FORBIDDEN SUBGRAPHS AND 3-COLORABILITY

A Thesis
Presented to
The Academic Faculty

by

Tianjun Ye

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics

Georgia Institute of Technology
August 2012

FORBIDDEN SUBGRAPHS AND 3-COLORABILITY

Approved by:

Professor Xingxing Yu, Committee Chair
School of Mathematics
Georgia Institute of Technology

Professor Xingxing Yu, Advisor
School of Mathematics
Georgia Institute of Technology

Professor William T. Trotter
School of Mathematics
Georgia Institute of Technology

Professor Robin Thomas
School of Mathematics
Georgia Institute of Technology

Professor Prasad Tetali
School of Mathematics
Georgia Institute of Technology

Professor Guantao Chen
Department of Mathematics and Statistics
Georgia State University

Date Approved: 26 May 2012

To my family and friends,

for their support and love

ACKNOWLEDGEMENTS

It's said that the age span of twenty to thirty is the "golden age" for a person. It's lucky for me to have been admitted as Ph.D. student at Georgia Tech; thus I could spend four years of my "golden age" here. Through four years of studying and working, I grew up from an aimless student to a person with a clear career plan, ready to accept the responsibility of work and life.

Georgia Tech was a great academic environment in which to study and to do research. I met many professors who provided me with knowledgeable mentor-ship and insightful instruction. First of all, I would like to express my sincere gratitude to my advisor, Professor Xingxing Yu, for his enthusiasm, patience and encouragement throughout my Ph.D. studies. Also, I would like to thank Professors William T. Trotter, Robin Thomas, Prasad Tetali and Guantao Chen for being members of my committee. I would like to convey my sincere thanks to Professors Asaf Shapira and Robin Thomas for their inspiring lectures which broadened my research scope. And, it is my pleasure to thank Professor Luca Dieci for recruiting me to Georgia Tech, so that I could meet those great people who may change my life orbit.

Georgia Tech also provides a vast platform enabling us to pursue careers in industry. It is my pleasure to thank Cathy Jacobson for her help on my English presentation, which is the first step to stepping off campus and on to a career. Also, thanks to Norfolk Southern Railway Company for giving me a summer internship opportunity, that introduced me to working in industry. I am also indebted to many of my colleagues for helpful suggestions and references, and for providing a stimulating environment in which to grow.

I love my parents, Minzeng Ye and Hongmei Gao, for giving birth to me. I love my boyfriend, Sijun Liu, for his solid support. Finishing a Ph.D. is like finishing a marathon, and love is always my motivation to fight and struggle. I love Georgia Tech; it carries my four-year memory of sorrow and joy. I love Atlanta; it's the witness of my family's love and support.

I know that finishing a Ph.D. is not the end of the story. Actually, it's the starting point of my career. Thanks to Georgia Tech, thanks to everyone who gave me help during these four years. To them, I dedicate this thesis.

SUMMARY

Classical vertex coloring problems ask for the minimum number of colors needed to color the vertices of a graph, such that adjacent vertices use different colors. Vertex coloring does have quite a few practical applications in communication theory, industry engineering and computer science. Such examples can be found in the book of Hansen and Marcotte [23].

Deciding whether a graph is 3-colorable or not is a well-known NP-complete problem, even for triangle-free graphs. Intuitively, large girth may help reduce the chromatic number. However, in 1959, Erdős [13] used the probabilistic method to prove that for any two positive integers g and k, there exist graphs of girth at least g and chromatic number at least k. Thus, restricting girth alone does not help bound the chromatic number. However, if we forbid certain tree structure in addition to girth restriction, then it is possible to bound the chromatic number. Randerath determined several such tree structures, and conjectured that if a graph is fork-free and triangle-free, then it is 3-colorable, where a *fork* is a star $K_{1,4}$ with two branches subdivided once.

The main result of this thesis is that Randerath's conjecture is true for graphs with odd girth at least 7. We also give a proof that Randerath's conjecture holds for graphs with maximum degree 4.

CHAPTER I

INTRODUCTION

We study colorablility of graphs in terms of forbidden subgraphs. In this chapter, we introduce notation and terminology that we use in this thesis, and review several classical coloring results. We will then discuss the notion of *Vizing bound*, and state a related conjecture of Randerath on 3-coloring. We conclude this chapter by stating our main result, and outlining its proof.

1.1 Notation and Terminology

We only consider simple graphs unless specified, as multiple edges don't effect coloring. Given a finite, simple, undirected graph G, we use $V(G)$ to denote the vertex set of G, and $E(G)$ to denote the edge set of G. Also, we use $|V(G)|$ to denote the number of vertices in G, and $|E(G)|$ to denote the number of edges in G.

Given a graph H, if $V(H) \subseteq V(G)$, and $E(H) \subseteq E(G)$, then H is a subgraph of G, denoted $H \subseteq G$. Given two initial graphs G_1 and G_2, we write $G = G_1 \cup G_2$ if $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$; $G = G_1 \cap G_2$ if $V(G) = V(G_1) \cap V(G_2)$ and $E(G) = E(G_1) \cap E(G_2)$. For $S \subseteq V(G)$, let $G[S]$ be the graph with vertex set S, and whose edges are those of G with both ends in S, and we call it the subgraph of G *induced* by S. A graph H is an *induced* subgraph of a graph G, if there exists a set $A \subseteq V(G)$ such that $G[A]$ is isomorphic to H. An induced subgraph that is a complete graph is called a *clique*. We use $\omega(G)$ to denote the maximum size of a clique in G. For convenience a graph G without an induced subgraph H is called H-free.

A *path* in G is a sequence of distinct vertices in which consecutive vertices are adjacent. A *cycle* is a sequence of vertices $v_1 v_2 \cdots v_k v_1$, such that consecutive vertices are adjacent. Let C_k denote a cycle with k vertices. A cycle with an odd number of vertices is called odd cycle. The *girth* of a graph is the length of a shortest cycle contained in the graph, and the *odd girth* is the length of a shortest odd cycle, denote by $og(G)$. A *cut* is a partition of the vertices of a graph into two disjoint subsets. The *edge-cut* of the cut is the set of edges whose end points are in different subsets of the partition. The *edge-connectivity* is the size of a smallest edge cut.

Let G be a graph. For any $u, v \in V(G)$, the distance between u and v in G is the number of edges in a shortest path connecting them, and we denote it by $d(u, v)$. For any $u \in V(G)$, let $N_i(u) = \{v \in V(G) : d(u, v) = i\}$ for any $i \in \mathbb{N}$. For any $H \subseteq G$ and $u \in V(G)$, define $d(H, u) = min\{u' \in V(H) : d(u', u)\}$, let $N_i(H) = \{v \in V(G) : d(H, v) = i\}$ for any $i \in \mathbb{N}$. For any graph H and G, define $G - H$ as the graph obtained from G by deleting $H \cap G$. For any $u \in V(G)$, $G - u$ is the graph obtained from G by deleting u and all the edges adjacent to u. For any $e \in E(G)$, $G - e$ is the graph obtained from G by deleting e. Given $u \in V(G)$, the degree of u is the number of edges incident to the vertex, and denoted by $d(u)$. Define $V_k = \{v \in V(G) : d(v) = k\}$ for all $k \in \mathbb{Z}$. For any $X \subseteq V(G)$, G/X is defined as contracting all vertices in X.

Let $\delta(G)$ denote the minimum degree of G, and $\Delta(G)$ denote the maximum degree of G. Given two vertices $u, v \in V(G)$, we use $u \sim v$ denote that u is adjacent to v, and $u \nsim v$ denote that u is not adjacent to v. Similarly, given $S \subseteq G$ and $u \in V(G)$, we use $u \sim S$ to denote that u is adjacent to at least one vertex in $V(S)$, and use $u \nsim S$ to denote that u is not adjacent to all vertices of $V(S)$. For terminology and notation not defined here we refer to [43].

A *vertex coloring* of G is an assignment of colors to the vertices of G such that no two adjacent vertices share the same color. A coloring using at most k colors is called a *k-coloring*. The smallest k such that G admits a k-coloring is called the *chromatic number* of G, and is denoted by $\chi(G)$. We say that the graph G is *k-chromatic* if $\chi(G) = k$; and *k-colorable* if $\chi(G) \leq k$.

Similarly, an *edge coloring* of a graph G is an assignment of colors to the edges of G

such that no two adjacent edges share the same color. An edge-coloring using at most k colors is called a *k-edge-coloring*. The smallest k such that G admits a k-edge-coloring is the *chromatic index* of G, and is denoted by $\chi'(G)$.

A graph is said to be *critical*, if $\chi(G) > \chi(H)$ for every proper induced subgraph H of G. Moreover, if $\chi(G) = k$ and G is critical, then G is *k-critical*.

The line graph $L(G)$ of the graph G is the graph with vertex set $E(G)$, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G are adjacent. Thus, by definition, $\chi(L(G)) = \chi'(G)$.

Let G be a graph and for each $v \in V(G)$, let $L(v)$ be a set of colors. A *list coloring* of G is a function that maps every vertex v of G to a color in the list $L(v)$, such that no two adjacent vertices receive the same color. A graph is *k-choosable* (or *k-list-colorable*) if it has a list coloring no matter how one assigns a list of k colors to each vertex. The *choosability* (or *list colorability* or *list chromatic number*) of G, denoted by $ch(G)$, is the least number k such that G is k-choosable. Obviously, $ch(G) \geq \chi(G)$, since a k-coloring of G is a k-list-coloring of G such that all vertices are assigned the same list of k colors.

It is easy to see that a tree has chromatic number at most 2. However, a graph may have large chromatic number even if it locally looks like a tree. We will be studying chromatic number of graphs with certain trees forbidden as induced subgraph.

1.2 Vertex Coloring

In general, it is NP-hard to decide $\chi(G)$ for an arbitrary graph G, even for triangle-free graphs. Therefore, much effort has been spent on determining bounds on $\chi(G)$ for various classes of graphs.

The most well known result on graph coloring is the Four Color Theorem [4, 5, 6, 7] that every planar graph is 4-colorable. The Four Color Theorem was first proven by Appel and Haken in 1976 using computer, and efficient algorithms have been found for 4-coloring maps in $O(n^2)$ time, where n is the number of vertices. In 1996, Robertson, Sanders, Seymour, and Thomas found a better proof, and improved a coloring algorithm based on Appel and Hakens proof [34, 35]. Chudnovsky, Robertson, Seymour and Thomas [11] also solved the famous conjecture *Strong Perfect Graph Conjecture* in 2002: a graph G is perfect if and only if neither G nor its complement $\overline{G}$ contains an induced odd cycle of order at least five.

In 1959, Grötzsch [18] proved that every triangle-free planar graph is 3-colorable. Later in 1963, Grünbaum generalized Grötzsch's theorem by showing that if a planar graph has at most three triangles then it is 3-colorable [19]. There exists graphs with four triangles whose chromatic number is 4, such as the complete graph K_4. Indeed, there exists a planar graph without 4-cycle whose chromatic number is 4.

From definition, we see that $ch(G) \geq \chi(G)$ for any graph G. Actually, $ch(G)$ cannot be bounded by any function of chromatic number. In 1996, Gravier [17] showed that there exist graphs with $\chi(G) = 2$ and arbitrarily large $ch(G)$. However, for a planar graph G, Thomassen [37, 38] showed that $ch(G) \leq 5$, and $ch(G) \leq 3$ if girth of G is at least 5. Alon and Tarsi proved that $ch(G) \leq 3$ if G is a bipartite planar graph [3].

Grötzsch's theorem cannot be generalized to nonplanar triangle-free graphs. There exist triangle-free nonplanar graphs G with $\chi(G) > 3$, such as the Grötzsch graph [42] and the Chvátal graph [41]. In 1955, Mycielski constructed triangle-free graphs with arbitrarily large chromatic number [29]. In 1959, Erdős [13] proved that for any two positive integers g and k, there exist graphs of girth at least g and chromatic number at least k. Therefore, restricting girth alone need not help to bound chromatic number of nonplanar graphs.

Chromatic number is related to other graph invariants. For example $\chi(G) \leq max\{\delta(H)|H \subseteq G\}+1$, and $\chi(G) \leq \frac{1}{2}+\sqrt{2|E(G)|+\frac{1}{4}}$, see [12]. While it's easy to show that $\chi(G) \leq \Delta(G)+1$ for any graph G, in 1941, Brooks showed the following (Lovász [28] and Bryant [10] gave simplified proof of Brooks' theorem separately):

Theorem 1.1 (Brooks [9])**.** *If G is neither a complete graph nor an odd cycle, then* $\chi(G) \leq \Delta(G)$.

Brooks' colorings can be found in linear time, and efficient algorithms are also known in parallel and distributed computing [16, 22]. Vizing [40] extended Brooks' Theorem to list coloring: $ch(G) \leq \Delta(G)$ if G is neither clique nor odd cycle. In 1999, Reed [33] proved that for large $\Delta(G)$, $\Delta(G) - 1$ colors suffice to color G if and only if the given graph is $\Delta(G)$-clique free.

The following result on critical graph will be useful:

Theorem 1.2 (Gallai [15])**.** *Let G be a k-critical graph and $Low(G)$ denote the graph of G induced by the vertices of G of degree $k-1$. Then every 2-connected induced subgraph of $Low(G)$ either is an odd hole (odd cycle of length greater than 3) or is complete.*

While it is difficult to get better bound on $\chi(G)$ for general graphs, Brooks' Theorem may be improved by forbidding triangles as subgraphs. For example, Brooks [9] showed that if G is a triangle-free and $K_{1,r}$-free graph, then G is r-colorable unless G is isomorphic to an odd cycle or a complete graph with at most two vertices. Also for triangle-free graphs, or more generally graphs in which the neighborhood of every vertex is sufficiently sparse, the chromatic number is $O(\Delta/log\Delta)$ [2].

If we forbid additional structures from the graph, we could extend triangle-free version of Brooks' Theorem. An *r-sunshade* (with $r \geq 3$) is a complete bipartite graph $K_{1,r}$ with one edge subdivided once. The 3-sunshade is called *chair* and the 4-sunshade *cross*. There are several results proved by Randerath [32]. Let G be a connected and triangle-free graph, If G is chair-free, then $\chi(G) \leq 3$; and the equality holds if and only if G is an odd hole [31]. If G is cross-free graph, then $\chi(G) \leq 3$ [32]. If G is r-sunshade free graph with $r \geq 3$, and G is not an odd cycle, then $\chi(G) \leq r$; $\chi(G) \leq 2$ if $\Delta(G) \geq 2r - 3$; $\chi(G) \leq r - 1$ if $r \in \{3, 4\}$ or $\Delta(G) \leq r - 1$, see [30].

1.3 Edge Coloring

Although the problem we will work on is about vertex coloring of graphs, it was motivated by edge-colorings of graphs. One of the classical theorems on edge-coloring is due to

Vizing:

Theorem 1.3 (Vizing [39])**.** *Let* G *be a multi-graph, then* $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$. *Here* $\mu(G)$ *denotes the maximum number of edges joining two vertices in* G*. In particular,* $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ *if* G *is simple.*

Vizing's proof provides a polynomial time algorithm to color a graph G using $\Delta(G) + \mu(G)$ colors. If G is bipartite or G is planar with large minimum degree, then $\chi'(G) = \Delta(G)$ [1]. However, the general problem of determining chromatic index of a graph is NP-complete. There are a number of practical applications of edge colorings, such as scheduling problems and frequency assignment for fiber optic networks, see [14].

By the definition of a line graph, we know that $\chi'(G) = \chi(L(G))$, and $\Delta(G) = \omega(L(G))$. Thus, Vizing's Theorem can be reformulated as: $\chi(L(G)) \leq \omega(L(G)) + 1$. Therefore, if $\chi(G) \leq \omega(G) + 1$ then G is said to satisfy the *Vizing Bound*. So line graphs satisfying Vizing bound.

In 1968, Beineke [8] gave an characterization of line graphs in terms of nine forbidden induced subgarphs. (A different characterization is given in [27]).

Theorem 1.4 (Beinek [8])**.** *A graph is a line graph if and only if no subset of its vertices induces one of the nine graphs in Fig.1.*

In [24], it is shown that graphs which are $K_{1,3}$-free and $(K_5 - e)$-free satisfy the Vizing bound. Randerath studied the problem of finding all pairs (A, B) of connected graphs, such that if G has no induced subgraph isomorphic to A or B, then G satisfies the Vizing bound. Such a pair (A, B) is called a Vizing pair.

1.4 Vizing Pair

Let A, B be connected graphs. We say that (A, B) is a *good vizing pair* if (A, B) is a Vizing Pair, and there are graphs that are A-free or B-free, but does not satisfy the Vizing bound. Moreover, a good Vizing-pair is *saturated*, if for every good Vizing-pair (A', B') with $A \subset$

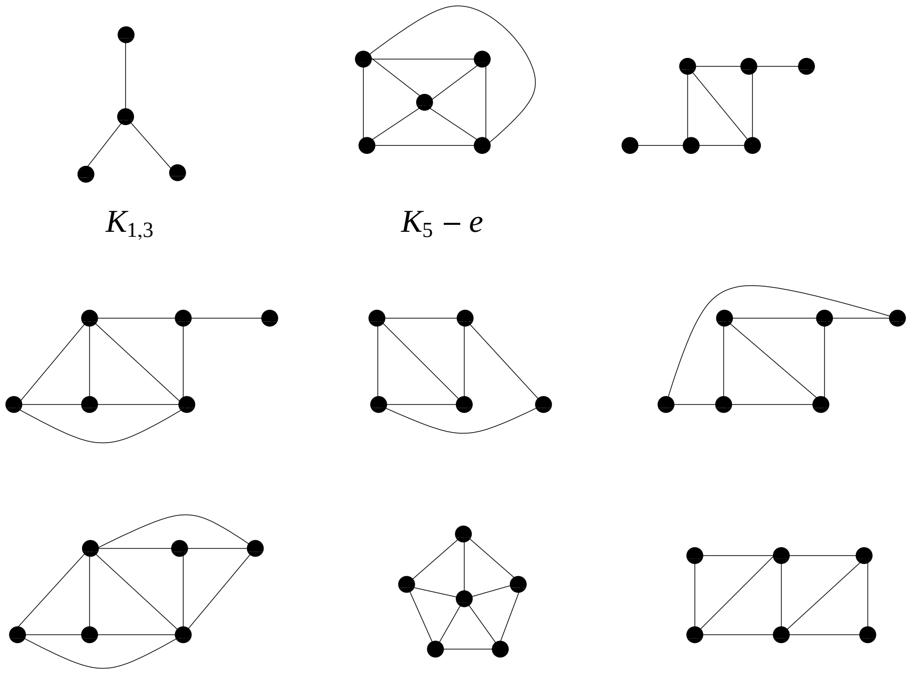

Figure 1: Beineke Graphs

A' and $B \subset B'$ we have $A \cong A'$ and $B \cong B'$. Those concepts are introduced in [31]. We summarize some of the results in [31].

Theorem 1.5. *If* (A, B) *is a good Vizing-pair, then* A *has to be a tree,* $A \subseteq P_4$ *and* $A \neq P_4$*, and* $B \in \{K_5 - e, HVN, K_4, K_3, P, D\}$*. See Fig.2.*

Let A *be a connected graph such that every* A*-free graph* G *with* $\omega(G) \leq 3$ *satisfies* $\chi(G) \leq \omega(G) + 1 \leq 4$*. Then* A *is an induced subgraph of the chair.*

Let B *be an induced subgraph of the* HVN *or the* $K_5 - e$ *and let* G *be a* B*-free and chair-free graph, then* $\chi(G) \leq \omega(G) + 1$.

In his Ph.D. thesis, Randerath examined good Vizing pairs for triangle-free graphs. Determine all pairs (A, B) of connected graphs such that, if G is A-free and B-free, then G is 3-colorable [32]. The absence of K_4 is a necessary condition for 3-colorability. Let B be an induced subgraph of K_4. Then $B \cong K_4$ and $B \cong K_3$ ($B = K_1, K_2$ is not interesting). By Theorem1.5, A must be a tree if (A, B) is to be a good pair. Randerath showed that (K_4, P_4) is a Vizing pair that is good and saturated [32].

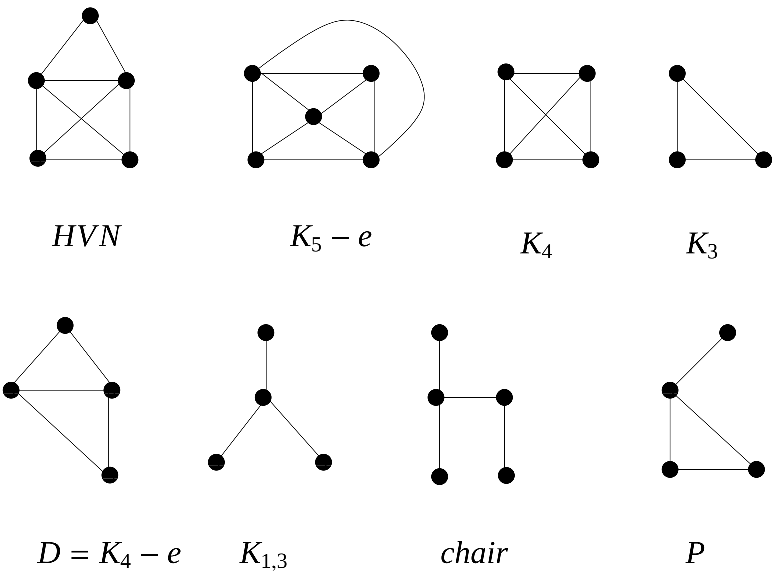

Figure 2: Extremal Graphs

Thus, the case $B \cong K_4$ is settled. Randerath proved several results for triangle-free graphs, by considering the graphs H, E, fork, and cross, given in Fig. 3.

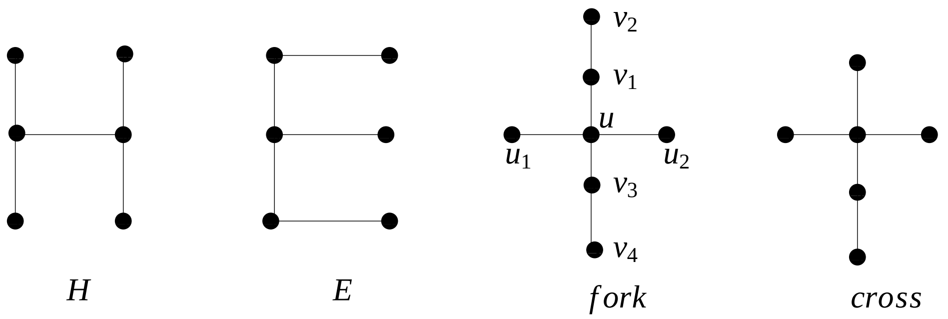

Figure 3: Forbidden Graphs

Theorem 1.6 ([32])**.** *Let* (A, B) *be a saturated pair of connected forbidden induced subgraphs implying 3-colorability. Then* $A \in \{K_3, K_4\}$ *and* $B \subset B' \in \{P_4, H, fork\}$*. Moreover, if* $A \cong K_4$*, then* $B \cong P_4$*. In case that* $A \cong K_3$*, then* $B \cong H$ *or* B *is an induced subgraph of the fork.*

Theorem 1.7 ([32])**.** *There are four good pair* (A, B) *of connected forbidden induced subgraphs implying 3-colorability:* (K_4, P_4)*,* (K_3, H)*,* (K_3, E)*,* $(K_3, cross)$*.*

Vizing bound is a special case of χ-bound. A family $\mathcal{F}$ of graphs is χ-bounded if there exists a function f, such that for any $G \in \mathcal{F}$, $\chi(G) \leq f(\omega(G))$. Gyárfás [20] and Sumner [36] independently conjectured that, for every forest T, the family of graphs which are T-free is χ-bounded. In 1980, Gyárfás, Szemerédi, and Tuza [21] proved the conjecture for triangle-free graphs when the radius of the tree T is at most two. Later in 1994, Kierstead and Penrice [25] proved this Gyárfás-Sumner conjecture for all graphs as long as the radius the tree T is at most two. Recently, Kierstead and Zhu [26] further extended this result by proving it for any radius three tree obtained from a radius two tree by making exactly one subdivision in every edge adjacent to the root.

1.5 Problem and main result

To complete the characterization of all saturated pairs (A, B) for 3-colorability, one need to decide if $(K_3, fork)$ is a good vizing pair. Thus, Randerath proposed the following conjecture.

Conjecture 1.8 ([32])**.** *Let G be a triangle-free and fork-free graph. Then* $\chi(G) \leq 3$.

The main result of this thesis is the following which proves Conjecture 1.8 for graph G with odd girth no less than 7.

Theorem 1.9. *Let G be a fork-free graph such that* $og(G) \geq 7$. *Then* $\chi(G) \leq 3$.

Let us give an outline of our proof. Suppose there is a minimum counterexample G. Then G is fork-free, $og(G) \geq 7$, and $\chi(G) \geq 4$. The main body of work is to show that G has two subgraphs H and K, such that $G = H \cup K$, and K contains a shortest odd cycle C of G with additional properties. $S := H \cap K$, $S \subseteq C$, and S consists of vertices of degree 3 in G. In Chapter 3, we show that C contains at least one degree 4 vertex. In Chapter 4, we show that any 4-cycle in G contains at least two degree 4 vertices. These two results will be used in Chapter 5 and 6 to determine the structure around C: given any vertex $u \in K$, the distance from u to C is at most 2 with some additional conditions. By the minimality of G, H has a 3-coloring, which induces a 3-coloring on S. In Chapter 2, we show that the 3-coloring on S could be extended to K, which means G is 3-colorable, a contradiction.

In Chapter 7, we also give a proof of Conjecture 1.8 for graphs with maximum degree 4.

CHAPTER II

WEAKLY DOMINATING CYCLE

Let G be a fork-free graph with $og(G) \geq 7$, and let $C = v_1 \dots v_g v_1$ be a shortest odd cycle in G. Suppose $V(G) = V(C) \cup N_1(C) \cup N_2(C)$, and we say that C is a *weakly* dominating cycle in G. Moreover, assume that for any $u \in N_2(C)$ there exist $1 \leq i \leq g$ and two paths uu_1v_{i-1} and uu_1v_{i+1}, and we say that u is *associated with* v_i. All operations in the subscript are modulo g. We derive properties (1)–(7) below about the structure of G.

(1) If $u \in N_2(C)$ is associated with v_i and if $w \in N(u) \cap N_1(C)$, then $N(w) \cap V(C) \subseteq \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$.

Let uu_1v_{i-1}, uu_1v_{i+1} be paths, and suppose there exist $w \in N(u) \cap N_1(C)$ and $v_j \in N(w) \setminus \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$. Then since $og(G) \geq 7$ and C is a shortest cycle in G, $w \neq u_1$ and $w \nsim \{v_{i-2}, v_i, v_{i+2}\}$. Then either $wuu_1v_{i+1}v_{i+2} \dots v_{j-1}v_jw$ or $wuu_1v_{i-1}v_{i-2} \dots v_{j+1}v_jw$ is an odd cycle shorter than C, a contradiction.

(2) For each $u \in N_2(C)$, if u is associated with v_i and v_j for some $i \neq j$ then $v_j \in \{v_{i-2}, v_{i+2}\}$.

For, suppose u is associated with v_i and v_j, and let uu_1v_{i-1}, uu_1v_{i+1}, uw_1v_{j-1}, and uw_1v_{j+1} be paths. By (1), $v_{j-1}, v_{j+1} \in \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$; so $v_j \in \{v_{i-2}, v_i, v_{i+2}\}$. Thus, if $v_j \neq v_i$ we have $v_j \in \{v_{i-2}, v_{i+2}\}$.

(3) No v_i can be associated with two distinct vertices in $N_2(C)$.

Suppose $u, w \in N_2(C)$ such that $u \neq w$ and u, w are associated with v_i. Let uu_1v_{i-1}, uu_1v_{i+1}, uw_1v_{i-1}, and uw_1v_{i+1} be paths in G. Then $u_1 \nsim w$ to avoid the fork $(u_1u, u_1w, u_1v_{i-1}v_{i-2}, u_1v_{i+1}v_{i+2})$, and $w_1 \nsim u$ to avoid the fork $(w_1u, w_1w, w_1v_{i-1}v_{i-2}, w_1v_{i+1}v_{i+2})$. In particular, $u_1 \neq w_1$. Now, using the minimality of C, it is easy to see that $(v_{i+1}v_i, v_{i+1}v_{i+2}, v_{i+1}u_1u, v_{i+1}w_1w)$ is a fork in G, a contradiction.

(4) Let $u, w \in N_2(C)$ be associated with v_i, v_j, respectively such that $u \neq w$. Then $v_j \notin \{v_{i-2}, v_i, v_{i+2}\}$, and if $u \sim w$ then $v_j \in \{v_{i-3}, v_{i-1}, v_i, v_{i+1}, v_{i+3}\}$.

Let $uu_1v_{i-1}, uu_1v_{i+1}, uw_1v_{j-1}, uw_1v_{j+1}$ be paths in G. By (3), $v_j \neq v_i$. If $v_j \in \{v_{i-2}, v_{i+2}\}$ then by symmetry let $v_j = v_{i+2}$; now $u_1 \neq w_1$ by the minimality of C, and hence $v_{i+1}v_i, v_{i+1}v_{i+2}$, $vu_1u, vw_1w)$ is a fork, a contradiction. So $v_j \notin \{v_{i+2}, v_{i-2}\}$. Thus, if $v_j \notin \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$, then it is easy to check that either $uu_1v_{i-1}v_{i-2} \dots v_{j+2}v_{j+1}w_1u$ (when $j-i$ is odd) or $uu_1v_{i+1}v_{i+2} \dots v_{j-2}v_{j-1}w_1u$ (when $j-i$ is even) is an odd cycle in G which is shorter than C, a contradiction.

(5) $G[N_2(C)]$ is a linear forest, i.e., each component of $G[N_2(C)]$ is a path.

Let xyz be a path in $G[N_2(C)]$, and assume that x, y, z are associated with v_i, v_j, v_k, respectively. By (4) and by symmetry, we may assume that $v_j \in \{v_{i+1}, v_{i+3}\}$. Then by (4) and (3), $v_k \in \{v_{j+1}, v_{j+3}\}$, and if $v_j = v_{i+1}$ then $v_k = v_{j+3}$. If $v_j = v_{i+3}$ then $v_k = v_{j+1}$; for if $v_k = v_{j+3}$ then let xu_1v_{i-1}, zw_1v_{k+1} be paths; now $xu_1v_{i-1}v_{i-2} \dots v_{k+2}v_{k+1}w_1zyx$ is an odd cycle shorter than C, a contradiction. So by symmetry, we may assume $v_j = v_{i+1}$ and $v_k = v_{j+3}$.

By (3) and (4), y has degree 2 in $G[N_2(C)]$. Therefore, $\Delta(G[N_2(C)]) \leq 2$. It remains to show that $G[N_2(C)]$ is acyclic.

Suppose D is a cycle in $G[N_2(C)]$, and let $D = x_1x_2x_3 \dots x_tx_1$. Then $t \geq 4$ as $og(G) \geq 7$. By the argument we have for xyz, we may assume without loss of generality that x_1, x_2 are associated with v_1, v_2, respectively. Then by (3) and (4) and apply the argument for xyz to $x_2x_3x_4$, x_3, x_4 are associated with v_5, v_6, respectively. If $t = 4$ then let $x_1u_1v_g$ and $x_4w_1v_7$ be paths; now $x_1x_4w_1, v_7v_8 \dots v_gu_1x_1$ is an odd cycle which is shorter than C, a contradiction. So $t \geq 5$, and by (4) and (3), x_5 is associated with v_9. Let $x_1u_1v_g$ and x_5wv_{10} be paths. Now $x_1x_2x_3x_4x_5wv_{10}v_{11} \dots v_gu_1x_1$ is an odd cycle shorter than C, a contradiction.

The argument in the proof of (5) actually proves the following.

(6) If $x_1x_2x_3 \dots x_t$, $t \geq 3$, is a component of $G[N_2(C)]$, then $t \leq 4$ and (by relabeling $x_1x_2 \dots x_t$ if necessary) we may assume that for some $1 \leq i \leq g$, x_1, x_2 are associated with v_i, v_{i+1}, respectively, x_3 is associated with v_{i+4}, and if $t = 4$ then x_4 is associated with v_{i+5}.

Our objective is to produce a 3-coloring of G, with certain vertices of C pre-colored. For this, we divide the neighbors of each v_i not on C into several groups. Let

$$X_{i,1} := \{v \in N(v_i) \setminus V(C) : N(v) \cap V(C) = \{v_i\} \text{ and } N(v) \cap N(\{v_{i-3}, v_{i+3}\}) = \emptyset\}$$

$$X^+_{i,2} := (N(v_i) \cap N(v_{i+2})) \setminus V(C), \quad X^-_{i,2} := (N(v_i) \cap N(v_{i-2})) \setminus V(C), \quad X_{i,2} := X^+_{i,2} \cup X^-_{i,2}$$

$$X^+_{i,3} := \{v \in N(v_i) \setminus V(C) : N(v) \cap (N(v_{i+3}) \setminus V(C)) \neq \emptyset\},$$

$$X^-_{i,3} := \{v \in N(v_i) \setminus V(C) : N(v) \cap (N(v_{i-3}) \setminus V(C)) \neq \emptyset\},$$

$$X_{i,3} := X^+_{i,3} \cup X^-_{i,3}$$

Let $X_1 := \bigcup_{i=1}^{g} X_{i,1}$, $X_2 := \bigcup_{i=1}^{g} X_{i,2}$, and $X_3 := \bigcup_{i=1}^{g} X_{i,3}$. Then

(7) $X_1 \cap (X_2 \cup X_3) = \emptyset$, and for $1 \leq i \leq g$, $N(v_i) \setminus V(C) = X_{i,1} \cup X_{i,2} \cup X_{i,3}$, $|X_{i,1}| \leq 1$, and $X^+_{i,j} \cap X^-_{i,k} = \emptyset$ for $j, k \in \{2, 3\}$.

It is easy to see from the definition that $X_1 \cap (X_2 \cup X_3) = \emptyset$.

Now let $v \in N(v_i) \setminus V(C)$ where $1 \leq i \leq g$ such that $v \notin X_{i,1}$. If $N(v) \cap N(\{v_{i-3}, v_{i+3}\}) \neq \emptyset$ then by definition, $v \in X_{i,3}$. So assume $N(v) \cap N(\{v_{i-3}, v_{i+3}\}) = \emptyset$. Then $N(v) \cap V(C) \neq \{v_i\}$ as $v \notin X_{i,1}$. Hence by the minimality of C, $N(v) \cap \{v_{i-2}, v_{i+2}\} \neq \emptyset$; so $v \in X_{i,2}$. Thus, $N(v_i) = X_{i,1} \cup X_{i,2} \cup X_{i,3}$.

If $|X_{i,1}| \geq 2$ then let $x, y \in X_{i,1}$ be distinct. We see that $(v_i x, v_i y, v_i v_{i-1} v_{i-2}, v_i v_{i+1} v_{i+2})$ is a fork, a contradiction. So $|X_{i,1}| \leq 1$.

Finally, it is easy to check, using the minimality of C, that $X^+_{i,j} \cap X^-_{i,k} = \emptyset$ for $1 \leq i \leq g$ and $j, k \in \{2, 3\}$.

Lemma 2.1. *Let G be a fork-free graph with $og(G) \geq 7$, and let $C = v_1 \ldots v_g v_1$ be a shortest odd cycle in G such that $V(G) = V(C) \cup N_1(C) \cup N_2(C)$ and each vertex in $N_2(C)$ is associated with some v_i. Let $S := V_2(G) \cap V(C)$ such that*

(*i*) *if some vertex in $N_1(v_i)$ is adjacent to two vertices in $N_2(C)$, one associated with one of $\{v_{i-3}, v_{i-1}\}$ and other associated with one of $\{v_{i+3}, v_{i+1}\}$, then $v_{i-1}, v_{i+1} \notin S$,*

(*ii*) *if $X_{i,1} \neq \emptyset$ and $v_j \in \{v_{i-1}, v_{i+1}\} \cap S$ then v_j is not associated with any vertex in $N_2(C)$,*

(iii) *if* v_i *is associated with some vertex in* $N_2(C)$ *which is adjacent to some vertex in* $X_{i+1,1} \cup X^+_{i+1,2} \cup X^+_{i+1,3}$ *(respectively,* $X_{i-1,1} \cup X^-_{i-1,2} \cup X^-_{i-1,3}$*) then* $v_i \notin S$ *or* $v_{i+3} \notin S$ *(respectively,* $v_{i-3} \notin S$*).*

Then any 3-coloring of $G[S]$ *can be extended to a 3-coloring of* G*.*

Proof. Let $c_S : S \to \{1, 2, 3\}$ be a 3-coloring of $G[S]$. We now extend c_S to a 3-coloring c of G in four steps: color C first, then $X_2 \cup X_3$, then X_1, and finally $N_2(C)$.

Step 1. If $S \neq \emptyset$ we simply extend c_S to a 3-coloring c of C so that each component of $C - S$, say $v_s v_{s+1} \dots v_t$, is 2-colored and uses at least one of the colors $\{c(v_{s-1}), c(v_{t+1})\}$. If $S = \emptyset$ then, since $|C|$ is odd and by (3), (4) and (6), there exists some v_i such that v_{i-1}, v_{i+1} are not associated with any vertex in $N_2(C)$. Without loss of generality, we may assume that v_1 is such a vertex (i.e. v_2, v_g are not associated with any vertex in $N_2(C)$). Let c be the 3-coloring on C such that $c(v_1) = 3$ and, for $2 \leq i \leq g$, $c(v_i) = 1$ if i is odd, and $c(v_i) = 2$ is i is even. Thus, c is a 3-coloring of C.

Step 2. We extend c to a 3-coloring of $G[V(C) \cup X_2 \cup X_3]$ as follows: for each $v \in X^+_{i,2} \cup X^+_{i,3}$, let $c(v) = c(v_{i+1})$; and for each $v \in X^-_{i,2} \cup X^-_{i,3}$, let $c(v) = c(v_{i-1})$.

By (7), we have $X^+_{i,j} \cap X^-_{i,k} = \emptyset$ for $j, k \in \{2, 3\}$; so c is well defined. We now prove that c is a 3-coloring of $G[V(C) \cup X_2 \cup X_3]$.

First, we show that for any $v \in X_2 \cup X_3$, if $v \sim v_i$ for some $1 \leq i \leq g$ then $c(v) \neq c(v_i)$. Assume $v \in X_2$. Then $v \in X^+_{j,2}$ for some $1 \leq j \leq g$. So by definition and by the minimality of C, $N(v) \cap V(C) = \{v_j, v_{j+2}\}$ and $i \in \{j, j+2\}$. Hence $c(v) = c(v_{j+1}) \neq c(v_i)$. Now assume that $v \notin X_2$. By symmetry we may further assume that $v \in X^+_{j,3}$ for some j. Then $j = i$ and so $c(v) = c(v_{j+1}) \neq c(v_i)$.

Next, we show that $c(v) \neq c(w)$ for any $v, w \in X_2 \cup X_3$ with $vw \in E(G)$. First, assume $v, w \in X_2$, and let $v \sim v_i$, $v \sim v_{i+2}$, $w \sim v_j$ and $w \sim v_{j+2}$ such that $1 \leq i \leq j \leq g$. Since $og(G) \geq 7$, $\{v_i, v_{i+2}\} \cap \{v_j, v_{j+2}\} = \emptyset$. If $v_j = v_{i-1}$ then $c(v) = c(v_{i+1}) \neq$

$c(v_i) = c(w)$, and if $v_j = v_{i+1}$ then $c(v) = c(v_{i+1}) \neq c(v_{i+2}) = c(w)$. Hence we may assume that $v_j \notin \{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$. Then $vv_{i+2}v_{i+3}\dots v_{j-1}v_jwv$ (when $j - i$ is even) or $vv_iv_{i-1}\dots v_{j+3}v_{j+2}wv$ (when $j - i$ is odd) is an odd cycle shorter than C, a contradiction.

Thus by symmetry, let $w \in X_3^+$ with $w \sim v_j$ and $w' \in N(w) \cap N(v_{j+3})$. If $v \sim v_{j+3}$ then $v \in X_{j+3,3}^-$ (since $v \sim w$); so $c(v) = c(v_{j+2}) \neq c(v_{j+1}) = c(w)$; and if $v \sim v_{j+1}$ then $c(v) \neq c(v_{j+1}) = c(w)$. So we may assume $v \nsim \{v_{j+1}, v_{j+3}\}$. Note that $v \nsim \{v_j, v_{j+2}\}$ as $og(G) \geq 7$. Let $v \sim v_i$; so $v_i \notin \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$. By symmetry, let $1 \leq j + 3 < i \leq g$. Then either $vwv_jv_{j-1}\dots v_{i+1}v_iv$ (when $i - (j + 3)$ is even) or $vww'v_{j+3}v_{j+4}\dots v_{i-1}v_iv$ (when $i - (j + 3)$ is odd) is an odd cycle shorter than C, a contradiction.

Step 3. We further extend c to a 3-coloring of $G[V(C) \cup N_1(C)]$ by coloring vertices in X_1. A *band* in G is a maximal sequence $v_sw_sv_{s+1}w_{s+1}\dots v_tw_t$ such that $w_i \in X_{i,1}$ for $i = s, s + 1, \dots, t$, and $w_i \sim w_{i+1}$ for $i = s, s + 1, \dots, t - 1$. Let $v_sw_sv_{s+1}w_{s+1}\dots v_tw_t$ be a band.

If $S \neq \emptyset$, then let $c(w_i) = c(v_{i-1})$ (respectively, $c(v_{i+1})$) when w_i is adjacent to some vertex in $N_2(C)$ that is associated with some $v_j \in \{v_{i-3}, v_{i-1}\}$ (respectively, $v_j \in \{v_{i+1}, v_{i+3}\}$), and otherwise let $c(w_i) = c(v_{i-1})$ for $i = s + 1, \dots, t$, and $c(w_s) = c(v_{s+1})$ (so if $t = 1$ then $c(w_t) = c(v_{t+1})$). Note that c is well defined, as by (i) and by the coloring in Step 1, $c(v_{i-1}) = c(v_{i+1})$ if one of $\{v_{i-1}, v_{i-3}\}$ and one of $\{v_{i+1}, v_{i+3}\}$ are associated with vertices in $N_2(C)$.

If $S = \emptyset$ then let $c(w_i) = 3$ if $i \in \{2, g\}$, $c(w_i) = 1$ if $i = 1$ and $N(w_i) \cap (X_{i+1,2}^+ \cup X_{i+1,3}^+) \neq \emptyset$, $c(w_i) = 2$ if $i = 1$ and $N(w_i) \cap (X_{i-1,2}^- \cup X_{i-1,3}^-) = \emptyset$, and $c(w_i) = c(v_{i-1})$ for all other i. Note that w_1 cannot be adjacent to two vertices in $N_2(C)$, one associated with v_2 and the other associated with v_g; for otherwise G would have a fork using v_1v_2, v_1v_g and two paths of length 2 from v_1 to $N(w_1) \cap N_2(C)$. Hence c is well defined. Also note that with the possible exceptions of w_g, w_1, w_2, all colors $c(w_i)$ alternate between 1 and 2.

We now show that on $G - N_2(C)$, c is a proper coloring. By (7), if v_i is adjacent to some vertex in X_1 then it is adjacent to exactly one such vertex, say w_i, and $c(v_i) \neq c(w_i)$ as by definition $c(w_i) \in \{c(v_{i-1}), c(v_{i+1})\}$.

Note that if two vertices in X_1 are adjacent, then they are contained in a band, say $v_s w_s v_{s+1} w_{s+1} \dots v_t w_t$. We need to show that $c(w_i) \neq c(w_{i+1})$. First, assume $S = \emptyset$. If $v_i = v_1$ then $c(w_{i+1}) = 3 = c(v_i) \neq c(w_i)$; and if $v_{i+1} = v_1$ then $c(w_i) = 3 = c(v_{i+1}) \neq c(w_{i+1})$. So by symmetry we may assume $v_1 \notin \{v_i, v_{i+1}, v_{i+2}\}$; so $c(v_i) = c(v_{i+2})$, and hence $c(w_{i+1}) = c(v_i) \neq c(w_i)$. Now assume $S \neq \emptyset$. If $c(w_{i+1}) = c(v_i)$ or $c(v_i) = c(v_{i+2})$ then we see that $c(w_{i+1}) \neq c(w_i)$. So we may assume that $c(w_{i+1}) = c(v_{i+2})$ and $c(v_{i+2}) \neq c(v_i)$. Thus by the definition of c, w_{i+1} is adjacent to some $x \in N_2(C)$ associated with v_{i+2} or v_{i+4}. If x is associated with v_{i+2} then by (ii) we have $v_{i+2} \notin S$, and so $c(v_{i+2}) = c(v_i)$, a contradiction. So x is associated with v_{i+4}, and let $xu_{i+1}v_{i+5}$ be a path. Similarly, we may assume that w_i is adjacent to some $y \in N_2(C)$ which is associated with v_{i-3}, and let yu_iv_{i-4} be a path. By (3), $v_{i-3} \neq v_{i+4}$. Thus $xw_{i+1}w_iyu_iv_{i-4}v_{i-5} \dots v_{i+6}v_{i+5}u_{i+1}x$ is an odd cycle shorter than C, a contradiction.

Now let $w \in X_2 \cup X_3$ such that $w \sim w_i$ where w_i belongs to some band $v_s w_s v_{s+1} w_{s+1} \dots v_t w_t$. Then $w \nsim \{v_{i-3}, v_{i+3}\}$, as otherwise $w_i \in X_3$, a contradiction. Thus, $w \sim \{v_{i-1}, v_{i+1}\}$ by the minimality of C and by the fact that $og(G) \geq 7$. By symmetry, let $w \sim v_{i+1}$. If $c(w) = c(v_i)$ then $c(w) \neq c(w_i)$. So assume $c(w) \neq c(v_i)$. So by definition, $c(w) = c(v_{i+2}) \neq c(v_i)$, and $w \in X^+_{i+1,2} \cup X^+_{i+1,3}$; and hence by the minimality of C, $X^-_{i+1,2} \cup X^-_{i+1,3} = \emptyset$. Thus, $v_{i+2} \in S$, or $S = \emptyset$ and $v_1 \in \{v_i, v_{i+2}\}$.

Suppose $S \neq \emptyset$. Then $c(w_i) \in \{c(v_{i-1}), c(v_{i+1})\}$. If $c(w_i) = c(v_{i+1})$ or $c(v_{i-1}) = c(v_{i+1})$ then $c(w_i) \neq c(w)$. Thus we may assume that $c(w_i) = c(v_{i-1}) \neq c(v_{i+1})$. Thus, $v_{i-1} \in S$. So by the coloring in Step 1, $c(v_i) = c(v_{i+2})$ or $c(v_{i-1}) = c(v_{i+1})$, a contradiction

Now assume $S = \emptyset$ and $v_1 \in \{v_i, v_{i+2}\}$. If $v_i = v_1$ then $c(w_i) = 1$ and $c(v_{i+2}) = 2$ by definition; so $c(w) \neq c(w_i)$. If $v_{i+2} = v_1$ then $c(w) \in \{1, 3\}$ while $c(w_i) = 2$ by definition. Again, $c(w_i) \neq c(w)$.

Step 4. We extend c to a 3-coloring of G by coloring all vertices in $N_2(C)$. Let $u \in N_2(C)$ be associated with v_i. Then by the minimality of C, $N(u) \cap N(v_{i+3}) = \emptyset$ or $N(u) \cap N(v_{i-3}) = \emptyset$, and $N(u)\cap(X^+_{i+1,2}\cup X^+_{i+1,3}\cup X^-_{i+3,2}\cup X^-_{i+3,3}) = \emptyset$ or $N(u)\cap(X^-_{i-1,2}\cup X^-_{i-1,3}\cup X^+_{i-3,2}\cup X^+_{i-3,3}) = \emptyset$. Moreover, also by the minimality of C, if $w \in N(u)\cap N(v_{i+3})$ then $w \in X^-_{i+3,2}\cup X^-_{i+3,3}$, and if $w \in N(u)\cap N(v_{i-3})$ then $w \in X^+_{i-3,2}\cup X^+_{i-3,3}$.

Case 1. $S = \emptyset$.

Let $u \in N_2(C)$ be associated with some v_i such that $|c(N(u)\cap N_1(C))|$ is maximum, and let $w \in N(u)\cap N_1(C)$. By (1), $N(w)\cap V(C) \subseteq \{v_{i-3}, v_{i-1}, v_{i+1}, v_{i+3}\}$.

Suppose $i = 1$. If $w \sim v_{i+3}$ then $c(w) = 2$ by definition of c. If $w \sim v_{i+1}$ then by definition of c, $c(w) = 3$ when $w \in X_1$, and $c(w) \in \{2,3\}$ when $w \in X_2 \cup X_3$. If $w \sim v_{i-3}$ then $c(w) = 1$ by definition of c. If $w \sim v_{i-1}$ then by definition of c, $c(w) = 3$ when $w \in X_1$, and $c(w) \in \{1,3\}$ when $w \in X_2 \cup X_3$. Since $N(u)\cap N(v_{i+3}) = \emptyset$ or $N(u)\cap N(v_{i-3}) = \emptyset$, and $N(u)\cap(X^+_{i+1,2}\cup X^+_{i+1,3}\cup X^-_{i+3,2}\cup X^-_{i+3,3}) = \emptyset$ or $N(u)\cap(X^-_{i-1,2}\cup X^-_{i-1,3}\cup X^+_{i-3,2}\cup X^+_{i-3,3}) = \emptyset$, we conclude that $|c(N(u)\cap N_1(C))| \le 2$.

Now assume that $i \neq 1$. By symmetry, we may assume that $1 \le i < g - i$. By definition of c, if $w \sim v_{i+3}$ then $c(w) = c(v_{i+2}) = c(v_i)$, and if $w \sim v_{i+1}$ then $c(w) \in \{c(v_i), c(v_{i+2})\}$; hence $c(w) = c(v_i)$. Similarly, if $v_{i-2} \neq v_1$ then $c(w) = c(v_i)$ when $w \sim \{v_{i-1}, v_{i-3}\}$, and we have $|c(N(u)\cap N_1(C))| = 1$. So assume $v_{i-2} = v_1$. Then $c(w) = 3$ when $w \in X_{i-1,1}\cup X_{i-3,1}\cup X^-_{i-1,2}\cup X^-_{i-1,3}\cup X^+_{i-3,2}\cup X^+_{i-3,3}$. So $|c(N(u)\cap N_1(C))| = 1$ or $c(N(u)\cap N_1(C)) \subseteq \{2,3\}$.

Suppose $|c(N(u)\cap N_1(C))| = 2$ and there is another $x \in N_2(C)$ such that $|c(N(x)\cap N_1(C))| = 2$. Then $i \neq 1$ by (3) and (4), and we may assume by symmetry that $v_{i-2} = v_1$, and x is associated with v_{i-4}. By (6), $u \nsim x$; so u and x are the ends of some component $x_1 \dots x_t$ of $G[N_2(C)]$, with $x_1 = u$ and $x_t = x$. Since $u \nsim x$, $t = 3$ or $t = 4$; hence, since $|C|$ is odd, $t = 4$. Note that $c(N(u)\cap N_1(C)) = \{2,3\}$ and $c(N(x)\cap N_1(C)) = \{1,3\}$. Clearly, the coloring c can be extended to G by letting $c(x_1) = c(x_3) = 1$ and $c(x_2) = c(x_4) = 2$.

So assume $|c(N(u)\cap N_1(C))| = 2$ and $|c(N(x)\cap N_1(C))| = 1$ for all $x \in N_2(c)\setminus\{u\}$. We can now extend the coloring c to $N_2(C)$. Let $P_1, \dots, P_k$ denote the components of $G[N_2(C)]$ such that $u \in P_1$, and let $P_1 = x_1 \dots x_s u x_{s+1} \dots x_y$. We greedily color $P_1, \dots, P_k$ in that order. For P_1 we greedily color $u, x_{s+1}, \dots, x_t, x_s, x_{s-1}, \dots, x_1$ in this order, and for each

$P_i \neq P_1$ we color the vertices in order starting from one end.

Case 2. $S \neq \emptyset$.

Let $x_1x_2\dots x_t$ be a component of $G[N_2(C)]$, and assume that x_1 is associated with v_i. Let $x_1u_1v_{i-1}, x_1u_1v_{i+1}$ be paths in G.

Note that for any $1 < j < t$, if x_j is associated with v_k then by (6), $v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2} \notin S$; hence $c(v_{k-2}) = c(v_k) = c(v_{k+2})$. For any $w \in N(x_j) \cap N_1(C)$, $N(w) \cap V(C) \subseteq \{v_{k-3}, v_{k-1}, v_{k+1}, v_{k+3}\}$ by (1). So by symmetry and by the minimality of C, we may assume $w \in X_{k+1,1} \cup X^+_{k+1,2} \cup X^+_{k+1,3} \cup X_{k+3,1} \cup X^-_{k+3,2} \cup X^-_{k+3,3}$. So by the definition of coloring in Steps 2 and 3, $c(w) = c(v_k)$. Hence, $|c(N(x_j) \cap N_1(C))| = 1$.

We now investigate $c(N(x_1) \cap N_1(C))$. Let $w \in N(x_1) \cap N_1(C)$. Then by the same argument as above, we may assume that $w \in X_{i+1,1} \cup X^+_{i+1,2} \cup X^+_{i+1,3} \cup X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$. So by the definition of coloring in Steps 2 and 3, if $w \in X^+_{i+1,2} \cup X^+_{i+1,3}$ then $c(w) = c(v_{i+2})$; if $w \in X_{i+1,1}$ then $c(w) = c(v_i)$; if $w \in X^-_{i+3,2} \cup X^-_{i+3,3}$ then $c(w) = c(v_{i+2})$. Thus, since, $N(u) \cap (X^+_{i+1,2} \cup X^+_{i+1,3} \cup X^-_{i+3,2} \cup X^-_{i+3,3}) = \emptyset$ or $N(u) \cap (X^-_{i-1,2} \cup X^-_{i-1,3} \cup X^+_{i-3,2} \cup X^+_{i-3,3}) = \emptyset$, we have $c(N(x_1) \cap N_1(C)) \subseteq \{c(v_i), c(v_{i+2})\}$.

Thus, if $t = 1$ then c can be extended by assigning x_1 a color not in $\{c(v_i), c(v_{i+2})\}$; if $c(v_i) = c(v_{i+2})$ then by (6), we can extend c by greedily coloring $x_t, x_{t-1}, \dots, x_1$ in the listed order.

Therefore, we may assume $c(v_i) \neq c(v_{i+2})$ and $t \geq 2$. Then $w \in X^+_{i+1,2} \cup X^+_{i+1,3} \cup X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$, and $v_i \in S$ or $v_{i+2} \in S$. Hence, x_2 cannot be associated with v_{i+1} and, by the minimality of C, x_2 is not associated with v_{i-3}.

Suppose x_2 is associated with v_{i-1}. Then, since $w \in X^+_{i+1,2} \cup X^+_{i+1,3} \cup X_{i+3,1} \cup X^-_{i+3,2} \cup X^-_{i+3,3}$, it follows from (6) and the minimality of C that $t = 2$, and $N(x_2) \cap N(v_{i-3}) = \emptyset$; so $|c(N(x_2) \cap N_1(C))| = 1$. Thus, c can extended to x_1x_2 by greedily coloring x_1 and then x_2.

So we may assume that x_2 is associated with v_{i+3}; so $v_{i+2} \notin S$ and $v_i \in S$. By (6), we see that $t = 2$ or $t = 3$. If $t = 2$ then $v_{i+3} \notin S$; otherwise $w \in X^+_{i+1,2} \cup X^+_{i+1,3}$, contradicting (iii). So assume $t = 3$. Then by (6), x_3 is associated with v_{i+4}. By applying the above argument for x_1 to x_2, we may assume that $|c(N(x_2) \cap N_1(C)| = 1$. So c can be extended to $x_1x_2x_3$ by greedily coloring x_1, x_2, x_3 in this order.

Therefore, we can extend c to a 3-coloring of G. □

The proof of Lemma 2.1 in fact implies the following.

Corollary 2.2. *Let G be a fork-free graph with $og(G) \geq 7$, and let $C = v_1 \ldots v_g v_1$ be a shortest odd cycle in G such that $V(G) = V(C) \cup N_1(C)$. Then any 3-coloring of C can be extended to a 3-coloring of G.*

CHAPTER III

PROPERTIES OF A MINIMUM COUNTEREXAMPLE

Suppose the assertion of Theorem 1.9 is not true. Then we may choose a graph G such that

(1) G is fork-free and $og(G) \geq 7$,

(2) $\chi(G) \geq 4$, and

(3) subject (1) and (2), $|G|$ is minimum.

We will derive useful properties about G.

Lemma 3.1. *G is 3-edge-connected.*

Proof. Suppose that G has a an edge-cut S with $|S| \leq 2$. Then we may write $G = G_1 \cup G_2$ such that $V(G_1) \cap V(G_2) = \emptyset$ and $E(G) = E(G_1) \cup S \cup E(G_2)$. Note that for each i, G_i is fork-free and $og(G_i) \geq 7$. So by the minimality of $|G|, \chi(G_i) \leq 3$. Let $c_i : V(G_i) \to \{1, 2, 3\}$ be a 3-coloring of G_i.

We now show that c_i may be chosen such that if we let $c(v) = c_i(v)$ for $v \in G_i$ and $i = 1, 2$ then c is a 3-coloring of G, a contradiction. This is certainly the case if $|S| = 0$. If $|S| = 1$ then let v_1v_2 be the edge in S with $v_i \in V(G_i)$. Now by exchanging color classes if necessary we may assume that $c_1(v_1) \neq c_2(v_2)$, and hence c would be a 3-coloring of G. So assume $|S| = 2$, and let $S = \{u_1u_2, v_1v_2\}$ with $u_i, v_i \in V(G_i)$. First, consider the case $c_1(u_1) = c_1(v_1)$. In this case we may exchange color classes for c_2 so that $c_1(u_1) \notin \{c_2(u_2), c_2(v_2)\}$, in which case c would be a 3-coloring of G. So assume $c_1(u_1) \neq c_1(v_1)$. Similarly we may assume $c_2(u_2) \neq c_2(v_2)$. Now it is easy to see that we may exchange color classes for c_1 and c_2 (if necessary) so that $c_1(u_1) \neq c_2(u_2)$ and $c_1(v_1) \neq c_2(v_2)$, and now c would be a 3-coloring of C. □

Lemma 3.2. *G is 4-color critical, every 2-connected induced subgraph of $G[V_3]$ is either an odd cycle of length at least 5 or a complete graph, and $N(u) \not\subseteq N(v)$ for any distinct $u, v \in V(G)$.*

Proof. By the minimality of G, $\chi(G - v) \leq 3$ for any $v \in V(G)$. But $\chi(G) \geq 4$, so G is 4-color critical. Thus, the second part of the assertion follows from Theorem 1.2. For the third part, let $u, v \in V(G)$ be distinct such that $N(u) \subseteq N(v)$. Now by the minimality of G, $\chi(G - v) \leq 3$, and let c be a 3-coloring of $G - v$. By assigning $c(v)$ to the vertex u, we extend c to a 3-coloring of G, a contradiction. □

We may view $u, v \in V(G)$ with $N(u) \subseteq N(v)$ as a *reducible configuration*, and thus Lemma 3.2 implies that G has no such reducible configuration. The next two results exclude two more reducible configurations.

Lemma 3.3. *Let $u_1v_1, u_2v_2 \in E(G)$ such that $\{u_1, v_1\} \cap \{u_2, v_2\} = \emptyset$, $u_1 \nsim u_2$, and $v_1 \nsim v_2$. If $N(u_1) - \{v_1\} \subseteq N(u_2) - \{v_2\}$ then $N(v_1) - \{u_1\} \not\subseteq N(v_2) - \{u_2\}$.*

Proof. For, suppose $N(u_1) - \{v_1\} \subseteq N(u_2) - \{v_2\}$ and $N(v_1) - \{u_1\} \subseteq N(v_2) - \{u_2\}$. Let G' be obtained from G by identifying u_1 with u_2 as u, and identifying v_1 with v_2 as v.

First, we show that $og(G') \geq 7$. For, suppose T is a C_3 or C_5 in G'. If $u, v \notin T$ then T is a cycle in G as well, a contradiction. If $u, v \in T$ then $uv \in T$ as T is induced; so $T - \{u, v\} + \{u_2, v_2, u_2v_2\} + \{u_2x : ux \in T \text{ for } x \in V(T) \setminus \{u, v\}\} + \{v_2x : vx \in T \text{ for } x \in V(T) \setminus \{u, v\}\}$ is a cycle in G of length $|T'|$, a contradiction. So without loss of generality, we may assume $u \in T$ and $v \notin T$. Then $T - u + \{u_2, u_2x : ux \in T \text{ for } x \in V(T) \setminus \{u, v\}\}$ is a cycle in G of length $|T'|$, a contradiction.

Next, we show that G' is fork-free. For, suppose F is a fork in G'. Then $u \in F$ or $v \in F$ as G is fork-free. Without loss of generality, let $u \in F$. If $uv \notin F$ then $v \notin F$ as F is induced; so it is easy to show that $F - u + \{u_2, u_2x : ux \in F \text{ for } x \in V(F) \setminus \{u\}\}$ is a fork in G. Thus $uv \in F$. Then $F - \{u, v\} + \{u_2, v_2, u_2v_2\} + \{u_2x, v_2y : ux, vy \in F \text{ for } x, y \in V(F) \setminus \{u, v\}\}$ is a fork in G, a contradiction.

