Universal recoverability in quantum information

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Main message

- Entropy inequalities established in the 1970s are a mathematical consequence of the postulates of quantum physics
- They have a number of applications: for determining the ultimate limits on many physical processes (communication, thermodynamics, uncertainty relations, cloning)
- Many of these entropy inequalities are equivalent to each other, so we can say that together they constitute a fundamental law of quantum information theory
- There has been recent interest in refining these inequalities, trying to understand how well one can attempt to reverse an irreversible physical process
- We discuss progress in this direction

Background — entropies

Umegaki relative entropy [Ume62]

The quantum relative entropy is a measure of dissimilarity between two quantum states. Defined for state ρ and positive semi-definite σ as

$$D(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log \rho - \log \sigma]\}$$

whenever $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $+\infty$ otherwise

Operational interpretation (quantum Stein's lemma) [HP91, NO00]

Given are n quantum systems, all of which are prepared in either the state ρ or σ. With a constraint of $\varepsilon \in (0,1)$ on the Type I error of misidentifying ρ, then the optimal error exponent for the Type II error of misidentifying σ is $D(\rho\|\sigma)$.

Fundamental law of quantum information theory

Monotonicity of quantum relative entropy [Lin75, Uhl77]

Let ρ be a state, let σ be positive semi-definite, and let $\mathcal{N}$ be a quantum channel. Then

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

"Distinguishability does not increase under a physical process"

Characterizes a fundamental irreversibility in any physical process

Proof approaches (among many)

- Lieb concavity theorem [L73]
- relative modular operator method (see, e.g., [NP04])
- quantum Stein's lemma [BS03]

Equality conditions [Pet86, Pet88]

When does equality in monotonicity of relative entropy hold?

- $D(\rho\|\sigma) = D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$ iff $\exists$ a recovery map $\mathcal{P}_{\sigma,\mathcal{N}}$ such that

$$\rho = (\mathcal{P}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho), \qquad \sigma = (\mathcal{P}_{\sigma,\mathcal{N}} \circ \mathcal{N})(\sigma)$$

- This "Petz" recovery map has the following explicit form [HJPW04]:

$$\mathcal{P}_{\sigma,\mathcal{N}}(\omega) \equiv \sigma^{1/2}\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2}$$

- Classical case: Distributions p_X and q_X and a channel $\mathcal{N}(y|x)$. Then the Petz recovery map $\mathcal{P}(x|y)$ is given by the Bayes theorem:

$$\mathcal{P}(x|y)q_Y(y) = \mathcal{N}(y|x)q_X(x)$$

where $q_Y(y) \equiv \sum_x \mathcal{N}(y|x)q_X(x)$

Approximate case

Approximate case would be useful for applications

Approximate case for monotonicity of relative entropy

- What can we say when $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = \varepsilon$?
- Does there exist a CPTP map $\mathcal{R}$ that recovers σ perfectly from $\mathcal{N}(\sigma)$ while recovering ρ from $\mathcal{N}(\rho)$ approximately? [WL12]

One-shot measure of similarity for quantum states

Fidelity [Uhl76]

Fidelity between ρ and σ is $F(\rho,\sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$. Has a one-shot operational interpretation as the probability with which a purification of ρ could pass a test for being a purification of σ.

New result of [Wil15, JSRWW15]

Recoverability Theorem

Let ρ and σ satisfy $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a channel. Then

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\int_{-\infty}^{\infty} dt\, p(t) \log\left[F\left(\rho, \mathcal{P}^{t/2}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\right)\right],$$

where $p(t)$ is a distribution and $\mathcal{P}^{t}_{\sigma,\mathcal{N}}$ is a rotated Petz recovery map:

$$\mathcal{P}^{t}_{\sigma,\mathcal{N}}(\cdot) \equiv \left(\mathcal{U}_{\sigma,t} \circ \mathcal{P}_{\sigma,\mathcal{N}} \circ \mathcal{U}_{\mathcal{N}(\sigma),-t}\right)(\cdot),$$

$\mathcal{P}_{\sigma,\mathcal{N}}$ is the Petz recovery map, and $\mathcal{U}_{\sigma,t}$ and $\mathcal{U}_{\mathcal{N}(\sigma),-t}$ are defined from $\mathcal{U}_{\omega,t}(\cdot) \equiv \omega^{it}(\cdot)\,\omega^{-it}$, with ω a positive semi-definite operator.

