MASS TRANSFER IN TWO-PHASE ANNULAR FLOW

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SUMMARY

Mass-transfer coefficients for two-phase annular flow are analytically determined for the laminar and turbulent flow situations.

The differential equations for three cases of laminar flow mass transfer, namely, gas-phase controlled, liquid-phase controlled, and interphase mass transfer, are solved by the method of separation of variables. The eigenfunctions are represented by infinite power series. The results show that Nusselt numbers for two-phase, gas-phase controlled mass transfer lie between the Nusselt numbers for single-phase mass transfer with parabolic and flat velocity profiles, 3.657 and 5.775, respectively. Nusselt numbers for two-phase, liquid-phase controlled mass transfer lie between 2.470 and 3.919. These are the Nusselt numbers for the falling film types of flow with flat and linear velocity profiles, respectively. Nusselt numbers for interphase mass transfer are practically constant for a certain range of the gas-liquid ratio. Within this range of gas-liquid ratios, the additivity-of-resistances principle is valid. However, the Nusselt numbers of interphase mass transfer are lower than this constant both below and above this critical range of gas-liquid ratios. Asymptotic solutions for the entrance region for these three cases are obtained by the method of combination of variables. They are in good agreement with the exact solutions in the entrance region.

For the case of turbulent two-phase, gas-liquid, annular flow, the mathematical model is set up by assuming the liquid film is

symmetrical and smooth and there is no effect of the entrainment on the flow. The overall effective diffusivities are expressed as

$$D_{ti} = \beta D_i + \epsilon_{Di} \qquad i = A, B.$$

where β is a correlating constant, D_i is the molecular diffusivity and ϵ_{Di} is the eddy diffusivity proposed by Kropholler and Carr (29). The effects of the entrainment and the interface instabilities on mass transfer are correlated by β. This mathematical model is solved also by the method of separation of variables. Eigenfunctions and eigenvalues are obtained numerically by use of a digital computer. β obtained by trial and error to fit the experimental data of Anderson (2) and Bollinger (9) is

$$\beta = 7.4 \times 10^6 \quad H^{0.21} \quad \left(\frac{\Delta P}{\Delta L}\right)_{\ell}^{0.83}$$

It is found that the Nusselt number resulting from this calculation can be expressed as

$$Nu = a \quad \left(\frac{\Delta P}{\Delta L}\right)_g^b \quad \left(\frac{\Delta P}{\Delta L}\right)_l^c \quad Sc_A^d \quad Sc_B^e \quad R_A^{+f} \quad R_B^{+g}$$

in which a is a function of ω and H, and the exponents b, c, d, e, f, and g vary significantly only with ω. When ω is small, b, e, and g are zero. When ω is large, d and f are zero.

Since the experimental data available for correlation are for horizontal pipes only, the results for turbulent flow mass transfer are applicable only for horizontal annular flow.

NOMENCLATURE

a	slope of velocity profile	hr^{-1}
a	constant coefficient	dimensionless
b	constant coefficient	dimensionless
C	constant coefficient	dimensionless
c	concentration	lb/ft^3
$< c >$	average concentration	lb/ft^3
D	diffusivity	ft^2/hr
D_t	overall effective diffusivity	ft^2/hr
E	liquid entrainment	lb/hr
E_1	constant defined by Eq. (2.1.10)	dimensionless
E_2	constant defined by Eq. (2.1.14)	dimensionless
E_3	constant defined by Eq. (2.1.11)	dimensionless
f	reduced diffusivity defined by Eq. (3.2.23)	dimensionless
G	gas volumetric flow rate	ft^3/hr
g	reduced velocity, $V/< V >$	dimensionless
H	parameter, $L/G\alpha$	dimensionless
I(x)	integer defined by Eq. (2.2.44)	dimensionless
J_0, J_1	Bessel functions of first kind	dimensionless
K	interphase mass-transfer coefficient	ft/hr
K_A	gas-phase controlled mass-transfer coefficient	ft/hr
K_B	liquid-phase controlled mass-transfer coefficient	ft/hr
k	radius fraction of interface position	dimensionless

L	liquid volumetric flow rate	ft^3/hr
N	function defined by Eq. (2.3.23) and Eq. (2.4.59)	dimensionless
Nu	Nusselt number for interphase mass transfer	dimensionless
Nu_A	gas-phase controlled Nusselt number	dimensionless
Nu_B	liquid-phase controlled Nusselt number	dimensionless
Nu_∞	Nusselt number for very large η	dimensionless
P	function defined by Eq. (2.3.22) and Eq. (2.4.58)	dimensionless
p	pressure	lb_f/ft^2
$(\frac{\Delta P}{\Delta L})$	pressure gradient defined by Eq. (2.1.6)	lb_f/ft^3
$(\frac{\Delta P}{\Delta L})_{TPF}$	pressure gradient of two-phase flow	lf_f/ft^3
$(\frac{\Delta P}{\Delta L})_g$	pressure gradient which would exist if gas flowed alone	lb_f/ft^3
$(\frac{\Delta P}{\Delta L})_\ell$	pressure gradient which would exist if liquid flowed alone	lb_f/ft^3
R	parameter defined by Eq. (3.1.5)	$lb_m ft^3/lb_f hr$
R	radius	ft
R^+	dimensionless radius, $\frac{R}{\nu}\sqrt{\frac{\tau_o}{\rho}}$	dimensionless
r	radial distance	ft
Sc	Schmidt number	dimensionless
u	function defined by Eq. (2.3.16) and Eq. (2.4.50)	dimensionless
v	function defined by Eq. (2.4.51)	dimensionless
V	velocity	ft/hr
$< V >$	average velocity	ft/hr
X	eigenfunction	dimensionless

X	parameter, $\left[\left(\frac{\Delta P}{\Delta L}\right)_\ell \Big/ \left(\frac{\Delta P}{\Delta L}\right)_g\right]^{\frac{1}{2}}$	dimensionless
x	reduced distance defined by Eq. (2.2.6) and Eq. (2.3.6)	dimensionless
Y	reduced concentration	dimensionless
$< Y >$	reduced average concentration	dimensionless
y	distance	ft
y^+	dimensionless distance parameter, $\frac{Y}{\nu}\sqrt{\frac{\tau_o}{\rho}}$	dimensionless
z	axial distance	ft

Greek Symbols

α	equilibrium constant	dimensionless
β	correlation parameter defined by Eq. (3.2.1)	dimensionless
γ	eigenvalue	dimensionless
ϵ	eddy diffusivity of momentum	ft^2/hr
ϵ_D	eddy diffusivity of mass transfer	ft^2/hr
ζ	dimensionless variable defined by Eq. (2.2.35), Eq. (3.4.75), and Eq. (3.4.76)	dimensionless
η	reduced axial distance	dimensionless
λ	eigenvalue	dimensionless
μ	viscosity	lb/ft hr
ν	kinematic viscosity	ft^2/hr
ρ	density	lb/ft^3
τ	shear stress	$lb/ft\text{-}hr^2$
Φ	dimensionless parameter defined by Eq.(2.4.21)	dimensionless
ψ	function of η	dimensionless
ω	dimensionless parameter defined by Eq.(2.3.8)	dimensionless

Subscript

A	gas phase
B	liquid phase
i	gas phase or liquid phase
O	initial, z = O
S	interface

CHAPTER I

INTRODUCTION

Two-phase, gas-liquid flow occurs frequently in engineering practice. The importance of this subject has stimulated considerable research effort in the past thirty years; over 5,000 articles have appeared in the literature (11,20,21,27,48). The flow pattern of gas-liquid cocurrent flow in a pipe depends on the flow rates of each phase, the physical properties of the fluids and the pipeline geometry (20,44).

For high mass flow rates of gas and relatively low liquid mass flow rates, the flow pattern established, both in horizontal and vertical pipes, is called annular flow. This type of flow is characterized by the gas flowing as a high speed core while the liquid flows as an annular film around the pipe wall and in most cases as droplets entrained in the gas core.

The purpose of this dissertation is to determine analytically, mass-transfer coefficients for annular cocurrent two-phase flow. It is divided into two parts: laminar flow mass transfer and turbulent flow mass transfer.

For the case of laminar flow, it is assumed that there are no liquid droplets in the gas core and that the annular liquid film is symmetrical and smooth. It is also assumed that the laminar velocity distributions of both phases are fully-developed and steady. Three cases of mass transfer are discussed: gas-phase controlled mass

transfer, liquid-phase controlled mass transfer, and interphase mass transfer. The gas-phase controlled mass transfer is a moving interface Graetz problem (19). The liquid-phase controlled mass transfer is similar to a falling-film tower problem. The interphase mass transfer is a two-region Sturm-Liouville problem.

Theoretical prediction of the mass-transfer coefficients by assuming laminar flow will provide a basic understanding of the mass transfer mechanism. The most probable case in which stable, laminar, annular flow will exist is the falling film case. However, it has been shown that a falling film is not in laminar flow even for Reynolds numbers as low as 300 (41). For gas-liquid cocurrent flow, the flow rates of both phases are very high when the flow pattern is annular. Mass-transfer coefficients for turbulent annular flow are desirable for practical designs.

Mass transfer between gas and liquid in annular flow has been shown to be of practical significance (9). However, very little work has been done in this area. In 1959 Anderson (2) reported a study on gas-phase controlled mass transfer in two-phase annular flow in horizontal pipes. Data on mass-transfer coefficients for the absorption by water of ammonia from ammonia-air mixtures were presented. Another experimental study on liquid-phase controlled mass transfer performed in the same laboratory was made by Bollinger (9) the following year. The solute used was carbon dioxide. Later, Hughmark (23) proposed that the momentum-mass transfer analogy be used to calculate mass-transfer coefficients in horizontal annular flow. The results check fairly well with the CO_2-water data of Bollinger and the NH_3-water data of Anderson.

Mass-transfer coefficients for physical and chemical absorption of carbon dioxide in annular and dispersed horizontal flow have also been measured by Wales (45). These are the only investigations concerning mass transfer in two-phase annular flow found in the literature.

For the case of turbulent flow, a model is proposed to calculate over-all mass-transfer coefficients in gas-liquid annular flow. In this model, the effects of the entrained liquid droplets and the interface ripples on the mass transfer are included by an effective diffusivity.

CHAPTER II

LAMINAR FLOW CASE

1. Velocity Profiles

The velocity profile for laminar, two-phase, (liquid-liquid) annular flow has been derived by Russell and Charles (37) assuming the two liquids have equal densities. For the case of gas-liquid annular flow treated in this chapter, consider the gas as phase A and the liquid as phase B each flowing laminarly in a vertical pipe of radius R with constant axial pressure gradient. Assuming that the properties of the fluids are constant, that the liquid forms a symmetrical annular film with the interface at kR and that the flow is fully developed and steady, the equation of motion can be written as

$$-\frac{dp}{dz} + \mu_i \frac{1}{r} \frac{d}{dr} \left(r \frac{dV_i}{dr}\right) + \rho_i g = 0 \qquad i = A, B, \qquad (2.1.1)$$

The boundary conditions are

$$r = R \qquad V_B = 0 \qquad (2.1.2)$$

$$r = kR \qquad V_A = V_B \qquad (2.1.3)$$

$$r = kR \qquad \mu_A \frac{dV_A}{dr} = \mu_B \frac{dV_B}{dr} \tag{2.1.4}$$

$$r = 0 \qquad \frac{dV_A}{dr} = 0 \tag{2.1.5}$$

Integrating Eq. (2.1.1) gives the velocity distributions

$$V_A = \frac{k^2 R^2}{4\mu_A} \left(\frac{\Delta P}{\Delta L}\right)_A \left(E_1^2 - \frac{r^2}{k^2 R^2}\right) \tag{2.1.6}$$

$$V_B = \frac{R^2}{4\mu_B} \left(\frac{\Delta P}{\Delta L}\right)_B \left(1 - \frac{r^2}{R^2} + E_3 \ln \frac{r}{R}\right) \tag{2.1.7}$$

where

$$\left(\frac{\Delta P}{\Delta L}\right)_i = \frac{dp}{dz} + \rho_i g \tag{2.1.8}$$

$$E_1^2 = 1 + \left(\frac{1}{k^2} - 1\right) \frac{\mu_A \left(\frac{\Delta P}{\Delta L}\right)_B}{\mu_B \left(\frac{\Delta P}{\Delta L}\right)_A} + \frac{2\mu_A}{\mu_B} \left[\frac{\left(\frac{\Delta P}{\Delta L}\right)_B}{\left(\frac{\Delta P}{\Delta L}\right)_A} - 1\right] \ln k \tag{2.1.9}$$

$$E_3 = 2\,k^2 \left[1 - \frac{\left(\frac{\Delta P}{\Delta L}\right)_A}{\left(\frac{\Delta P}{\Delta L}\right)_B} \right] \tag{2.1.10}$$

The average velocities are

$$\langle V_A \rangle = \frac{2\pi \int_o^{kR} V_A \, rdr}{2\pi \int_o^{kR} rdr} = \frac{k^2 R^2}{4\mu_A} \left(\frac{\Delta P}{\Delta L}\right)_A (E_1^2 - 0.5) \tag{2.1.11}$$

$$\langle V_B \rangle = \frac{2\pi \int_{kR}^{R} V_B \, rdr}{2\pi \int_{kR}^{R} rdr} = \frac{R^2}{4\mu_B E_2} \left(\frac{\Delta P}{\Delta L}\right)_B \tag{2.1.12}$$

where

$$E_2 = \left\{ \frac{1-k^2}{2} - \frac{k^2}{1-k^2} \left[1 - \frac{\left(\frac{\Delta P}{\Delta L}\right)_A}{\left(\frac{\Delta P}{\Delta L}\right)_B} \right] (1-k^2 + 2\,k^2 \ln k) \right\}^{-1} \tag{2.1.13}$$

Defining the dimensionless velocity as

$$g_i = \left\langle \frac{V_i}{V_i} \right\rangle, \quad i = A, B. \tag{2.1.14}$$

gives

$$g_A = \frac{E_1^2 - \frac{r^2}{k^2 R^2}}{E_1^2 - 0.5} \qquad (2.1.15)$$

$$g_B = E_2 \left(1 - \frac{r^2}{R^2} + E_3 \ln \frac{r}{R}\right) \qquad (2.1.16)$$

2. Gas-Phase Controlled Mass Transfer

In 1885 Graetz (19) published an analytical investigation of heat transfer to a fluid in fully developed laminar flow in a round tube of constant wall temperature. Since then, the so called "Graetz Problem" has been discussed in detail by a number of workers and an excellent review is given by Jakob (24).

The subject considered in this section will be the system in which a gas and a liquid, of constant properties, flow in steady laminar motion in a round tube. The flow pattern is annular and the velocity profile is fully developed. Up to a point along the tube ($z = 0$), the concentration of the gas phase is uniform. After this point a constant concentration is assumed at the interface, and it is desired to find the concentration distribution in the gas core and the mass-transfer coefficient. Although many extentions of the classical Graetz problem have appeared in the literature (16,39), none has considered the problem with a moving interface.

The differential equation which describes this case is

$$V_A \frac{\partial c_A}{\partial z} = D_A \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_A}{\partial r} \right) \right] \tag{2.2.1}$$

with boundary conditions

$$z = 0 \qquad c_A = c_{AO} \tag{2.2.2}$$

$$r = 0 \qquad \frac{\partial c_A}{\partial r} = 0 \tag{2.2.3}$$

$$r = kR \qquad c_A = c_{AS} \tag{2.2.4}$$

Defining the dimensionless variables

$$Y = \frac{c_A - c_{AS}}{c_{AO} - c_{AS}} \tag{2.2.5}$$

$$x = \frac{r}{kR} \tag{2.2.6}$$

$$\eta = \frac{D_A z}{k^2 R^2 < V_A >} \tag{2.2.7}$$

and substituting Eq. (2.1.6) into Eq. (2.2.1) yields

$$\frac{(E_1^2 - x^2)}{(E_1^2 - 0.5)} \frac{\partial Y}{\partial \eta} = \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial Y}{\partial x}\right) \tag{2.2.8}$$

Using the method of separation of variables, let

$$Y = X \psi \tag{2.2.9}$$

where X is a function of x and ψ is a function of η.