Hence, by the choice of G, $\chi(G') \leq 3$. Let c' be a 3-coloring of G'. Define c by setting $c(x) = c'(x)$ if $x \notin \{u_i, v_i\}$ for $i = 1, 2$, $c(x) = c(u)$ if $x \in \{u_1, u_2\}$, and $c(x) = c(v)$ if $x \in \{v_1, v_2\}$. Then c is a 3-coloring of G, a contradiction. □

Lemma 3.4. *Let* $v, w \in V(G)$ *and* $N(w) = \{v, w_1, \cdots, w_k\}$, *with* $k \geq 3$, *such that* $|N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})| \leq 1$. *Then either there exists* $x \in N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})$ *such that* $|N(x) \cap \{w_1, \cdots, w_k\}| \leq 1$, *or* $|N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})| = 0$ *and for some* $x \in N(\{w_1, \cdots, w_k\}) \setminus \{w\}$, $|N(x) \cap \{w_1, \cdots, w_k\}| \leq k - 2$.

Proof. Suppose on the contrary that either $|N(v) \cap (N(\{w_1, \cdots, w_k\}) \setminus \{w\})| = 0$ and for all $x \in N(\{w_1, \cdots, w_k\}) \setminus (\{w\} \cup N(v))$, $|N(x) \cap \{w_1, \cdots, w_k\}| \geq k-1$, or if there exists $x \in N(v) \cap N(\{w_1, \cdots, w_k\}) \setminus \{w\}$ then $|N(x) \cap \{w_1, \cdots, w_k\}| \geq 2$. Let $\{x_1, \cdots, x_s\} = N(\{w_1, \cdots, w_k\}) \setminus \{w\}$, $G' = (G - w)/\{w_1, \cdots, w_k\}$, and let x denote the identification of $w_1, \cdots, w_k$.

We claim that $og(G') \geq 7$. For suppose T' is a cycle in G' with $|T'| = 3$ or 5. Then $x \in T'$ as $og(G) \geq 7$. So without loss generality, we may assume $xx_1, xx_2 \in T'$. By the assumption above, there exists some $i \in \{1, 2\}$, such that $|N(x_i) \cap \{w_1, \cdots, w_k\}| \geq k - 1$ and $|N(x_{3-i}) \cap \{w_1, \cdots, w_k\}| \geq 2$. Hence, there exists some w_j such that $w_j \sim x_1$ and $w_j \sim x_2$. Now $T := (T' - x) + \{w_j, w_j x_1, w_j x_2\}$ is a cycle in G with $|T| = |T'|$, a contradiction.

Next we show that G' is fork-free. For, let F' be a fork in G'. Then $x \in F'$ as G is fork-free. If $d_{F'}(x) = 1$ then let $xx_1 \in F'$ and $w_1 \sim x_1$; now $F := (F' - x) + \{w_1, w_1 x_1\}$ is a fork in G, a contradiction. If $d_{F'}(x) = 2$ then let $xx_1, xx_2 \in F'$, and as in the previous paragraph, there exists some j such that $w_j \sim x_1$ and $w_j \sim x_2$; but then $F := (F' - x) + \{w_j, w_j x_1, w_j x_2\}$ is a fork in G, a contradiction. So $d_{F'}(x) = 4$ and, without loss of generality, let $F' = (xx_3, xx_4, xx_1y_1, xx_2y_2)$. By symmetry between x_1 and x_2, we may assume $x_2 \nsim v$. By the above assumption there exists w_i, say w_1, such that $w_1 \sim x_1$ and $w_1 \sim x_2$. If $x_3 \sim w_1$ then $F := (F' - x) + \{w_1, w, w_1 x_1, w_1 x_2, w_1 x_3, w_1 w\}$ is a fork in G. So $x_3 \nsim w_1$. Similarly, $x_4 \nsim w_1$. We may assume $w_2 \sim x_i$ for $i = 2, 3, 4$; so if $x_3 \nsim v$ and $x_4 \nsim v$ then $F := (F' - x) + \{v, w_2, w, w_2 x_2, w_2 x_3, w_2 x_4, w_2 w, wv\}$ is a fork in G, a contradiction. Thus, we may assume $x_3 \sim v$, and therefore $x_1 \nsim v$; so we may assume that $w_2 \sim x_i$ for $i = 1, 2, 4$.

Now $F := (F' - x) + \{w_2, w, w_2x_2, w_2x_4, w_2w\}$ is a fork in G, a contradiction.

Hence by the choice of G, $\chi(G') \leq 3$. Let $c' : V(G') \to \{1, 2, 3\}$ be a 3-coloring of G'. Define $c : V(G) \to \{1, 2, 3\}$ as follows: $c(u) = c'(u)$ if $u \in V(G' - x)$, $c(u) = c'(x)$ if $u \in \{w_1, \cdots, w_k\}$, $c(u) \in \{1, 2, 3\} - \{c(v), c(x)\}$ if $u = w$. Clearly c is a 3-coloring of G, a contradiction. ◻

We will show, in the next section, that $G[V_3]$ contains no shortest odd cycle of G. So we end this section with three lemmas concerning cycles in $G[V_3]$.

Lemma 3.5. *$G[V_3]$ contains no induced even cycles.*

Proof. By Lemma 3.2, the 2-connected components of $G[V_3]$ are either odd cycles of length at least 5 or complete graphs. An induced 4-cycle would contradict this fact. ◻

Lemma 3.6. *Let C be an induced cycle in $G[V_3]$. Then for any 3-coloring c of $G - C$ and for any $x, y \in N(C)$, $c(x) = c(y)$.*

Proof. Let c be a 3-coloring of $G - C$, $C = v_1 \cdots v_g v_1$, and $\{w_i\} = N(v_i) \setminus V(C)$ for $i = 1, \cdots, g$. Suppose there exist $1 \leq i \neq j \leq g$ such that $c(w_i) \neq c(w_j)$. Then there exists $s \in \{1, \cdots, g\}$ such that $c(w_s) \neq c(w_{s+1})$. Without loss of generality, we may assume $c(w_1) \neq c(w_2)$.

Define $c' : V(G) \to \{1, 2, 3\}$ as follows: $c'(v) = c(v)$ for all $v \notin C$, $c'(v_2) = c(w_1)$, and greedily color $v_3, v_4, \cdots, v_g, v_1$ in order. Then c' is a 3-coloring of G, a contradiction. ◻

Lemma 3.7. *Let $C = v_1v_2 \cdots v_gv_1$ be a shortest odd cycle in G, and assume $C \subseteq G[V_3]$. Then*

(1) *$(N(v_i) \setminus V(C)) \cap (N(v_j) \setminus V(C)) = \emptyset$, for $1 \leq i \neq j \leq g$*

(2) *$\bigcup_{i=1}^{g}(N(v_i) \setminus V(C))$ is independent*

(3) *for $1 \leq i \leq g$, $G - (C - \{v_i, v_{i+1}\})$ has a path from v_i to v_{i+1} of length 6.*

Proof. To prove (1), assume there exist $i \neq j$ such that $N(v_i) \setminus V(C) = N(v_j) \setminus V(C)$. By the minimality of C and without loss of generality, let $\{v\} = (N(v_1) \setminus V(C)) \cap (N(v_3) \setminus V(C))$. If $v \in V_3$ then $vv_1v_2v_3v$ is a 4-cycle in $G[V_3]$, contradicting Lemma 3.5. So let $v', v'' \in N(v) \setminus \{v_1, v_3\}$. Then $v', v'' \notin C$ by the minimality of C. Hence, $\{v', v''\} \sim \{v_4, v_g\}$ to avoid $(vv', vv'', vv_3v_4, vv_1v_g)$. Without loss of generality, we may assume $v' \sim v_g$. By the choice of G, $G - C$ has a 3-coloring c'. We can construct a 3-coloring c of G as follows: Let $c(x) = c'(x)$ for all $x \in V(G) \setminus V(C)$, let $c(v_g) = c(v)$, and then greedily color $v_{g-1}, v_{g-2}, \cdots, v_2, v_1$ in order. This is a contradiction. So (1) holds.

Next, assume that $\bigcup_{i=1}^{g}(N(v_i) \setminus V(C))$ is not independent. Then there exist $1 \leq i \neq j \leq g$, $x \in N(v_i) \setminus V(C)$, $y \in N(v_j) \setminus V(C)$, such that $x \sim y$. By the choice of G, $G - C$ has a 3-coloring c. However, $c(x) \neq c(y)$, contradicting Lemma 3.6. So (2) holds.

Now, suppose (3) fails. Then, without loss of generality, assume that $G-(C-\{v_1, v_2\})$ has no path from v_1 to v_2 of length 6. Let $w_i \in N(v_i) \setminus V(C)$ for $i = 1, 2$, and let $G' = G-C+w_1w_2$. Then $og(G') \geq 7$. Now, if F is a fork in G' then $w_1w_2 \in F$; otherwise F would be a fork in G. If w_1 or w_2 has degree 1 in F, say w_1, then $F - w_1 + w_2v_2$ is fork in G, a contradiction. So we may assume $d_F(w_1) = 4$ and $d_F(w_2) = 2$. Let $w \in F$ with $w \sim w_2$ and $w \neq w_1$. Now $F - \{w, w_2\} + \{v_1, v_2, w_1v_1, v_1v_2\}$ is a fork in G, a contradiction. So $\chi(G') \leq 3$ by the choice of G. Let c' be a 3-coloring of G'. Then c' is a 3-coloring of $G-C$ such that $c'(w_1) \neq c(w_2)$, contradicting Lemma 3.6. □

CHAPTER IV

EXCLUDING CERTAIN SHORTEST CYCLES

The objective of this section is to show that $G[V_3]$ does not contain any shortest odd cycle of G. Along the way we will exclude several more reducible configurations (based on C_4) from G.

First, we prove a lemma that deals with the case when $og(G) = 7$, which will be used to deal with the case when identifying two vertices results in a C_5.

Lemma 4.1. *Let* $C = v_1 v_2 \cdots v_7 v_1$ *be a shortest odd cycle in* G*, and assume that* G *contains an induced path* $P = v_i u_1 \cdots u_n$ *such that* $V(P \cap C) = \{v_i\}$*,* $d(u_n, C) \geq 2$*, and* $d(v_i) \geq 4$*. Further, assume that there exists* $v \in (N(v_j) \cap N(v_{j+2})) \setminus V(C)$ *for some* $1 \leq j \leq 7$ *(subscripts modular* 7*)). Then* $\{v_j, v_{j+1}, v_{j+2}\} \not\subseteq V_3$.

Proof. We choose P so that $|P|$ is minimal (subject to the conditions in the lemma). Without loss of generality, assume $i = 1$. Suppose for a contradiction that $\{v_j, v_{j+1}, v_{j+2}\} \subseteq V_3$. Then by Lemma 3.5, $v \notin V_3$. By symmetry, we may assume $j = 4$ or $j = 5$.

Case 1. $j = 5$.

In this case, $vv_5v_6v_7v$ is a 4-cycle. We distinguish cases according to the location of v.

Subcase 1.1. $v \in P$.

Let $v = u_s$. Then $s \geq 2$ and $s \neq 3$, since $og(G) \geq 7$. Moreover, u_{s+1} is defined as $d(u_n, C) \geq 2$. If $s \geq 4$ then $(vv_5, vu_{s+1}, vu_{s-1}u_{s-2}, vv_7v_1)$ is a fork in G, a contradiction. Thus $s = 2$.

Let $u \in N(u_3) \setminus N(u_1)$ (which is nonempty by Lemma 3.2), such that $u = u_4$ if u_4 is defined. Then $u \notin C$. Note that $v_4 \nsim \{u, u_1\}$ to avoid C_5. So $v_4 \sim u_3$ to avoid $(vv_7, vu_1, vv_5v_4, vu_3u)$. Hence, $u = u_4$.

Let $u' \in N(u_1) \setminus N(v_7)$. Then $u' \notin \{v_3, v_4\}$ and $u' \nsim v_3$, to avoid C_5. Hence, $u' \notin C$. Moreover, $u' \notin P$ as P is induced. So $u' \sim u_3$ to avoid $(u_2v_5, u_2v_7, u_2u_3u_4, u_2u_1u')$. Then $u_4 \sim v_3$ to avoid $(u_3u_4, u_3u', u_3v_4v_3, u_3vv_7)$. So u_5 exists as $d(u_n, C) \geq 2$. Therefore $(u_3u', u_3v_4, u_3vv_7, u_3u_4u_5)$ is a fork, a contradiction.

Subcase 1.2. $v \notin P$, but $N(v) \cap P \neq \emptyset$

Let s be maximum, such that $u_s \in N(v)$.

First, we show that $N(v) \cap V(P) = \{u_s\}$. For, otherwise, let $t < s$ such that $u_t \in N(v)$. Then $s = t + 2 \geq 3$ by the choice of P and the assumption $og(G) \geq 7$. If $t \geq 2$ then $t \geq 3$ (since $og(G) \geq 7$); so $(vv_5, vu_s, vv_7v_1, vu_tu_{t-1})$ is a fork, a contradiction. Thus, $t = 1$. Let $u \in N(u_3) \backslash N(u_1)$ such that $u = u_4$ if u_4 is defined. Then $u \notin C$ as $d(u_n, C) \geq 2$. Since $v_4 \nsim \{u_1, u\}$ (to avoid C_5), we have $v_4 \sim u_3$ to avoid $(vu_1, vv_7, vv_5v_4, vu_3u)$, which implies $u = u_4$. Since $u_2 \nsim v_3$ and $v \nsim v_3$ (to avoid C_5), $v_3 \sim u_4$ to avoid $(u_3u_4, u_3u_2, u_3vv_7, u_3v_4v_3)$. Therefore, u_5 is defined as $d(u_n, C) \geq 2$. By the choice of P, $v_3, v_4 \in V_3$. Now $(u_3v_4, u_3u_2, u_3vv_7, u_3u_4u_5)$ is fork, a contradiction.

Let $w_1 \in N(v_1) \setminus \{v_2, v_7, u_1\}$. Let $v' \in N(v) \setminus \{v_5, v_7, u_s\}$. Then $v' \notin C$ by minimality of C. Possibly, we have $v' = w_1$, in which case we have $s \leq 3$ by the choice of P. Since $og(G) \geq 7$, $w_1 \nsim v_4$ and $v' \nsim \{v_2, v_3\}$. Note that $v_6 \nsim \{u_s, v'\}$ to avoid $N(v_6) \subseteq N(v)$ (see Lemma 3.2).

Suppose $s \neq 1$. Then $s \geq 3$ since $og(G) \geq 7$. Note that $u_s \nsim v_4$; for, otherwise, by the choice of P, $v_4 \in V_3$ and u_{s+1} is defined, and hence $(u_su_{s+1}, u_sv_4, u_svv_7, u_su_{s-1}u_{s-2})$ is a fork, a contradiction. First, assume $v' \sim v_1$; thus we may assume $v' = w_1$. Now $w_1 \sim u_{s-1}$ to avoid $(vw_1, vv_7, vv_5v_4, vu_su_{s-1})$, and hence $s = 3$ by the choice of P. Let $u \in N(u_3) \setminus N(w_1)$ such that $u = u_4$ if u_4 is defined. Then $u \notin C$ by the definition of P, and $u \neq u_1$ since $og(G) \geq 7$. Hence $(vv_7, vw_1, vv_5v_4, vu_3u)$ is a fork, a contradiction. Therefore, $v' \nsim v_1$ for every choice of v'. Note that $v' \sim u_{s-1}$ to avoid $(vv', vv_5, vv_7v_1, vu_su_{s-1})$, $v' \sim v_4$ to avoid $(vv', vu_s, vv_7v_1, vv_5v_4)$. Let $u \in N(u_s) \setminus N(v')$. Then $u \notin C$ and $u \nsim v_1$ to avoid C_5. Hence, $(vv', vv_5, vv_7v_1, vu_su)$ is a fork, a contradiction.

Therefore, $s = 1$. Then, since $og(G) \geq 7$, $u_1 \nsim \{v_3, v_4\}$ and $v_4 \nsim \{u_1, u_2, w_1\}$. Moreover, $v' \nsim v_1$ for any choice of v'. For, otherwise, we may let $v' = w_1$. Then $w_1 \nsim v_3$ (to avoid C_5) and $w_1 \nsim v_6$ (to avoid $N(v_6) \subseteq N(v)$), and hence, $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$ is a fork, a contradiction.

Thus $v' \sim \{v_4, u_2\}$ to avoid $(vv', vv_7, vu_1u_2, vv_5v_4)$, and $w_1 \sim \{v_3, v_6\}$ to avoid $(v_1w_1, v_1u_1, v_1v_2v_3, v_1v_7v_6)$. Moreover, $u_2 \sim \{w_1, v_2\}$. This is clear when $w_1 \nsim v_6$ as we need to avoid $(v_1w_1, v_1v_2, v_1u_1u_2, v_1v_7v_6)$. When $w_1 \sim v_6$, then $w_1 \nsim v_3$ by the minimality of C; so $u_2 \sim \{w_1, v_2\}$ as we need to avoid $(v_1w_1, v_1v_7, v_1u_1u_2, v_1v_2v_3)$.

Suppose $u_2 \nsim w_1$. Then $u_2 \sim v_2$; so by the choice of P, $v_2 \in V_3$ and u_3 is defined. If $v' \sim u_2$, then $v' \nsim v_4$ (as $og(G) \geq 7$), and so $v' \nsim C$; hence let $v'' \in N(v') \setminus N(u_1)$, then $(vu_1, vv_7, vv'v'', vv_5v_4)$ is a fork, a contradiction. Thus, $v' \nsim u_2$, and $v' \sim v_4$. Let $v'' \in N(v') \setminus N(v_5)$. Then $v'' \sim u_1$ to avoid $(vv_5, vv_7, vu_1u_2, vv'v'')$. So $v'' \sim w_1$ to avoid $(u_1u_2, u_1v'', u_1vv_5, u_1v_1w_1)$, and $\{u_1, w_1\} \nsim v_3$ to avoid C_5. So $w_1 \sim v_6$. Moreover, $v'' \sim u_3$ to avoid $(u_1v'', u_1v_1, u_1vv_5, u_1u_2u_3)$. Hence, $(v''u_3, v''u_1, v''v'v_4, v''w_1v_6)$ is a fork, a contradiction.

Hence, $u_2 \sim w_1$. If $v' \nsim v_4$ then $v' \sim u_2$; so letting $v'' \in N(v') \setminus N(u_1)$, we see that $(vu_1, vv_7, vv'v'', vv_5v_4)$ is a fork, a contradiction. So $v' \sim v_4$.

Assume for the moment $v' \nsim u_2$. Let $v'' \in N(v') \setminus N(v_5)$. Then $u_1 \sim v''$ to avoid $(vv_5, vv_7, vv'v'', vu_1u_2)$, $v'' \nsim v_2$ to avoid C_5, and $v_2 \sim u_2$ to avoid $(u_1v'', u_1u_2, u_1vv_5, u_1v_1v_2)$. Hence by the choice of P, $v_2 \in V_3$, and u_3 is defined. So $v_3 \nsim w_1$ to avoid $N(v_2) \subseteq N(w_1)$ and, thus, $w_1 \sim v_6$ and $(u_2u_3, u_2v_2, u_2w_1v_6, u_2u_1v_5)$ is a fork, a contradiction.

Thus, $v' \sim u_2$. Therefore, $w_1 \nsim v_3$ and $u_2 \nsim v_2$ to avoid C_5, which implies $w_1 \sim v_6$.

We claim that $v' \in V_3$. For, otherwise, let $v'' \in N(v') \setminus \{u_2, v, v_4\}$. Note that $v'' \notin C$. Then $v'' \sim w_1$ to avoid $(v'v'', v'v_4, v'vv_7, v'u_2w_1)$, and so $v'' \nsim v_3$ to avoid C_5. Hence $(v'v'', v'u_2, v'vv_7, v'v_4v_3)$ is a fork, a contradiction.

We also claim that $v_4 \in V_3$. For, if there exists $x \in N(v_4) \setminus \{v', v_3, v_5\}$. Then $x \sim v_2$ to avoid $(v_4x, v_4v', v_4v_5v_6, v_4v_3v_2)$, and so $x \nsim u_2$ to avoid a 5-cycle. Thus, $(v_4x, v_4v_3,$

$v_4v_5v_6, v_4v'u_2)$ is a fork, a contradiction.

Moreover, $u_2 \in V_3$, as otherwise let $u \in N(u_2)\backslash\{u_1, v', w_1\}$, then $(u_2u, u_2u_1, u_2w_1v_6, u_2v'v_4)$ would be a fork. So $w_1 \notin V_3$ as $G[V_3]$ has no induced even cycle (by Lemma 3.5). Let $w \in N(w_1) \setminus \{u_2, v_1, v_6\}$. Then $(w_1w, w_1v_1, w_1u_2v', w_1v_6v_5)$ is a fork, a contradiction.

Subcase 1.3. $v \notin P$ and $N(v) \cap V(P) = \emptyset$ for any choice of P.

First, assume $v \nsim w_1$ for any choice of w_1. Let $v', v'' \in N(v) \setminus \{v_5, v_7\}$. Then $v_1 \nsim \{v', v''\}$, $v', v'' \notin C \cup P$, and $\{u_1, v_2, w_1\} \nsim \{v', v''\}$ to avoid C_5. So $v_4 \sim \{v', v''\}$ to avoid $(vv', vv'', vv_7v_1, vv_5v_4)$. Without loss of generality, let $v'' \sim v_4$. Let $v^* \in N(v') \setminus N(v'')$. Then $v^* \notin \{v_2, v_3\}$ to avoid C_5, and $v^* \nsim v_6$ to avoid $N(v_6) \subseteq N(v)$; so $v^* \notin C$. Hence $(vv'', vv_5, vv'v^*, vv_7v_1)$ is a fork, a contradiction.

Thus we may choose w_1 so that $v \sim w_1$. Then $w_1 \nsim u_2$, otherwise replacing P with $v_1w_1u_2 \ldots u_n$, we get back to Subcase 1.2. Also $w_1 \nsim \{v_3, v_4\}$ to avoid C_5, $w_1 \nsim v_6$ as $N(v_6) \nsubseteq N(v)$, and $u_1 \sim \{v_3, v_6\}$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$.

Now $u_2 \sim v_2$; for otherwise, since $u_1 \nsim v_3$ or $u_1 \nsim v_6$ (by the minimality of C), $(v_1w_1, v_1v_7, v_1v_2v_3, v_1u_1u_2)$ or $(v_1w_1, v_1v_2, v_1u_1u_2, v_1v_7v_6)$ would be a fork. Hence by the choice of P, $v_2 \in V_3$ and u_3 is defined. Let $w \in N(w_1) \setminus N(v_7)$; then $w \notin C$. Moreover, $w \nsim \{u_2, v_3, v_6\}$ to avoid C_5, and $w \notin P$ by the choice of P. If $w \nsim u_1$ then $u_1 \sim v_3$ to avoid $(v_1u_1, v_1v_7, v_1w_1w, v_1v_2v_3)$; so $u_1 \nsim v_6$ and $(v_1u_1, v_1v_2, v_1w_1w, v_1v_7v_6)$ is a fork, a contradiction. Thus $w \sim u_1$, and hence $w \nsim u_3$ (otherwise replacing P with $u_1w_1wu_3 \cdots u_n$ we get back to Subcase 1.2). If $u_1 \sim v_6$ then $(u_1v_1, u_1w, u_1v_6v_5, u_1u_2u_3)$ is a fork, a contradiction. So $u_1 \nsim v_6$. Thus $u_1 \sim v_3$, and $(u_1v_3, u_1w, u_1u_2u_3, u_1v_1v_7)$ is a fork, a contradiction.

Case 2. $j = 4$, that is, $vv_4v_5v_6v$ is a 4-cycle.

Subcase 2.1. $v \in P$, or $v \notin P$ and $N(v) \cap P \neq \emptyset$.

First, suppose $v \in P$. Let $v = u_s$. Then $s = 1$ or $s \geq 3$ to avoid C_5, and u_{s+1} is defined by the choice of P. Let $u \in N(u_{s+1}) \setminus N(u_{s-1})$ such that $u = u_{s+2}$ if $n \geq s + 2$. Then $u \notin C$ by the choice of P, and $u \notin P - u_{s+2}$ as P is induced. Now $s = 1$; otherwise, $(vv_4, vv_6, vu_{s+1}u, vu_{s-1}u_{s-2})$ is a fork, a contradiction. Thus, $u_2 \sim v_2$ to avoid

$(vv_4, vv_6, vu_2u, vv_1v_2)$, which implies that $u_2v_2v_3v_4vu_2$ is a C_5, a contradiction.

So $v \notin P$, and hence $N(v) \cap P \neq \emptyset$. Let $u_s \sim v$ with s maximum. Then $s \geq 2$ to avoid C_5. Let $u \in N(u_s) \setminus N(u_{s-2})$ such that $u = u_{s+1}$ if $n \geq s + 1$. Note that $u \notin C \cup P$ by the choice of P.

If v has another neighbor, say u_t, on P, then $2 \leq t \leq s - 2$ (since $og(G) \geq 7$). If $t = s - 2$, then $(vv_4, vv_6, vu_su, vu_tu_{t-1})$ is a fork, a contradiction. Otherwise, $t < s - 2$, and $(vv_4, vv_6, vu_su_{s-1}, vu_tu_{t-1})$ is a fork, a contradiction.

So $N(v) \cap V(P) = \{u_s\}$. Let $v' \in N(v) \setminus \{u_s, v_4, v_6\}$. Then $v' \notin C \cup P$. Note that $v' \nsim v_3$ or $v' \nsim v_7$ (to avoid C_5). Also note that $u_s \nsim v_3$; for otherwise, $v_3 \in V_3$ and $u = u_{s+1}$ (by the choice of P), and so $(u_su_{s+1}, u_sv_3, u_su_{s-1}u_{s-2}, u_svv_6)$ would be a fork.

Suppose $v' \nsim v_3$. Then $v_7 \sim \{v', u_s\}$ to avoid $(vv', vu_s, vv_4v_3, vv_6v_7)$. Suppose that $v_7 \sim u_s$. Then, $v_7 \in V_3$ and $u = u_{s+1}$ by the choice of P. If $s \neq 2$ then $(u_su_{s+1}, u_sv_7, u_su_{s-1}u_{s-2}, u_svv_4)$ is a fork, a contradiction. So $s = 2$. Let $u' \in N(u_3) \setminus N(u_1)$ such that $u' = u_4$ if $n \geq 4$. Then $u' \notin C$. So $u' \sim v$ to avoid $(u_2u_1, u_2v_7, u_2vv_4, u_2u_3u')$, and $u' \sim v_3$ to avoid $(vu', vv_6, vv_4v_3, vu_2u_1)$. Hence $v_3 \in V_3$ by the choice of P, and so $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$ is a fork in G, a contradiction. Thus, $v_7 \nsim u_s$; hence $v_7 \sim v'$, and so, $v_3 \nsim v'$. Then $v' \sim u_{s-1}$ to avoid $(vv', vv_6, vv_4v_3, vu_su_{s-1})$, and $v' \sim u$ to avoid $(vv', vv_6, vv_4v_3, vu_su)$. So $s = 2$; otherwise $(v'u, v'v_7, v'u_{s-1}u_{s-2}, v'vv_4)$ would be a fork. Then $u_1 \nsim v_3$ to avoid C_5; so $w_1 \sim v_3$ to avoid $(v_1w_1, v_1u_1, v_1v_7v_6, v_1v_2v_3)$. Hence $u_s \nsim \{v_2, w_1\}$ to avoid C_5. But then $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_7v_6)$ is a fork, a contradiction.

Hence $v' \sim v_3$. Then $v' \nsim v_7$. Note that $v_3 \nsim u_1$; otherwise, replacing P with $v_3u_1u_2 \cdots u_n$, we get back to Case 1. So $v_3 \sim w_1$ to avoid $(v_1w_1, v_1u_1, v_1v_2v_3, v_1v_7v_6)$. Then $w_1 \nsim u_2$; otherwise replacing P with $v_3w_1u_2 \cdots u_n$, we also get back to Case 1. Then $v' \nsim u_1$ to avoid C_5. Moreover, $v' \nsim u_i$ for $i \geq 2$; otherwise, $v' \sim u_2$ by the choice of P, and replacing P with $v_3v'u_2 \ldots u_n$, we get back to case 1 again. Hence $u_s \sim v_7$ to avoid $(vv', vv_4, vv_6v_7, vu_su_{s-1})$. Thus, by the choice of P, $v_7 \in V_3$ and $u = u_{s+1}$. Let $u' \in N(u_{s+1}) \setminus N(u_{s-1})$ such that $u' = u_{s+2}$ if $n \geq s + 2$. Then $u' \notin C$. Now

$(u_su_{s-1}, u_sv_7, u_su_{s+1}u', u_svv_4)$ is a fork in G, a contradiction.

Subcase 2.2. $v \notin P$, and $N(v) \cap P = \emptyset$.

Note that $\{u_1, w_1\} \sim v_3$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_7v_6)$. Let $v', v'' \in N(v) \setminus V(C \cup P)$. Then $\{v', v''\} \sim \{v_3, v_7\}$ to avoid $(vv', vv'', vv_4v_3, vv_6v_7)$.

First, suppose $v_3 \sim u_1$. Then $v_3 \in V_3$, otherwise, replacing P with $v_3u_1u_2 \cdots u_n$, we get back to Case 1. Now choose $v^* \in N(v') \setminus N(v'')$ or $v^* \in N(v') \setminus N(v'')$ such that $v^* \neq v_7$ (which exists because $N(v') \not\subseteq N(v'')$ and $N(v'') \not\subseteq N(v')$). By symmetry assume $v^* \in N(v') \setminus N(v'')$. Note that $v^* \neq v_5$, as otherwise we would have $N(v_5) \subseteq N(v)$. Thus, $(vv'', vv_6, vv'v^*, vv_4v_3)$ is a fork, a contradiction.

Hence, $v_3 \nsim u_1$, which implies $v_3 \sim w_1$. Moreover, we may assume $w_1 \nsim u_2$; for, otherwise, replacing P with $v_1w_1u_2 \cdots u_n$, we get back to the situation in the previous paragraph (with $v_3 \sim u_1$). So $u_2 \sim \{v_2, v_7\}$ to avoid $(v_1v_2, v_1w_1, v_1v_7v_6, v_1u_1u_2)$.

If $v_3 \nsim \{v', v''\}$ then as above we may assume there exists $v^* \in N(v') \setminus N(v'')$ such that $v^* \neq v_7$. Now $(vv'', vv_6, vv_4v_3, vv'v^*)$ is a fork, a contradiction.

So we may assume $v_3 \sim v'$. Note that $w_1 \nsim v_5$ since $og(G) \geq 7$. If $u_2 \sim v_2$, then $v_2 \in V_3$ (by the choice of P), and $u_2 \sim v'$ to avoid $(v_3w_1, v_3v', v_3v_4v_5, v_3v_2u_2)$; but then, replacing P with $v_3v'u_2 \cdots u_n$, we get back to Case 1. So $u_2 \nsim v_2$ and $u_2 \sim v_7$, and so $v_7 \in V_3$ (by the choice of P). Thus, $v_7 \nsim \{v', v''\}$. As before, we may assume that there exists $v^* \in N(v'') \setminus N(v')$ such that $v^* \neq v_3$. Now $(vv', vv_4, vv''v^*, vv_6v_7)$ is a fork, a contradiction. ☐

Corollary 4.2. *Let $C = v_1 \cdots v_7v_1$ be an odd cycle in G, $v \in V(G) \setminus V(C)$, and $1 \leq i \leq 7$ such that $v \sim v_i$ and $v \sim v_{i+2}$. Then $\{v_i, v_{i+1}, v_{i+2}\} \not\subseteq V_3$.*

Proof. Let $T := \{u \in V(G) \setminus V(C) : d(u, C) \geq 2\}$, and let H denote the subgraph of G obtained by taking the union of all paths P which are from vertices in T to C but internally disjoint from C.

Suppose $\{v_i, v_{i+1}, v_{i+2}\} \subseteq V_3$. Then by Lemma 4.1, $V(H) \cap V(C) \subseteq V_3$. Let $K := G \setminus (V(H) \setminus V(C))$.

If $T = \emptyset$, then $V(G) = V(C) \cup N_1(C)$; so $\chi(G) \leq 3$ by Corollary 2.2, a contradiction. So $T \neq \emptyset$. Then by the choice of G, $\chi(H) \leq 3$. Let $c : V(H) \to \{1, 2, 3\}$ be a 3-coloring of H. Then c induces a 3-coloring c' on $V(C) \cap V(H)$, and clearly c' can be extended to a 3-coloring of C (in a greedy way). Now by Corollary 2.2 again, c' may be extended further to a 3-coloring c^* of K. Thus c^* and c give a 3-coloring of G, a contradiction. $\square$

Next we give another reducible configuration that will be used frequently when we look for a fork.

Corollary 4.3. *For any 4-cycle C in G, $|V(C) \cap V_3| \leq 2$.*

Proof. Suppose $C = v_1v_2v_3v_4v_1$ is a 4-cycle in G such that $|V(C) \cap V_3| \geq 3$. Without loss of generality, let $v_i \in V_3$ for $i = 1, 2, 3$. Then $v_4 \notin V_3$ by Lemma 3.5. Let $G' := G/\{v_1, v_3\} - v_2$, and let v denote the identification of v_1 and v_3.

We claim that $og(G') \geq 7$. For, suppose on the contrary that G' has an odd cycle C with $|C| \leq 5$. Clearly, $v \in C$ and $|C| = 5$, as $og(G) \geq 7$. Let $C = u_1u_2u_3u_4u_5u_1$, with $u_1 = v$. If $vv_4 \in E(C)$ then assume by symmetry that $v_4 = u_2$ and $v_1u_5 \in E(G)$; so $v_1u_2u_3u_4u_5v_1$ is a C_5 in G, a contradiction. So $vv_4 \notin E(C)$. Then by symmetry assume $v_1u_1, v_3u_5 \in E(G)$. Now $v_1u_1 \cdots u_5v_3v_2$ is a 7-cycle, and $v_4v_1v_2v_3v_4$ is a 4-cycle, contradicting Corollary 4.2.

If G' is fork-free then $\chi(G') \leq 3$ by the choice of G. Let c' be a 3-coloring of G'. We may extend c' to a 3-coloring of G by coloring v_1 and v_3 with $c'(v)$, and v_2 with a color not used by its neighbors. This is a contradiction.

So let F' be a fork in G'. Then $v \in F'$, as otherwise, F' would be a fork in G. If v has degree 1 in F', then let $vx \in E(F')$; by symmetry we may assume that $v_1x \in E(G)$, and then $F' - v + \{v_1, v_1x\}$ is a fork in G, a contradiction. So v has degree 2 in F'. Let $vx, vz \in E(F')$, with $v_1x \in E(G)$. If $v_4 = z$ then $F' - v + \{v_1, v_1x, v_1v_4\}$ is a fork in G, a contradiction. So assume $v_3z \in E(G)$. By symmetry, we may assume that z has degree 1 in F'. Let $y \in N(v_2)\backslash\{v_1, v_3\}$. Now, since G has no C_5, $xy \in F'$ as otherwise $F' - v + \{v_1, v_2, xv_1, v_1v_2\}$ would be a fork in G. Also, $F' - v$ contains a neighbor of v_4; otherwise $F' - v + \{v_1, v_4, xv_1, v_1v_4\}$

would be a fork in G. So, since G contains no C_5, let $v' \in N(v_4) \setminus \{v_1, v_3\}$ such that $xv' \in E(F')$. Thus $z \nsim \{v', x, y\}$.

Let $G'' := G \setminus \{v_1, v_2, v_3\} + v_4y$. We claim that G'' is fork-free. For, let F'' be a fork in G''. Then $v_4y \in F''$ as otherwise F'' would be a fork in G. If y is of degree 1 in F'' then $x \notin F''$; so $F'' - y + \{v_1, v_4v_1\}$ is a fork in G, a contradiction. Suppose y is of degree 2 in F'', and let $yy' \in F''$ be the other edge of F'' incident with y. Note that if $y' \neq x$ then $x \notin F''$. So, regardless, $F'' \setminus \{y, y'\} + \{v_1, v_2, v_4v_1, v_1v_2\}$ is a fork in G, a contradiction. Thus y is of degree 4 in F''. Note that $z \notin F''$; for otherwise, since $yz \notin E(G'')$, F'' has a path $yz'z$, which gives the cycle $yz'zv_3v_2y$ in G, contradicting $og(G) \geq 7$. If v_4 is of degree 1 in F'' then $F'' - v_4 + \{v_2, yv_2\}$ is a fork in G, a contradiction. So let yv_4v^* be a path in F''. Then $F'' - \{v_4, v^*\} + \{v_2, v_3, yv_2, v_2v_3\}$ is a fork in G, a contradiction.

Suppose $og(G'') \geq 7$. Then by the choice of G, we have $\chi(G'') \leq 3$. Let c' be a 3-coloring of G''. We extend c' to a coloring of G by coloring v_2 with $c'(v_4)$, and then coloring v_1 and v_3 greedily.

So $og(G'') \leq 5$. Then, since $og(G) \geq 7$, $og(G'') = 5$ and any 5-cycle in G'' must contain v_4y. Let $C := v_4yy'wv''v_4$ be a 5-cycle in G''. Then, since $og(G) \geq 7$, we can show $x, v', z \notin C$ and $y' \nsim v'$. If $y \in V_3$ then we get a contradiction to Corollary 4.2, since in G, $yv_2v_1v_4v''wy'y$ is a 7-cycle, xyv_2v_1x is a 4-cycle, and $\{v_1, v_2, y\} \subseteq V_3$. So $y \notin V_3$. Let $y'' \in N(y) \setminus \{v_2, x, y'\}$. Then $y'' \sim v'$ to avoid $(yy'', yy', yv_2v_3, yxv')$. Now $w \nsim \{x, y''\}$ to avoid C_5. Hence, $(yx, yy'', yv_2v_3, yy'w)$ is a fork, a contradiction. □

To prove that $G[V_3]$ contains no shortest odd cycle of G, we need another lemma.

Lemma 4.4. *Let $C := v_1v_2 \cdots v_gv_1$ be a shortest odd cycle in G such that $V(C) \subseteq V_3$, and let $w_i \in N(v_i) \setminus V(C)$ for $1 \leq i \leq g$. Then $w_i \in V_3$ for $1 \leq i \leq g$.*

Proof. Suppose the contrary. Without loss of generality, we may assume that $w_1 \notin V_3$. By Corollary 4.2, $N(w_1) \cap V(C) = \{v_1\}$. Let $N(w_1) = \{v_1, x_1, \cdots, x_k\}$; so $k \geq 3$.

Now $\{v_2, v_g\} \nsim \{x_1, \ldots, x_k\}$ by (2) of Lemma 3.7; so $v_1 \nsim N(\{x_1, \cdots, x_k\}) \setminus \{w_1\}$. Thus by

Lemma 3.4, there exists some $v \in N(\{x_1, \ldots, x_k\}) \setminus \{w_1\}$ such that $|N(v) \cap \{x_1, \ldots, x_k\}| \le k-2$. So without loss of generality, assume that $v \nsim \{x_1, x_2\}$ and $v \sim x_3$. Note that $v \nsim v_2$, since $og(G) \ge 7$. Hence, $(w_1x_1, w_1x_2, w_1v_1v_2, w_1x_3v)$ is a fork, a contradiction. □

We can now show that $G[V_3]$ contains no shortest odd cycle in G.

Corollary 4.5. *If $V(C) = v_1 \cdots v_g v_1$ is a shortest odd cycle, then $V(C) \nsubseteq V_3$.*

Proof. Let $w_i \in N(v_i) \setminus V(C)$ for $i = 1, \ldots, g$, and let $G' := (G - C) + w_1w_2$.

We claim that G' is fork-free. For, let F' be a fork in G'. Then $w_1w_2 \in F'$ as G is fork-free. By symmetry, assume that in F', the degree of w_1 is larger than that of w_2. If the degree of w_2 in F' is 1, then $F' - w_2 + \{v_1, w_1v_1\}$ is a fork in G, a contradiction. So the degree of w_2 in F' is 2 and let w be the other neighbor of w_2 in F'. Then $F' - \{w_2, w\} + \{v_1, v_2, w_1v_1, v_1v_2\}$ is a fork in G, a contradiction.

If $og(G') \ge 7$ then $\chi(G') \le 3$ by the choice of G; so $G - C = G' - w_1w_2$ has a 3-coloring c' in which $c'(w_1) \ne c'(w_2)$, contradicting Lemma 3.6. Thus $og(G') \le 5$. Let T' be an odd cycle in G' such that $|T'| \le 5$. Then $w_1w_2 \in T'$ as $og(G) \ge 7$. Thus $T := (T' - w_1w_2) \cup w_1v_1v_2w_2$ is an odd cycle in G. Hence, $|T| = 7$ as $og(G) \ge 7$. So $|T'| = 5$, and let $T' - w_1w_2 = w_1xyzw_2$. Choose w_1xyzw_2 to minimize $|\{x, y, z\} \cap V_3|$.

Suppose $x, z \in V_3$. Then by Lemma 3.2, $y \notin V_3$. So let $y_1, y_2 \in N(y) \setminus \{x, z\}$. Then $\{y_1, y_2\} \sim \{w_1, w_2\}$ to avoid $(yy_1, yy_2, yxw_1, yzw_2)$. By symmetry, we may assume $y_1 \sim w_1$. So $y_1 \notin \{v_i, w_i : 1 \le i \le n\}$ (by Lemma 3.7(2)). By Corollary 4.3, $y_1 \notin V_3$ (because of the 4-cycle $y_1yxw_1y_1$). Thus, the path $w_1y_1yzw_2$ contradicts the choice of w_1xyzw_2.

So by symmetry, we may assume $x \notin V_3$. Let $N(x) \setminus \{w_1\} = \{x_1, \ldots, x_k\}$; then $k \ge 3$. We claim that there is a unique vertex $y_1 \in N(w_1) \cap N(\{x_1, \ldots, x_k\})$ such that $|N(y_1) \cap \{x_1, \ldots, x_k\}| = 1$. For, otherwise, by Lemma 3.4, for some $t \in N(\{x_1, \ldots, x_k\})$, $|N(t) \cap \{x_1, \ldots, x_k\}| \le k - 2$. Without loss of generality we may assume $t \sim x_1$ and $t \nsim \{x_2, x_3\}$. Now $(xx_2, xx_3, xx_1t, xw_1v_1)$ is a fork, a contradiction.

Let $y = x_k$. Now $z \sim \{x_1, x_2\}$ to avoid $(xx_1, xx_2, xw_1v_1, xyz)$. Thus, by the symmetry between y and $N(z) \cap \{x_1, x_2\}$, we may assume that $y_1 \sim x_1$. Then $y_1 \nsim N(x)\backslash\{w_1, x_1\}$.

Let $u \in N(x_2)\backslash N(y)$ and $v \in N(y)\backslash N(x_2)$. Then $u \notin \{v_1, v_2, w_1, w_2, x, y, y_1, z\}$ and $v \notin \{v_1, v_2, w_1, w_2, x, x_1, y_1\}$. Then $x_1 \sim u$ to avoid $(xx_1, xy, xw_1v_1, xx_2u)$, and $x_1 \sim v$ to avoid $(xx_1, xx_2, xw_1v_1, xyv)$.

Case 1. $v = z$ for any $v \in N(y) \setminus N(x_2)$ and for any choice of w_1xyzw_2.

Then $y \notin V_3$; otherwise $w_1xx_1zw_2$ contradicts the choice of w_1xyzw_2. Let $y', y'' \in N(y) \setminus \{x, z\}$. Then $w_2 \sim \{y', y''\}$ to avoid $(yy', yy'', yxw_1, yzw_2)$, and we may assume $w_2 \sim y'$ by symmetry. Then $z \notin V_3$, for otherwise, replacing w_1xyzw_2 with $w_1xyy'w_2$, we see that $|\{x, y, y'\} \cap V_3|$ smaller, or $v \neq y'$, a contradiction.

Let $z' \in N(z) \setminus \{w_2, x_1, y\}$. Note that $z' \notin C$ as $z \neq w_i$ for all i, and thus $z' \nsim v_2$. Now $z' \sim y_1$ to avoid $(zz', zy, zw_2v_2, zx_1y_1)$, $z' \sim u$ to avoid $(zz', zy, zw_2v_2, zx_1u)$, and $z' \sim y''$ to avoid $(zz', zx_1, zw_2v_2, zyy'')$. Hence, $(z'y'', z'u, z'y_1w_1, z'zw_2)$ is a fork in G, a contradiction.

Case 2. $v \neq z$ for some choice of v and w_1xyzw_2, and $x_1 \sim z$.

Then $y \notin V_3$, as otherwise $N(y) \subseteq N(x_1)$. So let $y' \in N(y)\backslash N(x_1)$. Note that $y' \notin \{v_1, v_2, w_1, w_2, x, y, z, x_1, x_2, u, y_1\}$. Hence $w_2 \sim \{v, y'\}$ to avoid (yv, yy', yxw_1, yzw_2). Also, $w_2 \sim \{u, v\}$ to avoid $(x_1u, x_1v, x_1xw_1, x_1zw_2)$. Therefore, since $w_2 \in V_3$, $w_2 \sim v$, $w_2 \nsim \{y', u\}$. Then $y' \nsim x_1$ to avoid $(x_1u, x_1y', x_1xw_1, xvw_2)$; so $y' \sim x_2$ to avoid $(xx_1, xx_2, xw_1v_1, xyy')$, and $z \nsim x_2$ to avoid $(x_2u, x_2y', x_2xw_1, x_2zw_2)$.

Let $w \in N(y_1)\backslash N(x)$. Then $w \nsim x_2$ to avoid C_5; so $w \sim \{v, z\}$ to avoid $(x_1v, x_1z, x_1xx_2, x_1y_1w)$, and $w \sim \{u, z\}$ to avoid $(x_1u, x_1x, x_1zw_2, x_1y_1w)$.

If $w \nsim z$, then $w \sim u$ and $w \sim v$. So $y' \sim w$ to avoid $(vw, vx_1, vyy', vw_2v_2)$. Hence, $(wu, wy', wvw_2, wy_1w_1)$ is a fork, a contradiction.

Hence, $w \sim z$. Then $w \sim u$ to avoid (zy, zw, zx_1u, zw_2v_2), and $w \sim y'$ to avoid $(zx_1, zw, zw_2v_2, zyy')$. Thus $(wu, wy', wzw_2, wy_1w_1)$ is a fork in G, a contradiction.

Case 3. $v \neq z$ for some choice of v and w_1xyzw_2, and $x_1 \nsim z$.

So $x_2 \sim z$ to avoid $(xx_1, xx_2, xw_1v_1, xyz)$. Thus we have symmetry between x_2 and y.

We claim that $N(x_2) \setminus \{u\} = N(y) \setminus \{v\}$. For, suppose, by symmetry between x_2 and y, that there exists $y' \in N(y) \setminus \{v\} \setminus (N(x_2) \setminus \{u\})$. Then $y' \sim x_1$ to avoid $(xx_1, xx_2, xw_1v_1, xyy')$, and $w_2 \sim \{y', v\}$ to avoid (yv, yy', yxw_1, yzw_2). By symmetry between v and y', we may assume $y' \sim w_2$. Then $(x_1u, x_1v, x_1xw_1, x_1y'w_2)$ is a fork in G, a contradiction.

Therefore, by Lemma 3.3, $N(u) \setminus \{x_2\} \nsubseteq N(v) \subseteq \{y\}$, and $N(v) \setminus \{y\} \nsubseteq N(u) \setminus \{x_2\}$. So let $u' \in N(u) \setminus \{x_2\} \setminus (N(v) \setminus \{y\})$, and $v' \in N(v) \setminus \{y\} \setminus (N(u) \setminus \{x_2\})$.

Note that $w_1 \notin \{u', v'\}$ as $w_1 \in V_3$, $v' \nsim x_2$ and $u' \nsim y$ to avoid C_5, $v' \nsim x$ to avoid $(xy, xv', xx_2u, xw_1v_1)$, and $u' \nsim x$ to avoid $(xx_2, xu', xyv, xw_1v_1)$. So $y_1 \sim \{u', v'\}$ to avoid $(x_1y_1, x_1x, x_1uu', x_1vv')$. By symmetry, we may assume $y_1 \sim u'$. By Corollary 4.3, $\{x_2, y, z\} \nsubseteq V_3$.

First, suppose $x_2 \notin V_3$. Let $w \in N(x_2) - \{x, z\}$; then $w \sim y$ by the above claim. If $w_2 \nsim w$ then, since $w_2 \nsim u$ or $w \nsim v$ (as $w_2 \in V_3$), (yv, yw, yxw_1, yzw_2) or $(x_2u, x_2w, x_2xw_1, x_2zw_2)$ is a fork in G, a contradiction. So $w_2 \sim w$, and we have symmetry between w and z. Note that $u' \sim \{w, z\}$ to avoid $(x_2w, x_2z, x_2xw_1, x_2uu')$. So by symmetry, assume $u' \sim w$. Then $(wx_2, wy, wu'y_1, ww_2v_2)$ is a fork in G, a contradiction.

Hence, $x_2 \in V_3$. Then by the above claim, $y \in V_3$, and hence $z \notin V_3$. Note that, if $w_2 \sim u$ then $u' \neq v_2$ and $u' \nsim v_2$; so $(uu', ux_2, ux_1v, uw_2v_2)$ is a fork in G, a contradiction. Hence $w_2 \nsim u$. Similarly, $w_2 \nsim v$.

Let $z' \in N(z) \setminus \{w_2, x_2, y\}$. Then $z' \sim u$ and $z' \sim v$ to avoid $(zz', zy, zx_2u, zw_2v_2)$ and $(zz', zx_2, zyv, zw_2v_2)$, respectively. Hence, $z' \notin \{u', v'\}$.

We claim that $\{x, z\} \subseteq V_4$. If $x \notin V_4$ and $x_3 \in N(x) \setminus \{w_1, x_1, x_2, y\}$, then $x_3 \sim z$ to avoid $(xx_1, xx_3, xw_1v_1, xyz)$, and $x_3 \sim u$ to avoid $(xy, xx_3, xw_1v_1, xx_2u)$, which implies $N(x_2) \subseteq N(x_3)$, contradicting Lemma 3.2. So $x \in V_4$. Now assume $z \notin V_4$, and let $z'' \in N(z) \setminus \{w_2, x_2, y, z'\}$. Then $(zz', zz'', zw_2v_2, zyx)$ is a fork, a contradiction. So $z \in V_4$.

Let $G' = G - \{v_1, v_2, w_1, w_2, x, x_2, y, z\}$. Then $\chi(G') \leq 3$ by the choice of G. So let c' be a 3-coloring of G'. We extend c' to a 3-coloring of G as follows. Let $c(x) = c'(x)$ for all $x \in V(G')$ and $c(x) = c(x_1)$ for $x \in \{x_2, y\}$. Then greedily color z, w_2, v_2, v_1, w_1, x in this order. This is a contradiction, completing the proof of this lemma. □

CHAPTER V

STRUCTURE AROUND A SHORTEST ODD CYCLE

In this section we derive certain useful properties about the structure of G around a shortest odd cycle.

Lemma 5.1. *Let $C := v_1 \cdots v_g v_1$ be a shortest odd cycle in G. Then there exist a vertex $z \notin V(C) \cup N_1(C)$ and a path Z from z to some v_i such that $V(Z) \cap V(C) = \{v_i\}$ and $d(v_i) \geq 4$.*

Proof. Let $T := V(G) \setminus (V(C) \cup N_1(C))$, and let H denote the subgraph of G obtained by taking the union of all paths P which are from vertices in T to C but internally disjoint from C. Now $T \neq \emptyset$; otherwise $V(G) = V(C) \cup N_1(C)$ and so $\chi(G) \leq 3$ by Corollary 2.2, a contradiction.

Now the assertion of the lemma fails. Then $V(H) \cap V(C) \subseteq V_3$. Let $K := G \setminus (V(H) \setminus V(C))$. By Corollary 4.5, $V(C) \nsubseteq V(H)$. So $H \neq G$. Then by the choice of $G, \chi(H) \leq 3$. Let $c : V(H) \to \{1, 2, 3\}$ be a 3-coloring of H. Then c induces a 3-coloring c' on $V(C) \cap V(H)$, and clearly c' can be extended to a 3-coloring of C (in a greedy way). Now by Corollary 2.2 again, c' may be extended to a 3-coloring c^* of K. Thus c^* and c give a 3-coloring of G, a contradiction.

□

Let $C := v_1 \cdots v_g v_1$ be a shortest odd cycle in G, such that $|V(C) \cap V_3|$ is maximum. By Lemma 5.1, there exist $z \in V(G) \setminus (V(C) \cup N_1(C))$, $i \in \{1, \ldots, g\}$, and path P from v_i to z such that $V(P \cap C) = \{v_i\}$ and $d(v_i) \geq 4$. We choose the triple (C, P, v_i) so that $|V(P)|$ is minimum.

Without loss of generality, we may assume $i = 1$, and let $P = v_1 u_1 \ldots u_n$, where $u_n = z$. Let $w_1 \in N(v_1) \setminus \{u_1, v_g, v_2\}$. Note that, by the minimality of P, P is an induced path.

Lemma 5.2. *If $u_1 \sim v_{g-1}$ then $v_g \nsim P - v_1$, and if $u_1 \sim v_3$ then $v_2 \nsim P - v_1$.*

Proof. Suppose the assertion is false. Then by symmetry, we may assume that $u_1 \sim v_{g-1}$ and $v_g \sim u_s$ for some $1 \leq s \leq n$. Then $s \geq 2$ and $s \neq 3$ (since $og(G) \geq 7$), u_{s+1} is defined (since $d(u_n, C) \geq 2$), $v_g \in V_3$ by the choice of P. So $s \geq 4$, as otherwise, $s = 2$ and $N(v_g) \subseteq N(u_1)$, contradicting Lemma 3.2.

Note that, by the choice of P, $v_{g-1} \nsim P - v_1$ and $w_1 \nsim P - \{v_1, u_2\}$. Moreover, $u_s \nsim v_2$; for otherwise, $v_2 \in V_3$ by the choice of P, and hence $(u_s u_{s+1}, u_s v_2, u_s v_g v_{g-1}, u_s u_{s-1} u_{s-2})$ would be a fork in G. Hence, $v_3 \sim w_1$ to avoid $(v_1 u_1, v_1 w_1, v_1 v_g u_s, v_1 v_2 v_3)$, and $u_2 \sim \{w_1, v_2\}$ to avoid $(v_1 w_1, v_1 v_2, v_1 u_1 u_2, v_1 v_g u_s)$. Thus, we consider two cases.

Case 1. $u_2 \sim w_1$.

Let $v \in N(v_2) \setminus N(w_1)$. Then $v \notin C$ by the minimality of C. If $v \in P$ then $v \neq u_s$ (as $u_s \nsim v_2$), $v \notin \{u_{s-1}, u_{s+1}\}$ (to avoid C_5), $v \neq u_1$ (to avoid C_3), and $v \neq u_2$ (as $v \nsim w_1$); so $(v_1 w_1, v_1 u_1, v_1 v_g u_s, v_1 v_2 v)$ is a fork in G, a contradiction. Hence, $v \notin P$.

Now $v \sim u_1$ to avoid $(v_1 u_1, v_1 w_1, v_1 v_g u_s, v_1 v_2 v)$, $v \nsim v_{g-2}$ to avoid the shorter odd cycle $v v_2 \cdots v_{g-2} v$, and $u_2 \nsim v_{g-2}$ as otherwise $u_2 w_1 v_3 \cdots v_{g-2} u_2$ is an odd cycle shorter than C. So $v \sim u_3$ to avoid $(u_1 v, u_1 v_1, u_1 v_{g-1} v_{g-2}, u_1 u_2 u_3)$. Hence, $v_2 \in V_3$ by choice of P. Therefore, as $v_1 w_1 v_3 v_2 v_1$ is a 4-cycle, $\{w_1, v_3\} \cap V_3 \neq \emptyset$ by Corollary 4.3.

Suppose $w_1 \notin V_3$. Let $w \in N(w_1) \setminus \{u_2, v_1, v_3\}$. Clearly, $w \notin P$, and $w \notin C$ by the minimality of C. Also $w \neq v$ as $v \nsim w_1$. Note that $v_3 \nsim u_3$ by the choice of P. So $w \sim u_3$ to avoid $(w_1 w, w_1 v_3, w_1 v_1 v_g, w_1 u_2 u_3)$. Since $u_2 \nsim v_4$ (to avoid C_5), we have $w \sim v_4$ to avoid $(w_1 w, w_1 u_2, w_1 v_1 v_g, w_1 v_3 v_4)$. Therefore $v_4 \in V_3$ by the choice of P; so $(u_3 u_4, u_3 u_2, u_3 w v_4, u_3 v v_2)$ is a fork in G, a contradiction.

Thus, $w_1 \in V_3$, and $v_3 \notin V_3$. Let $y \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $y \notin C$ and $y \neq u_1$ (by the minimality of C), $y \notin P - u_1$ by the choice of P, and $y \neq v$ as $og(G) \geq 7$. Note that $v_4 \nsim \{u_2, v\}$ by the minimality of C. If $y \sim v_5$ then $y \nsim \{v, u_2\}$ by the minimality of C; so $(v_3 y, v_3 v_4, v_3 v_2 v, v_3 w_1 u_2)$ is a fork in G, a contradiction. So $y \nsim v_5$, and hence $y \sim u_2$ to avoid $(v_3 y, v_3 v_2, v_3 v_4 v_5, v_3 w_1 u_2)$. Now $(u_2 y, u_2 w_1, u_2 u_3 u_4, u_2 u_1 v_{g-1})$ is a fork in G, a contradiction.

Case 2. $u_2 \nsim w_1$.

Hence, $u_2 \sim v_2$, and so $v_2 \in V_3$ by the choice of P. If $w_1 \in V_3$ then $C' := v_1 w_1 v_3 \cdots v_1$ is a shortest odd cycle in G such that $V(C') \cap V_3 = V(C) \cap V_3$; so (C', P, v_1) is a triple that gives the situation in Case 1. Hence, we may assume $w_1 \notin V_3$.

Let $w', w'' \in N(w_1) \setminus \{v_1, v_3\}$ be distinct, and let $w \in \{w', w''\}$. Then $w \notin P$ as $w_1 \nsim u_2$, $w_1 \nsim P - \{u_2, v_1\}$ by the choice of P, $w_1 \nsim u_1$ to avoid C_5, and $w \notin C$ by the minimality of C. Since $s \geq 4$ and by the choice of P, $w \nsim u_s$. So $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gu_s, v_1w_1w)$. By the minimality of C, $v_4 \nsim w$. Therefore, $(w_1w', w_1w'', w_1v_3v_4, w_1v_1v_g)$ is a fork in G, a contradiction. □

Lemma 5.3. *We may further choose C (while fixing P) so that if $w_1 \sim v_3$ then $v_2 \nsim P - v_1$, and if $w_1 \sim v_{g-1}$ then $v_g \nsim P - v_1$.*

Proof. Suppose this is not true. We may assume by symmetry that $w_1 \sim v_3$ and $v_2 \sim u_s$ for some $1 \leq s \leq n$. Then $s \geq 2$ to avoid C_3, and $s \neq 3$ to avoid C_5. Moreover, by the choice of P, $v_2 \in V_3$, and u_{s+1} is defined.

Note that $w_1 \nsim u_s$, as otherwise, $N(v_2) \subseteq N(w_1)$, contradicting Lemma 3.2. Moreover, by Lemma 5.2, $w_1 \nsim u_2$ and $u_1 \nsim v_3$. Thus by the choice of P, we see that $\{w_1, v_3\} \nsim P - v_1$.

We claim that $s = 2$. For, suppose $s \geq 4$. Then $u_s \nsim v_g$; otherwise, $v_g \in V_3$ by the choice of P, and so $(u_su_{s+1}, u_sv_g, u_su_{s-1}u_{s-2}, u_sv_2v_3)$ would be a fork in G. Also, $u_s \nsim v_{g-1}$ to avoid C_5. Hence $u_1 \sim v_{g-1}$ to avoid $(v_1w_1, v_1u_1, v_1v_2u_s, v_1v_gv_{g-1})$. Therefore, $v_g \nsim P - v_1$ by Lemma 5.2. So $(v_1v_g, v_1w_1, v_1u_1u_2, v_1v_2u_s)$ is a fork in G, a contradiction.

Case 1. $u_1 \sim v_{g-1}$.

By Corollary 4.3, $\{u_1, u_2\} \nsubseteq V_3$.

Subcase 1.1. $u_2 \in V_3$.

Then $u_1 \notin V_3$, and hence $v_g \in V_3$ by the choice of C. Let $u \in N(u_1) \setminus \{u_2, v_1, v_{g-1}\}$. Then $u \notin C \cup P$ by the choices of C and P. Note $u \nsim v_g$; for, otherwise, $N(v_g) \subseteq N(u_1)$, contradicting Lemma 3.2. Let $N(v_g) \setminus N(u_1) = \{v\}$. So $v \notin C$ by the minimality of C, and

$v \notin P$ by Lemma 5.2.

Then $w_1 \sim \{u, v\}$ to avoid $(v_1v_2, v_1w_1, v_1u_1u, v_1v_gv)$, $u \sim \{u_3, v_{g-2}\}$ to avoid $(u_1u, u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$, and $u \sim \{u_3, w_1\}$ to avoid $(u_1u, u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$. Hence, since $u \nsim w_1$ or $u \nsim v_{g-2}$ (by minimality of C), $u \sim u_3$. Moreover, $u \sim \{w_1, v_{g-2}\}$ to avoid $(u_1u, u_1u_2, u_1v_{g-1}v_{g-2}, u_1v_1w_1)$.