Two tools for proof

Rényi generalization of a relative entropy difference and the Stein–Hirschman operator interpolation theorem

Universal Recovery

Universal Recoverability Corollary

Let ρ and σ satisfy $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a channel. Then

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log F(\rho, \mathcal{R}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))),$$

where

$$\mathcal{R}_{\sigma,\mathcal{N}} \equiv \int_{-\infty}^{\infty} dt\, p(t)\, \mathcal{P}_{\sigma,\mathcal{N}}^{t/2}$$

(follows from concavity of logarithm and fidelity)

Universal Distribution

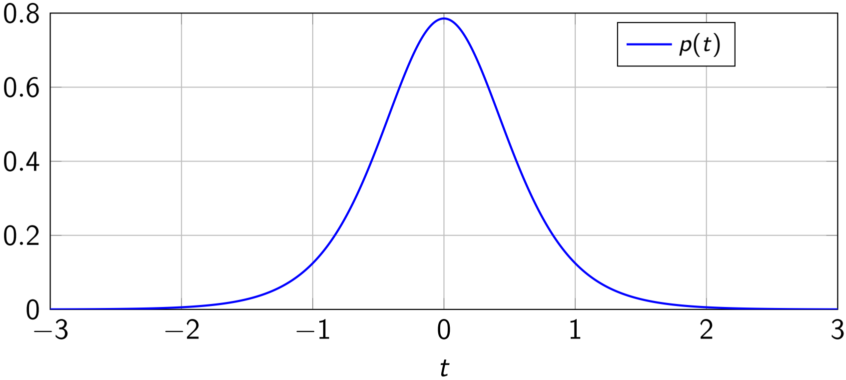

Figure: This plot depicts the probability density $p(t) := \frac{\pi}{2}\left(\cosh(\pi t) + 1\right)^{-1}$ as a function of $t \in \mathbb{R}$. We see that it is peaked around $t = 0$ which corresponds to the Petz recovery map.

Rényi generalizations of a relative entropy difference

Definition from [BSW14, SBW14]

$$\widetilde{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) \equiv \frac{2}{\alpha'} \log \left\| \left(\mathcal{N}(\rho)^{-\alpha'/2}\mathcal{N}(\sigma)^{\alpha'/2} \otimes I_E\right) U\sigma^{-\alpha'/2}\rho^{1/2} \right\|_{2\alpha},$$

where $\alpha \in (0,1) \cup (1,\infty)$, $\alpha' \equiv (\alpha - 1)/\alpha$, and $U_{S\to BE}$ is an isometric extension of $\mathcal{N}$.

Important properties

$$\lim_{\alpha\to 1} \widetilde{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) = D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)).$$

$$\widetilde{\Delta}_{1/2}(\rho,\sigma,\mathcal{N}) = -\log F(\rho, \mathcal{P}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))).$$

Stein–Hirschman operator interpolation theorem (setup)

Let $S \equiv \{z \in \mathbb{C} : 0 < \text{Re}\{z\} < 1\}$, and let $L(\mathcal{H})$ be the space of bounded linear operators acting on $\mathcal{H}$. Let $G : \overline{S} \to L(\mathcal{H})$ be an operator-valued function bounded on $\overline{S}$, holomorphic on S, and continuous on the boundary $\partial\overline{S}$. Let $\theta \in (0, 1)$ and define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \ ,$$

where $p_0, p_1 \in [1, \infty]$.

Stein–Hirschman operator interp. theorem (statement)

Then the following bound holds

$$\log \left\| G(\theta) \right\|_{p_\theta} \leq \int_{-\infty}^{\infty} dt \ \left(\alpha_\theta(t) \log \left[\left\| G(it) \right\|_{p_0}^{1-\theta} \right] + \beta_\theta(t) \log \left[\left\| G(1+it) \right\|_{p_1}^{\theta} \right] \right) \ ,$$

$$\text{where } \alpha_\theta(t) \equiv \frac{\sin(\pi\theta)}{2(1-\theta)\left[\cosh(\pi t) - \cos(\pi\theta)\right]},$$

$$\beta_\theta(t) \equiv \frac{\sin(\pi\theta)}{2\theta\left[\cosh(\pi t) + \cos(\pi\theta)\right]},$$

$$\lim_{\theta \searrow 0} \beta_\theta(t) = p(t).$$

Proof of Recoverability Theorem

Tune parameters

$$\text{Pick } G(z) \equiv \left([\mathcal{N}(\rho)]^{z/2} [\mathcal{N}(\sigma)]^{-z/2} \otimes I_E \right) U \sigma^{z/2} \rho^{1/2},$$

$$p_0 = 2, \;\; p_1 = 1, \;\; \theta \in (0,1) \;\; \Rightarrow p_\theta = \frac{2}{1+\theta}$$

