{L}{r_y}\right)^4 - p_1 \cdot \left(\frac{L}{r_y}\right)^2 - p_2 = 0 \qquad \text{(4-3)}$$

Thus

$$\left(\frac{L}{r_y}\right)_{cr} = \sqrt{\frac{p_1 \pm \sqrt{p_1^2 + 4p_2}}{2}} \qquad \text{(4-4)}$$

The larger root given by Equation (4-4) determines the critical value of L/r_y. With proper modification of the stiffness coefficients, Equation (4-4) can be applicable in the inelastic range.

One must be cautioned in using Equation (4-4) that this equation is only applicable as long as the flanges of the box beam have not fully yielded. This is because that if the flanges are not fully yielded, the remaining elastic core (see unshaded area of Figure 23a) still constitutes a closed box section. The non-uniform torsion differential equation [Equation (2-36)] with which Equation (4-4) is derived, is still applicable if one substitutes $(1-\alpha)\cdot t_f$ for t_f $(\alpha < 1)$ in computing the constants C_2 and C_3. To use Equation (4-4), one would first choose α to be some positive number which is less than 1. Once α is chosen, one determines the stress distribution (see Figure 23a), from which M_{cr} is calculated. Next, modified stiffness coefficients (S_x, S_y, C_1, C_2 and C_3) are computed based on the assumption that resistance to lateral-torsional buckling in the inelastic range is furnished by the unyielded elastic core of the cross-section. Substituting M_{cr} and the modified coefficients into Equations (4-2) and (4-3), we obtain the factors p_1 and p_2, which are then substituted into Equation (4-4) for computing the critical value of L/r_y.

When the flanges of the box beam are fully yielded, the warping resistance of the remaining elastic core is provided by two narrow rectangular sections (see Figure 23b). It is well known that the warping resistance of a narrow rectangular section is equal to zero [4]. The non-uniform torsion differential equation is now given by Equation (2-3), namely

$$M_z = C_1 \frac{d\phi}{dz} \qquad \text{(4-5)}$$

The constant C_1 is the uniform torsion coefficient* of the box section. According to Neal [5], the amount of yielding has no effect on the value of C_1 at the start of lateral buckling, so that the full elastic value of C_1, which is given by Equation (2-4), can be used for substitution into the buckling equations. The buckling problem is now governed by Equations (3-3), (3-4) and (4-5). Neal [4] presented the solution to this problem without a derivation (The derivation of his solution is given in Appendix C).

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*It is also referred to as St. Venant's torsional stiffness.

His solution can be rearranged into the following formulae:

$$M_{cr} = \sqrt{\frac{1}{(1-A)(1-B)}} \cdot \sqrt{\frac{\pi^2 E I_y}{L^2} \cdot C_1} \quad \text{(4-6)}$$

$$\left(\frac{L}{r_y}\right)_{cr} = \frac{\pi}{r_y \cdot M_{cr}} \cdot \sqrt{\frac{\pi^2 E I_y \cdot C_1}{(1-A)(1-B)}} \quad \text{(4-7)}$$

in which A and B are defined by Equations (3-8) and r_y is the radius of gyration of the original unyielded box section about the Y-axis. Equations (4-6) and (4-7) will be applicable once the flanges of the box beam are fully yielded. In using Equation (4-7), one would follow the same procedure as it has been explained in using Equation (4-4).

The Effective Stiffness Coefficients S_x, S_y, C_2, C_3 and C'_2

Once the amount of yielding over the cross-section of a box beam is known (or assumed), the effective value of S_x and S_y can be easily determined according to the shape of the remaining elastic core. If yielding has not yet covered the entire flanges of the box section, C_2 and C_3 can be computed according to the remaining elastic core using Equations (2-33) and (2-34). If yielding has extended beyond the flange-web junctions, then the coefficients C_2 and C_3 are no longer applicable, rather only coefficient C_1 is now useful. In each step, the computed values of these coefficients are substituted back to the correct buckling equations, which are valid in that particular instant, for computing M_{cr} or $(L/r_y)_{cr}$.

The Uniform Torsion Coefficient C_1

Neal [5] has demonstrated an argument theoretically and experimentally, that, if a beam has partially yielded under the action of pure bending moments, the torsional coefficient C_1 remains at its elastic value independent of the amount of yielding. For a complete discussion in this paper, Neal's argument is given as follows.

It has been postulated in the theory of plasticity that each component of strain at any point in a yielded body may be regarded as the sum of an elastic strain which is calculable according to elastic laws and is recoverable upon removal of the load, and a plastic strain which develops under the action of the applied stress system in accordance with certain criteria of plastic flow. When an increment of load is applied, the stress and strain change instantaneously according to the elastic laws, and then the material flows plastically in a manner governed by the changed stress system. It is assumed that the increment of load causes the body to remain yielded, so that the stress components must continue to obey some criterion for yield (for example, the Von Mises Hypothesis of constant elastic shear strain energy). With the above considerations, Hill, Lee and Tupper [41] showed, in discussing the theory of combined elastic and plastic deformation, that the differential relations between shear stress and shear strain in a yielded body are given by the following equations:

$$G\, d\gamma_{xy} = d\tau_{xy} + \tau_{xy} \cdot d\lambda \qquad (4\text{-}8)$$

$$G\,d\gamma_{yz} = d\tau_{yz} + \tau_{yz}\cdot d\lambda \quad \text{———} \quad (4\text{-}9)$$

$$G\,d\gamma_{zx} = d\tau_{zx} + \tau_{zx}\cdot d\lambda \quad \text{———} \quad (4\text{-}10)$$

In Equations (4-8), (4-9) and (4-10), the first term on the right-hand side corresponds to a recoverable elastic strain increment and the second term corresponds to an irrecoverable plastic strain increment. The term $d\lambda$is a non-dimensional proportionality introduced into the equation to express the degree of plastic flow. G is the shear modulus of rigidity.

Conceive now that a prismatic bar partially yielded under the action of pure bending moments, is subjected to a small twisting moment about its longitudinal axis. In the yielded regions the shear stresses τ_{yz} and τ_{zx} are negligibly small, so that we can neglect them from Equations (4-8) and (4-9). Hence

$$\begin{cases} G\,d\gamma_{xy} = d\tau_{xy} & \text{———} \quad (4\text{-}11) \\ G\,d\gamma_{yz} = d\tau_{yz} & \text{———} \quad (4\text{-}12) \end{cases}$$

The equation of equilibrium in the direction of the Z-axis is

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

It can be shown that $\frac{\partial \sigma_z}{\partial z} = 0$, such that

the above equation is reduced to the following form;

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \qquad \text{(4-13)}$$

One can see now that Equations (4-11), (4-12) and (4-13) are exactly the same equations governing the problem of elastic torsion of prismatic bars. It follows therefore that the uniform torsion coefficient C_1 retains its elastic value.

Part of Neal's experimental work involved applying small twisting moments to rectangular mild steel bars, which were already subjected to the action of pure bending moments. Flexural and torsional deformations were recorded at each increment of twisting moment. In two of his tests, the bars were entirely elastic, and in two other tests the bars had been partially yielded before the application of torsional moment. Test results showed that, in all cases, there was no sign of creep in the torsion readings during these tests, and when the applied twisting moments were removed the permanent set in rotational deformation was negligible. This was an indication that no appreciable amount of plastic flow had occurred. The greatest difference of the value of C_1 calculated between any two test results was about 1/3 of one percent. Considering the probability of contributing errors due to random sampling, Neal concluded that there was no reason to suppose that the value of C_1 changed because of partial yielding, and that if in fact there was a change, it was negligibly small.

Neal's work dealt with the problem of lateral-torsional buckling of prismatic bars. This problem is analogous to the problem of lateral-torsional buckling of rectangular box beams. Moreover, the assumptions with which Equations (4-8), (4-9) and (4-10) are derived, apply equally well to the type of stress strain curve

(Figure 20) used in this paper. It is logical, therefore, to incorporate Neal's argument in the inelastic solution of lateral-torsional buckling of rectangular box beams.

In summary, due to the complications created by a partially yielded condition, no simple, explicit equation is obtained for the critical buckling moment in the inelastic range. The method of solution is a numerical one, involving two sets of equations. The first set of equations [Equations (4-1) through (4-4)] is applicable if the flanges of the box beam are not yet fully yielded. The second set of equations [Equations (4-6) and (4-7)] are applicable if yielding has extended beyond the flange-web junctions. Examples of the utilization of these two sets of equations will be discussed in the next Article.

3. Numerical Examples

The M_{cr} versus $(L/r_y)_{cr}$ curves of three particular box sections are illustrated in Figures 24, 25a and 26. The box section shown in Figure 24 is similar to the box section used in Moran's experiments. The $(L/r_y)_{cr}$ ratio for this section at first yield is 1750. As yielding progresses toward the neutral axis of the section, the $(L/r_y)_{cr}$ ratio increases rapidly. This is an indication that the box beam is becoming more stable against lateral-torsional buckling as the moment is increased above the first yield moment value. The box section shown in Figure 25a has about the same area and M_p as that of a W8 x 31 section. The $(L/r_y)_{cr}$ ratio for this box section at first yield is equal to 4000. Again, the $(L/r_y)_{cr}$ ratio increases rapidly as the moment is increased above the first yield moment value.

The buckling curve of a W8 x 31 beam under pure bending is shown in Figure 25b. This curve is taken from the work of Galambos [8] and is shown here for the purpose of comparison with the box beam shown in Figure 25a. From these two figures it can be shown that a box beam is much stronger against lateral-torsional buckling than a wide-flange beam of comparable area and M_p. The box section in Figure 26 is intended to show the exceptional strength a box section may have against lateral-torsional buckling. The I_y/I_x ratio of this section is as small as 2%. The L_{cr}/r_y ratio at first yield is 575. There is a slight decrease of the $(L/r_y)_{cr}$ ratio when the moment is increased above the first yield moment value. But once the moment is increased to more than 0.92 M_p, the $(L/r_y)_{cr}$ ratio increases rapidly again.

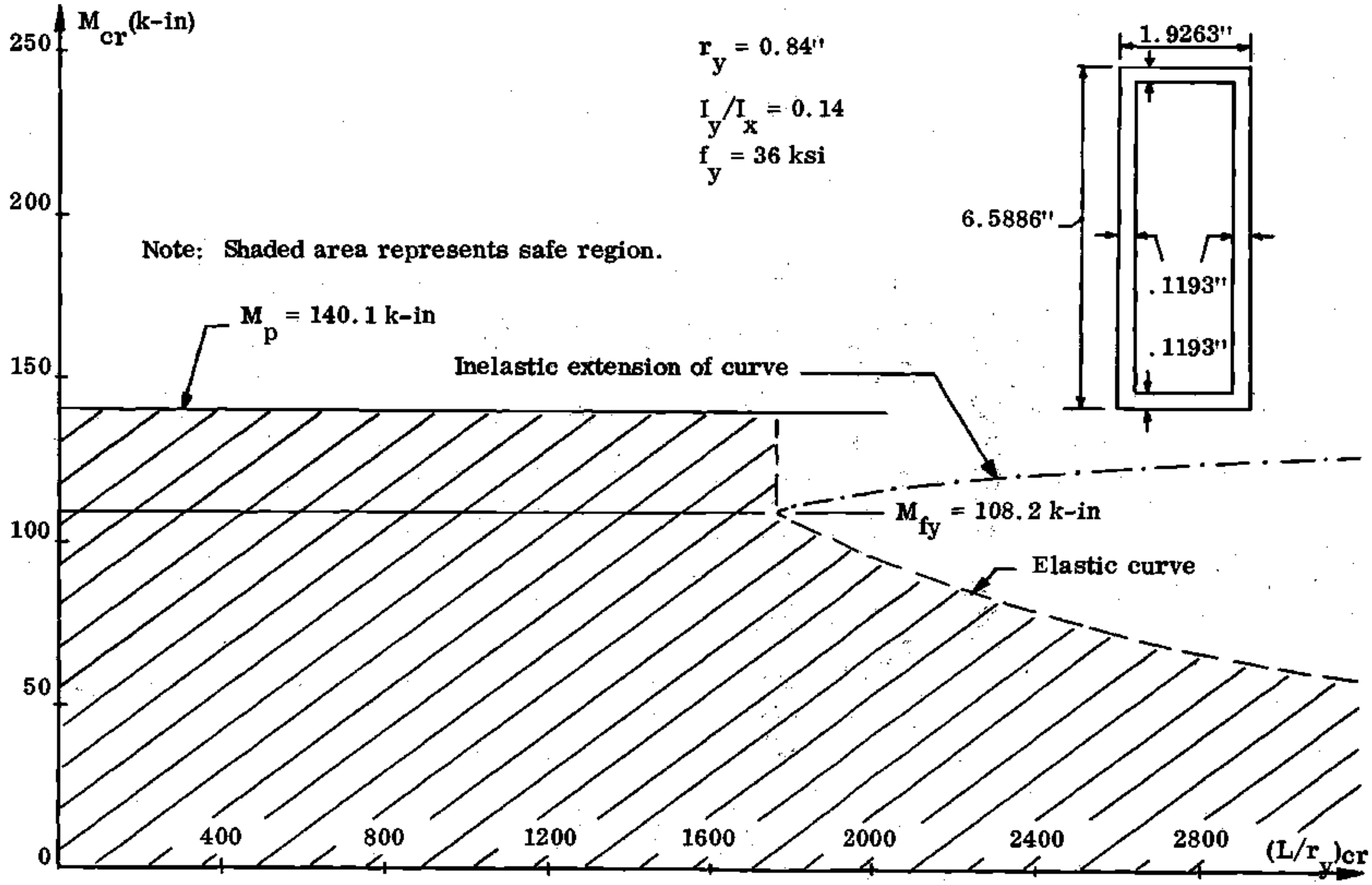

Figure 24. M_{cr} Versus $(L/r_y)_{cr}$ Curve of Box Beam Shown in Upper Right Hand Corner.