Suppose $u \sim w_1$. Then $u \nsim v_{g-2}$. First, assume $u_3 \notin V_3$, and let $u', u'' \in N(u_3) \setminus \{u, u_2\}$. Note that $v \nsim u_3$; otherwise let $v \neq u'$, and $(u_3u', u_3u, u_3u_2v_2, u_3vv_g)$ is a fork in G, a contradiction. Now $w_1 \sim \{u', u''\}$ to avoid $(u_3u', u_3u'', u_3u_2v_2, u_3uw_1)$. Without loss of generality, let $w_1 \sim u'$. Hence $v_{g-2} \nsim \{u, u'\}$ by the minimality of C. Also $u \nsim v_4$ by minimality of C. Now $u_1 \sim u'$ or $v \sim w_1$ to avoid $(v_1u_1, v_1v_2, v_1w_1u', v_1v_gv)$. If $u' \sim u_1$ then $(u_1u, u_1u', u_1v_1v_2, u_1v_{g-1}v_{g-2})$ is a fork in G, a contradiction. So $u' \nsim u_1$ and $v \sim w_1$. Then $(w_1v, w_1v_1, w_1v_3v_4, w_1uu_3)$ is a fork in G, a contradiction. Hence, $u_3 \in V_3$, and so $u \notin V_3$ by Corollary 4.3. Let $u' \in N(u) \setminus \{u_1, u_3, w_1\}$. Then $u' \neq v$ and $u' \nsim v_{g-2}$ to avoid C_5, $u' \nsim v_{g-3}$ by the minimality of C, and $u' \sim \{v_{g-1}, v_1\}$ to avoid $(u_1u_2, u_1v_1, u_1v_{g-1}v_{g-2}, u_1uu')$. If $u' \sim v_{g-1}$, then $(v_{g-1}u', v_{g-1}v_g, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}u_1u_2)$ is a fork in G, a contradiction. So $u' \nsim v_{g-1}$, and $u' \sim v_1$. Then $(v_1u', v_1w_1, v_1v_2u_2, v_1v_gv_{g-1})$ is a fork in G, a contradiction.

Thus, $u \nsim w_1$, and $u \sim v_{g-2}$. Hence, $v_{g-2} \in V_3$ by choice of P. Furthermore, $u_3 \nsim v_4$ to avoid C_5, so $u_3 \nsim C$ by the minimality of C and choice of P. If $u_3 \notin V_3$, then let $u', u'' \in N(u_3) \setminus \{u, u_2\}$; now $(u_3u', u_3u'', u_3uv_{g-2}, u_3u_2v_2)$ is a fork in G, a contradiction. So $u_3 \in V_3$, and hence $u \notin V_3$ by Corollary 4.3.

Let $u' \in N(u) \setminus \{u_1, u_3, v_{g-2}\}$. Then $u' \notin \{v, v_2\}$ to avoid C_5 and, by the minimality of C, $u' \notin C$, $u_3 \nsim v_{g-3}$, and $u_1 \nsim \{v_{g-2}, v_{g-3}\}$. So $u' \sim \{v_{g-1}, v_1\}$ to avoid $(u_1u_2, u_1v_{g-1}, u_1v_1w_1, u_1uu')$.

If $u' \nsim v_{g-1}$ then $u' \sim v_1$; so $(v_1u', v_1w_1, v_1v_2u_2, v_1v_gv_{g-1})$ is a fork in G, a contradiction. Hence, $u' \sim v_{g-1}$. Let $u'' \in N(u') \setminus N(u_1)$. Then $u' \sim v$ to avoid $(v_{g-1}v_{g-2}, v_{g-1}v_g, v_{g-1}u_1u_2, v_{g-1}u'u'')$. Since $u_2 \nsim v_{g-3}$ by the minimality of C, $u' \sim v_{g-3}$ to avoid $(v_{g-1}u', v_{g-1}v_g, v_{g-1}u_1u_2, v_{g-1}v_{g-2}v_{g-3})$. Hence $(u'v, u'v_{g-3}, u'vv_g, u'uu_3)$ is a fork in G, a contradiction.

Subcase 1.2. $u_2 \notin V_3$.

Let $u \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then $u \notin C$ (by the minimality of C and because of $u_2 \nsim b_g$ by Lemma 5.2), and $u \nsim w_1$ to avoid C_5. Let $u', u'' \in N(u_3) \setminus \{u_2\}$ such that $u' \notin N(u_1)$ and $u'' \notin N(u)$. Note that u', u'' need not be distinct, $\{u', u''\} \cap \{v_4, v_{g-2}\} = \emptyset$ to avoid C_5, and $u', u'' \notin C$ by the minimality of C, the choice of P, and Lemma 5.2.

Then $u \sim \{v_{g-1}, u'\}$ to avoid $(u_2u, u_2v_2, u_2u_3u', u_2u_1v_{g-1})$, $u \sim v_{g-1}$ or $u'' \sim u_1$ to avoid $(u_2u, u_2v_2, u_2u_3u'', u_2u_1v_{g-1})$, and $u \sim \{v_{g-1}, v_3\}$ to avoid $(u_2u, u_2u_3, u_2u_1v_{g-1}, u_2v_2v_3)$.

Suppose $u \nsim v_{g-1}$. Then $u \sim v_3$, $u \sim u'$, and $u'' \sim u_1$. So $u \sim v_1$ or $u'' \sim v_{g-2}$ to avoid $(u_1v_1, u_1u'', u_1v_{g-1}v_{g-2}, u_1u_2u)$, and $u \sim v_1$ or $u'' \sim w_1$ to avoid $(u_1u'', u_1v_{g-1}, u_1u_2u, u_1v_1w_1)$. By the minimality of C, $u'' \nsim w_1$ or $u'' \nsim v_{g-2}$; so $u \sim v_1$. This implies $N(v_2) \subseteq N(u)$, contradicting Lemma 3.2.

Therefore $u \sim v_{g-1}$; so $u \nsim v_3$ by the minimality of C. Thus, $u \sim u'$ to avoid $(u_2u, u_2u_1, u_2v_2v_3, u_2u_3u')$, and $u_1 \sim u''$ to avoid $(u_2u, u_2u_1, u_2v_2v_3, u_2u_3u'')$. Note that $v_g \nsim u''$, otherwise $v_g \in V_3$ by the choice of P, and so $N(v_g) \subseteq N(u_1)$, contradicting Lemma 3.2. Therefore, $u'' \sim v_{g-2}$ to avoid $(v_{g-1}v_g, v_{g-1}u, v_{g-1}u_1u'', v_{g-1}v_{g-2}v_{g-3})$, and so $v_{g-2} \in V_3$ by the choice of P. Also, $u' \sim v_g$ to avoid $(v_{g-1}u_1, v_{g-1}v_g, v_{g-1}uu', v_{g-1}v_{g-2}v_{g-3})$. So $v_g \in V_3$ by the choice of P, and $u'' \nsim w_1$ by the minimality of C. Hence, $u' \sim w_1$ to avoid $(v_1w_1, v_1v_2, v_1v_gu', v_1u_1u'')$. So $(u'u, u'v_g, u'u_3u'', u'w_1v_3)$ is a fork.

Case 2. $u_1 \nsim v_{g-1}$.

If $w_1 \in V_3$ then, since $w_1 \nsim P - v_1$, we see that the assertion of the lemma holds with $v_1w_1v_3 \cdots v_gv_1$ replacing C. Hence, we may assume $w_1 \notin V_3$. Let $w, w' \in N(w_1) \setminus \{v_1, v_3\}$.

Suppose $\{w, w'\} \nsim v_g$. Then $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w)$ and $w' \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w')$. Note $v_g \nsim u_3$ to avoid C_5. If $v_g \sim u_2$ then $v_g \in V_3$ by the choice of P, and hence $u_3 \sim w$ to avoid $(u_2u_3, u_2v_g, u_2v_2v_3, u_2u_1w)$; if $v_g \nsim u_2$ then $u_3 \sim \{w, w'\}$ to avoid $(u_1w, u_1w', u_1u_2u_3, u_1v_1v_g)$. Hence, by symmetry, we may assume $u_3 \sim w$. Then $w' \sim u_3$ to avoid $(w_1w', w_1v_3, w_1v_1v_g, w_1wu_3)$. Also, $v_4 \sim \{w, w'\}$ to avoid $(w_1w, w_1w', w_1v_1v_g, w_1v_3v_4)$, and we may assume $w' \sim v_4$. Then $v_4 \in V_3$ by the choice of P, and $u_3 \nsim v_{g-1}$ by the minimality of C. If $u_2 \nsim v_g$ then $(u_1u_2, u_1w, u_1w'v_4, u_1v_1v_g)$ is a fork in G, a contradiction; and if $u_2 \sim v_g$ then $(u_2u_3, u_2u_1, u_2v_3v_4, u_2v_gv_{g-1})$ is a fork in G, a contradiction.

Therefore, we may assume $w \sim v_g$.

Subcase 2.1. There exists $u \in N(u_1) \setminus (N(w_1) \cup \{u_2\})$.

Then $u \neq v_{g-1}$ as $u_1 \nsim v_{g-1}$; so $u \notin C$ by the minimality of C. Also, $u \notin P$ as P is induced. Now $u \sim v_g$ to avoid $(v_1w_1, v_1v_2, v_1u_1u, v_1v_gv_{g-1})$, and $u \sim v_{g-2}$ to avoid $(v_gu, v_gw, v_gv_1v_2, v_gv_{g-1}v_{g-2})$. Hence, $\{u, w\} \nsim u_3$ by the choice of P. By Corollary 4.3, $\{u_1, u_2\} \nsubseteq V_3$.

First, suppose $u_1 \notin V_3$. Let $u' \in N(u_1) \setminus \{u, u_2, v_1\}$. Then $u' \notin C \cup \{w_1\}$ and $u' \notin P$. Now $u' \sim \{u_3, w_1\}$ to avoid $(u_1u', u_1u, u_1u_2u_3, u_1v_1w_1)$, and $u' \sim \{u_3, v_{g-2}\}$ to avoid $(u_1u', u_1v_1, u_1u_2u_3, u_1uv_{g-2})$. By the minimality of C, $u' \nsim w_1$ or $u' \nsim v_{g-2}$; so $u' \sim u_3$. Hence, $u' \nsim v_{g-2}$ and $u' \neq w$, by the choice of P. If $u' \nsim w_1$ then $(u_1u', u_1u_2, u_1uv_{g-2}, u_1v_1w_1)$ is a fork in G, a contradiction; and if $u' \sim w_1$ $(w_1w, w_1v_1, w_1u'u_3, w_1v_3v_4)$ is a fork, a contradiction.

Hence, $u_1 \in V_3$. Therefore, $u_2 \notin V_3$, and let $u' \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then $u' \notin C$ by the minimality of C and the choice of P, $u' \notin \{w, u\}$ as $og(G) \geq 7$, and $u' \neq w_1$ as $w_1 \nsim u_2$. Since $u_3 \nsim \{u, v_3\}$ by the choice of P, $u' \sim \{u, v_3\}$ to avoid $(u_2u', u_2u_3, u_2u_1u, u_2v_2v_3)$.

If $u' \sim u$, then $u' \nsim \{v_3, v_{g-3}\}$ by the minimality of C, and $u' \sim w$ to avoid $(uu', uu_1, uv_gw, uv_{g-2}v_{g-3})$; so $(u_2u_3, u_2u_1, u_2u'w, u_2v_2v_3)$ is a fork in G, a contradiction. Hence $u' \nsim u$, and $u' \sim v_3$. Note that $u' \nsim v_5$ by the minimality of C. If $u' \nsim w$ then $(v_3v_2, v_3u', v_3w_1w, v_3v_4v_5)$ is a fork in G, a contradiction. So $u' \sim w$. Then $u' \sim v_1$ to avoid $(v_gu, v_gv_{g-1}, v_gwu', v_gv_1v_2)$. Now $(u'w, u'v_1, u'u_2u_3, u'v_3v_4)$ is a fork in G, a contradiction.

Subcase 2.2. $N(u_1) \subseteq N(w_1) \cup \{u_2\}$, and $u_1 \in V_3$ and $u_1 \sim w$.

Then $w \nsim u_3$. For, otherwise, $u_3 \sim v_{g-1}$ to avoid $(wu_3, wu_1, ww_1v_3, wv_gv_{g-1})$. So by the choice of P, $v_{g-1} \in V_3$ and u_4 is defined. Hence, $(u_3u_4, u_3v_{g-1}, u_3ww_1, u_3u_2v_2)$ is a fork in G, a contradiction.

Let $x \in N(u_3) \setminus N(u_1)$ such that $x = u_4$ whenever u_4 is defined. Then $x \notin C$, and $x \neq w_1$

by the choice of P. Note that $x \nsim v_g$ as otherwise $v_g x$ (when $x = u_4$) or $v_g x u_3$ (when $x \neq u_4$) contradicts the choice of P. So $x \nsim w_1$ to avoid $(v_1u_1, v_1v_2, v_1w_1x, v_1v_gv_{g-1})$.

Now $u_2 \nsim v_4$; for otherwise $v_4 \in V_3$ by choice of P, and hence $(u_2u_1, u_2v_2, u_2u_3x, u_2v_4v_5)$ is a fork, a contradiction. So $u_2 \nsim C$ by the minimality of C and the choice of P. By Corollary 4.3, $u_2 \notin V_3$. Let $u \in N(u_2) \setminus \{u_1, u_3, w\}$. Since $N(u_3) \nsubseteq N(u)$, we may further choose x so that if $x \neq u_4$ then $x \nsim u$. Note that $u \sim \{x, v_3\}$ to avoid $(u_2u, u_2u_1, u_2u_3x, u_2v_2v_3)$, $u \sim \{x, w\}$ to avoid $(u_2u, u_2v_2, u_2u_3x, u_2u_1w)$, and $u \sim \{w, v_3\}$ to avoid $(u_2u, u_2u_3, u_2u_1w, u_2v_2v_3)$.

Suppose $u \sim x$. Then $x = u_4$ by the choice of x. So $u \nsim v_3$ by the choice of P. Hence $u \sim w$, and so $u \sim v_{g-1}$ to avoid $(wu, wu_1, ww_1v_3, wv_gv_{g-1})$. Thus, $v_{g-1} \in V_3$ by the choice of P, and $x \sim v_{g-2}$ to avoid $(ux, uw, uu_2v_2, uv_{g-1}v_{g-2})$. By the choice of P again, u_5 is defined. Hence, $(uv_{g-1}, uw, uu_2v_2, uxu_5)$ is a fork in G, a contradiction.

Therefore, $u \nsim x$. Hence, $u \sim v_3$ and $u \sim w$; so $u \nsim v_1$ to avoid $(uv_1, uw, uu_2u_3, uv_3v_4)$.

We claim that $u \in V_3$. For, let $y \in N(u) \setminus \{u_2, w, v_3\}$. By the minimality of C, $y \notin C$. Then $y \nsim v_g$; otherwise $(v_gy, v_gw, v_gv_1v_2, v_gv_{g-1}v_{g-2})$ would be a fork. Also $w_1 \sim y$ to avoid $(ww_1, wu_1, wv_gv_{g-1}, wuy)$. Now $(v_1u_1, v_1v_2, v_1w_1y, v_1v_gv_{g-1})$ is a fork in G, a contradiction.

Now $w' \sim v_g$ to avoid $(v_1u_1, v_1v_2, v_1w_1w', v_1v_gv_{g-1})$. Hence $w' \nsim v_4$ by the minimality of C, and so $(w_1w', w_1v_1, w_1wu, w_1v_3v_4)$ is a fork in G, a contradiction.

Subcase 2.3. $N(u_1) \subseteq N(w_1) \cup \{u_2\}$, and $u_1 \in V_3$ and $u_1 \nsim w$.

Then we may assume $u_1 \sim w'$. Now $w' \nsim v_g$, otherwise we are back in Subcase 2.2. By Corollary 4.3, $u_2 \notin V_3$.

We claim that $u_2 \nsim v_4$. For, suppose $u_2 \sim v_4$. Then $v_4 \in V_3$ by the choice of P, and $u_3 \sim \{w', v_5\}$ to avoid $(u_2u_3, u_2v_2, u_2u_1w', u_2v_4v_5)$. If $w' \nsim u_3$ then $v_5 \sim u_3$; so $v_5 \in V_3$ and u_4 is defined, and hence $(u_2v_2, u_2v_4, u_2u_1w', u_2u_3u_4)$ is a fork, a contradiction. Hence, $w' \sim u_3$. Then $u_3 \sim w$ to avoid $(w_1w, w_1v_1, w_1v_3v_4, w_1w'u_3)$, and $u_3 \sim v_5$ to avoid $(u_2u_1, u_2v_2, u_2u_3w, u_2v_4v_5)$. Thus, the odd cycle $v_gwu_3v_5 \ldots v_g$ contradicts the minimality of C.

Thus, by the minimality of C and choice of P, $u_2 \nsim C \cup P$. Let $u \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Note $u \nsim \{w, w'\}$ as $og(G) \geq 7$. Let $x \in N(u_3) \setminus N(u_1)$ such that $x = u_4$ if u_4 is defined and, subject to this, $x \nsim u$ when possible. Then $x \notin C$ by definition, and $x \neq w_1$ to avoid C_5.

(2.3.1). Suppose $x \in \{w, w'\}$ for any choice of x.

Suppose $N(u_3) = \{u_2, w, w'\}$. Then $w_1 \in V_4$; for, if there exists $w_1' \in N(w_1) \setminus \{v_1, v_3, w, w'\}$ then $w_1' \notin C$ (by minimality of C), and so $(w_1w_1', w_1v_3, w_1wu_3, w_1v_1u_1)$ is a fork, a contradiction.

We claim that $v_1 \in V_4$. For, otherwise, let $v_1' \in N(v_1) \setminus \{u_1, v_2, v_g, w_1\}$. Then $v_1' \sim \{u_2, v_{g-1}\}$ to avoid $(v_1v_1', v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$, and $v_1' \sim \{w', v_{g-1}\}$ to avoid $(v_1v_1', v_1v_2, v_1v_gv_{g-1}, v_1w_1w')$. Since $N(u_1) \nsubseteq N(v_1')$, $v_1' \nsim u_2$ or $v_1' \nsim w'$; so $v_1' \sim v_{g-1}$. Thus, $v_1' \nsim v_3$ by the minimality of C, and hence $v_1' \sim w'$ to avoid $(v_1v_1', v_1v_g, v_1u_1w', v_1v_2v_3)$. Then $(w'u_3, w'u_1, w'v_1'v_{g-1}, w'w_1v_3)$ is a fork in G, a contradiction.

Then $u \sim v_3$; otherwise, let $u' \in N(u) \setminus N(u_3)$, and then $(u_2u_3, u_2u_1, u_2uu', u_2v_2v_3)$ would be a fork.

Suppose there exists $u' \in N(u) \setminus (N(u_3) \cup \{v_3\})$. Then $u \sim w'$ to avoid $(u_2u_3, u_2v_2, u_2u_1w', u_2uu')$, and so $u \nsim w$ (as $N(u_3) \nsubseteq N(u)$ by Lemma 3.2). Therefore, $u \sim v_5$ to avoid $(v_3u, v_3v_2, v_3w_1w, v_3v_4v_5)$. Now $(u_2u_1, u_2v_2, u_2u_3w, u_2uv_5)$ is a fork in G, a contradiction.

Hence, $N(u) \subseteq N(u_3) \cup \{v_3\}$. Since $N(u_3) \nsubseteq N(u)$ (by Lemma 3.2), $u \nsim w$ or $u \nsim w'$, so $u \in V_3$. In fact, $u \sim w$ to avoid $(v_3u, v_3v_2, v_3v_4v_5, v_3w_1w)$; and hence $u \nsim w'$. So $w' \sim v_4$ to avoid $(v_3v_2, v_3u, v_3v_4v_5, v_3w_1w')$. Note that $v_4, v_g \in V_3$ by the choice of P.

We claim that $u_2 \in V_4$. For, suppose there exists $u_2' \in N(u_2) \setminus \{u, u_1, u_3, v_2\}$. Then treating u_2' as u, we have $N(u_2') = \{u_2, v_3, w\} = N(u)$, contradicting Lemma 3.2.

Now $w \in V_4$. For, suppose there exists $y \in N(w) \setminus \{u_3, v_g, u, w_1\}$. By the minimality of C, $y \notin C$, and $y \nsim v_3$ or $y \nsim v_{g-1}$. If $y \nsim v_{g-1}$, then $(wy, ww_1, wu_3u_2, wv_gv_{g-1})$ is a fork, a contradiction. So $y \nsim v_3$; and hence $(wy, wv_g, wu_3u_2, ww_1v_3)$ is a fork, a contradiction.

Moreover, $v_3 \in V_4$; otherwise let $v_3' \in N(v_3) \setminus \{v_2, v_4, w_1, u\}$, and then $(v_3v_3', v_3v_4, v_3w_1w, v_3v_2u_2)$ is a fork, a contradiction. Finally $w' \in V_4$; as otherwise let $w'' \in N(w') \setminus$

$\{u_1, u_3, v_4, w_1\}$, and then $(w'w'', w'v_4, w'u_3w, w'w_1v_1)$ is a fork, a contradiction.

Therefore, $G - \{v_{g-1}v_g, v_4v_5\}$ is not connected, contradicting Lemma 3.1.

(2.3.2) x may be chosen so that $x \in\notin \{w, w'\}$.

Suppose $u \sim x$. Then $x = u_4$ by the choice of x, u and u_3 are symmetric, and $\{u, u_3\} \nsim \{v_1, v_3, v_g\}$ by the choice of P. Now $w' \sim \{u, u_3\}$ to avoid $(u_2u, u_2u_3, u_2v_2v_3, u_2u_1w')$, and we may assume $w' \sim u$ by symmetry. Then $u \sim w$ or $w' \sim v_4$ to avoid $(w_1w, w_1v_1, w_1w'u, w_1v_3v_4)$. If $u \sim w$, then $v_g \in V_3$ by the choice of P, so $(ux, uw', uu_2v_2, uwv_g)$ is a fork, a contradiction. Hence $u \nsim w$, and $w' \sim v_4$. Then $v_4 \in V_3$ by the choice of P, so $(w'v_4, w'u_1, w'ux, w'w_1w)$ is a fork, a contradiction.

So $u \nsim x$. Then $u \sim v_3$ to avoid $(u_2u, u_2u_1, u_2u_3x, u_2v_2v_3)$. Hence, $u \nsim v_1$ to avoid $N(v_2) \subseteq N(u)$, and $u \sim \{w, v_5\}$ to avoid $(v_3u, v_3v_2, v_3w_1w, v_3v_4v_5)$. If $u \sim v_5$ then $u_3 \nsim v_5$ by the choice of P, so $(u_2u_1, u_2v_2, u_2u_3x, u_2uv_5)$ is a fork in G, a contradiction. Thus, $u \nsim v_5$, and hence $u \sim w$. Then $u_3 \sim w$ to avoid $(u_2u_1, u_2v_2, u_2u_3x, u_2uw)$; so $v_g \in V_3$ by the choice of P. Now $w_1 \sim x$ or $u_3 \sim v_{g-1}$ to avoid $(wu, ww_1, wu_3x, wv_gv_{g-1})$. If $w_1 \sim x$ then $(v_1u_1, v_1v_2, v_1w_1x, v_1v_gv_{g-1})$ is a fork, a contradiction. Hence $w_1 \nsim x$, and $u_3 \sim v_{g-1}$. Then $v_{g-1} \in V_3$ by the choice of P; so $(u_3x, u_3v_{g-1}, u_3u_2v_2, u_3ww_1)$ is a fork in G, a contradiction.

Subcase 2.4. $N(u_1) \subseteq N(w_1) \cup \{u_2\}$, and $u_1 \notin V_3$.

First, suppose $u_1 \nsim w$. Let $w', w'' \in N(u_1) \cap N(w_1)$. Then $v_g \nsim \{w', w''\}$, to avoid $(v_gw, v_gw', v_gv_1v_2, v_gv_{g-1}v_{g-2})$ and $(v_gw, v_gw'', v_gv_1v_2, v_gv_{g-1}v_{g-2})$. So $v_4 \sim \{w', w''\}$ to avoid $(w_1w', w_1w'', w_1v_1v_g, w_1v_3v_4)$. Without loss of generality, let $v_4 \sim w''$. Then $w' \nsim v_4$ to avoid $(v_4w', v_4w'', v_4v_3v_2, v_4v_5v_6)$, and $u_3 \sim \{w', w''\}$ to avoid $(u_1w', u_1v_1, u_1w''v_4, u_1u_2u_3)$. If $u_3 \nsim w''$, then $u_3 \sim w'$, and $(w_1w'', w_1v_3, w_1w'u_3, w_1v_1v_g)$ is a fork, a contradiction. So $u_3 \sim w''$. Then $v_4 \in V_3$ by the choice of P, and $u_3 \sim w'$ to avoid $(w_1w', w_1v_3, w_1v_1v_g, w_1w''u_3)$. Hence, $u_3 \sim w$ to avoid $(w_1w, w_1v_1, w_1w'u_3, w_1v_3v_4)$. Now $(u_3w, w_3w', u_3u_2v_2, u_3w''v_4)$ is a fork, a contradiction.

Hence, $u_1 \sim w$. Note that $u_1 \nsim v_3$ since $N(v_2) \nsubseteq N(u_1)$ (by Lemma 3.2).

We claim that $u_3 \nsim w$. For, otherwise, $u_3 \sim v_{g-1}$ to avoid $(wu_3, wu_1, wv_gv_{g-1}, ww_1v_3)$.

Hence $v_g, v_{g-1} \in V_3$ by the choice of P, and u_4 is defined; so $(wv_g, wu_1, wu_3u_4, ww_1v_3)$ is a fork, a contradiction.

Next we show that $u_3 \nsim w'$ or $v_4 \nsim w'$. Suppose on the contrary that $u_3 \sim w'$ and $v_4 \sim w'$. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if u_4 is defined. Then $u \notin C$ by the choice of u, $u \nsim v_4$ by the choice of P, and $u_3 \sim v_5$ or $u \sim w_1$ to avoid $(w'w_1, w'u_1, w'v_4v_5, w'u_3u)$. If $u \sim w_1$ then $(w_1w, w_1v_1, w_1uu_3, w_1v_3v_4)$ is a fork in G, a contradiction. So $u \nsim w_1$, and $u_3 \sim v_5$. Then $v_5 \in V_3$ by the choice of P, and $(u_3u, u_3v_5, u_3w'w_1, u_3u_2v_2)$ is a fork, a contradiction.

Now $w' \nsim u_3$; for otherwise, $v_4 \nsim w'$, and so $(w_1w, w_1v_1, w_1v_3v_4, w_1w'u_3)$ is a fork, a contradiction. Let $w'' \in N(w') \setminus N(w)$. Then $w'' \neq u_3$. Moreover, $w'' \notin C$. For, if $w'' \in C$ then $w'' \in \{v_g, v_4\}$ by the minimality of C. If $w'' = v_g$ then $(v_gw, v_gw', v_gv_1v_2, v_gv_{g-1}v_{g-2})$ is a fork, a contradiction; and if $w'' = v_4$ then $(u_1v_1, u_1w, u_1u_2u_3, u_1w'v_4)$ is a fork, a contradiction.

Suppose $w'' \sim u_2$. Then $w'' \sim v_3$ to avoid $(u_2w'', u_2u_3, u_2u_1w, u_2v_2v_3)$. Hence $w'' \sim v_5$ to avoid $(v_3w'', v_3v_2, v_3w_1w, v_3v_4v_5)$. Now $(u_2u_3, u_2v_2, u_2u_1w, u_2w''v_5)$ is a fork in G, a contradiction.

Thus, $w'' \nsim u_2$. If $w'' \sim v_1$, then $w'' \sim v_3$ to avoid $(v_1v_g, v_1w'', v_1w_1v_3, v_1u_1u_2)$, and $w'' \sim v_{g-1}$ to avoid $(v_1w'', v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$; but then the odd cycle $v_3 \dots v_{g-1}w''v_3$ contradicts the choice of C. So $w'' \nsim v_1$. Then $(u_1w, u_1v_1, u_1w'w'', u_1u_2u_3)$ is a fork in G, a contradiction.

□

Now suppose C, P, v_1 are further chosen to satisfy Lemmas 5.2 and 5.3.

Lemma 5.4. *Suppose* $d(u_n, C) \geq 2$ *and* $n \geq 3$, $u_1 \sim v_{g-1}$, *and* $w_1 \sim v_3$. *Then for any* $v \in N(v_2) \cap N(v_g)$, $|N(v) \cap \{u_1, w_1\}| \neq 1$.

Proof. First, suppose there exists $v \in N(v_2) \cap N(v_g)$ such that $v \sim u_1$ and $v \nsim w_1$. Then $v_g \notin V_3$, as otherwise $N(v_g) \subseteq N(u_1)$ which contradicts Lemma 3.2. Hence, $v \nsim u_3$ by the choice of P. So $u_2 \sim v_{g-2}$ to avoid $(u_1v, u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$. Thus, $u_2 \nsim w_1$ by the minimality of C. Now, $(u_1v, u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$ is a fork in G, a contradiction.

Now suppose there exists $v \in N(v_2) \cap N(v_g)$ such that $v \nsim u_1$ and $v \sim w_1$. Then $v_2, w_1 \notin V_3$ as otherwise $N(v_2) \subseteq N(w_1)$ or $N(w_1) \subseteq N(v_2)$, contradicting Lemma 3.2. Let $w \in N(w_1) \setminus N(v_2)$. Then $w \notin C$ by the minimality of C, and $w \notin \{u_1, u_3\}$ since $og(G) \geq 7$. We claim that $w_1 \nsim u_2$ (so $w \neq u_2$); otherwise, with $v_1w_1u_2 \ldots n$ replacing P, we have the situation in the first paragraph.

Note that $w \sim \{u_1, v_g\}$ to avoid $(v_1v_2, v_1v_g, v_1u_1u_2, v_1w_1w)$. Suppose $w \sim u_1$. Then $u_2 \sim v_{g-2}$ to avoid $(u_1u_2, u_1w, u_1v_1v_2, u_1v_{g-1}v_{g-2})$, $w \sim u_3$ to avoid $(u_1w, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_3)$, and $u_3 \sim v$ to avoid $(w_1v, w_1v_1, w_1v_3v_4, w_1wu_3)$. Thus by the choice of P, $v_2 \in V_3$, a contradiction.

Hence, $w \nsim u_1$, and $w \sim v_g$. Let $v' \in N(v_2) \setminus N(w_1)$, which exists by Lemma 3.2. Then $v' \sim \{u_1, v_4\}$ to avoid $(v_2v', v_2v, v_2v_3v_4, v_2v_1u_1)$, and $v' \sim \{u_1, v_g\}$ to avoid $(v_1v_g, v_1w_1, v_1u_1u_2, v_1v_2v')$. Therefore, $v' \sim u_1$ by the minimality of C. So $(u_1v', u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$ is a fork, a contradiction. □

Lemma 5.5. *Suppose $d(u_n, C) \geq 2$ and $n \geq 3$, $u_1 \sim v_{g-1}$, and $w_1 \sim v_3$. Then for any $x \in N(v_g) \setminus N(u_1)$ and $y \in N(v_2) \setminus N(w_1)$, $x \sim v_2$ or $y \sim v_g$.*

Proof. Suppose there exist $x \in N(v_g) \setminus N(u_1)$ and $y \in N(v_2) \setminus N(w_1)$ such that $x \nsim v_2$ and $y \nsim v_g$. Note that $x, y \notin C$, and $x, y \notin P$ as $\{v_2, v_g\} \nsim P - v_1$ (by Lemma 5.2). Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Note that $u_2 \nsim v_4$ by minimality of C, and $u_1 \sim y$ or $w_1 \sim x$ to avoid $(v_1u_1, v_1w_1, v_1v_gx, v_1v_2y)$.

Case 1. $w_1 \nsim u_2$.

Then $y \sim u_1$ to avoid $(v_1w_1, v_1v_g, v_1u_1u_2, v_1v_2y)$, $y \sim u_3$ to avoid $(u_1y, u_1v_{g-1}, u_1v_1w_1, u_1u_2u_3)$, $u_2 \sim v_{g-2}$ to avoid to avoid $(u_1u_2, u_1y, u_1v_1w_1, u_1v_{g-1}v_{g-2})$, and $x \sim w_1$ to avoid $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_gx)$.

Suppose $v_3 \notin V_3$. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \notin C \cup \{u_1\}$ by the minimality of C, $v \notin P - \{u_1, u_2, v_1\}$ by the choice of P, and $v \notin \{u_2, x, y\}$ since $og(G) \geq 7$. Now $v \sim \{y, v_5\}$ to avoid $(v_3v, v_3w_1, v_3v_2y, v_3v_4v_5)$. If $v \nsim v_5$ and $v \sim y$ then replacing P with $v_3vyu_3 \ldots u_n$, we get a contradiction to Lemma 5.3. So $v \sim v_5$. Then $v \nsim \{y, u_2\}$ by the minimality of C; so $(v_3v, v_3v_4, v_3v_2y, v_3w_1x)$ is a fork in G, a contradiction.

So $v_3 \in V_3$ and $w_1 \notin V_3$. Let $w \in N(w_1) \setminus \{v_1, v_3, x\}$. Then $w \notin P - u_1$, since $w \nsim u_2$ and by the choice of P, $w \notin C$ by the minimality of C, and $w \neq y$ as $w_1 \nsim y$. Note that $w \sim \{u_1, v_4\}$ to avoid $(w_1w, w_1x, w_1v_1u_1, w_1v_3v_4)$. If $w_1 \sim u_1$ then $w \sim u_3$ to avoid $(u_1w, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_3)$; so $(u_3u, u_3w, u_3yv_2, u_3u_2v_{g-2})$ is a fork, a contradiction. Thus $w \nsim u_1$. Then $w \sim v_4$, and hence $w \nsim v_g$ by the minimality of C. Then $(v_1v_g, v_1v_2, v_1u_1u_2, v_1w_1w)$ is a fork, a contradiction.

Case 2. $w_1 \sim u_2$.

So there is symmetry between w_1 and u_1, and we may assume $u_1 \sim y$. If $u_2 \sim v_{g-2}$ then $v_{g-2} \in V_3$ by the choice of P, and $w_1 \sim u$ to avoid $(u_2v_{g-2}, u_2u_1, u_2w_1v_3, u_2u_3u)$; hence, $(w_1u, w_1v_3, w_1v_1v_g, w_1u_2v_{g-2})$ is a fork, a contradiction. Thus, $u_2 \nsim v_{g-2}$, and so $y \sim u_3$ to avoid $(u_1y, u_1v_1, u_1v_{g-1}v_{g-2}, u_1u_2u_3)$. Hence, $v_2 \in V_3$ by the choice of P. Therefore, by Lemma 4.3, $\{w_1, v_3\} \not\subseteq V_3$.

Suppose $v_3 \notin V_3$. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then as in Case 1, $v \notin C \cup \{u_1\}$, $v \notin P - \{u_1, u_2, v_1\}$, $v \notin \{u_2, x, y\}$, $v \sim v_5$. Then $v \nsim \{y, u_2\}$ by the minimality of C; so $(v_3v, v_3v_4, v_3v_2y, v_3w_1u_2)$ is a fork in G, a contradiction.

Thus, $v_3 \in V_3$ and $w_1 \notin V_3$. Let $w \in N(w_1) \setminus \{u_2, v_1, v_3\}$. Then $w \notin P$ by the choice of P, $w \notin C$ by the minimality of C, and $w \neq y$ as $w_1 \nsim y$. If $w_1 \sim x$ then $x \sim u_3$ to avoid $(w_1x, w_1v_1, w_1u_2u_3, w_1v_3v_4)$; so replacing P with $v_1w_1xu_3 \ldots u_n$, we get a contradiction to Lemma 5.3. Hence, $w_1 \nsim x$. By Lemma 5.2, $u_2 \nsim \{v_2, v_g\}$. Hence $w \sim \{v_4, v_g\}$ to avoid $(w_1w, w_1u_2, w_1v_1v_g, w_1v_3v_4)$. If $w \sim v_4$ then $w \nsim \{u_1, v_g\}$ by the minimality of C; so $(v_1u_1, v_1v_2, v_1w_1w, v_1v_gx)$ is a fork in G, a contradiction. Thus, $w \nsim v_4$, and $w \sim v_g$. So $w \nsim u_3$ by the choice of P, and $(w_1w, w_1v_1, w_1u_2u_3, w_1v_3v_4)$ is a fork, a contradiction. □

Lemma 5.6. *Suppose $d(u_n, C) \geq 2$ and $n \geq 3$, $w_1 \sim v_3$, and $u_1 \sim v_{g-1}$. Then $(N(v_2) \setminus N(w_1)) \cap (N(v_g) \setminus N(u_1)) = \emptyset$.*

Proof. Suppose on the contrary there exists $v \in (N(v_2) \setminus N(w_1)) \cap (N(v_g) \setminus N(u_1))$. Then $v \neq u_2$ by Lemma 5.2, and $v \nsim u_2$ since $og(G) \geq 7$. Thus, since $v_1 \notin V_3$, $v \notin V_3$ by the choice of C.

We claim that $u_1, w_1 \notin N(v)$. For, otherwise, it follows from Lemma 5.4 that $u_1, w_1 \in N(v)$. Then $u_3 \sim v$ to avoid $(u_1v, u_1v_1, u_1v_{g-1}v_{g-2}, u_1u_2u_3)$. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin C$. Hence $(vu_1, vv_g, vu_3u, vv_2v_3)$ is a fork in G, a contradiction.

Let $v', v'' \in N(v) \setminus \{v_2, v_g\}$ be distinct. Then $\{v', v''\} \sim \{v_3, v_{g-1}\}$ to avoid $(vv', vv'', vv_2v_3, vv_gv_{g-1})$. By symmetry, let $v' \sim \{v_3, v_{g-1}\}$. Then $v' \notin P - \{u_1, u_2v_1\}$ by the choice of P. Also note that $v' \notin \{u_1, w_1\}$ as $v \nsim \{u_1, w_1\}$, and $v' \notin \{u_1, u_2, v_1\}$ and $v' \nsim \{u_1, w_1\}$ as $og(G) \geq 7$.

We claim that $w_1 \nsim u_2$. For, suppose $w_1 \nsim u_2$. By symmetry let $v' \sim v_3$. Then $u_2 \sim v'$ to avoid $(v_3v', v_3v_2, v_3v_4v_5, v_3w_1u_2)$ and $v \sim u_3$ to avoid $(u_2u_3, u_2w_1, u_2v'v, u_2u_1v_{g-1})$. Then by the choice of P, $\{v_2, v_g\} \subseteq V_3$. Let $u \in N(u_3) \setminus N(v')$ such that $u = u_4$ if $n \geq 4$. Thus $u \notin C$ by its definition, and $u \nsim \{v_1, v_{g-2}\}$ by the choice of P. So $(vv_2, vv', vu_3u, vv_gv_{g-1})$ is a fork in G, a contradiction.

Let $w \in N(w_1) \setminus N(v_2)$. Then $w \notin C$ by the minimality of C, $w \notin P - \{u_1, u_2, v_1\}$ by the choice of P, $w \neq u_2$ as $w_1 \nsim u_2$, and $w \notin \{u_1, v', v''\}$ since $og(G) \geq 7$. Now $w \sim \{u_1, v_g\}$ to avoid $(v_1v_2, v_1v_g, v_1u_1u_2, v_1w_1w)$. If $w \sim u_1$ then, since $w \nsim v_2$, $w \sim u_3$ to avoid $(u_1w, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_3)$; hence, replacing P with $v_1w_1wu_3 \ldots u_n$, we get back a contradiction to the claim that $w_1 \nsim u_2$. Thus, $w \nsim u_1$ and $w \sim v_g$.

Hence, by the choice of C, $w \nsim v_3$ and $u_1 \notin V_3$. Let $u \in N(u_1) \setminus \{u_2, v_1, v_{g-1}\}$. Then $u \notin C \cup P \cup \{v, w, w_1\}$, by the minimality of C, the choice of P, and the fact that $u_1 \nsim \{v, w\}$.

Case 1. $u \nsim u_3$.

Then $u \sim w_1$ to avoid $(u_1u, u_1v_{g-1}, u_1u_2u_3, u_1v_1w_1)$, and $u \sim v_2$ to avoid $(u_1u, u_1v_{g-1}, u_1u_2u_3, u_1v_1v_2)$. So $u \nsim v_{g-2}$ by the minimality of C, and $u_2 \sim v_{g-2}$ to avoid $(u_1u, u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$. Hence, by choice of P, $v_{g-2} \in V_3$. If $u_2 \in V_3$ then $v_{g-1} \in V_3$ by the choice of C, contradicting Corollary 4.3. Hence $u_2 \notin V_3$.

Let $u' \in N(u_2) \setminus \{u_1, u_3, v_{g-2}\}$. Then $u' \notin C \cup P$ by the choice of P and the minimality of C, $u' \notin \{u, v, w\}$ since $og(G) \geq 7$, and $u' \neq w_1$ as $u_2 \nsim w_1$. Let $z \in N(u_3) \setminus N(u_1)$ such that $z \notin C$. Note that such z does exist as otherwise u_4 is defined an we choose $z = u_4$. Note that $z \notin \{v, w_1\}$ by the choice of P.

Suppose $u' \nsim z$. Then $u' \sim v_1$ to avoid $(u_2u', u_2v_{g-2}, u_2u_3z, u_2u_1v_1)$. So $u' \sim v_{g-1}$ to avoid $(v_1v_2, v_1w_1, v_1u'u_2, v_1v_gv_{g-1})$, and $u' \sim w$ to avoid $(v_{g-1}u', v_{g-1}u_1, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}v_gw)$. Now $(u'v_{g-1}, u'w, u'v_1v_2, u'u_2u_3)$ is a fork, a contradiction.

So $u' \sim z$ for all $z \in N(u_3) \setminus N(u_1)$ such that $z \notin C$. Let $u_3' \in N(u_3) \setminus N(u')$. Then by the choice of z, $u_3' \sim u_1$ or $u_3' = v_{g-3}$. If $u_3' \sim u_1$ then $(u_1u, u_1v_1, u_1u_3'u_3, u_1v_{g-1}v_{g-2})$ is a fork, a contradiction. So $u_3' \nsim u_1$ and $u_3' = v_{g-3}$, and we may let $z = u_4$. Now u' is symmetric to u_3. So for any $u'' \in N(u') \setminus N(u_1)$, $u'' = v_{g-3}$, a contradiction as $v_{g-3} \in V_3$ (by the choice of P).

Case 2. $u \sim u_3$.

So u and u_2 are symmetric. Note that $u \nsim \{v_2, v_g\}$ by Lemma 5.2 (with $v_1u_1uu_3 \ldots u_n$ in place of P). Hence $v_{g-2} \sim \{u, u_2\}$ to avoid $(u_1u, u_1u_2, u_1v_1v_2, u_1v_{g-1}v_{g-2})$. By symmetry, we may assume $u \sim v_{g-2}$. Then by the choice of P, $v_{g-2} \in V_3$. Note that $u \nsim V(C - v_g) \cup \{w_1\}$ by the minimality of C, and $u \nsim \{v, w\}$ as $og(G) \geq 7$. If $u \in V_3$ then $v_{g-1} \in V_3$ by the choice of C, contradicting Corollary 4.3. Hence, $u \notin V_3$.

Let $u' \in N(u) \setminus \{u_1, u_3, v_{g-2}\}$. Then $u' \notin C \cup \{v, w, w_1\}$, $u' \neq u_2$ since $og(G) \geq 7$, and $u' \notin P$ by the choice of P. Let $z \in N(u_3) \setminus N(u_1)$ such that $z = u_4$ if $n \geq 4$. Then $z \notin C$, and $z \notin \{u_1, v, w_1\}$ by the choice of P. So $u' \sim \{z, v_1\}$ to avoid $(uu', uv_{g-2}, uu_3z, uu_1v_1)$.

If $u' \nsim z$, then $u' \sim v_1$. So $u' \nsim v_{g-3}$ by the minimality of C, $u' \sim v_{g-1}$ to avoid $(v_1w_1, v_1v_2, v_1u'u, v_1v_gv_{g-1})$, and $v \sim u'$ to avoid $(v_{g-1}u', v_{g-1}u_1, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}v_gv)$. Then $(u'v_{g-1}, u'v, u'v_1w_1, u'uu_3)$ is a fork, a contradiction.

So $u' \sim z$. Note that $z \nsim \{v_1, v_2, v_3, v_{g-1}, v_g\}$ by the choice of P. Let $u_2' \in N(u_2) \setminus N(u)$. Then $u_2' \notin C$ by the choices of C and P, $u_2' \notin \{v, w_1\}$ since $u_2 \notin \{v, w_1\}$, and $u_2' \nsim \{w, z\}$ since $og(G) \geq 7$.

Now $u_2' \sim \{v_1, v_{g-1}\}$ to avoid $(u_1u, u_1v_{g-1}, u_1v_1v_2, u_1u_2u_2')$. If $u_2' \sim v_1$ then $u_2' \sim v_3$

to avoid $(v_1u_2', v_1v_g, v_1u_1u, v_1v_2v_3)$; but then $(v_3w_1, v_3v_2, v_3u_2'u_2, v_3v_4v_5)$ is a fork, a contradiction. So $u_2' \nsim v_1$ and $u_2' \sim v_{g-1}$. Then $u_2' \nsim v_{g-3}$; for otherwise replacing P with $v_{g-1}u_1uu_3 \dots u_n$, we get a contradiction to Lemma 5.3. Hence $u_2' \sim w$ to avoid $(v_{g-1}u_2', v_{g-1}u_1, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}v_gw)$, $u' \nsim v_1$ to avoid $(v_1u_1, v_1v_g, v_1u'z, v_1v_2v_3)$, and $u' \sim \{u_2, v_{g-1}\}$ to avoid $(u_1u_2, u_1v_{g-1}, u_1v_1v_2, u_1uu')$. If $u' \sim u_2$ then $u' \sim w$ to avoid $(u_2u', u_2u_3, u_2u_1v_1, u_2u_2'w)$; and so $(u'z, u'u_2, u'uv_{g-2}, u'wv_g)$ is a fork in G, a contradiction. So $u' \nsim u_2$ and $u' \sim v_{g-1}$. Then $u' \sim v_{g-3}$ to avoid $(v_{g-1}u', v_{g-1}v_g, v_{g-1}u_1u_2, v_{g-1}v_{g-2}v_{g-3})$. Now, replacing P with $v_{g-1}u_1uu_3 \dots u_n$, we get a contradiction to Lemma 5.3.

□

CHAPTER VI

FINAL REDUCTION

We now show that $d(u_n, C) \leq 1$ or $n \leq 2$, and then complete the proof using Lemma 2.1 to derive a final contradiction. First, we need the following lemma.

Lemma 6.1. *Suppose* $d(u_n, C) \geq 2$ *and* $n \geq 3$. *If* $w_1 \sim \{v_3, v_{g-1}\}$ *and* $u_1 \nsim \{v_3, v_{g-1}\}$. *Then* $w_1 \nsim u_2$.

Proof. By symmetry assume $w_1 \sim v_3$. Suppose $w_1 \sim u_2$. We distinguish two cases according to whether or not $w_1 \in V_3$.

Case 1. $w_1 \notin V_3$.

Let $w \in N(w_1) \setminus \{u_2, v_1, v_g\}$. Then $w \notin C$ by the minimality of C, and $w \notin P$ by the choice of P and the fact that $og(G) \geq 7$.

Subcase 1.1. $w \nsim u_3$.

Then $\{w, u_2\} \sim v_4$ to avoid $(w_1w, w_1v_1, w_1u_2u_3, w_1v_3v_4)$. First, suppose $w \sim v_4$. Then $w \nsim v_g$ by the minimality of C. Hence, $u_2 \sim v_g$ to avoid $(w_1w, w_1v_3, w_1u_2u_3, w_1v_1v_g)$. So $v_g \in V_3$ by the choice of P. Now $u_3 \sim v_{g-1}$ to avoid $(u_2u_3, u_2u_1, u_2w_1v_3, u_2v_gv_{g-1})$. So u_4 is defined by the choice of P. Now $(u_2v_g, u_2u_1, u_2u_3u_4, u_2w_1v_3)$ is a fork in G, a contradiction.

Thus, $w \nsim v_4$ and $u_2 \sim v_4$. Hence $v_4 \in V_3$ by the choice of P. Suppose $u_3 \nsim v_5$, and let $u_3' \in N(u_3) \setminus N(u_1)$; then $u_3' \sim w_1$ to avoid $(u_2u_1, u_2w_1, u_2v_4v_5, u_2u_3u_3')$, and so $(w_1w, w_1v_1, w_1v_3v_4, w_1u_3'u_3)$ is a fork in G, a contradiction. Hence $u_3 \sim v_5$ and, by the choice of P, $v_5 \in V_3$ and u_4 is defined. Now $w \sim u_1$ to avoid $(u_2u_1, u_2v_4, u_2u_3u_4, u_2w_1w)$. Let $u \in N(u_1) \setminus N(w_1)$, which exists by Lemma 3.2. Note that $u \notin C$ since $u_1 \nsim \{v_3, v_{g-1}\}$ and by the minimality of C, and $u \notin P - \{u_1, u_2, v_1\}$ by the choice of P. Now $u \sim u_3$ to avoid $(u_2w_1, u_2v_4, u_2u_3u_4, u_2u_1u)$. Then $u_4 \sim v_6$ to avoid $(u_3u_4, u_3u, u_3u_2w_1, u_3v_5v_6)$. Hence $v_6 \in V_3$ and u_5 is defined; so $(u_3u, u_3v_5, u_3u_4u_5, u_3u_2w_1)$ is a fork in G, a contradiction.

Subcase 1.2. $w \sim u_3$.

Then $w \nsim v_2$; for otherwise, replacing P with $v_1w_1wu_3 \dots u_n$, we get a contradiction to Lemma 5.2.

We claim that $u_2 \nsim v_g$. For, suppose $u_2 \sim v_g$. Then $v_g \in V_3$ by the choice of P. Now $v_{g-1} \sim u_3$ to avoid $(u_2u_3, u_2u_1, u_2w_1v_3, u_2v_gv_{g-1})$; so by the choice of P, $v_{g-1} \in V_3$ and u_4 is defined. Then $(u_2u_1, u_2v_g, u_2u_3u_4, u_2w_1v_3)$ is a fork in G, a contradiction.

We also claim that $u_2 \nsim v_4$. For, suppose $u_2 \sim v_4$. Then $v_4 \in V_3$ by the choice of P. Let $u \in N(u_3) \setminus N(w_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin C$, and $u \neq u_1$ since $og(G) \geq 7$. Suppose $u \sim u_1$. Then $u \nsim v_2$ (otherwise with $v_1u_1uu_3 \dots u_n$ replacing P, we get a contradiction to Lemma 5.3). So $u \sim v_g$ to avoid $(v_1w_1, v_1v_2, v_1u_1u, v_1v_gv_{g-1})$ and hence $v_g \in V_3$; so $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w)$. But then $(u_1u, u_1w, u_1u_2v_4, u_1v_1v_2)$ is a fork, a contradiction. Hence $u \nsim u_1$, and so $u_3 \sim v_5$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2v_4v_5)$. Then by the choice of P, $v_5 \in V_3$ and $u = u_4$. If $w \nsim u_1$ then $u_4 \sim v_6$ to avoid $(u_3u_4, u_3w, u_3u_2u_1, u_3v_5v_6)$ and, hence, by the choice of P, $v_6 \in V_3$ and u_5 is defined; so $(u_3w, u_3v_5, u_3u_4u_5, u_3u_2u_1)$ is a fork, a contradiction. Thus $w \sim u_1$. Let $w' \in N(w) \setminus N(u_2)$ which exists by Lemma 3.2. Then $w' \notin (C \setminus \{v_2, v_g\}) \cup P$ by the minimality of C and by the choice of P. By Lemma 5.2, with $v_1w_1wu_3 \cdots u_n$ replacing P, we see $w' \neq v_2$. Also, $w \nsim v_g$, as otherwise, $wu_3v_5 \dots v_gw$ is an odd cycle shorter than C, a contradiction. So $w' \notin C$. Now If $w' \nsim u_4$ then $u_4 \sim v_6$ to avoid $(u_3u_2, u_3u_4, u_3v_5v_6, u_3ww')$; so u_5 is defined and $(u_3u_2, u_3v_5, u_3u_4u_5, u_3ww')$ is a fork, a contradiction. Hence, $w' \sim u_4$. Then $w' \nsim \{v_1, v_3\}$ by the choice of P, and so $(w_1u_2, w_1v_3, w_1ww', w_1v_1v_g)$ is a fork, a contradiction.

Next, we show $w \nsim v_4$. For, suppose $w \sim v_4$. Then by the choice of P, $v_4 \in V_3$. Note that $w \nsim v_g$ by the minimality of C; so $w \sim u_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1w)$. If $u_3 \nsim v_5$, then let $u \in N(u_3) \setminus N(w_1)$ such that $u = u_4$ if $n \geq 4$; now $u \notin C$, and $(wu_1, ww_1, wu_3u, wv_4v_5)$ is a fork, a contradiction. Hence, $u_3 \sim v_5$ and, by the choice of P, so $v_5 \in V_3$ and u_4 is defined. Let $u_1' \in N(u_1) \setminus N(w_1)$ which exists by Lemma 3.2. Then $u_1' \notin C \cup P$ by the minimality of C and the choice of P, and $u_1' \sim u_3$ to avoid $(wv_4, ww_1, wu_3u_4, wu_1u)$. Now

$(u_3u_1', u_3u_4, u_3u_1w_1, u_3v_5v_4)$ is a fork, a contradiction.

Hence, $w \sim v_g$ to avoid $(w_1w, w_1u_2, w_1v_3v_4, w_1v_1v_g)$; so $v_g \in V_3$ by the choice of P. Note that $w \nsim u_1$; for otherwise, $u_3 \sim v_{g-1}$ to avoid $(wu_1, wu_3, wv_gv_{g-1}, ww_1v_3)$, and hence u_4 is defined by the choice of P; so $(wu_1, wv_g, wu_3u_4, ww_1v_3)$ is a fork, a contradiction. By Lemma 4.3, $\{u_2, u_3, w\} \nsubseteq V_3$. Let $u \in N(u_3) \setminus \{u_2, w\}$ such that $u = u_4$ if $n \geq 4$. Then $u \notin C$.

First, suppose $u_3 \notin V_3$, and let $u' \in N(u_3) \setminus \{u_2, u, w\}$. Note that $u' \nsim w_1$; for otherwise, $u' \notin C$ and $u' \sim v_4$ to avoid $(w_1u', w_1u_2, w_1v_1v_g, w_1v_3v_4)$, and so $v_4 \in V_3$ (by the choice of P) and $(u_3u, u_3u_2, u_3u'v_4, u_3wv_g)$ is a fork, a contradiction. Also $u \nsim w_1$; for otherwise, $u \sim v_4$ to avoid $(w_1u, w_1u_2, w_1v_1v_g, w_1v_3v_4)$, and so $(u_3u', u_3u_2, u_3uv_4, u_3wv_g)$ is a fork, a contradiction. Then $u_1 \sim \{u, u'\}$ to avoid $(u_3u, u_3u', u_3wv_g, u_3u_2u_1)$. If $u_1 \sim u$ then $u \nsim v_2$ by Lemma 5.3 (with $v_1u_1uu_3 \dots u_n$ replacing P); so $(v_1v_2, v_1w_1, v_1u_1u, v_1v_gv_{g-1})$ is a fork, a contradiction. So $u_1 \sim u'$. Then $u' \nsim v_2$ by Lemma 5.3 (with $v_1u_1u'u_3 \dots u_n$ replacing P); so $(v_1v_2, v_1w_1, v_1u_1u', v_1v_gv_{g-1})$ is a fork, a contradiction.

Now suppose $u_3 \in V_3$ and $u_2 \notin V_3$. Then $u \nsim w_1$ by Lemma 3.2. Let $u' \in N(u_2) \setminus \{u_1, u_3, w_1\}$. Then $u' \notin C$ by Lemma 5.2 and the minimality of C, and $u' \notin P$ by the choice of P. If $u' \nsim \{v_1, v_3\}$ then $u' \sim w$ to avoid $(w_1w, w_1v_1, w_1v_3v_4, w_1u_2u')$ and $u' \nsim u$ by Lemma 3.2; so $(wu', wv_g, ww_1v_3, wu_3u)$ is a fork, a contradiction. If $u' \nsim v_3$ and $u' \sim v_1$ then, since $w \nsim u_1$, $(v_1u_1, v_1u', v_1v_2v_3, v_1v_gw)$ is a fork, a contradiction. Thus $u' \sim v_3$. If $u' \sim w$ then $u' \nsim \{u, v_{g-1}\}$ by Lemma 3.2, and so $(wu', ww_1, wv_gv_{g-1}, wu_3u)$ is a fork, a contradiction. So $u' \nsim w$. Then $u' \sim v_5$ to avoid $(v_3u', v_3v_2, v_3w_1w, v_3v_4v_5)$, and $u \sim \{u', u_1\}$ to avoid $(u_2u_1, u_2w_1, u_2u'v_5, u_2u_3u)$. If $u \sim u_1$ then $u \nsim v_2$ by Lemma 5.3 (with $v_1u_1uu_3 \dots n$ replacing P); so $(v_1v_2, v_1w_1, v_1v_gv_{g-1}, v_1u_1u)$ is a fork in G, a contradiction. Hence $u \nsim u_1$ and $u \sim u'$. Then $u \nsim v_2$ by Lemma 5.3 (with $v_3u'uu_3 \dots n$ replacing P), and $u \sim v_4$ to avoid $(v_3v_4, v_3v_2, v_3u'u, v_3w_1w)$. So $v_4 \in V_3$ by the choice of P, which implies $N(v_4) \subseteq N(u')$, contradicting Lemma 3.2.

Thus, $u_2, u_3 \in V_3$ and $w \notin V_3$. Let $w' \in N(w) \setminus \{u_3, v_g, w_1\}$. Then $w' \notin C \cup P \cup \{w_1\}$, and $w' \nsim v_5$ by the minimality of C. So $w' \sim \{v_3, v_{g-1}\}$ to avoid $(ww', wu_3, ww_1v_3, wv_gv_{g-1})$.

If $w' \sim v_3$ then $(v_3w', v_3v_2, v_3v_4v_5, v_3w_1u_2)$ is a fork, a contradiction. Hence $w' \nsim v_3$, and $w' \sim v_{g-1}$. Thus $w' \sim u$ to avoid $(ww', wv_g, ww_1v_3, wu_3u)$, and $w' \sim v_1$ to avoid $(w_1u_2, w_1v_1, w_1ww', w_1v_3v_4)$. But then $N(v_g) \subseteq N(w')$, contradicting Lemma 3.2.

Case 2. $w_1 \in V_3$.

Then $v_2 \in V_3$ by the choice of C, and $v_3 \notin V_3$ by Lemma 4.3. Let $x \in N(v_2) \setminus \{v_1, v_3\}$ and $y \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Note that $x \notin C \cup P \cup w_1\}$ by the minimality of C and Lemma 5.2; and $y \notin C \cup P \cup \{w_1, x\}$ by the minimality of C, the choice of P, and the assumption that $v_3 \nsim u_1$. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$. Then $u \notin C$ by the choice of P, and $u \notin \{u_1, w_1, y\}$ as $og(G) \geq 7$.

We claim that $u_2 \nsim C$. For, suppose $u_2 \sim C$. Then $u_2 \sim \{v_4, v_g\}$ by the minimality of C. If $u_2 \sim v_g$ then $v_g \in V_3$ by the choice of P; so $(u_2u_1, u_2v_g, u_2u_3u, u_2w_1v_3)$ is a fork, a contradiction. So $u_2 \nsim v_g$, and $u_2 \sim v_4$. Then $v_4 \in V_3$ by the choice of P. Now $u_3 \sim v_5$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2v_4v_5)$. So by the choice of P, $v_5 \in V_3$ and $u = u_4$. Let $z \in N(u_4) \setminus \{u_3\}$ such that $z = u_5$ if $n \geq 5$ (z is arbitrary if $n = 4$). Then $z \notin C$, $z \notin \{u_1, u_2, w_1\}$ by the choice of P, and $z \nsim u_2$ to avoid $(u_2u_1, u_2w_1, u_2zu_4, u_2v_4v_5)$. Since $u_3 \notin V_3$ by Lemma 4.3, let $u' \in N(u_3) \setminus \{u_2, u_4, v_5\}$. Then $u' \notin C \cup P$ by the choice of P and the minimality of C. Hence $z \sim u'$ to avoid $(u_3u', u_3v_5, u_3u_2w_1, u_3u_4z)$, $z = u_5$ to avoid $N(u_4) \subseteq N(u')$ (by Lemma 3.2 and the fact that z is arbitrary when $n = 4$), and $v_6 \sim \{u', u_4\}$ to avoid $(u_3u', u_3u_4, u_3v_5v_6, u_3u_2w_1)$. By the symmetry between u' and u_4, let $u' \sim v_6$. Let $u'' \in N(u_4) \setminus N(u')$ (by Lemma 3.2). Then $u'' \sim u_2$ to avoid $(u_3u', u_3v_5, u_3u_2w_1, u_3u_4u'')$. So $(u_2u'', u_2v_4, u_2u_3u', u_2w_1v_1)$ is a fork, a contradiction.