Substituting Eq. (2.2.9) into Eq. (2.2.8) gives

$$\frac{d\psi}{\psi (E_1^2 - 0.5) d\eta} = \frac{1}{x (E_1^2 - x^2) X} \frac{d}{dx} \left(x \frac{dX}{dx}\right) \tag{2.2.10}$$

which must be valid for all values of x and η. Therefore, each side of this equation must equal a constant, say $-\lambda^2$. This yields

$$\frac{d\psi}{d\eta} = - (E_1^2 - 0.5) \lambda^2 \psi \tag{2.2.11}$$

and

$$x \frac{d^2X}{dx^2} + \frac{dX}{dx} + \lambda^2 x (E_1^2 - x^2) X = 0 \tag{2.2.12}$$

Integrating Eq. (2.2.11) gives

$$\psi = C \exp\left[- \lambda^2 \left(E_1^{\,2} - 0.5 \right) \eta \right] \tag{2.2.13}$$

where C is a constant of integration.

The solution of Eq. (2.2.12) is given by the power series

$$X = \sum_{i=0}^{\infty} a_i \, x^{2i} \tag{2.2.14}$$

where

$$a_o = 1$$

$$a_1 = - \frac{E_1^{\,2} \lambda^2}{4}$$

$$a_i = \frac{\lambda^2}{4i^2} \left(a_{i-2} - E_1^{\,2} \, a_{i-1} \right), \qquad i \geq 2 \tag{2.2.15}$$

In order to satisfy Eq. (2.2.4), the constant λ must be the root of the following equation

$$\sum_{i=0}^{\infty} a_i = 0 \tag{2.2.16}$$

Thus, the complete solution of Eq. (2.2.8) is

$$Y = \sum_{n=1}^{\infty} C_n X_n \exp\left[-\lambda_n^2 \left(E_1^2 - \tfrac{1}{2}\right) \eta\right] \tag{2.2.17}$$

where X_n is the eigenfunction that satisfies Eq. (2.2.12) which in turn corresponds to the eigenvalue $\lambda = \lambda_n$, the nth root of Eq. (2.2.16).

Because of the orthogonal property of eigenfunctions

$$\int_0^1 x \left(E_1^2 - x^2\right) X_m X_n \, dx = 0, \quad \text{when } m \neq n \tag{2.2.18}$$

and the coefficients C_n can be determined by a procedure similar to that of a Fourier analysis by introducing the initial condition, $Y = 1$ at $\eta = 0$. C_n then becomes

$$C_n = \frac{\int_0^1 x \left(E_1^2 - x^2\right) X_n \, dx}{\int_0^1 x \left(E_1^2 - x^2\right) X_n^2 \, dx} \tag{2.2.19}$$

It is interesting to point out here that any given symmetric function of c_A at $z = 0$ will give the same eigenvalues and eigenfunctions, the only difference being the coefficients C_n.

The average concentration is defined as

$$\langle c_A \rangle = \frac{2\pi \int_0^{kR} c_A V_A \, r\,dr}{2\pi \int_0^{kR} V_A \, r\,dr} \qquad (2.2.20)$$

Substituting Eq. (2.1.8) in Eq. (2.2.20) yields

$$\langle Y \rangle = \frac{\langle c_A \rangle - c_{AS}}{c_{AO} - c_{AS}} = \frac{2}{E_1^2 - \frac{1}{2}} \int_0^1 x\,(E_1^2 - x^2)\,Y\,dx \qquad (2.2.21)$$

From Eq. (2.2.17),

$$\langle Y \rangle = \frac{2}{E_1^2 - \frac{1}{2}} \sum_{n=1}^{\infty} C_n \exp\left[-\lambda_n^2 \left(E_1^2 - \tfrac{1}{2}\right)\eta\right] \int_0^1 x\,(E_1^2 - x^2)\,X_n\,dx \qquad (2.2.22)$$

The local gas-phase controlled mass-transfer coefficient, K_A, is defined by

$$K_A\, 2\pi\, kR\, dz \left[c_{AS} - \langle c_A \rangle\right] = \pi k^2 R^2 \langle V_A \rangle\, d\langle c_A \rangle$$

$$= -D_A\, 2\pi\, kR\, dz \left.\frac{\partial c_A}{\partial r}\right|_{r=kR} \qquad (2.2.23)$$

The Nusselt number may be expressed as

$$Nu_A = \frac{2\,kR\;K_A}{D_A} = \frac{\frac{d\langle Y\rangle}{d\eta}}{-\langle Y\rangle} \qquad (2.2.24)$$

Substituting Eq. (2.2.22) into the expression above, the Nusselt number becomes

$$Nu_A = \frac{(E_1^2 - 0.5)\sum_{n=1}^{\infty} C_n \lambda_n^2 \exp\left[-\lambda_n^2 (E_1^2 - 0.5)\eta\right]\int_0^1 x\,(E_1^2 - x^2)\,X_n\,dx}{\sum_{n=1}^{\infty} C_n \exp\left[-\lambda_n^2 (E_1^2 - 0.5)\eta\right]\int_0^1 x\,(E_1^2 - x^2)\,X_n\,dx} \qquad (2.2.25)$$

When z is very large, the above equation gives

$$Nu_{A\infty} = (E_1^2 - 0.5)\,\lambda_1^2 . \qquad (2.2.26)$$

$Nu_{A\infty}$ is shown in Figure 1 and Table 1 as a function of E_1^2. When the interfacial velocity is zero, i.e. $E_1^2 = 1$, this problem is exactly the one investigated by Graetz (19). Therefore, this gas-phase controlled mass-transfer problem may be considered as an extension of the classical Graetz problem.

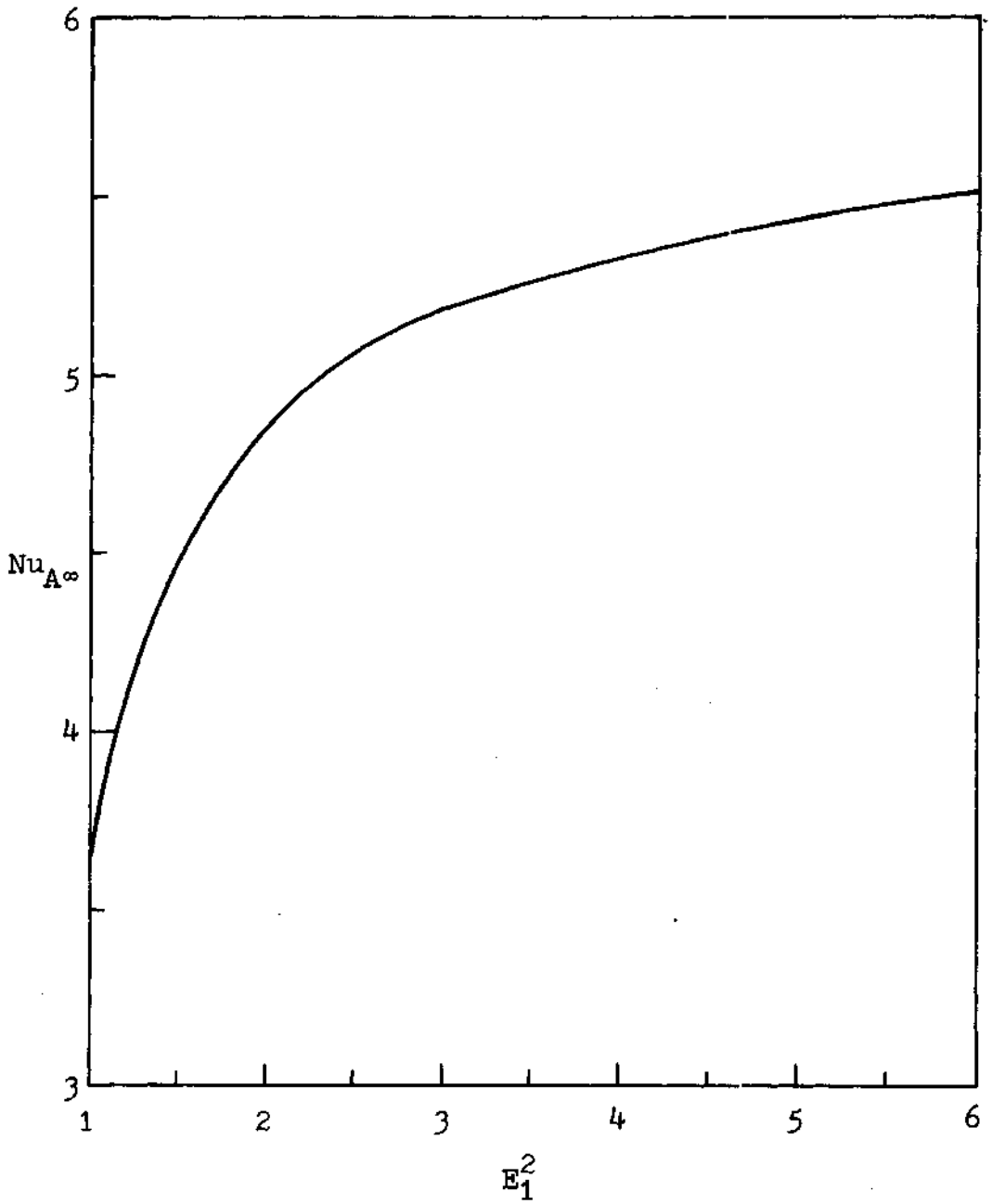

Figure 1. $Nu_{A\infty}$ as a Function of E_1^2.

Table 1. $Nu_{A\infty}$ and λ_1 as Functions of E_1^2.

E_1^2	λ_1	$Nu_{A\infty}$
1.00	2.704	3.657
1.05	2.624	3.786
1.10	2.549	3.900
1.30	2.305	4.251
2.00	1.800	4.858
3.00	1.441	5.193
5.00	1.100	5.441

Asymptotic Solution for Large Eigenvalues

The calculation of terms with large eigenvalues in these eigenfunctions, X_n, although straightforward, is time-consuming even by an electronic computer. Furthermore, the roundoff errors of the computer become so serious that the accuracy of the coefficients is poor. It is therefore desirable to have an asymptotic solution for large eigenvalues.

Using a technique similar to that of Sellars, et al. (39), the following asymptotic solution for large values of n is obtained.

$$X_n = \left(\frac{2}{\pi\lambda_n x}\right)^{\frac{1}{2}} (E_1^2 - x^2)^{-\frac{1}{4}} \cos\left[\frac{\lambda_n}{2}\left(x\sqrt{E_1^2 - x^2} + E_1^2 \sin^{-1}\frac{X}{E_1}\right) - \frac{\pi}{4}\right] \tag{2.2.27}$$

The boundary condition $X_n(1) = 0$ gives the eigenvalues

$$\lambda_n = \frac{2\pi\left(n - \frac{1}{4}\right)}{\left(E_1^2 - 1\right)^{\frac{1}{2}} + E_1^2 \sin^{-1}\frac{1}{E_1}} \qquad (2.2.28)$$

Comparison of the eigenvalues calculated from this asymptotic solution with those from Eq. (2.2.16) is shown in Table 2.

Table 2. Comparison of Eigenvalues for Gas-Phase Controlled Mass-Transfer Calculated From the Asymptotic Solution Eq. (2.2.28), With Those From the Exact Solution Eq. (2.2.16).

	$E_1 = 1.05$		$E_1 = 3$	
	Eq. (2.2.16)	Eq. (2.2.28)	Eq. (2.2.16)	Eq. (2.2.28)
λ_1	2.624	2.870	1.441	1.445
λ_2	6.448	6.697	3.369	3.372
λ_3	10.29	10.52	5.297	5.299
λ_4	14.12	14.35	7.224	7.226
λ_5	17.96	18.18	9.152	9.153

Asymptotic Solution for the Entrance Region

In the entrance region, where z is small, the series of Eq. (2.2.17) does not converge rapidly. Therefore, an asymptotic solution for small z is desirable. For mass transfer during brief exposure

of a gas and a liquid in confined flow, it is permissible to consider only the region very near the interface and to neglect the effect of the confining wall upon the mass transfer process.

For the problem under consideration, Eq. (2.2.1) reduces to

$$(V_{AS} + ay) \frac{\partial c_A}{\partial z} = D_A \frac{\partial^2 c_A}{\partial y^2} \tag{2.2.29}$$

with the boundary conditions

$$Y = 0, \qquad c_A = c_{AS} \tag{2.2.30}$$

$$Y = \infty, \qquad c_A = c_{AO} \tag{2.2.31}$$

$$z = 0, \qquad c_A = c_{AO} \tag{2.2.32}$$

where a is the slope of the gas velocity profile at the interface.

This problem has been solved by Beek and Bakker (7) using the Laplace transformation. They found two asymptotic solutions for local mass-transfer coefficients based on the initial driving force. Their solution for short exposures ($a^2 D_A z / V_{AS}^3 \ll 1$) is

$$K \left(\frac{z}{V_{AS} D_A}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} + \frac{1}{4} \left(\frac{a^2 D_A z}{V_{AS}^3}\right)^{\frac{1}{3}} \tag{2.2.33}$$

while for long exposures ($a^2 D_A z / V_{AS}^3 \gg 1$), their solution is

$$K \left(\frac{z}{V_{AS} D_A} \right)^{\frac{1}{2}} = 0.538 \left(\frac{a^2 D_A z}{V_{AS}^3} \right)^{\frac{1}{6}} \left[1 + 0.375 \left(\frac{V_{AS}^3}{a^2 D_A z} \right)^{\frac{1}{3}} \right] \qquad (2.2.34)$$

Since it is the entrance region that is being considered, Eq. (2.2.34) has no meaning here.

Bayer and King (10) obtained a numerical solution for this problem.

An analytical solution for this problem can be obtained by the method of combination of variables. Defining

$$\zeta = \frac{V_{AS} + ay}{(9 a^2 D_A z)^{\frac{1}{3}}} \qquad (2.2.35)$$

and substituting in Eq. (2.2.29) gives

$$\frac{d^2 Y}{d\zeta^2} + 3 \zeta^2 \frac{dY}{d\zeta} = 0 \qquad (2.2.36)$$

The boundary conditions are

$$\zeta = \frac{V_{AS}}{(9 a^2 D_A z)^{\frac{1}{3}}}, \qquad Y = 0 \qquad (2.2.37)$$

$$\zeta = \infty , \qquad Y = 1 \tag{2.2.38}$$

The solution of Eq. (2.2.36) with the above boundary conditions is

$$Y = \frac{\displaystyle\int_{\frac{V_{AS}}{(9\,a^2\,D_A\,z)^{\frac{1}{3}}}}^{\zeta} \exp\,(-\zeta^3)\,d\zeta}{\displaystyle\int_{\frac{V_{AS}}{(9\,a^2\,D_A\,z)^{\frac{1}{3}}}}^{\infty} \exp\,(-\zeta^3)\,d\zeta} \tag{2.2.39}$$

Since $< Y >$ is approximately unity when z is small, the Nusselt number can be expressed as

$$Nu_A = 2\,kR\,\left.\frac{\partial Y}{\partial y}\right|_{y=0} = \frac{2\,kR\,a\,\exp\left(\dfrac{-V_{AS}^{\,3}}{9\,D_A\,a^2\,z}\right)}{(9\,D_A\,a^2\,z)^{\frac{1}{3}}\displaystyle\int_{\frac{V_{AS}}{(9\,D_A\,a^2\,z)^{\frac{1}{3}}}}^{\infty} \exp\,(-\zeta^3)\,d\zeta} \tag{2.2.40}$$

Since

$$V_{AS} = \frac{(E_1^{\,2} - 1) < V_A >}{E_1^{\,2} - 0.5} \tag{2.2.41}$$

$$a = \frac{2 < V_A >}{kR\ (E_1^{\,2} - 0.5)} \tag{2.2.42}$$

the Nusselt number for small η can be written as

$$Nu_A = \frac{4 \exp \left[\dfrac{-(E_1^{\,2} - 1)^3}{36\ (E_1^{\,2} - 0.5)\eta} \right]}{\left[36\ (E_1^{\,2} - 0.5)\eta\right]^{\frac{1}{3}} \left\{ \dfrac{\Gamma(\frac{1}{3})}{3} - I\left(\dfrac{E_1^{\,2} - 1}{\left[36(E_1^{\,2} - 0.5)\eta\right]^{\frac{1}{3}}} \right) \right\}} \tag{2.2.43}$$

in which

$$I\ (x) = \int_0^x \exp\ (-\zeta^3)\ d\zeta \tag{2.2.44}$$

This integral has been tabulated by Abramowitz (1).