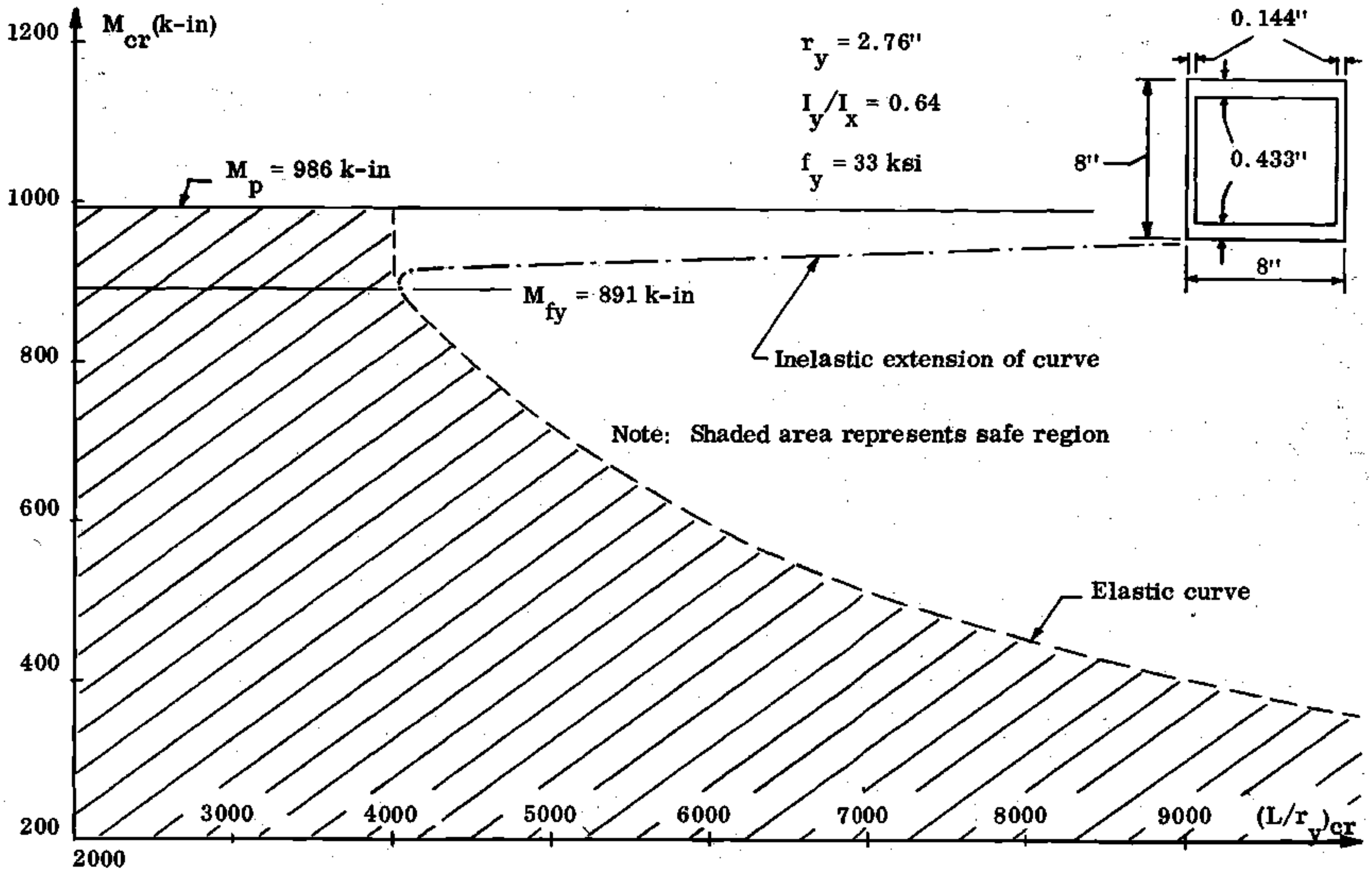

Figure 25a. M_{cr} Versus $(L/r_y)_{cr}$ Curve of Box Beam Shown in Upper Right Hand Corner.

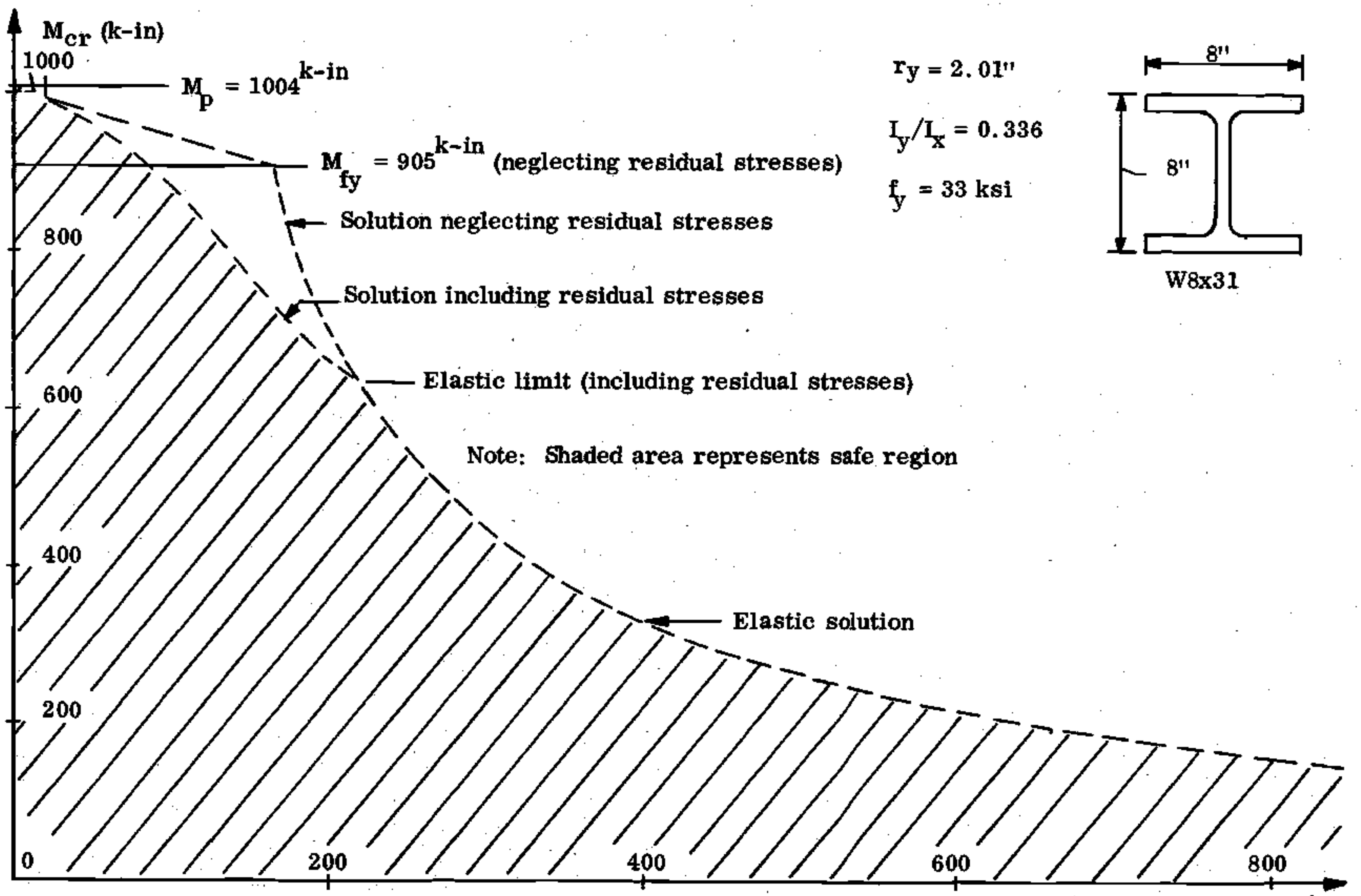

Figure 25b. M_{cr} Versus $(L/r_y)_{cr}$ Curve of A W8x31 Beam Subjected to Pure Bending (Due to Galambos [8]).

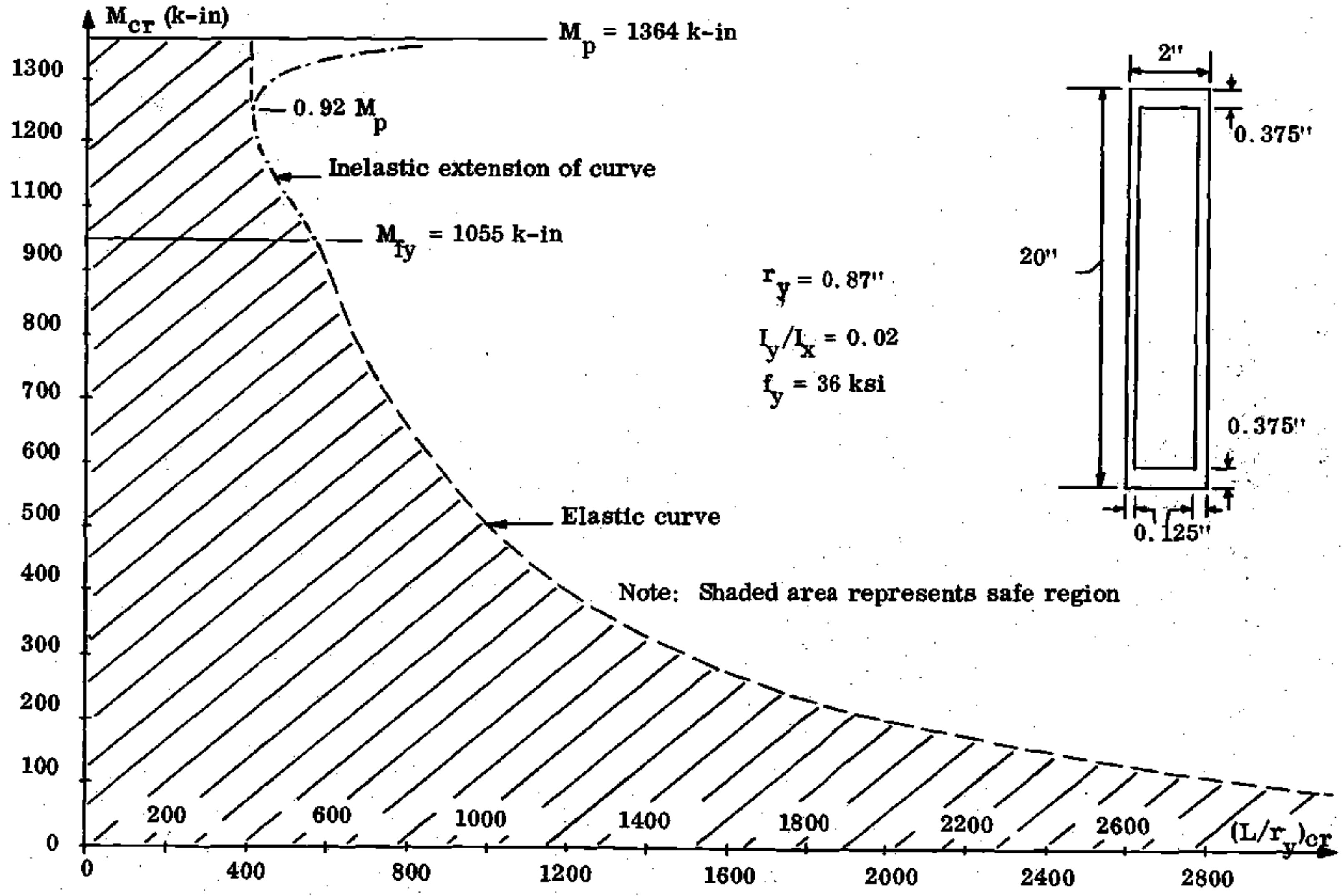

Figure 26. M_{cr} Versus $(L/r_y)_{cr}$ Curve of Box Beam Shown in Upper Right Hand Corner.

CHAPTER V

CONCLUSIONS AND RECOMMENDATION

The problem of lateral torsional buckling of box beams has been solved in this dissertation. Solutions have been obtained both in the elastic and in the inelastic range. A solution in the elastic range is derived neglecting the effect of deflections in the plane of the primary bending moment. This solution is represented by Equations (2-72) and (2-73). A second solution in the elastic range is derived considering the effect of deflections in the plane of the primary bending moment. This solution is represented by Equations (3-23) and (3-24). The solution in the inelastic range is a numerical one, involving two sets of equations. The first set consists of Equation (4-1) through Equation (4-4). These equations are applicable if the flanges of the box section are not yet fully yielded. The second consists of Equations (4-6) and (4-7). This set of equations is applicable if yielding has extended beyond the flange-web junctions. Numerical examples have been given in the form of M_{cr} versus L_{cr}/r_y curves as shown in Figures 24, 25 and 26.

Three conclusions of practical usefulness can be drawn from the work of this dissertation.

(1) A box beam with an unsupported length conceivable in practical applications generally does not buckle lateral-torsionally under the action of pure bending moments. In Figure 26 it is seen that with an I_y/I_x ratio as small as

0.02, the lowest L_{cr}/r_y ratio is equal to 400, corresponding to a buckling moment equal to 0.92 M_p.

(2) Unlike a wide-flange beam, a box beam does not gradually lose its strength against lateral-torsional buckling due to penetration of yielding over the cross-section. A box beam can become more stable against lateral-torsional buckling if the moment is artificially increased above the value at first yield.* An explanation for this phenomenon can be given as follows: It was shown at the end of Article 1 of Chapter III that the lateral stability of a box beam is to a large degree controlled by the I_y/I_x ratio of the section. The I_y/I_x ratio of the elastic core of a partially yielded box beam increases rapidly as yielding spreads gradually from the outer fibers toward the neutral axis of the cross-section. The increased value of the I_y/I_x ratio results in the increase of lateral stability of the box beam in the inelastic range.

Only for box sections with a very small I_y/I_x ratio (Figure 26) does the lateral stability of the section show a small decrease after yielding has begun. This small decrease of lateral stability will be recovered since the I_y/I_x ratio will be on the increase once yielding has penetrated to a certain extent; eventually, the section will gain more stability as the applied moment approaches the plastic moment of the cross-section.

(3) Since lateral-torsional buckling is not likely to occur, the stability of a box beam subject to pure bending is mainly controlled by local buckling of the

---

*Obviously, it will fail in lateral-torsional buckling in the elastic range before any yielding.

compression flange. This conclusion is evident also in the experiments carried out by Moran [1] and Leddick [30]. In practically all of their tests of box beams with unusually long unsupported span lengths, failure of the box beams was initiated by local buckling of the compression flange.

Finally, there are two exceptional cases of box sections governed by the following conditions:

$$b_f t_f - b_w t_w = 0$$

$$b_f t_w - b_w t_f = 0$$

In either case, the torsion of the box section is reduced to a case of uniform torsion. The critical buckling moment is given by Equation (4-6) and the critical value of L_{cr}/r_y is given by Equation (4-7). At the end of Chapter II, it was shown that the effect of warping on the lateral stability of the box beam is very small. The writer recommends that the effect of warping can be neglected in determining the lateral stability of box beams. The result will be conservative.

APPENDIX A

WARPING OF A RECTANGULAR BOX SECTION UNDER THE CONDITION OF UNIFORM TORSION

The problem of uniform torsion of thin-walled rectangular box beams was solved by Williams [20] using the so called semi-inverse method due to St. Venant and Bredt. Timoshenko [19] presented solutions by the same method, in the case of uniform torsion of open sections. The material contained in this Appendix is due to Timoshenko and Williams. It is given here for a complete discussion.

Let it be assumed that the deformations of a twisted rectangular box beam as shown in Figure 5 consists i) of rotations of cross-sections of the beam as in the case of a circular bar under uniform torsion and ii) of warping displacements of the cross-section which are the same for all cross-sections under the condition of uniform torsion.

Taking the origin of coordinates at an end section (Figure 5), the displacements corresponding to rotation of the cross-section are

$$u = -\theta \cdot z \cdot y$$

$$v = \theta \cdot z \cdot x$$

where u is the displacement of a point in the X-direction and v is the displacement of the same point in the Y-direction. $\theta = d\phi/dz$ is the unit twist along the rotation axis Z. The value of θ is a constant in the case of uniform torsion. Thus $\theta \cdot z$

gives the angle of rotation (ϕ) at a distance z from the origin.