Now we show $y \nsim v_5$. For, suppose $y \sim v_5$. If $u_2 \sim y$ then $u \sim y$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2yv_5)$, and hence $u \neq u_4$ by the choice of P; so $(yu, yv_5, yu_2u_1, yv_3v_2)$ is a fork, a contradiction. Therefore, $u_2 \nsim y$. Also, $u_2 \nsim v_4$; for, otherwise, $v_4 \in V_3$ by the choice of P, and $(u_2u_1, u_2w_1, u_2v_4v_5, u_2u_3u)$ is a fork, a contradiction. Hence, $x \sim \{v_4, y\}$ to avoid $(v_3y, v_3v_4, v_3w_1u_2, v_3v_2x)$. If $x \sim v_4$ then let $v \in N(y) \setminus N(v_4)$; now $(v_3v_2, v_3v_4, v_3w_1u_2, v_3yv)$ is a fork, a contradiction. If $x \sim y$ then let $v \in N(v_4) \setminus N(y)$, and $(v_3v_2, v_3y, v_3w_1u_2, v_3v_4v)$ is a fork, a contradiction.

Then $y \sim u_2$ to avoid $(v_3y, v_3v_2, v_3v_4v_5, v_3w_1u_2)$, and $x \sim \{y, v_4\}$ to avoid $(v_3y, v_3w_1, v_3v_4v_5, v_3v_2x)$. If $N(y) = \{u_2, v_3, x\}$ for all $y \in N(v_3) \setminus \{v_2, v_4, w_1\}$, then $v_3 \in V_4$, and v_5, v_4 contradicts Lemma 3.4. So let y be chosen so that there exists $y' \in N(y) \setminus \{u_2, v_3, x\}$. Then $y' \neq v_5$ (since $y \nsim v_5$), $y \nsim v_1$ (so $y' \neq v_1$) to avoid $N(w_1) \subseteq N(y)$ (by Lemma 3.2), and $y' \sim v_4$ to avoid $(v_3w_1, v_3v_2, v_3v_4v_5, v_3yy')$.

We claim that $x \nsim v_4$ (and hence $x \sim y$). For, suppose $x \sim v_4$. Then $u_3 \nsim y'$ by the choice of P, $y' \sim v_6$ to avoid $(v_4y', v_4x, v_4v_3w_1, v_4v_5v_6)$, and $u \sim y$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2yy')$. So $(yu, yu_2, yv_3v_2, yy'v_6)$ is a fork, a contradiction.

Note that $u_3 \nsim x$; otherwise with $v_3yxu_3 \dots u_n$ replacing P we get a contradiction to Lemma 5.3. If $y \sim u$ then $u \neq u_4$ by the choice of P (hence $u_3 \nsim C$), $y' \sim u_3$ to avoid $(yy', yx, yv_3w_1, yuu_3)$, and $y' \sim u_1$ to avoid $(yy', yu, yv_3v_2, yu_2u_1)$; so $(y'y, y'u_3, y'v_4v_5, y'u_1v_1)$ is a fork, a contradiction. So $y \nsim u$. Then $x \sim u_1$ to avoid $(u_2u_1, u_2w_1, u_2u_3u, u_2yx)$. Moreover, $y' \sim u_3$; for, otherwise, $y' \sim u_1$ to avoid $(u_2u_1, u_2w_1, u_2yy', u_2u_3u)$, and so $(u_1v_1, u_1x, u_1u_2u_3, u_1y'v_4)$ is a fork in G, a contradiction.

We now show that $u_1 \in V_3$. For, suppose $u_1 \notin V_3$, and let $u' \in N(u_1) \setminus \{u_2, v_1, x\}$. Then, $u' \notin C \cup P \cup \{w_1, y\}$ by the choices of C and P and the fact that $og(G) \geq 7$. Moreover, $u' \neq y'$ to avoid $(y'y, y'v_4, y'u_3u, y'u_1v_1)$. Now $u' \sim \{u_3, y\}$ to avoid $(u_2y, u_2w_1, u_2u_3u, u_2u_1u')$. If $u' \sim u_3$ then $u' \nsim v_4$ by the choice of P; so $(u_3u', u_3u, u_3u_2w_1, u_3y'v_4)$ is a fork, a contradiction. If $u' \sim y$ then $(v_3v_2, v_3w_1, v_3v_4v_5, v_3yu')$ is a fork, a contradiction.

If $v_1 \in V_4$, then we see that v_g, v_1 contradict Lemma 3.4. So $v_1 \notin V_4$, and let $v \in N(v_1) \setminus \{v_2, v_g, u_1, w_1\}$. Then $v \notin C \cup P$ by the choice of C and P, $v \notin \{x, y'\}$ since $og(G) \geq 7$, and $v \neq y$ (to avoid $N(w_1) \subseteq N(y)$). Now $v \sim \{u_2, v_{g-1}\}$ to avoid $(v_1v, v_1v_2, v_1u_1u_2, v_1v_gv_{g-1})$, and $v \sim \{u_2, v_3\}$ to avoid $(v_1v, v_1v_g, v_1u_1u_2, v_1v_2v_3)$. By the minimality of C, $v \nsim v_3$ or $v \nsim v_{g-1}$; so $v \sim u_2$. Then $v \nsim v_3$ (to avoid $N(w_1) \subseteq N(v)$), and so $(u_2v, u_2u_1, u_2u_3u, u_2w_1v_3)$ is a fork, a contradiction. □

Lemma 6.2. *Suppose* $n \geq 2$ *and* $d(u_n, C) \geq 3$. *If* $w_1 \sim \{v_3, v_{g-1}\}$ *then* $u_1 \sim \{v_3, v_{g-1}\}$.

Proof. For, suppose $w_1 \sim \{v_3, v_{g-1}\}$ and $u_1 \nsim \{v_3, v_{g-1}\}$. By symmetry assume $w_1 \sim v_3$. Then $w_1 \nsim u_2$ by Lemma 6.1. So $w_1 \nsim P - v_1$ by the choice of P. Also, $v_2 \nsim P - v_1$ by Lemma 5.3. Therefore, $u_2 \sim v_g$ to avoid $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$. So by the choice of P, $v_g \in V_3$.

Let $u \in N(u_3) \setminus N(u_1)$ (by Lemma 3.2) such that $u = u_4$ if $n \geq 4$. Then $u \notin C$, and $u \neq w_1$ since $og(G) \geq 7$. Let $x_1 \in N(w_1) \setminus N(v_2)$, and $x_2 \in N(v_2) \setminus N(w_1)$. Then $x_1, x_2 \notin C \cup P$. Note that $u_1 \sim x_1$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1x_1)$, and $u_1 \sim x_2$ to avoid $(v_1u_1, v_1w_1, v_1v_gv_{g-1}, v_1v_2x_2)$. So $u \notin \{x_1, x_2\}$ as $u \notin N(u_1)$.

Claim 1. $N(w_1) \setminus \{x_1\} = N(v_2) \setminus \{x_2\}$.

First, suppose there exists $v \in N(v_2) \setminus \{x_2\} \setminus N(w_1)$. Then $u_1 \sim v$ to avoid $(v_1u_1, v_1w_1, v_1v_gv_{g-1}, v_1v_2v)$, and $v_4 \sim \{x_2, v\}$ to avoid $(v_2x_2, v_2v, v_2v_1v_g, v_2v_3v_4)$. By symmetry, let $v_4 \sim v$. Note that $u_3 \nsim \{v, x_2\}$ by the choice of P. So $v_4 \sim x_2$ to avoid $(u_1v_1, u_1x_2, u_1u_2u_3, u_1vv_4)$. Now $(v_4x_2, v_4v, v_4v_3w_1, v_4v_5v_6)$ is a fork, a contradiction.

Next, suppose there exists $v \in N(w_1) \setminus \{x_1\} \setminus N(v_2)$. Then $u_1 \sim v$ to avoid $(v_1u_1, v_1v_2, v_1v_gv_{g-1}, v_1w_1v)$, and $v_4 \sim \{x_1, v\}$ to avoid $(w_1x_1, w_1v, w_1v_1v_g, w_1v_3v_4)$. By symmetry, let $v \sim v_4$. Now $v_4 \nsim x_1$ to avoid $(v_4x_1, v_4v, v_4v_3v_2, v_4v_5v_6)$. So $u_3 \sim \{x_1, v\}$ to avoid $(u_1x_1, u_1v_1, u_1u_2u_3, u_1vv_4)$. If $u_3 \nsim v$ then $u_3 \sim x_1$; so $(w_1v, w_1v_3, w_1x_1u_3, w_1v_1v_g)$ is a fork, a contradiction. Hence $u_3 \sim v$ and, by the choice of P, $v_4 \in V_3$. If $u_3 \sim v_5$ then $u = u_4$ and $v_5 \in V_3$ (by the choice of P); so $(u_3u_4, u_3v_5, u_3u_2v_g, u_3vw_1)$ is a fork, a contradiction. Hence, $u_3 \nsim v_5$. Then $u \sim w_1$ to avoid $(vu_1, vw_1, vu_3u, vv_4v_5)$. Now $(w_1x_1, w_1u, w_1v_1v_g, w_1vv_4)$ is a fork, a contradiction.

By Claim 1 and Lemma 3.3, we have $N(x_1) \setminus \{w_1\} \nsubseteq N(x_2) \setminus \{v_2\}$ and $N(x_2) \setminus \{v_2\} \nsubseteq N(x_1) \setminus \{w_1\}$. Let $x \in N(x_2) \setminus \{v_2\} \setminus N(x_1)$.

Claim 2. $x \notin C$.

For, assume $x \in C$. Then $x = v_4$ by the minimality of C, and so $v_4 \nsim x_1$. Hence, $u_3 \sim \{x_1, x_2\}$ to avoid $(u_1x_1, u_1v_1, u_1x_2v_4, u_1u_2u_3)$, and $u_3 \nsim x_1$ or $u_3 \nsim x_2$ to avoid $(u_3u, u_3x_1, u_3x_2v_4, u_3u_2v_g)$. Then $u_3 \nsim x_2$ and $u_3 \sim x_1$; for, otherwise, $u_3 \sim x_2$ and $u_3 \nsim x_1$, and

$v_2, v_4 \in V_3$ by the choice of P; so $(x_2v_2, x_2v_4, x_2u_1x_1, x_2u_3u)$ is a fork in G, a contradiction.

Next, we show $x_2 \notin V_3$. For, assume $x_2 \in V_3$. Then $v_3 \in V_3$ by the choice of C, and so $v_4 \notin V_3$ by Corollary 4.3. Let $v \in N(v_4) \setminus \{v_3, v_5, x_2\}$. Then $v \nsim w_1$ to avoid $(w_1v, w_1v_3, w_1x_1u_3, w_1v_1v_g)$, $v \sim u_1$ to avoid $(v_4v, v_4v_5, v_4v_3w_1, v_4x_2u_1)$, and $v \nsim v_6$ by the minimality of C. Hence, $(v_4v, v_4x_2, v_4v_5v_6, v_4v_3w_1)$ is a fork, a contradiction.

Thus, let $x' \in N(x_2) \setminus \{u_1, v_2, v_4\}$. Note that $x' \notin C \cup (P - u_3)$ by the choice of C and P, and $x' \neq x_1$ since $og(G) \geq 7$, and $x' \notin \{u_3, w_1\}$ since $x_2 \nsim \{u_3, w_1\}$. Now $x' \sim \{u_2, v_5\}$ to avoid $(x_2x', x_2v_2, x_2v_4v_5, x_2u_1u_2)$.

Suppose $x' \sim u_2$. Then $x' \nsim \{v_5, v_{g-1}\}$ since $og(G) \geq 7$. Now $x' \sim u$ or $u_3 \sim v_{g-1}$ to avoid $(u_2x', u_2u_1, u_2u_3u, u_2v_gv_{g-1})$. If $x' \sim u$, then $(x_2u_1, x_2v_2, x_2x'u, x_2v_4v_5)$ is a fork, a contradiction. Moreover, $x' \nsim u$, and $u_3 \sim v_{g-1}$. Now by the choice of P, $v_{g-1} \in V_3$ and $u = u_4$. Hence $u_4 \sim v_{g-2}$ to avoid $(u_3u_4, u_3u_2, u_3x_1w_1, u_3v_{g-1}v_{g-2})$; so by the choice of P, u_5 is defined. Now $(u_3v_{g-1}, u_3u_2, u_3u_4u_5, u_3x_1w_1)$ is a fork, a contradiction.

Hence, $x' \nsim u_2$, and $x' \sim v_5$. Let $x'' \in N(x') \setminus N(v_4)$ which exists by Lemma 3.2. Then $x'' \notin C$ by the minimality of C, $x'' \neq u_2$ as $x' \nsim u_2$, $x'' \notin \{u_1, u_3\}$ as $og(G) \geq 7$, and $x'' \neq u$ to avoid $(x_2v_2, x_2v_4, x_2x'u, x_2u_1u_2)$. Now $x'' \sim \{u_1, v_2\}$ to avoid $(x_2v_2, x_2v_4, x_2u_1u_2, x_2x'x'')$. If $x'' \sim u_1$ then $x'' \sim w_1$ to avoid $(u_1u_2, u_1x'', u_1x_2v_4, u_1v_1w_1)$; hence $x'' \sim v_2$ by Claim 1, and $x'' \nsim u_3$ by the choice of P. If $x'' \nsim u_1$ and $x'' \sim v_2$ then $x'' \sim w_1$ by Claim 1, and $x'' \nsim u_3$ by the choice of P. In both cases, $(w_1x'', w_1v_3, w_1v_1v_g, w_1x_1u_3)$ is a fork, a contradiction.

Claim 3. $x \notin P$.

Suppose $x \in P$. Then $x = u_3$; so $v_2 \in V_3$ by the choice of P, and $w_1 \in V_3$ by Claim 1. Then $u_3 \nsim x_1$ to avoid $(u_3u, u_3u_2, u_3x_2v_2, u_3x_1w_1)$, and $v_3 \notin V_3$ by Corollary 4.3. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$ be arbitrary. Then $v \notin C \cup P$ by the choices of C and P, and $v \notin \{x_1, x_2\}$ since $og(G) \geq 7$.

If $v \sim x_2$ then replacing P with $v_3vx_2u_3 \dots u_n$ we get a contradiction to Lemma 5.3. So $v \nsim x_2$. Then $v_4 \nsim x_2$; otherwise, $v_4 \in V_3$ by the choice of P, and so $(x_2v_2, x_2v_4, x_2u_1x_1, x_2u_3u_4)$ is a fork, a contradiction. So $v \sim v_5$ to avoid $(v_3v, v_3w_1, v_3v_2x_2, v_3v_4v_5)$.

Note that $x_1 \sim \{v, v_4\}$ to avoid $(v_3v, v_3v_4, v_3v_2x_2, v_3w_1x_1)$. If $x_1 \sim v$ then let $v' \in N(v_4) \setminus N(v)$; now $(v_3w_1, v_3v, v_3v_2x_2, v_3v_4v')$ is a fork, a contradiction. So $x_1 \sim v_4$, and let $v' \in N(v) \setminus N(v_4)$. Then $v' \notin C$ by the minimality of C (and since $v \sim v_5$); so $(v_3w_1, v_3v_4, v_3v_2x_2, v_3vv')$ is a fork, a contradiction.

Claim 4. $x \nsim v_1$.

Suppose $x \sim v_1$. Then $x \sim v_{g-1}$ to avoid $(v_1x, v_1v_2, v_1w_1x_1, v_1v_gv_{g-1})$, and $x \sim u_2$ to avoid $(v_1x, v_1v_2, v_1w_1x_1, v_1v_gu_2)$. Now, replacing P with $v_1xu_2 \dots u_n$, we get a contradiction to Lemma 5.2.

Claim 5. $x \nsim u_2$.

Suppose $x \sim u_2$. Then $x \nsim v_5$ by the minimality of C, and $x \sim u$ or $x_1 \sim u_3$ to avoid $(u_2x, u_2v_g, u_2u_3u, u_2u_1x_1)$.

Case 1. $u = u_4$ and $x \sim u_4$.

Then $v_2 \in V_3$ by the choice of P, and so $w_1 \in V_3$ by Claim 1. Hence, $v_3 \notin V_3$ by Lemma 4.3. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$ be arbitrary. Then $v \notin C \cup (P - u_1) \cup \{x\}$ by the minimality of C and the choice of P, $v \neq u_1$ as $v_3 \nsim u_1$, and $v \neq x_1$ and $v \nsim x$ since $og(G) \geq 7$.

Suppose $v \sim x_2$ for all $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \sim x_1$ to avoid $(x_2v, x_2v_2, x_2xu_4, x_2u_1x_1)$, and $v \nsim v_5$ to avoid $(x_2v_2, x_2u_1, x_2xu_4, x_2vv_5)$. If $v \in V_3$ for all $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$ then $v_3 \in V_4$ by Lemma 3.2; so v_4, v_3 contradict Lemma 3.4. Hence, $v \notin V_3$ for some choice of v. Let $v' \in N(v) \setminus \{v_3, x_1, x_2\}$. Then $v' \sim v_4$ to avoid $(v_3w_1, v_3v_2, v_3v_4v_5, v_3vv')$. So $v' \nsim x$; otherwise $v_4 \in V_3$ by the choice of P, and $(xu_4, xu_2, xx_2v_2, xv'v_4)$ would be a fork. Hence $v' \sim u_1$ to avoid $(x_2v_2, x_2u_1, x_2xu_4, x_2vv')$. But then $(u_1v', u_1x_1, u_1u_2x, u_1v_1v_2)$ is a fork, a contradiction.

Thus, $v \nsim x_2$ for some $v \in N(v_3) \setminus \{v_3, x_1, x_2\}$. Note that $v_4 \nsim x_2$; otherwise, $v_4 \in V_3$ by the choice of C, and $(x_2v_2, x_2v_4, x_2xu_4, x_2u_1x_1)$ would be a fork. Hence $v \sim v_5$ to avoid $(v_3v, v_3w_1, v_3u_2x_2, v_3v_4v_5)$. Thus, $x_1 \sim \{v, v_4\}$ to avoid $(v_3v, v_3v_4, v_3v_2x_2, v_3w_1x_1)$. If $x_1 \sim v$, then let $v' \in N(v_4) \setminus N(v)$; now $(v_3v, v_3w_1, v_3v_2x_2, v_3v_4v')$ is a fork, a contradiction. So $x_1 \sim v_4$. Let $v' \in N(v) \setminus N(v_4)$. Then $v' \notin C$ (since $v \sim v_5$), and $(v_3v_4, v_3w_1, v_3v_2x_2, v_3vv')$ is a fork, a contradiction.

Case 2. $x \nsim u_4$ or $u \neq u_4$.

Then we can always choose u so that $x \nsim u$; for otherwise, $n = 3$ and $N(u_3) \subseteq N(x)$, contradicting Lemma 3.2. Thus, $x_1 \sim u_3$, $u_2 \nsim v_{g-2}$ to avoid $(u_2v_{g-2}, u_2x, u_2u_3u, u_2u_1v_1)$, and $v_{g-1} \sim \{u_3, x\}$ to avoid $(u_2x, u_2u_1, u_2u_3u_4, u_2v_gv_{g-1})$.

Suppose $v_{g-1} \sim u_3$. Then, by the choice of P, $v_{g-1} \in V_3$ and $u = u_4$; so $u \sim v_{g-2}$ to avoid $(u_3u_4, u_3x_1, u_3u_2x, u_3v_{g-1}v_{g-12}$. Hence by the choice of P, $v_{g-2} \in V_3$ and u_5 is defined. Thus, $(u_3v_{g-1}, u_3x_1, u_3u_4u_5, u_3u_2x)$ is a fork, a contradiction. Hence, $v_{g-1} \nsim u_3$ and $v_{g-1} \sim x$. So $x_2 \nsim v_2$ by the minimality of C.

We claim that $w_1 \in V_3$. For, otherwise, let $w \in N(w_1) \setminus \{v_1, v_3, x_1\}$. Then $w \sim v_2$ by Claim 1, and $w \nsim u_3$ by the choice of P. Hence $(w_1w, w_1v_3, w_1x_1u_3, w_1v_1v_g)$ is a fork, a contradiction.

Therefore, $v_2 \in V_3$ by Claim 1. So $v_3 \notin V_3$ by Corollary 4.3. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$.

Suppose $v \sim x_2$ for each $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \sim x_1$ to avoid $(x_2v, x_2v_2, x_2xv_{g-1}, x_2u_1x_1)$. If $v \in V_3$ for all $v \in N(v) \setminus \{v_2, v_4, w_1\}$, then $v_3 \in V_4$ by Lemma 3.2, and hence v_4, v_3 contradict Lemma 3.4. So $v \notin V_3$ for some choice of v, and let $v' \in N(v) \setminus \{v_3, x_1, x_2\}$. Now $v \nsim v_5$ (so $v' \neq v_5$) by the minimality of C, and $v' \sim v_4$ to avoid $(v_3w_1, v_3v_2, v_3v_4v_5, v_3vv')$. Hence $v' \nsim x$ by the minimality of C, and $v' \sim u_1$ to avoid $(x_2v_2, x_2u_1, x_2xv_{g-1}, x_2vv')$. But then $(u_1u_2, u_1x_1, u_1v'v_4, u_1v_1v_2)$ is a fork, a contradiction.

So $v \nsim x_2$ for some $v \in N(v_3) \setminus \{v_3, x_1, x_2\}$. Then $v \sim v_5$ to avoid $(v_3w_1, v_3v, v_3v_2x_2, v_3v_4v_5)$, and $x_1 \sim \{v, v_4\}$ to avoid $(v_3v, v_3v_4, v_3u_2x_2, v_3w_1x_1)$. If $x_1 \sim v$ then let $v' \in N(v_4) \setminus N(v)$ (by Lemma 3.2); now $(v_3v, v_3w_1, v_3v_2x_2, v_3v_4v')$ is a fork, a contradiction. If $x_1 \sim v_4$ then let $v' \in N(v) \setminus N(v_4)$ (by Lemma 3.2); now $(v_3v_4, v_3w_1, v_3v_2x_2, v_3vv')$ is a fork, a contradiction.

Claim 6. $x_2 \nsim u_3$.

For, suppose $x_2 \sim u_3$. Then by the choice of P, $v_2 \in V_3$. Now $x \sim u$ or $u_3 \sim x_1$, to avoid $(x_2x, x_2v_2, x_2u_3u, x_2u_1x_1)$. If $x_1 \sim u_3$ then $(u_3u, u_3x_1, u_3x_2v_2, u_3u_2v_g)$ is a fork, a contradiction. So $x_1 \nsim u_3$; and hence $x \sim u$ for any choice of u. Hence, $x \sim v_3$ to avoid

$(x_2x, x_2u_3, x_2v_2v_3, x_2u_1x_1)$. Now $x \sim v_5$; for otherwise $(v_3x, v_3v_2, v_3w_1x_1, v_3v_4v_5)$ (when $u \sim v_4$) or $(v_3v_2, v_3w_1, v_3xu, v_3v_4v_5)$ (when $u \nsim v_4$) would be a fork. Then $(xv_5, xu, xv_3w_1, xx_2u_1)$ is a fork, a contradiction.

By Claims 4, 5 and 6, $x \nsim \{u_2, v_1\}$ and $x_2 \nsim u_3$. So $u_3 \sim x_1$ to avoid $(u_1v_1, u_1x_1, u_1u_2u_3, u_1x_2x)$.

Suppose $w_1 \notin V_3$, and let $w \in N(w_1) \setminus \{v_1, v_3, x_1\}$. Then $w \notin C \cup (P - \{u_1, u_2\})$ by the choices of C and P, $w \notin \{u_2, x_2\}$ as $w_1 \nsim \{u_2, x_2\}$, and $w \notin \{u_1, x\}$ as $og(G) \geq 7$. Since $w \sim v_2$ (by Claim 1), $w \nsim u_3$ by the choice of P. So $(w_1w, w_1v_3, w_1x_1u_3, w_1v_1v_g)$ is a fork, a contradiction.

Therefore, $w_1 \in V_3$. So $v_2 \in V_3$ by Claim 1, and hence $v_3 \in V_3$ by Lemma 4.3. Let $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$ be arbitrary. Then $v \notin C \cup (P - u_1)$ by the choices of C and P, $v \notin \{x_1, x_2\}$ since $og(G) \geq 7$, and $v \neq u_1$ as $u_1 \nsim v_3$.

Suppose $v \sim x_1$ for all $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$. Then $v \neq x$ as $x \nsim x_1$. If $v \nsim x_2$ then $v \sim u$ to avoid $(x_1v, x_1w_1, x_1u_1x_2, x_1u_3u)$, $v \nsim v_5$ to avoid $(x_1u_3, x_1w_1, x_1u_1x_2, x_1vv_5)$, $x_2 \sim v_4$ to avoid $(v_3v, v_3w_1, v_3v_2x_2, v_3v_4v_5)$, and $u \sim v_4$ to avoid $(v_3v_2, v_3w_1, v_3vu, v_3v_4v_5)$; so $(v_3w_1, v_3v, v_3v_2x_2, v_3v_4v_5)$ is a fork, a contradiction. Thus, $v \sim x_2$. If $v \in V_3$ for all $v \in N(v_3) \setminus \{v_2, v_4, w_1\}$ then $v_3 \in V_4$ by Lemma 3.2, and so, v_4, v_3 contradict Lemma 3.4. Thus $v \notin V_3$ for some choice of v. Let $v' \in N(v) \setminus \{v_3, x_1, x_2\}$. If $v \sim v_5$ then $v \sim u$ to avoid $(x_1w_1, x_1u_1, x_1u_3u, x_1vv_5)$; so $(vv_5, vu, vx_1u_1, vv_3v_2)$ is a fork, a contradiction. Hence $v \nsim v_5$ (so $v' \neq v_5$); and so $v' \sim v_4$ to avoid $(v_3w_1, v_3v_2, v_3v_4v_5, v_3vv')$. If $v' \nsim \{u_3, u_1\}$ then $v \sim u$ to avoid $(x_1u_1, x_1w_1, x_1u_3u, x_1vv')$, and hence $(vv', vu, vx_1u_1, vv_3v_2)$ is a fork, a contradiction. If $v' \sim u_3$ then $(u_3u, u_3v', u_3u_2v_g, u_3x_1w_1)$ is a fork, a contradiction. So $v' \nsim u_3$ and $v' \sim u_1$. Then $(u_1v', u_1x_2, u_1u_2u_3, u_1v_1w_1)$ is a fork, a contradiction.

Thus $v \nsim x_1$ (possibly $v = x$) for some v. If $v_4 \sim x_1$ then $v_4 \in V_3$ by the choice of P; so $(x_1u_1, x_1w_1, x_1u_3u, x_1v_4v_5)$ is a fork, a contradiction. Hence $v_4 \nsim x_1$, and so $v \sim v_5$ to avoid $(v_3v, v_3v_2, v_3w_1x_1, v_3v_4v_5)$, and $x_2 \sim \{v, v_4\}$ to avoid $(v_3v, v_3v_4, v_3v_2x_2, v_3w_1x_1)$. If $x_2 \sim v$ then let $v' \in N(v_4) \setminus N(v)$ (by Lemma 3.2); now $(v_3v, v_3v_2, v_3w_1x_1, v_3v_4v')$ is a fork, a contradiction. If $x_2 \sim v_4$ then let $v' \in N(v) \setminus N(v_4)$ (by Lemma 3.2); now $(v_3v_4, v_3v_2, v_3w_1x_1, v_3vv')$ is a fork, a contradiction. □

Lemma 6.3. *$n \leq 2$ or $d(u_n, C) \leq 1$.*

Proof. Suppose $n \geq 3$ and $d(u_n, C) \geq 2$.

Case 1. $w_1 \sim \{v_3, v_{g-1}\}$.

Then by Lemma 6.2, $u_1 \sim \{v_3, v_{g-1}\}$.

If $w_1 \sim v_3$ and $u_1 \sim v_{g-1}$ then by Lemma 5.6, $(N(v_g) \setminus N(u_1)) \cap (N(v_2) \setminus N(w_1)) = \emptyset$. Let $x \in N(v_2) \setminus N(w_1)$ and $y \in N(v_g) \setminus N(u_1)$; so $x \neq y$. By Lemma 5.5, $x \sim v_g$ or $y \sim v_2$. If $x \sim v_g$ then $x \sim u_1$, contradicting Lemma 5.4 (as $x \sim w_1$). So $y \sim v_2$. Hence $y \sim v_2$, contradicting Lemma 5.4 again.

Similarly, if $w_1 \sim v_{g-1}$ and $u_1 \sim v_3$, we also get a contradiction to Lemma 5.4.

Thus by symmetry, we may assume that $v_3 \sim u_1$, $v_3 \sim w_1$ and $v_{g-1} \nsim \{u_1, w_1\}$. Then $v_2 \nsim P - v_1$ by Lemma 5.2. If $w_1 \nsim u_2$ then $u_2 \sim v_4$ to avoid $(v_3v_2, v_3w_1, v_3v_4v_5, v_3u_1u_2)$, and $u_2 \nsim v_g$ by minimality of C; so $(v_1v_2, v_1w_1, v_1u_1u_2, v_1v_gv_{g-1})$ is a fork, a contradiction. Thus $w_1 \sim u_2$, and hence u_1 and w_1 are symmetric. Let $u \in N(u_1) \setminus N(w_1)$ and $w \in N(w_1) \setminus N(u_1)$. Then $u, w \notin C$ by the minimality of C, and $u, w \notin P$ by the choice of P.

Note that $w \sim \{v_2, v_4\}$ to avoid $(v_3v_2, v_3u_1, v_3v_4v_5, v_3w_1w)$, and $w \sim \{v_2, v_g\}$ to avoid $(v_1v_2, v_1u_1, v_1v_gv_{g-1}, v_1w_1w)$. Hence, $w \sim v_2$, since $w \nsim v_4$ or $w \nsim v_g$ (by minimality of C). Similarly, $u \sim v_2$. Also note that $\{x, w\} \sim \{v_4, v_g\}$ to avoid $v_2u, v_2w, v_2v_3v_4, v_2v_1v_g)$. So by symmetry let $w \sim v_4$; hence $w \nsim v_g$ by the minimality of C.

Now $v_g \sim u_2$ to avoid $(w_1w, w_1v_3, w_1u_2u_3, w_1v_1v_g)$. Thus, $v_g \in V_3$ (by the choice of P), and $u_3 \sim v_{g-1}$ to avoid $(u_2w_1, u_2u_3, u_2u_1u, u_2v_gv_{g-1})$. So u_4 is defined by the choice of P. Now $(u_2u_1, u_2v_g, u_2u_3u_4, u_2w_1w)$ is a fork in G, a contradiction.

Case 2. $w_1 \nsim \{v_3, v_{g-1}\}$.

Then $u_1 \sim \{v_3, v_{g-1}\}$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_gv_{g-1})$. If $w_1 \sim u_2$ then by replacing P with $v_1w_1u_2 \ldots u_n$ we get back to Case 1. So $w_1 \nsim u_2$. By symmetry assume $u_1 \sim v_{g-1}$.

Then $v_g \nsim P - v_1$ by Lemma 5.2. So $u_2 \sim v_2$ to avoid $(v_1v_g, v_1w_1, v_1v_2v_3, v_1u_1u_2)$. Hence, $v_2 \in V_3$ by the choice of P, and so $u_1, u_2 \notin V_3$ by Corollary 4.3. Let $u \in N(u_3) \setminus N(u_1)$ such that $u = u_4$ if $n \geq 4$ and u is arbitrary otherwise. Note that $u \notin C$, and $u \neq w_1$ by the choice of P.

Subase 2.1. $u_2 \notin V_3$.

Let $u' \in N(u_2) \setminus \{u_1, u_3, v_2\}$. Then $u' \notin C$ by the minimality of C, $u' \neq u$ since $og(G) \geq 7$, and $u' \sim \{u, v_{g-1}\}$ to avoid $(u_2u', u_2v_2, u_2u_3u, u_2u_1v_{g-1})$. If $u' \sim v_{g-1}$ then replacing P with $v_{g-1}u'u_2 \cdots u_n$, we get back to Case 1. So $u' \nsim v_{g-1}$ and $u' \sim u$ (for all choice of u). Thus, $u = u_4$ to avoid $N(u_3) \subseteq N(u')$.

So we have symmetry between u and u_3, and thus we also have $u_3 \nsim v_{g-1}$. Hence, $v_3 \sim \{u', u_3\}$ to avoid $(u_2u', u_2u_3, u_2u_1v_{g-1}, u_2v_2v_3)$. By symmetry, let $u' \sim v_3$. Then $v_3 \in V_3$ by the choice of P; so $u' \notin V_3$ by Corollary 4.3.

Let $u'' \in N(u') \setminus \{u_2, u, v_3\}$. Then by the choices of C and P, $u'' \notin C \cup P$ and $u'' \neq w_1$. Let $u_3' \in N(u_3) \setminus N(u')$ (by Lemma 3.2). Then by the choices of C and P, $u_3' \notin C \cup P$ and $u_3' \neq w_1$. Now $u_3' \sim u_1$ to avoid $(u_2u', u_2v_2, u_2u_3u_3', u_2u_1v_{g-1})$, and $u_3' \sim w_1$ to avoid $(u_1u_3', u_1v_{g-1}, u_1u_2u', u_1v_1w_1)$. But then, replacing P with $v_1w_1u_3'u_3 \cdots u_n$, we get back to Case 1.

Subcase 2.2. $u_2 \in V_3$ and $u_1 \notin V_3$.

Let $u' \in N(u_1) \setminus \{u_2, v_1, v_{g-1}\}$. Then $u' \notin C \cup P$ by the choices of C and P, $u' \neq u$ as $u \nsim u_1$, $u' \neq w_1$ as $og(G) \geq 7$, and $u' \sim \{u_3, v_{g-2}\}$ to avoid $(u_1u', u_1v_1, u_1u_2u_3, u_1v_{g-1}v_{g-2})$. Also note that $v_g \in V_3$ by the choice of C; so $u' \nsim v_g$ by Lemma 3.2.

Note that $u' \sim w_1$ to avoid $(v_1w_1, v_1v_g, v_1v_2v_3, v_1u_1u')$. So $u' \nsim u_3$ as otherwise, replacing P with $v_1w_1uu_3 \ldots u_n$, we get back to Case 1. Thus, $u' \sim v_{g-2}$.

We claim that $u' \in V_3$. For, suppose there exists $u'' \in N(u') \setminus \{u_1, v_{g-2}, w_1\}$. Then $u'' \neq u$ since $og(G) \geq 7$, $u'' \notin C$ by the minimality of C, and $u'' \notin P$ by the choice of P and the fact $u' \nsim u_3$. Note that $u'' \sim \{v_1, v_{g-1}\}$ to avoid $(u_1v_1, u_1v_{g-1}, u_1u_2u_3, u_1u'u'')$. If $u'' \sim v_1$ then $u'' \sim v_{g-1}$ to avoid $(v_1u'', v_1w_1, v_1v_2v_3, v_1v_gv_{g-1})$; and so $(v_{g-1}u'', v_{g-1}v_g, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}u_1u_2)$

is a fork, a contradiction. So $u'' \nsim v_1$ and $u'' \sim v_{g-1}$. Then $u'' \sim v_{g-3}$ to avoid $(v_{g-1}u'', v_{g-1}v_g, v_{g-1}v_{g-2}v_{g-3}, v_{g-1}u_1u_2)$. Now, replacing P with $v_{g-1}u_1u_2 \cdots u_n$, we get back to Case 1.

Thus $v_{g-1} \in V_3$ as otherwise $v_2 \dots v_{g-2}u'u_1u_2v_2$ would contradict the choice of C. Hence, $v_{g-2} \notin V_3$ by Corollary 4.3. Let $v \in N(v_{g-2}) \setminus \{v_{g-1}, v_{g-3}, u'\}$. Then $v \notin C \cup (P - u_1)$ by the choices of C and P, $v \notin \{u_1, w_1\}$ since $og(G) \geq 7$, $v \sim \{v_{g-4}, v_g\}$ to avoid $(v_{g-2}v, v_{g-2}u', v_{g-2}v_{g-1}v_g, v_{g-2}v_{g-3}v_{g-4})$, and $v \sim \{v_g, w_1\}$ to avoid $(v_{g-2}v, v_{g-2}v_{g-3}, v_{g-2}u'w_1, v_{g-2}v_{g-1}v_g)$. Hence, $v \sim v_g$ (for any choice of v); otherwise, $v \sim v_{g-4}$ and $v \sim w_1$, contradicting the minimality of C. Thus, $v_{g-2} \in V_4$ as $v_g \in V_3$, and $v \sim \{u_1, w_1\}$ to avoid $(v_1u_1, v_1w_1, v_1v_2v_3, v_1v_gv)$. Note that $v \nsim u_1$ since $N(v_{g-1}) \not\subseteq N(v)$ (by Lemma 3.2). So we have $v \sim w_1$.

If $v \in V_3$ then v_{g-3}, v_{g-2} contradicts Lemma 3.4. So $v \notin V_3$, and let $v' \in N(v) \setminus \{v_{g-2}, v_g, w_1\}$. Then $v' \notin C \cup (P - \{u_1, u_2\})$ by the choices of C and P, $v' \neq u'$ as $og(G) \geq 7$, and $v' \notin \{u_1, u_2\}$ as $v \nsim \{u_1, u_2\}$. So $v' \sim v_{g-3}$ to avoid $(v_{g-2}u', v_{g-2}v_{g-1}, v_{g-2}vv', v_{g-2}v_{g-3}v_{g-4})$. Now $v \in V_4$; for otherwise, let $v'' \in N(v) \setminus \{v_{g-2}, v_g, v', w_1\}$, then $(vv', vv'', vv_gv_{g-1}, vw_1u')$ is a fork, a contradiction.

Suppose there exists $w \in N(w_1) \setminus \{u', v, v_1\}$. Then $w \notin C \cup (P - \{u_1, u_2\})$ by the choices of C and P, $w \neq u_1$ since $og(G) \geq 7$, and $w \neq u_2$ as $w_1 \nsim u_2$. So $w \sim u_1$ to avoid $(v_1u_1, v_1v_g, v_1v_2v_3, v_1w_1w)$, and $w \sim u_3$ to avoid $(u_1w, u_1u', u_1v_1v_g, u_1u_2u_3)$. Now, replacing P with $v_1w_1wu_3 \cdots u_n$, we get back to Case 1.

Thus $w_1 \in V_3$. By the choice of G, $G' := G - \{u', v, v_{g-1}, v_{g-2}, v_g, w_1\}$ is 3-colorable. Let c' be a 3-coloring of G'. Let $c(z) = c'(z)$ for all $z \in V(G')$; $c(v_{g-2}) = c'(v')$; and greedily color $\{u', v, v_{g-1}\}$ (with one single color for all three), v_g, w_1 in order, we get a 3-coloring of G, a contradiction. □

Finally, we show that $\chi(G) \leq 3$, contradicting the assumption that G is a counterexample. Let $C = v_1 \dots v_g v_1$ be a shortest cycle in G such that the assertion of Lemma 6.3 holds.

By Lemma 2.2, we see that $N_2(C) \neq \emptyset$. Let T denote the set of vertices u in $G - C$ such that if P is a path in G from u to some v_i with $V(P) \cap V(C) = \{v_i\}$ then $v_i \in V_3$, and let H

denote the subgraph of G obtained by taking the union of all paths P from T to C such that $|V(P) \cap V(C)| = 1$. Define $S = V(H) \cap V(C)$; so $S \subseteq V_3$. Let $K = G - (H - S)$.

By Lemma 4.5, $S \neq V(C)$. So by the minimality of $G, \chi(H) \leq 3$. Let c_H be a 3-coloring of H, which induces a 3-coloring c_S on $G[S]$. We now use Lemma 2.1 to extend c_S to a coloring c_K of K, and so we need to verify the conditions of that lemma.

By Lemma 6.3, we see that if $u \in N_2(C) \cap V(K)$ then there is a path uu_1v_i in K such that $v_i \notin V_3$. Let $w_1 \in N(v_i) \setminus \{u_1, v_{i-1}, v_{i+1}\}$. Then $\{u_1, w_1\} \sim \{v_{i-2}, v_{i+2}\}$ to avoid $(v_iu_1, v_iw_1, v_iv_{i-1}v_{i-2}, v_iv_{i+1}v_{i+2})$. By symmetry and by the minimality of C, assume $w_1 \nsim v_{i+2}$. If $u_1 \sim \{v_{i-2}, v_{i+2}\}$ then u is associated with v_{i-1} or v_{i+1}. On the other hand, if $u \nsim \{v_{i-2}, v_{i+2}\}$ then $u \sim w_1$ to avoid $(v_iw_1, v_iv_{i-1}, v_iv_{i+1}v_{i+2}, v_iu_1u)$, and $w_1 \sim v_{i-2}$ to avoid $(v_iw_1, v_iv_{i+1}, v_iu_1u, v_iv_{i-1}v_{i-2})$; so u is associated with v_{i-1}. So we have shown that in K, every vertex in $N_2(C) \cap V(K)$ is associated with a vertex of C.

Next we show that (i), (ii) and (iii) of Lemma 2.1 holds.

Suppose $w \in N_1(v_i)$ and $w \sim x_1, x_2 \in N_2(C)$, such that x_1 is associated with one of $\{v_{i-3}, v_{i-1}\}$ and x_2 is associated with one of $\{v_{i+3}, v_{i+1}\}$. We show that $v_{i-1}, v_{i+1} \notin S$. By the minimality of C and by symmetry we may assume x_1 is associated with v_{i-1}. Let $x_1u_1v_i$ be a path. If $v_{i+1} \in S$ then let $v \in N(v_{i+1}) \setminus \{v_i, v_{i+2}\}$; now $(v_iu_1, v_iv_{i-1}, v_iwx_2, v_iv_{i+1}v)$ is a fork, a contradiction. If $v_{i-1} \in S$ then let $v \in N(v_{i-1}) \setminus \{v_i, v_{i-2}\}$; now $(v_iu_1, v_iv_{i+1}, v_iwx_2, v_iv_{i-1}v)$ is a fork, a contradiction. So we have $v_{i-1}, v_{i+1} \notin S$.

Now suppose $X_{i,1} \neq \emptyset$ and $v_j \in \{v_{i-1}, v_{i+1}\} \cap S$. We show that v_j cannot be associated with any vertex in $N_2(C)$. For, assume without loss of generality that v_{i+1} is associated with a vertex $u \in N_2(C)$, and let uu_1v_i be a path. Let $w \in X_{i,1}$. If $v_{i+1} \in S$ then let $v \in N(v_{i+1}) \setminus \{v_i, v_{i+2}\}$; now $(v_iw, v_iu_1, v_iv_{i-1}v_{i-2}, v_iv_{i+1}v)$ is a fork. So $v_{i+1} \notin S$.

Finally, assume that v_i is associated with some vertex $u \in N_2(C)$ and $u \sim w \in X_{i+1,1} \cup X^+_{i+1,2} \cup X^+_{i+1,3}$ (respectively, $X_{i-1,1} \cup X^-_{i-1,2} \cup X^-_{i-1,3}$). We claim that $v_i \notin S$ or $v_{i+3} \notin S$ (respectively, $v_{i-3} \notin S$). Otherwise, by symmetry assume that $u \sim w \in X_{i+1,1} \cup X^+_{i+1,2} \cup X^+_{i+1,3}$, and $v_i, v_{i+3} \in S$. Let $v \in V(H) \setminus V(K)$ such that $v \sim v_i$. Let uu_1v_{i+1} be a path. Then

$(v_{i+1}u_1, v_{i+1}w, v_{i+1}v_{i+2}v_{i+3}, v_{i+1}v_iv)$ is a fork, a contradiction.

Hence, by Lemma 2.1, c_S can be extended to a 3-coloring of c_K of K. Let $c(v) = c_K(v)$ if $v \in V(K)$ and $c(v) = c_H(v)$ if $v \in V(H)$. We see that c is a 3-coloring of G, a contradiction. This completes the proof of Theorem 1.9.

CHAPTER VII

MAXIMUM DEGREE IS 4

In the previous chapter, we finished the proof of our main result, i.e., Conjecture 1.8 holds when the odd girth of G is greater than 5. In this chapter we provide further evidence to Conjecture 1.8 by presenting a proof that Conjecture 1.8 holds when $\Delta(G) \leq 4$.

Theorem 7.1. *Any fork-free and triangle-free graph with maximum degree at most 4 is 3-colorable.*

Let G be a graph. For any $u \in V(G)$, let $S(u) = \{v \in N_2(u) : |N_1(v) \cap N_1(u)| = 1\}$.

Suppose Theorem 7.1 fails to hold. Recall from Brooks' Theorem that any graph G with $\Delta(G) \leq 4$ has $\chi(G) \leq 4$. So there exists a graph G such that

(1) G is fork-free and triangle-free,

(2) $\Delta(G) \leq 4$ and $\chi(G) = 4$, and

(3) subject (1) and (2), $|G|$ is minimum.

7.1 Properties of minimum counter example

By (3), $G - v$ is 3-colorable for any $v \in V(G)$. Let c be a 3-coloring of $G - v$. If $d(v) \leq 2$, there is a color, say α, not used by $c(u)$ for any $u \in N_1(v)$. Now assign α to v, we produce a 3-coloring of G, contradicting (3). Thus we have

Lemma 7.2. *The minimum degree of G is at least 3.*

Now suppose there exist $u, v \in V(G)$ such that $N_1(v) \subseteq N_1(u)$. By (3), $G - v$ admits a 3-coloring. Assigning v the color of u, we get a 3-coloring of G, contradicting (3). So we have

Lemma 7.3. *For any distinct* $u, v \in V(G)$, $N_1(u) \nsubseteq N_1(v)$.

Lemma 7.4. *G is 3-connected.*

Proof. First, G must be 2-connected. For, if G is not 2-connected, then there are subgraphs G_1, G_2 of G such that $|G_1 \cap G_2| = 1$, $G = G_1 \cup G_2$, and $|G_i| \geq 2$ for $i = 1, 2$. Now G_i are induced subgraphs of G, and hence, are triangle-free and fork-free. Thus, by (3), each G_i is 3-colorable. Let u denote the vertex in $G_1 \cap G_2$. We may choose a 3-cloring c_i of G_i (for each i) such that $c_1(u) = c_2(u)$. For each $x \in V(G)$, let $c(x) = c_i(x)$ whenever $x \in V(G_i)$. Then c is a 3-coloring of G, contradicting (3).

Now suppose G is not 3-connected. Then G has a 2-cut, say $\{u, v\}$. Let G_1, G_2 be subgraphs of G such that $G = G_1 \cap G_2$, $G_1 \cap G_2 = \{u, v\}$, and $E(G_1 \cap G_2) = \emptyset$.

We claim that $uv \notin E(G)$. For, suppose $uv \in E(G)$. Then $G_i' := G_i + uv$, $i = 1, 2$, are induced subgraphs of G. By (3), each G_i' is 3-colorable. Let c_i be a 3-edge-coloring of G_i'. Since $uv \in E(G_i')$, $c_i(u) \neq c_i(v)$. Thus we may choose c_i, $i = 1, 2$, so that $c_1(u) = c_2(u)$ and $c_1(v) = c_2(v)$. For each $x \in V(G)$, let $c(x) = c_i(x)$ whenever $x \in V(G_i)$. Then c is a 3-coloring of G, a contradicting (3).

Suppose u has at least two neighbors in each G_i, say u_i, v_i, for $i = 1, 2$. Since $N(u_i) \nsubseteq N(v_i)$, there is a vertex $w_i \neq v$ such that $w_i \in N(u_i) - N(v_i)$ or $w_i \in N(v_i) - N(u_i)$. We may assume that the notation is chosen so that $w_i \in N(u_i) - N(v_i)$ for $i = 1, 2$. Then $(uv_1, uv_2, uu_1w_1, uu_2w_2)$ is a fork in G, a contradicting(1).

So u has only one neighbor in G_i for some $i \in \{1, 2\}$. Similarly, v has only one neighbor in G_j for some $j \in \{1, 2\}$. This implies that G has a 2-edge-cut $\{x'x'', y'y''\}$, with $u, v \in \{x', x'', y', y''\}$. Since G is 2-connected, $G - \{x'x'', y'y''\}$ has exactly two components, say G' and G''. By renaming vertices if necessary, we may assume $x', y' \in G'$ and $x'', y'' \in G''$.

Since G', G'' are induced subgraphs of G, G' and G'' are 3-colorable by (3). So let c' be a 3-coloring of G' and c'' a 3-coloring of G''. By a simple case analysis, we can choose c' and c'' so that $c'(x') \neq c''(x'')$ and $c'(y') \neq c''(y'')$. We obtain a 3-coloring c of G by setting $c(x) = c'(x)$ if $x \in V(G')$ and $c(x) = c''(x)$ if $x \in V(G'')$. This contradicts (3), and completes the proof of the lemma. □

Lemma 7.5. *Let u be a 4-vertex in G, and let v_1 and v_2 be two vertices in $S(u)$ without common neighbor in $N(u)$. Then $v_1 \sim v_2$. Furthermore, the vertices of $S(u)$ have at most two neighbors in $N(u)$.*

Proof. Suppose that $N(u) = \{u_1, u_2, u_3, u_4\}$, and suppose by symmetry that $v_i \sim u_i$ for $i = 1, 2$. If $v_1 \nsim v_2$, then there exists a fork $(uu_1v_1, uu_2v_2, uu_3, uu_4)$. Therefore, $v_1 \sim v_2$. If the vertices of $S(u)$ have three neighbors in $N(u)$, then the subgraph induced by $S(u)$ has a triangle by the former conclusion. □

Let u be a 4-vertex of G. Throughout the paper, we always suppose that $N(u) = \{u_1, u_2, u_3, u_4\}$. Let $S_1 = \{u_2, u_3, u_4\}$.

If $|S(u)| \geq 6$, then by Lemma 7.5, the vertices in $S(u)$ have exact two neighbors in $N(u)$ that implies u is a cut vertex of G, a contradiction to Lemma 7.4.

Suppose that $|S(u)| = 5$ and let $S(u) = \{v_1, v_2, v_3, v_4, v_5\}$. Without loss of generality, we suppose that $v_1/v_2/v_3 \sim u_1$ and $v_4/v_5 \sim u_2$. If $N(u_2) \cap N(u_3) = \{u\} = N(u_2) \cap N(u_4)$, then $N(u_3) = N(u_4)$. Otherwise, we may suppose that u_2 and u_3 have a common neighbor, say v_6, other than u, and then $N(u_4) \subseteq N(u_3)$. Both contradict Lemma 7.3.

Therefore, we suppose that $0 \leq |S(x)| \leq 4$ for each vertex x of G.

7.2 $|S(u)| = 0$

The objective of this section is to show that for any $u \in V_4$, we have $S(u) \neq \emptyset$.

Suppose there exists some vertex $u \in V_4$ such that $S(u) = \emptyset$. Let $N_1(u) = \{u_1, u_2, u_3, u_4\}$.

Since $S(u) = \emptyset$, each vertex in $N_2(u)$ has at least 2 neighbors in $N_1(u)$. Hence, since $\Delta(G) \leq 4$, $2|N_2(u)| \leq 3|N_1(u)|$, which implies $|N_2(u)| \leq 6$.

On the other hand, we claim that $|N_2(u)| \geq 4$. To see this, assume $|N_2(u)| \leq 3$. Then $d_G(u_i) = 3$ for $1 \leq i \leq 4$; as otherwise, by Lemma 7.2, $d(u_i) = 4$ for some $i \in \{1, 2, 3, 4\}$

and, hence, $N_2(u) \subseteq N_1(u_i)$, which implies $N_1(u_j) \subseteq N_1(u_i)$ for all $j \neq i$, contradicting Lemma 7.3. So each u_i has precisely two neighbors in $N_2(u)$. Now there are 6 different pairs $\{u_i, u_j\}$ ($i \neq j$), and there are at most three pairs from $N_2(u)$. By the pigeon-hole principle, there must be a pair $\{u_i, u_j\}$ such that u_i, u_j have the same two neighbors in $N_2(u)$. This implies $N_1(u_i) = N_1(u_j)$, contradicting Lemma 7.3.

So we consider the three cases: $|N_2(u)| = 4$, $|N_2(u)| = 5$, and $|N_2(u)| = 6$. In each case, we will derive a contradiction.

Case 1: $|N_2(u)| = 4$.

We consider three subcases according to $|N_1(u) \cap V_4|$. In each case we force a structure surrounding u (see Figure 4), and derive a contradiction using that structure. Let $N_2(u) = \{v_1, v_2, v_3, v_4\}$.

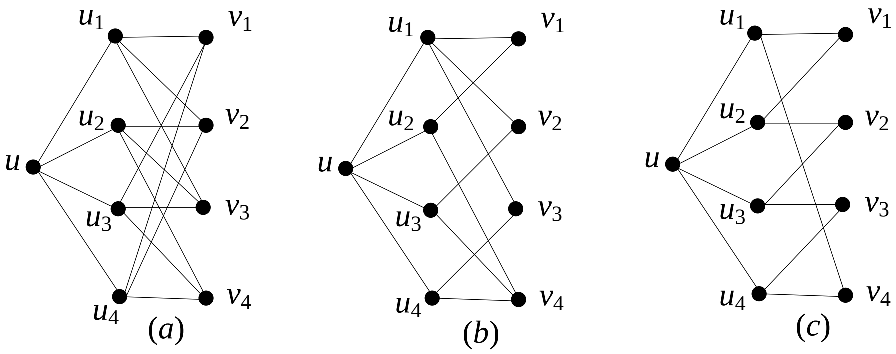

Figure 4: $|N_2(u)| = 4$.

Subcase 1.1. $|N_1(u) \cap V_4| \geq 2$.

Without loss of generality, let $u_1, u_2 \in V_4$. By Lemma 7.3, $N_1(u_1) \not\subseteq N_1(u_2) \not\subseteq N_1(u_1)$; so by symmetry, we may assume that $v_1, v_2, v_3 \in N_1(u_1)$ and $v_2, v_3, v_4 \in N_1(u_2)$. Also by Lemma 7.3, $N_1(u_j) \not\subseteq N_1(u_i)$ for any $i \in \{1, 2\}$ and $j \in \{3, 4\}$; so $v_1, v_4 \in N_1(u_3) \cap N_1(u_4)$. Again by Lemma 7.3, $N_1(u_3) \not\subseteq N_1(u_4) \not\subseteq N_1(u_3)$; so by symmetry we may assume $v_2 \in N_1(u_3)$ and $v_3 \in N_1(u_4)$. See Figure 4(a).

Since G is triangle-free, $N_2(u)$ is independent in G. Let $G' := G - (N_1(u) \cup N_2(u) \cup \{u\}$. If $G' = \emptyset$ then $\chi(G) = 2$, contradicting (2). So $G' \neq \emptyset$. Hence by the choice of G, G' admits a 3-coloring, say c'. Let α, β, γ denote the colors used by c. We now have a contradiction to (2) by deriving a 3-coloring c of G as follows: $c(x) = c'(x)$ for all $x \in V(G')$, $c(v_i) \in \{\alpha, \beta\}$ (for $i = 1, 2, 3, 4$) and is different from its neighbor in G' (at most one such neighbor exists), $c(u_i) = \gamma$ for $i = 1, 2, 3, 4$, and $c(u) = \alpha$.

Subcase 1.2. $|N_1(u) \cap V_4| = 1$.

Without loss of generality, let $u_1 \in V_4$, and assume that $v_1, v_2, v_3 \in N_1(u_1)$. By Lemma 7.3, we have $N(u_i) \not\subseteq N(u_1)$ for $i = 2, 3, 4$; so $v_4 \in N(u_i)$ for $i = 2, 3, 4$. Recall that each $i \in \{2, 3, 4\}$, since u_i has exactly two neighbors in $N_2(u)$. So by Lemma 7.3 and symmetry, we may assume that $u_2v_1, u_3v_2, u_4v_3 \in E(G)$. See Figure 4($b$).

Since G is triangle-free, $N_2(u)$ is independent in G. Let $G' := G/\{u_1, u_2, u_3, u_4\} - \{u, v_4\}$, and let u' denote the vertex resulted from identification. Then G' is triangle-free and fork-free (since u', v_1, v_2, v_3 all have degree at most 3 in G'). So by (3), G' admits a 3-coloring c', using colors α, β, γ. We now have a contradiction by deriving a 3-coloring c of G as follows: $c(x) = c'(x)$ for all $x \in V(G') - \{u'\}$, $c(v_4)$ is a color not used by its (at most two) neighbors contained in G', $c(u_i) = c'(u')$ for $i = 1, 2, 3, 4$, and $c(u) = c'(v_1)$.

Subcase 1.3. $|N_1(u) \cap V_4| = 0$.

Then the subgraph of G, denoted by H, induced by the edges between $N_1(u)$ and $N_2(u)$ is a 2-regular bipartite graph. If H is not connected, then it consists of two disjoint cycles of length 4; but one easily sees that Lemma 7.3 is violated. So H is connected, and hence H is a cycle of length 8. By renaming the vertices if necessary, we may assume that $H = u_1v_1u_2v_2u_3v_3u_4v_4u_1$. See Figure 4($c$).

Suppose $v_2 \sim v_4$ and $v_1 \sim v_3$. Then $|N_2(u) \cap V_4| \geq 1$, for otherwise $V(G) = \{u\} \cup N_1(u) \cup N_2(u)$ and hence $\chi(G) = 3$, contradicting (2). By symmetry, we may assume that $v_1 \in V_4$ and let $w_1 \in N_1(v_1) - (\{u\} \cup N_1(u) \cup N_2(u))$. Then $w_1 \sim v_2$, to forbid the fork $(v_1u_2v_2, v_1v_3u_4, v_1u_1, v_1w_1)$. Since G is triangle-free $w_1 \nsim v_3$ and $w_1 \nsim v_4$. But then $(v_2u_3v_3, v_2v_4u_1, v_2u_2, v_2w_1)$ is a fork in G, a contradiction.

So by symmetry, we may assume $v_2 \nsim v_4$. Let G' be obtained from $G - u$ by identifying

u_1 with u_2 as u_{12}, and u_3 with u_4 as u_{34}. Since G is triangle-free and $v_2 \nsim v_4$, G' is triangle-free. If G' is also fork-free then by (3), G' admits a 3-coloring, say c'. We now obtain a 3-coloring c of G by setting: $c(x) = c'(x)$ for $x \in V(G) - (N_1(u) \cup \{u\})$, $c(u_1) = c(u_2) = c'(u_{12})$, $c(u_3) = c(u_4) = c'(u_{34})$ and $c(u) = c'(v_2)$ (note that $c'(v_2) \notin \{c'(u_{12}), c'(u_{34})\}$). But this contradicts (3).

So G' has fork, say F', and let x be the center of F'. Note that $x \notin \{u_{12}, u_{34}, v_1, v_3\}$ as all these vertices have degree 3 in G'. So $x \in \{v_2, v_4\}$ or $x \in V(G') - (N_2(u) \cup \{u_{12}, u_{34}\})$.

Suppose $x \in \{v_2, v_4\}$. By the symmetry between v_2 and v_4, we may assume $x = v_2$. Note that $v_2u_{34}v_4$ cannot be an arm of F', as $u_{12} \sim v_4$. Similarly, $v_2u_{12}v_4$ cannot be an arm of F'. We now derive a contradiction by producing a fork in G. If v_2u_{12} and v_2u_{34} are arms of F' then replacing them with v_2u_2, v_2u_3 we obtain a fork in G. If v_2u_{12} and $v_2u_{34}v_3$ are arms of F', then replacing them by v_2u_2 and $v_2u_3v_3$ we produce fork in G. If v_2u_{34} and $v_2u_{12}v_1$ are arms of F' then replacing them by v_2u_3 and $v_2u_2v_1$ we obtain a fork in G. If $v_2u_{12}v_1$ and $v_2u_{34}v_3$ are arms of F' then replacing them by $v_2u_2v_1$ and $v_2u_3v_3$ we obtain a fork in G.

So $x \in V(G') - (N_2(u) \cup \{u_{12}, u_{34}\})$. If $u_{12}, u_{34} \notin F'$ then F' is a fork in G, a contradiction. So we may assume by symmetry that $u_{12} \in F'$. If $u_{34} \in F'$ then xv_1u_{12}, xv_3u_{34} must be arms of F', and $v_2, v_4 \notin F'$; in this case, replacing xv_1u_{12}, xv_3u_{34} by xv_1u_1, xv_3u_3 we produce from F' a fork in G, a contradiction. So $u_{34} \notin F'$. Now the arm of F' containing u_{12} is xv_iu_{12} for some $i \in \{1, 2\}$, and $v_{3-i}v_4 \notin F'$. Then replacing xv_iu_{12} by xv_iu_i we obtain from F' a fork in G, a contradiction.

Case 2. $|N_2(u)| = 5$.

Let $N_2(u) = \{v_1, v_2, v_3, v_4, v_5\}$. Since $|S(u)| = 0$, we have $\sum_{i=1}^{4}(d(u_i) - 1) \geq 10$; so $|N_1(u) \cap V_4| \geq 2$.

Subcase 2.1. $|N_1(u) \cap V_4| = 2$.

Let $u_1, u_2 \in N_1(u) \cap V_4$; then $u_3, u_4 \in V_3$. Without loss of generality, assume $N_1(u_1) = \{u, v_1, v_2, v_3\}$. Since $|N_2(u)| = 5$, we have $|N_1(u_2) \cap N_1(u_1)| \geq 2$; and by Lemma 7.3, $|N_1(u_2) \cap N_1(u_1)| \geq 3$. If $|N_1(u_2) \cap N_1(u_1)| = 3$, we may assume by symmetry that

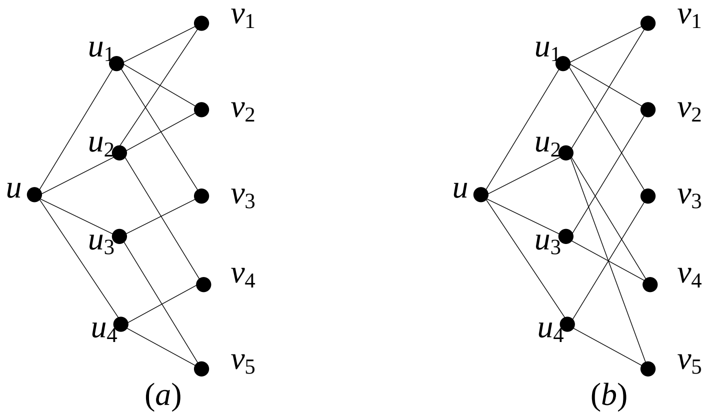

Figure 5: Two graphs.