Furthermore, the penetration model, which assumes a flat velocity profile near the interface, yields

$$Nu_A = \left[\frac{4\ (E_1^{\,2} - 1)}{\pi\ (E_1^{\,2} - 0.5)\eta} \right]^{\frac{1}{2}} \tag{2.2.45}$$

As shown in Figure 2, the asymptotic solution for the entrance region, Eq. (2.2.43), together with the solution for very large z, Eq. (2.2.26), represent the complete solution, Eq. (2.2.25) very well. Beek's solution, Eq. (2.2.33), and the solution of penetration model, Eq. (2.2.45), are good only when z is very small.

When the interfacial velocity equals zero, i.e. $E_1^2 = 1$, Eq. (2.2.43) becomes

$$Nu_A = 1.71\ \eta^{-\frac{1}{3}} \tag{2.2.46}$$

This is the same result derived by Leveque (24) for single-phase mass transfer. Eq. (2.2.45) shows that Nu_A is proportional to $\eta^{-\frac{1}{2}}$ rather than $\eta^{-\frac{1}{3}}$ at very small η.

3. Liquid-Phase Controlled Mass Transfer

When the solubility of the solute in the liquid phase is small, or when the gas phase is a pure gas soluble in the liquid, the process of mass transfer is liquid-phase controlled. Previous workers dealing with this subject considered only cases in which curvature effects were negligible (17,26,42).

The equation of mass transfer for this problem is

$$V_B \frac{\partial c_B}{\partial z} = D_B \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_B}{\partial r}\right)\right] \tag{2.3.1}$$

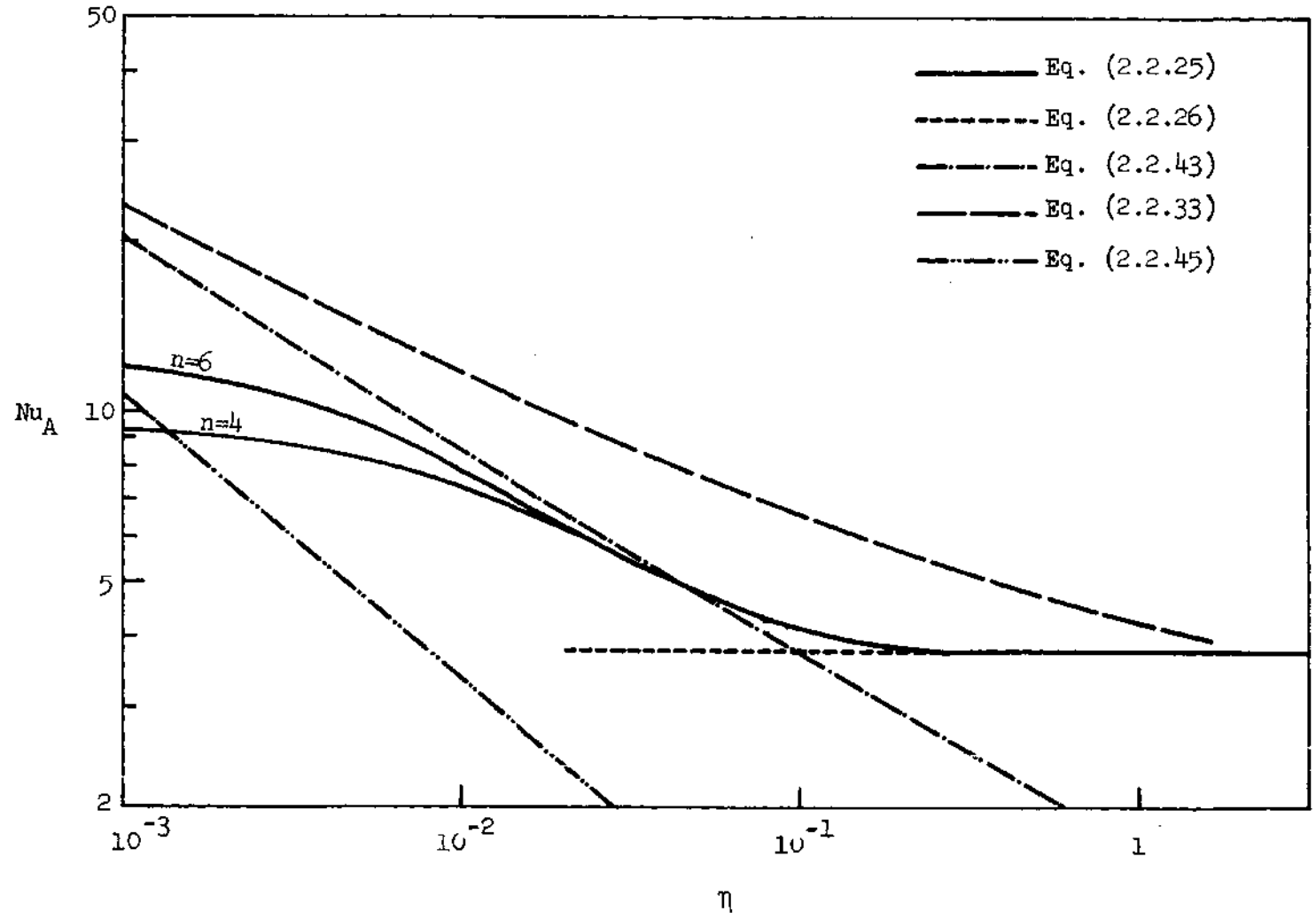

Figure 2. Nu_A as a Function of η at $E_1^2 = 1.05$.

with the boundary conditions

$$z = 0 \qquad c_B = c_{BO} \tag{2.3.2}$$

$$r = R \qquad \frac{\partial c_B}{\partial r} = 0 \tag{2.3.3}$$

$$r = kR \qquad c_B = c_{BS} \tag{2.3.4}$$

Defining the dimensionless variables

$$Y = \frac{c_B - c_{BS}}{c_{BO} - c_{BS}} \tag{2.3.5}$$

$$x = \frac{1 - \frac{r}{R}}{1 - k} \tag{2.3.6}$$

and substituting into Eq. (2.3.1), gives

$$\omega\, g_B \frac{\partial Y}{\partial \eta} = \frac{1}{1 - (1-k)\, x} \frac{\partial}{\partial x} \left\{ \left[1 - (1-k)\, x \right] \frac{\partial Y}{\partial x} \right\} \tag{2.3.7}$$

where g_B and η are defined by Eq. (2.1.14) and Eq. (2.2.7) respectively and

$$\omega = \frac{(1-k)^2 < V_B > D_A}{k^2 < V_A > D_B} \tag{2.3.8}$$

By the method of separation of variables, set $Y = X(x)$. $\psi(\eta)$ and obtain

$$\psi = C \exp\left[-\frac{\lambda^2 \eta}{\omega}\right] \tag{2.3.9}$$

The eigenfunction X is given by

$$\frac{d}{dx}\left\{\left[1 - (1-k)x\right]\frac{dX}{dx}\right\} + \lambda^2\left[1 - (1-k)x\right]\varepsilon_B X = 0 \tag{2.3.10}$$

with the boundary conditions

$$X'(0) = 0 \tag{2.3.11}$$

$$X(1) = 0 \tag{2.3.12}$$

Since there is a logarithmic term in Eq. (2.3.10), it is difficult to approximate X by a power series and therefore the "perturbation method" (34) will be used.

Let $u(x)$ be the solution of the problem assuming ideal flow, i.e. let u be the solution of the following equation

$$\frac{d}{dx}\left\{\left[1 - (1-k)x\right]\frac{du}{dx}\right\} + \gamma^2\left[1 - (1-k)x\right]u = 0 \tag{2.3.13}$$

with the boundary conditions that

$$u'(0) = 0 \tag{2.3.14}$$

$$u(1) = 0 \tag{2.3.15}$$

The function of u is found to be

$$u = \sum_{j=0}^{\infty} b_j x^j \tag{2.3.16}$$

where $b_0 = 1$, $b_1 = 0$, $b_2 = -\gamma^2/2$ and

$$b_j = \frac{1}{j(j-1)} \left[(1-k)(j-1)^2 b_{j-1} + \gamma^2 (1-k) b_{j-3} \right] \tag{2.3.17}$$

for $j \geq 3$

The eigenvalues γ^2 are the roots of

$$\sum_{j=0}^{\infty} b_j = 0 \tag{2.3.18}$$

Now, set $X_n = \sum_{i=1}^{\infty} a_{ni} u_i$ (2.3.19)

where u_i are functions shown by Eq. (2.3.16) with γ_i^2 corresponding to the i^{th} root of Eq. (2.3.18). By doing this, X_n will satisfy the

boundary conditions Eq. (2.3.11) and Eq. (2.3.12). Substituting Eq. (2.3.19) and Eq. (2.3.13) into Eq. (2.3.10) yields

$$\sum_{i=1}^{\infty} a_{ni} \left[1 - (1-k)\, x\right] g_B\, u_i = \sum_{i=1}^{\infty} a_{ni} \frac{\gamma_i^2}{\lambda_n^2} \left[1 - (1-k)x\right] u_i \qquad (2.3.20)$$

When this equation is multiplied by u_k and integrated from 0 to 1 with respect to x, it yields

$$\sum_{i=1}^{\infty} a_{ni}\, P_{ki} = \frac{\gamma_k^2}{\lambda_n^2}\, a_{nk}\, N_k \qquad (2.3.21)$$

In which

$$P_{ki} = \int_0^1 \left[1 - (1-k)\, x\right] g_B\, u_i\, u_k\, dx \qquad (2.3.22)$$

$$N_k = \int_0^1 \left[1 - (1-k)\, x\right] u_k^2\, dx \qquad (2.3.23)$$

Here the orthogonal relation that

$$\int_0^1 \left[1 - (1-k)\, x\right] u_m\, u_n\, dx = 0 \qquad \text{when } m \neq n \qquad (2.3.24)$$

has been applied.

When the infinite series in Eq. (2.3.21) is truncated to a finite number of terms, it yields a homogeneous set of linear algebraic equations with a_{ni} as unknowns. This set of equations is then solved to obtain a_{ni} and λ_n^2 as the eigenvectors and eigenvalues. The complete solution of the concentration profile is

$$Y = \sum_{n=1}^{\infty} C_n X_n \exp \frac{-\lambda_n^2 \eta}{\omega} \tag{2.3.25}$$

where X_n is given in Eq. (2.3.19) and

$$C_n = \frac{\int_0^1 \left[1 - (1-k)\, x\right] g_B X_n \, dx}{\int_0^1 \left[1 - (1-k)\, x\right] g_B X_n^2 \, dx} \tag{2.3.26}$$

The average concentration is

$$\langle Y \rangle = \frac{2}{1+k} \sum_{n=1}^{\infty} C_n \exp \left[\frac{-\lambda_n^2 \eta}{\omega}\right] \int_0^1 \left[1 - (1-k)\, x\right] g_B X_n \, dx \tag{2.3.27}$$

If the liquid-phase controlled local mass-transfer coefficient, K_B, is defined as

$$K_B \, 2\pi \, kR \, dz \left[c_{BS} - \langle c_B \rangle \right] = \pi \, (1-k^2) \, R^2 \langle V_B \rangle \, d \langle c_B \rangle \qquad (2.3.28)$$

then the Nusselt number is

$$Nu_B = \frac{(1-k) \, R \, K_B}{D_B} = \frac{-\omega(1-k) \, \frac{d \langle Y \rangle}{d\eta}}{2 \, k \langle Y \rangle} \qquad (2.3.29)$$

Substituting Eq. (2.3.27) into the above equation gives

$$Nu_B = \frac{(1+k) \sum_{n=1}^{\infty} C_n \lambda_n^2 \exp\left(\frac{-\lambda_n^2 \eta}{\omega}\right) \int_0^1 \left[1 - (1-k) \, x\right] g_B \, X_n \, dx}{2k \sum_{n=1}^{\infty} C_n \exp\left(\frac{-\lambda_n^2 \eta}{\omega}\right) \int_0^1 \left[1 - (1-k) \, x\right] g_B \, X_n \, dx} \qquad (2.3.30)$$

When η is very large,

$$\lim_{\eta \to \infty} Nu_B = Nu_{B\infty} = \frac{(1+k) \, \lambda_1^2}{2k} \qquad (2.3.31)$$

Figure 3 shows $Nu_{B\infty}$ as a function of k and E_3.

In a manner similar to that used in deriving Eq. (2.2.27), the asymptotic solution for large eigenvalues of the liquid-phase controlled mass-transfer problem can be obtained. The result is

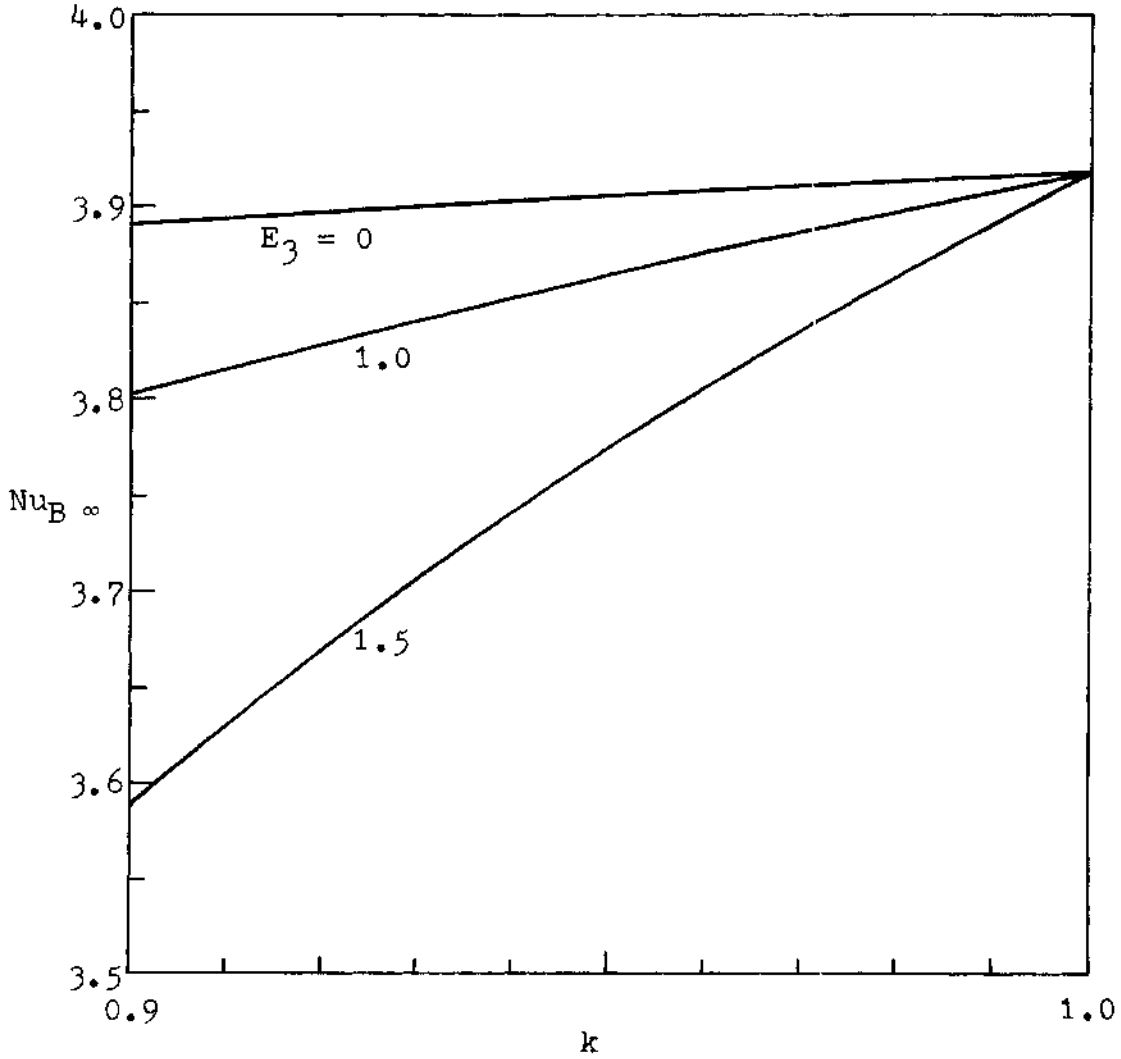

Figure 3. $Nu_{B\infty}$ as a Function of k and E_3.

$$X_n = \left\{ \frac{3}{\pi\lambda_n \left[1-(1-k)x\right]} \right\}^{\frac{1}{2}} \left\{ 2(1-k)x-(1-k)^2x^2 + E_3 \ell n \left[1-(1-k)x\right] \right\}^{-\frac{1}{4}}$$

$$\cos \left\{ \lambda_n \int_0^x g_B^{\frac{1}{2}} \, dx - \frac{\pi}{12} \right\} \qquad (2.3.32)$$

This asymptotic solution gives

$$\lambda_n = \xi \left(n - \frac{5}{12}\right) \qquad (2.3.33)$$

where

$$\xi = \frac{\pi}{\int_0^1 g_B^{\frac{1}{2}} \, dx} \qquad (2.3.34)$$

ξ is plotted in Figure 4.