It is further assumed that the warping of the cross-section is defined by

$$w = \theta \cdot \varphi(x, y)$$

where w is the warping displacement parallel to the longitudinal axis (Z) of the beam.

Substituting these displacements into the fundamental relationships between strains and displacements [19], we obtain

$$\begin{cases} \epsilon_x = \dfrac{\partial u}{\partial x} = 0 \\[2ex] \epsilon_y = \dfrac{\partial v}{\partial y} = 0 \\[2ex] \epsilon_z = \dfrac{\partial w}{\partial z} = 0 \\[2ex] \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \\[2ex] \gamma_{xz} = \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} = \theta\left[\dfrac{\partial \varphi}{\partial x} - y\right] \\[2ex] \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} = \theta\left[\dfrac{\partial \varphi}{\partial y} + x\right] \end{cases}$$

since

$$
\begin{cases}
\sigma_x = \dfrac{\nu E}{(1+\nu)(1-2\nu)} e + \dfrac{E}{1+\nu} \epsilon_x \\[2ex]
\sigma_y = \dfrac{\nu E}{(1+\nu)(1-2\nu)} e + \dfrac{E}{1+\nu} \epsilon_y \\[2ex]
\sigma_z = \dfrac{\nu E}{(1+\nu)(1-2\nu)} e + \dfrac{E}{1+\nu} \epsilon_z \\[2ex]
\tau_{xy} = G \gamma_{xy} \\[2ex]
\tau_{yz} = G \gamma_{yz} \\[2ex]
\tau_{zx} = G \gamma_{zx}
\end{cases}
$$

where $\quad e = \epsilon_x + \epsilon_y + \epsilon_z$

we have,

$$
\begin{cases}
\sigma_x = 0 \\[2ex]
\sigma_y = 0 \\[2ex]
\sigma_z = 0 \\[2ex]
\tau_{xy} = 0 \\[2ex]
\tau_{yz} = G\theta \left[\dfrac{\partial \varphi}{\partial y} + x \right] \\[2ex]
\tau_{zx} = G\theta \left[\dfrac{\partial \varphi}{\partial x} - y \right]
\end{cases}
$$

The equation of equilibrium without body force can be written as

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{xz}}{\partial z} = 0 \\[2ex] \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial z} = 0 \\[2ex] \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = 0 \end{cases}$$

Substituting with the expressions for stresses, it can be shown that the first and second of the equilibrium equations are indentically satisfied and the third equation becomes

$$G\theta \frac{\partial^2 \varphi}{\partial x^2} + G\theta \frac{\partial^2 \varphi}{\partial y^2} = 0$$

Since θ does not vanish for the entire length of the beam, we have

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \text{———————} \qquad \text{(A-1)}$$

Thus the function φ with which one defined the warping displacement must satisfy Equation (A-1).

The boundary conditions under the condition of no body force can be written as follows.

$$\begin{cases} \sigma_x \cos(NX) + \tau_{xy} \cos(NY) + \tau_{xz} \cos(NZ) = 0 \\ \sigma_y \cos(NY) + \tau_{yz} \cos(NZ) + \tau_{xy} \cos(NX) = 0. \\ \sigma_z \cos(NZ) + \tau_{xz} \cos(NX) + \tau_{yz} \cos(NY) = 0 \end{cases}$$

where N represents the normal direction at a point on the boundary (Figure 27).

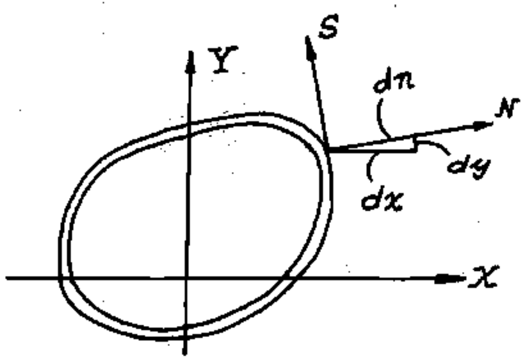

Figure 27. Relationship Between the Normal and the Rectangular Coordinates at the Boundary.

The direction cosines of N are given by

$$\begin{cases} \cos(NX) = \dfrac{dx}{dn} \\ \cos(NY) = \dfrac{dy}{dn} \\ \cos(NZ) = \dfrac{dz}{dn} = 0 \end{cases}$$

Substituting these values into the boundary equations, one finds that the first two of the boundary conditions are identically zero, and the third gives

$$G\theta\left[\frac{\partial\varphi}{\partial x} - y\right]\cos(NX) + G\theta\left[\frac{\partial\varphi}{\partial y} + x\right]\cos(NY) = 0$$

$$\frac{\partial\varphi}{\partial x}\cos(NX) + \frac{\partial\varphi}{\partial y}\cos(NY) = y\cdot\cos(NX) - x\cdot\cos(NY)$$

Since
$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial x}{\partial n} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial y}{\partial n}$$

$$= \frac{\partial\varphi}{\partial x}\cos(NX) + \frac{\partial\varphi}{\partial y}\cos(NY)$$

the boundary condition remains

$$\frac{\partial\varphi}{\partial n} = y\cdot\cos(NX) - x\cdot\cos(NY) \qquad \text{(A-2)}$$

Thus the problem of torsion reduces to the problem of finding a function φ satisfying Equation (A-1) and the boundary condition (A-2). In the case of a thin-walled hollow cylinder the variation of the warping displacement w across the thickness of the wall can be neglected. Only its variation along the wall in the s-direction needs to be considered, as shown in Figure 28. From the definition introduced for warping displacements, we have

$$\frac{\partial w}{\partial s} = \theta\cdot\frac{\partial\varphi}{\partial s}$$

Consider the difference of the warping displacements between any two points Q and R in the cross-section, we have

$$W_R - W_Q = \int_Q^R \frac{\partial w}{\partial s}\, ds = \theta \int_Q^R \frac{\partial \varphi}{\partial s}\, ds \quad \text{———} \quad (A-3)$$

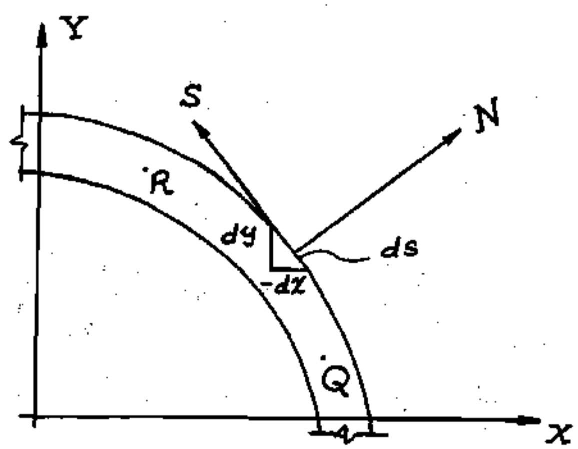

Figure 28. Relationship Between Arc Length and the Rectangular Coordinates at the Boundary.

From equation (A-1) one finds that $\varphi(x, y)$ has continuous second derivations and satisfies the Laplace equation. It follows that the conjugate function $\psi(x, y)$ of $\varphi(x,y)$ exists and also satisfies the Laplace equation and that $\psi(x, y)$ is given by the Cauchy-Riemann equations [31]. Thus

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \psi}{\partial y} \\[2ex] \dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

The condition of $\psi(x,y)$ on the boundary is determined as follows. From Figure 28, we have

$$\begin{cases} \cos(NX) = \dfrac{dy}{ds} \\[2ex] \cos(NY) = -\dfrac{dx}{ds} \end{cases}$$

Comparing these results with those obtained with respect to Figure 27, we have

$$\begin{cases} \dfrac{dx}{dn} = \dfrac{dy}{ds} \\[2ex] \dfrac{dy}{dn} = -\dfrac{dx}{ds} \end{cases}$$

Since

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial x}{\partial n} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial y}{\partial n} = \frac{\partial\psi}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial\psi}{\partial x}\cdot\frac{\partial x}{\partial s}$$

we have

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\psi}{\partial s}$$

Similarly,

$$\frac{\partial\varphi}{\partial s} = \frac{\partial\varphi}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= -\frac{\partial\psi}{\partial y}\cdot\frac{\partial y}{\partial n} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial x}{\partial n}$$

$$\frac{\partial\varphi}{\partial s} = -\frac{\partial\psi}{\partial n}$$

Furthermore, from Equation (A-2), we have

$$\frac{\partial \psi}{\partial s} = y \cdot \frac{dy}{ds} + x \cdot \frac{dx}{ds}$$

$$\partial \psi = y \cdot dy + x \cdot dx$$

$$\psi - \frac{1}{2}(x^2 + y^2) = \text{Constant} \qquad \text{(A-4)}$$

Equation (A-4) must hold true at all points along the boundary.

Let

$$\Psi = \psi - \frac{1}{2}(x^2 + y^2) \qquad \text{(A-5)}$$

then Ψ must be constant along the boundary.

Differentiating Equation (A-5) by n, and with reference to Figure 29, we have

$$\frac{\partial \psi}{\partial n} = \frac{\partial \Psi}{\partial n} + \frac{1}{2} \cdot \frac{\partial}{\partial n}(r^2)$$

$$= \frac{\partial \Psi}{\partial n} + r \cdot \frac{\partial r}{\partial n} = \frac{\partial \Psi}{\partial n} + r \cdot \sin \beta$$

$$= \frac{\partial \Psi}{\partial n} + p$$

where r and p are the respective distances of the origin O* from any point P in the boundary and of the origin O* from the tangent to the boundary at P; β is then the angle between OP and the tangent at P (Figure 29). From Equation (A-3),

*The origin O shall be chosen coincident with the shear center of the cross-section.

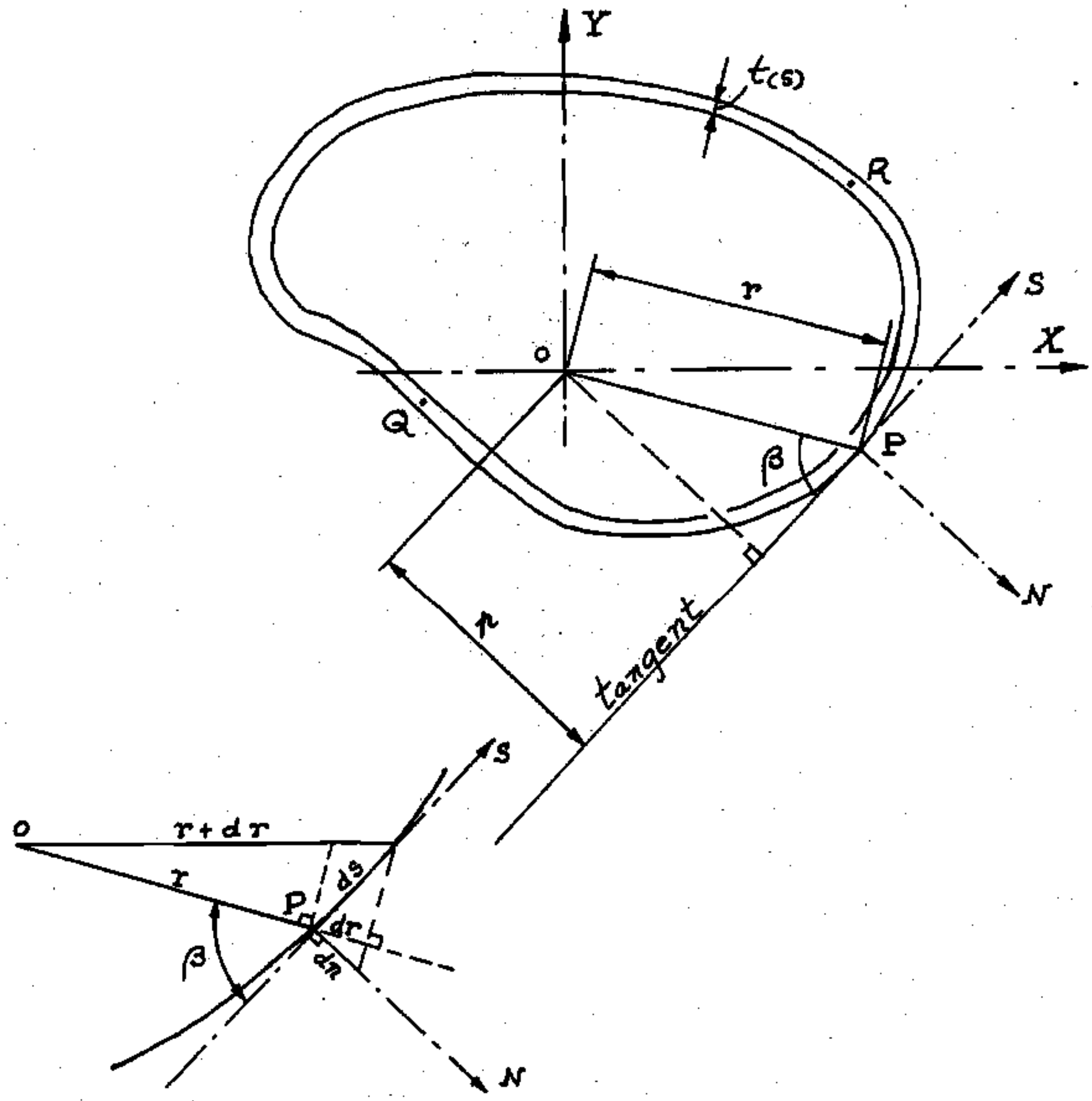

Figure 29. Defining the Relative Position of

A Point at the Boundary.

we have

$$W_R - W_Q = - \theta \int_Q^R \frac{\partial \psi}{\partial n} ds$$

$$W_R - W_Q = -\theta \int_Q^R \left(\frac{\partial \Psi}{\partial n} + p \right) ds \qquad \text{(A-6)}$$

For a multiply connected region with i closed boundries, the function Ψ must be constant at each boundary. Thus

$$\Psi_i = k_i$$

where k_i is the constant value of the function Ψ at the ith boundary. In the case of a box section as shown in Figure 6, we have

$$\begin{cases} \Psi_0 = k_0 \\ \Psi_1 = k_1 \end{cases}$$

where Ψ_0 is the value at the inner boundary and Ψ_1 is the value at the outer boundary. k_0 and k_1 are some constants. Now that the wall of the box section is thin one can arbitrarily choose $\Psi_0 = 0$, and approximately express

$$\frac{\partial \Psi}{\partial n} = \frac{\Psi_1}{t} \qquad \text{(A-7)}$$

This is legitimate as only the quantity $\frac{\partial \Psi}{\partial n}$ is important. Thus

$$W_R - W_Q = -\theta \left[\Psi_1 \int_Q^R \frac{ds}{t} + \int_Q^R p \cdot ds \right]$$

$$W_R - W_Q = -\theta \left[\Psi_1 \int_Q^R \frac{ds}{t} + 2A' \right] \qquad \text{(A-9)}$$

in which A' is the area included between the periphery QR as defined by the mean curve between the inner and outer boundaries and the terminal radii.