$N_1(u_2) = \{u, v_1, v_2, v_4\}$, and then we must have $v_5 \sim u_3$ and $v_5 \sim u_4$; and by symmetry we may assume $v_3 \sim u_3$ and $v_4 \sim u_4$, see Figure 6(a). When $|N_1(u_2) \cap N_1(u_1)| = 2$, we let $N_1(u_2) = \{u, v_1, v_4, v_5\}$, and by Lemma 7.3 and by symmetry, we may assume $v_2 \sim u_3$, $v_3 \sim u_4$, $v_4 \sim u_3$ and $v_5 \sim u_4$ (see Figure 6(b)).

First, we deal with graph in Figure 6(a). We have two cases: $v_3 \nsim v_4$, and $v_3 \sim v_4$.

Suppose $v_3 \nsim v_4$. Let G' be obtained from $G - u$ by identifying (u_1 with u_2) as u_{12}, and identifying (u_3 with u_4) as u_{34}. Clearly G' is triangle-free. If G' is also fork-free, then by (3), G' admits a 3-coloring, say c', which gives rise to a 3-coloring c of G, a contradiction. So let F' be a fork in G'. We now derive a contradiction by constructing from F' a fork in G. This is easily done if the center of F' is in $G' - \{u_{12}, u_{34}, v_1, v_2, v_3, v_4, v_5\}$. Note that u_{34}, v_1, v_2, v_5 each have degree at most 3 in G'; so none can serve as the center of F'. If v_3 is the center of F', then for some $i \in \{1, 2\}$, $v_3u_{12}v_i$ and $v_3u_{34}v_5$ are the long arms of F'; and replacing the long arms in F' with $v_3u_1v_i$ and $v_3u_3v_5$ we obtain a fork in G. If v_4 is the center of F', then for some $i \in \{1, 2\}$, $v_4u_{12}v_i$ and $v_4u_{34}v_5$ are the long arms of F'; and replacing the long arms in F' with $v_4u_1v_i$ and $v_4u_4v_5$ we obtain a fork in G. So u_{12} is the center of F'. If $u_{12}u_4$ is a short arm of F' then replacing u_{12}, v_4 in F' by u_1, u_4 respectively we obtain a fork in G. So F' has a long arm $u_{12}v_4w_4$. Then $(u_2v_4w_4, u_2uu_3, u_2v_1, u_2v_2)$ is a fork in G.

Now assume $v_3 \sim v_4$. Suppose $v_1 \sim v_5$. Then $v_2 \sim v_5$ to avoid the fork $(u_1v_1v_5, u_1v_3v_4, u_1u, u_1v_2)$. By Lemma 7.4, $v_1 \in V_4$ or $v_3 \in V_4$ (to avoid the 2-cut $\{v_2, v_4\}$). If $v_1 \in V_4$ let $w_1 \in N_1(v_1) - \{u_1, u_2, v_5\}$; then $(v_1v_5u_3, v_1u_2v_4, v_1u_1, v_1w_1)$ is a fork in G. If $v_3 \in V_4$ then let $w_3 \in N_1(v_3) - \{u_1, u_3, v_4\}$; then $(v_3u_3v_5, v_3v_4u_2, v_3u_1, v_3w_3)$ is a fork in G, a contradiction.

Thus $v_1 \nsim v_5$. By symmetry between v_1 and v_2, we also have $v_2 \nsim v_5$. So by Lemma 7.3, there exists $w_1 \in N_1(v_1) - (\{u\} \cup N_1(u) \cup N_2(u))$ such that $w_1 \nsim u_2$. Since G is triangle-free, $w_1 \nsim v_3$ or $w_1 \nsim v_4$; hence $(u_1v_1w_1, u_1uu_4, u_1v_2, u_1v_3)$ or $(u_2v_1w_1, u_2uu_3, u_2v_2, u_2v_4)$ is a fork in G, a contradiction.

Next, we deal with the situation depicted in Figure 6(b).

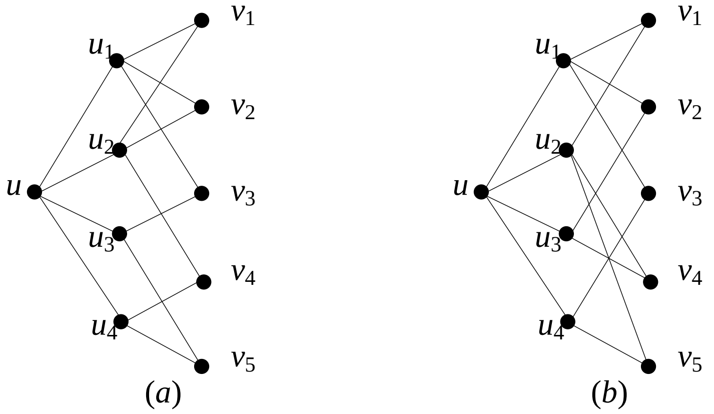

Figure 6: Two graphs.

Notice that in this graph, v_2, v_3, v_4, v_5 are symmetric, also, $(v_2, v_3), (v_4, v_5)$ are symmetric. If we can show that at least one of v_2, v_3 has degree 3, then by symmetry, at least one of v_4, v_5 has degree 3. Then, we can identify u_1, u_2, u_3, u_4 as u', delete u and all vertices which have degree 3 in G, the new graph is fork free since u' and the remaining v_i has degree at most 3; if we can also show that the new graph is triangle free, then, G is reducible, contradiction.

Claim 1: For any vertex w, w cannot be adjacent to four of $(v_1, v_2, v_3, v_4, v_5)$. If w is adjacent to three of $(v_1, v_2, v_3, v_4, v_5)$, there are only 2 possible situations: $w \sim v_1, v_2, v_4$, $w \sim v_1, v_3, v_5$.

By our assumption, $w \in V_3$ or V_4. If $N_1(w) \subset N_2(u)$, we can identify u_1, u_2, u_3, u_4 as u', delete $u, w, N_1(w), u'$ in order, the new graph is fork free, triangle free, i.e. G is reducible, contradiction. That means w cannot be adjacent to four of $(v_1, v_2, v_3, v_4, v_5)$; and if w is adjacent to three of $(v_1, v_2, v_3, v_4, v_5)$, w must have degree 4, and the fourth neighbor w' is not in $N_2(u)$, we can check that there are only 3 possible such situations: $w \sim v_1, v_2, v_4$, $w \sim v_1, v_3, v_5$, $w \sim v_2, v_3, v_5$. Otherwise, for example, if w is adjacent to v_1, v_2, v_3, then there is a fork in G: $(wv_1u_2, wv_3u_4, wv_2, ww')$, contradiction.

Claim 2: If $v_2 \in V_4$, then $v_2 \nsim v_i$, $i \neq 2$. By symmetry, if $v_3/v_4/v_5 \in V_4$, then $v_3/v_4/v_5 \nsim v_i$, $i \neq 3/4/5$.

Here, $v_2 \nsim v_i$, $i \neq 2, 5$ since G is triangle free. Now suppose $v_2 \in V_4$, and $v_2 \sim v_5$, denote the fourth neighbor of v_2 by w_2. Then $w_2 \sim v_4$ to forbid the fork $(v_2v_5u_4, v_2u_3v_4, v_2u_1, v_2w_2)$; also $w_2 \sim v_1$ to forbid the fork $(v_2v_5u_4, v_2u_1v_1, v_2u_3, v_2w_2)$; also $w_2 \sim v_3$ to forbid the fork $(v_2v_5u_2, v_2u_1v_3, v_2u_3, v_2w_2)$. By *Claim 1*, it's impossible.

Claim 3: At least one of v_2, v_3 has degree 3.

Suppose not, both v_2, v_3 have degree 4, by *Claim 2*, $v_2 \nsim v_i, i \neq 2$, $v_3 \nsim v_i, i \neq 3$. Then there are 3 possible cases:

Case 1: v_2, v_3 have the common neighborhoods w_1, w_2. Then $w_1 \nsim v_1, v_4$ by *Claim 1*, by symmetry, $w_2 \nsim v_1, v_4$. Then there is a fork $(v_2u_1v_1, v_2u_3v_4, v_2w_1, v_2w_2)$ in G, contradiction.

Case 2: v_2, v_3 have one common neighbor w_1, and the fourth neighbor of v_2 is w_2, the fourth neighbor of v_3 is w_3. Sine $w_1 \nsim v_1, v_4$ by *Claim 1*, $w_2 \sim v_1$ or v_4 to forbid the fork $(v_2u_1v_1, v_2u_3v_4, v_2w_2, v_2w_2)$.

First, suppose $w_2 \sim v_1$, now let's prove that in this case, $v_1 \in V_3$. Suppose not, if the fourth neighbor is w, $w \sim v_4, v_5$ to forbid the fork $(v_1u_2v_4/v_5, v_1u_1v_3, v_1w_2, v_1w)$, but by *Claim 1*, it's impossible. Also, by *Claim 1*, $v_1 \nsim w_1$, thus, if $v_1 \in V_4$, $v_1 \sim w_3$, now, we can identify v_1, v_2, v_3 as v', delete u, u_1, u_3, u_4 in order, the only possible fork in new graph is the one centered on v', however, we can check that any fork centered on v' can be revered to a fork in G centered on $v_1/v_2/v_3$. For example, if the new fork is $(v'u_2v_4, v'w_1*, v'w_2, v'w_3)$, the there is a fork $(v_2u_3v_4, v_2w_1*, v_2u_1, v_2w_2)$, contradiction.Thus, we have shown that $v_1 \in V_3$. Next, let's show that $w_1 \in V_3$. Suppose not, $w_1 \nsim v_4, v_5$ by *Claim 1*, then there is a fork centered on w_1 with long arms $(w_1v_2u_3, w_1v_3u_4)$, contradiction. Also, $w_2 \in V_3$, to see that, we only need to prove $w_2 \nsim v_4$, then play the same trick above. Suppose $w_2 \sim v_4$, then at most v_4 has another neighbor that is not w_1, w_3. We can identify v_1, v_2, v_3, v_4 as v', delete $u, u_1, u_2, u_3, u_4, w_1, w_2$ in order, now, v' has at most 2 neighbors, we can also delete v' to make sure that there is no triangle. That means G is reducible, contradiction. Now, we have shown that $w_1, w_2, v_1 \in V_3$, then we can identify v_1, v_2, v_3 as v', delete $u, u_1, u_3, u_4, w_1, w_2$ in order, the new graph is fork free, triangle free, contradiction.

Next, suppose that $w_2 \sim v_4$. We have shown that $w_2 \nsim v_1$, by symmetry, $w_3 \nsim v_1$, thus $w_2 \sim w_3$ to forbid the fork $(u_1v_2w_2, u_1v_3w_3, u_1v_1, u_1u)$. Also, $w_3 \sim v_5$ to forbid the fork $(v_3u_1v_1, v_3u_4v_5, v_3w_1, v_3w_3)$, since $w_1 \nsim v_1, v_5$ by *Claim 1*, $w_3 \nsim v_1$ by symmetry. Next, let's show that $w_1, w_2, w_3 \in V_3$. Suppose $w_2 \in V_4$, $w_2 \nsim w_1, v_5$ since triangle free, then there is a fork centered on w_2 with long arms $(w_2v_2u_1, w_2v_4u_2)$. Thus, $w_2 \in V_3$, by symmetry, $w_3 \in V_3$. Now look at w_1, $w_1 \nsim v_1, v_4, v_5$ by *Claim 1*, then if $w_1 \in V_4$, there is a fork centered on w_1 with long arms $w_1v_2u_3, w_1v_3u_4$. Thus $w_1 \in V_3$. Now we can identify v_2, v_3, v_4, v_5 as v', delete $u, u_1, u_2, u_3, u_4, w_1, w_2, w_3$ in order, then v' has at most 2 neighbors (the possible fourth neighbor of v_2, v_5), then we can also delete v' to destroy the possible triangle. Now the new graph is fork free, triangle free, contradiction.

Case 3: v_2, v_3 have no common neighbor, the neighbors of v_2 are w_2, w_2', and the neighbors of v_3 are w_3, w_3'.

Then there are several subcases depend on the connection between v_1 and w_2, w_2', w_3, w_3', first, v_1 must be adjacent to at least one of w_2, w_2', w_3, w_3', otherwise, denote the third neighbor of v_1 by w_1, then w_1 must be adjacent to all of w_2, w_2', w_3, w_3' to forbid the fork centered on u_1, contradiction to the assumption that $\Delta(G) \leq 4$.

Subcase 1: v_1 is adjacent to only one of w_2, w_2', w_3, w_3', by symmetry, we can only consider the case that $v_1 \sim w_2$, then $w_2' \sim w_3, w_3'$ to forbid the fork centered on u_1. Also,

$v_1 \in V_3$, otherwise, denote the fourth neighbor of v_1 by w_1, $w_1 \sim w_2', w_3, w_3'$ to forbid the fork centered on u_1, then there is a fork $v_1u_1v_3, v_1w_1w_2', v_1u_2, v_1w_2$, contradiction.

Next, look at the fork $(v_3u_4v_5, v_3u_1v_1, v_3w_3, v_3w_3')$, to forbid this fork, $v_5 \sim w_3$ or w_3', by symmetry, we can only consider the case that $v_5 \sim w_3'$; also, look at the fork $(v_2u_1v_3, v_2u_3v_4, v_2w_2, v_2w_2')$, to forbid this fork, $v_4 \sim w_2$ or w_2', however, $v_4 \nsim w_2'$ otherwise there is a fork $(w_2'v_2u_1, w_2'v_4u_2, w_2'w_3, w_2'w_3')$, thus $v_4 \sim w_2$.

Here, $v_5 \nsim w_2$ by *Claim 1*, and $v_5 \nsim v_4, w_2'$ since triangle free.

We also claim that $v_5 \nsim w_3$. Suppose not, $v_5 \sim w_3$, then for w_3, either $w_3 \in V_3$, or $w_3 \sim w_2$ or $w_3 \sim v_4$, otherwise there is a fork centered on w_3 with long arms $(w_3v_5u_2, w_3w_2'v_2)$, so is w_3' by symmetry. Then, we can identify v_1, v_2, v_3, v_4, v_5 as v', delete $u, u_1, u_2, u_3, u_4, w_2$, notice that after we delete w_2, w_3, w_3' has at most 2 degree, we can delete them, too. Now, v' has at most 2 degree, delete it to destroy possible triangle. Now, the new graph is fork free, triangle free, contradiction.

We have shown that $v_5 \nsim w_2, w_2', w_3, w_3', v_4$, we also claim that $v_5 \in V_3$, otherwise, denote the fourth neighbor of v_5 by w_5, $w_5 \sim w_2'$ to forbid the fork $(v_5u_2v_1, v_5w_3'w_2', v_5u_4, v_5w_5)$, then there is a fork $(v_2u_1v_3, v_2w_2'w_5, v_2w_2, v_2u_3)$ in G, contradiction.

Thus, we can identify u_1, u_2, u_3, u_4 as u', delete u, v_1, v_5, the new graph is fork free, triangle free, contradiction.

Subcase 2: v_1 is adjacent to two of w_2, w_2', w_3, w_3'. In this case, v_1 cannot be adjacent to both w_2, w_2', otherwise, $w_2, w_2' \nsim v_5$ by *Claim 1*, then there is a fork $v_1u_1v_3, v_1u_2v_5, v_1w_2, v_2w_2'$. Thus, there is only one possible case by symmetry, $v_1 \sim w_2, v_1 \sim w_3$, also $w_2' \sim w_3'$ to forbid the fork centered on u_1.

In this subcase, look at w_2, either $w_2 \sim v_4$ or v_5, or $w_2 \in V_3$, otherwise, there is a fork centered on w_2 with long arms $(w_2v_1u_2, w_2v_2u_3)$, by symmetry, so is w_3. That mean if we identify $N_2(u)$, w_2, w_3 can be deleted.

To forbid the fork $(v_2u_1v_3, v_2u_3v_4, v_2w_2, v_2w_2')$, $v_4 \sim w_2$ or w_2'; also, to forbid the fork $(v_3u_1v_2, v_3u_4v_5, v_3w_3, v_3w_3')$, $v_5 \sim w_3$ or w_3'. Thus, there are 4 combinations:

Combination 1. $v_4 \sim w'_2$, $v_5 \sim w'_3$. Look at w'_2, either $w'_2 \sim v_5$, or $w'_2 \in V_3$, otherwise there is a fork centered on w'_2 with long arms $(w'_2v_2u_1, w'_2v_4u_2)$, by symmetry, so is w'_3. That means, after we identify v_1, v_2, v_3, v_4, v_5 as v', we can delete $u, u_1, u_2, u_3, u_4, w_2, w'_2, w_3, w'_3$, now v' has at most 2 degree(if both $v_4, v_5 \in V_4$), we can delete v' to destroy possible triangle, the new graph is fork free, triangle free, contradiction.

Combination 2. $v_4 \sim w_2, v_5 \sim w'_3$. By the similar argument above, we know that either $w'_3 \sim v_4$, or $w'_3 \in V_3$, i.e, after we identify $N_2(u)$, w'_3 can be deleted.

Now, look at v_4, we claim that $v_4 \in V_3$. $v_4 \nsim w'_2$, otherwise it's the case the same with combination 1; $v_4 \nsim w_3, w'_3$ by *Claim 1*. Suppose, the fourth neighbor of v_4 is w_4, then $w_4 \sim v_5$ or w'_3 to forbid the fork $(u_2v_4w_4, u_2v_5w'_3)$. If $w_4 \sim w'_3$, then there is a fork $(w'_3v_5u_2, w'_3v_3u_1, w'_3w'_2, w'_3w_4)$, contradiction; if $w_4 \sim v_5$, then $w_4 \in V_3$, otherwise there is a fork centered on w_4 with long arms $(w_4v_4u_3, w_4v_5u_4)$, thus we can identify $N_2(u)$ as v', delete w_2, w_3, w'_3, w_4, the new graph is triangle free, fork free, contradiction. Thus $v_4 \in V_3$.

That means, after identifying $N_2(u)$ as v', we can delete w_2, w_3, w'_3, and v' has at most degree 2(w'_2 and the fourth neighbor of v_5), we can delete v', the new graph is fork free, triangle free, contradiction.

Combination 3. $v_4 \sim w'_2, v_5 \sim w_3$. By symmetry, this combination is the same with Combination 1.

Combination 4. $v_4 \sim w_2, v_5 \sim w_3$, in this combination, $w_2, w_3 \in V_4$, and $w_2 \nsim v_5$, $w_3 \nsim v_4$ by *Claim 1*. thus, there are 5 possibilities:

1. $w_2 \sim w'_3$, $w_3 \sim w'_2$. Look at w'_3, either $w'_3 \sim v_4$ or v_5, or $w'_3 \in V_3$, otherwise there is a fork centered on w'_3 with long arms $(w'_3v_3u_4, w'_3w_2v_1)$; by symmetry, so is w'_2. Then either this is the graph G, which is 3-colorable, or v_4, v_5 is a 2-cut set, contradiction.

2. $w_3 \sim w'_2$, the fourth neighbor of w_2 is x. Then $x \sim w'_2$ to forbid the fork $(v_2u_1v_3, v_2w_2x, v_2u_3, v_2w'_2)$. Then there is a fork $(w'_2w_3v_1, w'_2v_2u_3, w'_2x, w'_2w'_3)$ in G, contradiction.

3. $w_2 \sim w'_3$, the fourth neighbor of w_3 is x. It's symmetric with the previous situation.

4. w_3, w_2 has the common fourth neighbor x. Then $x \sim w'_2$ to forbid the fork $(v_2u_1v_3, v_2w_2x, v_2w'_2, v_2u$

also $x \sim w'_3$ to forbid the fork $(v_3u_1v_2, v_3w_3x, v_3u_4, v_3w'_3)$. Now there is a fork centered on x: $(xw_2v_4, xw_3v_5, xw'_2, xw'_3)$, to forbid this fork, $v_4 \sim w'_2$, or $v_4 \sim w'_3$, or $v_5 \sim w'_2$, or $v_5 \sim w'_3$, in either case, there is a two cut set, $(v_4/v_5, w'_2/w'_3)$ in G, contradiction.

5. The fourth neighbor of w_2 is x_1, the fourth neighbor of w_3 is x_2. Then $x_1 \sim w'_2$ to forbid the fork $(v_2u_1v_3, v_2w_2x_1, v_2w'_2, v_2u_3)$; $x_2 \sim w'_3$ to forbid the fork $(v_3u_1v_2, v_3w_3x_2, v_3u_4, v_3w'_3)$; also $x_1 \sim x_2$ to forbid the fork $(v_1w_2x_1, v_1w_3x_2, v_1u_1, v_1u_2)$. For w'_2, either $w'_2 \sim v_4$ or v_5, or $w'_2 \in V_3$, otherwise there is a fork centered on w'_2 with long arms $(w'_2v_2u_3, w'_2w'_3v_3)$; by symmetry, so is w'_3; for x_1, either $x_1 \sim v_5$, or $x_1 \in V_3$, otherwise there is a fork centered on x_1 with long arms $(x_1w_2v_4, x_1x_2w_3)$; by symmetry, so is x_2, either $x_1 \sim v_4$, or $x_1 \in V_3$. Anyway, either this is the graph G, which is 3-colorable; or v_4, v_5 is a 2 cut set, contradiction.

So far, we have shown that Case 3 is impossible.

We have exhausted all the possible neighbors around v_2, v_3 if both of them have degree 4, every case leads to a contradiction. Thus, at least one of v_2, v_3 has degree 3. By symmetry, at least one of v_4, v_5 has degree 3.

Now we can identify u_1, u_2, u_3, u_4 as u', delete u, and 2 of v_i which have degree 3. Now the graph is fork free since u' has at most 3 degree. Also, we claim that this graph is triangle free. Suppose not, then the only possible edges between v_i are $v_2 \sim v_5$, $v_3 \sim v_4$. for example, if the remaining v_i are v_1, v_2, v_5, and $v_2 \sim v_5$, then $v_2, v_5 \in V_3$ by *Claim 2*, we can also delete them, then the new graph is fork free, triangle free, contradiction; also, if the remaining v_i are v_1, v_3, v_4, we can apply the same argument.

Subcase 2.2. All of $N_1(u)$ has degree 4, then there are three types of graph by symmetry. See Figure 7.

For graph (a), let us play stick trick on the graph: stick u_1,u_2 as on vertex u_{12}, and stick u_3, u_4 as on vertex u_{34}. We can check there is no triangles in this new graph G', otherwise it will also be a triangle in G.

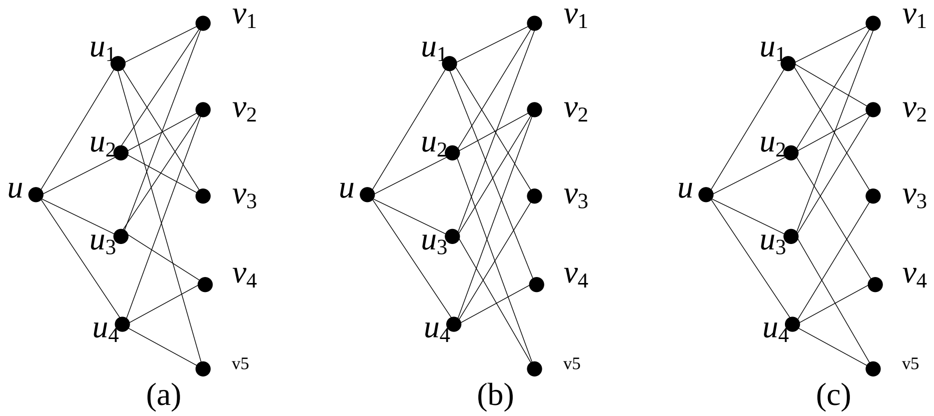

Figure 7: Three graphs.

Next, let's show that there is no fork in G'. If there exits a fork, the fork must center on $u_{12}, u_{34}, v_1, v_2, v_5$, since v_3, v_4 has degree less than 4 in G'. If the fork centers on $v_i, i = 1, 2, 5$, the only possible fork is the one with long arms: $(v_i u_{12} v_3, v_i u_{34} v_4)$, it's also a fork in G centered on v_i. If the fork centered on u_{12}, the long arms must be $(u_{12} v_2 v'_2, u_{12} v_5 v'_5)$, otherwise there will be a fork in G centered on u_{12} or u_{34}. Now we know that v'_2 cannot be adjacent to v_3, v_5 since it's a fork in G', then we can find a fork centered on v_2 in G: $(v_2 u_2 v_3, v_2 u_4 v_5, v_2 u_3, v_2 v'_2)$, a contradiction. We can apply similar argument if the fork centered on u_{34}, the long arms must be $(u_{34} v_1 v'_1, u_{34} v_5 v'_5)$, then we can find a fork centered on v_1 in G: $(v_1 u_1 v_5, v_1 u_3 v_4, v_1 u_2, v_1 v'_1)$.

Thus, G' has no fork, it's 3-colorable by induction, then G is also 3-colorable, contradiction.

Here $N_1(u_2) = N_1(u_3) = \{v_1, v_2, v_5\}$, we have shown that it must be 3-colorable.

For graph (b), we also play stick trick on this graph, but this time, we stick $\{u, u_1, u_2, u_3, u_4\}$ as u', and delete v_1, v_2. We can see that in the new graph G', the degrees of $\{u', v_3, v_4, v_5\}$ are less than 4, thus there shouldn't be any fork in G', and also G' has no triangles, then G' is 3-colorable by induction. Then we can also give v_1, v_2 a proper color since v_1, v_2 has only 2 neighborhoods in G'. Finally, we can get a 3-coloring of G, contradiction.

Case 3: $|N_2(u)| = 6$. Since every vertex in $N_2(u) \notin S(u)$, $N_2(u)$ has at least $2 \times 6 = 12$

edges connecting with $N_1(u)$; on the other hand, $N_1(u)$ has at most $3 \times 4 = 12$ edges connecting with $N_2(u)$ since $\Delta(G) \leq 4$. Thus, there are 12 edges between $N_1(u)$ and $N_2(u)$, there is only one graph by symmetry:

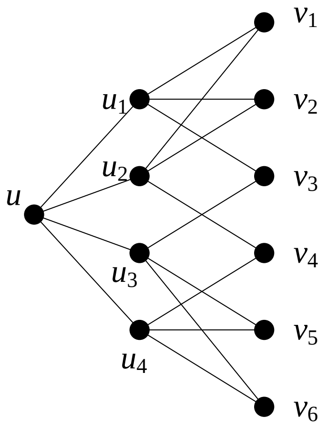

Figure 8: $|N_2(u)| = 6$.

Notice that in this graph, v_3, v_4 are symmetric, v_1, v_2, v_5, v_6 are symmetric.

Claim 1: v_3 must have degree 3, and be adjacent to v_4. By symmetry, v_4 must have degree 3 and be adjacent to v_3.

Proof. First notice that v_3 cannot be adjacent to v_i ($i \neq 4$), since G is triangle free. Suppose the claim is not true, v_3 has another neighbor w_3, w_3 must be adjacent to v_1 or v_2 to forbid the fork $(u_1v_3w_3, u_1uu_4, u_1v_1, u_1v_2)$. However, w_3 cannot be adjacent to both v_1, v_2, otherwise, w_3 also need to be adjacent to one of v_5, v_6 to forbid the fork $(u_3uu_2, u_3v_3w_3, u_3v_5, u_3v_6)$, then we can identify u_1, u_2, u_3, u_4 as u', delete $u, w_3, v_1, v_2, v_3, v_5/v_6$(the fourth neighbor of w_3), u' in order, the new graph is fork free, triangle free, i.e. G is reducible, contradiction. Thus w_3 must be adjacent to only one of v_1, v_2, by symmetry, suppose that w_3 is adjacent to v_2, then w_3 must be adjacent to v_4 to forbid fork $(u_2v_2w_3, u_2uu_3, u_2v_1, u_2v_4)$, also w_3 must be adjacent to one of v_5, v_6 to forbid the fork $(u_3uu_2, u_3v_3w_3, u_3v_5, u_3v_6)$, then we can play the same trick as the previous case to show that G is reducible, contradiction. Thus v_3 must be only adjacent to v_4, have no other neighbor. □

Claim 2: v_1 is not adjacent to v_6, by symmetry, v_1 is not adjacent to v_5, and v_2 is not adjacent to v_5 or v_6.

Proof. Suppose v_1 is adjacent to v_6, v_6 must be adjacent to v_2 to forbid the fork $(u_1v_1v_6, u_1v_3v_4, u_1v_2, u_1u)$; also v_1 must be adjacent to v_5 to forbid the fork $(u_4v_6v_1, u_4v_4v_3, u_4v_5, u_4u)$; then v_2 must be adjacent to v_5 to forbid the fork $(u_3v_6v_2, u_3v_3v_4, u_3v_5, u_3u)$. Now all the vertices are saturated, this graph is G, it's 3-colorable, contradiction. □

By *Claim 1,2*, the third or fourth neighbor of v_1, v_2 must be the same, otherwise, if there exists some $w_1 \in N_1(v_1), w_1 \notin N_1(v_2)$, then there is a fork $(u_1v_1w_1, u_1uu_4, u_1v_2, u_1v_3)$. Then we can see that $N_1(v_1) = N_1(v_2)$ in G, a contradiction to our assumption.

7.3 $|S(u)| = 1$

Proposition 7.6. *For any $u \in V_4$, we have $|S(u)| > 1$.*

Suppose not, then there exists some vertex $u \in V_4$, such that $S(u) = 1$. Let $N_1(u) = \{u_1, u_2, u_3, u_4\}$, $v_1 \in S(u)$, and assume $v_1 \sim u_1$.

Since $|S(u)| = 1$, each vertex in $N_2(u) - \{v_1\}$ has at least 2 neighbors in $N_1(u)$. Hence, since $\Delta(G) \leq 4$, u_1 has at most 2 neighbors in $N_2(u) - \{v_1\}$. Thus $2|N_2(u) - 1| \leq 3|N_1(u) - 1| + 2$, which implies $|N_2(u)| \leq 6$.

On the other hand, $|N_2(u)| \geq 4$; for, otherwise, since $v_1 \in S(u)$ and $\delta(G) \geq 3$, we have $N_1(u_1) = N_1(u_2) = N_1(u_3)$, a contradiction. In fact, we must have $|N_2(u)| \geq 5$. For, suppose $N_2(u) = \{v_1, v_2, v_3, v_4\}$. Then $u_2, u_3, u_4 \in V_3(G)$ since $v_1 \in S(u)$, and $G[\{u_2, u_3, u_4, v_2, v_3, v_4\}] \cong C_6$ (cycle of length 6). In $G' := G/\{u_2, u_3, u_4\} - u$, all vertices in $(N_1(u) \cup N_2(u)) - \{v_1\}$ are of degree at most 3; so G' is fork-free (and clearly triangle-free), a contradiction.

Thus we consider two cases: $|N_2(u)| = 5$ or $|N_2(u)| = 6$.

Case 1. $|N_2(u)| = 5$.

Let $N_2(u) = \{v_1, v_2, v_3, v_4, v_5\}$.

Subcase 1.1: $u_2, u_3, u_4 \in V_3(G)$.

Then $u_1 \in V_4(G)$, and we must have the local structure shown in Figure 9. Note the symmetry between v_2 and v_5, and the symmetry between v_3 and v_4.

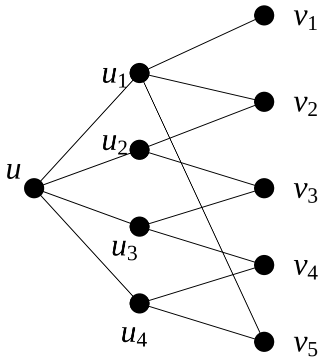

Figure 9: graph 1.2.1

Claim 1. $v_2 \in V_4(G)$ or $v_5 \in V_4(G)$.

Suppose otherwise, then $v_2, v_5 \in V_3(G)$. There are five cases to consider, and in each case we will derive a contradiction.

Case 1. $v_2 \sim v_4$, and $v_3 \sim v_5$. Then $N_1(v_1) \subseteq N_1(v_2) \cup N_1(v_5)$; otherwise let $v \in N_1(v_1) - (N_1(v_2) \cup N_1(v_5))$ then $(u_1uu_3, u_1v_1v, u_1v_2, u_1v_5)$ would be a fork in G. Hence $N_1(v_1) = \{u_1, v_3, v_4\}$ since $v_1 \in V_3(G)$. Now all the vertices are saturated, and we can check that this graph is 3-colorable, contradictory to our assumption.

Case 2. $v_2 \sim v_4$, and there exists $w_5 \in N_1(v_5) - N_1(u) \cup N_2(u) \cup \{u\}$. Then $w_5 \sim v_1$, to avoid the fork $(u_1uu_3, u_1v_5w_5, u_1v_2, u_1v_1)$. Also, $N_1(v_1) \subseteq \{u_1, w_5, v_3, v_4\}$; otherwise let $v \in N_1(v_1) - \{u_1, w_5, v_3, v_4\}$ then $(u_1uu_3, u_1v_1v, u_1v_2, u_1v_5)$ would be a fork in G. If $v_1 \sim v_3$ and $v_1 \sim v_4$ then $\{v_3, w_5\}$ would be a 2-cut in G; if $v_1 \sim v_4$ and $v_1 \nsim v_3$ then $\{v_3, w_5\}$ would be a 2-cut in G. So $v_1 \sim v_3$ and $v_1 \nsim v_4$. Then $v_3 \in V_4(G)$ or else $\{v_4, w_5\}$ would be a 2-cut in G. Let $w_3 \in N_1(v_3) - \{u_3, v_1\}$. Then $w_3 \sim v_4$ to avoid the fork $(v_3v_1u_1, v_3u_3v_4, v_3u_2, v_3w_3)$, which implies that $\{w_3, w_5\}$ is a 2-cut in G, a contradiction.

Case 3. $v_3 \sim v_5$, and there exists $w_2 \in N_1(v_2) - N_1(u) \cup N_2(u) \cup \{u\}$. This is symmetric to Case 2.

Case 4. v_2 and v_5 has the same neighbor w. then the neighbor of v_1 must be chosen from (w, v_3, v_4), otherwise there is fork centered on u_1. If $v_1 \sim w, v_3, v_4$, then there is a fork $(v_1v_3u_2, v_1v_4u_4, v_1u_1, v_1w)$. If $v_1 \sim v_3, w$, then $w \nsim v_3$ and $w \nsim v_4$ (since G is 3-connected), and hence $w \in V_3(G)$ (to avoid the fork centered on w with long arms (wv_2u_2, wv_5u_4)), which implies that $\{v_3, v_4\}$ is a 2-cut in G. The case when $v_1 \sim w$ and $v_1 \sim v_4$ can be dealt with similarly. If $v_1 \sim v_3$ and $v_1 \sim v_4$, then v_4 must have another neighbor w_4 (otherwise $\{w, v_3\}$ would be a 2-cut in G), and v_3 must have another neighbor w_3 for the same reason. $w_3 \nsim w$ and $w_3 \nsim w_4$, otherwise there is a fork $(v_3v_1u_1, v_3w_3*, v_3u_2, v_3u_3)$. $w_4 \nsim w$, otherwise there is a fork $(v_4v_1u_1, v_4w_4w, v_4u_3, v_4u_4)$. Hence $\{w, w_3, w_4\}$ is independent, we can apply Lemma 7.5.

Case 5. The third neighbor of v_5 is w_5, the third neighbor of v_2 is w_2, then v_1 must adjacent to w_2, w_5 to forbid fork centered on u_1 with long arm $(u_1uu_3, u_1v_2w_2)$, or $(u_1uu_3, u_1v_5w_5)$. Here, v_1 must be adjacent to v_3 or v_4, otherwise, we can identify u_1, u_2, u_3, u_4, delete u, v_2, v_5, the new graph is fork free, triangle free, G is reducible. Suppose that v_1 is adjacent to v_3, we will show that v_3 has degree 3, then we can we can identify u_1, u_2, u_3, u_4, delete u, v_2, v_5, v_3, the new graph is fork free, triangle free, G is reducible. Suppose the fourth neighbor of v_3 is w_3(v_3 is not adjacent to w_2, w_5 since triangle free), w_3 must be adjacent to v_4 to forbid fork $(v_3v_1u_1, v_3u_3v_4, v_3u_2, v_3w_3)$, and w_3 must be adjacent to w_2 or w_5 to forbid fork $(v_1v_3w_3, v_1u_1u, v_1w_2, v_1w_5)$. If w_2 is adjacent to w_3, w_2 must have degree 3, otherwise there is a fork centered on itself with long arms $(w_2v_2u_2, w_2w_3v_4)$, also, w_3 has degree 3, otherwise there is a fork centered on itself with long arms $(w_3v_3u_2, w_3v_4u_4)$, then v_4, w_5 is a 2-cut set, contradiction. Thus v_3 has degree 3, i.e. G is reducible. By symmetry, v_1 cannot be adjacent to v_4, either. This completes the proof of *Claim 1*.

Claim 2. $v_2 \in V_4(G)$ and $v_5 \in V_4(G)$.

Suppose not. Because of the symmetry between v_2 and v_5, we may assume $v_2 \in V_4(G)$ and $v_5 \in V_3(G)$. Let w_2, w_2' be the other neighbors of v_2, and let w_5 be the third neighbor of v_5. There are six cases to consider.

Case 1. $v_5 \sim v_3$ and $v_2 \sim v_4$.

Then $v_1 \sim w_2$ to avoid the fork $(u_1uu_3, u_1v_2w_2, u_1v_1, u_1v_5)$. Other neighbors of v_1 must be from (v_3, v_4), otherwise there is a fork centered on u_1 with long arms (u_1uu_3, u_1v_1*). In any event, $\{v_3, w_2\}$ or $\{v_4, w_2\}$ would be a 2-cut in G.

Case 2. $v_2 \sim v_4$ and $v_5 \sim w_2$. Then other neighbors of v_1 must be from (w_2, v_3, v_4). $v_1 \nsim v_4$, otherwise $\{v_3, w_2\}$ would be a 2-cut in G. Thus $v_1 \sim v_3$ and $v_1 \sim w_2$ since $d(v_1) \leq 3$. $w_2 \nsim v_3$ or $w_2 \nsim v_4$ since G is triangle-free; so $w_2 \in V_3(G)$, as otherwise there would be a fork in G centered on itself with long arms $(w_2v_2u_2, w_2v_5u_4)$. Now, if $v_3 \in V_3(G)$ and $v_4 \in V_3(G)$ then G is 3-colorable, and otherwise $\{v_3, v_4\}$ is a 2-cut in G, a contradiction.

Case 3. $v_2 \sim v_4$.

In this case, $(v_2u_1v_5, v_2v_4u_3, v_2u_2, u_2w_2)$ is a fork in G, a contradiction.

Case 4. $v_5 \sim v_3$.

Then $v_1 \sim w_2$ and $v_1 \sim w_2'$ to avoid the fork centered on u_1. $v_1 \nsim v_4$, to avoid the fork $(v_1v_4u_3, v_1u_1v_5, v_1w_2, v_1w_2')$; $v_1 \nsim v_3$, to avoid the fork $(v_3u_2v_2, v_3u_3v_4, v_3v_1, v_3v_5)$. Thus, $H := G/\{u_1, u_2, u_3, u_4\} - \{u, v_5, v_3\}$ is both triangle-free and fork-free. By the choice of G, $\chi(H) \leq 3$, which implies $\chi(G) \leq 3$, a contradiction.

Case 5. $v_5 \sim w_2$.

If $v_1 \sim v_3$ and $v_1 \sim v_4$ then $(v_1v_4u_4, v_1v_3u_2, v_1u_1, v_1w_2')$ would be a fork in G. If $v_1 \sim v_3$ and $v_1 \sim w_2'$ then $(v_1w_2'v_5, v_1v_3u_2, v_1u_1, v_1w_2')$ would be a fork in G. If $v_1 \sim w_2'$ and $v_1 \sim v_4$ then $(v_1v_4u_4, v_1w_2'v_5, v_1u_1, v_1w_2')$ would be a fork in G. So $v_1 \sim w_2'$, and other neighbors of v_1 must belong to $\{w_2, v_3, v_4\}$, and $v_1 \in V_3(G)$.

Suppose $v_1 \sim v_4$. If $v_4 \in V_3(G)$ then $H := G/\{u_1, u_2, u_3, u_4\} - \{u, v_5, v_4\}$ is both triangle-free and fork-free; hence $\chi(H) \leq 3$ (by the choice of G), which implies $\chi(G) \leq 3$, a contradiction. So $v_4 \in V_4(G)$. If $v_4 \sim w_2$ then $w_2 \in V_3(G)$ to avoid the fork in G centered on w_2 with long arms $(w_2v_2u_2, w_2v_4u_3)$ (here $w_2 \nsim v_3$, for otherwise $\{w_2', v_3\}$ would be a 2-cut in G), but then $\{w_2', v_3\}$ is a 2-cut in G, a contradiction. Thus $v_4 \nsim w_2$, and let w_4 denote the fourth neighbor of v_4. $w_4 \sim v_3$ to avoid the fork $(v_4v_1u_1, v_4u_3v_3, v_4u_4, v_4w_4)$; $w_4 \sim w_2'$, to avoid the fork $(v_4w_4*, v_4v_1u_1, v_4u_3, v_4u_4)$. If $w_4 \sim w_2'$, then $w_4 \in V_3(G)$ to avoid the fork

$(v_4w_4*, v_4v_1u_1, v_4u_3, v_4u_4)$. next we will show that w_2' also has degree 3, then (v_3, w_2) is a 2-cut set. Suppose $w_2' \in V_4(G)$, and denote the fourth neighbor of w_2' by w_2''. Then $w_2 \nsim v_3$ otherwise w_2 is a cut vertex in G. Then $w_2'' \sim w_2$ to avoid the fork $(v_2w_2'w_2'', v_2)$, since $v_3 \nsim w_2, w_2''$ (by 3-connectedness of G). Then $w_2 \in V_3(G)$ to avoid the fork centered on w_2 with long arms $(w_2v_2u_2, w_2v_5u_4)$. But then $\{v_3, w_2''\}$ is a 2-cut in G, contradiction.

So $v_1 \nsim v_4$. By applying the very similar argument, we can also prove that $v_1 \nsim v_3$. Thus, $v_1 \sim w_2'$, and $w_2 \sim v_3$ or $w_2 \sim v_4$, or $w_2 \in V_3(G)$; otherwise there is a fork centered on itself with long arms $(w_2v_2u_2, w_2v_5u_4)$. Now $H := G/\{u_1, u_2, u_3, u_4\} - \{u, v_5, w_2, v_2\}$ is triangle-free and fork-free. So $\chi(H) \leq 3$ by the choice of G, which implies $\chi(G) \leq 3$, a contradiction.

So $v_5 \nsim w_2$. Then $v_1 \sim w_2, w_2', w_5$ to forbid the fork centered on u_1, and $\{v_1, v_2, v_3, v_4, v_5\}$ is independent. Let H be obtained from $G - u$ by identifying v_1 and v_2 as u' and identifying v_3, v_4 and v_5 as u''. The only possible fork in H must be centered on v_3. However, any fork on v_3 gives rise to a fork in G, since $v_1 \sim w_2$, $v_1 \sim w_2'$, the long arm $v_3u'v_1$ is also the long arm $v_3u_2v_2$ in G. So H is triangle-free and fork-free. By the choice of G, $\chi(H) \leq 3$, which implies $\chi(G) \leq 3$, a contradiction.

Case 1. $v_2 \sim v_4$, $v_5 \sim v_3$, and the fourth neighbor of v_2, v_5 is w, the neighbour of v_1 belong to v_3, v_4, w.

$v_1 \nsim v_i$ for $i \in \{3, 4\}$; otherwise $\{v_i, w\}$ would be a 2-cut in G. This implies $d(v_1) = 2$, a contradiction.

Case 2. $v_2 \sim v_4$, $v_5 \sim v_3$, and the fourth neighbor of v_2, v_5 is w_2, w_5 respectively.

In this case, $(v_5v_3u_3, v_5u_1v_2, v_5w_5, v_5u_4)$ is a fork in G, a contradiction.

Case 3. $v_2 \sim v_4$, v_2, v_5 share the same neighbor w_2, the fourth neighbor of v_5 is w_5.

Then $v_1 \sim w_5$ to avoid the fork centered on u_1, and $v_1 \sim w_2$ to avoid the fork $(v_2v_4u_3, v_2u_1v_1, v_2u_2, v_2w_2)$. Here, $w_2 \nsim v_4$ (since G is triangle-free). Also, $w_2 \nsim v_3$; otherwise $H := G/\{u_1, u_2, u_3, u_4\} - \{u, w_2, v_2, v_5\}$ is triangle-free and fork-free; so $\chi(H) \leq 3$ which implies $\chi(G) \leq 3$, a contradiction. Thus $w_2 \in V_3(G)$, otherwise there is a fork centered on itself with long arms $(w_2v_2u_2, w_2v_5u_4)$. Now $H := G/\{u_1,u_2,u_3,u_4\}-\{u,w_2,v_2,v_5\}$ is triangle-free and fork-free.

Case 4. $v_2 \sim v_4$, the fourth neighbor of v_2 is w_2, w_5, w_5' are the neighbor of v_5.

Then $v_1 \sim w_2, w_5, w_5'$ to avoid the fork centered on u_1. However, $(v_2v_4u_3, v_2u_1v_5, v_2u_2, v_2w_2)$ is a fork in G, a contradiction.

Case 5. $v_2 \sim w_2, w_2'$, and $v_5 \sim v_3, w_2$.

Then $v_1 \sim w_2'$ to avoid the fork centered on u_1. Here $v_3 \nsim w_2'$, to avid the fork $(v_3v_5u_4, v_3w_2'v_1, v_3u_2, v_3u_3)$. Also $v_3 \nsim w_2$ by triangle-freeness. So $v_3 \in V_3(G)$, otherwise there is a fork centered on v_3 with long arms $(v_3v_5u_4, v_3u_2v_2)$. Then $H := G/\{u_1,u_2,u_3,u_4\}-\{u,v_3,v_5\}$ is triangle-free and fork-free, i.e. G is reducible.

Case 6. $v_2 \sim w_2, w_2'$, $v_5 \sim v_3$, and the fourth neighbor of v_5 is w_5.

Then there is a fork $(v_5v_3u_3, v_5u_1v_2, v_5u_4, v_5w_5)$ in G.

Case 7. $v_2 \sim w_2, w_2'$, and $w_5 \sim w_5, w_5'$.

Then $v_1 \sim w_2, w_2', w_5, w_5'$ in order to avoid the fork centered on u_1. This is impossible since $d(v_1) \leq 4$.

Case 8. $v_2 \sim w_2, w_2'$, and $v_5 \sim w_2, w_2'$.

Then $N_1(v_1) \subseteq \{u_1, w_2, w_2', v_3, v_4\}$. First, suppose $v_1 \sim w_2$ and $v_1 \sim w_2'$. Then $w_2 \sim v_3$, to avoid the fork $(w_2v_3u_3, w_2v_5u_4, w_2v_1, w_2v_2)$; $w_2 \nsim v_4$, to avoid the fork $(w_2v_2u_2, w_2v_4u_3, w_2v_1, w_2v_5)$; and hence $w_2 \in V_3(G)$ to avoid the fork centered on w_2 with long arms $(w_2v_2u_2, w_2v_5u_4)$. By symmetry, $w_2' \in V_3(G)$. Then $\{v_3, v_4\}$ is a 2-cut set in G. Hence $v_1 \nsim w_2$ or $v_1 \nsim w_2'$,

Next, we show $v_1 \nsim v_3$ or $v_1 \nsim v_4$. Suppose $v_1 \sim v_3$ and $v_1 \sim v_4$. Then $v_1 \in V_3(G)$, to avoid the fork centered on v_1 with long arms $(v_1v_3u_2, v_1v_4u_4)$; also $v_3 \nsim w_2$, otherwise $w_2 \in V_3(G)$ to avoid the fork with long arms $(w_2v_3u_3, w_2v_5u_4)$ (notice that $w_2 \nsim v_4$, otherwise w_2' would be a cut vertex in G, and $\{v_4, w_2'\}$ is a 2-cut in G. Similarly, $v_3 \nsim w_2'$. Hence $v_3 \in V_3(G)$; otherwise let w_3 denote the fourth neighbor of v_3, then $w_3 \sim v_4$ to forbid fork $(v_3v_1u_1, v_3u_3v_4, v_3u_2, v_3w_3)$, w_3 must have another neighbor w_3', which implies the fork

$(v_4v_1u_1, v_4w_3w_3', v_4u_3, v_4u_4)$ in G, a contradiction.

In this graph, since v_3, v_4 are symmetric, w_2, w_2' are symmetric, we only need to show that v_1 cannot be adjacent to both v_3, w_2. Suppose not, then v_1 cannot be adjacent to w_2' or v_4 by the previous proof. Here, w_2 is not adjacent to v_3 since triangle free, also w_2 is not adjacent to v_4, otherwise there is a fork $(w_2v_2u_2, w_2v_4u_3)$, then w_2 must have degree 3, otherwise, there is a fork centered on itself with long arms $(w_2v_2u_2, w_2v_5u_4)$. Then we can identify (u_1, u_2, u_3, u_4), delete $u, w_2, v_1, v_3, v_5, v_2$ in order, the new graph is fork free, triangle free, i.e. G is reducible. That means, v_1 can only adjacent to one vertex of (w_2, w_2', v_3, v_4), i.e. v_1 has degree at most 2, contradictory to our assumption.

Case 9: v_2 has neighbor w_2, w_2', and v_5 is adjacent to w_2, the fourth neighbor of v_5 is w_5, here v_1 must be adjacent to w_2', w_5 to forbid fork centered on u_1, then v_1 is adjacent at most one of v_3, v_4 since $deg(v_1) \leq 4$, by symmetry, we can assume that v_1 is not adjacent to v_3, then we can identify u_1, u_2 as u', u_3, u_4 as u'', delete u the graph is Figure 10:

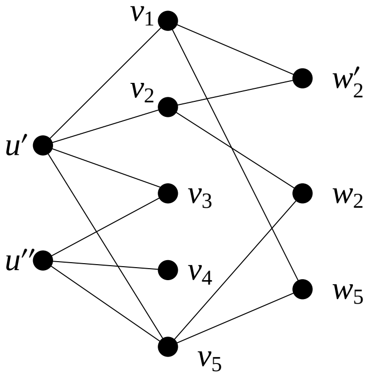

Figure 10: graph H 1.2.1

Obviously, H is triangle free; if H is fork free, then G is reducible. Actually, there are two possible new fork in H which is not the fork in G:

Potential Fork 1: centered on u', with long arms $(u'v_1v_4, u'v_3w_3)$(w_3 is the neighbor of v_3). Then look at graph G, in G, v_4 is not adjacent to w_5 since triangle free; also we claim that v_4 is not adjacent to w_2, suppose not, then v_3 is also adjacent to w_2 to forbid fork $(v_4u_3v_3, v_4v_1u_1, v_4w_2, v_4u_4)$, then we can identify (u_1, u_2, u_3, u_4) in G, delete

$u, w_2, v_4, v_3, v_2, v_5$ in order, the new graph is fork free, triangle free, i.e. G is reducible. Thus v_4 is not adjacent to w_2 or w_5, then v_4 must have degree 3 in G, otherwise there will be a fork centered on itself with long arms $(v_4v_1u_1, v_4u_4v_5)$. Then v_4 has degree 2 in H, we can delete it in H to destroy this potential fork.

Potential Fork 2: centered on v_3, one of its long arm is $v_3u'v_1$, and $v_3u'v_2$ cannot be the long arm, otherwise this fork can be reverted to a fork in G, that means v_3 is adjacent to w_2. Now let's look at graph G, w_2 cannot be adjacent to v_4, otherwise we can identify (u_1, u_2, u_3, u_4) in G, delete $u, w_2, v_2, v_3, v_4, v_5$ in order, the new graph is fork free, triangle free, i.e. G is reducible. Then w_2 must have degree 3 in G, otherwise there is a fork centered on itself with long arms $(w_2v_3u_3, w_2v_5u_4)$, that means in H, $N_1(w_2) \subset N_1(u')$, thus we can delete w_2 in H, destroying this potential fork.

By far, we have exhausted all the possible cases in Figure 9, and every case lead to a contradiction, thus Figure 9 does not exist.

Subcase 1.2. $\{u_2, u_3, u_4\} \cap V_4(G) \neq \emptyset$, and $u_1 \in V_3(G)$.

Then the local structure of G around u is given in Figure 11.

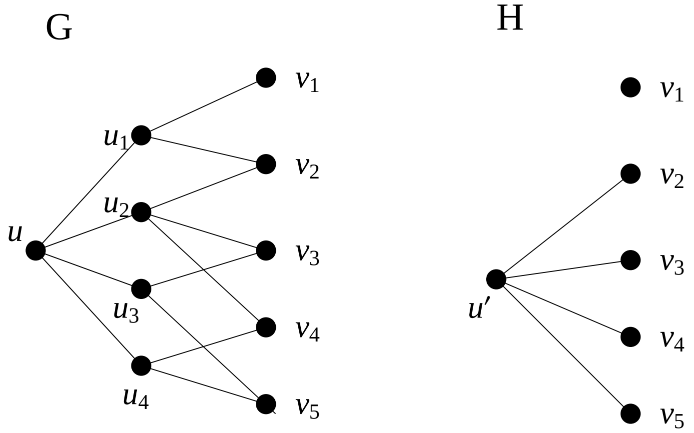

Figure 11: graph 1.2.2

Let $H = G/\{u_2, u_3, u_4\} - \{u, u_1\}$, and let u' denote the vertex resulted from identification. Then by the choice of G, H contains a triangle or a fork.

Claim 1: H is triangle-free. Suppose H has a triangle. Then $v_2 \sim v_5$. Now $v_2 \in V_4(G)$;

otherwise, $H - \{v_2\}$ is both triangle-free and fork-free, and so $\chi(H') \leq 3$, which implies $\chi(G) \leq 3$, a contradiction.

Let $v_2' \in N_1(v_2) - \{u_1, u_2, v_4\}$. Then $v_2' \notin \{v_1, v_3, v_4\}$ because G is triangle-free, $v_2' \sim v_4$ to avoid the fork $(v_2v_5u_3, v_2u_2v_4, v_2u_1, v_2v_2')$, and $v_2' \sim v_3$ to avoid the fork $(v_2v_5u_4, v_2u_2v_3, v_2u_1, v_2v_2')$. Since $N_1(v_2') \nsubseteq N_1(u_2)$, $v_2' \in V_4(G)$; so let $v_2'' \in N_1(v_2') - (N_2(u) - \{v_1, v_5\})$. Then $(v_2'v_3u_3, v_2'v_4u_4, v_2'v_2, v_2'v_2'')$ is a fork in G, a contradiction.

So H must contain a fork, say F, which must be centered on u'. If for some permutation ijk of $\{2, 3, 4\}$, $F = (u'v_iw_i, u'v_jw_j, u'v_k, u'v_5)$ then $(u_2v_iw_i, u_2v_jw_j, u_2w_k, u_2u)$ would be fork in G. So there exists $v_5' \in N_1(v_5) - (N_1(u) \cup N_2(u))$ such that $u'v_5v_5'$ is a long arm of F. This implies that $v_5' \nsim v_i$ for $i = 2, 3, 4$.

Also, $\{v_2, v_3, v_4, v_5\} \subseteq V_4(G)$. For, suppose for some $i \in \{2, 3, 4, 5\}$, $v_i \in V_3(G)$. Then $H - v_i$ both triangle-free and fork-free, and hence $\chi(H - v_i) \leq 3$ by the choice of G; which implies $\chi(H) \leq 3$, a contradiction.

Let $w_1, w_2 \in N_1(v_2) - \{u_1, u_2\}$. Then by Claim 1, $w_1, w_2 \notin N_2(u)$.

Claim 2. For $i \in \{1, 2\}$, $|N_1(w_i) \cap \{v_3, v_4, v_5\}| \leq 1$.

It suffices to prove Claim 2 for $i = 1$. First, suppose $w_1 \sim v_3$ and $w_1 \sim v_4$. Then $w_1 \in V_4(G)$, as otherwise $N_1(w_1) \subseteq N_1(v_2)$, a contradiction. So let $w_1' \in N_1(w_1) - \{v_2, v_3, v_4\}$. If $w_1' \neq v_5$ then $(w_1v_3u_3, w_1v_4u_4, w_1v_2, w_1w_1')$ would be a fork in G; if $w_1' = v_5$ then $(w_1v_2u_1, w_1v_3u_3, w_1v_4, w_1v_5)$ would be a fork in G. So $w_1 \nsim v_3$ or $w_1 \nsim v_4$.

Now assume $w_1 \sim v_3$ and $w_1 \sim v_5$. Then $w_1 \sim v_1$ to avoid the fork $(w_1v_2u_1, w_1v_5u_4, w_1v_3, w_1v_4)$. Let $H' := G/\{u_1, u_2, u_3, u_4\} - \{w_1, u, v_2, v_3, v_5\}$, and let u'' denote the vertex resulted from identification. Since $N_1(w_1) \subset N_1(u'')$, H' is triangle-free and fork-free, and $\chi(H') \leq 3$ by the choice of G. So $\chi(H) \leq 3$, and hence $\chi(G) \leq 3$, a contradiction. So $w_1 \nsim v_3$ or $w_1 \nsim v_5$.

By symmetry between v_3 and v_4,$w_1 \nsim v_4$ or $w_1 \nsim v_5$.

Claim 3. $w_1 \sim v_1$ or $w_2 \sim v_1$.

Suppose $w_1 \nsim v_1$ and $w_2 \nsim v_1$. Then we may assume (by symmetry between w_1 and w_2) that $w_1 \sim v_3$ to avoid the fork $(v_2u_1v_1, v_2u_2v_3, v_2w_1, v_2w_2)$. Then by Claim 2, $w_1 \nsim v_4$. So $w_2 \sim v_4$ to avoid the fork $(v_2u_1v_1, v_2u_2v_4, v_2w_1, v_2w_2)$. The by Claim 2 again, $w_2 \nsim v_3$. Then $F = (u'v_5, v_5', u'v_3w_3, u'v_2, u'v_4)$ or $F' = (u'v_5, v_5', u'v_4w_4, u'v_2, u'v_4)$, where ($w_3 \in N_1(v_3) - \{u_2, u_3, w_1\}$) and $w_4 \in N_1(v_4) - \{u_3, u_4, w_2\}$. By the symmetry between v_3 and v_4, let $F = (u'v_5, v_5', u'v_3w_3, u'v_2, u'v_4)$. Then $(v_3u_2v_4, v_3u_3v_5, v_3w_1, v_3w_3)$ is a fork in G, a contradiction.

By Claim 3 and by symmetry, we may assume that $w_1 \sim v_1$.

Claim 4. $v_1 \nsim v_3$ or $v_1 \nsim v_4$.

Noting the symmetry between v_3 and v_4, it suffices to prove $v_1 \nsim v_4$. Suppose for a contradiction that $v_1 \sim v_4$. Then $v_4 \in V_4(G)$; otherwise, $H - v_4$ is both triangle-free and fork-free, and $\chi(H - v_4) \leq 3$ by the choice of G; so $\chi(H) \leq 3$ (and hence $\chi(G) \leq 3$), a contradiction.

Let $w_4 \in N(v_4) - \{u_2, u_4, v_1\}$. Then $w_4 \neq w_2$; otherwise by Claims 2 and 3, $(v_4v_1u_1, v_4u_2v_3, v_4w_4, v_4u_4)$ would be fork in G. Also, $w_4 \sim v_3$ and $w_4 \sim v_5$ to avoid the forks $(v_4v_1u_1, v_4u_2v_3, v_4u_4, v_4w_4)$ and $(v_4v_1u_1, v_4u_4v_5, v_4u_2, v_4w_4)$, respectively.

Now $v_1 \nsim v_3$, or else $(v_1v_3u_3, v_1v_4u_4, v_1u_1, v_1w_1)$ would be a fork in G. So $v_1 \sim w_2$, or $w_2 \sim v_3$, as otherwise $(u_2v_2w_2, u_2v_4v_1, u_2v_3, u_2u)$ would be a fork in G.

Suppose $v_1 \sim w_2$. Then $w_4 \sim w_1$ or $w_4 \sim w_2$ to avoid the fork $(v_1u_1u, v_1v_4w_4, v_1w_1, v_1w_2)$. By symmetry between w_1 and w_2, we may assume $w_4 \sim w_2$. Suppose $v_3 \in V_4(G)$, and let $w_3 \in N_1(v_3) - \{u_2, u_3, w_4\}$. If $w_3 \neq w_1$ then $(u_2v_3w_3, u_2v_4v_1, u_2u, u_2v_2)$ would be a fork in G; and if $w_3 = w_1$ then $(v_1w_1v_3, v_1v_4u_4, v_1w_2, v_1u_1)$ would be a fork in G. So $v_3 \in V_3(G)$. Now $\{w_1, w_2, v_5\}$ is a cut in G, and is independent (since G is 3-connected and triangle-free). This is a contradiction to Lemma 7.5.