Comparison of values of λ_n calculated from Eq. (2.3.21) and Eq. (2.3.33) is shown in Table 3.

At the entrance region, where η is small, the Nusselt number will approach the result obtained by Beek and Bakker from the penetration model (7)

$$Nu_B = \varphi \quad \omega^{\frac{1}{2}} \quad \eta^{-\frac{1}{2}} \qquad (2.3.35)$$

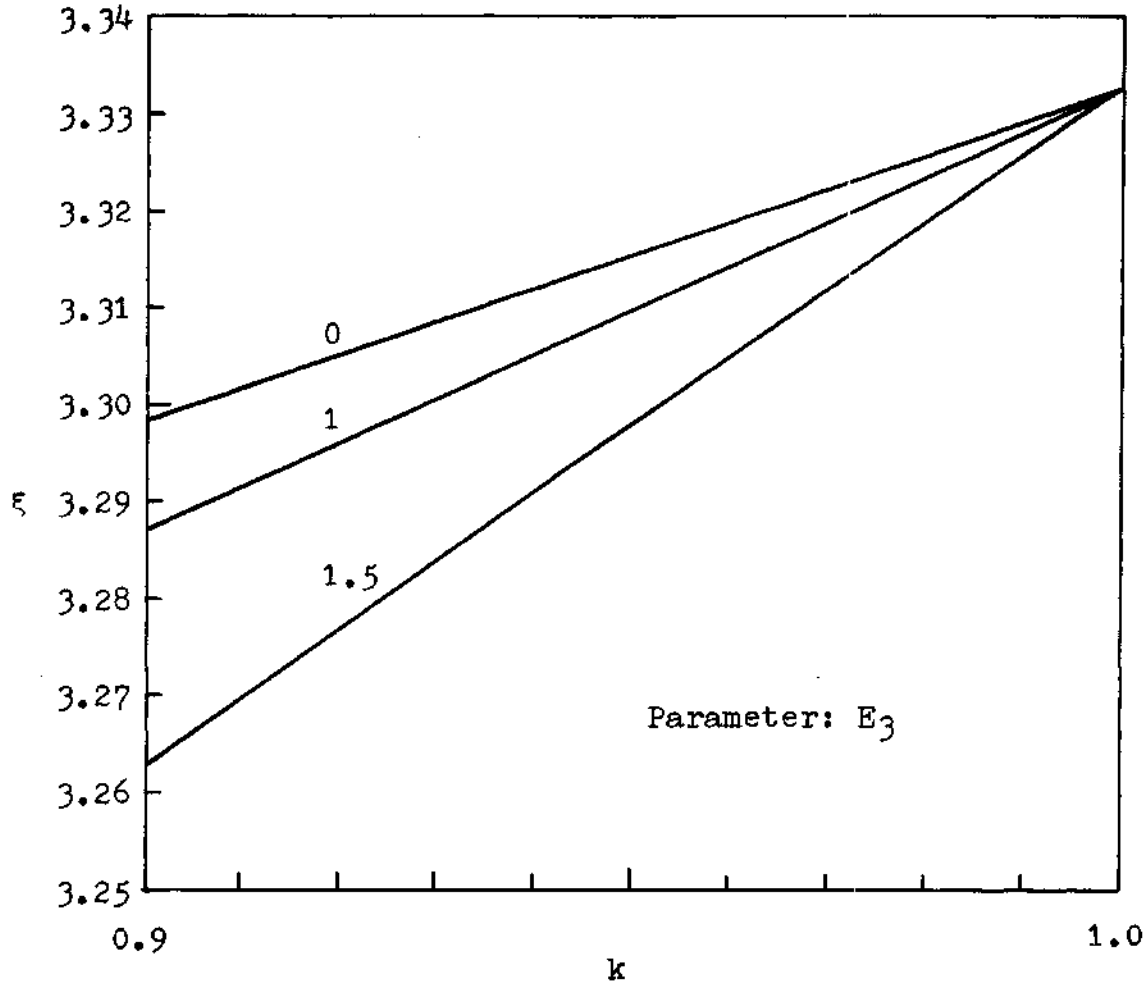

Figure 4. ξ as a Function of k and E_3.

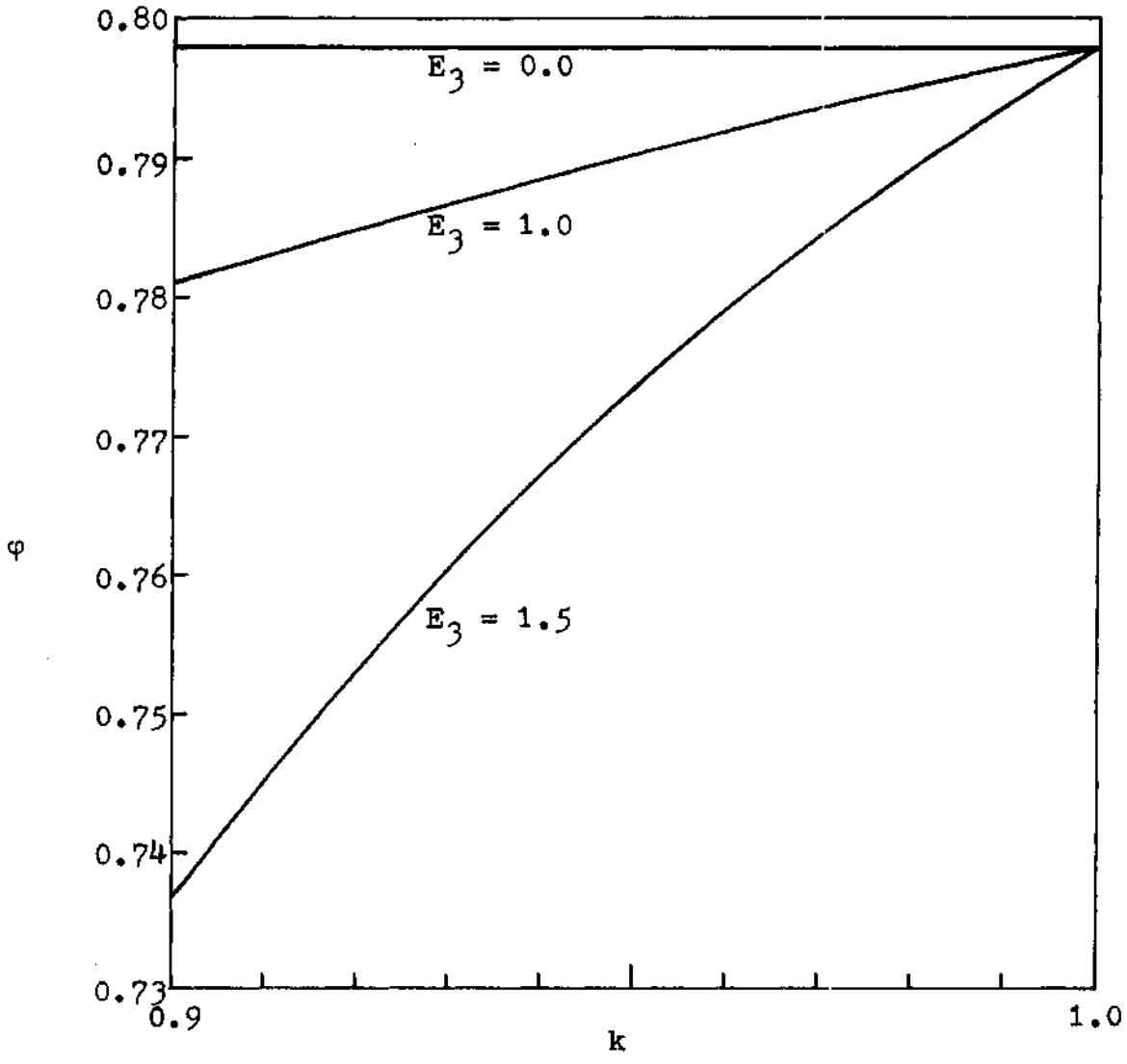

Figure 5. φ as a Function of k and E_3.

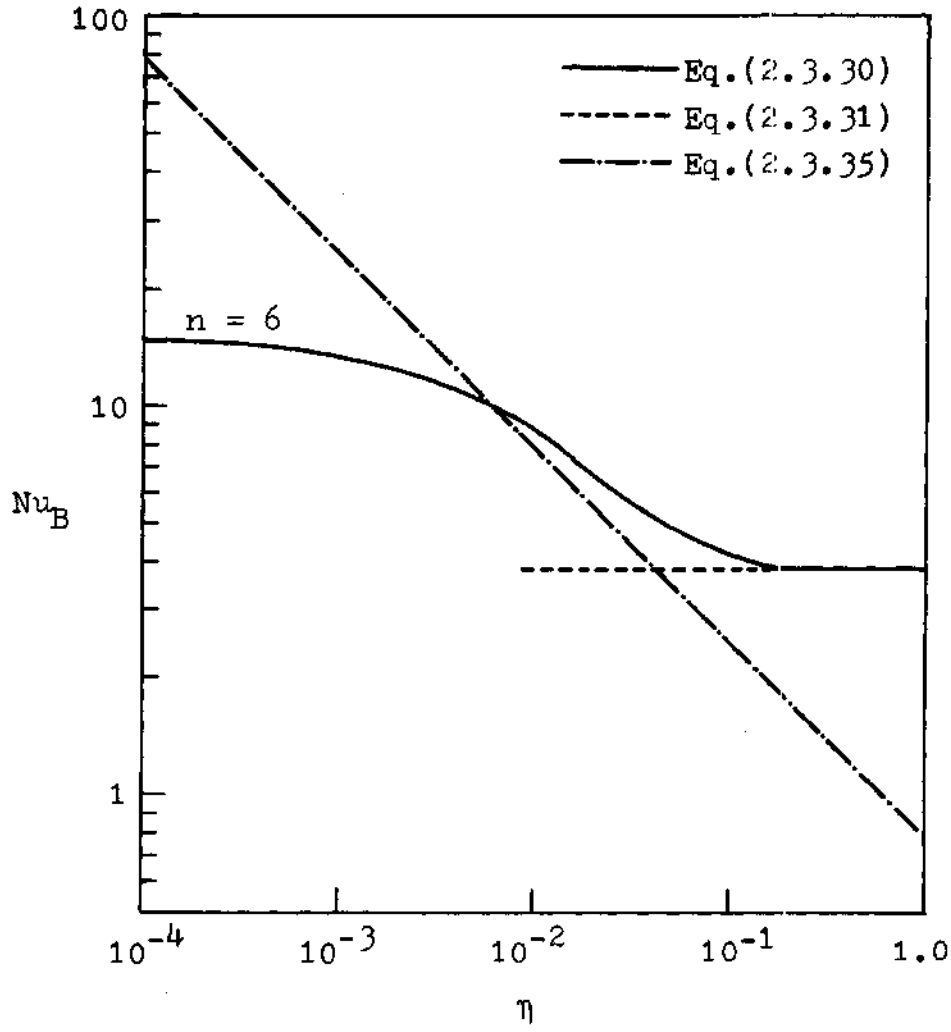

Figure 6. Nu_B as a Function of η at $k = 0.9$, $\omega = 1$ and $E_3 = 1$.

Table 3. Comparison of Eigenvalues for Liquid-Phase Controlled Mass Transfer Calculated From the Asymptotic Solution Eq. (2.3.33), With Those From the Exact Solution, Eq. (2.3.21), at k = 0.9 and E_3 = 1.2.

	Eq. (2.3.21)	Eq. (2.3.33)
λ_1	1.8804	1.9140
λ_2	5.1606	5.1952
λ_3	8.4342	8.4764
λ_4	11.706	11.658
λ_5	14.982	15.039

where

$$\varphi = \left[\frac{E_2 \, (1-k^2 + E_3 \, \ell n \, k)}{\pi} \right]^{\frac{1}{2}} \qquad (2.3.36)$$

φ is plotted in Figure 5.

Typical curves of Eq. (2.3.30), Eq. (2.3.31) and Eq. (2.3.35) are shown in Figure 6.

4. Interphase Mass Transfer

Interphase mass transfer in two-phase cocurrent laminar flow in a rectangular channel has been investigated by Byers and King (10) and Apelblat and Katchalsky (5). The present study is concerned with the case of annular flow in a round duct. Considerations are restricted to low concentrations and low net fluxes of mass. Physical properties are

assumed constant. It is also assumed that the concentrations at the interface are in equilibrium and that the laminar velocity profiles are fully developed. The equation of mass transfer is

$$V_i \frac{\partial c_i}{\partial z} = D_i \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_i}{\partial r}\right)\right], \quad i = A, B. \tag{2.4.1}$$

The boundary conditions are

$$z = 0, \qquad c_A = c_{AO} \tag{2.4.2}$$

$$z = 0, \qquad c_B = c_{BO} \tag{2.4.3}$$

$$r = 0, \qquad \frac{\partial c_A}{\partial r} = 0 \tag{2.4.4}$$

$$r = kR, \qquad c_A = \alpha c_B \tag{2.4.5}$$

$$r = kR, \qquad D_A \frac{\partial c_A}{\partial r} = D_B \frac{\partial c_B}{\partial r} \tag{2.4.6}$$

$$r = R, \qquad \frac{\partial c_B}{\partial r} = 0 \tag{2.4.7}$$

where α is the equilibrium constant.

When the method of separation of variables is applied to this problem, a two-region Sturm-Liouville system results. Stein (40) has successfully extended the conventional mathematical techniques to solve

this type of problem for the double pipe heat exchanger. As will be seen, these techniques are equally well applied to this mass-transfer problem.