To determine the value Ψ_1, one finds that on integrating completely round the boundary,

$$w_R - w_Q = 0 = -\theta \oint \frac{\partial \psi}{\partial n} \cdot ds$$

where $\oint$ represents the integral along a closed boundary. Since θ is not identically zero, we have

$$\oint \frac{\partial \psi}{\partial n} \cdot ds = 0$$

$$\oint [\Psi_1 \cdot \frac{ds}{t} + \rho \cdot ds] = 0$$

$$\Psi_1 \oint \frac{ds}{t} + 2A = 0$$

$$\Psi_1 = -\frac{2A}{\oint \frac{ds}{t}} \qquad \text{(A-10)}$$

where A is total area enclosed by the box periphery. Applying Equation (A-10) to the box section as shown in Figure 6, we have

$$\Psi_1 = -\frac{2b_w b_f}{\frac{b_f}{t_f} + \frac{b_w}{t_w} + \frac{b_f}{t_f} + \frac{b_w}{t_w}} = -\frac{b_w b_f t_w t_f}{(b_w t_f + b_f t_w)} \qquad \text{(A-11)}$$

And considering the difference of warping displacements between points D and E in Figure 6, we have, when using Equations (A-6) and (A-7) and recognizing that s is negative in passing from E to D,

$$W_D - W_E = -\theta \int_E^D \left(\frac{\partial \Psi}{\partial n} + p \right)(-ds)$$

$$W_D - W_E = \theta \left[\Psi_1 \cdot \frac{\left(\frac{b_f}{2}\right)}{t_f} + \left(\frac{b_w}{2}\right)\left(\frac{b_f}{2}\right) \right]$$

Thus

$$W_D - W_E = \theta \cdot \left[\frac{b_w b_f (b_w t_f - b_f t_w)}{4(b_w t_f + b_f t_w)} \right] \text{—————} \quad \text{(A-12)}^*$$

Similarly,

$$W_H - W_D = \theta \cdot \left[\Psi_1 \frac{\left(\frac{b_w}{2}\right)}{t_w} + \left(\frac{b_f}{2}\right)\left(\frac{b_w}{2}\right) \right]$$

$$W_H - W_D = \theta \cdot \left[\frac{-b_w b_f (b_w t_f - b_f t_w)}{4(b_w t_f + b_f t_w)} \right] \text{—————} \quad \text{(A-13)}^*$$

Adding Equations (A-12) and (A-13) we obtain

$$W_H - W_E = 0$$

So that points H and E and therefore also points F and G remain in the same plane which is perpendicular to the longitudinal axis Z of the box beam. It is also evident from Equations (A-8) and (A-11) that the warping displacement varies

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*Equations (A-12) and (A-13) were obtained by Williams [20].

linearly from E to D and from D to H, etc. One can now conclude that points D and C are displaced out of the plane of the cross-section by an equal and opposite amount of warping displacement. By symmetry, one can conclude that points B and A have identical displacements as points D and C respectively. In Figure 30 is shown a schematic picture of the relative warping displacements of a box section under uniform torsion (assuming that $b_w t_f > b_f t_w$). The rotational displacement of the box section is omitted from the figure.

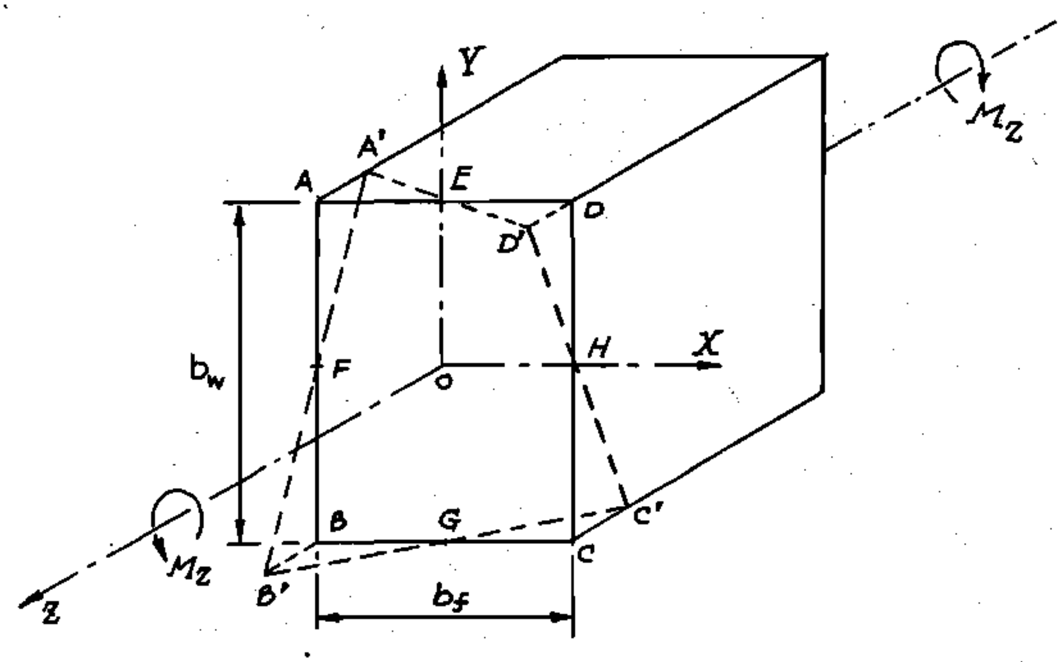


Figure 30. Warping Displacements of A Box Beam

Due to Uniform Torsion ($b_w t_f > b_f t_w$).

It is of interest to point out a special case of torsion of box beams in which

$b_w t_f = b_f t_w$. It is seen from Equations (A-12) and (A-13) that under this condition we shall obtain the following results:

$$W_D - W_E = 0$$

and

$$W_H - W_D = 0$$

Thus points H, D and E (Figure 10) remain in the same plane as they were before loading. Since warping displacement varies linearly from H to D and from D to E, the entire quadrant of the box beam therefore remains in the same plane. This result is also obtained for the other three quadrants. One can conclude therefore, that for box beams with $b_w t_f = b_f t_w$ there is no warping of the cross-section under torsion i.e. plane cross-sections remain plane before and after the application of torsional moment. The reason that warping of the cross-section does not occur is that the shear deformations in the planes of the four plates are equal to each other. The possibility that this case of torsion may occur has been cited in the discussions of Article 2 of Chapter I (Figure 3).

APPENDIX B

DERIVATION OF KIRCHHOFF'S EQUILIBRIUM EQUATIONS OF A BENT AND TWISTED BEAM

In finding a solution of lateral-torsional buckling of box beams considering the effect of deflections in the plane of the primary bending moment, one has to make use of Kirchhoff's [38, 39] equilibrium equations of a bent and twisted beam in space. Although Kirchhoff's equations have been known, due to reasons of notations and sign conventions and for the purpose of a complete discussion, the derivation of these equations is given in this appendix.

To derive Kirchhoff's equations, one has to establish the relationships between the components of curvature of a space curve with respect to a set of moving axes whose origin moves along the space curve at unit velocity (Figure 34). Consider a space curve as shown in Figure 31 at any point A, one can construct the Unit Tangent Vector $\vec{u}$.

$$\vec{u} = \frac{d\vec{r}}{ds}$$

where $\vec{r}$ represents the directional vector of the space curve and s represents the arc length of the space curve. The curvature K of the space curve at any point A is a function of the arc length s and is defined by

$$K \cdot \vec{p} = \frac{d\vec{u}}{ds}$$

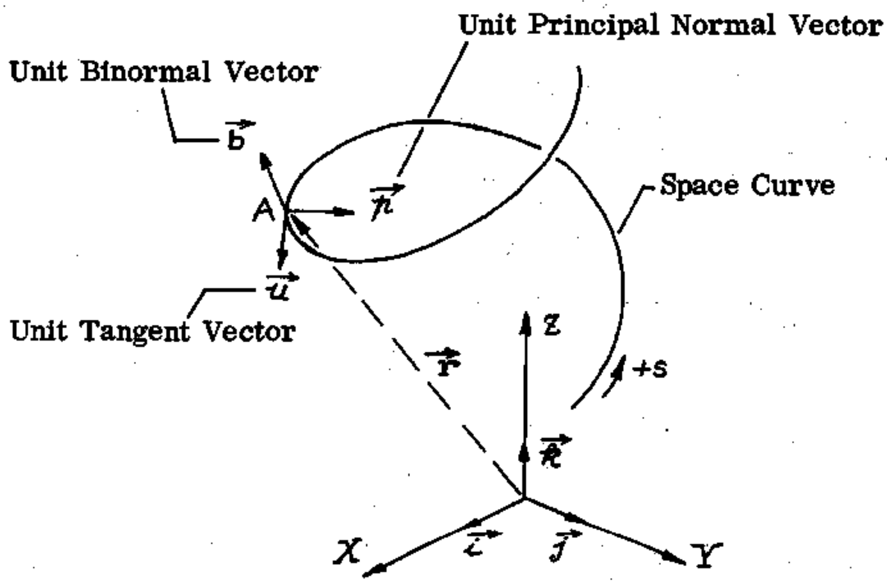


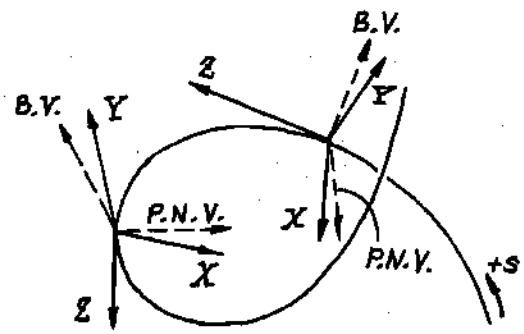
Figure 31. Coordinate System for A Space Curve.

where $\vec{p}$ is the Unit Principal Normal Vector at point A. By this definition one can see that the magnitude of the Principal Normal Vector at A is equal to the curvature of the space curve at A. If $K \neq 0$, $\vec{u}$ and $d\vec{u}/ds$ are two orthogonal vectors. The positive direction of the Principal Normal Vector is taken to be pointing toward the center of curvature of the space curve. With respect to the Unit Tangent Vector $\vec{u}$ and the Unit Principal Normal Vector $\vec{p}$, one can define the Unit Binormal Vector $\vec{b}$ at any point on the space curve as

$$\vec{b} = \vec{u} \times \vec{p}$$

such that the vectors $\vec{u}$, $\vec{p}$ and $\vec{b}$ constitute a right-handed triple of orthogonal unit vectors (Figure 31).

Since the Principal Normal Vector of a space curve is not always easily defined, one shall consider the curvature of a space curve in a different manner. Consider now that the origin of a set of three orthogonal axes X, Y, Z is moving along a space curve at unit velocity. Suppose that the Z-axis is always tangent to the curve but the other two axes X and Y may or may not coincide with the directions of the Principal Normal Vector and the Binormal Vector respectively (Figure 32). Then across an infinitesimal arc length ds, the instantaneous angular speed ($d\phi_y/ds$) with which the ZX plane rotates about the Y-axis gives the component of curvature of the space curve on the ZX plane. This component of curvature can be represented by the magnitude of a velocity vector* which is pointing in the positive direction of the Y-axis. In the case of a plane curve, for instance, if one



B. V. = Binormal Vector

P. N. V. = Principal Normal Vector

Figure 32. Relationship Between Unit Vectors and Moving Axes on A Space Curve.

*It is a velocity vector which defines the instantaneous rate with which the ZX plane is rotating about the Y-axis.

considers that the Z-axis and the X-axis are chosen to be coincident with the Tangent Vector and the Principal Normal Vector respectively, then this velocity vector acts in the direction of the Binormal Vector (Figure 33). Similarly, the instantaneous angular speed ($d\phi_x/ds$) with which the ZY plane rotates about the X-axis gives the component of curvature of the space curve on the ZY plane. This component of curvature can be represented by the magnitude of a velocity vector acting in the positive direction of the X-axis. The resultant of these two vectors gives the curvature of the space curve. Notice that the resultant vector acts in the direction of the Binormal Vector of the space curve at the point of discussion. It is also to be noted that as the set of axes X, Y, Z moves along the space curve in unit velocity, the XY plane is rotating about the Z-axis. The instantaneous speed of this rotation ($d\phi_z/ds$) is called the twist of the curve.

Next, one shall establish the kinematic relations among the components of curvature and the twist of the space curve. Let X, Y, Z be a set of orthogonal

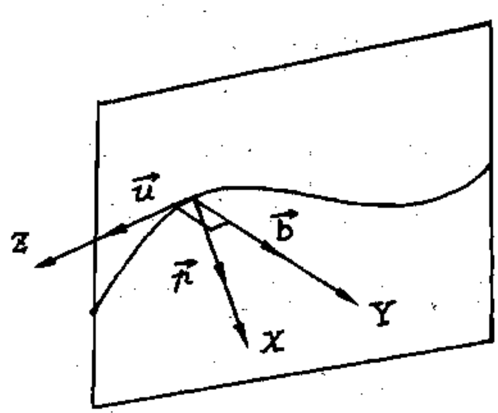


Figure 33. Coordinate System for A Plane Curve.

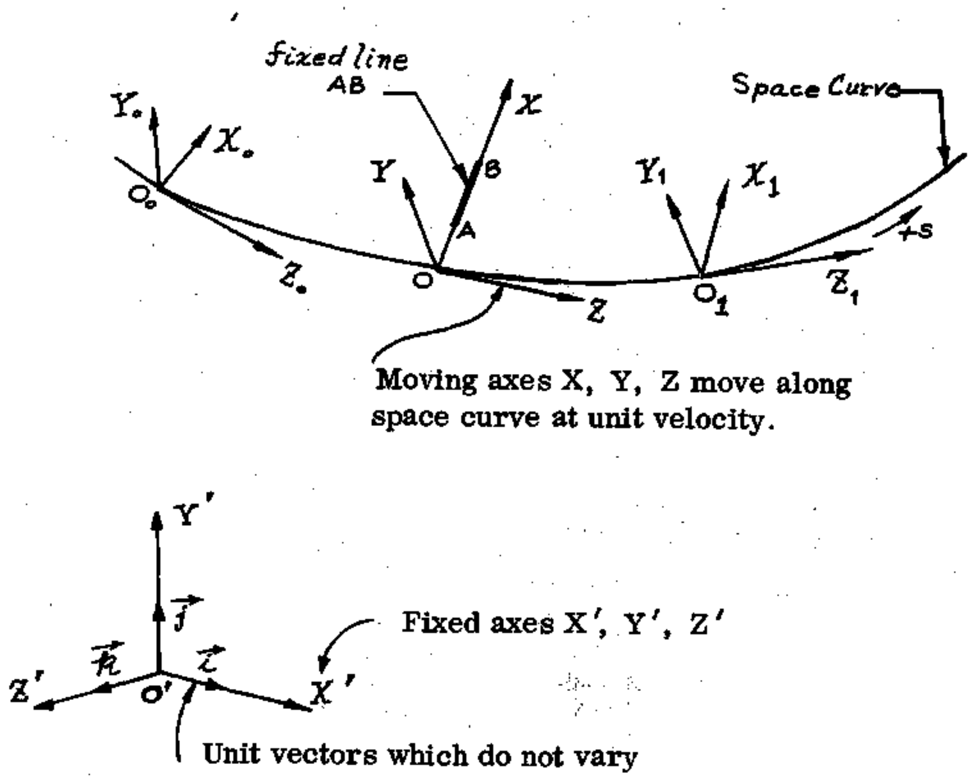


Figure 34. Relationship Between A Set of Fixed Axes in Space and A Set of Moving Axes on A Space Curve.

axes whose origin moves along the space curve at unit velocity. The axis Z is always tangent to the space curve. The other two axes X and Y may or may not coincide with the Principal Normal Vector and Binormal Vector respectively (Figure 34). At point O_1 on the space curve, the set of moving axes X, Y, Z may have taken up the positions X_1, Y_1, Z_1. But at an arbitrary point O, the positions of the axes are designated by X, Y, Z. Let $\vec{R}$ be any vector in space and $\vec{u}$, $\vec{v}$, $\vec{w}$ be its components projected on the moving axes X, Y, Z respectively.