Evaluate norms

$$\|G(it)\|_2 = \left\| \left(\mathcal{N}(\rho)^{it/2} \mathcal{N}(\sigma)^{-it/2} \otimes I_E \right) U \sigma^{it/2} \rho^{1/2} \right\|_2 \leq \left\| \rho^{1/2} \right\|_2 = 1,$$

$$\|G(1+it)\|_1 = \left[F\left(\rho, \mathcal{P}^{t/2}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho)) \right) \right]^{1/2}.$$

Proof of Recoverability Theorem (ctd.)

Apply the Stein–Hirschman theorem

$$\log \left\| \left([\mathcal{N}(\rho)]^{\theta/2} [\mathcal{N}(\sigma)]^{-\theta/2} \otimes I_E \right) U \sigma^{\theta/2} \rho^{1/2} \right\|_{2/(1+\theta)} \leq \int_{-\infty}^{\infty} dt \ \beta_\theta(t) \log \left[F\left(\rho, (\mathcal{P}_{\sigma,\mathcal{N}}^{t/2} \circ \mathcal{N})(\rho)\right)^{\theta/2} \right].$$

Final step

Apply a minus sign, multiply both sides by $2/\theta$, and take the limit as $\theta \searrow 0$ to conclude.

Specializing to the Holevo Bound

- Specializing to the Holevo bound leads to a refinement. Given

$$\rho_{XB} \equiv \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_B^x, \qquad \omega_{XY} \equiv \sum_y \langle \varphi^y|_B \rho_{XB} |\varphi^y\rangle_B |y\rangle\langle y|_Y .$$

- Then the following inequality holds

$$I(X;B)_\rho - I(X;Y)_\omega \geq -2 \log \sum_x p_X(x) \sqrt{F}(\rho_B^x, \mathcal{E}_B(\rho_B^x)),$$

- where $\mathcal{E}_B$ is an entanglement-breaking map of the form

$$\mathcal{E}_B(\cdot) \equiv \int_{-\infty}^{\infty} dt\ \beta_0(t)\ \sum_y \langle \varphi_y|_B (\cdot) |\varphi_y\rangle_B \frac{\rho_B^{(1+it)/2} |\varphi_y\rangle\langle \varphi_y|_B \rho_B^{(1-it)/2}}{\langle \varphi_y|_B \rho_B |\varphi_y\rangle_B}.$$

Applying to Entropy

Special case: Entropy gain (also called Entropy Production)

- Specializing to entropy gives the following bound for a unital quantum channel $\mathcal{N}$:

$$H(\mathcal{N}(\rho)) - H(\rho) \geq -\log F(\rho, \mathcal{N}^\dagger(\mathcal{N}(\rho)))$$

- A different approach [BDW16] gives a stronger bound and applies to more general maps. For $\mathcal{N}$ a positive, subunital, trace-preserving map:

$$H(\mathcal{N}(\rho)) - H(\rho) \geq D(\rho \| \mathcal{N}^\dagger(\mathcal{N}(\rho))) \geq 0$$

Application to entropy uncertainty relations [BWW15]

- Let ρ_{ABE} be a state for Alice, Bob, and Eve, and let $\mathbb{X} \equiv \{P_A^x\}$ and $\mathbb{Z} = \{Q_A^z\}$ be projection-valued measures for Alice's system
- Define the post-measurement states:

$$\sigma_{XBE} \equiv \sum_x |x\rangle\langle x|_X \otimes \sigma_{BE}^x \qquad \text{where}$$

$$\sigma_{BE}^x \equiv \mathrm{Tr}_A\{(P_A^x \otimes I_{BE})\rho_{ABE}\}$$

$$\omega_{ZBE} \equiv \sum_z |z\rangle\langle z|_Z \otimes \omega_{BE}^z \qquad \text{where}$$

$$\omega_{BE}^z \equiv \mathrm{Tr}_A\{(Q_A^z \otimes I_{BE})\rho_{ABE}\}$$

- Then

$$H(Z|E)_\omega + H(X|B)_\sigma \geq -\log \max_{x,z} \|P_A^x Q_A^z\|_\infty^2 - \log F(\rho_{AB}, \mathcal{R}_{XB\to AB}(\sigma_{XB}))$$

Case of quantum Gaussian channels

- If σ is a Gaussian state and $\mathcal{N}$ is a Gaussian channel, then the Petz recovery map $\mathcal{P}_{\sigma,\mathcal{N}}$ is a Gaussian channel (result with Lami and Das).
- We have an explicit form for the Petz recovery map in terms of its action on the mean vector and covariance matrix of a quantum Gaussian state.
- We have the same for rotated Petz recovery maps.