In order to render the variables dimensionless, define

$$Y_A = \frac{c_A - c_{AO}}{\alpha\, c_{BO} - c_{AO}} \tag{2.4.8}$$

$$Y_B = \frac{\alpha\, c_B - c_{AO}}{\alpha\, c_{BO} - c_{AO}} \tag{2.4.9}$$

$$x = \frac{r}{kR} \quad \text{for phase A} \tag{2.4.10}$$

$$x = \frac{1 - \frac{r}{R}}{1-k} \quad \text{for phase B} \tag{2.4.11}$$

$$\eta = \frac{D_A\, z}{k^2\, R^2 < V_A >} \tag{2.4.12}$$

Thus Eq. (2.4.1) becomes

$$g_A \frac{\partial Y_A}{\partial \eta} = \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial Y_A}{\partial x}\right) \quad \text{for phase A} \tag{2.4.13}$$

and

$$\omega \; g_B \; \frac{\partial Y_B}{\partial \eta} = \frac{1}{1 - (1-k)x} \; \frac{\partial}{\partial x} \left\{ \left[1 - (1-k)x\right] \frac{\partial Y_B}{\partial x} \right\} \text{ for phase B} \tag{2.4.14}$$

The boundary conditions are

$$Y_A (x,0) = 0 \tag{2.4.15}$$

$$Y_B (x,0) = 1 \tag{2.4.16}$$

$$\frac{\partial Y_A (0,\eta)}{\partial x} = 0 \tag{2.4.17}$$

$$\frac{\partial Y_B (0, \eta)}{\partial x} = 0 \tag{2.4.18}$$

$$Y_A (1,\eta) = Y_B (1,\eta) \tag{2.4.19}$$

$$\Phi \; \frac{\partial Y_A (1,\eta)}{\partial x} + \frac{\partial Y_B (1,\eta)}{\partial x} = 0 \tag{2.4.20}$$

where

$$\Phi = \frac{\alpha (1-k) D_A}{k D_B} \tag{2.4.21}$$

The average concentrations are

$$< Y_A > = 2 \int_0^1 g_A \, Y_A \, x dx \tag{2.4.22}$$

$$< Y_B > = \frac{2}{1+k} \int_0^1 g_B \, Y_B \left[1 - (1-k)x \right] dx \tag{2.4.23}$$

The overall mass balance yields

$$< Y_A > = H \, (1 - < Y_B >) \tag{2.4.24}$$

When $z \to \infty$, phase A and phase B are in equilibrium.

$$\lim_{z \to \infty} Y_A = \lim_{z \to \infty} Y_B = \frac{H}{1+H} \tag{2.4.25}$$

where $H = \frac{L}{G\alpha}$ and (2.4.26)

G and L are total volumetric flow rates of phases A and B.

$$G = \pi \, k^2 \, R^2 < V_A > \tag{2.4.27}$$

$$L = \pi \, (1-k^2) \, R^2 < V_B > \tag{2.4.28}$$

The mass-transfer coefficient, K, is defined as

$$2\pi kR\,K\,(\langle Y_B\rangle - \langle Y_A\rangle)\,dz = G\,d\langle Y_A\rangle \qquad (2.4.29)$$

Thus, the Nusselt number becomes

$$Nu = \frac{2kRK}{D_A} = \frac{\dfrac{d\langle Y_A\rangle}{d\eta}}{1 - \dfrac{1+H}{H}\langle Y_A\rangle} = \frac{2\left.\dfrac{\partial Y_A}{\partial x}\right|_{x=1}}{1 - \dfrac{1+H}{H}\langle Y_A\rangle} \qquad (2.4.30)$$

Applying the method of separation of variables, set

$$Y_i = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n X_{in} \exp(-\lambda_n^2 \eta), \quad i = A, B. \qquad (2.4.31)$$

Substituting this into Eq. (2.4.13) and Eq. (2.4.14) yields

$$x\frac{d^2 X_{An}}{dx^2} + \frac{d X_{An}}{dx} + \lambda_n^2\, x\, g_A X_{An} = 0 \qquad (2.4.32)$$

$$\frac{d}{dx}\left\{\left[1 - (1-k)x\right]\frac{d X_{An}}{dx}\right\} + \lambda_n^2\,\omega\left[1 - (1-k)x\right] g_B X_{Bn} = 0 \qquad (2.4.33)$$

The boundary conditions become

$$\eta = 0, \qquad \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n X_{An} = 0 \tag{2.4.34}$$

$$\eta = 0, \qquad \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n X_{Bn} = 1 \tag{2.4.35}$$

$$x = 0, \qquad \frac{d X_{An}}{dx} = 0 \tag{2.4.36}$$

$$x = 0, \qquad \frac{d X_{Bn}}{dx} = 0 \tag{2.4.37}$$

$$x = 1, \qquad X_{An} - X_{Bn} = 0 \tag{2.4.38}$$

$$x = 1, \qquad \Phi \frac{d X_{An}}{dx} + \frac{d X_{Bn}}{dx} = 0 \tag{2.4.39}$$

The coefficients C_n of Eq. (2.4.31) can be determined by the boundary conditions, Eq. (2.4.34) and Eq. (2.4.35), such that

$$C_n = \frac{\int_0^1 \frac{H}{1+k} \left[1 - (1-k)x\right] g_B X_{Bn} \, dx}{\int_0^1 \left\{ x \, g_A X_{An}^2 + \frac{H}{1+k} \left[1 - (1-k)x\right] g_B X_{Bn}^2 \right\} dx} \tag{2.4.40}$$

Here in determining C_n, the orthogonality condition

$$\int_0^1 \left\{ x\, g_A\, X_{Am}\, X_{An} + \frac{H}{1+k} \left[1 - (1-k)x\right] g_B\, X_{Bm}\, X_{Bn} \right\} dx = 0 \qquad (2.4.41)$$

when $m \neq n$

has been applied.

Note that $\lambda_0 = 0$ with the corresponding $X_{i0} = 1$ must be included. The corresponding C_0 is $H/(1+H)$.

Now the problem remains to solve the differential equations Eq. (2.4.32) and Eq. (2.4.33) with the boundary conditions Eq. (2.4.36) through Eq. (2.4.39). Since the logarithmic term in g_B for laminar Newtonian flow makes it difficult to approximate X_{Bn} by use of polynominal approximations, the case of ideal flow that $g_A = g_B = 1$ will be solved first. And then, by use of the perturbation method similar to that used in the previous section, the solutions for the case of laminar velocity profiles for Newtonian flow will be approximated by the solutions for the ideal flow case.

The Ideal Flow Case

When $g_A = g_B = 1$, the solution of Eq. (2.4.32) with boundary condition Eq. (2.4.36) is

$$X_{An} = C_1\, J_0\, (\lambda_n x) \qquad (2.4.42)$$

The solution of Eq. (2.4.33) with boundary condition Eq. (2.4.37) is

$$X_{Bn} = C_2 \sum_{i=0}^{\infty} b_i\, x^i \qquad (2.4.43)$$

C_1 and C_2 here are arbitrary constants, J_0 is the zero order Bessel function of the first kind, $b_0 = 1$, $b_1 = 0$, $b_2 = -\omega \lambda_n^2/2$ and

$$b_i = \frac{1}{i(i-1)} \left[(1-k)\,(i-1)^2\, b_{i-1} - \omega \lambda_n^2\, b_{i-2} + (1-k)\,\omega \lambda_n^2\, b_{i-3} \right] \tag{2.4.44}$$

for $i \geq 3$. Set the constants

$$C_1 = \sum_{i=1}^{\infty} i\, b_i \tag{2.4.45}$$

$$C_2 = \Phi \lambda_n J_1 (\lambda_n) \tag{2.4.46}$$

so that Eq. (2.4.39) is satisfied. According to Eq. (2.4.38), the eigenvalues λ_n are the roots of

$$J_0 (\lambda_n) \sum_{i=1}^{\infty} i\, b_i - \Phi \lambda_n J_1 (\lambda_n) \sum_{i=0}^{\infty} b_i = 0 \tag{2.4.47}$$

The Newtonian Fluid-Velocity-Profile Case

For the velocity profiles where

$$g_A = \frac{E_1^2 - x^2}{E_1^2 - 0.5} \tag{2.4.48}$$

$$g_B = E_2 \left\{ (1-k)\, x \left[2 - (1-k)\, x \right] + E_3 \ell n \left[1 - (1-k)\, x \right] \right\} \qquad (2.4.49)$$

the perturbation method is applied to solve differential equations Eq. (2.4.32) and Eq. (2.4.33) with boundary conditions Eq. (2.4.36) through Eq. (2.4.39). Set

$$u_j = \left(\sum_{i=1}^{\infty} i\, b_i \right) J_0 (\gamma_j x) \qquad (2.4.50)$$

$$v_j = \Phi\, \gamma_j\, J_1 (\gamma_j) \sum_{i=0}^{\infty} b_i\, x^i \qquad (2.4.51)$$

That is, set u_j and v_j equal to the j^{th} eigenfunctions of the ideal flow case, which are shown in Eq. (2.4.42) and Eq. (2.4.43), respectively. γ_j are the eigenvalues of the ideal flow case. Then represent the eigenfunctions of the laminar Newtonian flow case as a linear sum of the ideal flow eigenfunctions

$$X_{An} = \sum_{j=0}^{\infty} a_{nj}\, u_j \qquad (2.4.52)$$

$$X_{Bn} = \sum_{j=0}^{\infty} a_{nj}\, v_j \qquad (2.4.53)$$

Using similar arguments discussed in the previous section, substitute Eq. (2.4.52) and Eq. (2.4.53) into Eq. (2.4.32) and Eq. (2.4.33) respectively to give

$$\sum_{j=0}^{\infty} a_{nj} \left(\lambda_n^2 g_A - \gamma_j^2\right) u_j = 0 \qquad (2.4.54)$$

$$\sum_{j=0}^{\infty} a_{nj} \left(\lambda_n^2 g_B - \gamma_j^2\right) v_j = 0 \qquad (2.4.55)$$

The orthogonal relation for the ideal flow case is

$$\int_0^1 \left\{ x\, u_m\, u_n + \frac{H}{1+k} \left[1 - (1-k)\, x\right] v_m\, v_n \right\} dx = 0 \quad \text{when } m \neq n \qquad (2.4.56)$$

Taking advantage of this relation, multiplying Eq. (2.4.54) by xu_k/λ_n^2 and Eq. (2.4.55) by $H\left[1 - (1-k)\, x\right] v_k/(1+k)\,\lambda_n^2$, adding them together and then integrating from 0 to 1 with respect to x, yields

$$\sum_{j=0}^{\infty} a_{nj}\, P_{kj} = \frac{\gamma_k^2}{\lambda_n^2}\, a_{nk}\, N_k \qquad (2.4.57)$$

where

$$P_{kj} = \int_0^1 \left\{ x\, g_A\, u_k\, u_j + \frac{H}{1+k} \left[1 - (1-k)\, x \right] g_B\, v_k\, v_j \right\} dx \qquad (2.4.58)$$

and

$$N_k = \int_0^1 \left\{ x u_k^{\,2} + \frac{H}{1+k} \left[1 - (1-k)\, x \right] v_k^{\,2} \right\} dx \qquad (2.4.59)$$

Note that $\gamma_0 = 0$ with the corresponding eigenfunctions $u_0 = v_0 = 1$ should be included. Once again the series in Eq. (2.4.57) is truncated to a finite number of terms and the resulting set of algebraic equations is solved to get a_{nj} and $\lambda_n^{\,2}$. Thus the complete solution is

$$Y_i = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n\, X_{in}\, \exp\left(-\lambda_n^{\,2}\, \eta\right), \quad i = A,\ B. \qquad (2.4.60)$$

where

$$X_{An} = \sum_{j=0}^{\infty} a_{nj} \left(\sum_{i=1}^{\infty} i\, b_i \right) J_0\, (\gamma_j\, x) \qquad (2.4.61)$$

$$X_{Bn} = \sum_{j=0}^{\infty} a_{nj}\, \Phi\, \gamma_j\, J_1\, (\gamma_j) \sum_{i=0}^{\infty} b_i\, x^i \qquad (2.4.62)$$

C_n are expressed by Eq. (2.4.40), J_0 and J_1 are Bessel functions of the first kind, b_i are indicated by Eq. (2.4.44), γ_j are the eigenvalues for the ideal flow case and a_{nj} and λ_n are the eigenvectors and eigenvalues of Eq. (2.4.57).

According to Eq. (2.4.30), the Nusselt number is

$$Nu = \frac{H \sum_{n=1}^{\infty} C_n \lambda_n^2 \exp(-\lambda_n^2 \eta) \int_0^1 2x\, g_A\, X_{An}\, dx}{(1+H) \sum_{n=1}^{\infty} C_n \exp(-\lambda_n^2 \eta) \int_0^1 2x\, g_A\, X_{An}\, dx} \qquad (2.4.63)$$

When η is very large, the above equation becomes

$$Nu_\infty = \frac{H \lambda_1^2}{1+H} \qquad (2.4.64)$$

Nu_∞ is presented in Figure 7 as a function of five variables, Φ, H, k, E_1^2, and E_3. Some data are also given in Table 4 and Table 5 to show the effects of different velocity profiles, i.e. the effects of E_1^2 and E_3 on Nu_∞.

Asymptotic Solution for the Entrance Region

Again when η is small, the series in Eq. (2.4.63) does not converge rapidly. Because of the brief contact of the gas and the liquid in the entrance region, an asymptotic solution for small η for the interphase mass-transfer case can be obtained by assuming that the two phases are two semi-infinite media with a moving interface.

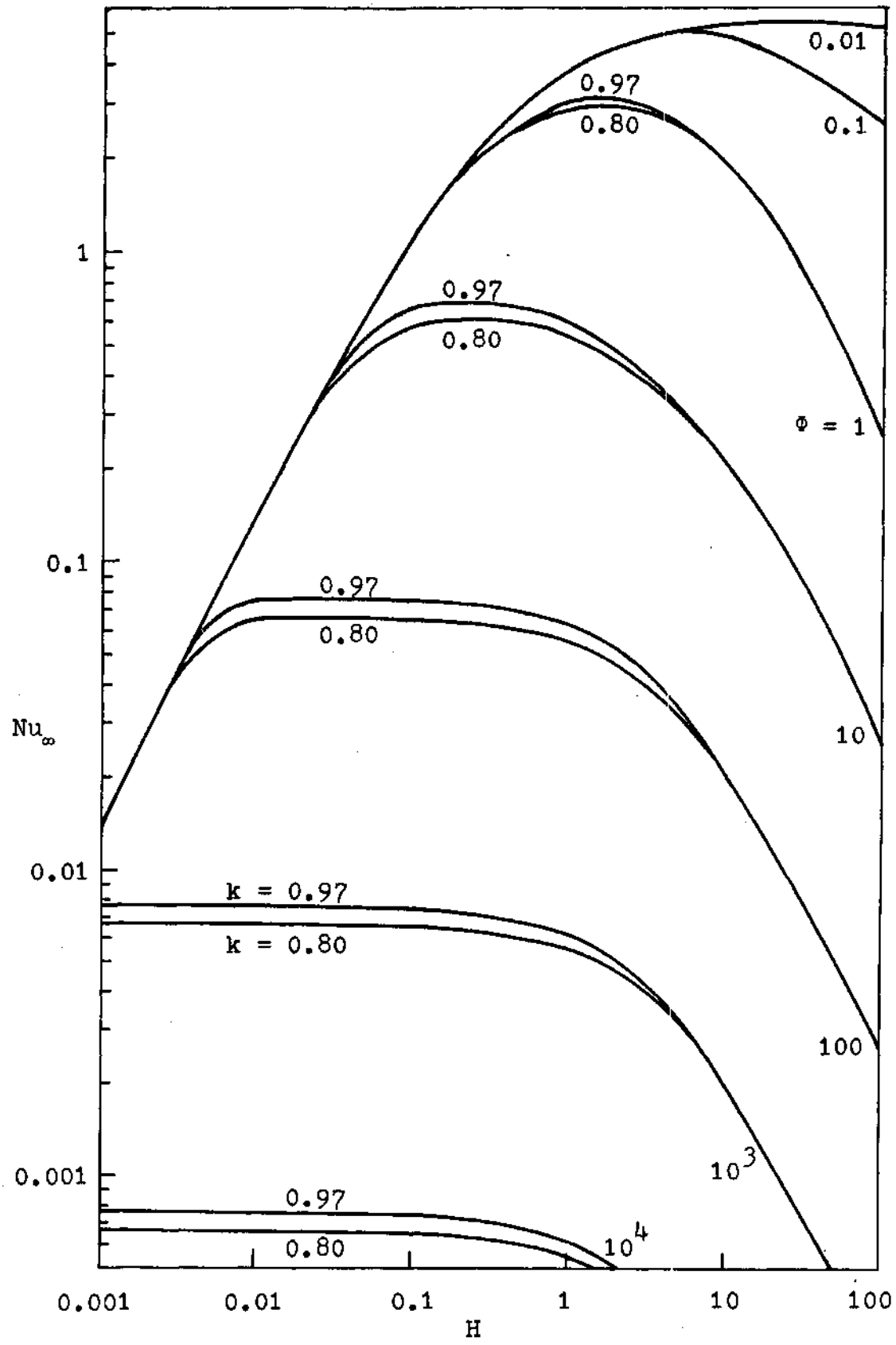

Figure 7. Nu_∞ as a Function of H, k and Φ at $E_1^2 = 3$ and $E_3 = 1.4$.

Table 4. Effect of E_1^2 on Nu_∞ at k = 0.9 and E_3 = 1.4.