Let AB be any straight line fixed in space and that the direction cosines of AB with respect to the moving axes X, Y, Z at any point O can be designated by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively. Let $\vec{R}_1$ be the projection of $\vec{R}$ on AB. Hence,

$$\vec{R}_1 = \vec{u}\cos\alpha + \vec{v}\cos\beta + \vec{w}\cos\gamma$$

As the set of moving axes moves along the space curve at unit velocity, the positions of the moving axes have changed. The space vector $\vec{R}$ may have also varied. Differentiating $\vec{R}_1$ with respect to s, we have

$$\frac{d\vec{R}_1}{ds} = \frac{d\vec{u}}{ds}\cos\alpha + \frac{d\vec{v}}{ds}\cos\beta + \frac{d\vec{w}}{ds}\cos\gamma$$

$$- \vec{u}\cdot\sin\alpha\cdot\frac{d\alpha}{ds} - \vec{v}\cdot\sin\beta\cdot\frac{d\beta}{ds} - \vec{w}\cdot\sin\gamma\cdot\frac{d\gamma}{ds} \qquad \text{(B-1)}$$

Since AB is any fixed line in space, let it be so chosen that the moving axis OX coincides with it at point O. Thus

$$\begin{cases} \alpha = 0 \\ \beta = \pi/2 \\ \gamma = \pi/2 \end{cases} \qquad \text{(B-2)}$$

β is the angle between AB and OY of the moving axes. $d\beta/ds$ expresses the angular speed with which the XY plane rotates about the Z axis as the set of moving axes moves along the space curve at unit velocity. Using the symbol ϕ for angle of rotation as in the previous discussions, we can replace the term $\frac{d\beta}{ds}$ by $\frac{d\phi_z}{ds}$.

Similarly, the term $\frac{d\gamma}{ds}$ expresses the angular speed with which the ZX plane is rotating about the Y axis and can be replaced by $-\frac{d\phi_y}{ds}$. The negative sign was used because this rotation diminishes the angle between the fixed line AB and the moving axis OY. Substituting this new term and Equations (B-2) into Equation (B-1), we have

$$\frac{d\vec{R}_x}{ds} = \frac{d\vec{u}}{ds} - \vec{v}\cdot\frac{d\phi_z}{ds} + \vec{w}\cdot\frac{d\phi_y}{ds} \tag{B-3}$$

Two more equations similar to Equation (B-3) can be obtained if we choose the fixed line AB to coincide with axes OY and OZ respectively. We have

$$\left\{\begin{aligned} \frac{d\vec{R}_x}{ds} &= \frac{d\vec{u}}{ds} - \vec{v}\cdot\frac{d\phi_z}{ds} + \vec{w}\cdot\frac{d\phi_y}{ds} \\ \frac{d\vec{R}_y}{ds} &= \frac{d\vec{v}}{ds} - \vec{w}\cdot\frac{d\phi_x}{ds} + \vec{u}\cdot\frac{d\phi_z}{ds} \\ \frac{d\vec{R}_z}{ds} &= \frac{d\vec{w}}{ds} - \vec{u}\cdot\frac{d\phi_y}{ds} + \vec{v}\cdot\frac{d\phi_x}{ds} \end{aligned}\right. \tag{B-4}$$

It has been explained in the previous discussions that $d\phi_y/ds$ represents the component of curvature of the space curve projected on the ZX plane. Let this component of curvature be denoted by K_{zx}. Similarly, let the component of curvature on the ZY plane be denoted by K_{zy} and the twist of the curve be denoted by τ_z. Equation (B-4) can be reduced to the following form.

$$\begin{cases} \dfrac{d\vec{R_x}}{ds} = \dfrac{d\vec{u}}{ds} - \vec{v}\cdot\tau_z + \vec{w}\cdot K_{zx} \\[2ex] \dfrac{d\vec{R_y}}{ds} = \dfrac{d\vec{v}}{ds} - \vec{w}\cdot K_{zy} + \vec{u}\cdot\tau_z \\[2ex] \dfrac{d\vec{R_z}}{ds} = \dfrac{d\vec{w}}{ds} - \vec{u}\cdot K_{zx} + \vec{v}\cdot K_{zy} \end{cases} \qquad \text{(B-5)}^*$$

Suppose that the moving axes X, Y, Z are connected to an arbitrary set of fixed axes X′, Y′, Z′ in space with the following orthogonal relations (Figure 34):

$$\begin{array}{cccc} & X & Y & Z \\ X' & l_1 & l_2 & l_3 \\ Y' & m_1 & m_2 & m_3 \\ Z' & n_1 & n_2 & n_3 \end{array} \qquad \text{(B-6)}$$

In Relations (B-6), l_1, l_2, l_3 are the direction cosines of the fixed axis X′ referred to the instantaneous position of the moving axes X, Y, Z, and so on. Hence the direction cosines are functions of s.

In order to utilize the results of Equations (B-5), one conceives now that the space vector $\vec{R}$ being a non-varying unit vector in the direction of the fixed axis X′ (Figure 34). Since $\vec{R}_x$, $\vec{R}_y$, $\vec{R}_z$ are projections of the non-varying unit vector

*Similar relations have been obtained by Kelvin and Tait [35], and Routh [36].

on the fixed line AB when AB is chosen to coincide with the directions of X, Y, Z respectively, $\vec{R}_x$, $\vec{R}_y$, $\vec{R}_z$ do not change with respect to s. Hence

$$\begin{cases} \dfrac{d\vec{R}_x}{ds} = 0 \\ \\ \dfrac{d\vec{R}_y}{ds} = 0 \\ \\ \dfrac{d\vec{R}_z}{ds} = 0 \end{cases} \qquad \text{(B-7)}$$

Moreover, it is observed that in Equations (B-5),

$$\begin{cases} \vec{u} = \vec{i} \cdot \ell_1 \;, & \dfrac{d\vec{u}}{ds} = \dfrac{d\vec{i}}{ds} \cdot \ell_1 + \vec{i} \cdot \dfrac{d\ell_1}{ds} \\ \\ \vec{v} = \vec{i} \cdot \ell_2 \;, & \dfrac{d\vec{v}}{ds} = \vec{i} \cdot \dfrac{d\ell_2}{ds} \\ \\ \vec{w} = \vec{i} \cdot \ell_3 \;, & \dfrac{d\vec{w}}{ds} = \vec{i} \cdot \dfrac{d\ell_3}{ds} \end{cases}$$

Thus considering the space vector $\vec{R}$ being a non-varying unit vector in the X'-direction, one reduces Equations (B-5) to the following form:

$$\begin{cases} \dfrac{dl_1}{ds} = l_2 \cdot \tau_z - l_3 \cdot K_{zx} \\[2ex] \dfrac{dl_2}{ds} = l_3 \cdot K_{zy} - l_1 \cdot \tau_z \\[2ex] \dfrac{dl_3}{ds} = l_1 \cdot K_{zx} - l_2 \cdot K_{zy} \end{cases} \qquad \text{(B-8)}$$

Similarly considering the space vector $\vec{R}$ being a non-varying unit vector in the Y$'$-direction, we have

$$\begin{cases} \vec{u} = \vec{j} \cdot m_1 , \quad \dfrac{d\vec{u}}{ds} = \vec{j} \cdot \dfrac{dm_1}{ds} \\[2ex] \vec{v} = \vec{j} \cdot m_2 , \quad \dfrac{d\vec{v}}{ds} = \vec{j} \cdot \dfrac{dm_2}{ds} \\[2ex] \vec{w} = \vec{j} \cdot m_3 , \quad \dfrac{d\vec{w}}{ds} = \vec{j} \cdot \dfrac{dm_3}{ds} \end{cases}$$

and

$$\begin{cases} \dfrac{dm_1}{ds} = m_2 \cdot \tau_z - m_3 \cdot K_{zx} \\[2ex] \dfrac{dm_2}{ds} = m_3 \cdot K_{zy} - m_1 \cdot \tau_z \\[2ex] \dfrac{dm_3}{ds} = m_1 \cdot K_{zx} - m_2 \cdot K_{zy} \end{cases} \qquad \text{(B-9)}$$

And considering the space vector $\vec{k}$ being a non-varying unit vector in the Z′-direction, we have

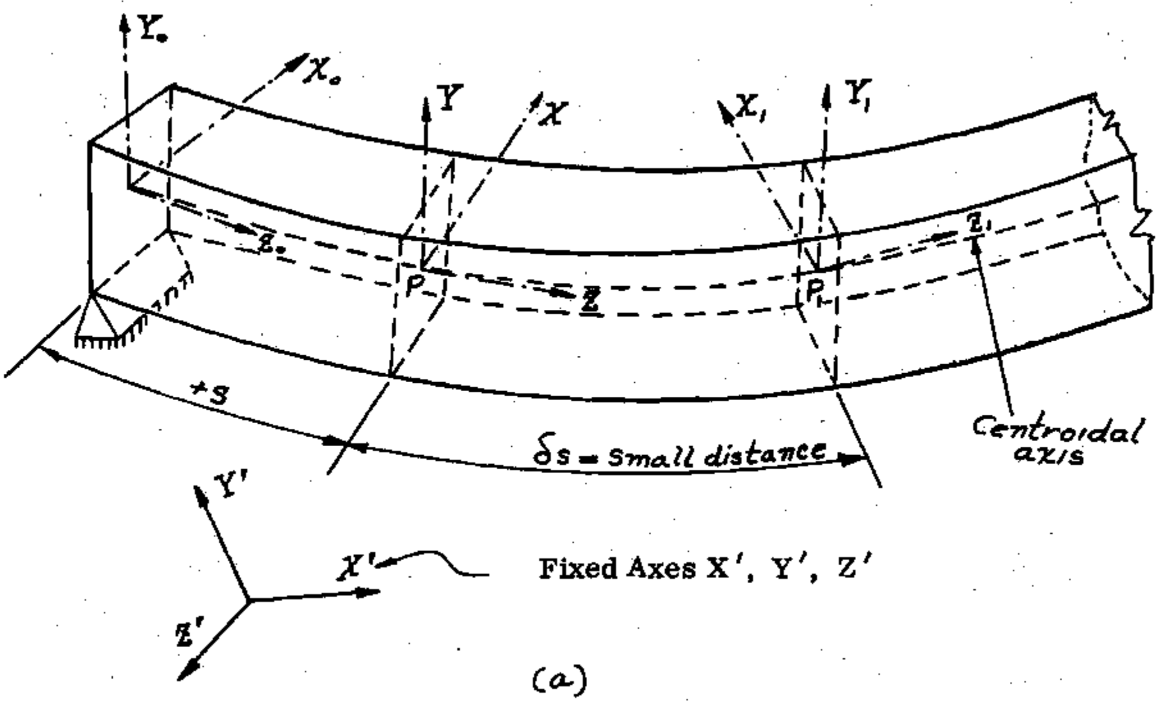
$$\begin{cases} \vec{u} = \vec{k}\cdot n_1 , & \dfrac{d\vec{u}}{ds} = \vec{k}\cdot\dfrac{dn_1}{ds} \\ \vec{v} = \vec{k}\cdot n_2 , & \dfrac{d\vec{v}}{ds} = \vec{k}\cdot\dfrac{dn_2}{ds} \\ \vec{w} = \vec{k}\cdot n_3 , & \dfrac{d\vec{w}}{ds} = \vec{k}\cdot\dfrac{dn_3}{ds} \end{cases}$$

and

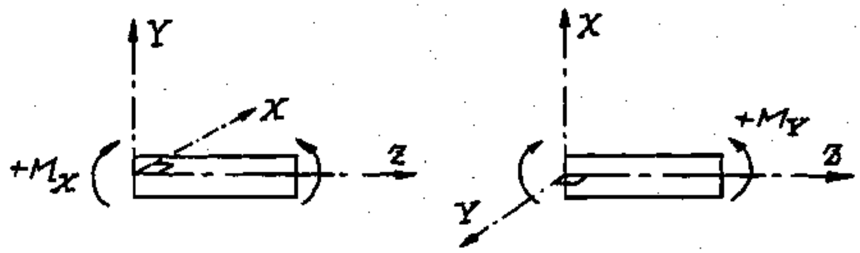
$$\begin{cases} \dfrac{dn_1}{ds} = n_2\cdot\tau_z - n_3\cdot K_{zx} \\ \dfrac{dn_2}{ds} = n_3\cdot K_{zy} - n_1\cdot\tau_z \\ \dfrac{dn_3}{ds} = n_1\cdot K_{zx} - n_2\cdot K_{zx} \end{cases} \qquad \text{(B-10)}$$

When a beam is bent and twisted, the forces acting on the face of a cross-section can be resolved into the following six forces and moments acting at the centroid of the cross-section, namely,

$$\begin{cases} F_z = \iint \sigma_z \, dx\,dy \,, & M_x = \iint -(y \cdot \sigma_z)\, dx\,dy \\ V_y = \iint \tau_{yz}\, dx\,dy \,, & M_y = \iint -(x \cdot \sigma_z)\, dx\,dy \\ V_x = \iint \tau_{xz}\, dx\,dy \,, & M_z = \iint (x \cdot \tau_{yz} - y \cdot \tau_{xz})\, dx\,dy \end{cases} \qquad \text{(B-11)}$$



(a)



(b)

Figure 35. Coordinate Systems for A Bent and Twisted Box Beam.

In Equation (B-11), F_z is the resultant axial force, positive if in tension. V_x and V_y are the resultant shear forces along the X and Y axes respectively, positive if acting in the positive directions of these axes. M_x and M_y are bending moments about the X and Y axes respectively, their positive directions are indicated in Figure 35b. M_z is the twisting moment about the Z-axis, its positive direction is according to the right-hand rule.