So it remains to consider $w_2 \sim v_3$. Recall definition of v_5'. Note that $v_5' \neq w_1$ or else $(v_2w_1v_5, v_2u_2v_4, v_2u_1, v_2w_2)$ would be a fork in G. Moreover, $v_5' \neq w_2$, $v_5' \neq v_1$, and $v_5' \sim v_1$ or $v_5' \sim w_2$; for otherwise $H' := H/\{v_3, v_4, v_5\} - \{u', w_4\}$ is both triangle-free and fork-free, and so $\chi(H') \leq 3$, which implies $\chi(H) \leq 3$ (and hence $\chi(G) \leq 3$), a contradiction. So $v_5' \notin N_1(u) \cup N_2(u) \cup \{u, w_1, w_2, w_4\}$.

On the other hand, if $v_5' \sim v_1$ then $(v_1v_5'v_5, v_1v_4u_2, v_1u_1, v_1w_1)$ would be a fork in G; and if $v_5' \sim w_2$ then $(v_2w_2v_5', v_2u_2v_4, v_2u_1, v_2w_1)$ would be a fork in G.

Claim 5: $v_1 \nsim v_5$. Suppose $v_1 \sim v_5$. Then possibly $v_5' = v_1$. Since $v_5' \in V_4(G)$, let $w_5 \in N_1(v_5) - \{u_3, u_4, v_1\}$. Then $w_5 \sim v_3$ to avoid the fork $(v_5v_1u_1, v_5u_3v_3, v_5u_4, v_5w_5)$, and $w_5 \sim v_4$ to avoid the fork $(v_5v_1u_1, v_5u_4v_4, v_5u_3, v_5w_5)$. So by claim 2, $w_5 \notin \{w_1, w_2\}$.

Recall that $v_3, v_4 \in V_4(G)$. If $v_3 \sim w_1$ and $v_4 \sim w_2$, or $v_3 \sim w_2$ and $v_4 \sim w_1$, then by Claim 3, we see that H has no fork centered at u', contradicting the existence of F. So by symmetry between v_3 and v_4, let $w_3 \in N_1(v_3) - (N_1(u) \cup N_2(u) \cup \{u, w_1, w_2\})$.

Let $w_4 \in N_1(v_4) - \{u_2, u_4, w_5\}$. Clearly, $w_4 \notin N_1(u) \cup \{u, v_2, v_3, v_4\}$. By Claim 4, $w_4 \neq v_1$. Suppose $w_4 \notin \{w_1, w_2, w_3\}$. Then since G is triangle-free, there is a permutation ijk of $\{2, 3, 4\}$ such that $w_i \nsim w_j$. Then $(u_2v_iw_i, u_2v_jw_j, u_2u, u_2v_k)$ is a fork in G, a contradiction. So $w_4 \in \{w_1, w_2, w_3\}$.

Suppose $v_4 \sim w_3$. Then $w_3 \nsim v_1$ otherwise $(v_1w_3v_4, v_1v_5u_3, v_1w_1, v_1u_1)$ would be a fork in G. Now $H' := H/\{v_3, v_4, v_5\} - \{u', w_5\}$ is both triangle-free and fork-free; so $\chi(H') \leq 3$ by the choice of G. Hence $\chi(H) \leq 3$, and so, $\chi(G) \leq 3$, a contradiction.

Now suppose $v_4 \sim w_1$. Then $w_2 \sim w_3$ to avoid the fork $(u_2v_4w_2, u_2v_3w_3, u_2u, u_2v_4)$. Now $v_1 \in V_3(G)$; otherwise let $v_1' \in N_1(v_1) - \{v_1, v_5, w_1\}$ then $(v_1w_1v_4, v_1v_5u_3, v_1u_1, v_1v_1')$ would be a fork in G. Then $w_5 \in V_4(G)$, as otherwise $H - w_5$ is triangle-free and fork-free, and $\chi(H - w_5) \leq 3$ implies $\chi(H) \leq 3$ (and hence $\chi(G) \leq 3$), a contradiction. However, $w_5 \nsim w_2$; otherwise $w_2 \in V_4(G)$ (or else $\{w_1, w_3\}$ would be a 2-cut in G), and let $w_2' \in N_1(w_2) - \{v_2, w_3, w_5\}$ then $(w_2v_2u_1, w_2w_5v_5, w_2w_3, w_2w_2')$ would be a fork in G. Thus there exists $w_5' \in N_1(w_5) - N_1(u) \cup N_2(u) \cup \{u, w_1, w_2, w_3\}$, Now $(v_5v_1u_1, v_5w_5w_5', v_5u_3, v_5u_4)$ is a fork in G, a contradiction.

So we must have $v_4 \sim w_2$. Then $w_1 \sim w_3$ to avoid the fork $(u_2v_2w_1, u_2v_4w_5, u_2u, u_2v_3)$. If $\{v_1, w_2, w_3\}$ is independent in G, then $H' := H/\{v_3, v_4, v_5\} - \{u', w_5\}$ is triangle-free and fork-free, and hence $\chi(H') \leq 3$ by the choice of G, which implies $\chi(H) \leq 3$ and $\chi(G) \leq 3$, a contradiction. So $\{v_1, w_2, w_3\}$ is not independent in G.

Now $v_1 \nsim w_2$ to avoid the fork $(v_1v_5u_3, v_1w_2v_4, v_1u_1, v_1w_1)$; and $v_1 \nsim w_3$ to avoid the fork $(v_1v_5u_4, v_1w_3v_3, v_1u_1, v_1w_1)$. So $w_2 \sim w_3$. Then $w_2 \in V_3(G)$, otherwise let $w_2' \in N_1(w_2) - \{v_1, v_4, w_3\}$ then $(w_2w_3v_3, w_2v_4u_4, w_2v_1, w_2w_2')$ would be a fork in G. Also $w_3 \in V_3(G)$, for, if $w_3' \in N_1(w_3) - \{v_3, w_1, w_2\}$ then $(w_3w_1v_1, w_3w_2v_4, w_3w_3', w_3v_3)$ would be a fork in G. Moreover, $w_1 \in V_3(G)$; for if $w_1' \in N_1(w_1) - \{v_1, v_2, w_5\}$ then $(w_1v_2u_1, w_1v_4u_4, w_1w_3, w_1w_1')$ would be a fork in G. But this implies that (v_1, w_5) is a 2-cut in G, a contradiction.

From Claims 4 and 5, we see that $N_2(u)$ is an independent set in G. Recall that $u'v_5v_5'$ is a long arm of F. The other long arm of F is $u'v_2w_1$, or $u'v_2w_2$, or $u'v_3w_3$, or $u'v_4w_4$.

Claim 6. $F \neq (u'v_5v_5', u'v_2w_p, u'u_3, u'u_4)$, for $p \in \{1, 2\}$.

Proof. Suppose $F \neq (u'v_5v_5', u'v_2w_p, u'u_3, u'u_4)$ for some $p \in \{1, 2\}$. Then $v_5' \notin \{w_1, w_2\}$. If $v_5 \sim w_i$ for some $i \in \{1, 2\}$ then, since $w_i \nsim v_j$ for some $j \in \{3, 4\}$, $(v_5w_iv_2, v_5u_jv_j, v_5u_{7-j}, v_5v_5')$ would be a fork in G. So Further, $v_5 \nsim w_i$ for $i = 1, 2$.

Since $v_5 \in V_4(G)$, let $w_5 \in N_1(v_5) - \{u_3, u_4, v_5'\}$. If $w_5 \nsim v_3$ and $w_5 \nsim v_4$ then $(v_5u_3v_3, v_5u_4v_4, v_5v_5', v_5w_5)$ is a fork in G, a contradiction. So by the symmetry between v_3 and v_4, we may assume that $w_5 \sim v_4$. If $w_5 \nsim v_3$ then $w_5 \sim w_i$ for $i = 1, 2$, to avoid the fork $(u_2v_2w_i, u_2v_4w_5, u_2u, u_2v_3)$; but then $(w_5v_4u_2, w_5v_5u_3, w_5w_1, w_5w_2)$ is a fork in G, a contradiction. So $w_5 \nsim v_3$.

Recall that $v_3, v_4 \in V_4(G)$. So let $w_i \in N_1(v_i) - \{u_2, v_i, w_5\}$ for $i = 3, 4$. If $\{w_3, w_4, w_5\}$ is independent in G then $H' := H/\{v_3, v_4, v_5\} - \{u', w_5'\}$ is both triangle-free and fork-free, and $\chi(H') \leq 3$ by the choice of G; so $\chi(H) \leq 3$ (and hence $\chi(G) \leq 3$, a contradiction. So $\{w_3, w_4, w_5\}$ is not independent in G. Since G is triangle-free, we must have $w_3 \neq w_4$ and $w_3 \sim w_4$. Then $\{w_1, w_2\} \neq \{w_3, w_4\}$. Because of F, $w_p \notin \{w_3, w_4\}$. So by symmetry between w_3 and w_4, we may assume $w_{3-p} = w_3$. Then $w_3 \in V_3(G)$; otherwise let $w_3' \in N_1(w_3) - \{v_2, v_3, w_4\}$, and we see that $(w_3v_3u_3, w_3w_4v_4, w_3v_2, w_3w_3')$ would be a fork in G.

This, in particular, implies $p = 1$.

So $w_1 \sim w_4$; otherwise $(u_2v_2w_1, u_2v_4w_4, u_2u, u_2v_3)$ would be a fork in G. Also $w_4 \in V_3(G)$, otherwise let $w_4' \in N_1(w_4) - \{v_3, v_4, w_1\}$ and we see that $(w_4w_3v_3, w_4v_4u_4, w_4w_p, w_4w_4')$ would be a fork in G.

Moreover, $w_5 \in V_3(G)$. Otherwise, let $w_5' \in N(w_5) - \{v_3, v_4, v_5\}$. Then $w_5' \neq w_1$; or else $\{v_1, v_5\}$ would be a 2-cut in G. Then $(v_3w_5w_5', v_3w_3w_4, v_3u_3, v_3u_3)$ is a fork in G, a contradiction.

By Claim 6 and by the symmetry between v_3 and v_4, we may assume $F = (u'v_5v_5', u'v_3w_3, u'v_2, u'v_4)$, here w_3 is a neighbor of v_3. So $w_3 \nsim \{v_2, v_3, v_4, v_5, v_5'\}$, and in particular, $w_3 \notin \{w_1, w_2\}$.

Recall that $v_4 \in V_4(G)$. We claim there exists $w_4 \in N_1(v_4) - (N_1(u) \cup N_2(u) \cup \{u, w_1, w_2, w_3, v_5'\}$. For, otherwise, $v_4 \sim w_1$ and $v_4 \sim w_2$. Then by Claim 2, for $i = 1, 2$, $w_i \nsim v_3$ and $w_i \nsim v_5$. So $(v_4u_2v_3, v_4u_4v_5, v_4w_1, v_4w_2)$ is a fork in G, a contradiction.

Suppose $w_4 \nsim v_3$. Then $v_4 \sim w_i$ for $i = 1, 2, 3$; for otherwise, $(u_2v_2w_i, u_2v_4w_4, u_2u, u_2v_3)$ (for $i = 1, 2$) or $(u_2v_3w_3, u_2v_4w_4, u_2u, u_2v_2)$ would be a fork in G. Since $w_i \nsim v_4$ some $i \in \{1, 2\}$, $(u_2v_2w_i, u_2v_3w_3, u_2u, u_2v_4)$ is a fork in G, a contradiction.

Thus $w_4 \sim v_3$ for any $w_4 \in N_1(v_4) - (N_1(u) \cup N_2(u) \cup \{u, w_1, w_2, w_3, v_5'\}$, and hence $w_4 \sim v_5$ to avoid the fork $(v_3u_2v_2, v_3u_3v_5, v_3w_3, v_3w_4)$. Hence, $v_4 \sim w_i$ for some $i \in \{1, 2\}$; so $w_{3-i} \nsim v_j$ for $j = 3, 4, 5$. So by Claim 6, $w_{3-i} \sim v_5'$. Moreover, $w_{3-i} \sim w_3$, to avoid the fork $(u_2v_2w_{3-i}, u_2v_3w_3, u_2u, u_2v_4)$.

Suppose $i = 2$. Then $(w_1w_3v_3, w_1v_5'v_5, w_1v_1, w_1v_2)$ is a fork in G, a contradiction. So $i = 1$. Then $(v_2w_2v_5', v_2u_2v_3, v_2w_1, v_2u_1)$ is a fork in G, a contradiction.

Subcase 1.3: $u_1 \in V_4(G)$ and $u_1 \sim v_3$ in G. See Figure 12.

Let $H := G/\{u_1, u_2, u_3, u_4\} - \{u, v_3\}$, and let u' denote the vertex resulted from identification. Clearly, $\chi(H) \leq 3$ implies that $\chi(G) \leq 3$. Therefore, since G is irreducible, H contains a triangle or a fork.

Since $N_1(v_3) \nsubseteq N_1(u)$, $v_3 \in V_4(G)$; so let w_3 be the unique member of $N_1(v_3) - N_1(u)$.

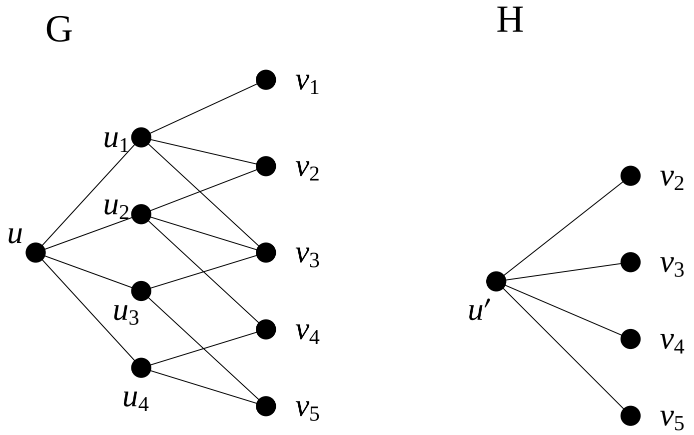

Figure 12: graph 1.2.3

Claim 1. $v_2 \sim v_5$.

Suppose $v_2 \nsim v_5$. First, assume $w_3 \sim v_5$. Then $w_3 \sim v_4$ to avoid the fork $(v_5v_2u_1, v_5u_4v_4, v_5u_3, v_5w_3)$, and $w_3 \sim v_1$ to avoid the fork $(u_1uu_4, u_1v_3w_3, u_1v_1, u_1v_2)$ (note that $w_3 \nsim v_2$ since G is triangle-free). Thus $\{v_1, v_2, v_4\}$ is a cut in G, and $v_2 \nsim v_1$ and $v_2 \nsim v_4$ since G is triangle-free. So $v_1 \nsim v_4$, as otherwise, $v_4 \in V_4(G)$ and hence $\{v_1, v_2\}$ is a 2-cut in G, a contradiction. Thus $\{v_1, v_2, v_4\}$ is an independent set in G; so by Lemma 7.5, G is 3-colorable, contradiction.

So $w_3 \nsim v_5$. Hence, $w_3 \sim v_4$ to avoid the fork $(v_3u_2v_4v_3u_3v_5, v_3u_1, v_3w_3)$.

Next assume $w_3 \sim v_2$. Then $w_3 \sim v_1$ to avoid the fork $(v_2u_1v_1, v_2v_5u_4, v_2u_2, v_2w_3)$. Hence $\{v_1, v_4, , v_5\}$ is a cut in G. Moreover, $\{v_1, v_4, v_5\}$ is independent in G: $v_4 \nsim v_5$ since G is triangle-free, $v_1 \nsim v_4$ since $\{v_1, v_5\}$ is not a cut in G, and $v_1 \nsim v_5$ since $\{v_1, v_4\}$ is not a cut in G. By Lemma 7.5, G is 3-colorable, a contradiction. So we have $w_3 \nsim v_2$.

Since $w_3 \nsim v_2$, $w_3 \sim v_1$ to avoid the fork $(u_1uu_4, u_1v_3w_3, u_1v_1, u_1v_2)$. Now $w_3 \in V_3(G)$, for otherwise, let $w_3' \in N_1(w_3) - N_2(u)$ then $(w_3v_3u_3, w_3v_4u_4, w_3v_1, w_3w_3')$ would be a fork in G. Also, $v_2 \in V_3(G)$, for, let $v_2' \in N_1(v_2) - N_2(u)$ then $(v_2u_2v_4, v_2v_5v_3, v_2u_1, v_2v_2')$ would be a fork in G. Then $\{v_1, v_4, v_5\}$ is an independent cut in G; and by the same argument as above we arrive at a contradiction to Lemma 7.5, completing the proof of Claim 1.

Claim 2. $v_1 \nsim v_4$.

Suppose $v_1 \sim v_4$. First, assume $w_3 \sim v_4$; so $w_3 \nsim v_1$ since G is triangle-free. Then $w_3 \sim v_5$ to avoid the fork $(v_4v_1u_1, v_4u_4v_5, v_4u_2, v_4w_3)$, and $w_3 \sim v_2$ to avoid the fork $(u_1uu_4, u_1v_3w_3, v_1, u_1v_2)$. Hence the same argument in the proof of Claim 1 shows that $\{v_1, v_2, v_3\}$ is independent and leads to a contradiction to Lemma 7.5.

So $w_3 \nsim v_4$. Then $w_3 \sim v_5$ to avoid the fork $(v_3u_2v_4v_3u_3v_5, v_3u_1, v_3w_3)$.

Now assume $w_3 \sim v_1$. If $v_1, w_3 \in V_3(G)$ then $\{v_2, v_4, v_5\}$ is independent in G, and we obtain a contradiction by Lemma 7.5. So $v_1 \in V_4(G)$ or $w_3 \in V_4(G)$. First, assume $v_1 \in V_4(G)$, and let w_1 denote the unique member of $N_1(v_1) \setminus (N_1(u) \cup N_2(u) \cup \{w_3\})$, which exists since G is triangle-free. Then $w_1 \sim v_5$ to avoid the fork $(v_1v_4u_2, v_1w_3v_5, v_1u_1, v_1w_1)$, and $w_1 \sim v_2$ to avoid the fork $(u_1uu_4, uv_1w_1, uv_2, uv_3)$. If $w_1 \in V_4(G)$ then let $w_1' \in N_1(w_1) \setminus \{u_1, v_4, w_3\}$ and $(w_1v_2u_2, w_1v_5u_3, w_1v_1, w_1w_1')$ would be a fork in G. So $w_1 \in V_3(G)$. Then by Lemma 7.5, G is 3-colorable, contradiction. Thus $v_1 \in V_3(G)$. So $w_3 \in V_4(G)$, and let $w_3' \in N_1(w_3) - \{v_1, v_3, v_5\}$. Note that $w_3 \nsim v_2$, as otherwise $\{v_2, v_4, v_5\}$ is an independent set and we derive a contradiction to Lemma 7.5. Thus $(w_3v_3u_2, w_3v_5u_4, w_3v_1, W_3W_3')$ is a fork in G, a contradiction.

So $w_3 \nsim v_1$. Then $w_3 \sim v_2$ to avoid the fork $(u_1v_3w_3, u_1uu_4, u_1v_1, u_1v_2)$. Since G is 3-connected, $\{v_1, w_3\}$ cannot be a 2-cut in G; so $\{v_2, v_4, v_5\} \not\subseteq V_3(G)$.

Suppose $v_2 \in V_4(G)$, and let $w_2 \in N_1(v_2) \setminus \{u_1, u_2, w_3\}$. Then $w_2 \sim v_1$ to avoid the fork $(u_1uu_4, u_1v_2w_2, u_1v_1, u_1v_3)$, and $w_2 \sim v_5$ to avoid the fork $(v_2w_3v_5, v_2u_2v_4, v_2u_1, v_2w_2)$. So $w_2 \in V_3(G)$, as otherwise, let $w_2' \in N_1(w_2) \setminus \{v_1, v_2, v_5\}$ then $(w_2v_2u_2, w_2v_5u_4, w_2v_1, w_2w_2')$ would be a fork in G. Then $v_4 \in V_3(G)$ for if there exists $w_4 \in N_1(v_4) \setminus \{u_2, u_4, v_1\}$ then $(v_4v_1u_1, v_4u_4v_5, v_4v_1, v_4w_4)$ would be a cut in G. But now (v_1, w_3) is a 2-cut in G, and this contradiction shows that $v_2 \in V_3(G)$.

Now suppose $v_4 \in V_4(G)$, and let $w_4 \in N_1(v_4) - \{u_2, u_4, v_1\}$. Then $w_4 \sim v_5$ to avoid the fork $(v_4v_1u_1, v_4u_4v_5, v_4w_4, v_4u_2)$. Hence, $\{v_1, w_3, w_4\}$ is a cut in G and is also independent. So we derive a contradiction by Lemma 7.5.

Therefore, $v_4 \in V_3(G)$. This then implies that v_5 has a neighbor $w_5 \notin \{u_3, u_4, w_3\}$, and

$(v_5u_4v_4, v_5w_3v_2, v_5u_3, v_5w_5)$ is a fork in G, a contradiction.

Claim 3. $v_1 \nsim v_5$.

Suppose $v_1 \sim v_5$. First, consider the case when $w_3 \sim v_5$. Note that $w_3 \nsim v_1$ since G is triangle-free. Then $w_3 \sim v_4$ to avoid the fork $(v_5v_1u_1, v_5u_4v_4, v_5u_3, v_5w_3)$, $w_3 \sim v_2$ to avoid the fork $(u_1uu_4, u_1v_3w_3, u_1v_1, u_1v_2)$. Now $\{v_1, v_2, v_4\}$ is an independent cut in G, contradicting Lemma 7.5.

So $w_3 \nsim v_5$. Then $w_3 \sim v_4$ to avoid the fork $(v_3u_2v_4, v_3u_3v_5, v_3w_3, v_3u_1)$.

If $v_5 \in V_3(G)$ then $H - v_5$ is triangle-free and fork-free, and so $\chi(H - v_5) \leq 3$ by the choice of G; which implies that $\chi(G) \leq 3$, a contradiction.

So $v_5 \in V_4(G)$, and let $w_5 \in N_1(v_5) - \{u_3, u_4, v_1\}$. Then $w_5 \sim v_4$ to avoid the fork $(v_5v_1u_1, v_5u_4v_4, v_5u_3, v_5w_5)$. Also $w_3 \sim v_i$ for some $i \in \{1, 2\}$, to avoid the fork $(u_1uu_4, u_1v_3w_3, u_1v_1, u_1v_2)$. If $w_3 \in V_4(G)$ then let $w_3' \in N_1(w_3) - \{v_3, v_4, v_i\}$; now $(w_3v_3u_3, w_3v_4u_4, w_3v_i, w_3w_3')$ is a fork in G, a contradiction. So $w_3 \in V_3(G)$. Then $\{v_1, v_2, w_5\}$ is a cut in G. By Lemma 7.5, $\{v_1, v_2, w_5\}$ is not independent in G; so $v_2 \sim w_5$. Then $i = 1$, otherwise, $\{v_1, w_5\}$ would be a 2-cut in G. Also $w_5 \in V_4(G)$, otherwise $\{v_1, v_2\}$ would be a 2-cut in G. So let $w_5' \in N_1(w_5) - \{u_3, u_4, v_1\}$. Then $(w_5v_5u_3, w_5v_4w_3w_5v_2, w_5w_5')$ is a fork in G, a contradiction.

By Claims 1-3, we know that H is triangle-free. So H contains a fork, say F. Since v_2, v_3, v_4, v_5 all have degree at most 3 in H, they cannot be the center of F. Also, it is easy to see that since G contains no fork, F must center on u'.

Suppose for some $i \in \{1, 2\}$, $u'v_iw_i$ is a long arm of F. Then $w_i = w_3$; for otherwise, $(u_1v_iw_i, u_1uu_4, u_1v_{3-i}, u_1v_3)$ would be fork in G. Now $(v_3u_2v_4, v_3u_3v_5, v_3u_1, v_3w_i)$ is a fork in G, a contradiction.

Thus $F = (u'v_4w_4, u'v_5w_5, u'v_1, u'v_2)$, where $w_4 \in N_1(v_4)$ and $w_5 \in N_1(v_5)$. Now $w_3 \notin \{w_4, w_5\}$; for if $w_3 = w_4$ then $(v_3u_1v_1, v_3u_3v_5, v_3u_2, v_3w_4)$ would be a fork in G, and if $w_3 = w_5$ then $(v_3u_1v_1, v_3u_2v_4, v_3u_3, v_3w_5)$ would be a fork in G.

If $w_3 \nsim v_1$ and $w_3 \nsim v_4$ then $(v_3u_1v_1, v_3u_2v_4, v_3u_3, v_3w_3)$ would be a fork in G; if $w_3 \nsim v_1$

and $w_3 \nsim v_5$ then $(v_3u_1v_1, v_3u_3v_5, v_3u_2, v_3w_3)$ would be a fork in G; if $w_3 \nsim v_4$ and $w_3 \nsim v_5$ then $(v_3u_2v_4, v_3u_3v_5, v_3u_3, v_3w_3)$ would be a fork in G.

So $|N_1(w_3) \cap \{v_1, v_4, v_5\}| \geq 2$.

If $w_3 \sim v_1$ and $w_3 \sim v_4$ then $(v_4u_2v_2, v_4w_3v_1, v_4u_4, v_4w_4)$ (when $w_3 \nsim v_2$) or $(v_4w_3v_1, v_4u_4v_5, v_4u_2, v_4w_4)$ (when $w_3 \nsim v_5$) would be a fork in G. So $w_3 \nsim v_1$ or $w_3 \nsim v_4$.

Now if $w_3 \sim v_1$ and $w_3 \sim v_5$ then $w_3 \nsim v_4$, and hence $(v_5w_3v_1, v_5u_4v_4, v_5u_3, v_5w_5)$ is a fork in G. So $w_3 \nsim v_1$ or $w_3 \nsim v_5$. Then $w_3 \sim v_4$, $w_3 \sim v_5$ and $w_3 \nsim v_1$. Now $H' := H/\{v_4, v_5\} - w_3$ is triangle-free and fork-free. So $\chi(H') \leq 3$. This implies $\chi(H) \leq 3$, and so $\chi(G) \leq 3$, a contradiction.

Subcase 1.4. $u_1 \in V_4(G)$ and $u_1 \sim v_5$. See Figure 13.

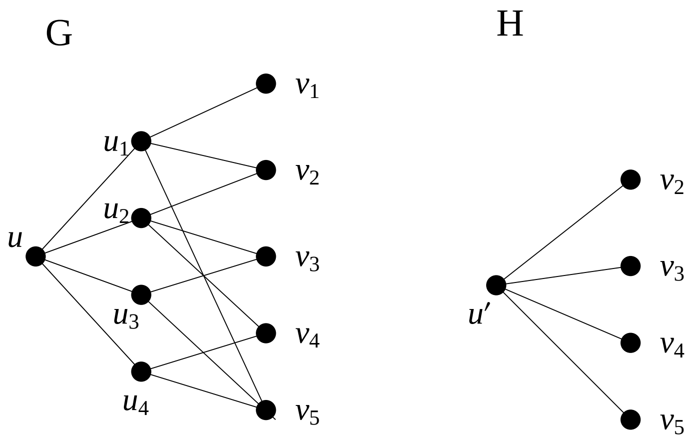

Figure 13: graph 1.2.4

Consider $H = G/\{u_1, u_2, u_3, u_4\} - \{v_5, u\}$, and let u' denote the vertex resulted from the contraction. Since $N_1(v_5) \nsubseteq N_1(u)$, we may let $w_5 \in N_1(v_5) - \{u_1, u_3, u_4\}$. Since G is triangle-free, $w_5 \notin \{u\} \cup N_1(u) \cup N_2(u)$.

Claim 1. $v_1 \nsim v_3$ or $v \nsim v_4$.

Suppose $v_1 \sim v_3$ and $v_1 \sim v_4$. If v_1 has a neighbor, say w_1, different from u_1, v_3, v_4, then $(v_1v_3u_3, v_1v_4u_4, v_1u_1, v_1w_1)$ is a fork in G, a contradiction. So $v_1 \in V_3(G)$.

If $w_5 \nsim v_3$ and $w_5 \nsim v_4$ then $(v_5u_4v_4, v_5u_3v_3, v_5u_1, v_5w_5)$ would be a fork in G. So $w_5 \sim v_3$ or $w_5 \sim v_4$. Moreover, $w_5 \nsim v_3$ or $w_5 \nsim v_4$; as otherwise, $\{v_2, w_5\}$ would be a 2-cut in G. Then, $w_5 \sim v_2$; for, otherwise, $\{v_2, v_4, w_5\}$ or $\{v_2, v_3, w_5\}$ is an independent cut in G, contradicting Lemma 7.5. So $w_5 \sim v_2$.

Suppose $w_5 \sim v_3$; then $w_5 \nsim v_4$. Then there exists $w'_5 \in N_1(w_5) - \{v_2, v_3, v_5\}$, otherwise $\{v_2, v_4\}$ would be 2-cut in G. Now $(w_5v_3v_1, w_5v_5u_4, w_5v_2, w_5w'_5)$ is a fork in G, a contradiction.

Similarly, if $w_5 \sim v_4$ then $w_5 \nsim v_3$, and there exists $w'_5 \in N_1(w_5) - \{v_2, v_4, v_5\}$; hence $(w_5v_4v_1, w_5v_5u_3, w_5v_2, w_5w'_5)$ is a fork in G, a contradiction.

Claim 2. $v_1 \nsim v_3$ and $v_1 \nsim v_4$.

Note the symmetry between v_3 and v_4; so it suffices to prove $v_1 \nsim v_3$. Suppose for a contradiction that $v_1 \sim v_3$. Then $v_1 \nsim v_4$ by Claim 1.

First, assume $w_5 \sim v_3$. Then $w_5 \sim v_4$ to avoid the fork $(v_3v_1u_1, v_3u_2v_4, v_3u_3, v_3w_5)$. Moreover, there exists $w'_5 \in N_1(w_5) - N_2(u)$; otherwise $H - \{w'_5, v_3\}$ is triangle-free and fork-free, and $\chi(H - \{w'_5, v_3\}) \leq 3$, which implies $\chi(G) \leq 3$, a contradiction. On the other hand, $w'_5 \sim v_i$ for $i = 1, 2$, as otherwise $(v_5u_1v_i, v_5w_5w'_5, v_5u_3, v_5u_4)$ would be a fork in G. If $w'_5 \in V_3(G)$ then $\{v_1, v_2, v_4\}$ is an independent cut in G, contradicting to Lemma 7.5. So $w'_5 \in V_4(G)$, and let $w''_5 \in N_1(w_5) - \{v_1, v_2, w_5\}$. Then $(w'_5v_2u_2, w'_5w_5v_5, w'_5v_1, w'_5w''_5)$ is a fork in G, a contradiction.

So $w_5 \nsim v_3$. Then $w_5 \sim v_2$ to avoid the fork $(v_5u_1v_2, v_5u_3v_3, v_5u_4, v_5w_5)$, and $w_5 \sim v_4$ to avoid the fork $(v_5u_3v_3, v_5u_4v_4, v_5u_1, v_5w_5)$. We may assume that there exist $w'_5 \in N_1(w_5) - N_2(u)$; otherwise, $H' := H - \{w_5, v_4\}$ is triangle-free and fork-free, and so $\chi(H') \leq 3$, which implies $\chi(G) \leq 3$, a contradiction. Now $w_5 \sim v_1$ to avoid the fork $(v_5w_5w'_5, v_5u_1v_1, v_5u_3, v_5u_4)$. But then $(w_5w'_5v_1, w_5v_5u_3, w_5v_2, w_5v_4)$ is a fork in G, a contradiction.

By Claim 2, H is triangle free. Then by the choice of G, H must contain a fork, say F. If there exists a permutation ijk of $\{2, 3, 4\}$ such that $F = (u'v_iw_i, u'v_jw_j, u'v_k, u'v_1)$ then $(u'v_iw_i, u'v_jw_j, u'v_k, u'u)$ would be a fork in G. So there exists $i \in \{2, 3, 4\}$ such that $F = (u'v_1w_1, u'v_iw_i, u'v_j, u'v_k)$, where $\{j, k\} = \{2, 3, 4\} - \{i\}$.

Suppose $i = 2$. Then $v_5 \in w_p$ for some $p \in \{1, 2\}$, to avoid the fork $(u_1v_1w_1, u_1v_2w_2,$

$u_1u, u_1v_5)$. Then $(v_5u_4v_4, v_5u_1v_{3-p}, v_5u_3, v_5w_p)$ is a fork in G, a contradiction.

So $i \neq 2$. Note the symmetry between v_3 and v_4. So we may assume $i = 3$. If $w_5 = w_p$ for some $p \in \{1, 3\}$ then $(v_5u_4v_4, v_5w_5w_p, v_5u_1, v_5u_3)$ would be a fork in G; so $w_5 \notin \{w_1, w_3\}$.

If $N(w_5) \subseteq N_2(u)$ then let $w_5 \sim w_p$ for some $p \in \{2, 3, 4\}$; and hence $H' := H - \{w_5, v_p\}$ is both triangle-free and fork-free, and $\chi(H') \leq 3$ implies $\chi(H) \leq 3$ (and so $\chi(G) \leq 3$), a contradiction.

So w_5 is adjacent to at most two of $\{v_1, v_2, v_3, v_4\}$. If $w_5 \nsim v_3$ and $w_5 \nsim v_4$ then $(v_5u_3v_3, v_5u_4v_4, v_5u_1, v_5w_5)$ would be a fork in G; if $w_5 \nsim v_1$ and $w_5 \nsim v_3$ then $(v_5u_1v_1, v_5u_3v_3, v_5u_4, v_5w_5)$ would be a fork in G; if $w_5 \nsim v_1$ and $w_5 \nsim v_4$ then $(v_5u_1v_1, v_5u_4v_4, v_5u_3, v_5w_5)$ would be a fork in G; if $w_5 \nsim v_2$ and $w_5 \nsim v_3$ then $(v_5u_1v_2, v_5u_3v_3, v_5u_4, v_5w_5)$ would be a fork in G; if $w_5 \nsim v_2$ and $w_5 \nsim v_4$ then $(v_5u_1v_2, v_5u_4v_4, v_5u_3, v_5w_5)$ would be a fork in G. So $N(w_5) \cap \{v_1, v_2, v_3, v_4\} = \{v_3, v_4\}$. Then $w_5 \sim w_1$ to avoid the fork $(u_1v_1w_1, u_1v_5w_5, u_1u, u_1v_2)$. But then $(w_5v_3w_3, w_5w_1v_1, w_5v_4, w_5v_5)$ is a fork in G, a contradiction. □

If two of u_2, u_3, u_4 has degree 4, the structure between (u_2, u_3, u_4) and (v_2, v_3, v_4, v_5) is fixed, as in Figure 14:

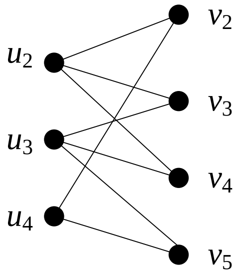

Figure 14: graph H 1.2.5

then the degree of u_1 cannot be 3, otherwise we can identify (u_2, u_3, u_4) as u', delete u, u_1 and the another vertex adjacent to u_1, the new graph is fork free, triangle free, i.e. G is reducible, contradiction.

When u_1 has degree 4, since v_2, v_5 are symmetric, v_3, v_4 are symmetric, there are 3 types of graph by symmetry:

Subcase 1.5: u_1 is adjacent to v_2, v_3 in Figure 14: For this graph, we can identify

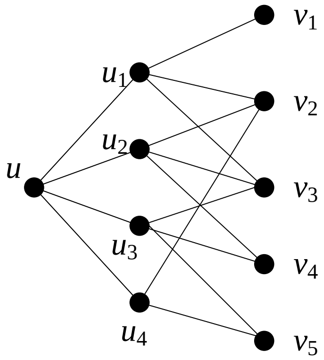

Figure 15: graph G 1.2.5

u_2, u_3, u_4 as u', delete u, denote the new graph by H, obviously H is triangle free then the only potential fork in H is the one centered on u' with long arms $(u'v_2w_2, u'v_5w_5)$(here, w_2 is the vertex adjacent to v_2, w_5 is the vertex adjacent to v_5), other forks in H can be converted to a fork in G. However, this potential fork doesn't exist, otherwise there is a fork in G: $(v_2u_4v_5, v_2u_2v_4, v_2u_1, v_2w_2)$. Thus H is fork free triangle free, i.e G is reducible.

Subcase 1.6: u_1 is adjacent to v_3, v_4 in Figure 14: In this graph, since $N_1(v_3) \not\subseteq N_1(u)$,

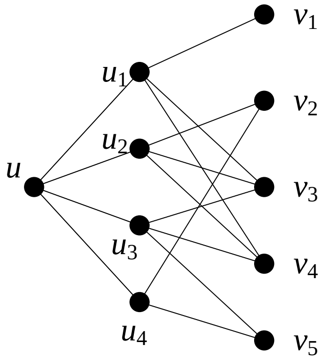

Figure 16: graph 1.2.6

$N_1(v_4) \not\subseteq N_1(u)$, and $N_1(v_3) \neq N_1(v_4)$, v_3 must have the fourth neighbor w_3, v_4 must have the fourth neighbor w_4, then v_1 must be adjacent to w_3, w_4 to forbid fork centered on u_1 with one long arm u_1uu_4. Then let's identify u_2, u_3, u_4 as u', delete u, denote the new graph by H, obviously, H is triangle free, we can see that the potential new fork in H must be centered on u', since v_2, v_3, v_4, v_5 have degree at most 3 in H.

Claim 1 There is no fork centered on u' in H.

Proof. suppose not, then two long arms of the fork must be $u'v_2w_2, u'v_5w_5$(w_2 is adjacent to v_2, w_5 is adjacent to v_5), otherwise, it's also a fork in G centered on u_2/u_3. If two long arms of the fork are $u'v_2w_2, u'v_5w_5$, v_1 cannot be adjacent to v_2(here v_1 has at most one more neighbor since $deg(v_1) \leq 4$), otherwise there is a fork $(v_2u_2v_3, v_2u_4v_5, v_2v_1, v_2w_2)$; also, v_1 cannot be adjacent to v_5 by symmetry. Thus, if we identify u_1, u_2, u_3, u_4 in G, delete u, v_3, v_4, the new graph is fork free triangle free, i.e. G is reducible, contradiction.

□

By *Claim 1* and previous analysis, H is fork free, triangle free, i.e. G is reducible.

Subcase 1.7: u_1 is adjacent to v_2, v_5 in Figure 14: For this graph, we can apply exact same argument with *Subcase 2.5*

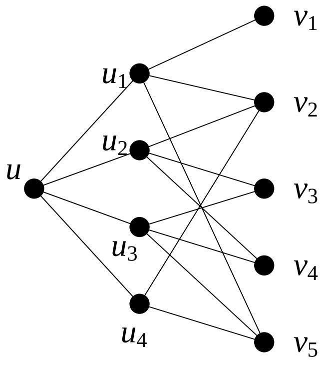

Figure 17: graph 1.2.7

Case 3: $|N_2(u)| = 6$, $N_2(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $v_1 \in S_1(u)$ then we have several subcases depending on the degree of $N_1(u)$:

If $u_2, u_3, u_4 \in V_4(G)$, then $G[\{u_2, u_3, u_4, v_2, v_3, v_4, v_5, v_6\}]$ is isomorphic to H_1 or H_2 in Figure 18. Notice that in H_2, v_3 and v_4 are symmetric, and v_5 and v_6 are symmetric.

Subcase 3.1: $G[\{u_2, u_3, u_4, v_2, v_3, v_4, v_5, v_6\}] \cong H_1$.

Then, since $S_1(u) = \{v_1\}$, $u_1 \sim v_2$ and $u_1 \sim v_6$. See Figure 19.

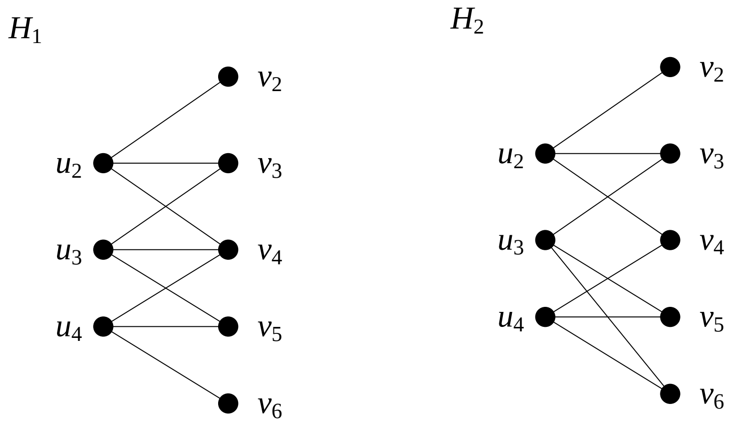

Figure 18: Two Graphs

Claim 1: $v_3 \nsim v_6$, and $v_2 \nsim v_5$ Note the symmetry between v_2 and v_6 and the symmetry between v_3 and v_5. So we need only prove $v_3 \nsim v_6$. Suppose, for a contradiction, that $v_3 \sim v_6$. Then $v_3 \nsim v_1$, to avoid the fork $(v_3u_2v_2, v_3v_6u_4, v_3u_3, v_3v_1)$.

Assume $v_3 \in V_4(G)$, and let $w_3 \in N_1(v_3) - \{u_2, u_4, v_6\}$. Then $w_3 \sim v_2$, to avoid the fork $(v_3v_6u_4, v_3u_2v_2, v_3u_3, v_3w_3)$; hence $w_3 \sim v_1$ as otherwise $(u_1v_2w_3, u_1uu_3, u_1v_1, u_1v_6)$ would be a fork in G. But then $(v_3v_6u_4, v_3w_3v_1, v_3u_2, v_3u_3)$ is a fork in G, a contradiction.

So $v_3 \in V_3(G)$. Then $v_2 \nsim v_5$, or else there would be a fork $(u_3v_3v_6, u_3v_5v_2, u_3u, u_3v_4)$ in G; and hence $v_1 \nsim v_5$, as otherwise $(u_1v_1v_5, u_1v_6v_3, u_1v_2, u_1u)$ would be a fork in G. Then, $H := G/\{u_1, u_2, u_3, u_4\} - \{u, v_3, v_4, v_6\}$ is triangle-free and fork-free, and $\chi(H) \leq 3$ implies $\chi(G) \leq 3$. However, the choice of G implies $\chi(H) \leq 3$, a contradiction.

Now $v_4 \nsim v_1$, for otherwise $(v_4u_4v_6, v_4u_2v_2, v_4u_3, v_4v_1)$ would be a fork in G. Since $N_1(v_4) \not\subseteq N_1(u)$, we have $v_4 \in V_4(G)$; so let $w_4 \in N_1(v_4) - \{u_2, u_3, u_4\}$. Then since G is triangle-free and $v_4 \nsim v_1$, $v_2 \notin N_1(u) \cup N_2(u) \cup \{u\}$. So $w_4 \sim v_3$ or $w_4 \sim v_5$ to avoid the fork $(u_3uu_1, u_3v_4w_4, u_3v_3, u_3v_5)$. So by symmetry between v_3 and v_5, we may assume that $w_4 \sim v_5$.

Also $w_4 \sim v_2$ or $w_4 \sim v_6$, to avoid the fork $(v_4u_2v_2, v_4u_4v_6, v_4u_3, v_4w_4)$. So by symmetry between v_2 and v_6, we may assume $w_4 \sim v_6$. Then $w_4 \nsim v_1$ to avoid the fork $(v_4u_2v_2, v_4w_4v_1, v_4u_3, v_4u_4)$; hence $w_4 \sim v_2$ to avoid the fork $(u_1uu_3, u_1v_6w_4, u_1v_1, u_1v_2)$. Now

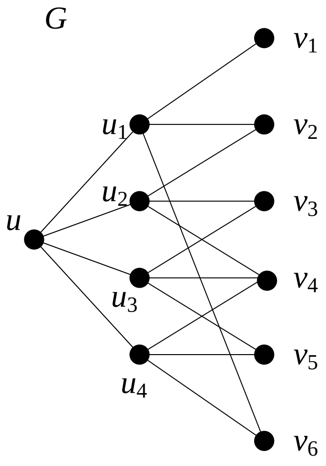

Figure 19: graph 1.3.1

$H := G/\{u_2, u_3, u_4\} - \{u, w_4, v_4, v_5\}$ is triangle-free ($v_3 \nsim v_6$ by Claim 1) and fork-free. So $\chi(H) \leq 3$ by the choice of G, which implies $\chi(G) \leq 3$, a contradiction.

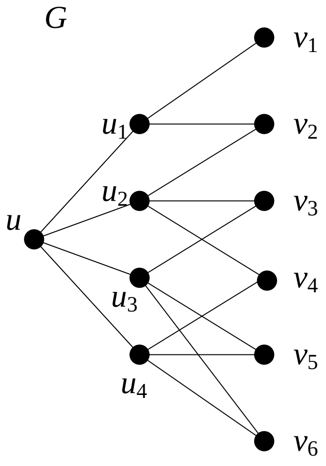

Figure 20: graph 1.3.2

Subcase 3.2. $G[\{u_2, u_3, u_4, v_2, v_3, v_4, v_5, v_6\}]$ is the graph in Figure 20, where $u_1 \in V_3(G)$ and $u_1 \sim v_2$.

Notice that there is symmetry between v_5 and v_6, and between v_3 and v_4. Let $v \in N_1(\{v_3, v_4, v_5, v_6\}) - \{v_1\}$. Then $v \neq v_2$ since G is triangle-free.

First, suppose $v \sim v_i$ for some $i \in \{5, 6\}$. Then $v \sim v_k$ for $k = 5, 6$, or $v \sim v_j$ for $j = 3, 4$. Otherwise, by symmetry between v_5 and v_6, we may assume $v \sim v_5$ and $v \sim$

v_6. Then $v \sim v_3$ to avoid the fork $(u_3uu_1, u_3v_5v, u_3v_3, u_3v_6)$, and $v \sim v_4$ to avoid the fork $(u_4uu_1, u_4v_5v, u_4v_4, u_4v_6)$.

Now suppose $v \sim v_i$ for some $i \in \{3, 4\}$. By symmetry between v_3 and v_4, assume $v \sim v_3$. Then $v \sim v_k$ for some $k \in \{5, 6\}$, to avoid the fork $(u_3uu_1, u_3v_3v, u_3v_5, u_3v_6)$. So by the conclusion in the above paragraph, we have $|N_1(v) \cap \{v_3, v_4, v_5, v_6\}| \geq 3$.

From the conclusions of the above two paragraphs, we derive that $|N_1(\{v_3, v_4, v_5, v_6\}) - \{v_1\}| \leq 2$ and $N_1(\{v_3, v_4, v_5, v_6\}) - \{v_1\}$ is independent in G.

Since G is 3-connected, let $N_1(\{v_3, v_4, v_5, v_6\}) - \{v_1\} = \{w_1, w_2\}$.

We claim that $N_1(\{v_3, v_4, v_5, v_6\})$ must be independent in G. Otherwise, $v_1 \sim v_i$ for some $i \in \{3, 4, 5, 6\}$, and $v_1 \sin w_1$ or $v_1 \sim w_2$. Without loss of generality, assume $v_1 \sim w_1$. Suppose $v_1 \sim v_3$. Then $w_1 \nsim v_3$, and hence $w_1 \sim v_5$ and $v \sim v_6$. (More details.)

Now $H := G/\{v_3, v_4, v_5, v_6\} - (\{u, v_2\} \cup N_1(u))$ is triangle-free and fork-free, and $\chi(H) \leq 3$ implies $\chi(G) \leq 3$. But by the choice of G, $\chi(H) \leq 3$, a contradiction.

Subcase 3.3: $G[\{u_2, u_3, u_4, v_2, v_3, v_4, v_5, v_6\}]$ is the graph in Figure 21.

In this case $u_1 \sim v_3$. Since $N_1(v_3) \nsubseteq N_1(u)$, let $w_3 \in N_1(v_3) - N_1(u)$. Since G is triangle-free, $w_3 \notin N_1(u) \cup N_2(u) \cup \{u\}$.

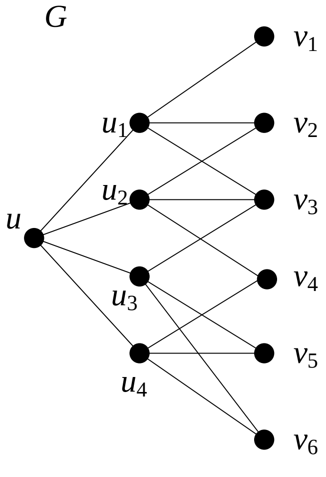

Figure 21: graph1.3.3

We claim that $w_3 \sim v_i$ for some $i \in \{1, 2\}$; as otherwise $(u_1uu_4, u_1v_3w_3, u_1v_1, u_1v_2)$ would be a fork in G. Also $w_3 \sim v_j$ for some $j \in \{5, 6\}$; or else $(u_3uu_1, u_3v_3w_3, u_3v_5, u_3v_6)$ would be a fork in G. Now $w_3 \sim v_4$ or $w_3 \sim v_{11-j}$, to avoid the fork $(u_4uu_1, u_4v_6w_3, u_4v_4, u_4v_5)$. This implies $w_3 \in V_4(G)$ and $N_1(w_3) \subseteq N_2(u)$.

Further, $w_3 \sim v_1$. For, suppose $w_3 \nsim v_1$. Then $N_1(w_3) \subseteq \{v_2, v_3, v_4, v_5, v_6\}$; $H := G/u_2, u_3, u_4 - \{u, w_3, v_3, u_1, v_2\}$ is both triangle-free and fork-free. So by the choice of G, $\chi(H) \leq 3$. However, this implies $\chi(G) \leq 3$, a contradiction.

So $N_1(w_3)$ is one of the following: $\{v_1, v_3, v_4, v_5\}$, $\{v_1, v_3, v_4, v_6\}$, or $\{v_1, v_3, v_5, v_6\}$.

Suppose $N_1(w_3) = \{v_1, v_3, v_4, v_5\}$ or $N_1(w_3) = \{v_1, v_3, v_4, v_6\}$. By symmetry between v_5 and v_6, we only consider $N_1(w_3) = \{v_1, v_3, v_4, v_6\}$. If $v_5 \nsim v_1$ and $v_5 \nsim v_2$, then $H := G/\{u_1, u_2, u_3, u_4\} - \{u, w_3, v_3, v_4, v_6\}$ is both triangle-free and fork-free; so $\chi(H) \leq 3$ by the choice of G, which implies $\chi(G) \leq 3$, a contradiction. Suppose $v_1 \sim v_5$. If $v_1 \in V_3(G)$ then $H := G - N_1(u) \cup \{u, w_3, v_1, v_3, v_4, v_6\}$ is both triangle-free and fork-free; so $\chi(H) \leq 3$ by the choice of G, which implies $\chi(G) \leq 3$, a contradiction. So $v_1 \in V_4(G)$ and let $w_1 \in N_1(v_1) - \{u_1, v_5, w_3\}$. Then $w_1 \sim v_4$ to avoid the fork $(v_1v_5u_3, v_1w_3v_4, v_1w_1, v_1u_1)$; $w_1 \sim v_2$ to avoid the fork $(u_1uu_4, u_1v_1w_1, u_1v_2, u_1v_3)$, and $w_1 \sim v_5$ or $w_1 \sim v_6$ to avoid the fork $(u_4uu_1, u_4v_4w_1, u_4v_5, u_4v_6)$. Then $H := G - N_1(u) \cup \{u, v_1, v_3, v_4, v_6, w_3\}$ is triangle-free and fork-free; and hence $\chi(H) \leq 3$ by the choice of G, which implies $\chi(G) \leq 3$, a contradiction. So we have shown $v_1 \nsim v_5$. Apply the very similar argument we can prove that $v_2 \nsim v_5$.

The case $N_1(w_3) = \{v_1, v_3, v_4, v_5\}$ is similar by the symmetry between v_5 and v_6

Finally assume $N_1(w_3) = \{v_1, v_3, v_5, v_6\}$. We can apply very similar argument above to show that G is reducible, contradiction.

Subcase 3.4: u_1 is also adjacent to v_5 in Figure 20. The graph is in Figure 22.

If one of u_2, u_3, u_4 has degree 3, then there are 2 cases by symmetry.

Subcase 3.5:

Lemma 7.7. *In this graph, for any vertex x, x cannot be adjacent to four of* $(v_2, v_3, v_4, v_5, v_6)$.

Proof. Suppose not, then we can identify $N_1(u)$ as u', delete u, x, $N_1(x)$ in order. Notice

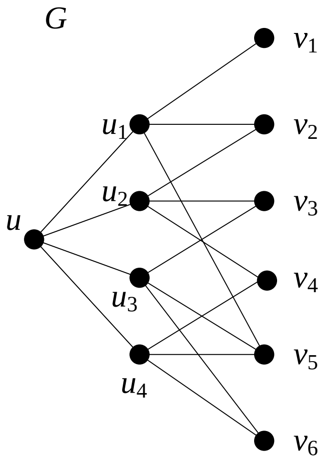

Figure 22: graph 1.3.4

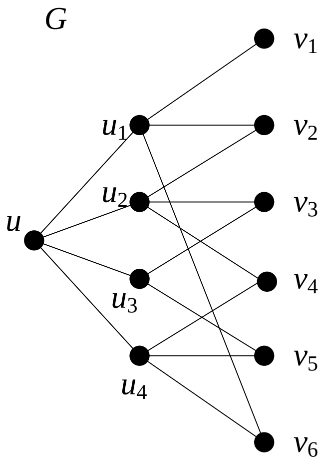

Figure 23: graph 1.3.5

that after we delete x, every vertex in $N_1(x)$ has at most degree 2, thus we can delete them. Now, u' also has degree 2, we can delete it to destroy possible triangle. Then the new graph is fork free, triangle free, i.e. G is reducible, contradiction. □

First, notice that in this graph, $v_1 \nsim v_4$, otherwise, there is a fork $u_1uu_3, u_1v_1v_4, u_1v_2, u_1v_6$ in G, contradiction.

Second, notice that in this graph, v_2, v_6 are symmetric, v_3, v_5 are symmetric. Also, u_1 has degree 4, and u_1 has a potential long arm which cannot be forbidden: u_1uu_3, thus we can start the proof by exhausting all the possible structure on $N_2(u_1)$:

Case 1: $v_2 \sim v_5, v_3 \sim v_6$. It's impossible since there is a fork $(u_4v_5v_2, u_4v_6v_3, u_4v_4, u_4u)$. Notice that here $v_2 \nsim v_4, v_6$, $v_3 \nsim v_4, v_5$ since triangle free.

Case 2: v_2, v_6 has degree 3, and $v_2 \sim v_5$, the third neighbor of v_6 is w_6, then $w_6 \sim v_1$ since here $u_1v_6w_6$ is another potential long arm of u_1. Then $w_6 \sim v_4$ or v_5 to forbid the fork $(u_4v_5v_2, u_4v_6w_6, u_4v_4.u_4u)$.

If $w_6 \sim v_5$, then v_5 has degree 4 and has 3 potential long arms $(v_5v_2u_1, v_5u_4v_4, v_5u_3v_3)$, w_6 must be adjacent to both v_3, v_4 to forbid the fork centered on v_5, i.e. w_6 must have degree 5, contradiction to our assumption. Thus $w_6 \nsim v_5$.

If $w_6 \sim v_4$, $w_6 \sim v_3$ to forbid the fork $(u_2v_2v_5, u_2v_4w_6, u_2v_3, u_2u)$. Then we can identify u_1, u_2, u_3, u_4 as u', delete $u, w_6, v_2, v_3, v_4, v_5, v_6$, the new graph is fork free, triangle free, i.e. G is reducible, contradiction.

Case 3: v_2, v_6 has degree 3, and $v_6 \sim v_3$, the third neighbor of v_2 is w_2. This case is symmetric with Case 2.

Case 4: v_2, v_6 has degree 3, and the third neighbor of v_2 is w_2, the third neighbor of v_6 is w_6, then $v_1 \sim w_2, w_6$ to forbid the fork centered on u_1.

Claim 1: $v_1 \nsim v_3$, by symmetry, $v_1 \nsim v_5$.

Suppose not, v_3 must have degree 4, otherwise, we can identify u_1, u_2, u_3, u_4, delete u, v_2, v_3, v_6, the new graph is fork free, triangle free, i.e. G is reducible. Denote the fourth degree of v_3 by w_3, then $w_3 \sim v_4, v_5$ to forbid the forks $(v_3v_1u_1, v_3u_2v_4, v_3u_3, v_3w_3)$ and

$(v_3v_1u_1, v_3u_3v_5, v_3u_2, v_3w_3)$, and $w_3 \sim w_2$ or w_6 to forbid the fork $(v_1v_3w_3, v_1u_1u, v_1w_6, v_1w_2)$.

If $w_3 \sim w_6$, then let's show that v_4 must have degree 3. $v_4 \nsim w_6$ since triangle free, and $v_4 \nsim w_2$, otherwise there is a fork $(v_4w_2v_1, v_4u_4v_6, v_4u_2, v_4w_3)$; denote the fourth neighbor of v_4 by w_4, then there is a fork $(v_4u_2v_2, v_4u_4v_6, v_4w_3, v_4w_4)$, contradiction. Next, let's show that v_5 also has degree 3. $v_5 \nsim w_6$ since triangle free, and $v_5 \nsim w_2$, otherwise, either there is a two cut set (w_2, w_6), or this is the graph G, and G is 3-colorable, contradiction. denote the fourth neighbor of v_5 by w_5, then $w_5 \sim w_6$ to forbid the fork $(u_4v_5w_5, u_4v_6w_6, u_4v_4, u_4u)$, then (w_2, w_5) is a 2-cut set, contradiction. Now, we have shown that v_4, v_5 have degree 3, then we can identify u_1, u_2, u_3, u_4 as u', delete $u, v_1, v_4, v_5, v_6, u'$, the new graph is fork free, triangle free, i.e. G is reducible, contradiction. Thus $w_3 \nsim w_6$.

If $w_3 \sim w_2$, we can apply very similar argument as above, prove that v_4, v_5 have degree 3, and G is reducible, contradiction.

Claim 2: $v_1 \nsim v_4$

Suppose not, v_4 must have degree 4, otherwise, we can identify u_1, u_2, u_3, u_4, delete u, v_2, v_4, v_6, G is reducible. Also, the fourth neighbor of v_4 cannot be w_2 or w_6 since triangle free, then there is a fork centered on v_4 with long arms $(v_4u_2v_2, v_4u_4v_6)$, contradiction.

Now we have shown that v_1, v_3, v_4, v_5 are independent. Also, v_1 must have degree 3, otherwise, denote the fourth neighbor of v_1 by w_1, there is a fork $(u_1uu_1, u_1v_1w_1, u_1v_2, u_1v_6)$, contradiction.

Claim 3: v_4 has degree 3.

Suppose not, there are 4 situations:

If $v_4 \sim w_2$, and the fourth neighbor is w_4, there is a fork $(v_4w_2v_1, v_4u_4v_6, v_4w_4, v_4u_2)$, contradiction;

If $v_4 \sim w_6$, and the fourth neighbor is w_4, there is a fork $(v_4w_6v_1, v_4u_2v_2, v_4w_4, v_4u_4)$, contradiction;

If $v_4 \sim w_2, w_6$, then w_2 or w_6 must be adjacent to v_3 or v_5 to forbid the fork $(v_4u_2v_3, v_4u_4v_5, v_4w_2, v_4w_6)$. However, for example, if $w_2 \sim v_3$, then we can identify $N_1(u)$, delete $u, w_2, v_2,$

v_3, v_4, then new graph is fork free, triangle free, contradiction(notice that we have shown that v_1, v_5, v_6 are independent); similarly, if $w_2 \sim v_5$ or $w_6 \sim v_3/v_5$, we can apply the same argument which leads to a contradiction.

Otherwise, denote the third and fourth neighbor by w_4, w_4', there is a fork $(v_4u_2v_2, v_4u_4v_6, v_4w_4, v_4w_4')$, contradiction.

Thus, in graph G, v_2, v_4, v_6 have degree 3, and v_1, v_3, v_5 are independent, we can identify u_1, u_2, u_3, u_4, delete u, v_2, v_4, v_6, the new graph is fork free, triangle free, i.e. G is reducible, contradiction.

Case 5: $v_2, v_6 \in V_3$, and v_2, v_6 have a common neighbor w.

In this case, the $N_1(v_1) - u_1 \subset (v_3, v_5, w)$, otherwise there will be a fork centered on u_1 with long arms (u_1uu_3, u_1v_1*). First, v_1 cannot be adjacent to all of (v_3, v_5, w), otherwise there is a fork $(v_1v_3u_2, v_1v_5u_4, v_1w, v_1u_1)$. That means $v_1 \in V_3$, thus there are 2 possibilities:

Subcase 1: $v_1 \sim w, v_3$, or $v_1 \sim w, v_5$. Here, $w \in V_4$, otherwise $N_1(w) \subset N_1(u_1)$, contradictory to our assumption. However, the fourth neighbor of w must be one of v_3, v_4, v_5, otherwise there is a fork centered on w, with long arms (wv_2u_2, wv_6u_4). Then we can identify $N_1(u)$, delete $u, w, N_1(w)$(notice that here we can delete v_1 since $v_1 \in V_3$), the new graph is fork free, triangle free, contradiction.

Subcase 2: $v_1 \sim v_3, v_5$. Now, let's prove that $v_3 \in V_3$. First $v_3 \nsim w$, otherwise $w \sim v_4$ to forbid the fork $(v_3v_1u_1, v_3u_2v_4, v_3u_3, v_3w)$, but it's impossible by Lemma 7.3. Now let's denote the fourth neighbor of v_3 by w_3, then $w_3 \sim v_4$ to forbid the fork $(v_3v_1u_1, v_3u_2v_4, v_3u_3, v_3w_3)$; and $w_3 \sim v_5$ to forbid the fork $(u_4v_5v_1, u_4v_4w_3, u_4v_6, u_4u)$; and $w \sim v_4$ to forbid the fork $(u_2v_2w, u_2v_3v_1, u_2v_4, u_2u)$. Now, either this is the graph G, which is 3-colorable; or w_3, w is a 2-cut set, contradiction.