E_1^2	Nu_∞ H=100, Φ = 0.1	Nu_∞ H=1, Φ = 10	Nu_∞ H=0.1, Φ = 0.1
1	2.457	0.5480	1.026
2	2.489	0.5637	1.102
3	2.492	0.5665	1.139
4	2.494	0.5677	1.158
5	2.494	0.5683	1.169
6	2.495	0.5687	1.176

Table 5. Effect of E_3 on Nu_∞ at K = 0.9 and E_1^2 = 2.

E_3	Nu_∞ H=100, Φ = 0.1	Nu_∞ H=1, Φ = 10	Nu_∞ H=0.1, Φ = 0.1
1.2	2.507	0.5743	1.102
1.3	2.499	0.5699	1.102
1.4	2.489	0.5637	1.102
1.5	2.474	0.5546	1.102
1.6	2.456	0.5352	1.102

Apelblat, et al. (5) obtained a solution using this method and assuming Couette flow velocity distributions. Unfortunately they made a mistake in the calculation which led to an invalid solution.

The differential equations under these conditions are

$$(V_{AS} + ay) \frac{\partial c_A}{\partial z} = D_A \frac{\partial^2 c_A}{\partial y^2} \tag{2.4.65}$$

$$V_{AS} \frac{\partial c_B}{\partial z} = D_B \frac{\partial^2 c_B}{\partial y^2} \tag{2.4.66}$$

where y is distance from the interface. The boundary conditions are

$$z = 0, \qquad c_A = c_{AO} \tag{2.4.67}$$

$$z = 0, \qquad c_B = c_{BO} \tag{2.4.68}$$

$$y = 0, \qquad c_A = \alpha c_B \tag{2.4.69}$$

$$y = 0, \qquad D_A \frac{\partial c_A}{\partial y} = D_B \frac{\partial c_B}{\partial y} \tag{2.4.70}$$

$$y = \infty, \qquad c_A = c_{AO} \tag{2.4.71}$$

$$y = -\infty, \qquad c_B = c_{BO} \tag{2.4.72}$$

Defining the dimensionless variables

$$Y_A = \frac{c_A - c_{AO}}{\alpha\, c_{BO} - c_{AO}} \tag{2.4.73}$$

$$Y_B = \frac{\alpha\, c_B - \alpha\, c_{BO}}{\alpha\, c_{BO} - c_{AO}} \tag{2.4.74}$$

$$\zeta_A = \frac{V_{AS} + ay}{(9\, D_A\, a^2\, z)^{\frac{1}{3}}} \tag{2.4.75}$$

$$\zeta_B = \frac{y}{2\left(\frac{D_B\, z}{V_{AS}}\right)^{\frac{1}{2}}} \tag{2.4.76}$$

and substituting into Eq. (2.4.65) and Eq. (2.4.66), gives

$$Y_A'' + 3\,\zeta_A^2\, Y_A' = 0 \tag{2.4.77}$$

$$Y_B'' + 2\,\zeta_B^2\, Y_B' = 0 \tag{2.4.78}$$

The boundary conditions are

$$\zeta_A = \infty, \qquad Y_A = 0 \tag{2.4.79}$$

$$\zeta_B = -\infty, \qquad Y_B = 0 \tag{2.4.80}$$

$$\zeta_A = \frac{V_{AS}}{(9\, a^2\, D_A\, z)^{\frac{1}{3}}} \quad \text{and} \quad \zeta_B = 0,\ Y_A = Y_B + 1 \tag{2.4.81}$$

$$\zeta_A = \frac{V_{AS}}{(9\, a^2\, D_A\, z)^{\frac{1}{3}}} \quad \text{and} \quad \zeta_B = 0,\ 2\alpha\left(\frac{a^2\, D_A{}^4\, z}{81\, V_{AS}{}^3\, D_B{}^3}\right)^{\frac{1}{6}} Y_A{}' = Y_B{}' \tag{2.4.82}$$

Integrating these equations and using the above boundary condition gives

$$Y_A = \frac{\displaystyle\int_{\zeta_A}^{\infty} \exp\,(-\zeta_A{}^3)\, d\,\zeta_A}{\displaystyle\int_{\frac{V_{AS}}{(9\, D_A a^2 z)^{\frac{1}{3}}}}^{\infty} \exp\,(-\zeta_A{}^3)\, d\,\zeta_A \;+\; 2\alpha\left(\frac{D_A{}^4\, a^2\, z}{81\, D_B{}^3 V_{AS}{}^3}\right)^{\frac{1}{6}} \exp\left(\frac{-V_{AS}{}^3}{9\, D_A a^2 z}\right)}$$

$$\int_0^{\infty} \exp\,(-\zeta_B{}^2)\, d\,\zeta_B \tag{2.4.83}$$

Consequently, the Nusselt number is

$$Nu = \frac{2kRK}{D_A} = \frac{d < Y_A >}{d\eta}$$

$$= \left\{ 2\alpha \left[\frac{\pi(E_1^2 - \frac{1}{2})\, D_A \eta}{(E_1^2 - 1) D_B} \right]^{\frac{1}{2}} + \left[\frac{9(E_1^2 - \frac{1}{2})\eta}{16} \right]^{\frac{1}{3}} \left[\frac{\Gamma(\frac{1}{3})}{3} - I\left(\frac{E_1^2 - 1}{\left[36(E_1^2 - \frac{1}{2})\eta \right]^{\frac{1}{3}}} \right) \right] \right.$$

$$\left. \exp\left[\frac{(E_1^2 - 1)^3}{36(E_1^2 - \frac{1}{2})\eta} \right] \right\}^{-1} \qquad (2.4.84)$$

Here I(x) is the function defined by Eq. (2.2.44).

The system of Eqs. (2.4.65-72) has been solved by Byers, et al. (10) using the Laplace transformation which leads to the Nusselt number for very small η, $\left(\frac{4\,(E_1^2 - 0.5)\,\eta}{(E_1^2 - 1)^3} \ll 1 \right)$

$$Nu = \frac{2}{\alpha(\frac{D_A}{D_B})^{\frac{1}{2}} + 1} \left[\frac{E_1^2 - 1}{\pi(E_1^2 - 0.5)\eta} \right]^{\frac{1}{2}} + \frac{1}{(E_1^2 - 1)\left[\alpha\, (\frac{D_A}{D_B})^{\frac{1}{2}} + 1 \right]^2} \qquad (2.4.85)$$

Typical curves of Eq. (2.4.63), Eq. (2.4.64), Eq. (2.4.84) and Eq. (2.4.85) are presented in Figure 8 for comparison.

5. Discussion

(a) Studying mass-transfer problems by assuming laminar flow

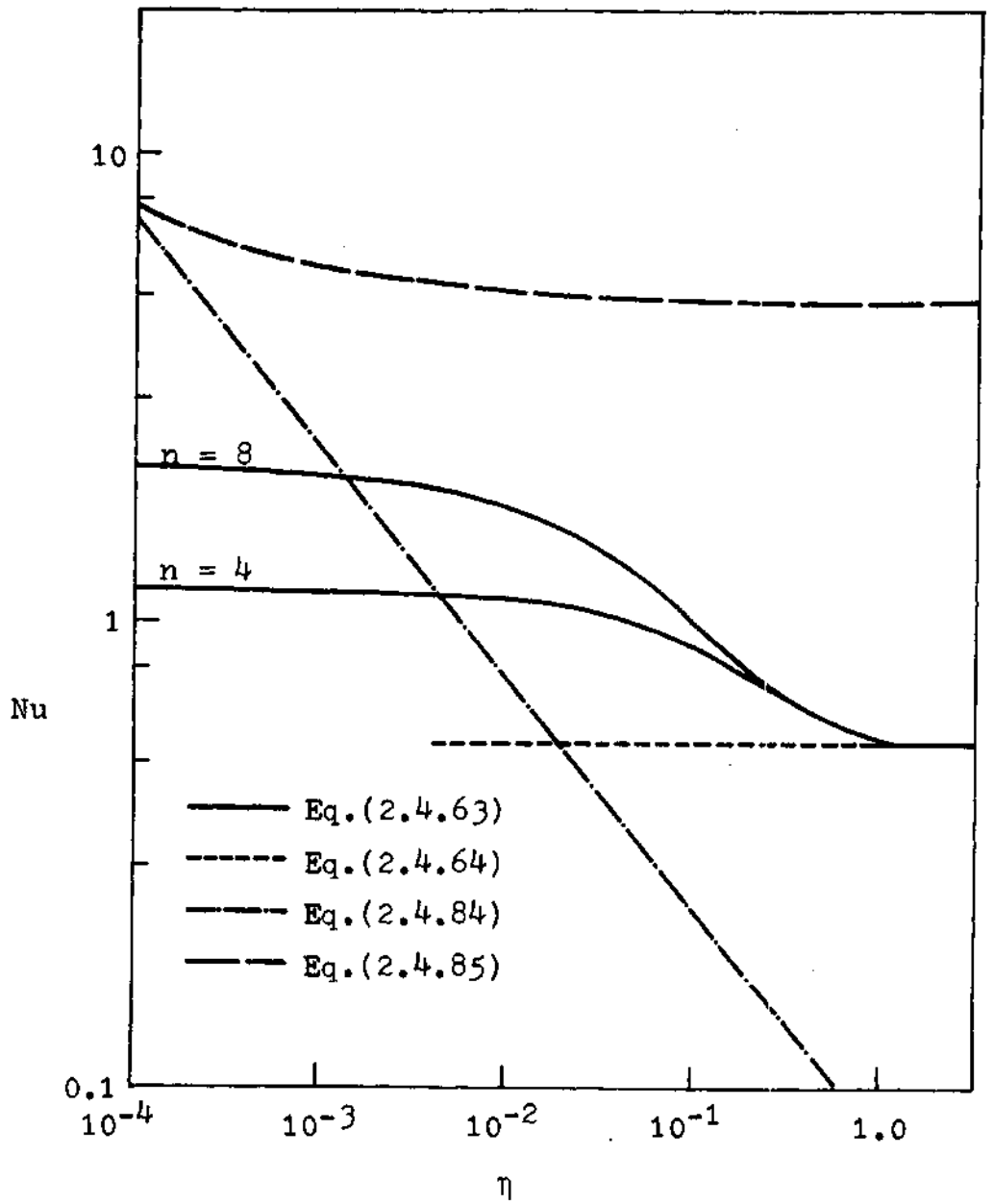

Figure 8. Nu as a Function of η at k = 0.9, H = 1, $\Phi = 10$, $E_1^2 = 1.05$ and $E_3 = 1.4$.

conditions makes it possible to get exact solutions for theoretical models. These solutions yield basic details of the transfer mechanism. It should be noted that studies have shown that the most probable laminar annular flow, a falling film, is not laminar even for Reynolds numbers as low as 300 (41). The question of interfacial instability has not been considered in this chapter concerned with laminar flow. The solutions given are applicable only so long as the interface is, in fact, stable. Wallis has pointed out that a falling film is always unstable (47). However, disturbances may be very small at low Reynolds numbers.

(b). In the problem concerned with gas-phase controlled mass transfer, the solution of Eq. (2.2.12) would be

$$X = J_0 (\lambda x) \qquad (2.5.1)$$

when E_1^2 is very large, i.e. when the velocity profile is flat. In this case, $Nu_{A\infty}$ will equal 5.775, the square of the first zero of J_0. Figure 1 shows that $Nu_{A\infty}$ will approach 5.775 when E_1^2 is large. When $E_1^2 = 1$, $Nu_{A\infty}$ equals 3.675 as in the classical Graetz problem. The conclusion is that the value of $Nu_{A\infty}$ lies between Nusselt numbers for the single-phase Graetz problem with a parabolic velocity profile, 3,675, and a flat velocity profile, 5.775.

(c). The velocity profile of the liquid phase is similar to the velocity profile of single-phase flow when $E_3 = 0$. When $E_3 = 2k^2$, it is identical to the velocity profile of a free-falling cylindrical film. Results of the liquid-phase controlled mass transfer presented in

Figure 3 show that $Nu_{B\infty}$ will decrease when E_3 increases. Galiullin, et al. who neglected the effect of the curvature of the pipe wall, have reached similar conclusions that the flow with a more concaved velocity profile will result in more intense mass transfer (17). The effect of the pipe wall curvature is not negligible, however. Figure 3 shows that $Nu_{B\infty}$ drops from 3.919 at k = 1 to 3.589 at k = 0.9 when $E_3 = 1.5$. This effect will decrease when E_3 decreases. When k approaches 1, $Nu_{B\infty}$ for all values of E_3 approach 3.919. This is the value of $Nu_{B\infty}$ for a linear velocity profile without curvature effect, which was also calculated by Galiullin, et al. (17).

(d). The Nusselt number of interphase mass transfer for large Graetz number, Nu_∞, is a function of E_1^2, E_3, k, H, and Φ. The effects of velocity profiles and film thickness, i.e. the effects of E_1^2, E_3, and k, on Nu_∞ are small in comparison with the effects of H and Φ. Nu_∞ is proportional to H when H is small and is inversely proportional to H when H is large. Nu_∞ also is inversely proportional to Φ when Φ is large, but it is independent of Φ when Φ is small. These characteristics are clearly shown in Figure 7.

(e). Figure 2 shows Nu_A as a function of the Graetz number, η, for $E_1^2 = 1.05$. The poor convergence of the series in Eq. (2.2.25) at low values of η is demonstrated by the two curves given in this figure. One curve uses six terms in the series and the other curve uses four terms. However, it also shows that the applicability of the asymptotic solution, Eq. (2.2.43), is excellent at low values of η. Figure 6 and Figure 8 give similar comparisons for Nu_B and Nu. The asymptotic solution for the entrance region for liquid-phase controlled mass

transfer derived by the method of combination of variables is similar to that derived by the Laplace transformation. Both solutions indicate that Nu_B is proportional to the -1/2 power of η as shown in Eq. (2.3.35). Figure 6 shows that this result is in good agreement with the exact solution, Eq. (2.3.30), when η is small. For the case of interphase mass transfer, the asymptotic solution for the entrance region derived by the method of combination of variables, Eq. (2.4.84), is in better agreement with the exact solution than the result derived by the Laplace transformation, Eq. (2.4.85). This comparison has been presented in Figure 8.

(f). Results of the asymptotic solution for large eigenvalues are excellent for the case of gas-phase controlled mass transfer and the case of liquid-phase controlled mass transfer. Comparisons of the eigenvalues calculated from the asymptotic solutions with those calculated from exact solutions are shown in Table 2 and Table 3. It can be seen that the asymptotic solutions are in good agreement with the exact solutions even to the first eigenvalues. The case of interphase mass transfer is so complicated that no asymptotic solution for large eigenvalues has been found.

(g). The equation of the additivity of individual phase resistances in mass transfer derived from the two-film theory is

$$\frac{1}{Nu} = \frac{1}{Nu_A} + \frac{\Phi}{2\ Nu_B} \qquad (2.5.2)$$

This additivity principle should be valid for cocurrent flow mass transfer

in the entrance region, since Nu, Nu_A and Nu_B all vary directly with $\eta^{-\frac{1}{2}}$ for very small values of η. Actually when the asymptotic solutions for the entrance region, Eq. (2.2.43), Eq. (2.3.35), and Eq. (2.4.80), are substituted into Eq. (2.5.2), this equation holds exactly. When η is large, the gas-liquid ratio H becomes the controlling factor in interphase mass transfer. But both the gas-phase controlled mass transfer and the liquid-phase controlled mass transfer are independent of H because of the assumption of constant interfacial concentrations. Thus the additivity principle is no longer valid. For example, when $E_1^2 = 3$, $E_3 = 1.4$, $k = 0.97$ and $\Phi = 1000$, Nu_∞ calculated from the additivity principle is 0.0077. Figure 7 shows that $Nu_\infty = 0.0077$ only when $0.001 < H < 0.1$. Nu_∞ is proportional to H when H is small and inversely proportional to H when H is large. For a certain range of value of H in between, Nu_∞ is a constant. This is the range where the additivity principle is applicable.

CHAPTER III

TURBULENT FLOW CASE

It is desired to predict the interphase mass-transfer coefficients for two-phase gas-liquid annular flow. Because of the limited availability of experimental data, the calculations presented will be applicable to horizontal pipes only.