The external forces applied to the portion of the beam between sections drawn through points P and P_1 in Figure 35a can be resolved to a force and a moment acting at P. Let $\bar{V}_x$, $\bar{V}_y$, $\bar{F}_z$ be the components of the resultant external force per unit length of the centroidal axis, and $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ be the components of the resultant external moment per unit length of the centroidal axis. Now that the forces applied to the portion of the beam between sections drawn through P and P_1 balance the internal resisting forces and moments acting on the two cross-sections. Let δ denote the difference of the value of any quantity belonging to the section through P_1 compared to the value belonging to the section through P. Let X, Y, Z be the coordinate axes of any point on the centroidal axis, the Z-axis is taken always to be tangent to the centroidal axis, the X and Y axes are respectively the primary and the secondary bending axes of the cross-section. At P the axes X, Y, Z will take the positions X, Y, Z and at P_1 they will take the positions X_1, Y_1, Z_1. Also let the axes X, Y, Z be referred to an arbitrary set of fixed axes X', Y', Z' in space by the orthogonal scheme as shown in Relations (B-6) (see Figure 35).

Considering the equilibrium of forces in the X'-direction ($\Sigma F_{x'} = 0$), we have, when assuming P and P_1 are only at a small distance apart (see Figure 36),

$$(V_x + \delta V_x)(\ell_1 + \delta \ell_1) - V_x \cdot \ell_1 + (V_y + \delta V_y)(\ell_2 + \delta \ell_2) - V_y \cdot \ell_2$$

$$+ (F_z + \delta F_z)(\ell_3 + \delta \ell_3) - F_z \cdot \ell_3 + \int_s^{s+\delta s} (\overline{V}_x \cdot \ell_1 + \overline{V}_y \cdot \ell_2 + \overline{F}_z \cdot \ell_3) \cdot ds = 0 \qquad \text{(B-12)}$$

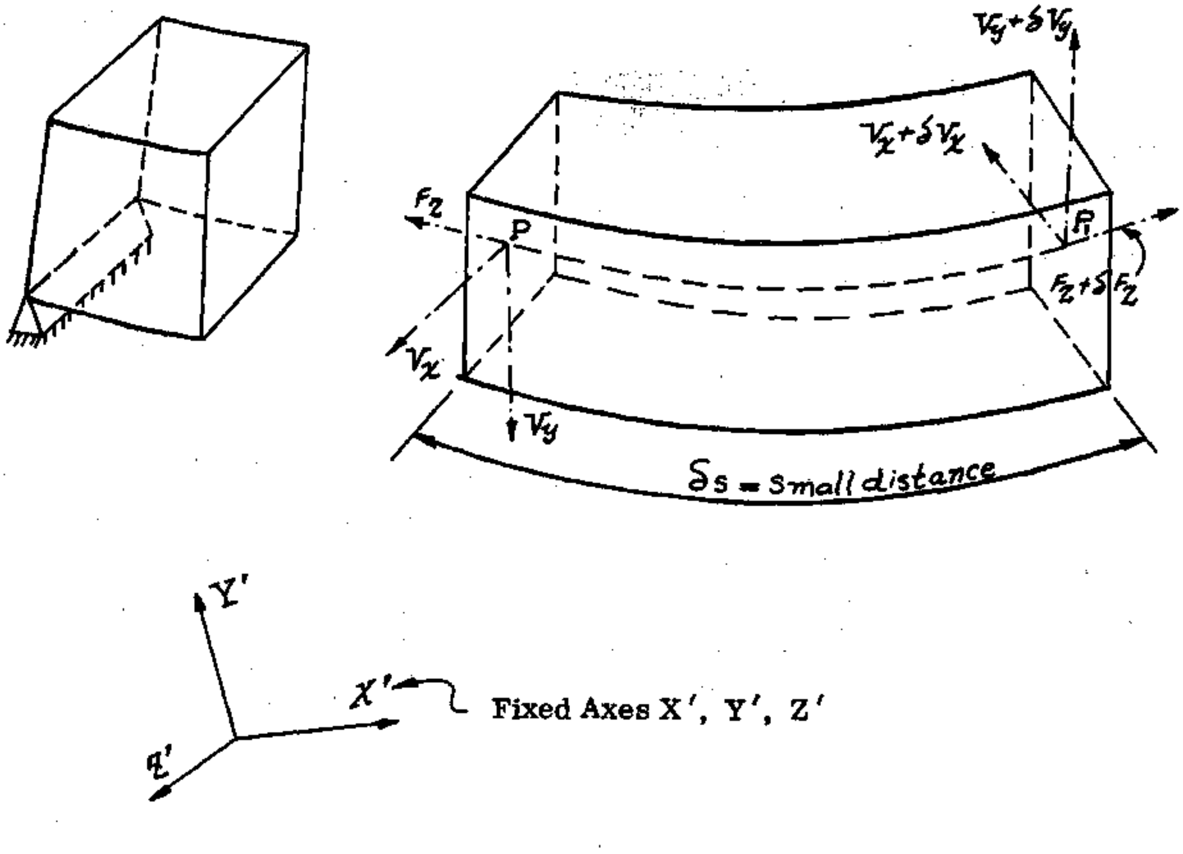


Figure 36. Free Body Diagrams of A Bent and Twisted Box Beam.

Dividing the left hand side of the above equation by δs and taking the limit as $\delta s \to 0$, we have, when neglecting higher order terms,

$$\lim_{\delta s \to 0} \left[\frac{\delta V_x}{\delta s} \cdot l_1 + V_x \cdot \frac{\delta l_1}{\delta s} + \frac{\delta V_y}{\delta s} \cdot l_2 + V_y \cdot \frac{\delta l_2}{\delta s} \right.$$

$$\left. + \frac{\delta F_z}{\delta z} \cdot l_3 + F_z \frac{\delta l_3}{\delta s} + \frac{1}{\delta s} \int_s^{s+ds} (l_1 \cdot \overline{V_x} + l_2 \cdot \overline{V_y} + l_3 \cdot \overline{F_z}) \, ds \right] = 0$$

$$l_1 \cdot \frac{dV_x}{ds} + V_x \cdot \frac{dl_1}{ds} + l_2 \cdot \frac{dV_y}{ds} + V_y \cdot \frac{dl_2}{ds} + l_3 \cdot \frac{dF_z}{ds} + F_z \cdot \frac{dl_3}{ds}$$

$$+ (l_1 \overline{V_x} + l_2 \overline{V_y} + l_3 \overline{F_z}) = 0 \qquad \text{(B-13)}$$

Since the set of axes X′, Y′, Z′ is an arbitrary set of axes in space, we can assume the axes X′, Y′, Z′ to be coincident with X, Y, Z. Thus $l_1 = 1$, $l_2 = 0$, $l_3 = 0$. Also, from Equation (B-8), we have

$$\begin{cases} \dfrac{d\ell_1}{ds} = \ell_2 \cdot \tau_z - \ell_3 \cdot K_{zx} = 0 \\ \\ \dfrac{d\ell_2}{ds} = \ell_3 \cdot K_{zy} - \ell_1 \cdot \tau_z = -\tau_z \\ \\ \dfrac{d\ell_3}{ds} = \ell_1 \cdot K_{zx} - \ell_2 \cdot K_{zy} = K_{zx} \end{cases}$$

Substituting these results into Equation (B-13) we have

$$\frac{dV_x}{ds} - V_y \cdot \tau_z + F_z \cdot K_{zx} + \overline{V_x} = 0 \qquad \text{(B-14)}$$

Considering the equilibrium of forces in the Y′ direction ($\Sigma F_{y'} = 0$), we have

$$(V_x + \delta V_x)(m_1 + \delta m_1) - V_x \cdot m_1 + (V_y + \delta V_y)(m_2 + \delta m_2) - V_y \cdot m_2$$

$$+ (F_z + \delta F_z)(m_3 + \delta m_3) - F_z \cdot m_3 + \int_s^{s+\delta s} (\overline{V_x} \cdot m_1 + \overline{V_y} \cdot m_2 + \overline{F_z} \cdot m_3) ds = 0$$

$$\lim_{\delta s \to 0} \Big[\frac{\delta V_x}{\delta s} \cdot m_1 + V_x \cdot \frac{\delta m_1}{\delta s} + \frac{\delta V_y}{\delta s} \cdot m_2 + V_y \cdot \frac{\delta m_2}{\delta s}$$

$$+ \frac{\delta F_z}{\delta s} \cdot m_3 + F_z \cdot \frac{\delta m_3}{\delta s} + \frac{1}{\delta s} \int_s^{s+\delta s} (\overline{V_x} \cdot m_1 + \overline{V_y} \cdot m_2 + \overline{F_z} \cdot m_3) ds \Big] = 0$$

$$m_1 \cdot \frac{dV_x}{ds} + V_x \cdot \frac{dm_1}{ds} + m_2 \cdot \frac{dV_y}{ds} + V_y \cdot \frac{dm_2}{ds} + m_3 \cdot \frac{dF_z}{ds} + F_z \cdot \frac{dM_3}{ds}$$

$$+ (\overline{V_x} \cdot m_1 + \overline{V_y} \cdot m_2 + \overline{F_z} \cdot m_3) = 0$$

Since by choosing X′, Y′, Z′ to coincide with X, Y, Z, we have $m_1 = 0$; $m_2 = 1$, $m_3 = 0$. And from Equations (B-9),

$$\begin{cases} \dfrac{dm_1}{ds} = m_2 \cdot \tau_z - m_3 \cdot K_{zx} = \tau_z \\ \dfrac{dm_2}{ds} = m_3 \cdot K_{zy} - m_1 \cdot \tau_z = 0 \\ \dfrac{dm_3}{ds} = m_1 \cdot K_{zx} - m_2 \cdot K_{zy} = -K_{zy} \end{cases}$$

Thus

$$\frac{dV_y}{ds} + V_x \cdot \tau_z - F_z \cdot K_{zy} + \overline{V_y} = 0 \qquad \text{(B-15)}$$

Considering the equilibrium of forces in the Z′-direction ($\Sigma F_{z'} = 0$), we have

$$(V_x + \delta V_x)(n_1 + \delta n_1) - V_x \cdot n_1 + (V_y + \delta V_y)(n_2 + \delta n_2) - V_y \cdot n_2$$

$$+ (F_z + \delta F_z)(n_3 + \delta n_3) - F_z \cdot n_3 + \int_s^{s+\delta s} (\overline{V_x} \cdot n_1 + \overline{V_y} \cdot n_2 + \overline{F_z} \cdot n_3) ds = 0$$

$$\lim_{\delta s \to 0} \left[\frac{\delta V_x}{\delta s} \cdot n_1 + V_x \cdot \frac{\delta n_1}{\delta s} + \frac{\delta V_y}{\delta s} \cdot n_2 + V_y \cdot \frac{\delta n_2}{\delta s} + \frac{\delta F_z}{\delta s} \cdot n_3 + F_z \cdot \frac{\delta n_3}{\delta s} \right.$$

$$\left. \frac{1}{\delta s} \int_s^{s+\delta s} (\overline{V_x} \cdot n_1 + \overline{V_y} \cdot n_2 + \overline{F_z} \cdot n_3) ds \right] = 0$$

$$n_1 \frac{dV_x}{ds} + V_x \cdot \frac{dn_1}{ds} + n_2 \cdot \frac{dV_y}{ds} + V_y \cdot \frac{dn_2}{ds} + n_3 \cdot \frac{dF_z}{ds} + F_z \cdot \frac{dn_3}{ds}$$

$$+ (\overline{V_x} \cdot n_1 + \overline{V_y} \cdot n_2 + \overline{F_z} \cdot n_3) = 0$$

Since $n_1 = 0$, $n_2 = 0$, $n_3 = 1$ when X′, Y′ Z′ are coincident with X, Y, Z. And from Equation (B-10), we have

$$\begin{cases} \dfrac{dn_1}{ds} = n_2 \cdot \tau_z - n_3 \cdot K_{zx} = - K_{zx} \\[2ex] \dfrac{dn_2}{ds} = n_3 \cdot K_{zy} - n_1 \cdot \tau_z = K_{zy} \\[2ex] \dfrac{dn_3}{ds} = n_1 \cdot K_{zx} - n_2 \cdot K_{zy} = 0 \end{cases}$$

Thus

$$\frac{dF_z}{ds} + V_y \cdot K_{zy} - V_x \cdot K_{zx} + \overline{F_z} = 0 \qquad \text{(B-16)}$$

Next, consider the equilibrium of moments about the X′-axis ($\Sigma M_{x'} = 0$) in the following manner:

a) Projecting the moment vectors M_x, M_y, M_z at sections through P_1 and P_2:

$$(M_x + \delta M_x)(l_1 + \delta l_1) - M_x \cdot l_1 + (M_y + \delta M_y)(l_2 + \delta l_2)$$

$$- M_y \cdot l_2 + (M_z + \delta M_z)(l_3 + \delta l_3) - M_z \cdot l_3$$

$$= M_x \cdot \delta l_1 + \delta M_x \cdot l_1 + \delta M_x \cdot \delta l_1 + M_x \cdot l_1 - M_x \cdot l_1$$

$$+ M_y \cdot \delta l_2 + \delta M_y \cdot l_2 + \delta M_y \delta l_2 + M_y \cdot l_2 - M_y \cdot l_2$$

$$+ M_z \cdot \delta l_3 + \delta M_z \cdot l_3 + \delta M_z \cdot \delta l_3 + M_z \cdot l_3 - M_z \cdot l_3$$

$$= M_x \cdot \delta l_1 + \delta M_x \cdot l_1 + M_y \cdot \delta l_2 + \delta M_y \cdot l_2 + M_z \delta l_3 + \delta M_z \cdot l_3$$

b) Taking moment about the X′-axis, for the components of V_x and ($V_x + \delta V_x$) on to the Z′-direction. Here, we assume the coordinates of P are x, y, z and the coordinates of P_1 are $x_1 = x + \delta x$, $y_1 = y + \delta y$, $z_1 = z + \delta z$ when referred to the axes X′, Y′, Z′.