Quantum cloning, partial trace, and recovery [LW16]

- Let $\omega^{(n)}$ be a state with support in the symmetric subspace of $(\mathbb{C}^d)^{\otimes n}$, let $\pi_{\text{sym}}^{d,n}$ denote the maximally mixed state on this symmetric subspace, let $\mathcal{C}_{k\to n}$ denote a universal quantum cloning machine, and $\mathcal{P}_{n\to k}$ the symmetrize partial trace. Then

$$D(\omega^{(n)}\|\pi_{\text{sym}}^{d,n}) \geq D(\mathcal{P}_{n\to k}(\omega^{(n)})\|\mathcal{P}_{n\to k}(\pi_{\text{sym}}^{d,n})) + D(\omega^{(n)}\|(\mathcal{C}_{k\to n}\circ\mathcal{P}_{n\to k})(\omega^{(n)})).$$

- With the same notation, the following inequality holds

$$D(\omega^{(k)}\|\pi_{\text{sym}}^{d,k}) \geq D(\mathcal{C}_{k\to n}(\omega^{(k)})\|\mathcal{C}_{k\to n}(\pi_{\text{sym}}^{d,k})) + D(\omega^{(k)}\|(\mathcal{P}_{n\to k}\circ\mathcal{C}_{k\to n})(\omega^{(k)})).$$

- So cloning machines and partial trace are dual to each other in the above sense.

Generality of approach [DW15]

- Technique is very general and can be used to prove inequalities for norms of multiple operators chained together (called "Swiveled Renyi Entropies" in [DW15], due to presence of "unitary swivels")
- Example: The following quantity

$$\widetilde{L}'_{\alpha}\left(\rho_{A_1\cdots A_l}\right) \equiv \frac{2}{\alpha'} \max_{\left\{V_{\rho_S}\right\}_S} \log \left\| \left[\prod_{S\in\mathcal{P}'} \rho_S^{-a_S\alpha'/2} V_{\rho_S}\right] \rho_{A_1\cdots A_l}^{1/2} \right\|_{2\alpha},$$

  where $\alpha' = (\alpha - 1)/\alpha$ is monotone increasing in α for $\alpha \in [1/2, \infty]$.
- Another example: for positive semi-definite operators $C_1, \ldots, C_L$, a unitary V_{C_i} commuting with C_i, and $p \geq 1$, the quantity

$$\max_{V_{C_1},\ldots,V_{C_L}} \left\| C_1^{1/p} V_{C_1} \cdots C_L^{1/p} V_{C_L} \right\|_p^p$$

  is monotone decreasing in p for $p \geq 1$. (See also [Wil16])

Generality of approach (ctd.) [DW15]

- Another example: Let $C_1, \ldots, C_L$ be positive semi-definite operators, and let $p > q \geq 1$. Then the following holds [DW15, Wil16]:

$$\log \left\| C_1^{1/p} C_2^{1/p} \cdots C_L^{1/p} \right\|_p^p \leq \int_{-\infty}^{\infty} dt \ \beta_{q/p}(t) \ \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \cdots C_L^{(1+it)/q} \right\|_q^q .$$

- By taking a limit: Let $C_1, \ldots, C_L$ be positive definite operators, and let $q \geq 1$. Then the following inequality holds [DW15, Wil16]:

$$\log \operatorname{Tr}\left\{\exp\left\{\log C_1 + \cdots + \log C_L\right\}\right\} \leq \int_{-\infty}^{\infty} dt \ \beta_0(t) \ \log \left\| C_1^{(1+it)/q} C_2^{(1+it)/q} \cdots C_L^{(1+it)/q} \right\|_q^q .$$

Conclusions

- The result in [Wil15, JSRWW15] applies to relative entropy differences, has a brief proof, and yields a universal recovery map (depending only on σ and $\mathcal{N}$).
- Applications in a variety of areas, including entropy gain [BDW16], entropic uncertainty [BWW15], quantum cloning [LW16], quantum Gaussian channels, etc.
- Later results of [DW15] clarify how the approach is very general and leads to many other inequalities
- It has been conjectured that the recovery map can be the Petz recovery map alone (not a rotated Petz map), but it is unclear whether this will be true.

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