Thus, we have shown that $v_3 \in V_3$, we can identify $N_1(u)$, delete u, v_2, v_3, v_6, v_1 in order, notice that after we delete v_3, v_1 has degree 2, and can be deleted. The new graph is fork free, triangle free, contradiction.

From Case 1 $\sim$ 5, we know that at least one of v_2, v_6 has degree 4.

Case 6: v_2 has degree 4, and $v_2 \sim v_5$, the fourth neighbor of v_2 is w_2.

In this case, $w_2 \sim v_3$ to forbid the fork $(v_2v_5u_4, v_2u_2v_3, v_2w_6, v_2u_1)$; also, $w_2 \sim v_4$ to forbid the fork $(v_2v_5u_3, v_2u_2v_4, v_2w_6, v_2u_1)$. Then $w_2 \in V_4$, otherwise $N_1(w_2) \subset N_1(u_2)$, contradictory to our assumption. However, by Lemma 7.3, $w_2 \nsim v_6$, denote the fourth neighbor of w_2 by w_2', then there is a fork centered on w_2: $(w_2v_3u_3, w_2v_4u_4, w_2v_2, w_2w_2')$, contradiction.

That means, if $v_2 \in V_4$, $v_2 \nsim v_5$. By symmetry, if $v_6 \in V_4$, $v_6 \nsim v_3$.

Case 7: $v_2 \in V_4$, $v_6 \in V_3$, v_2, v_6 has a common neighbor w_6, and the fourth neighbor of v_2 is w_2.

In this case, $v_1 \sim w_2$ to forbid the fork centered on u_1. Other neighbors of v_1 must be in (w_6, v_3, v_5),(recall that $v_1 \nsim v_4$) otherwise there will be a fork centered on u_1.

Claim 1: $v_1 \nsim w_6$.

Suppose not, $v_1 \sim w_6$, then $w_6 \in V_4$, otherwise, $N_1(w_6) \subset N_1(u_1)$. If the fourth neighbor of w_6 is v_3, then there is a fork $(w_6v_3u_3, w_6v_6u_4, w_6v_2, w_6v_1)$, contradiction; if the fourth neighbor of w_6 is v_5, then there is a fork $(w_6v_5u_3, w_6v_2u_2, w_6v_5, w_6v_1)$, contradiction; if the fourth neighbor of w_6 is some other vertex x, then there is a fork $(w_6v_2u_2, w_6v_6u_4, w_6v_1, w_6x)$, contradiction. Thus, the only possible is, $w_6 \sim v_4$.

Then, we claim that $v_1 \nsim v_3$. Suppose not, $w_2 \sim v_4$ to forbid the fork $(v_1v_3u_3, v_1w_6v_4, v_1u_1, v_1w_2)$, then $w_2 \sim v_5$ to forbid the fork $(v_4u_2v_3, v_4u_4v_5, v_4w_2, v_4w_6)$(notice that here w_6 is saturated, $w_2 \nsim v_3$ since triangle free). Then we can identify $N_1(u)$, delete $u, w_2, w_6, v_1, v_2, v_4, v_5, v_6$ in order, the new graph is fork free, triangle free, contradiction. Thus, $v_1 \nsim v_3$, we can apply very similar argument to show that $v_1 \nsim v_5$. That means v_1, v_3, v_5 are independent.

Now we can identify $N_1(u)$ as u', delete u, w_6, v_2, v_4, v_6 in order, the new graph is fork free since $u' \in V_3$, and triangle free, since v_1, v_3, v_5 are independent, contradiction. Thus $v_1 \nsim w_6$.

By our assumption, v_1 has at least 3 degree, thus v_1 need to be adjacent to at least one of v_3, v_5. At the same time, v_1 cannot be adjacent to both v_3, v_5, otherwise there is a fork $(v_1v_3u_2, v_1v_5u_4, v_1u_1, v_1w_2)$, contradiction. Thus there are 2 subcases:

Subcase 1: $v_1 \sim v_3, v_1 \in V_3$. Here, $w_6 \nsim v_3$, otherwise, $v_3 \in V_4$, and has one long arm that cannot be forbidden: $v_3v_1u_1$, and two potential long arms: $v_3u_2v_4, v_3u_3v_5$, that means w_6 need to be adjacent to both v_4, v_5 to forbid the fork centered on v_3, the degree of w_6 is 5, contradictory to our assumption. Thus $w_6 \nsim v_3$, then $w_6 \sim v_4$ to forbid the fork $(u_2v_3v_1, u_2v_2w_6, u_2v_4, u_2u)$.

Next, let's show that $v_3 \in V_3$. Suppose not, $v_3 \nsim w_2$ since triangle free, $v_3 \nsim w_6$ by Lemma 7.3, denote the fourth neighbor of v_3 by w_3, then $w_3 \sim v_4, v_5$, since v_3 has one long arm that cannot be forbidden: $v_3v_1u_1$, and two potential long arms $(v_3u_2v_4, v_3u_3v_5)$. Here, $w_3 \in V_4$, otherwise, we can identify $N_1(u)$, delete $u, w_3, v_3, v_4, v_5, v_6$ in order, the new graph is fork free, triangle free, contradiction. Then let's look at the fourth neighbor of w_3, $w_3 \nsim w_6$ since triangle free. We also claim that $w_3 \nsim w_2$. Suppose $w_2 \sim w_3$, then either $w_2 \in V_3$, or the fourth neighbor of w_2, denote by x, is adjacent to v_5 to forbid the fork $(w_2w_3v_5, w_2v_2u_2, w_2v_1, w_2x)$, in either case, there is a two cut set $(w_6, v_5/x)$, or this is the graph G, but it's 3-colorable. Thus $w_3 \nsim w_2$. Let's denote the fourth neighbor of w_3 by m, then there is a fork $(v_3v_1u_1, v_3w_3m, v_3u_2, v_3u_3)$, contradiction.

We have shown that $v_3 \in V_3$, then we can identify $N_1(u)$, delete u, v_3, v_6, v_1 in order, the new graph is fork free, triangle free, contradiction.

Subcase 2: $v_1 \sim v_5$. We can apply very similar argument to show that $w_6 \sim v_4$ and $v_5 \in V_3$, then identify $N_1(u)$, delete u, v_5, v_6, v_1 in order, the new graph is fork free, triangle free, contradiction.

Thus, Case 7 is impossible.

Case 8: $v_2 \in V_4$, $v_6 \in V_3$, the third neighbor of v_6 is w_6, the third and fourth neighbor of v_2 is w_2, w_2'.

In this case, $v_1 \sim w_6, w_2, w_2'$ to forbid the fork centered on u_1, then v_1 is saturated, i.e.v_i, $i = 1...6$ are independent.

Claim 1: Any vertex x, cannot be adjacent to four of $N_2(u)$.

Suppose not, we can identify $N_1(u)$ as u', delete u, x, and at least 3 of $N_1(x)$.(if $v_1 \in N_1(x)$, we cannot delete v_1.) Then the graph is fork free, since u' has at most 3 degree; the new graph is also triangle free, since v_i are independent. That means G is reducible, contradiction.

Now look at v_2, there is a long arm on v_2 which cannot be forbidden: $v_2u_1v_6$; also, there are 2 potential long arms on v_2, $v_2u_2v_3/v_4$. There are 2 possible ways to forbid the fork centered on v_2. The first way is $w_2 \sim v_3, v_4$. It's impossible by *Claim 1*. By symmetry, w_2' cannot be adjacent to both v_3, v_4. The second way is $w_2 \sim v_3, w_2' \sim v_4$. If so, $w_6 \sim v_3$ or v_4 to forbid the fork $(v_1w_2v_3, v_1w_2'v_4, v_1u_1, v_1w_6)$. (recall that $w_2 \nsim v_4$, $w_2' \nsim v_3$ by *Claim 1*.). If $w_6 \sim v_3$, $w_6 \nsim v_4, v_5$, $w_2' \nsim v_5$ by *Claim 1*, $w_2' \nsim w_6$ since triangle free. then there is a fork $(u_4v_4w_2', u_4v_6w_6, u_4u, u_4v_5)$, contradiction. If $w_6 \sim v_4$, $w_6, w_2' \nsim v_3, v_5$ by *Claim 1*, then there is a fork $(v_4u_2v_3, v_4u_4v_5, v_4w_2', v_4w_6)$, contradiction.

Case 9: $v_2 \in V_4$, $v_6 \in V_3$, $v_6 \sim v_3$, the third and fourth neighbor of v_2 is w_2, w_2'.

In this case, $v_1 \sim w_2, w_2'$ to forbid the fork centered on u_1. Here, $w_2 \nsim v_3$, otherwise there is a fork $(v_3v_6u_4, v_3w_2v_1, v_3u_2, v_3u_3)$; by symmetry, $w_2' \nsim v_3$. Then, $v_4 \sim w_2, w_2'$ to forbid the forks $(u_2v_3v_6, u_2v_2w_2/w_2', u_2v_4, u_2u)$. Then $w_2 \sim v_5$ or $w_2' \sim v_5$ to forbid the fork $(v_4u_2v_3, v_4u_4v_5, v_4w_2, v_4w_2')$.(recall that we have shown $w_2., w_2' \nsim v_3$). By symmetry, we can only consider the case $w_2 \sim v_5$, that means after identify $N_1(u)$ as u', we can delete u, w_2, v_2, v_4, v_5, v_6, u' in order, the new graph is fork free, triangle free, contradiction.

From Case 1, and $6 \sim 9$, we know that both of v_2, v_6 have degree 4. From Case 6, we know that $v_2 \nsim v_5$, $v_6 \nsim v_3$.

Case 10: $v_2 \in V_4$, $v_6 \in V_4$, and they have common neighbors w_1, w_2.

In this case, the neighbor of v_1 must be in v_3, v_5, w_1, w_2, otherwise, there is a fork centered on u_1. (recall that there is a potential long arm u_1uu_3, and $v_1 \nsim v_4$).

Notice that in this graph w_1, w_2 are symmetric, v_3, v_5 are symmetric. Since v_1 has at least 3 neighbors, there are several possible cases by symmetry as follows:

Subcase 1. $v_1 \sim w_1, w_2$. Here, $w_1, w_2 \in V_4$, otherwise $N_1(w_1), N_1(w_2) \subset N_1(u_1)$. The fourth neighbor of w_1/w_2 must be one of (v_3, v_4, v_5), otherwise there is a fork centered on w_1/w_2 with long arms $(w_1/w_2v_2u_2, w_1/w_2v_6u_4)$. That means, after we identify u_1, u_2, u_3, u_4 as u', we can delete u, w_1, w_2, v_1(after delete w_1, w_2, v_1 has at most degree 2), v_2, v_6, and the fourth neighbor of w_1/w_2, then u' has at most degree 2, we can delete it, too. The new graph is fork free, triangle free, contradiction.

Subcase 2. $v_1 \sim w_1$. From subcase 1, we know that $v_1 \nsim w_2$. Now we claim that $v_1 \in V_3$. Suppose not, if $v_1 \sim v_3, v_5$, there is a fork $(v_1v_3u_2, v_1v_5u_4, v_1w_1, v_1u_1)$.

Thus $v_1 \in V_3$. Now look at w_1, $w_1 \in V_4$, otherwise $N_1(w_1) \subset N_1(u_1)$, contradiction to our assumption. However, the fourth neighbor of w_1 must be one of v_3, v_4, v_5, otherwise there is a fork centered on w_1 with long arms $(w_1v_2u_2, w_1v_6u_4)$. That means, after identify $N_1(u)$, we can delete u, w_1 and $N_1(w_1)$, (recall that here $v_1 \in V_3$, thus we can delete it.), then the new graph is fork free, triangle free, contradiction.

Subcase 2 shows that $v_1 \nsim w_1$, by symmetry, $v_1 \nsim w_2$. But v_1 has at least 3 degree, thus v_1 is and is only adjacent to both v_3, v_5.

Subcase 3: $v_1 \in V_3$, and $v_1 \sim v_3, v_5$. Here $w_1 \nsim v_3$, otherwise, to forbid the fork $(v_3v_1u_1, v_3u_3v_5, v_3u_2, v_3w_1)$, $w_1 \sim v_5$, but it's impossible by Lemma 7.3. By symmetry, $w_1 \nsim v_5$, $w_2 \nsim v_3, v_5$. Thus, $w_1, w_2 \sim v_4$ to forbid the fork $(u_2v_2w_1/w_2, u_2v_3v_1, u_2v_4, u_2u)$. Then there is a fork $(v_4u_2v_3, v_4u_4v_5, v_4w_1, v_4w_2)$, contradiction.(recall that here $w_1, w_2 \nsim v_3, v_5$ by Lemma 7.3)

Case 11: $v_2 \in V_4$, $v_6 \in V_4$, and they have one common neighbor w, $v_2 \sim w_2$, $v_6 \sim w_6$.

In this case, $v_1 \sim w_2, w_6$ to forbid the fork centered on u_1.

Claim 1: $v_1 \nsim v_3$. By symmetry, $v_1 \nsim v_5$.

Suppose not, $v_1 \sim v_3$, first we claim that $w \nsim v_3$. Suppose not, by Lemma 7.3, $w \nsim v_5$, then there is a fork $(v_3v_1u_1, v_3u_3v_5, v_3u_2, v_3w)$, contradiction.

Thus $w \nsim v_3$, then $w \sim v_4$ to forbid the fork $(u_2v_3v_1, u_2v_2w, u_2v_4, u_2u)$.

Next we will show that $v_3 \in V_3$. First, $v_3 \nsim w_2, w_6$ since triangle free; $v_3 \nsim w$ by

Lemma 7.3. Denote the fourth neighbor of v_3 by w_3, then $w_3 \sim v_4, v_5$ to forbid the fork $(v_3v_1u_1, v_3u_2v_4, v_3u_3, v_3w_3)$, and the fork $(v_3v_1u_1, v_3u_3v_5, v_3u_2, v_3w_3)$. Here, $w_3 \in V_4$, otherwise, we can identify $N_1(u)$, delete u, w_3, v_3, v_4, v_5 in order, then the new graph is fork free, triangle free, contradiction. Also, the fourth neighbor of w_3 can only be w_2, otherwise there is a fork $(v_3v_1u_1, v_3w_3*, v_3u_2, v_3u_3)$, thus $w_3 \sim w_2$. Now there are 4 possible unsaturated vertices: w_2, w, w_6, v_5. Then look at w_2, there are two possible long arms centered on w_2: $w_2v_2u_2, w_2w_3v_5$, that means either $w_2 \in V_3$, or w_2 share a common neighbor with v_5. If $w_2 \in V_3$, then either $v_5 \sim w_6$, or $v_5 \in V_3$, otherwise there is a fork $(v_5u_4v_6, v_5w_3w_2, v_5u_3, v_5*)$, (recall that $v_5 \nsim w$ by Lemma 7.3), in each situation, w, w_6 is a two cut set, contradiction; if w_2 share a common neighbor with v_5, denote it by x, then $x \sim w_6$ to forbid the fork $(v_1w_2x, v_1v_3u_2, v_1u_1, v_1w_6)$. Here there are only 3 possible unsaturated vertices: w, x, w_6, $w_6 \in V_3$ otherwise there is a fork $(w_6v_1v_3, w_6v_6u_4, w_6x, w_6*)$, contradiction. That means x, w is a 2-cut set, contradiction. Thus, we have shown that $v_3 \in V_3$.

Next, we will show that $v_4 \in V_3$. First $v_4 \nsim w_2$, otherwise there is fork $(v_1w_2v_4, v_1v_3u_3, v_1u_1, v_1w_6)$; also $v_4 \nsim w_6$, otherwise, $v_5 \sim w_6$ to forbid the fork $(v_4u_4v_5, v_4u_2v_3, v_4w, v_4w_6)$, recall that we have proved $v_3 \in V_3$ and $w \nsim v_5$ by Lemma 7.3, then there is a fork $(w_6v_5u_3, w_6v_4u_2, w_6v_1, w_6v_6)$, contradiction. Let's denote the fourth neighbor of v_4 by w_4, then there is a fork $(u_2v_4w_4, u_2v_3v_1, u_2v_2, u_2u)$, a contradiction.

Now, we have prove that $v_3, v_4 \in V_3$, let's identify $N_1(u)$ as u', delete u, v_3, v_4 in order. Since we have known all the neighbor of v_1, v_2, v_6, we can see none of them can use as a long arm for fork centered on u', thus there's no fork centered on u', the new graph is fork free, triangle free, contradiction. So far, we have finished the proof of *Claim 1*.

Claim 2: For any vertex x, x cannot be adjacent to 4 of $N_2(u)$

If x is adjacent to 4 of v_2, v_3, v_4, v_5, v_6, it's impossible by Lemma 7.3; if $x \sim v_1$, and 3 of v_2, v_3, v_4, v_5, v_6, we can identify $N_1(u)$ as u', delete u, w, and $N_1(w)$ except v_1, by *Claim 1*, this graph is triangle free, also it's fork free since $u' \in V_3$, contradiction.

Claim 3: $v_1 \in V_3$.

Suppose not, the fourth neighbor of v_1 must be one of v_3, v_5, w, otherwise there is a fork centered on u_1. By *Claim 1*, the only possibility is $v_1 \sim w$. Then $w \in V_4$, otherwise $N_1(w) \subset N_1(u_1)$; also, the fourth neighbor of w cannot be one of v_3, v_4, v_5 by *Claim 2*, then there is a fork centered on w with long arms (wv_2u_2, wv_6u_4), contradiction.

Claim 4: v_3 cannot be adjacent to 2 of w_2, w_6, w.

Suppose not, v_3 has 2 potential long arms: $(v_3u_2v_4, v_3u_3v_5)$, then at least one of w_2, w_6, w is adjacent to 4 of $N_2(u)$, by *Claim 2*, it's impossible.

Claim 5: If $v_3 \in V_4$, $v_3 \nsim$ any of w_2, w_6, w.

Suppose not, there are 3 possibilities by *Claim 4*:

1. $v_3 \in V_4$, $v_3 \sim w_2$, and the fourth neighbor of v_3 is w_3. Here, $w_3 \sim v_4, v_5$ to forbid the fork $(v_3w_2v_1, v_3u_2v_4, v_3u_3, v_3w_3)$, and the fork $(v_3w_2v_1, v_3u_3v_5, v_3w_3, v_3u_2)$. Now, look at v_4, either $v_4 \in V_3$, or $v_4 \sim$ one of w_2, w_6, w, otherwise there is a fork centered on v_4 with long arms $(v_4u_2v_2, v_4u_4v_6)$; then look at w, for w, either it's adjacent to one of v_3, v_4, v_5, or $w \in V_3$, otherwise there is a fork centered on w with long arms (wv_2u_2, wv_6u_4). That mean, after we identify $N_2(u)$ as v', we can delete $u, N_1(u), w_2, w_3, w$, since they have at most degree 2; then v' has at most degree 2, w_6 and the fourth neighbor of v_5(recall we have shown that v_4 doesn't have the fourth neighbor other than w, w_2, w_6), then we can delete v', too. The new graph is fork free, triangle free, contradiction.

2. $v_3 \in V_4$, $v_3 \sim w_6$, and the fourth neighbor of v_3 is w_3. It's symmetric with previous case.

3. $v_3 \in V_4$, $v_3 \sim w$, and the fourth neighbor of v_3 is w_3. It's not symmetric with case 1, but we can apply very similar argument, first show that $w_3 \sim v_4, v_5$ to forbid the fork $(v_3wv_6, v_3u_2v_4, v_3u_3, v_3w)$, and the fork $(v_3wv_6, v_3u_3v_5, v_3u_2, v_3w)$. And after identify $N_2(u)$, we can delete all the vertices with degree less than 2, and get a fork free, triangle free graph, contradiction.

Claim 6: $v_3 \in V_3$, by symmetry, $v_5 \in V_3$.

Suppose not, by *Claim 4* and *Claim 5*, we can denote the third and fourth neighbor of v_3

by x_1, x_2, then x_1 or x_2 must be adjacent to v_5 to forbid the fork $v_3u_3v_5, v_3u_2v_2$, by symmetry, we can only consider the case $x_1 \sim v_5$.

First we will prove that $w_2 \nsim v_4$. Suppose not, by *Claim 2*, $w_2 \nsim v_5$; also $w_6 \nsim v_4$, otherwise by *Claim 2*, $w_6 \nsim v_5$, then there is a fork $(v_4u_2v_3, v_4u_3v_5, v_4w_2, v_4w_6)$, contradiction. Then $w_6 \sim v_5$ to forbid the fork $(u_4v_4w_2, u_4v_6w_6, u_4v_5, u_4u)$(notice that here $w_2 \nsim w_6$ since triangle free.). However, if $w_6 \sim v_5$, then $v_4 \sim x_1$ to forbid the fork $v_5u_4v_4, v_5w_6v_1, v_5u_3, v_5x_1$(notice that here we have proved that $v_4 \nsim w_6$). Then we can identify $N_2(u)$ as v', delete $u, N_1(u), w_2, w_6, x_1, v'$ in order, the new graph is fork free, triangle free, contradiction.

Second, we will prove that $x_2 \nsim v_4$. Suppose not, then to forbid the fork $(u_4v_4x_2, u_4v_5x_1, u_4v_6, u_4u)$, either $x_2 \sim v_5$, or $x_1 \sim v_4$, notice that here $x_1 \nsim x_2$ since triangle free. However, if $x_1 \sim v_4$ there is a fork $v_4u_2v_2, v_4u_4v_6, v_4x_1, v_4x_2$, contradiction, that means $x_2 \sim v_5$. Now we claim that $N_1(N_2(u)) = N_1(u), x_1, x_2, w_2, w_6, w$. Actually, we only need to check v_4, which is the only unsaturated vertex. If the fourth neighbor of v_4 is w_4, then there is a fork $(v_4u_2v_2, v_4u_4v_6, v_4x_2, v_4w_4)$, contradiction. Also, for w, either $w \in V_3$, or three of $N_1(w) \in V_2(u)$, otherwise there is a fork centered on w with long arms (wv_2u_2, wv_6u_4), contradiction; similarly, for x_1, either $x_1 \in V_3$, or three of $N_1(x_1) \in V_2(u)$, otherwise there is a fork centered on x_1 with long arms $x_1v_3u_2, x_1v_5u_4$, contradiction. That means, after we identify $N_2(u)$ as v', we can delete $u, N_1(u), w, x_1, x_2, v'$ in order, then the new graph is fork free, triangle free, contradiction.

Now we have shown that $w_2, x_2 \nsim v_4$, then $w_2 \sim x_2$ to forbid the fork $(u_2v_2w_2, u_2v_3x_2, u_2v_4, u_2u)$.

Next, let's show that $x_1 \nsim w_2$. If $x_1 \sim w_2$, then $v_5 \sim x_2$ to forbid the fork $(w_2v_2u_2, w_2x_1v_5, w_2v_1, w_2x_2)$ (recall that here $v_5 \nsim v_1$ by *claim 3*); also, x_1 or x_2 must be adjacent to w_6 to forbid the fork $(w_2v_1w_6, w_2v_2u_2, w_2x_1, w_2x_2)$. If $x_1 \sim w_6$, then there is a fork $(x_1w_2v_2, x_1w_6v_6, x_1v_3, x_1v_5)$, a contradiction; if $x_2 \sim w_6$, then there is a fork $(x_2w_2v_2, x_2v_5u_4, x_2v_3, x_2w_6)$, contradiction. Thus $x_1 \nsim w_2$.

Now, we have shown that $w_2, x_2 \nsim v_4$, $x_1 \nsim w_2$, then to forbid the fork $(v_3u_2v_4, v_3x_2w_2, v_3x_1, v_3u_3)$, $x_1 \sim v_4$.

Next, let's show that $w \nsim x_2$. Suppose $w \sim x_2$, then there are 4 ways to forbid the fork $(v_6wx_2, v_6u_4v_5, v_6u_1, v_6w_6)$: $x_2 \sim w_6$, $x_2 \sim v_5$, $v_5 \sim w$, $v_5 \sim w_6$. However, $x_2 \nsim v_5$, otherwise there is a fork $(x_2v_3u_2, x_2v_5u_4, x_2w, x_2w_2)$. If $x_2 \sim w_6$, then $x_1 \sim w_6$ to forbid the fork $(v_3u_2v_2, v_3x_2w_6, v_3u_3, v_3x_1)$, here all the vertices are saturated except w, w_2, v_4, v_5, but w cannot have a new neighbor, otherwise there is a fork centered on w with long arms (wv_2u_2, wv_6u_4), and v_4 cannot have a new neighbor, otherwise there is a fork centered on v_4 with long arms $(v_4u_2v_2, v_4u_4v_6)$, that means either this is the graph G, which is 3-colorable, or w_2, v_5 is a 2-cut set, contradiction. If $v_5 \sim w$, then there is a fork $(wv_2u_2, , wv_5u_3, wv_6, wx_2)$, contradiction. If $v_5 \sim w_6$, we can see that $N_1(N_2(u)) = w_2, w_6, w, x_1, x_2$(here, we only need to check the possible fourth neighbor of v_4, v_4 cannot have a new vertex as its neighbor, otherwise there is a fork centered on v_4 with long arms $v_4u_2v_2, v_4u_4v_6$, contradiction), also, either $w \in V_3$, or $w \sim v_4$ or v_5, otherwise there is a fork centered on w with long arms $(wv_2u_2, , wv_5u_3)$, contradiction. That means we can identify $N_2(u)$ as v', delete $N_1(u), u, x_1, w, w_6, v'$ in order, the new graph is fork free, triangle free, contradiction.

So far, we have shown that $w \nsim x_2$, then $v_4 \sim w$ to forbid the fork $(v_2u_2v_4, v_2w_2x_2, v_2u_1, v_2w)$ (recall that we have shown that $w_2, x_2 \nsim v_4$).

Now look at w, w must have degree 4, otherwise, we can identify $N_1(u)$, delete u, w, v_2, v_4, v_6 in order, the new graph is fork free, triangle free, contradiction. Also, $w \nsim w_2, x_1, w_6$ since triangle free, and $w \nsim v_5$ by Lemma 7.3, $w \nsim x_2$ as we have shown, then w must have a new vertex as its fourth neighbor, denote it by w', then $w_2 \sim w'$ to forbid the fork $(v_2u_2v_3, v_2ww', v_2u_1, v_2w_2)$.

Now, look at the fork centered on w_2: $(w_2v_1w_6, w_2x_2v_3, w_2w', w_2v_2)$, there are 2 ways to forbid it: $w_6 \sim x_2$, or $w_6 \sim w'$.

If $w_6 \sim x_2$, then $v_5 \sim x_2$ or $v_5 \sim w_6$ to forbid the fork $(v_6w_6x_2, v_6u_4v_5, v_6w, v_6u_1)$, however, if $v_5 \sim w_6$, then we can identify $N_2(u)$ as v', delete $N_1(u), u, w, x_1, w_6, v'$ in order, the new graph is fork free, triangle free, contradiction; if $v_5 \sim x_2$, then there is a fork $(v_5u_4v_6, v_5x_2w_2, v_5u_3, v_5x_1)$, contradiction. Thus $w_6 \nsim x_2$.

If $w_6 \sim w'$, now look at x_2, $x_2 \nsim w', x'$ since triangle free; $x_2 \nsim w_6$, otherwise there is a fork $(v_3u_2v_2, v_3x_2w_6, v_3x_1, v_3u_3)$, contradiction; $x_2 \nsim v_5$, otherwise there is a fork $(u_4v_5x_2, u_4v_6w_6, u_4v_4, u_4u)$, since we have shown that $x_2 \nsim w_6$. Thus, x_2 must have a new vertex as its neighbor, denote is by x, $x \sim x_1$ to forbid the fork $(v_3u_2v_2, v_3x_2x, v_3u_3, v_3x_1)$, and then $x \sim w'$ to forbid the fork $(v_4ww', v_4x_1x, v_4u_2, v_4u_4)$, now there are only 3 unsaturated vertices in the graph x, x_2, v_5, w_6, but here, $x_2 \in V_3$, otherwise the fourth neighbor of x_2 must be v_5, to forbid the fork centered on x_2 with long arms $(x_2w_2v_2, x_2v_3u_3)$, then there is a fork $(u_4v_5x_2, u_4v_6w_6, u_4v_4, u_4u)$, contradiction; also we claim that $v_5 \in V_3$. First $v_5 \nsim w_6, x$, otherwise either this is the graph G, or there is a cut vertex w_6 or x in this graph, contradiction, suppose v_5 has the fourth neighbor w_5, then $w_5 \sim w_6$ to forbid the fork $(u_4v_5w_5, u_4v_6w_6, u_4v_4, u_4u)$, then w_5, x is a 2-cut set, contradiction. So far, we have shown that $v_5, x_2 \in V_3$, then either is this the graph, which is 3-colorable, or w_6, x is a 2 cut vertex, contradiction.

So far, we have finished *Claim 6*, $v_3 \in V_3$, by symmetry, $v_5 \in V_3$. Then we can identify $N_1(u)$ as u', delete u, v_3, v_5, by *Claim 3*, the new graph is triangle free; then we can check $N_1(v_1, v_2, v_6)$(since $N_1(v_1, v_2, v_6)$ has been fixed), we can see that there is no fork centered on u'; also there is no fork centered on v_1, v_2, v_4, v_6 since their degrees are less than 4. Thus, the new graph is fork free, triangle free, contradiction, i.e. Case 11 is impossible.

Case 12: $v_2 \in V_4$, $v_6 \in V_4$, and they do not have common neighbors, i.e. $v_2 \sim w_2, w_2'$, $v_6 \sim w_6, w_6'$.

This case is impossible since $v_1 \sim w_2, w_2', w_6, w_6'$ to forbid the fork centered on u_1, contradictory to our assumption that $\Delta G \leq 4$.

By far, we have exhausted all the possible situation for $N_2(u_1)$, each case leads to a contradiction. Thus, this graph doesn't exist.

Subcase 3.6:

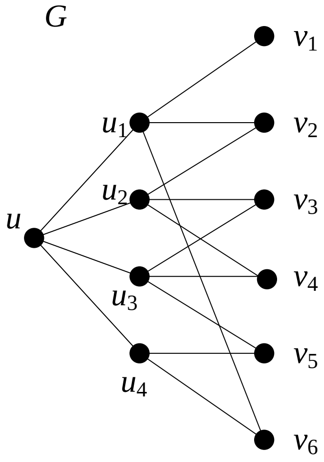

Figure 24: graph1.3.6

7.4 $|S(u)| = 4$

We use $\mathcal{G}$ to denote the family of triangle-free and fork-free graphs with maximum degree at most four.

Let G be a graph in $\mathcal{G}$, and let $S_1, S_2, \ldots, S_l$ be disjoints independent sets of $V(G)$. We use $G(x_1, x_2, \ldots, x_l) = G[S_1, S_2, \ldots, S_l]$ to denote the graph obtained from G by identifying the vertices in S_i into a new vertex x_i, $1 \leq i \leq l$, and removing all multi-edges. A sequence of vertices $(x_1, x_2, \ldots, x_k)$ of G is *reducible* if $d_G(x_1) \leq 2$, and $d_{G-\{x_1,x_2,\ldots,x_i\}}(x_{i+1}) \leq 2$ for each $1 \leq i \leq k-1$. We say that G is *reducible* if there exist disjoint independent sets $S_1, S_2, \ldots, S_l$ of G such that the graph obtained from $G(x_1, x_2, \ldots, x_l) = G[S_1, S_2, \ldots, S_l]$ by removing a reducible sequence is still in $\mathcal{G}$. If G is reducible, a reducible sequence of $G(x_1, x_2, \ldots, x_l)$ is also called a reducible sequence of G for convenience, and the new graph obtained from G by above procedure is called a *reduction* of G.

Let G be a minimum counterexample to Theorem 7.1. Suppose that $|S(x)| > 1$ for each 4-vertex of G.

Lemma 7.8. *G is not reducible.*

Proof. Suppose that G is reducible. Let G' be a reduction of G. Then G' is 3-colorable by the minimality of G, and any 3-coloring ϕ' of G' yields a 3-coloring of G by extending ϕ' in the reverse order of its reducible sequence. □

Let $S(u) = \{v_1, v_2, v_3, v_4\}$. There are two cases by symmetry, one is that $v_1/v_2/v_3 \sim u_1$ and $v_4 \sim u_2$, and the other is that $v_1/v_2 \sim u_1$ and $v_3/v_4 \sim u_2$.

We consider the former case first. Since each vertex in $N_2(u) \setminus S(u)$ has at least two neighbors in $N(u)$, and since $N(x) \nsubseteq N(y)$ for any two vertices x and y, we may suppose by symmetry that $v_5 \sim u_2/u_3$, $v_6 \sim u_3/u_4$, and $v_7 \sim u_2/u_4$. Then, $u_4 \nsim v_5$ for avoiding $N(u_3) \subseteq N(u_4)$, and $u_3 \nsim v_7$ for avoiding $N(u_4) \subseteq N(u_3)$.

Since $N(v_1) \neq N(v_3)$, v_1 has a neighbor, say w_1, not adjacent to v_3. Suppose that $w_1 \nsim v_2$. Then, $w_1 \neq v_5$ for avoiding $(u_1uu_4, u_1v_1w_1, u_1v_2, u_1v_3)$, $w_1 \neq v_6$ for avoiding $(u_1uu_2, u_1v_1w_1, u_1v_2, u_1v_3)$, and $w_1 \neq v_7$ for avoiding $(u_1uu_3, u_1v_1w_1, u_1v_2, u_1v_3)$. But this implies that G has a fork $(u_1uu_2, u_1v_1w_1, u_1v_2, u_1v_3)$. Therefore, $w_1 \sim v_2$, and thus $|N(v_1) \cup N(v_2) \cup N(v_3)| \leq 5$ by symmetry.

Let $G(u', v') = G[S_1, \{v_1, v_2, v_3]$. Then, G is reducible as (u, u_1, v_4) is reducible, contradicting Lemma 7.8.

Now, we consider the latter case. Again, since each vertex in $N_2(u) \setminus S(u)$ has at least two neighbors in $N(u)$, and since $N(x) \nsubseteq N(y)$ for any two vertices x and y, we may suppose by symmetry that $v_5 \sim u_1/u_3$, $v_6 \sim u_3/u_4$, $v_7 \sim u_2/u_4$, and $u_4 \nsim v_5$ and $u_3 \nsim v_7$.

If $v_1 \sim v_6$, then $(u_1uu_2, u_1v_1v_6, u_1v_2, u_1v_5)$ forces $v_2 \sim v_6$, and thus $N(v_1) = N(v_2)$ contradicting Lemma 7.3. Therefore, we suppose by symmetry that $v_6 \nsim v_1/v_2/v_3/v_4$.

By Lemma 7.3, $N(x) \setminus N(y) \neq \emptyset$ for any two vertices x and y. Let w_1 (resp. w_2, w_3, w_4) be the vertex in $N(v_1) \setminus N(v_2)$ (resp. $N(v_2) \setminus N(v_1)$, $N(v_3) \setminus N(v_4)$, $N(v_4) \setminus N(v_3)$). If $w_1 \neq v_7$ (resp. $w_2 \neq v_7$), then $(u_1uu_2, u_1v_1w_1, u_1v_2, u_1v_5)$ (resp. $(u_1uu_2, u_1v_2w_2, u_1v_1, u_1v_5)$) forces $w_1 \sim v_5$ (resp. $w_2 \sim v_5$). The same argument shows that each vertex in $\{w_3, w_4\}$ is adjacent to v_7 unless it is v_5. We distinguish two cases.

Case 1. Suppose that $w_1 = v_7$. Then, $w_2 \neq v_7$, and hence $w_2 \sim v_5$. If $v_5 \notin \{w_3, w_4\}$, then

$w_3/w_4 \sim v_7$ contradicting $d(v_7) \leq 4$. So, we suppose, by symmetry, that $w_3 = v_5$. Then, $w_4 \sim v_7$, and $(v_2u_1u, v_2v_4w_4, v_2w_2, v_2v_3)$ forces $w_2 \sim w_4$.

If $w_2 \in V_4$, let $x_2 \notin \{v_2, v_5, w_4\}$ be a neighbor of w_2, then $x_2 \in \{u_3, u_4\}$ to forbid $(v_2u_1u, v_2w_2x_2, v_2v_3, v_2v_4)$. But this implies $w_2 \in N_2(u) \setminus S(u)$ and thus $w_2 \sim u_3/u_4$, contradicting $w_2 \in V_4$. Therefore, we suppose by symmetry that $w_2, w_4 \in V_3$, and then $\{u_3, u_4\}$ forms a 2-cut of G, contradicting Lemma 7.4.

Case 2. By symmetry, we suppose that $w_1 \neq v_7 \neq w_2$ and $w_3 \neq v_5 \neq w_4$. Then, $w_1/w_2 \sim v_5$, and $w_3/w_4 \sim v_7$. Let $G_1 = G(v_1', v_3') = G[\{v_1, v_2, v_5\}, \{v_3, v_4, v_7\}]$. Then, G is reducible since (u_1, u_2, u) is reducible, contradicting Lemma 7.8.

7.5 $|S(u)| = 3$

In this section, we suppose that for every 4-vertex x of G, $1 < |S(x)| \leq 3$. Suppose that $S(u) = \{v_1, v_2, v_3\}$. We consider two possibilities by symmetry, one is that $v_1/v_2/v_3 \sim u_1$, and the other is that $v_1/v_2 \sim u_1$ and $v_3 \sim u_2$. Note that we set $S_1 = \{u_2, u_3, u_4\}$.

First suppose $v_1/v_2/v_3 \sim u_1$. Since each vertex in $N_2(u) \setminus S(u)$ has two neighbors in $N(u)$, by Lemma 7.3 we may suppose that $v_4 \sim u_2/u_3$, $v_5 \sim u_3/u_4$ and $v_6 \sim u_2/u_4$. If $u_2, u_3, u_4 \in V_3$, let $G(u') = G[S_1]$, then G is reducible since u itself is reducible, contradicting Lemma 7.8. Therefore, there are exactly two 4-vertices in S_1. Note that $S(u_1) = \{u_2, u_3, u_4\}$. We may suppose by symmetry that $u_3, u_4, v_1, v_2 \in V_4$, and $v_3, u_2 \in V_3$.

Suppose that $v_6' \sim u_3/u_4$, $w_1/w_1' \sim v_1/v_2$, $w_2 \sim v_2/v_3$, and $w_3 \sim v_1/v_3$. If $\{w_1, w_1', w_2, w_3\} \cap \{v_4, v_5, v_6, v_6'\} \neq \emptyset$, suppose $w_1 = v_4$ for instance, let $G(u', v') = G[S_1, \{v_1, v_2, v_3\}]$, then G is reducible as (u, u_1, w_1) is reducible, contradicting Lemma 7.8. Therefore, $\{w_1, w_1', w_2, w_3\} \cap \{v_4, v_5, v_6, v_6'\} = \emptyset$. Now, $\{u, u_2, u_3, u_4\} \cap S(v_1) = \{u\}$. Since each vertex in $S(v_1) \setminus \{u\}$ has to be adjacent to u by Lemma 7.5, $|S(v_1)| = 1$, contradicting our assumption.

Now, we suppose that $v_1/v_2 \sim u_1$ and $v_3 \sim u_2$. Without loss of generality, we suppose that if $|S(x)| = 3$ for a 4-vertex x, then the three vertices in $S(x)$ are not adjacent to a same vertex in $N(x)$. Let $w_1 \in N(v_1) \setminus N(v_2)$ and $w_2 \in N(v_2) \setminus N(v_1)$. Let $S_2 = \{v_1, v_2\}$. We consider two cases depending on $d(u_1)$.

Case 1. In this case, we suppose $u_1 \in V_3$. Since $N(x) \nsubseteq N(y)$ for any distinct x, y, we suppose that $v_4 \sim u_2/u_3$, $v_5 \sim u_3/u_4$, and $v_6 \sim u_2/u_4$. Since $S(u) = \{v_1, v_2, v_3\}$, either $u_3, u_4 \in V_3$, or $u_3, u_4 \in V_4$ implying u_3 and u_4 that have a common neighbor, say v'_5, other than u and v_5.

We first consider the situation that $u_3, u_4 \in V_4$. Then, $u_1 \in S(u_2) \cap S(u_3) \cap S(u_4)$, and thus $S(u_2) \cup S(u_3) \cup S(u_4) \subseteq \{u_1, v_1, v_2\}$ by Lemma 7.5. Let w_5 and w'_5 be vertices in $N(v_5) \setminus N(v'_5)$ and $N(v'_5) \setminus N(v_5)$, respectively. Note that $w_5 \neq w'_5$.

We show that

$$\text{except } v_3 \text{ no } v_i \text{ can be adjacent to both } v_1, v_2. \tag{1}$$

Since $|S(u_2)| \geq 2$, there exists at most one edge from v_4 (resp. v_6) to S_2. If $v_5 \sim v_1/v_2$, then $w'_5 \notin \{v_1, v_2\}$, and hence $w'_5 \sim v_6$ for avoiding $(u_4uu_1, u_4v'_5w'_5, u_4v_5, u_4v_6)$, $w'_5 \sim v_4$ for avoiding $(u_3uu_1, u_3v'_5w'_5, u_3v_4, u_3v_5)$. Note that if $v_6 \nsim v_1/v_2$, then its neighbor other than u_2, u_4 and w'_5 has to be adjacent to v'_5 and v_4. Let $G(u', v'_1, v'_4) = G[S_1, S_2, \{v_4, v'_5, v_6\}]$. Then, G is reducible as $(u, u_1, v_5, w'_5, u', v_3, v'_1)$ is reducible, contradicting Lemma 7.8. The same happens if $v'_5 \sim v_1/v_2$. Therefore, (1) holds.

Note that $|S(u_3)| \geq 2$. We suppose by symmetry that $v_1 \in S(u_3)$. Then $v_2 \notin S(u_3)$ by Lemma 7.5 and (1). We consider three possibilities upon $\{v_1, v_2\} \cap \{w_5, w'_5\}$.

First, suppose $w_5 = v_1$ and $w'_5 = v_2$ by symmetry. Then, $v_2 \sim v_4$ for otherwise $v_2 \in S(u_3)$, and hence $v_2 \in S(u_4)$. So, $v_1 \sim v_6$ for otherwise $\{u_1, v_1, v_2\} \subseteq S(u_4)$. But then, both v_1 and v_2 are not in $S(u_2)$ implying that $|S(u_2)| = 1$, a contradiction to our assumption.

Next, suppose that $w_5 = v_1$ and $w'_5 \neq v_2$. Since $w'_5 \nsim v_5$, $w'_5 \sim v_6$ for avoiding $(u_4uu_1, u_4v'_5w'_5, u_4v_5, u_4v_6)$, and $w'_5 \sim v_4$ for avoiding $(u_3uu_1, u_3v'_5w'_5, u_3v_4, u_3v_5)$. If $v_2 \sim v'_5$, then $v_1 \nsim v'_5$ by (1), and hence v_2 can be viewed as w'_5. So, we suppose that $v_2 \nsim v'_5$. If $v_3 \sim v'_5$, there would be a fork $(u_3uu_1, u_3v'_5v_3, u_3v_4, u_3v_5)$. So, we suppose that $v_2/v_3 \nsim v_5/v'_5$. If $v_1 \sim v_6$, then $v_2 \nsim v_6$ by (1), and hence $|S(u_4)| = 1$ as $v_2 \nsim v_5/v'_5$, a contradiction. Note that $v_1 \nsim v_5'/v_6$ and $v_2 \nsim v_5/v_5'$, $v_2 \sim v_6$ implies that $S(u_4) = \{u_1, v_1, v_2\}$, also a contradiction. Therefore, $v_6 \nsim v_1/v_2$. With the same arguments as used in the proof of (1), one can show that G is reducible.

The third, we suppose $\{w_5, w_5'\} \cap \{v_1, v_2\} = \emptyset$, i.e., each of v_1 and v_2 is either adjacent to both v_5 and v_5', or adjacent to non of them. Then, $(u_3uu_1, u_3v_5w_5, u_3v_4, u_3v_5')$ forces $w_5 \sim v_4$, $(u_4uu_1, u_4v_5w_5, u_4v_5', u_4v_6)$ forces $w_5 \sim v_6$, and the similar arguments show that $w_5' \sim v_4/v_6$. Thus, $v_1 \notin S(u_3)$, a contradiction.

Now, we suppose that $u_3, u_4 \in V_3$, and distinguish two subcases depending upon $d(v_3)$.

Subcase 1.1. We first consider the subcase that $v_3 \in V_4$. Let w_3 be the vertex in $N(v_3) \setminus \{u_2, v_1, v_2\}$. Since $v_3 \sim v_5$ producing a fork $(u_2uu_1, u_2v_3v_5, u_2v_4, u_2v_6)$, $w_3 \neq v_5$. So,

$$w_3 \notin \{v_1, \ldots, v_6\} \tag{2}$$

as G is triangle-free. Forced by $(u_2uu_1, u_2v_3w_3, u_2v_4, u_2v_6)$, we suppose by symmetry that $w_3 \sim v_4$. Then, $u \in S(v_3)$ and $N(u) \cap S(v_3) = \emptyset$, and hence each vertex in $S(v_3)$ must be adjacent to u_2 by Lemma 7.5, and thus $v_6 \in S(v_3)$ by our assumption $|S(v_3)| \geq 2$. So,

$$v_6 \nsim v_1/v_2/w_3. \tag{3}$$

Since $w_1 \nsim v_2$ and $w_2 \nsim v_1$, $w_1/w_2 \sim w_3$ forced by $S(v_3)$. Let $w_6 \notin \{u_2, u_4\}$ be a neighbor of v_6. Then, $w_6 \sim v_4$ for avoiding $(u_2uu_1, u_2v_6w_6, u_2v_3, u_2v_4)$, and thus $v_6 \in V_3$, and

$$v_1/v_2 \nsim v_4 \tag{4}$$

since $N(v_4) = \{u_2, u_3, w_3, w_6\}$.

If $w_1 = v_5$, let $G(u', v_1', v_4') = G[S_1, S_2, \{v_4, v_5, v_6\}]$, then G is reducible since $(u, u_1, u', v_3, w_3, v_4')$ is reducible, contradicting Lemma 7.8. Therefore, $w_1 \neq v_5 \neq w_2$ by symmetry, and hence following (2), (3) and (4),

$$\{w_1, w_3, w_3\} \cap \{v_1, v_2 \ldots, v_6\} = \emptyset.$$

Since $u \in S(v_3)$, $N(v_1) \cap S(v_3) = \emptyset = N(v_2) \cap S(v_3)$ by (3). Either $v_1, v_2 \in V_3$, or they have another common neighbor, say w'_1, other than u_1 and v_3. In the former situation, let $G(v'_1) = G[S_2]$, then G is reducible since u_1 itself forms a reducible sequence, contradicting Lemma 7.8. So, we suppose the latter occurs. Since $v_6 \in V_3$, $w'_1 \neq v_6$. Note that $u, u_2 \in S(v_1) \cap S(v_2)$. Each vertex in $(N(w'_1) \cup N(w_1) \cup N(w_2)) \setminus \{v_1, v_2, w_3\}$ is adjacent to all of w'_1, w_1, w_2. By letting $G(v'_1, w') = G[S_2, \{w'_1, w_1, w_2\}]$, one can check that G is reducible. This contradicts Lemma 7.8 and completes the proof of this subcase.

Subcase 1.2. Suppose that $v_3 \in V_3$. We consider two possibilities depending on whether there exist edges between $\{v_1, v_2\}$ and $\{v_4, v_6\}$.

First we suppose that there exists no edge between $\{v_1, v_2\}$ and $\{v_4, v_6\}$. If $v_4 \in V_3$, let $w_4 \notin \{u_2, u_3\}$ be a neighbor of v_4. Otherwise, let $w_4, w'_4 \notin \{u_2, u_3\}$ be the other neighbors of v_4. Then, $w_4 \sim v_6$ to avoid $(u_2uu_1, u_2v_4w_4, u_2v_3, u_2v_6)$, and $w'_4 \sim v_6$ by symmetry. Furthermore, we suppose, by symmetry, that $w_4 \sim v_5$ to forbid $(v_4u_2v_3, v_4u_3v_5, v_4w_4, v_4w'_4)$ while $v_4 \in V_4$. Whenever $v_4 \in V_4$, let $G(u', v') = G[S_1, \{v_4, v_5, v_6\}]$, then G is reducible as $(u, u_1, u', v_3, v_1, v_2, w_4, v')$ is reducible, contradicting Lemma 7.8. So, we suppose by symmetry that $v_4, v_6 \in V_3$. Note that $v_5 \in V_4$ implies that $v_5 \sim w_4$ to avoid the fork with long arms $v_5u_3v_4$ and $v_5u_4v_6$. Let $G(u'', v'') = G[S_1, \{v_4, v_5, v_6\}]$. Then, G is reducible as $(u, u_1, u'', v_3, v_1, v_2, v'')$ is reducible, contradicting Lemma 7.8 again.

Next, we suppose that there exists edge between $\{v_1, v_2\}$ and $\{v_4, v_6\}$. If $v_4 \sim v_1/v_2$, then $w_1 \sim w_2$ forced by $(v_4v_1w_1, v_4v_2w_2, v_4u_2, v_4u_3)$, and hence a fork $(v_1u_1u, v_1w_1w_2, v_1v_3, v_1v_4)$ occurs in G. Therefore, v_4 (resp. v_6) is adjacent to at most one of v_1 and v_2. Without loss of generality, we suppose that $v_1 \sim v_4$, i.e., $w_1 = v_4$. If $v_4 \in V_3$, let $G(u', v') = G[S_1, \{v_3, v_4\}]$, then G is reducible as (u, u_1, v_1, v', u') is reducible, contradicting Lemma 7.8. So, we suppose that $v_4 \in V_4$. Let $w_4 \notin \{u_2, u_3, v_1\}$ be a neighbor of v_4. Then, $w_4 \neq v_2$ since $w_1 \nsim v_2$. The fork $(v_4v_1u_1, v_4u_3v_5, v_4u_2, v_4w_4)$ forces $w_4 \sim v_5$, the fork $(u_2uu_1, u_2v_4w_4, u_2v_3, u_2v_6)$ forces $w_4 \sim v_6$. If $v_6 \in V_4$, let $w_6 \notin \{u_2, u_4, w_4\}$ be a neighbor of v_6, then $(u_2uu_1, u_2v_6w_6, u_2v_3, u_2v_4)$ forces $w_6 \in \{v_1, v_2\}$. Whatever $v_6 \in V_4$ or not, let $= G(u', v') = G[S_1, \{v_4, v_5, v_6\}]$. Then, G is reducible as $(u, u_1, u', v_3, v_1, v_2, v')$ is reducible, contradicting Lemma 7.8. This completes the proof of Case 1.

Case 2. Now, we suppose $u_1 \in V_4$.

If u_1 and u_2 have another common neighbor other than u, then one can check easily that either $N(u_3) \subseteq N(u_4)$ or $N(u_4) \subseteq N(u_3)$, both contradict Lemma 7.3. Therefore, we suppose that $\{u\} = N(u_1) \cap N(u_2)$, and suppose by symmetry that u_1 and u_3 have another common neighbor other than u. We discuss in two subcases depending upon $N(u_2) \cap N(u_3) = \{u\}$ or not.

Subcase 2.1. We first suppose that $N(u_2) \cap N(u_3) \neq \{u\}$. Let $v_4 \neq u$ be a common neighbor of u_2 and u_3, $v_5 \neq u$ be a common neighbor of u_1 and u_3. Then, one can check easily by Lemma 7.3 that u_3 and u_4 have another common neighbor, say $v_6 \neq u$, and u_2 and u_4 have another common neighbor, say $v_7 \neq u$.

Claim 7.9. $u_4 \nsim v_4$.

Proof. Suppose to the contrary that $u_4 \sim v_4$. Then, $u_2, u_4 \in S(u_1)$, and any other vertex in $S(u_1)$ should be adjacent to both u_2 and u_4. Since $|S(u_1)| \leq 3$, $|\{w_1, w_2\} \cap S(u_1)| \leq 1$. By symmetry, we may suppose $w_1 \in S(u_1)$ and $w_2 \notin S(u_1)$ whenever $|\{w_1, w_2\} \cap S(u_1)| = 1$. Then, $w_2 \sim v_5$ for avoiding $(u_1uu_2, u_1v_2w_2, u_1v_1, u_1w_5)$, and $w_1 \in \{v_4, v_7\}$ whenever $w_1 \in S(u_1)$.

We consider $w_1 \in S(u_1)$ first.

Suppose that $w_1 = v_4$. Then $(u_3v_4v_1, u_3v_5w_2, u_3u, u_3v_6)$ forces either $v_1 \sim v_6$ or $w_2 \sim v_6$. If the former occurs, then $v_2 \sim v_6$ since $v_6 \notin S(u_1)$, let $G(u', v') = G[S_1, S_2]$, one can easily check that G is reducible as $(u, u_1, v_4, v_6, v', v_3, u')$ is reducible, contradicting Lemma 7.8. So, we assume the latter situation, i.e., $w_2 \sim v_6$ and $v_1 \nsim v_6$. If $v_2 \sim v_6$, then $v_2 \in S(u_4)$ implying $v_1 \notin S(u_4)$ by Lemma 7.5 and thus $v_1 \sim v_7$, and one can check easily that G is reducible by letting $G(u, v_1', v_5') = G[S_1, S_2, \{v_5, v_6, v_7\}]$ and reducing $(u, u_1, v_4, u', v_3, v_1', w_2, v_5')$. So, we suppose that $v_1/v_2 \nsim v_6$. If $v_6 \in V_4$, let $w_6 \notin \{u_3, u_4, w_2\}$

be a neighbor of v_6, then $v_7 \sim w_6$ for avoiding $(u_4uu_1, u_4v_6w_6, u_4v_4, u_4v_7)$, $v_3 \sim w_6$ for avoiding $(u_2uu_1, u_2v_7w_6, u_2v_3, u_2v_4)$, $w_2 \sim w_6$ for avoiding $(v_3u_2u, v_3v_2w_2, v_3v_1, v_3w_6)$, $v_1 \sim v_5$ for avoiding $(u_3v_4v_1, u_3v_6w_6, u_3u, u_3v_5)$, and thus $\{v_2, v_7\}$ is a 2-cut of G contradicting Lemma 7.3. So, $v_6 \in V_3$. Then, one sees that G is reducible by letting $G(u', v') = G[S_1, S_2]$ and reducing (u, u_1, v_4, v_6).

Next, we suppose that $w_1 = v_7$. We show that

$$\text{besides } v_1v_3, v_2v_3, v_1v_7 \text{ there is no edge } v_iv_j \text{ for } 1 \le i < j \le 7. \tag{5}$$

Since $v_7 = w_1 \in S(u_1)$, $v_7 \nsim v_2/v_5$. Since G is triangle free, $v_4 \nsim v_3/v_5/v_6/v_7$.

Since v_1 and v_2 cannot be adjacent to v_4 both, if $v_1 \sim v_4$ then v_4 can be viewed as w_1. By the above discussion, we may suppose that $v_1 \nsim v_4$. If $v_1 \sim v_6$, then $v_2 \sim v_6$ since $v_6 \notin S(u_1)$, and hence $\{u, u_2, u_3, u_4\} \subseteq S(v_2)$ contradicting $|S(v_2)| \le 3$. Therefore, $v_1 \nsim v_4/v_5/v_6$.

If $v_2 \sim v_4$, note $v_5 \nsim v_7$ as $v_7 = w_1 \in S(u_1)$, a fork $(u_1v_1v_7, u_1v_2v_4, u_1u, u_1v_7)$ occurs. If $v_2 \sim v_6$, then $v_1 \sim v_6$ implying $\{u, u_2, u_3, u_4\} \subseteq S(v_2)$ contradicting $|S(v_2)| \le 3$. Therefore, $v_2 \nsim v_4/v_5/v_6/v_7$.

If $v_3 \sim v_5$, then $w_2 \sim v_6$ for avoiding $(v_5v_3u_2, v_5u_3v_6, v_5u_1, v_5w_2)$, and furthermore either $w_2 \sim v_4$ or $w_2 \sim v_7$ for avoiding $(u_4uu_1, u_4v_6w_2, u_4v_4, u_4v_7)$. By discussing on $S(u_1)$ and $S(u_4)$, we see that $v_6 \in V_3$ as $v_6 \nsim v_1/v_2$, and $v_7 \in V_3$ whenever $w_2 \sim v_4$. In both cases, let $G(u', v_1', v_5') = G[S_1, S_2, \{v_5, v_6, v_7\}]$. Then, G is reducible as $(u, u_1, v_4, u, v_3, w_2, v_5', v_1')$ is reducible, contradicting Lemma 7.8. If $v_3 \sim v_6$, then $w_2 \sim v_6$ for avoiding $(v_3u_2u, v_3v_2w_2, v_3v_1, v_3v_6)$, and furthermore $w_2 \sim v_4$ or $w_2 \sim v_7$ for avoiding $(u_4uu_1, u_4v_6w_2, u_4v_4, u_4v_7)$, the same argument as just used on the case $v_3 \sim v_5$ yields a contradiction. Therefore, $v_3 \nsim v_4/v_5/v_6/v_7$.

We have proved (5). Let $w_6 \notin \{u_3, u_4\}$ be a neighbor of v_6. Then, $w_6 \neq v_i$ for any i by (5). Since $w_6 \sim v_3$ producing a fork $(v_3u_2u, v_3w_6v_6, v_3v_1, v_3v_2)$, $w_6 \nsim v_3$. Since $u_1 \in S(u_4)$ and $w_6 \nsim u_1$, $w_6 \notin S(u_4)$ by Lemma 7.5, and hence $w_6 \sim v_4$ or $w_6 \sim v_7$ for avoiding $(u_4uu_1, u_4v_6w_6, u_4v_4, u_4v_7)$. If $w_6 \sim v_4$, then $w_6 \sim v_5$ for avoiding $(v_4u_2v_3, v_4u_3v_5, v_4u_4, v_4w_6)$

(note $w_6 \nsim v_3$), and thus for avoiding $(u_1uu_4, u_1v_5w_6, u_1v_1, u_1v_2)$, either $w_6 \sim v_1$ producing a fork $(v_1u_1u, v_1w_6v_4, v_1v_3, v_1v_7)$, or $w_6 \sim v_2$ producing a fork $(v_2u_1u, v_2w_6v_4, v_2v_3, v_1w_2)$, both are contradiction. Otherwise, we have $w_6 \sim v_7$. Then, $w_6 \sim v_4$ for avoiding $(u_2uu_1, u_2v_7w_6, u_2v_3, u_2v_4)$, furthermore $w_6 \sim v_5$ for avoiding $(v_4u_2v_3, v_4u_3v_5, v_4u_4, v_4w_6)$ (note $w_6 \nsim v_3$), and thus a fork $(u_1uu_4, u_1v_5w_6, u_1v_1, u_1v_2)$ occurs.

Finally, we suppose that $w_1 \notin S(u_1)$, i.e., $|\{w_1, w_2\} \cap S(u_1)| = 0$. By symmetry, we may suppose further that $N(v_1) \cap S(u_1) = \emptyset = N(v_2) \cap S(u_1)$. Then, $w_1/w_2 \sim v_5$.

If $v_1 \sim v_i$ for $i = 4, 6, 7$, then $v_2 \sim v_i$ for otherwise $v_i \in S(u_1)$ contradicting $N(v_1) \cap S(u_1) = \emptyset$. So, $v_1/v_2 \nsim v_4$. If $v_1 \sim v_6$, let $G(u', v') = G[S_1, S_2]$, then G is reducible as (u, v_4, v_6, u_1) is a reducible sequence, contradicting Lemma 7.8. The same happens if $v_1 \sim v_7$. Therefore, we suppose that $v_1/v_2 \nsim v_4/v_5/v_6/v_7$.

If $v_3 \sim v_6$, then for $i = 1, 2$, $w_i \sim v_6$ for avoiding $(v_3v_iw_i, v_3u_2u, v_3u_6, v_3v_{3-i})$, contradicting $d(v_6) \leq 4$. Since $v_6, v_7 \in N_2(v_4)$, $v_4 \nsim v_6/v_7$.

We have shown that

$$\text{besides } v_1v_3, v_2v_3 \text{ there is no edge } v_iv_j \text{ for } 1 \leq i < j \leq 7. \tag{6}$$

Let $w_6 \notin u_3, u_4\}$ be a neighbor of v_6. Then, $w_6 \neq v_i$ for any i by (5). Since $w_6 \sim v_3$ producing a fork $(v_3u_2u, v_3w_6v_6, v_3v_1, v_3v_2)$, $w_6 \nsim v_3$. To forbid $(u_4uu_1, u_4v_6w_6, u_4v_4, u_4v_7)$, $w_6 \sim v_4$ or $w_6 \sim v_7$. But for $i \in \{4, 7\}$, $w_6 \sim v_i$ and $w_6 \nsim v_{11-i}$ will produce a fork $(u_2uu_1, u_2v_iw_6, u_2v_3, u_2v_{11-i})$. Therefore, $w_6 \sim v_4$ and $w_6 \sim v_7$, and thus $v_6 \in V_3$. Let $G(u', v') = G[S_1, S_2]$. Then, G is reducible as $(u, u_1, v_4, v_6, w_6, v_7, u, v_3, v_5)$ is reducible, contradicting Lemma 7.8.

This completes the proof of Claim 7.9. □

Claim 7.10. $u_4 \nsim v_5$.