$$(V_x+\delta V_x)(n_1+\delta n_1)(y+\delta y)-V_x\cdot n_1\cdot y$$

$$=V_x\cdot n_1\cdot y+V_x\cdot\delta n_1\cdot y+\delta V_x\cdot n_1\cdot y+\delta V_x\cdot\delta n_1\cdot y+V_x\cdot n_1\cdot\delta y$$

$$+V_x\cdot\delta n_1\cdot\delta y+\delta V_x\cdot n_1\cdot\delta y+\delta V_x\cdot\delta n_1\cdot\delta y-V_x\cdot n_1\cdot y$$

$$=(\delta y)(V_x\cdot\delta n_1+\delta V_x\cdot n_1)+(\delta y)(n_1\cdot V_x)$$

$$+(y)[\delta V_x\cdot n_1+V_x\cdot\delta n+\delta V_x\cdot\delta n_1]$$

Taking moment about X′-axis for components of V_x and $(V_x+\delta V_x)$ onto the Y′-direction,

$$-(V_x+\delta V_x)(m_1+\delta m_1)(z+\delta z)+(V_x\cdot m_1\cdot z)$$

$$=-V_x\cdot m_1\cdot z-V_x\cdot\delta m_1\cdot z-\delta V_x\cdot m_1\cdot z-\delta V_x\cdot\delta m_1\cdot z-V_x\cdot m_1\cdot\delta z$$

$$-V_x\cdot\delta m_1\cdot\delta z-\delta V_x\cdot m_1\cdot\delta z-\delta V_x\cdot\delta m_1\cdot\delta z+V_x\cdot m_1\cdot z$$

$$=-(\delta z)(V_x\cdot\delta m_1+\delta V_x\cdot m_1)-(\delta z)(V_x\cdot m_1)$$

$$-(z)(\delta V_x\cdot m_1+V_x\cdot\delta m_1+\delta V_x\cdot\delta m_1)$$

c) Taking moment about X'-axis for the components of V_y and $(V_y + \delta V_y)$ onto the Z'-direction,

$$(V_y + \delta V_y)(n_2 + \delta n_2)(y + \delta y) - V_y \cdot n_2 \cdot y$$

$$= (\delta y)(V_y \cdot \delta n_2 + \delta V_y \cdot n_2) + (\delta y)(n_2 \cdot V_y)$$

$$+ (y)(\delta V_y \cdot n_2 + V_y \cdot \delta n_2 + \delta V_y \cdot \delta n_2)$$

Taking moment about X'-axis for the components of V_y and $(V_y + \delta V_y)$ onto the Y'-direction,

$$-(V_y + \delta V_y)(m_2 + \delta m_2)(z + \delta z) + V_y \cdot m_2 \cdot z$$

$$= -(\delta z)(V_y \cdot \delta m_2 + \delta V_y \cdot m_2) - (\delta z)(V_y \cdot m_2)$$

$$-(z)(\delta V_y \cdot m_2 + V_y \cdot \delta m_2 + \delta V_y \cdot \delta m_2)$$

d) Taking moment about X'-axis for the components of F_z and $(F_z + \delta F_z)$ onto Z'-direction,

$$(F_z + \delta F_z)(n_3 + \delta n_3)(y + \delta y) - F_z \cdot n_3 \cdot y$$

$$= (\delta y)(F_z \cdot \delta n_3 + \delta F_z \cdot n_3) + (\delta y)(n_3 \cdot F_z)$$

$$+ (y)(\delta F_z \cdot n_3 + F_z \cdot \delta n_3 + \delta F_z \cdot \delta n_3)$$

Taking moment about X′-axis for the components of F_z and $(F_z + \delta F_z)$ onto the Y′-direction,

$$-(F_Z + \delta F_Z)(m_3 + \delta m_3)(Z + \delta Z) + F_Z \cdot m_3 \cdot Z$$

$$= -(\delta Z)(F_Z \cdot \delta m_3 + \delta F_Z \cdot m_3) - (\delta Z)(F_Z \cdot m_3)$$

$$-(Z)(\delta F_Z \cdot m_3 + F_3 \cdot \delta m_3 + \delta F_Z \cdot \delta m_3)$$

e) Taking moment about X′-axis for the components of the external loads $\bar{V}_x$, $\bar{V}_y$, $\bar{F}_z$ onto the Z′-direction.

$$\int_s^{s+\delta s} \left[(\overline{V_x} ds)(n_1) + (\overline{V_y} ds)(n_2) + (\overline{F_Z} ds)(n_3)\right](y - y_1)$$

$$= \int_s^{s+\delta s} (y - y_1)(n_1 \cdot \overline{V_x} + n_2 \cdot \overline{V_y} + n_3 \cdot \overline{F_Z}) \cdot ds$$

Here x_1, y_1, z_1, denote the coordinates at P_1 and x, y, z denote those at any point on the centroidal axis. Taking moment about X′-axis for the components of the external loads $\bar{V}_x$, $\bar{V}_y$, $\bar{F}_z$ onto the Y′-direction, we have

$$-\int_s^{s+\delta s} \left[(\overline{V_x} ds)(m_1) + (\overline{V_y} ds)(m_2) + (\overline{F_Z} ds)(m_3)\right](Z - Z_1)$$

$$= -\int_s^{s+\delta s} (Z - Z_1)(m_1 \overline{V_x} + m_2 \overline{V_y} + m_3 \cdot \overline{V_Z}) ds$$

f) Projecting the external moment vectors $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ onto the X'-direction,

$$\int_s^{s+\delta s} (\bar{M}_x \cdot l_1 + \bar{M}_y \cdot l_2 + \bar{M}_z \cdot l_3) ds$$

Summing up all the terms in the equation $\Sigma M_{x'} = 0$, we have

$$M_x \cdot \delta l_1 + \delta M_x \cdot l_1 + M_y \cdot \delta l_2 + \delta M_y \cdot l_2 + M_z \cdot \delta l_3 + \delta M_z \cdot l_3$$

$$+ (\delta y)(V_x \cdot \delta n_1 + \delta V_x \cdot n_1) + (\delta y)(V_x \cdot n_1) + (y)[\delta V_x \cdot n_1 + V_x \cdot \delta n_1 + \delta V_x \cdot \delta n_1]$$

$$- (\delta z)(V_x \cdot \delta m_1 + \delta V_x \cdot m_1) - (\delta z)(V_x m_1) - (z)[\delta V_x \cdot m_1 + V_x \cdot \delta m_1 + \delta V_x \cdot \delta m_1]$$

$$+ (\delta y)(V_y \cdot \delta n_2 + \delta V_y \cdot n_2) + (\delta y)(V_y \cdot n_2) + (y)[\delta V_y \cdot n_2 + V_y \cdot \delta n_2 + \delta V_y \cdot \delta n_2]$$

$$- (\delta z)(V_y \cdot \delta m_2 + \delta V_y \cdot m_2) - (\delta z)(V_y \cdot m_2) - (z)[\delta V_y \cdot m_2 + V_y \cdot \delta m_2 + \delta V_y \cdot \delta m_2]$$

$$+ (\delta y)(F_z \cdot \delta n_3 + \delta F_z \cdot n_3) + (\delta y)(F_z \cdot n_3) + (y)[\delta F_z \cdot n_3 + F_z \cdot \delta n_3 + \delta F_z \cdot \delta n_3]$$

$$- (\delta z)(F_z \cdot \delta m_3 - \delta F_z \cdot m_3) - (\delta z)(F_z m_3) - (z)[\delta F_z \cdot m_3 + F_z \cdot \delta m_3 + \delta F_z \cdot \delta m_3]$$

$$+\int_{s}^{s+\delta s}\left[(y-y_1)(n_1\overline{V_x}+n_2\overline{V_y}+n_3\overline{F_z})-(z-z_1)(m_1\overline{V_x}+m_2\overline{V_y}+m_3\overline{V_z})\right]ds$$

$$+\int_{s}^{s+ds}(M_x\cdot l_1+M_y\cdot l_2+M_z\cdot l_3)\,ds=0$$

Divide both sides of the above equation by δs, and pass to a limit by diminishing δs indefinitely. Meanwhile, one chooses the arbitrary fixed axes X′, Y′, Z′ to coincide with the axis X, Y, Z at P. Thus one can write $x = 0$, $y = 0$, $z = 0$ and

$$\begin{cases} l_1 = 1 & l_2 = 0 & l_3 = 0 \\ m_1 = 0 & m_2 = 1 & m_3 = 0 \\ n_1 = 0 & n_2 = 0 & n_3 = 1 \end{cases}$$

One observes now that all terms multiplying by x, y or z will vanish and when using the results of Equations (B–8), we have

$$\lim_{\delta s\to 0}\left[\frac{M_x\cdot\delta l_1+\delta M_x\cdot l_1+M_y\cdot\delta l_2+\delta M_y\cdot l_2+M_z\cdot\delta l_3+\delta M_z\cdot l_3}{\delta s}\right]$$

$$= M_x \cdot \frac{dl_1}{ds} + \frac{dM_x}{ds}\cdot l_1 + M_y \cdot \frac{dl_2}{ds} + \frac{dM_y}{ds}\cdot l_2 + M_z \cdot \frac{dl_3}{ds} + \frac{dM_z}{ds}\cdot l_3$$

$$= M_x(l_2 \cdot \tau_z - l_3 \cdot K_{zx}) + \frac{dM_x}{ds} + M_y(l_3 \cdot K_{zy} - l_1 \cdot \tau_z) + M_z(l_1 \cdot K_{zx} - l_2 \cdot K_{zy})$$

$$= \frac{dM_x}{ds} - M_y \cdot \tau_z + M_z \cdot K_{zx}$$

Since X and Y axes are normal to the arc length ds and the Z-axis is tangent to ds, hence

$$\begin{cases} \lim\limits_{\delta s \to 0} \dfrac{\delta x}{\delta s} = \dfrac{dx}{ds} = 0 \\[2em] \lim\limits_{\delta s \to 0} \dfrac{\delta y}{\delta s} = \dfrac{dy}{ds} = 0 \\[2em] \lim\limits_{\delta s \to 0} \dfrac{\delta z}{\delta s} = \dfrac{dz}{ds} = 1 \end{cases}$$

Furthermore,

$$
\left\{
\begin{aligned}
&\lim_{\delta s\to 0} \frac{(\delta y)(V_x\cdot\delta n_1+\delta V_x\cdot n_1)+(\delta y)(V_x\cdot n_1)}{\delta s}\\
&\quad=\left(\lim_{\delta s\to 0}\frac{\delta y}{\delta s}\right)\left[\lim_{\delta s\to 0}(V_x\cdot\delta n_1+\delta V_x\cdot n_1)\right]+\lim_{\delta s\to 0}V_x\cdot n_1\cdot\frac{\delta y}{\delta s}=0\\
&\lim_{\delta s\to 0} \frac{(\delta y)(V_y\cdot\delta n_1+\delta V_y\cdot n_2)+(\delta y)(V_y\cdot n_2)}{\delta s}=0\\
&\lim_{\delta s\to 0} \frac{(\delta y)(F_z\cdot\delta n_3+\delta F_z\cdot n_3)+(\delta y)(F_z\cdot n_3)}{\delta s}=0
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&\lim_{\delta s\to 0} -\frac{(\delta z)(V_x\cdot\delta m_1+\delta V_x\cdot m_1)+(\delta z)(V_x\cdot m_1)}{\delta s}\\
&\quad=\left(\lim_{\delta s\to 0}-\frac{\delta z}{\delta s}\right)\left[\lim_{\delta s\to 0}(V_x\cdot\delta m_1+\delta V_x\cdot m_1)\right]-\lim_{\delta s\to 0}V_x\cdot m_1\cdot\frac{\delta z}{\delta s}=0\\
&\lim_{\delta s\to 0} -\frac{(\delta z)(V_y\cdot\delta m_2+\delta V_y\cdot m_2)+(\delta z)(V_y\cdot m_2)}{\delta s}\\
&\quad=\left(\lim_{\delta s\to 0}-\frac{\delta z}{\delta s}\right)\left[\lim_{\delta s\to 0}(V_y\cdot\delta m_2+\delta V_y\cdot m_2)\right]-\lim_{\delta s\to 0}V_y\, m_2\cdot\frac{\delta z}{\delta s}=-V_y\\
&\lim_{\delta s\to 0} -\frac{(\delta z)(F_z\cdot\delta m_3+\delta F_z\cdot m_3)+(\delta z)(F_z\cdot m_3)}{\delta s}\\
&\quad=\left(\lim_{\delta s\to 0}-\frac{\delta z}{\delta s}\right)\left[\lim_{\delta s\to 0}(F_z\cdot\delta m_3+\delta F_z\cdot m_3)\right]-\lim_{\delta s\to 0}F_z\cdot m_3\cdot\frac{\delta z}{\delta s}=0
\end{aligned}
\right.
$$

$$\lim_{\delta s \to 0} \frac{1}{\delta s}\int_{s}^{s+ds}\Big[(y-y_1)(n_1\cdot\overline{V_x}+n_2\cdot\overline{V_y}+n_3\cdot\overline{F_z})$$

$$-(z-z_1)(m_1\cdot\overline{V_x}+m_2\cdot\overline{V_y}+m_3\cdot\overline{F_z})\Big]ds = 0$$

and

$$\lim_{\delta \to 0} \frac{1}{\delta s}\int_{s}^{s+\delta s}(\overline{M_x}\cdot l_1+\overline{M_y}\cdot l_2+\overline{M_z}\cdot l_3)ds$$

$$= \overline{M_x}\cdot l_1 + \overline{M_y}\cdot l_2 + \overline{M_z}\cdot l_3 = \overline{M_x}$$

Thus, the equilibrium equation for $\Sigma M_{x'} = 0$ is reduced to the following form:

$$\frac{dM_x}{ds} - M_y\cdot\tau_z + M_z\cdot K_{zx} - V_y + \overline{M_x} = 0 \qquad \text{(B-17)}$$

Considering the equilibrium of moments about the Y'-axis ($\Sigma M_{y'} = 0$), one shall obtain

$$M_x\cdot\delta m_1 + \delta M_x\cdot m_1 + M_y\cdot\delta m_2 + \delta M_y\cdot m_2$$

$$+ M_z\cdot\delta m_3 + \delta M_z\cdot m_3 + (\delta z)(V_x\cdot\delta l_1 + \delta V_x\cdot l_1)$$

$$+ (\delta z)(V_x \cdot l_1) + (z)(\delta V_x \cdot l_1 + V_x \cdot \delta l_1 + \delta V_x \cdot \delta l_1)$$

$$- (\delta x)(V_x \cdot \delta n_1 + \delta V_x \cdot n_1) - (\delta x)(V_x \cdot n_1)$$

$$- (x)(\delta V_x \cdot n_1 + V_x \cdot \delta n_1 + \delta V_x \cdot \delta n_1) + (\delta z)(V_y \cdot \delta l_2 + \delta V_y \cdot l_2)$$

$$+ (\delta z)(V_y \cdot l_2) + (z)(\delta V_y \cdot l_2 + V_y \cdot \delta l_2 + \delta V_y \cdot \delta l_2)$$

$$- (\delta x)(V_y \cdot n_2 + \delta V_y \cdot n_2) - (\delta x)(V_y \cdot n_2) - (x)(\delta V_y \cdot n_2$$

$$+ V_y \cdot \delta n_2 + \delta V_y \cdot \delta n_2) + (\delta z)(F_z \cdot \delta l_3 + \delta F_z \cdot \delta l_3)$$

$$+ (\delta z)(F_z \cdot l_3) + (z)(\delta F_z \cdot l_3 + F_z \cdot \delta l_3 + \delta F_z \cdot \delta l_3)$$

$$- (\delta x)(F_z \cdot \delta n_3 + \delta F_z \cdot \delta n_3) - (\delta x)(F_z \cdot n_3)$$

$$- (x)(\delta F_z \cdot n_3 + F_z \cdot \delta n_3 + \delta F_z \cdot \delta n_3)$$

$$+ \int_S^{S+\delta S} [(z - z_1)(\overline{V_x} \cdot l_1 + \overline{V_y} \cdot l_2 + \overline{F_z} \cdot l_3)$$

$$-(x-x_1)(\overline{V_x}\cdot n_1+\overline{V_y}\cdot n_2+\overline{F_z}\cdot n_3)]ds$$

$$+\int_s^{s+ds}(\overline{M_x}\cdot m_1+\overline{M_y}\cdot m_2+\overline{M_z}\cdot m_3)ds = 0$$