For stable, turbulent, fully developed, annular flow, the gas flow rate is very high. There are waves on the gas-liquid interface (44,47). The liquid film is continuously impinged by liquid droplets while new droplets are being generated from the liquid film (38). Therefore, both phases of two-phase annular flow are most likely turbulent. Unlike the laminar flow case discussed in Chapter II, which was devoted to the basic understanding of the mass transfer mechanism across a moving interface, the investigation of the turbulent flow case in this chapter will be of more practical significance.

Turbulent mass transfer in a circular tube with single-phase flow has been well investigated (6,8,28,29,36). In the case of two-phase annular flow, Hughmark (23) estimated gas and liquid phase mass-transfer coefficients in horizontal annular flow using momentum-mass transfer analogy. No other attempts to calculate interphase mass-transfer coefficients were found in the literature. The difficulty lies in setting up an applicable model.

1. Flow Characteristics

Certain characteristics of flow must be known before the mass transfer rates can be predicted. Because of its practical importance, two-phase flow has been attracting a great deal of research effort in recent years. Although more efforts are necessary for better understanding of the flow properties, some useful analyses have been made.

Pressure Drop

In the analysis of two-phase annular flow, the Martinelli (32) correlation method still provides good results in comparison with other methods of analysis. After some comparison, Wallis's suggestion (46) is chosen for the calculation of pressure drops. That is

$$\left(\frac{\Delta P}{\Delta L}\right)_{TPF} = \left(\frac{\Delta P}{\Delta L}\right)_g (1 + X^{0.572})^{3.5} \qquad (3.1.1)$$

where the parameter X for turbulent-turbulent two-phase flow is defined as

$$X = \left[\frac{\left(\frac{\Delta P}{\Delta L}\right)_\ell}{\left(\frac{\Delta P}{\Delta L}\right)_g}\right]^{\frac{1}{2}} = \left(\frac{\mu_B}{\mu_A}\right)^{0.1} \left(\frac{\rho_B}{\rho_A}\right)^{0.4} \left(\frac{\langle V_B \rangle}{\langle V_A \rangle}\right)^{0.9} \qquad (3.1.2)$$

$\left(\frac{\Delta P}{\Delta L}\right)_g$ is the pressure gradient that would exist if the gas flows alone in the pipe and $\left(\frac{\Delta P}{\Delta L}\right)_\ell$ is the pressure gradient that would exist if the liquid flows alone in the pipe.

Film Thickness

Some existing methods of measuring the film thickness have been

given by Collier and Hewitt (13). Possibly because of the asymmetric film thickness (38) in horizontal annular flow, most the available experimental data on film thickness measurements are for vertical annular flow. A few studies on the prediction of the film thickness are available in literature (30,47). Again, the empirical correlation proposed by Wallis (46) for the prediction of the void fraction of horizontal annular flow will be used to calculate the average film thickness. The equation is

$$k = (1 + X^{0.8})^{-0.187} \qquad (3.1.3)$$

Since the velocity of the entrained liquid is much greater than the velocity of the film, calculating the average film thickness from the void fraction will not lead to significant errors when the percentage of the entrained liquid is low. However, Eq. (3.1.3) will be applied through this work because the effect of the entrainment on mass transfer is included in the effective diffusivity.

Liquid Entrainment

Liquid entrainment is another important characteristic of two-phase annular flow. In comparison with others (22,30,33), the correlation on liquid entrainment proposed by Wicks and Dukler (49) is considered to be the most successful. This can be expressed as

$$R = 200\, X^{2.1} \qquad (3.1.4)$$

for $X < 0.008$. The parameter R is defined as

$$R = \frac{L\ We_c E}{G \left(\frac{\Delta P}{\Delta L}\right)_g} \qquad (3.1.5)$$

where We_c is the critical Weber number and E is the liquid entrainment in lb/hr.

Eddy Diffusivity and Velocity Profiles

Velocity and eddy diffusivity distributions are essential in calculating mass-transfer coefficients. However, very little work has been done in this area. Although the flow mechanism of horizontal annular flow is quite complicated, approximate relationships for eddy diffusivity and velocity profiles can be obtained under the following simplifying assumptions:

i. Neglecting gravity effects, the flow is symmetrical and the shear stress distribution is

$$\tau = \frac{\tau_o}{R} r \qquad (3.1.6)$$

where τ_o is the shear stress on the pipe wall.

ii. Since the behavior of the entrainment and the interfacial wave motion in annular flow are not clearly understood and the velocity distributions do not have a significant effect on the mass transfer, it is assumed that the gas-liquid interface is smooth and that the velocity profiles are not influenced by the entrainment.

iii. Assume the flow pattern is fully developed and steady. Note that it is doubtful that the flow can even reach a fully-developed state

because of the unavoidable axial pressure drop (4,18).

Film thickness is small in comparison with the diameter in two-phase annular flow. Generally, investigators assume the gas phase is flowing in a rough pipe with the liquid film as roughness to get the gas core velocity distribution (14,22,35). Under this assumption, the gas core eddy diffusivity will be the same as that of a single-phase flow because the eddy diffusivity is independent of the roughness (28). Dhanak (14) has indicated that the eddy diffusivity remains fairly constant at its maximum value in the main portion of the turbulent core. Therefore, the modified universal profile of eddy diffusivity proposed by Kropholler and Carr (29) is chosen to be the eddy diffusivity profile for the gas phase. It gives a constant diffusivity in the main portion of the turbulent core and has been successfully applied to single-phase mass transfer. The eddy diffusivity, ϵ, is defined as

$$(\epsilon + \nu) \frac{dV}{dy} = \frac{\tau}{\rho} \tag{3.1.7}$$

The equations are

$$\text{for } 0 \le y^+ \le 5, \qquad \frac{\epsilon}{\nu} = (y^+/14.5)^3 \tag{3.1.8}$$

$$\text{for } 5 < y^+ < 33, \qquad \frac{\epsilon}{\nu} = \frac{y^+}{5} - 0.959 \tag{3.1.9}$$

$$\text{for } 33 < y^+ < R^+/5, \qquad \frac{\epsilon}{\nu} = \frac{y^+}{2.5}\left(1 - \frac{y}{R}\right) \tag{3.1.10}$$

$$\text{for } y^+ > R^+/5, \qquad \frac{\epsilon}{\nu} = \frac{R^+}{14.4} \tag{3.1.11}$$

where $y^+ = \frac{y}{\nu}\sqrt{\frac{\tau_o}{\rho}}$ is the dimensionless distance from the interface.

The velocity profile of the gas core can be obtained by integrating Eq. (3.1.7). The boundary condition is $V = V_S$ at $y = 0$. V_S is the interfacial velocity which will be given by the liquid film velocity distribution. An example of the velocity profile is shown in Fig. 10.

Most of the studies on annular flow assume an eddy diffusivity profile for the liquid film as if it were only a portion of a single-phase flow (3,15). This always results in eddy diffusivities near the interface being much higher than experimental values (9,41). Because of the discontinuity of properties at the interface, the flow mechanism of the liquid film should be similar to that of Couette flow. That is, the gas core of annular flow is acting like a moving solid with respect to the liquid film. This argument has been supported by Hughmark (23). He applied the momentum-mass transfer analogy to calculate mass-transfer coefficients in horizontal annular flow by the method of Kropholler and Carr. This implies that there is a laminar sublayer close to the interface. Kropholler and Carr's Eqs. (3.1.8-11) are applied to both sides of the liquid film. The dimensionless distance, y^+, is measured either from the pipe wall or from the interface. These two eddy diffusivity profiles starting from different sides of the liquid film will meet at the center as shown in Figure 9. These equations are then substituted into Eq. (3.1.7) and integrated to yield the velocity profile.

2. Interphase Mass Transfer

Entrainment is unavoidable in horizontal annular flow. New droplets are continuously being generated at the gas-liquid interface while droplets which have been in the gas core return to the film. Since the

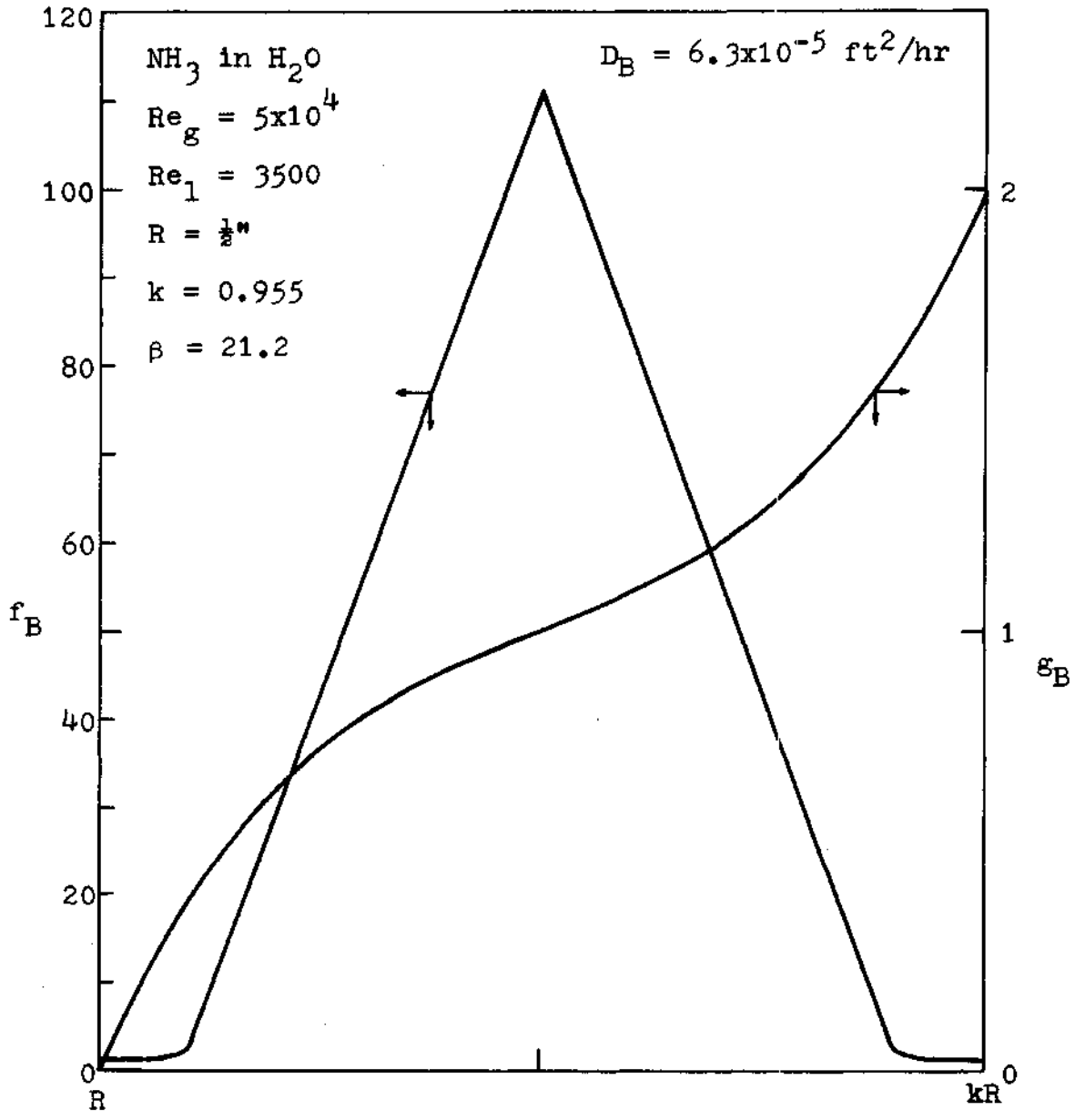

Figure 9. Liquid Film Velocity and Diffusivity Profiles.

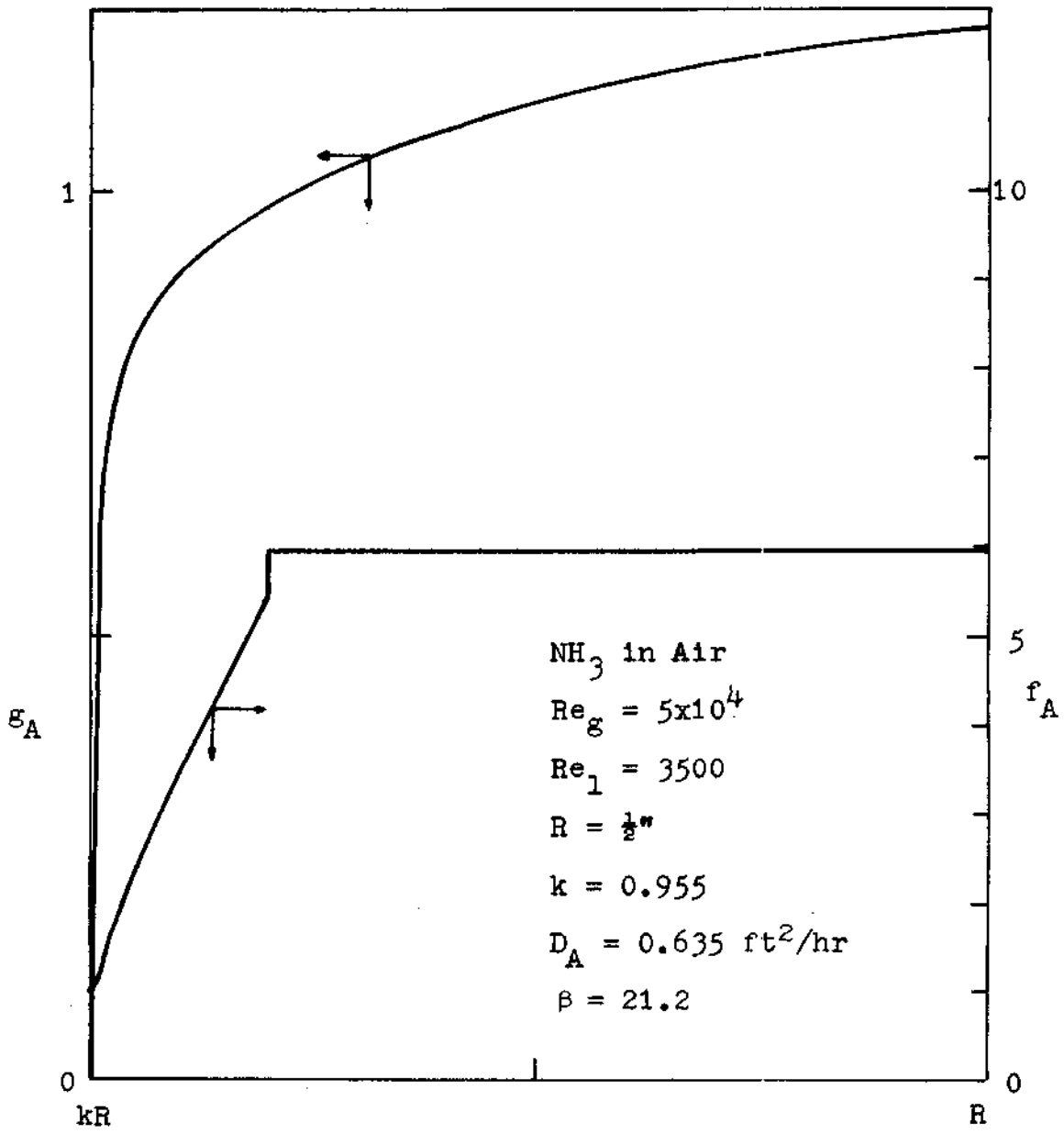

Figure 10. Gas Core Velocity and Diffusivity Profiles

droplets may travel across the gas core and since the impinging droplets may penetrate into the liquid film a significant distance, the net effect of the entrainment on mass transfer can be considered as a rise in the molecular diffusivity. Therefore, the overall diffusivity of mass transfer in annular flow including the effects of entrainment can be expressed as

$$D_{ti} = \beta D_i + \epsilon_{Di}, \qquad i = A, B. \tag{3.2.1}$$

In which β is a correlating constant, D_i is molecular diffusivity and ϵ_D is the eddy diffusivity for mass transfer which is assumed to be equal to the eddy diffusivity for momentum by Kropholler and Carr (29).

Other factors that influence the mass-transfer process, such as the effective interfacial surface area and the wave motion of the liquid film, are also included in the correlation of β.