Proof. Suppose to the contrary that $u_4 \sim v_5$. Since $v_5 \sim u_1/u_3/u_4$, at most one of w_1 and w_2 may be adjacent to v_5, and thus $|\{w_1, w_2\} \cap S(u_1)| \geq 1$. Since $u_2 \in S(u_1)$, if $w_1, w_2 \in S(u_1)$, then $u_2w_1w_2u_2$ is a triangle by Lemma 7.5. Therefore, we may suppose, by symmetry, that $w_1 \in S(u_1)$ and $w_2 \notin S(u_1)$. Then, $w_1 \in \{v_4, v_7\}$ for otherwise $w_1 \nsim u_2$ contracting Lemma 7.5, and $w_2 \sim v_5$ for avoiding $(u_1uu_2, u_1v_2w_2, u_1v_1, u_1v_5)$. By symmetry, we suppose that $w_1 = v_4$. Then, $v_1 \sim v_6$, or $w_2 \sim v_4$, or $w_2 \sim v_6$ for avoiding $(u_3v_4v_1, u_3v_5w_2, u_3u, u_3v_6)$.

If $v_1 \sim v_6$, then $v_2 \sim v_6$ since $v_6 \notin S(u_1)$, $w_2 \sim v_7$ for avoiding $(u_4v_5w_2, u_4v_6v_1, u_4u, u_4v_7)$, and $w_2 \sim v_4$ for avoiding $(u_2uu_1, u_2v_7w_2, u_2v_3, u_2v_4)$. But then, $\{v_3, v_7\}$ is a 2-cut contradicting Lemma 7.4. So, we suppose that $v_1 \nsim v_6$.

If $w_2 \sim v_4$, then $w_2 \sim v_7$ for avoiding $(u_2uu_1, u_2v_4w_2, u_2v_2, u_2v_7)$. Let $w_6 \notin \{u_3, u_4\}$ be a neighbor of v_6. If $w_6 = v_3$, there would be a fork $(u_4v_6v_3, u_4v_7w_2, u_4u, u_4v_5)$. If $w_6 = v_2$, there would be a fork $(u_1uu_2, u_1v_2v_6, u_1v_1, u_1v_5)$ (note that $v_1 \nsim v_6$). Therefore, $w_6 \notin \{v_1, v_2, \ldots, v_7\}$. Then, $w_6 \sim v_1$ for avoiding $(u_3v_4v_1, u_4v_6w_6, u_3u, u_3v_5)$, $w_6 \sim v_2$ for avoiding $(u_1uu_2, u_1v_1w_6, u_1v_2, u_1v_5)$, and hence a fork $(v_2v_3u_2, v_2w_6v_6, v_2u_1, v_2w_2)$ occurs. So, we further suppose that $w_2 \nsim v_4$.

The only remaining situation is that $w_2 \sim v_6$. Then, $w_2 \sim v_7$ for avoiding $(v_5u_3v_4, v_5u_4v_7, v_5u_1, v_5w_2)$, and thus produces a fork $(u_2uu_1, u_2v_7w_2, u_2v_3, u_2v_4)$. This contradiction completes the proof of Claim 7.10. □

By Claims 7.9 and 7.10, $u_4 \nsim v_4/v_5$. Then, $u_4 \in V_3$. We show that

$$\{w_1, w_2\} \cap \{v_4, v_5, v_6, v_7\} = \emptyset. \tag{7}$$

Since G is triangle-free, $w_1 \neq v_5 \neq w_2$. Since $w_1 = v_4$ producing a fork $(u_1uu_4, u_1v_1v_4, u_1v_2, u_1v_5)$, and $w_1 = v_6$ producing a fork $(u_1uu_2, u_1v_1v_6, u_1v_2, u_1v_5)$, we have $\{w_1, w_2\} \cap \{v_4, v_6\} = \emptyset$ by symmetry.

To prove (7), we need only to show that $w_1 \neq v_7$ by symmetry. Suppose to the contrary that $w_1 = v_7$. Then, $w_2 \sim v_5$ for avoiding $(u_1uu_4, u_1v_2w_2, u_1v_1, u_1v_5)$. Let $w_4 \notin \{u_2, u_3\}$ be a neighbor of v_4. Since G is triangle-free, $w_4 \notin \{v_3, v_5, v_6, v_7\}$. We prove that $w_4 \neq v_1, v_2$.

Suppose that $w_4 = v_1$. Then, $v_4 \sim v_2$ by (7). If $v_7 \in V_3$, let $G(u', v') = G[S_1, S_2]$, then G is reducible as $(u, v_4, v_7, u_1, v_5, v', v_3)$ is reducible. Otherwise, suppose that $v_7 \in V_4$ and let

$w_7 \neq \{u_2, u_4, v_1\}$ be a neighbor of v_7. Since $v_7 \sim v_5$ producing a fork $(u_3v_4v_2, u_3v_5v_7, u_3u, u_3v_6)$, $v_7 \nsim v_5$. Then, $w_7 \sim v_3$ for avoiding $(u_2uu_1, u_2v_7w_7, u_2v_3, u_2v_4)$, $w_2 \sim w_7$ for avoiding $(v_3u_2u, v_3v_2w_2, v_3v_1, v_3w_7)$, and furthermore $w_2 \sim v_6$ for avoiding $(v_7w_7w_2, v_7u_4v_6, v_7u_2, v_7v_1)$, and thus $\{v_6, w_7\}$ is a 2-cut, contradicting Lemma 7.4. Therefore,

$$w_4 \notin \{v_1, v_2, \ldots, v_7\}.$$

If $w_4 = w_2$, then $w_2 \sim v_7$ for avoiding $(u_2uu_1, u_2v_4w_2, u_2v_3, u_2v_7)$, and hence forcing a fork $(v_7u_4v_6, v_7w_2v_2, v_7u_2, v_7v_1)$. So, we suppose that $w_4 \neq w_2$. Then, either $w_4 \sim v_3$ or $w_4 \sim v_7$ for avoiding $(u_2uu_1, u_2v_4w_4, u_2v_3, u_2v_7)$. In the former case, $w_4 \sim v_7$ for avoiding $(v_3u_2u, v_3v_1v_7, v_3v_2, v_3w_4)$, and $w_2 \sim w_4$ for avoiding $(v_3u_2u_1v_3v_2w_2, v_3v_1, v_3w_4)$, and thus a fork $(w_4v_4u_3, w_4v_7u_4, w_4v_3, w_4w_2)$ occurs. In the latter case, $w_4 \sim v_6$ for avoiding $(v_7u_4v_6, v_7v_1u_1, v_7u_2, v_7w_4)$, then by letting $G(v_1', v_4') = G[\{v_1, v_2, v_5\}, \{v_4, v_6, v_7\}]$, one sees that G is reducible as $(u_1, u_4, w_4, u, u_2, u_3, v_3)$ is reducible. This proves (7).

To avoid $(u_1uu_4, u_1v_1w_1, u_1v_2, u_1v_5)$, $w_1 \sim v_5$. The same argument shows that $w_2 \sim v_5$. Thus, either $v_1, v_2 \in V_3$, or they have a common neighbor w_1' other than u_1 and v_3. If $v_3 \in V_3$, let $G(v_1') = G[\{v_1, v_2, v_5\}]$, then G is reducible as (u_1, v_3) is reducible. So, $v_3 \in V_4$. Let $w_3 \notin \{u_2, v_1, v_2\}$ be a neighbor of v_3. Since $w_3 = v_6$ implies $w_1/w_2 \sim v_6$ for avoiding $(v_3u_2u, v_3v_1w_1, v_3v_2, v_3v_6)$ and $(v_3u_2u, v_3v_2w_2, v_3v_1, v_3v_6)$ that contradict $d(v_6) \leq 4$,

$$w_3 \notin \{v_1, v_2, \ldots, v_7\}.$$

Since $u \in S(v_3)$, $w_1, w_2 \notin S(v_3)$, and thus $w_1/w_2 \sim w_3$ for avoiding $(v_3u_2u, v_3v_1w_1, v_3v_2, v_3w_3)$ and $(v_3u_2u, v_3v_2w_2, v_3v_1, v_3w_3)$. To avoid $(u_2uu_1, u_2v_3w_3, u_2v_4, u_2v_7)$, $w_3 \sim v_4$ or $w_3 \sim v_7$.

If $w_3 \sim v_4$, then by (7), either $v_7 \sim v_1/v_2$, or $v_7 \in V_3$ and v_7 and v_4 have a common neighbor other than u_2 for avoiding a fork at u_2. In the former case, let $G(u', v_1', v_4', w') = G[\{S_1, S_2, \{v_4, v_6\}, \{w_1, w_2\}]$, then G is reducible as $((u, v_7, u_1, v_5, u', v_3, v_1', w_3)$ is reducible. In the latter case, let $G(u', v_1', v_4', w') = G[\{S_1, S_2, \{v_4, v_6, v_7\}, \{w_1, w_2\}]$, then G is reducible as $(u, u_1, v_5, u', v_3, v_1', w_3)$ is reducible.

Suppose that $w_3 \sim v_7$. The same as above, either $v_4 \sim v_1/v_2$, or $v_4 \in V_3$ and v_4 and v_7 have a common neighbor other than u_2 for avoiding a fork at u_2, one gets a contradiction.

Subcase 2.2. Now, we suppose that $N(u_2) \cap N(u_3) = \{u\}$. Let $v_4 \neq u$ be a common neighbor of u_1 and u_3, $v_5 \neq u$ be a common neighbor of u_3 and u_4, and $v_6 \neq u$ be a common neighbor of u_2 and u_4. If $\{u_2, u_3, u_4\} \nsubseteq V_3$, then either $d(u_3) = d(u_4) = 4$, or $d(u_2) = d(u_4) = 4$. In the former case, suppose that $N(u_3) \cap N(u_4) = \{u, v_5, v_5'\}$. In the latter case, suppose that $N(u_2) \cap N(u_4) = \{u, v_6, v_6'\}$.

Since $u_2, u_4 \in S(u_1)$, if $|S(u_1)| = 3$, then $v_6 \in S(u_1)$ by Lemma 7.5. Since $v_5 \nsim v_4$ and $v_5 \notin S(u_1)$, $w_1 \neq v_5 \neq w_2$.

Suppose that $w_1 = v_6$. Then, $w_2 \sim v_4$ for avoiding $(u_1uu_2, u_1v_2w_2, u_1v_1, u_1v_4)$. If $u_2, u_3, u_4 \in V_3$, let $G(u_2', v_1', v_3') = G[\{u_2, u_4\}, \{v_1, v_2, v_4\}, \{v_3, v_6\}]$. If $d(u_3) = d(u_4) = 4$, by considering the possible forks with center u_4, one sees that either v_5 (resp. v_5') has degree 3 or it is adjacent to both v_1 and v_2, then let $G(u_2', v_1', v_3', v_5') = G[\{u_2, u_4\}, \{v_1, v_2, v_4\}, \{v_3, v_6\}, \{v_5, v_5'\}]$. If $d(u_2) = d(u_4) = 4$, by considering the forks with center u_2, one sees that each neighbor of v_6' is adjacent to v_3 or v_6, then let $G(u_2', v_1', v_3') = G[\{u_2, u_4\}, \{v_1, v_2, v_4\}, \{v_3, v_6, v_6'\}]$. In each situation, it is easy to check that G is reducible. Therefore, be symmetry, we suppose that $\{w_1, w_2\} \cap \{v_4, v_5, v_5', v_6, v_6'\} = \emptyset$. Hence

$$w_1/w_2 \sim v_4 \tag{8}$$

for avoiding $(u_1uu_2, u_1v_1w_1, u_1v_2, u_1v_4)$ and $(u_1uu_2, u_1v_2w_2, u_1v_1, u_1v_4)$, and

$$\text{either } v_1, v_2 \in V_3, \text{ or } \text{ they have a common neighbor other than } u_1, v_3. \tag{9}$$

If $v_3 \in V_4$, let $w_3 \notin \{u_2, v_1, v_2\}$ be a neighbor of v_3. If $w_3 = v_5$, there would be either a fork $(v_3u_2u, v_3v_1w_1, v_3v_2, v_3v_5)$ or a fork $(v_3u_2u, v_3v_2w_2, v_3v_1, v_3v_5)$. Then, $v_5 \neq w_3 \neq v_5'$ by symmetry. Therefore, w_3 is not adjacent to u_2, u_3 and u_4. To avoid forks with center v_3, $w_1/w_2 \sim w_3$.

If $d(u_2) = 4$, then $N(u_2) \cap N(u_4) = \{u, v_6, v_6'\}$. To avoid the fork with one long arm u_2uu_1, one sees that $N(v_6) = N(v_6')$, a contradiction to Lemma 7.3.

So, we suppose that $d(u_2) = 3$. If $d(v_3) = 3$, let $G(u', v') = G[S_1, S_2]$, then G is reducible as (u, v_3, u_1) is reducible. If $d(v_3) = 4$, let $G(u'_3, v'_1, w') = G[\{u_3, u_4\}, \{v_1, v_2, v_4\}, \{w_1, w_2\}]$, then G is reducible as (u_1, u, u_2, v_3, w_3) is reducible. This contradiction completes the proof of Section 2.

7.6 $|S(u)| = 2$

In this section, we always suppose that $|S(x)| = 2$ for every 4-vertex x.

Suppose $S(u) = \{v_1, v_2\}$. There are two possibilities by symmetry: one is that $v_1/v_2 \sim u_1$, and the other is that $v_1 \sim u_1$ and $v_2 \sim u_2$. Note that $S_1 = \{u_2, u_3, u_4\}$, and $S_2 = \{v_1, v_2\}$. We distinguish two cases accordingly.

Case 1. Suppose that $v_1/v_2 \sim u_1$.

First, we consider the situation that $u_1 \in V_3$. Let $v_3 \neq u$ be a common neighbor of u_2 and u_3, $v_4 \neq u$ be a common neighbor of u_3 and u_4, and $v_5 \neq u$ be a common neighbor of u_2 and u_4. If $\{u_2, u_3, u_4\} \subseteq V_3$, let $G(u') = G[S_1]$, then G is reducible as (u, u_1) is reducible. If there exists a vertex, say v_3, in $N_2(u)$ who is adjacent to each vertex of S_1, one sees that G is reducible by letting $G(u') = G[S_1]$. So, we suppose that each vertex in $N_2(u)$ is adjacent to at most two vertices of S_1. Then, $\{u_2, u_3, u_4\} \nsubseteq V_4$. By symmetry, we suppose that $u_2 \in V_3$ and $u_3, u_4 \in V_4$. Let $v'_4 \notin \{u, v_4\}$ be a common neighbor of u_3 and u_4, let $w_4 \in N(v_4) \setminus N(v'_4)$ and $w'_4 \in N(v'_4) \setminus N(v_4)$.

If $w_4 = v_1$ and $w'_4 = v_2$, then $v_1 \sim v_3$ or $v_2 \sim v_3$ for avoiding $(u_3v_4v_1, u_3v'_4v_2, u_3u, u_3v_3)$, and $v_1 \sim v_5$ or $v_2 \sim v_5$ for avoiding $(u_4v_4v_1, u_4v'_4v_2, u_3u, u_4v_5)$. Whatever which occurs, one sees, by the possible forks at u_3 and u_4, that either v_4 and v'_4 have a common neighbor other than u_3, u_4, or one of them has degree 3 and the other has a common neighbor with v_3 and v_5. Let $G(u', v'_1, v'_3, v') = G[S_1, S_2, \{v_3, v_5\}, \{v_4, v'_4\}]$, then G is reducible as (u, u_1, u', v') is reducible.

If $w_4 = v_1$ and $w'_4 \neq v_2$, then $w'_4 \sim v_3/v_5$ for avoiding $(u_3uu_1, u_3v'_4w'_4, u_3v_3, u_3v_4)$ and $(u_4uu_1, u_4v'_4w'_4, u_4v_3, u_4v_5)$. Note that $u_1 \in S(u_3) \cap S(u_4)$ and $|S(u_3)| = |S(u_4)| = 2$, by analyzing the possible forks at u_3 and u_4, either $v_3, v_4, v'_4, v_5 \in V_3$, or there exists a new vertex adjacent to at least two of them. Let $G(u', v'_3) = G[S_1, \{v_3, v_4, v'_4, v_5\}]$. Then G is reducible as (u, u', u_1, w'_4) is reducible.

Therefore, we assume by symmetry that $\{v_1, v_2\} \cap \{w_4, w'_4\} = \emptyset$. Then, $w_4/w'_4 \sim v_3/v_5$ for avoiding forks with center u_3 or u_4. By letting $G(u', v'_3) = G[S_1, \{v_3, v_4, v'_4, v_5\}]$, one sees that G is reducible.

Now, we suppose that $u_1 \in V_4$. Let $v_3 \neq u$ be a common neighbor of u_1 and u_2, $v_4 \neq u$ be a common neighbor of u_2 and u_3, $v_5 \neq u$ be a common neighbor of u_3 and u_4, and $v_6 \neq u$ be a common neighbor of u_2 and u_4. For $i = 1, 2$, let $w_i \in N(v_i) \setminus N(v_{3-i})$. We distinguish five subcases by symmetry.

Subcase 1.1. Suppose that $u_3 \sim v_3$ and $u_4 \nsim v_4$. Then, $u_4 \in V_3 \cap S(u_1)$, and $u_2, u_3 \notin S(u_1)$.

Let $S(u_1) = \{u_4, x\}$. Then, $x \sim u_4$ by Lemma 7.5, and thus $x \in \{v_5, v_6\}$. Since G is triangle-free, $v_3 \nsim v_4/v_5/v_6$. Since u_2 and u_3 take the symmetric roles, we may suppose that $x = v_5 = w_1$. Then, $w_2 \sim v_3$ as $S(u_1) = \{u_4, v_5\}$, and thus $w_2 \notin \{v_1, v_2, \ldots, v_6\}$. To avoid $(v_3u_1v_1, v_3u_2v_6, v_3u_3, v_3w_2)$, either $v_1 \sim v_6$ or $w_2 \sim v_6$. We distinguish two possibilities accordingly.

(a) $v_1 \sim v_6$. Then $v_2 \sim v_6$ as $v_6 \notin S(u_1)$, and thus for avoiding $\{v_3, v_4, w_2\} \subseteq S(v_6)$, $v_4 \sim v_1/v_2$ as $v_4 \notin S(u_1)$ that forces $\{v_5, w_2\}$ to be a 2-cut, contradicting Lemma 7.4.

(b) $w_2 \sim v_6$. Since $w_2 \sim v_5$ implying $S(v_3) = \{v_1\}$, $w_2 \nsim v_5$, and thus $v_5 \in S(v_3)$. If $w_2 \in V_3$, then $w_2 \in S(u_3)$ implying $v_1 \notin S(u_3)$ as $v_1 \nsim w_2$, and thus $v_1 \sim v_4$ implying $v_2 \sim v_4$ as $v_4 \notin S(u_1)$. Let $G(u', v'_1) = G[S_1, S_2]$. Then G is reducible as $(u, v_4, u_1, v_3, u', v_5, v_6, w_2, v'_1)$ is reducible. Otherwise $w_2 \in V_4$, then $w_2 \sim v_4$ for otherwise the fourth neighbor of w_2 together with v_1 and v_5 makes $|S(v_3)| \geq 3$, and so $u_4 \in S(w_2)$. Since $v_5 \in S(u_1)$, $v_2 \nsim v_5$ implying $v_5 \notin S(w_2)$, and thus $S(w_2) = \{u_4\}$ contradicting $|S(w_2)| = 2$.

Subcase 1.2. Suppose that $u_3 \sim v_3$ and $u_4 \sim v_4$. Let $G(u') = G[S_1]$. Then, G is reducible as (u, v_4) is reducible.

After Subcases 1.1 and 1.2, we suppose, by symmetry, that

$$u_3/u_4 \nsim v_3. \tag{10}$$

Then, $S(u_1) = \{u_3, u_4\}$, and hence $w_1/w_2 \sim v_3$. Let $w_3 \neq u$ be a common neighbor of v_1 and v_2.

Subcase 1.3. $u_3, u_4 \in V_3$.

If $\{w_1, w_2, w_3\} \cap \{v_4, v_5, v_6\} \neq \emptyset$, we suppose, as an instance, that $w_3 = v_4$, let $G(u', v') = G[S_1, S_2]$, then G is reducible as (u, v_4, u_1) is reducible. So, we suppose that

$$\{w_1, w_2, w_3\} \cap \{v_4, v_5, v_6\} = \emptyset.$$

Since $S(u_1) = \{u_3, u_4\}$, if $v_1 \in V_4$, then v_1 and v_2 have another common neighbor other than u_1 and w_3, and hence $\{u\} = S(v_1) = S(v_2)$ by Lemma 7.5. Therefore, $v_1, v_2 \in V_3$.

If there does not exist edge between $\{w_1, w_2\}$ and $\{v_4, v_6\}$, then $S(u_2) = \{w_1, w_2\}$ and $S(v_3) = \{v_4, v_6\}$, and hence each neighbor of w_1 other than v_1 must be adjacent to w_2, and each neighbor of v_6 other than u_4 must be adjacent to v_4. Let $G(u', v_1', v_4', w') = G[S_1, \{v_1, v_2, v_3\}, \{v_4, v_6\}, \{w_1, w_2\}]$. Then, G is reducible as (u, u_1) is reducible. So, we suppose that there must be edges between $\{w_1, w_2\}$ and $\{v_4, v_6\}$. Without loss of generality, suppose $w_1 \sim v_6$.

If $w_1 \sim v_4$, let $G(u', v_1', v_4') = G[S_1, \{v_1, v_2, v_3\}, \{v_4, v_6\}]$, then G is reducible as (u, u_1, w_1) is reducible. Therefore, we suppose that each of w_1 and w_2 has at most one neighbor in $\{v_4, v_6\}$, and vice versa. We distinguish two possibilities.

(a) $w_2 \sim v_4$. Then the two vertices in $S(v_3)$, say x_1 and x_2, are adjacent to w_1 and w_2 respectively, and the two vertices in $S(u_2)$, say w_4 and w_6, are adjacent to v_4 and v_6 respectively. Suppose $x_1 \sim w_1$, $x_2 \sim w_2$, $w_4 \sim v_4$, and $w_6 \sim v_6$.

If $x_1 = w_4$, then $v_4 \in S(w_1)$, and $w_2 \in S(w_1)$ as $w_2 \nsim v_6/x_1/v_1$, contradicting $u_4 \in S(w_1)$ by Lemma 7.5. By symmetry, we may suppose that $|\{x_2, x_3, w_4, w_6\}| = 4$. Then, $S(w_1) = \{w_2, u_4\}$, contradicting Lemma 7.5.

(b) Suppose that $w_2 \nsim v_4$. If $w_2 \in V_3$, let $x_2 \notin \{v_2, v_3\}$ be a neighbor of w_2. Otherwise, let $x_2, x_2' \notin \{v_2, v_3\}$ be the neighbors of w_2.

If $w_2 \in V_4$ and $x_2/x_2' \nsim w_1$, then $S(v_3) = \{v_4, x_2, x_2'\}$ as $w_1 \nsim v_4$, contradicting $|S(v_3)| = 2$.

If $w_2 \in V_4$ and $x_2' \sim w_1$, then $S(v_3) = \{x_2, v_4\}$, and hence $v_4 \sim x_2$ by Lemma 7.5. Since $u_4 \in S(w_1)$, $S(w_1) = \{u_4, v_5\}$ and thus $v_5 \sim x_2'$. Since $w_3 \notin S(w_1)$, $w_3 \sim v_6$ or $w_3 \sim x_2'$. In the former situation, $w_3 \sim v_4$ as $S(u_2) = \{w_2, x_2\}$, this forces $S(v_4) = \{v_2, w_2\}$ that further forces $v_5 \sim x_2$ for otherwise $v_5 \in S(v_4)$. But then, $\{x_2, x_2'\}$ is a 2-cut. Therefore, we suppose that $w_3 \nsim v_6$ and $w_3 \sim x_2'$. Then, a fork $(x_2'v_5u_4, x_2'w_1v_6, x_2'w_2, x_2'w_3)$ occurs, a contradiction.

So, we suppose by symmetry that $w_2, v_4 \in V_3$. Note that $v_4 \in S(v_3)$ as $v_4 \nsim w_1/w_2$. If $x_2 \sim w_1$, then $x_2 \notin S(v_3)$ and thus $S(v_3) = \{v_4\}$, a contradiction. So, we assume that $x_2 \nsim w_1$. Then, $x_2 \in S(v_3)$ and thus $v_4 \sim x_2$ by Lemma 7.5. Now, $w_1, v_6 \in V_3$ for otherwise $|S(v_3)| > 2$ or $|S(u_2)| > 2$. Let $G(u', v') = G[S_1, \{v_1, v_2, v_3\}]$. Then G is reducible as $(u, u_1, v_6, v_5, w_1, w_2, u', v')$ is reducible.

Subcase 1.4. $u_3, u_4 \in V_4$. By (10), this happens if and only if u_3 and u_4 have another common neighbor $v_5' \neq u$. Using the same arguments as that used in Subcase 1.3, one shows that $\{w_1, w_2, w_3\} \cap \{v_4, v_5, v_5', v_6\} = \emptyset$, and $v_1, v_2 \in V_3$. Let $S(u_3) = \{u_1, x\}$. Then, $x \in \{v_1, v_2, v_3\}$ by Lemma 7.5. But this means $\{w_1, w_2, w_3\} \cap \{v_4, v_5, v_5', v_6\} \neq \emptyset$, a contradiction.

Subcase 1.5. The only remaining subcase is that $u_3 \nsim v_3$ but $u_4 \sim v_4$. Using the same arguments as that used in Subcase 1.3, one shows that $\{w_1, w_2, w_3\} \cap \{v_4, v_5, v_6\} = \emptyset$, and $v_1, v_2 \in V_3$. Let $S(u_4) = \{u_1, x\}$. Then, $x \in \{v_1, v_2, v_3\}$ by Lemma 7.5. But this means $\{w_1, w_2, w_3\} \cap \{v_4, v_5, v_6\} \neq \emptyset$, a contradiction.

Case 2. Suppose that $v_i \sim u_i$ for $i = 1, 2$. Then, $v_1 \sim v_2$ by Lemma 7.5. Without loss of generality, we suppose that

(*) for each 4-vertex x, the two vertices in $S(x)$ have no common neighbor.

Since $N(u_3) \not\subseteq N(u_4)$, $|N(u_1) \cap N(u_2)| \leq 2$, and $N(u_i) \cap N(u_j) \neq \{u\}$ for some $i \in \{1, 2\}$ and $j \in \{3, 4\}$. By symmetry, we suppose that $v_3 \neq u$ be a common neighbor of u_1 and u_3.

Subcase 2.1. Suppose that $N(u_1) \cap N(u_2) = \{u, v_1'\}$. Let $v_4 \neq u$ be a common neighbor of u_3 and u_4, and $v_5 \neq u$ be a common neighbor of u_2 and u_4. Since $N(u_3) \nsubseteq N(u_4)$ and $N(u_4) \nsubseteq N(u_3)$, $u_3 \nsim v_5$ and $u_4 \nsim v_3$. Either $u_3, u_4 \in V_3$, or u_3 and u_4 have another common neighbor, say $v_4' \notin \{u, v_4\}$.

First, we suppose that $u_3, u_4 \in V_3$. Let $S(u_1) = \{u_4, x\}$. Then, $x \in \{v_4, v_5\}$ by Lemma 7.5, and $v_2 \sim v_3$ as $v_2 \notin S(u_1)$. The same argument by discussing on $S(u_2)$ (note $u_3 \in S(u_2)$), we have $v_1 \sim v_5$. If $v_5 \notin S(u_1)$, then $v_5 \sim v_3$ that forces $u_4 \in S(v_3)$, and thus $S(v_3) \supseteq \{u_4, v_1', v_4\}$ contradicting $|S(v_3)| = 2$. So, $S(u_1) = \{u_4, v_5\}$. Let $w_1' \notin \{u_1, u_2\}$ be a neighbor of v_1'. Since $w_1' \notin S(u_1)$, $w_1' \sim v_1$ or $w_1' \sim v_3$. If $w_1' \sim v_1$, then $w_1' = v_4$ for otherwise $S(u_2) = \{u_3, w_1'\}$ as $w_1' \nsim v_5$, contradicting Lemma 7.5. So, $w_1' \sim v_3$. To forbid $(v_3v_2u_2, v_3u_3v_4, v_3u_1, v_3w_1')$, $v_2 \sim v_4$ or $w_1' \sim v_4$. If $v_2 \sim v_4$, then $u \in S(v_2)$ forcing $v_1', w_1' \notin S(v_2)$ by Lemma 7.5, and hence $v_1' \sim v_4$ and $w_1' \sim v_1$ that imply $\{v_5, w_1'\}$ is a 2-cut of G. If $w_1' \sim v_4$, since $u_2 \in S(v_3)$ forces $S(v_3) = \{u_2, v_5\}$, $v_5 \sim w_1'$, then $S(u_1) = \{u_4, v_5\}$ and $S(u_2) = \{u_3, v_3\}$ force either $v_1', v_1, v_2 \in V_3$ that implies v_4 is a cut vertex, or they have a common neighbor x that implies $\{v_4, x\}$ is a 2-cut of G, both contradict Lemma 7.4.

Suppose, u_3 and u_4 have another common neighbor, say $v_4' \notin \{u, v_4\}$. Since $u_4 \in S(u_1)$, $S(u_1)$ contains one of v_4, v_4', v_5, and then $v_2 \notin S(u_1)$ that forces $v_2 \sim v_3$. Since $u_3 \in S(u_2)$, $S(u_2)$ contains one of v_3, v_4, v_4', and then $v_1 \notin S(u_2)$ that forces $v_1 \sim v_5$. If $v_5 \notin S(u_1)$, then $v_5 \sim v_3$ that forces $u_4 \in S(v_3)$, contradicting $v_1' \in S(v_3)$ by Lemma 7.5. So, we suppose that $S(u_1) = \{u_4, v_5\}$. The same arguments as above one can deduce contradictions.

From now on to the end of the proofs, we always suppose that

$$N(u_1) \cap N(u_2) = \{u\}.$$

Note that the number of edges between $N(u)$ and $N_2(u) \setminus \{v_1, v_2\}$ is at most 14. Since each vertex in $N_2(u) \setminus \{v_1, v_2\}$ has at least two neighbors in $N(u)$, $|N_2(u)| \leq 7$.

If $d(u_1) = 4$ and $N(u_1) \cap N(u_3) = \{u\}$, then $\{u_2, u_3\} \subseteq S(u_1)$, contradicting our assumption (*). So, we suppose by symmetry that

(**) for $i \in \{1,2\}$, if $d(u_i) = 4$ then $N(u_i) \cap N(u_j) \neq \{u\}$ for $3 \leq j \leq 4$.

Subcase 2.2. First suppose that each vertex in $N_2(u) \setminus \{v_1, v_2\}$ has exactly two neighbors in $N(u)$. Then, $6 \leq |N_2(u)| \leq 7$. Note that $N(u_1 \cap N(u_2) = \{u\}$. By our assumption (**), we suppose that $N(u_1) \cap N(u_3) = \{u, v_3\}$, $N(u_2) \cap N(u_3) = \{u, v_4\}$, $N(u_1) \cap N(u_4) = \{u, v_5\}$, $N(u_2) \cap N(u_4) = \{u, v_6\}$, and suppose that $N(u_3) \cap N(u_4) = \{u, v_7\}$ while $|N_2(u)| = 7$.

We show that

$$\text{there is no edge } v_iv_j \text{ with } 3 \leq i < j \leq 7. \tag{11}$$

Since G is triangle-free, the only possible edges are v_3v_6 and v_4v_5. Since $v_3 \sim v_6$ and $v_4 \sim v_5$ produce a fork $(u_1v_3v_6, u_1v_4v_5, u_1u, u_4v_1)$, we suppose by symmetry that $v_3 \sim v_6$ and $v_4 \nsim v_5$.

If $v_6 \notin S(u_1)$, then $v_6 \sim v_1$ that forces $u_3, v_5 \in S(v_6)$, contradicting Lemma 7.5. So, we suppose that $v_6 \in S(u_1)$, i.e., $S(u_1) = \{u_2, v_6\}$. Since $v_2 \notin S(u_1)$, $v_2 \sim v_3$ or $v_2 \sim v_5$. If $v_2 \sim v_5$ and $v_2 \nsim v_3$, then $S(u_2) \supseteq \{u_1, v_3, v_5\}$ contradicting $|S(u_2)| = 2$. So, we suppose that $v_2 \sim v_3$. Then $v_2 \sim v_5$ for avoiding $(v_3u_1v_5, v_3u_3v_4, v_3v_2, v_3v_6)$, and $S(v_2) = \{u, u_4\}$, and $v_4 \sim v_1$ as $v_4 \notin S(v_2)$ and $v_4 \nsim v_5$, but this implies $\{u_2, v_6, v_4\} \subseteq S(u_1)$ contradicting $|S(u_1)| = 2$.

We have proved (11). Next, we show that

$$\text{there is no edge } v_iv_j \text{ for } 1 \leq i \leq 2 \text{ and } 3 \leq j \leq 7. \tag{12}$$

If $v_1 \sim v_7$, then $S(u_1) = \{u_2, v_7\}$ as $v_7 \nsim v_3/v_5$ by (11), contradicting Lemma 7.5. The same happens if $v_2 \sim v_7$. So,

$$v_1/v_2 \nsim v_7. \tag{13}$$

To prove (12), we only need to show that $v_1 \nsim v_4$ by symmetry.

Suppose to the contrary that $v_1 \sim v_4$. Then, $S(u_2) = \{u_1, v_3\}$ or $\{u_1, v_5\}$. By rearranging u_3 and u_4, one sees that v_3 and v_5 take the same roles. So, we suppose

$$S(u_2) = \{u_1, v_3\}. \tag{14}$$

By (11)

$$v_3 \sim v_2, \text{ and } v_5 \nsim v_2. \tag{15}$$

We further distinguish two possibilities upon $d(v_4)$.

(a) Assume $v_4 \in V_4$. Let $w_4 \notin \{u_2, u_3, v_1\}$ be a neighbor of v_4. Then, $w_4 \neq v_i$ for any $1 \leq i \leq 7$ by (11), and $w_4 \sim v_2$ or $w_4 \sim v_6$ as $w_4 \notin S(u_2)$. Now, we have

$$S(u_1) = \{u_2, v_4\}. \tag{16}$$

(a-1) Suppose $w_4 \sim v_2$. Then $S(v_2) = \{u, u_3\}$, and $v_6 \notin S(v_2)$ forces $v_6 \sim v_1$ or $v_6 \sim w_4$ by (11). Since $v_6 \sim v_1$ forces $v_6 \in S(u_1)$ contradicting (16), $v_6 \sim w_4$ that forces $S(v_4) = \{u_1, v_3\}$. Let $w_5 \neq \{u_1, u_4\}$ be a neighbor of v_5. Then $w_5 \neq v_i$ for any i, and $w_5 \sim v_1$ or $w_5 \sim v_3$ for otherwise $w_5 \in S(u_1)$ contradicting (16).

First suppose $v_5 \in V_3$. If $w_5 \sim v_1$, then $S(v_1) = \{u, u_3\}$, and $w_5 \sim v_3$ or $w_5 \sim w_4$ as otherwise the third neighbor of w_5 would be in $S(v_1)$. In the former situation, $w_5 \sim v_3$, then $S(v_3) = \{u_2, v_4\}$ forcing $w_4 \sim w_5$ as otherwise $w_4 \in S(v_3)$, and this implies that $\{v_6, v_7\}$ is a 2-cut contradicting Lemma 7.4. So, we suppose the latter situation, i.e., $w_5 \sim w_4$ and $w_5 \nsim v_3$. Then, $w_5 \in V_3$ as otherwise the fourth neighbor of w_5 would be in $S(v_1)$ contradicting Lemma 7.5 as $u \in S(v_1)$. But this forces $v_3 \in V_3$ as otherwise the fourth neighbor of v_3 would be in $S(u_1)$ contradicting (16), and thus $\{v_6, v_7\}$ is a 2-cut again. By using almost the same argument, one can deduce the same contradiction if $w_5 \nsim v_3$.

Therefore, suppose $v_5 \in V_4$. Let $w_5' \neq \{u_1, u_4, w_5\}$ be the fourth neighbor of v_5. Following above analysis, we may suppose by symmetry that $w_5 \sim v_1$ and $w_5' \sim v_3$. Since $d(w_5) \geq 3$, $w_5 \sim w_4$ for otherwise the third neighbor of w_5 would be in $S(v_1)$ contradicting Lemma 7.5 as $u \in S(v_1)$. Then, $S(v_3) = \{u_2, w_4\}$, again contradicting Lemma 7.5.

(a-2) Suppose that $w_4 \sim v_6$ and $w_4 \nsim v_2$. Then, $S(v_4) = \{u_1, v_3\}$ or $\{u_1, v_5\}$.

(a-2-1) Suppose that $S(v_4) = \{u_1, v_5\}$. Then, $v_5 \sim w_4$, $v_3 \sim w_4$ as $v_3 \notin S(v_4)$, and $|N_2(u)| = 6$ as otherwise $v_7 \in S(v_4)$ as $v_7 \nsim v_1$ by (13). By (15), $v_5 \nsim v_2$.

If $v_5 \in V_4$, let $w_5 \notin \{u_1, u_4, w_4\}$ be the fourth neighbor of v_5, then $w_5 \sim v_1$ as $w_5 \notin S(u_1)$, and thus $S(v_1) = \{u, u_3\}$. Let $x \notin \{v_1, v_5\}$ be a neighbor of w_5, then $x \sim v_2$ as $x \notin S(v_1)$, and thus $x \neq v_6$ as G is triangle-free. Since $S(u_2) = \{u_1, v_3\}$ and $S(v_1) = \{u, u_3\}$, $v_6, w_5 \in V_3$ that forces x to be a cut vertex, contradicting Lemma 7.4. The same contradiction would occur if $v_6 \in V_4$. Therefore, $v_5, v_6 \in V_3$ that forces $\{v_1, v_2\}$ to be a 2-cut contradicting Lemma 7.4.

(a-2-2) Suppose that $S(v_4) = \{u_1, v_3\}$. Again, by (15), $v_5 \nsim v_2$. Let $w_5 \notin \{u_1, u_4\}$ be a neighbor of v_5. Then, $w_5 \sim v_1$ or $w_5 \sim v_3$ as $w_5 \notin S(u_1)$.

(a-2-2-1) Suppose that $w_5 \sim v_1$. Then, $S(v_1) = \{u, u_3\}$. Note that $w_4 \nsim v_2$, $w_4 \sim w_5$ as $w_4 \notin S(v_1)$.

If $|N(u)| = 7$, then, $v_7 \sim w_4$ as $v_7 \notin S(v_4)$ and $v_7 \nsim v_1$ by (13). Thus, $v_5 \in S(w_4)$ by (11). Let $S(w_4) = \{v_5, x\}$. Then $x \sim v_5$ by Lemma 7.5, and $x \sim v_6$ or $x \sim v_7$ by our assumption (*). Since $x \notin S(u_1)$, $x \sim v_3$, and thus $S(v_3) = \{u_2, v_4\}$. If $x \sim v_6$, then $x \in S(u_2)$ contradicting Lemma 7.5. If $x \sim v_7$, then $w_5 \in S(v_7)$. Let $S(v_7) = \{w_5, y\}$. Then $y \sim x$, $y \sim v_2$ as $y \notin S(v_3)$, and $y \sim w_5$ by Lemma 7.5. Therefore, $\{v_6, y\}$ is a 2-cut, contradicting Lemma 7.4.

So, we suppose that $|N(u)| = 6$. If $v_5 \in V_4$, let $w_5' \notin \{u_1, u_4, w_5\}$ be a neighbor of v_5, then $w_5' \sim v_3$ as $w_5' \notin S(u_1)$, and hence $S(v_3) = \{u_2, v_4\}$. Let $x \notin \{v_3, v_5\}$ be a neighbor of w_5'. Then, $x \sim v_2$ as $x \notin S(v_3)$, and $x \sim v_6$ as $x \notin S(u_2)$, and further $w_5 \sim x$ as otherwise $S(v_2) = \{u, u_3, w_5\}$. But then w_5' is a cut vertex contradicting Lemma 7.4. The same contradiction would occur if $v_6 \in V_4$. Therefore, $v_5, v_6 \in V_3$, and hence $v_2, v_3 \in V_3$ for otherwise the fourth neighbor of v_3 would be in $S(u_1)$ contradicting $S(v_1) = \{u, u_3\}$, and the fourth neighbor of v_2 would be in $S(u_2)$ contradicting $S(u_2) = \{u_1, v_3\}$. Then either $w_4 \in V_3$ or $w_4 \sim v_3$ for otherwise the fourth neighbor of w_4 would be in $S(v_1)$. It is easy to check that G is 3-colorable.

(a-2-2-2) We suppose that $w_5 \sim v_3$ and $w_5 \nsim v_1$. Then, $S(v_3) = \{u_2, v_4\}$. Without loss of generality, we may suppose that $v_5 \in V_3$ for otherwise the fourth neighbor of v_5 must be adjacent to v_1, and thus we reduce it to case (a-2-2-1). Then, $v_1 \in V_3$ also as otherwise the fourth neighbor of v_1 would be in $S(u_1)$. Let $x \notin \{v_4, v_6\}$ be a neighbor of w_4.

If $|N_2(u)| = 6$, then $x \in S(v_4)$ as $w_4 \nsim v_2$ contradicting our assumption that $S(v_4) = \{u_1, v_3\}$. Therefore, $|N_2(u)| = 7$. Then, $v_7 \sim w_4$ as $v_7 \notin S(v_4)$, i.e., $x = v_7$. If $w_4 \in V_4$, the fourth neighbor of w_4 would be in $S(v_4)$ again, then $w_4 \in V_3$.

If $v_6 \in V_4$, let $w_6 \notin \{u_2, u_4\}$ be a neighbor of v_6. Since $v_6 \sim v_1$ forces $v_6 \sim S(u_1)$ contradicting (16), $w_6 \neq v_i$ for any $1 \leq i \leq 7$. Then, $w_6 \sim v_2$ as $w_6 \notin S(u_2)$, and $S(v_2) = \{u, u_3\}$. To forbid $w_5 \in S(v_2)$, $w_5 \sim w_6$. By discussing on $S(v_2)$ and $S(v_3)$, it is easy to see that $w_5, w_6 \in V_3$. Then, v_7 is a cut vertex. This contradiction completes the proof of the case (a). So, we suppose that $v_6 \in V_3$. Hence $v_2 \in V_3$ for otherwise the fourth neighbor of v_2 would be in $S(u_2)$ contradicting (14). But then, $\{v_7, w_5\}$ is a 2-cut contradicting Lemma 7.4.

(b) Assume $v_4 \in V_3$. Since $d(v_6) \geq 3$ and the third neighbor of v_6 must be adjacent to v_2 as $S(u_2) = \{u_1, u_3\}$ by our assumption (14), $v_2 \sim V_4$. Let $w_2 \notin \{u_2, v_1, v_3\}$ be the fourth neighbor of v_2. Then, $w_2 \neq v_i$ for $1 \leq i \leq 7$ by (11), and $w_2 \sim v_6$ as $v_4 \in V_3$, and $v_6 \in V_3$ as otherwise its fourth neighbor would be in $S(u_2)$.

First suppose that $|N_2(u)| = 6$. If $v_5 \in V_3$, let $G(u', v_1') = G[S_1, \{v_1, v_3\}]$, then G is reducible as $(u, v_4, v_6, u_1, v_5, u', v_2)$ is reducible. So, we suppose that $v_5 \in V_4$, let $w_5, w_5' \notin \{u_1, u_4\}$ be two neighbors of v_5. Then, $v_6 \in S(v_5)$ forcing $S(v_5) = \{v_6, w_2\}$. Since $w_5, w_5' \notin S(u_1)$, we suppose that $w_5 \sim v_1$ and $w_5' \sim v_3$ by symmetry. Thus $S(v_1) = \{u, u_3\}$ that forces $w_2 \sim w_5$ as $w_2 \notin S(v_1)$. Let $x \notin \{v_3, v_5\}$ be a neighbor of w_5'. Then, $x \neq w_2$ as $w_2 \in S(v_5)$, $x \sim w_5$ as $x \notin S(v_5)$. Therefore, $w_5' \in V_3$ for otherwise its fourth neighbor would be in $S(v_5)$, and then $\{w_2, x\}$ is a 2-cut of G.

Next, we suppose that $|N_2(u)| = 7$, and suppose that $v_3 \in V_4$. Let $w_3 \notin \{u_1, u_3, v_2\}$ be a neighbor of v_3. Then, $w_3 \neq v_i$ for any i by (11), $S(v_3) = \{u_2, v_4\}$, and thus $v_5 \notin S(v_3)$ forcing $v_5 \sim w_3$ as $v_5 \nsim v_2$ by (15), and $v_7 \notin S(v_3)$ forcing $v_7 \sim w_3$. By (13) and (15), $v_2 \in S(u_3)$. Let $S(u_3) = \{v_2, x\}$. Then $x \sim v_2$ by Lemma 7.5. If $x \sim v_7$, then $x = w_2$ (by (13)) forcing $\{u_3, v_1, w_2\} \subseteq S(u_3)$, contradicting $|S(u_3)| = 2$. Therefore, $x \nsim v_7$ and thus $v_7 \in V_3$. Since $S(v_3) = \{u_2, v_4\}$, $w_2 \notin S(v_3)$ forcing $w_2 \sim w_3$. Now we consider $S(w_3)$.

Note that $v_6 \in S(w_3)$. Then, $S(w_3) = \{v_6\}$ as $v_6 \in V_3$ and $u_2, u_4 \notin S(w_3)$, contradicting $|S(w_3)| = 2$.

Finally, we suppose that $|N_2(u)| = 7$ and $v_3 \in V_3$. Then, $v_5 \in V_3$ for otherwise one of its neighbor would be in $S(u_1)$ contradicting (16). Let $w_5 \notin \{u_1, u_4\}$ be a neighbor of v_5. Then, $w_5 \sim v_1$ as $S(u_2) = \{u_1, v_3\}$, and $w_5 \neq v_2$ by (15). Then, $S(v_2) = \{u, u_3\}$, $w_5 \notin S(v_2)$ that forces $w_5 \sim w_2$. If $v_7 \sim w_5$, then $\{v_7, w_6\}$ is a 2-cut. If $v_7 \sim w_2$, then $\{v_7, w_5\}$ is a 2-cut. So, $v_7 \nsim w_2/w_5$, and hence $d(v_7) = 2$ or $|S(u_4)| \geq 3$ as $v_5, v_6 \in V_3$. This completes the proof of case (b).

We have proved (12).

Since $S(u_1) = \{u_2, v_2\}$ and $S(u_2) = \{u_1, v_1\}$, for each vertex $w \in N_3(u)$, either $|N(w) \cap \{v_1, v_2, \dots, v_6\}| = 0$ or $|N(w) \cap \{v_1, v_2, \dots, v_6\}| \geq 2$, and further $|N(w) \cap \{v_1, v_2, \dots, v_6\}| = 4$ if $w \sim v_{2i}/v_{2j-1}$ for some $1 \leq i, j \leq 3$. Let $W = \{w_1, \dots, w_l\} \subseteq N_3(u)$ be set of vertices that have four neighbors $\{v_1, v_2, \dots, v_6\}$, and let $G(u', v'_1, v'_2) = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}]$. Then G is reducible as $(u, u_1, w_1, w_2, \dots, w_l, u')$ is reducible.

Subcase 2.3. Suppose now that $v_3/v_4 \sim u_4$. Since $N(u_3) \nsubseteq N(u_4)$ and vice versa, we may suppose that $v_5 \sim u_1/u_3$ and $v_6 \sim u_2/u_4$. We show that

$$\text{except } v_1v_2 \text{ there exists no edge } v_iv_j \text{ for } 1 \leq i < j \leq 6. \tag{17}$$

If $v_1 \sim v_4$, then $v_1 \in V_3$ as otherwise $\{u, u_3, u_4\} \subseteq S(v_1)$ contradicting $|S(v_1)| = 2$. Let $G(u') = G[S_1]$, then G is reducible as (u, v_4, v_1, u_1, v_3) is reducible. By symmetry, we suppose that $v_1 \nsim v_4$ and $v_2 \nsim v_3$.

Suppose that $v_1 \sim v_6$. Then, $S(u_2) = \{u_1, v_5\}$ as $v_3 \nsim v_2/v_4/v_6$, and hence $v_5 \sim v_2$ or $v_5 \sim v_6$. If $v_5 \sim v_6$, then $S(u_3) = \{v_6\}$ contradicting $|S(v_6)| = 2$. So, we suppose that $v_2 \sim v_5$. Then $S(u_1) = \{u_2, v_6\}$ as $v_4 \nsim v_1/v_3/v_5$, and $S(u_2) = \{u_1, v_5\}$ as $v_3 \nsim v_2/v_4/v_6$. By considering $S(u_1)$, one sees that $\{v_1, v_2, v_3\} \cap V_3 = \emptyset$ only if v_1, v_2, v_3 have a common neighbor in $N_3(u)$. The same happens to v_2, v_4 and v_6. Further more, if a vertex, say $w \in N_3(u)$, has neighbors in both $\{v_1, v_2, v_3\}$ and $\{v_2, v_4, v_6\}$, then w has four neighbors in $N_2(u)$.

It is easy to check that G is reducible by letting $G(u', v'_1, v'_2) = G[S_1, \{v_1, v_2, v_3\}, \{v_2, v_4, v_6\}]$. By symmetry, we suppose further that $v_1 \nsim v_6$ and $v_2 \nsim v_5$.

We have showed that $v_1 \nsim v_4/v_6$ and $v_2 \nsim v_3/v_5$. Then, $S(u_1) = \{u_2, v_2\}$, and hence $v_5 \nsim v_6$ as otherwise $v_6 \sim S(u_1)$. This proves (17).

By (17), every vertex in $N_3(u)$ has at least two neighbors in either $\{v_1, v_3, v_5\}$ or in $\{v_2, v_4, v_6\}$, and if a vertex, say w, in $N_3(u)$ has neighbors in both $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$, then w has four neighbors in $N_2(u)$. Let $G(u', v'_1, v'_2) = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}]$. It is easy to check that G is reducible.

Subcase 2.4. Suppose now that $v_3 \sim u_4$ but $v_4 \nsim u_4$. Since $N(u_4) \nsubseteq N(u_3)$, u_4 has a neighbor, say v_5, not adjacent to u_3. There are two possibilities: one is that $v_5 \sim u_1$, and the other is $v_5 \sim u_2$. We prove the former one. The latter can be dealt with almost the same arguments.

(a) Suppose that $v_5 \sim u_1$.

First we consider the situation that $u_3 \sim v_5$. If $u_2, u_4 \in V_4$, let $v_6 \neq u$ be a common neighbor of u_2 and u_4. Let w_3, w_5 be the vertices in $N(v_3) \setminus N(v_5)$ and $N(v_5) \setminus N(v_3)$, respectively. If $w_3 = v_2$, then $v_4 \sim v_1$ or $v_6 \sim v_1$ as otherwise $S(u_1) = \{u_2\}$. Let $G(u', v'_1, v'_2) = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4\}]$ while $u_2, u_4 \in V_3$, and let $G(u', v'_1, v'_2) = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}]$ while $u_2, u_4 \in V_4$. It is easy to see that G is reducible. So, $w_3 \neq v_2$. The same arguments show that $w_5 \neq v_2$. Therefore, $\{w_3, w_5\} \cap N_2(u) = \emptyset$.

If $w_3 \sim w_5$, then at most one of w_3 and w_5,say w_3, may be adjacent to v_1, and thus $w_5 \in S(u_1)$ contradicting Lemma 7.5 as $u_2 \in S(u_1)$. So we suppose that $w_3 \nsim w_5$. Then, $w_3/w_5 \sim v_1$ as $w_3, w_5 \notin S(u_1)$, one of w_3 and w_5, say w_5, is adjacent to v_4 for avoiding $(u_3v_3w_3, u_3w_5v_5, u_3u, u_3v_4)$, and thus $w_5 \sim v_2$ or $w_5 \sim v_6$ while $d(u_2) = 4$ as $w_5 \notin S(u_2)$. Let $G(u', v') = G[S_1, \{v_3, v_5\}]$ while $u_2, u_4 \in V_3$, and let $G(u', v'_1, v'_2) = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4, v_6\}]$ while $u_2, u_4 \in V_4$. In both cases, one sees that G is reducible.

Next, we suppose that $u_3 \nsim v_5$. There are four possibilities. Let w_5 be a vertex in

$N(v_5) \setminus N(v_3)$, and let w_3 be the fourth neighbor of v_3 while $v_3 \in V_4$.

(a-1) Suppose that $u_2, u_3, u_4 \in V_3$.

If $w_5 = v_2$, then $v_4 \sim v_5$ or $v_4 \sim v_1$ as otherwise $S(u_1) = \{u_2\}$, and it is easy to see that G is reducible by letting $G(u', v_1', v_2') = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4\}]$. If $w_5 = v_4$, then $v_2 \sim v_3$ or $v_2 \sim v_5$ or $v_4 \sim v_1$ as $|S(u_1)| = 2$, and one sees that G is reducible by letting $G(u', v_1', v_2') = G[S_1, \{v_1, v_3, v_5\}, \{v_2, v_4\}]$. So, $w_5 \notin N_2(u)$. The same arguments show that $w_3 not \in N_2(u)$. Thus $w_5 \sim v_1$, and $w_3 \sim v_1$ or $w_3 \sim v_5$, as $w_3, w_5 \notin S(u_1)$. Let $G(u', v') = G[S_1, \{v_3, v_5\}$. Then, G is reducible as (u, u_1) is reducible.

(a-2) Suppose that u_2 and u_4 have another common neighbor other than u and v_4. By using the similar arguments as that used in the case (a-1), we can show that $w_5 \notin N_2(u)$, $w_3 \notin N_2(u)$ if it exists, and thus $w_5 \sim v - 1$, and $w_3 \sim v_1$ or $w_3 \sim v_5$.

Let w_4, w_4', respectively, be the in $N(v_4) \setminus N(v_4')$ and $N(v_4') \setminus N(v_4)$. By using the same arguments as that used on w_5, we can show that $w_4, w_4' \notin N_2(u)$, and thus $w_4/w_4' \sim v_2$ as $w_4, w_4' \notin S(u_2)$. By letting $G(u', v_2', v_3') = G[S_1, \{v_2, v_4, v_4'\}, \{v_3, v_5\}]$, one sees that G is reducible.

The remaining two possibilities, one that u_3 and u_4 have a common neighbor other than u, and the other that u_2 and u_4 have a common neighbor other than u, can be treated with the same arguments, and we omit the details.

REFERENCES

[1] Alon, N., "A simple algorithm for edge-coloring bipartite multigraphs," *Information Processing Letters*, vol. 85, no. 6, pp. 301–302, 2003.

[2] Alon, N., Krivelevich, M., and Sudakov, B., "Coloring graphs with sparse neighborhoods," *Journal of Combinatorial Theory, Series B*, vol. 77, no. 1, pp. 73–82, 1999.

[3] Alon, N. and Tarsi, M., "Colorings and orientations of graphs," *Combinatorica*, vol. 12, no. 2, pp. 125–134, 1992.

[4] Appel, K. and Haken, W., "Every planar map is four colorable. part i: Discharging," *Illinois Journal of Mathematics*, vol. 21, no. 3, pp. 429–490, 1977.

[5] Appel, K. and Haken, W., "The solution of the four-color-map problem," *Scientific American*, vol. 237, pp. 108–121, 1977.

[6] Appel, K., Haken, W., and Koch, J., "Every planar map is four colorable. part ii: Reducibility," *Illinois Journal of Mathematics*, vol. 21, no. 3, pp. 491–567, 1977.

[7] Appel, K. and Haken, W., *Every planar map is four colorable*, vol. 98. Amer Mathematical Society, 1989.

[8] Beineke, L., "Derived graphs and digraphs," *Beitrage zur Graphentheorie*, pp. 17–23, 1968.

[9] Brooks, R., "On colouring the nodes of a network," *Classic Papers in Combinatorics*, pp. 118–121, 1987.

[10] Bryant, V., "A characterisation of some 2-connected graphs and a comment on an algorithmic proof of brooks' theorem," *Discrete Mathematics*, vol. 158, no. 1, pp. 279–281, 1996.

[11] Chudnovsky, M., Robertson, N., Seymour, P., and Thomas, R., "The strong perfect graph theorem," *Annals of Mathematics*, pp. 51–229, 2006.

[12] Diestel, R., "Graph theory. 2005," *Grad. Texts in Math*, 2005.

[13] Erdős, P., "Graph theory and probability," *Canad. J. Math.*, vol. 11, pp. 34–38, 1959.

[14] Favrholdt, L., Stiebitz, M., and Toft, B., *Graph edge colouring: Vizing's theorem and Goldberg's conjecture*. Institut for Matematik og Datalogi, Syddansk Universitet, 2006.

[15] Gallai, T., *Kritische Graphen 1*. Akad. Kiadó, 1963.

[16] Grable, D. and Panconesi, A., "Fast distributed algorithms for brooks-vizing colourings," in *Proceedings of the ninth annual ACM-SIAM symposium on Discrete algorithms*, pp. 473–480, Society for Industrial and Applied Mathematics, 1998.

[17] Gravier, S., "A hajós-like theorem for list coloring," *Discrete Mathematics*, vol. 152, no. 1, pp. 299–302, 1996.

[18] Grötzsch, H., "Zur theorie der diskreten gebilde, vii," *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, pp. 109–120.

[19] Grünbaum, B., "Grötzsch's theorem on 3-colorings.," *The Michigan Mathematical Journal*, vol. 10, no. 3, pp. 303–310, 1963.

[20] Gyárfás, A., "On ramsey covering numbers," *Infinite and Finite Sets*, vol. 2, pp. 801–816, 1975.

[21] Gyárfás, A., Szemerédi, E., and Tuza, Z., “Induced subtrees in graphs of large chromatic number,” *Discrete Mathematics*, vol. 30, no. 3, pp. 235–244, 1980.

[22] Hajnal, P. and Szemerédi, E., “Brooks coloring in parallel,” *SIAM journal on Discrete Mathematics*, vol. 3, p. 74, 1990.

[23] Hansen, P. and Marcotte, O., *Graph colouring and applications*, vol. 23. Amer Mathematical Society, 1999.

[24] Kierstead, H., “On the chromatic index of multigraphs without large triangles,” *Journal of Combinatorial Theory, Series B*, vol. 36, no. 2, pp. 156–160, 1984.

[25] Kierstead, H. and Penrice, S., “Radius two trees specify χ-bounded classes,” *Journal of Graph Theory*, vol. 18, no. 2, pp. 119–129, 1994.

[26] Kierstead, H. and Zhu, Y., “Radius three trees in graphs with large chromatic number,” *SIAM Journal on Discrete Mathematics*, vol. 17, p. 571, 2004.

[27] Krauz, J., “Demonstration nouvelle d’un theoreme de whitney sur les reseaux, mat,” *Fiz. Lapok*, vol. 50, pp. 75–89, 1943.

[28] Lovász, L., “Three short proofs in graph theory,” *Journal of Combinatorial Theory, Series B*, vol. 19, no. 3, pp. 269–271, 1975.

[29] Mycielski, J., “Sur le coloriage des graphes,” in *Colloq. Math*, vol. 3, p. 9, 1955.

[30] Randerath, B. and Schiermeyer, I., “A note on brooks’ theorem for triangle-free graphs,” *Australasian Journal of Combinatorics*, vol. 26, pp. 3–10, 2002.

[31] Randerath, B. and Schiermeyer, I., “Vertex colouring and forbidden subgraphs–a survey,” *Graphs and Combinatorics*, vol. 20, no. 1, pp. 1–40, 2004.

[32] Randerath, H., *The Vizing bound for the chromatic number based on forbidden pairs*. Shaker, 1998.

[33] Reed, B., "A strengthening of brooks' theorem," *Journal of Combinatorial Theory, Series B*, vol. 76, no. 2, pp. 136–149, 1999.

[34] Robertson, N., Sanders, D., Seymour, P., and Thomas, R., "The four-colour theorem," *Journal of Combinatorial Theory, Series B*, vol. 70, no. 1, pp. 2–44, 1997.

[35] Robertson, N., Sanders, D., Seymour, P., and Thomas, R., "A new proof of the four-colour theorem," *Electron. Res. Announc. Amer. Math. Soc*, vol. 2, no. 1, pp. 17–25, 1996.

[36] Sumner, D., "Subtrees of a graph and chromatic number," *The Theory and Applications of Graphs, G. Chartrand (ed.), John Wiley, New York*, 1981.

[37] Thomassen, C., "Every planar graph is 5-choosable," *Journal of Combinatorial Theory, Series B*, vol. 62, no. 1, pp. 180–181, 1994.

[38] Thomassen, C., "A short list color proof of grötzsch's theorem," *Journal of Combinatorial Theory, Series B*, vol. 88, no. 1, pp. 189–192, 2003.

[39] Vizing, V., "On an estimate of the chromatic class of a p-graph," *Diskret. Analiz*, vol. 3, no. 7, pp. 25–30, 1964.

[40] Vizing, V., "Vertex colorings with given colors," *Metody Diskret. Analiz*, vol. 29, pp. 3–10, 1976.

[41] Weisstein, E. W., "Chvátal graph.," *From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/ChvatalGraph.html.*

[42] Weisstein, E. W., "Grötzsch graph.," *From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/GroetzschGraph.html.*

[43] West, D., *Introduction to graph theory*. Prentice Hall Upper Saddle River, NJ.:, 2 ed., 2001.