By dividing both sides of the above equation by δ s and taking the limit as δ s approaches to zero, while at the same time choosing X', Y', Z' to coincide with X, Y, Z respectively, we have, when using the results of Equations (B-9),

$$\lim_{\delta s\to 0}\left[\frac{M_x\cdot\delta m_1+\delta M_x\cdot m_1+M_y\cdot\delta m_2+\delta M_y\cdot m_2+M_z\cdot\delta m_3+\delta M_z\cdot m_3}{\delta s}\right]$$

$$= M_x\cdot\frac{dm_1}{ds}+\frac{dM_x}{ds}\cdot m_1+M_y\cdot\frac{dm_2}{ds}+\frac{dM_y}{ds}\cdot m_2+M_z\cdot\frac{dm_3}{ds}+\frac{dM_z}{ds}\cdot m_3$$

$$= M_x(m_2\cdot\tau_z-m_3\cdot K_{zx})+M_y(m_3\cdot K_{zy}-m_1\cdot\tau_z)+\frac{dM_y}{ds}$$

$$+M_z(m_1\cdot K_{zx}-m_2\cdot K_{zy})$$

$$= \frac{dM_y}{ds}-M_z\cdot K_{zy}+M_x\cdot\tau_z$$

$$\lim_{\delta s\to 0}\left[\frac{(\delta z)(V_x\cdot\delta l_1+\delta V_x\cdot l_1)+(\delta z)(V_x\cdot l_1)}{\delta s}\right]$$

$$=\left(\lim_{\delta s\to 0}\frac{\delta z}{\delta s}\right)\left[\lim_{\delta s\to 0}(V_x\cdot\delta l_1+\delta V_x\cdot l_1)\right]+\lim_{\delta s\to 0}V_x\cdot l_1\cdot\frac{\delta z}{\delta s}=V_x$$

$$\lim_{\delta s\to 0}\frac{1}{\delta s}\int_s^{s+\delta s}(\overline{M_x}\cdot m_1+\overline{M_y}\cdot m_2+\overline{M_z}\cdot m_3)\cdot ds$$

$$=\overline{M_x}\cdot m_1+\overline{M_y}\cdot m_2+\overline{M_z}\cdot m_3=\overline{M_y}$$

The rest of the terms are equal to zero. Thus the equilibrium equation $\Sigma M_{y'} = 0$ is reduced to the following form:

$$\frac{dM_y}{ds}-M_z\cdot K_{zy}+M_x\cdot\tau_z+V_x+\overline{M_y}=0 \qquad \text{(B-18)}$$

Finally, by considering the equilibrium of moments about the Z′-axis ($\Sigma M_{z'} = 0$), we have

$$M_x\cdot\delta n_1+\delta M_x\cdot n_1+M_y\cdot\delta n_2+\delta M_y\cdot n_2+M_z\cdot\delta n_3+\delta M_z\cdot n_3$$

$$+(\delta x)(V_x\cdot\delta m_1+\delta V_x\cdot m_1)+(\delta x)(V_x\cdot m_1)$$

$$+(x)(\delta V_x \cdot m_1 + V_x \cdot \delta m_1 + \delta V_x \cdot \delta m_1) - (\delta y)(V_x \cdot \delta l_1 + \delta V_x \cdot l_1)$$

$$-(\delta y)(V_x \cdot l_1) - (y)(\delta V_x \cdot l_1 + V_x \cdot \delta l_1 + \delta V_x \cdot \delta l_1)$$

$$+(\delta x)(V_y \cdot \delta m_2 + \delta V_y \cdot m_2) + (\delta x)(V_y \cdot m_2)$$

$$+x(\delta V_y \cdot m_2 + V_y \cdot \delta m_2 + \delta V_y \cdot \delta m_2)$$

$$-(\delta y)(V_y \cdot \delta l_2 + \delta V_y \cdot l_2) - (\delta y)(V_y \cdot l_2) - (y)(\delta V_y \cdot l_2$$

$$+V_y \cdot \delta l_2 + \delta V_y \cdot \delta l_2) + (\delta x)(F_z \cdot \delta m_3 + \delta F_z \cdot m_3) + (\delta x)(F_z \cdot m_3)$$

$$+(x)(\delta F_z \cdot m_3 + F_z \cdot \delta m_3 + \delta F_z \cdot \delta m_3) - (\delta y)(F_z \cdot \delta l_3 + \delta F_z \cdot l_3)$$

$$-(\delta y)(F_z \cdot l_3) - (y)(\delta F_z \cdot l_3 + F_z \cdot \delta l_3 + \delta F_z \cdot \delta l_3)$$

$$+\int_s^{s+\delta s} [(x-x_1)(V_x \cdot m_1 + V_y \cdot m_2 + F_z \cdot m_3) - (y-y_1)(V_x \cdot l_1 + V_y \cdot l_2 + F_z \cdot l_3)]\,ds$$

$$+\int_s^{s+\delta s} (\bar{M}_x \cdot n_1 + \bar{M}_y \cdot n_2 + \bar{M}_z \cdot n_3)\,ds = 0$$

Again, divide both sides of the above equation by δs and take the limit as $\delta s \to 0$. Choosing the axes X′, Y′, Z′ to coincide with X, Y, Z respectively and using the results of Equations (B-10), we have

$$\lim_{\delta s \to 0} \left[\frac{M_x \cdot \delta n_1 + \delta M_x \cdot n_1 + M_y \cdot \delta n_2 + \delta M_y \cdot n_2 + M_z \cdot \delta n_3 + \delta M_z \cdot n_3}{\delta s} \right]$$

$$= M_x \cdot \frac{dn_1}{ds} + \frac{dM_x}{ds} \cdot n_1 + M_y \cdot \frac{dn_2}{ds} + \frac{dM_y}{ds} \cdot n_2$$

$$+ M_z \cdot \frac{dn_3}{ds} + \frac{dM_z}{ds} \cdot n_3$$

$$= M_x \cdot (n_2 \cdot \tau_z - n_3 \cdot K_{zx}) + M_y \cdot (n_3 \cdot K_{zy} - n_1 \cdot \tau_z)$$

$$+ M_z \cdot (n_1 \cdot K_{zx} - n_2 \cdot K_{zy}) + \frac{dM_z}{ds}$$

$$= \frac{dM_z}{ds} - M_x \cdot K_{zx} + M_y \cdot K_{zy}$$

$$\lim_{\delta s \to 0} \frac{1}{\delta s} \int_s^{s+\delta s} (\bar{M}_x \cdot n_1 + \bar{M}_y \cdot n_2 + \bar{M}_z \cdot n_3) \cdot ds$$

$$= \overline{M}_x \cdot n_1 + \overline{M}_y \cdot n_2 + \overline{M}_z \cdot n_3 = \overline{M}_z$$

The rest of the terms vanish. The equilibrium equation $\Sigma M_{z'} = 0$ is thus reduced to the following form:

$$\frac{dM_z}{ds} - M_x \cdot K_{zx} + M_y \cdot K_{zy} + \overline{M}_z = \qquad \text{(B-19)}$$

Equations (B-14) through (B-19) are the equilibrium equations* for a bent and twisted beam in space. These equations are conveniently grouped in the following manner:

$$\left\{\begin{aligned}
&\frac{dV_x}{ds} - V_y \cdot \tau_z + F_z \cdot K_{zx} + \overline{V}_x = 0 \\
&\frac{dV_y}{ds} + V_x \cdot \tau_z - F_z \cdot K_{zy} + \overline{V}_y = 0 \\
&\frac{dF_z}{ds} + V_y \cdot K_{zy} - V_x \cdot K_{zx} + \overline{F}_z = 0 \\
&\frac{dM_x}{ds} - M_y \cdot \tau_z + M_z \cdot K_{zx} - V_y + \overline{M}_x = 0 \\
&\frac{dM_y}{ds} - M_z \cdot K_{zy} + M_x \cdot \tau_z + V_x + \overline{M}_y = 0 \\
&\frac{dM_z}{ds} - M_x \cdot K_{zx} + M_y \cdot K_{zy} + \overline{M}_z = 0
\end{aligned}\right. \qquad \text{(B-20)}$$

---

*These equations are referred to as Kirchhoff's equilibrium equations. See Reissner [37] and Love [38].

APPENDIX C

Lateral-Torsional Buckling of Beams of Narrow Rectangular Cross-Section

The case of lateral-torsional buckling of narrow rectangular beam, in which the effect of primary bending curvature is considered, had been solved by Neal. In his paper [5], Neal presented the solution without showing the development which led to his equation. In the following discussion, a detailed derivation of his equation is given. It has been shown in Article 2 of Chapter IV that this buckling problem is defined by Equations (3-3), (3-4) and (4-5). Namely,

$$\begin{cases} \dfrac{dK_{zx}}{dz} = \dfrac{M}{(EI_x)(EI_y)} \cdot M_z - \dfrac{M}{EI_y} \cdot \dfrac{d\phi}{dz} & \text{(C-1)} \\[2ex] \dfrac{dM_z}{dz} = M \cdot \left(1 - \dfrac{EI_y}{EI_x}\right) \cdot K_{zx} & \text{(C-2)} \\[2ex] M_z = C_1 \cdot \dfrac{d\phi}{dz} & \text{(C-3)} \end{cases}$$

Differentiating Equation (C-2) with respect to z, we have

$$\frac{d^2M_z}{dz^2} = M \cdot \left(1 - \frac{EI_y}{EI_x}\right) \cdot \frac{dK_{zx}}{dz} \qquad \text{(C-4)}$$

Substituting Equation (C-1) into (C-4),

$$\frac{d^2 M_z}{dz^2} = \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M^2}{EI_x \cdot EI_y}\right) M_z - \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M^2}{EI_y}\right)\frac{d\phi}{dz} \quad \text{—— (C-5)}$$

Substituting Equation (C-3) into (C-5),

$$C_1 \frac{d^3\phi}{dz^3} = \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M^2}{EI_x \cdot EI_y}\right) \cdot C_1 \frac{d\phi}{dz} - \left(1 - \frac{EI_y}{EI_x}\right)\left(\frac{M^2}{EI_y}\right) \cdot \frac{d\phi}{dz}$$

$$\frac{d^3\phi}{dz^3} + \left(1 - \frac{EI_y}{EI_x}\right)\left(1 - \frac{C_1}{EI_x}\right)\left(\frac{M^2}{EI_y \cdot C_1}\right) \cdot \frac{d\phi}{dz} = 0$$

Using the expressions

$$\begin{cases} A = \dfrac{EI_y}{EI_x} \\[2ex] B = \dfrac{C_1}{EI_x} \end{cases} \quad \text{———————— (C-6)}$$

we have

$$\frac{d^3\phi}{dz^3} + (1-A)(1-B) \cdot \frac{M^2}{EI_y \cdot C_1} \cdot \frac{d\phi}{dz} = 0 \quad \text{———— (C-7)}$$

Assuming that the value of EIx is relatively large when compared to the values of EI_y and C_1, we have

$$\begin{cases} (1-A) > 0 \\ (1-B) > 0 \end{cases} \quad \text{———————— (C-8)}$$

Let

$$\alpha = (1-A)(1-B)\cdot\frac{M^2}{E\,I_y\cdot C_1} > 0 \qquad \text{(C-9)}$$

Equation (C-7) then becomes

$$\frac{d^3\phi}{dz^3} + \alpha\cdot\frac{d\phi}{dz} = 0 \qquad \text{(C-10)}$$

The auxiliary equation of Equation (C-10) is

$$p^3 + \alpha\, p = 0$$

giving the roots $p_1 = 0$, $p_2 = i\sqrt{\alpha}$, and $p_3 = -i\sqrt{\alpha}$. Thus the general solution of Equation (C-10) can be written as follows

$$\phi = A_1\cdot e^{i\sqrt{\alpha}} + A_2\cdot e^{-i\sqrt{\alpha}} + A_3$$

Here the constant A_3 represents a rigid body rotation which is equal to zero, since we do not consider the beam to have rigid body movements. Hence

$$\phi = A_1\cdot e^{i\sqrt{\alpha}} + A_2\cdot e^{-i\sqrt{\alpha}}$$

$$= (A_1 + A_2)\cos\sqrt{\alpha}\cdot z + (A_1 - A_2)\sin\sqrt{\alpha}\cdot z$$

Substituting a new constant A_1 for $A_1 + A_2$ and A_2 for $A_1 - A_2$, we have

$$\phi = A_1 \cos \sqrt{\alpha} \cdot Z + A_2 \sin \sqrt{\alpha} \cdot Z$$

Using the boundary condition $\begin{cases} \phi = 0 \\ z = 0 \end{cases}$, we have

$$A_1 = 0$$

Hence

$$\phi = A_2 \sin\sqrt{\alpha} \cdot z$$

Using the boundary condition $\begin{cases} \phi = 0 \\ z = L \end{cases}$, we have

$$0 = A_2 \sin\sqrt{\alpha} \cdot L$$

For a non-trivial solution, A_2 must not equal to zero. Thus

$$\sin\sqrt{\alpha} \cdot L = 0 \qquad \text{(C-11)}$$

The smallest root of Equation (C-11) is

$$\sqrt{\alpha} \cdot L = \pi$$

$$\sqrt{\alpha} = \frac{\pi}{L} \qquad \text{(C-12)}$$

Substituting Equation (C-9) into (C-12), we have

$$\sqrt{(1-A)(1-B)} \cdot \frac{M_{cr}}{\sqrt{E I_y \cdot C_1}} = \frac{\pi}{L}$$

$$M_{cr} = \frac{\pi}{L} \cdot \sqrt{\frac{EI_y \cdot C_1}{\left(1 - \frac{EI_y}{EI_x}\right)\left(1 - \frac{C_1}{EI_x}\right)}} \qquad \text{(C-13)}$$

This is the equation obtained by Neal [5]. With further rearrangements, Equation (C-13) becomes

$$M_{cr} = \frac{1}{\sqrt{(1-A)(1-B)}} \cdot \sqrt{\frac{\pi^2 EI_y}{L^2} \cdot C_1} \qquad \text{(C-14)}$$

And

$$\left(\frac{L}{r_y}\right)_{cr} = \frac{\pi}{r_y \cdot M_{cr}} \cdot \sqrt{\frac{EI_y \cdot C_1}{(1-A)(1-B)}} \qquad \text{(C-15)}$$

It is to be noted that although Equations (C-14) and (C-15) are derived for narrow rectangular beams, they are also applicable to box beams in which there is no warping resistance. At the end of Article 2 of Chapter II, it has been shown that a box section has no warping resistance under either of the conditions:

$$b_w t_f = b_f t_w$$

$$b_w t_w = b_f t_f$$

In each case Equations (C-14) and (C-15) are applicable.

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