It seems reasonable to expect β to be a function of the flow rates, the entrainment, and H.

$$\beta = \beta\ (G, L, E, H) \tag{3.2.2}$$

According to Eq. (3.1.4), the entrainment, E, is roughly proportional to $\frac{G}{L}\left(\frac{\Delta P}{\Delta L}\right)_\ell$. Therefore, the following form is suggested for β.

$$\beta = a \left(\frac{\Delta P}{\Delta L}\right)_\ell^b H^c \tag{3.2.3}$$

where a, b, and c are constants.

Under the assumptions made in the previous section, the differential equations for mass transfer can be written as

$$V_i \frac{\partial c_i}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left(r D_{ti} \frac{\partial c_i}{\partial r}\right), \quad i = A, B. \tag{3.2.4}$$

The boundary conditions are

$$z = 0 \qquad c_A = c_{AO} \tag{3.2.5}$$

$$z = 0, \qquad c_B = c_{BO} \tag{3.2.6}$$

$$r = 0, \qquad \frac{\partial c_A}{\partial r} = 0 \tag{3.2.7}$$

$$r = kR, \qquad c_A = \alpha c_B \tag{3.2.8}$$

$$r = kR, \qquad D_{tA} \frac{\partial c_A}{\partial r} = D_{tB} \frac{\partial c_B}{\partial r} \tag{3.2.9}$$

$$r = R, \qquad \frac{\partial c_B}{\partial r} = 0 \tag{3.2.10}$$

Defining dimensionless variables

$$Y_A = \frac{c_A - c_{AO}}{\alpha c_{BO} - c_{AO}} \tag{3.2.11}$$

$$Y_B = \frac{\alpha c_B - c_{AO}}{\alpha c_{BO} - c_{AO}} \tag{3.2.12}$$

$$x = \frac{r}{kR} \quad \text{for phase A} \tag{3.2.13}$$

$$x = \frac{1 - \frac{r}{R}}{1-k} \quad \text{for phase B} \tag{3.2.14}$$

$$\eta = \frac{\beta D_A z}{k^2 R^2 < V_A >} \tag{3.2.15}$$

and using the method of separation of variables, the solution of Eq. (3.2.4) should be of the following form

$$Y_i = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n X_{in} \exp(-\lambda_n^2 \eta) \tag{3.2.16}$$

Here, $X_{in}(x)$ should satisfy the following equations

$$\frac{d}{dx}\left[x f_A \frac{dX_{An}}{dx}\right] + \lambda_n^2 x g_A X_{An} = 0 \tag{3.2.17}$$

$$\frac{d}{dx}\left\{\left[1 - (1-k)x\right] f_B \frac{dX_{Bn}}{dx}\right\} + \lambda_n^2 \omega \left[1 - (1-k)x\right] g_B X_{Bn} = 0 \tag{3.2.18}$$

with the boundary conditions

$$x = 0, \qquad \frac{dX_{An}}{dx} = 0 \tag{3.2.19}$$

$$x = 0, \qquad \frac{dX_{Bn}}{dx} = 0 \tag{3.2.20}$$

$$x = 1, \qquad X_{An} - X_{Bn} = 0 \tag{3.2.21}$$

$$x = 1, \qquad \Phi \frac{dX_{An}}{dx} + \frac{dX_{Bn}}{dx} = 0 \tag{3.2.22}$$

In the differential equations,

$$f_i = 1 + \frac{\varepsilon_{Di}}{\beta D_i} \tag{3.2.23}$$

C_n are similar to those given by Eq. (2.4.41)

$$C_n = \frac{\int_0^1 \frac{H}{1+k} \left[1 - (1-k)x\right] g_B X_{Bn} \, dx}{\int_0^1 \left\{x \, g_A X_{An}^2 + \frac{H}{1+k}\left[1 - (1-k)x\right] g_B X_{Bn}^2\right\} dx} \tag{3.2.24}$$

Numerical solutions of λ_n and X_{in} were obtained with the aid of a digital computer. Computer programs are presented in the Appendix of this work.

The mass-transfer coefficient, K, is defined by Eq. (2.4.29). The resulting Nusselt number is

$$Nu = \frac{2kRK}{D_A} = \frac{\beta \dfrac{d < Y_A >}{d\eta}}{1 - \dfrac{1+H}{H} < Y_A >} \qquad (3.2.25)$$

According to the definition in Eq. (2.2.20), the average concentration is

$$< Y_A > = \frac{H}{1+H} + 2 \sum_{n=1}^{\infty} C_n \exp(-\lambda_n^2 \eta) \int_0^1 x\, g_A\, X_{An}\, dx \qquad (3.2.26)$$

The series in this equation converges very rapidly. For mass transfer in annular or annular mist flow, the summation of the terms after n = 1 is less than two per cent of the total when the length of the pipe is 20 feet or more. Therefore the mass-transfer coefficient can be approximated as

$$K = \frac{D_A\, Nu}{2\, kR} = \frac{\beta\, D_A\, H\, \lambda_1^2}{2\, kR\, (1+H)} \qquad (z \geq 20 \text{ ft}) \qquad (3.2.27)$$

Comparing the values of mass-transfer coefficients calculated from the above equation with the experimental data of gas-phase

controlled mass transfer obtained by Anderson (2) and data of liquid-phase controlled mass transfer obtained by Bollinger (9), the best fitting function of β was found by trial and error to be

$$\beta = 7.4 \times 10^6 \left(\frac{\Delta P}{\Delta L}\right)_\ell^{0.83} H^{0.21} \qquad (3.2.28)$$

Figure 11 and Figure 12 compare the values of mass-transfer coefficients obtained from these calculations with the experimental data. These calculated values show an average absolute deviation of 15.3 per cent from Anderson's data and 19.2 per cent from Bollinger's data.

3. Results and Discussions

These calculations were applied to the whole region of horizontal annular and annular mist flow for all gas-liquid systems. The Nusselt number is a function of ten variables

$$Nu = \frac{\beta H \lambda_1^2}{1+H} = Nu\ (Re_g,\ Re_\ell\ \ \nu_A,\ \nu_B,\ \rho_A,\ \rho_B,\ \alpha,\ D_A,\ D_B,\ R) \qquad (3.3.1)$$

where Re_g and Re_ℓ are superficial Reynolds numbers. Since the interfacial velocities are small in comparison with the gas core velocities and since the velocity profiles have little effect on the mass-transfer coefficients, the interfacial velocity may be arbitrarily assumed equal to zero in the calculation of the gas core velocity profiles. By doing this, the Nusselt number can be rearranged to become a function of eight variables

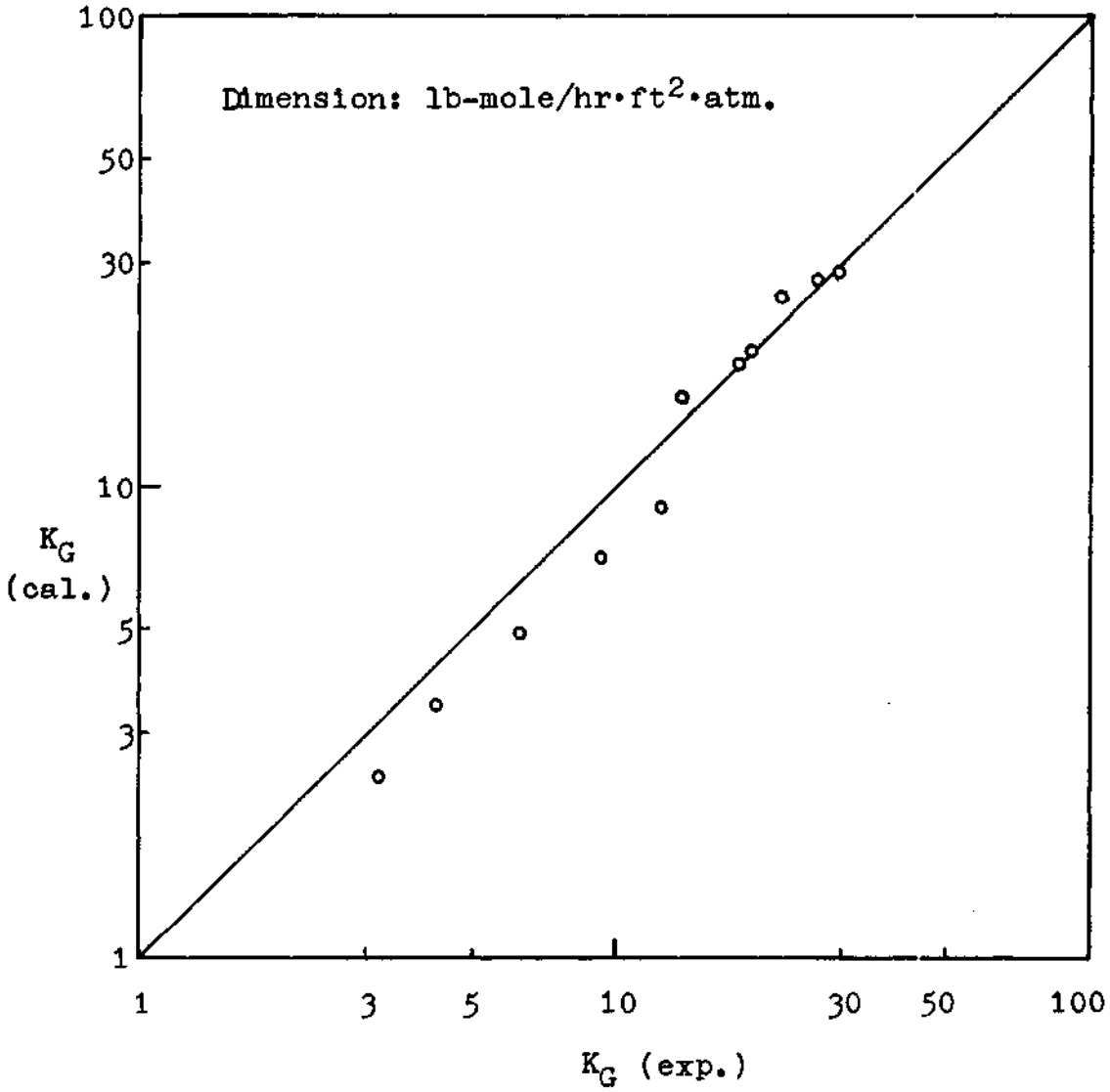

Figure 11. Comparison of Anderson's Data(2) of K_G with Calculated Values.

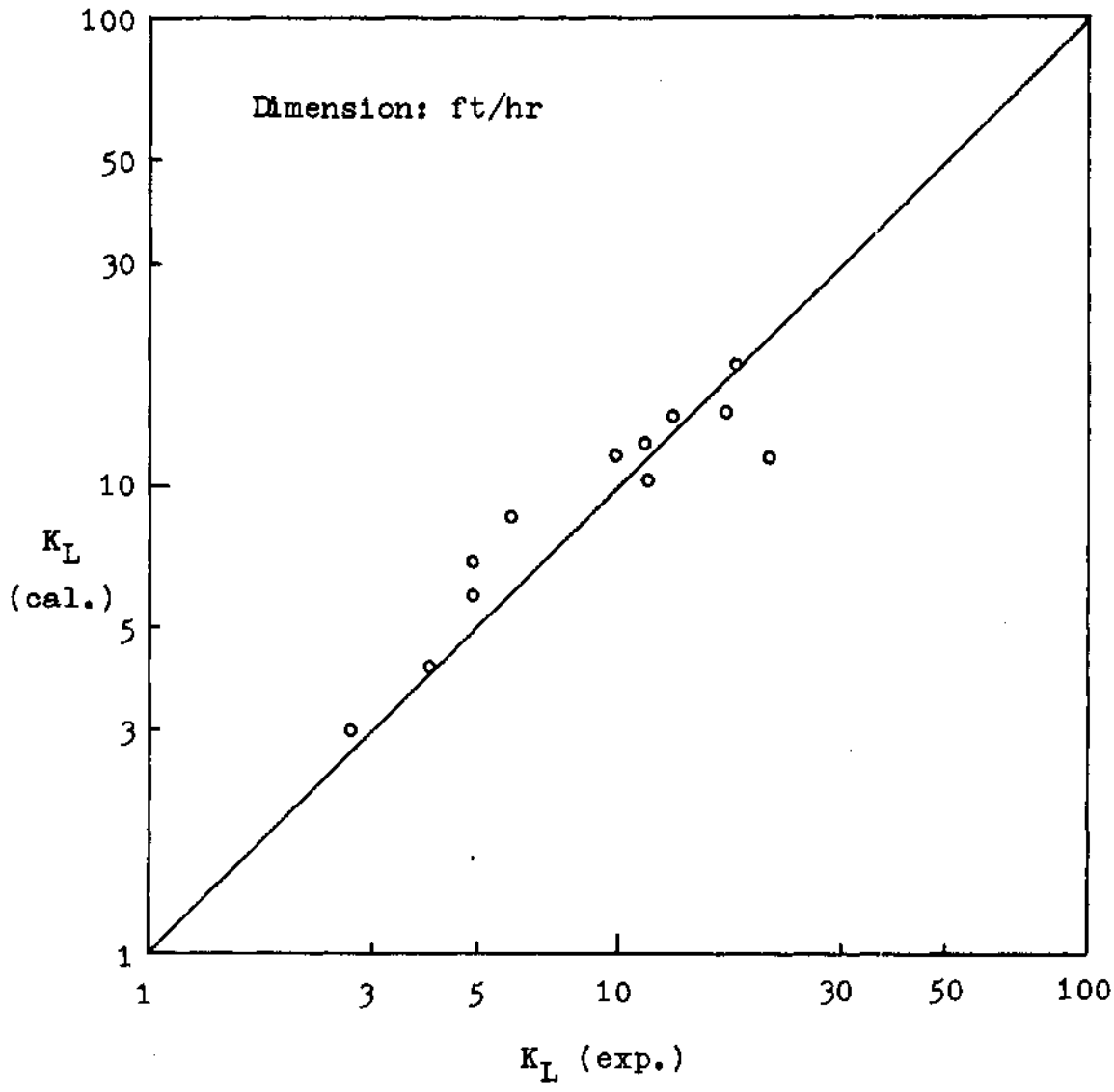

Figure 12. Comparison of Bollinger's Data(9) of K_L with Calculated Values.

$$Nu = Nu\ (\omega,\ H,\ (\frac{\Delta P}{\Delta L})_g,\ (\frac{\Delta P}{\Delta L})_\ell,\ Sc_A,\ Sc_B,\ R_A^+,\ R_B^+) \qquad (3.3.2)$$

where $R_i^+ = \frac{R}{\nu_i}\sqrt{\frac{\tau_o}{\rho_i}}$ and Sc is the Schmidt number.

The following ranges of values of these variables will cover most gas-liquid systems in two-phase annular flow mass transfer

ω:	$0.05 \approx 10$
H:	$0.01 \approx 5$
$(\frac{\Delta P}{\Delta L})_g$:	$8 \times 10^8 \approx 10^{10}$
$(\frac{\Delta P}{\Delta L})_\ell$:	$4 \times 10^7 \approx 4 \times 10^8$
Sc_A:	$0.5 \approx 2.5$
Sc_B:	$300 \approx 1200$
R_A^+:	$3000 \approx 8000$
R_B^+:	$10^3 \approx 10^4$

Nusselt numbers calculated for the range described above are presented in Figure 13 through Figure 20.

The equation of resistances-additivity-principle is

$$\frac{1}{Nu} = \frac{1}{Nu_A} + \frac{\omega}{H}\ \frac{1}{Nu_B} \qquad (3.3.3)$$

This equation shows that the process of mass transfer is gas-phase controlled when $\omega/H \ll 1$. When $\omega/H \gg 1$, the process is liquid-phase controlled. The results presented in Figure 13 show that Nu is independent of ω when ω/H is small. When ω/H is large, Nu is inversely

proportional to ω. This is a natural conclusion because neither Nu_A nor Nu_B is a function of ω, which is defined by Eq. (2.3.8). However, as was discussed in Chapter II, Eq. (3.3.3) is valid only under certain conditions.

Figure 14 indicates that Nu increases when H increases. It approaches a limiting value of about 1200 when H is very large.

The effect of the other variables, $\left(\frac{\Delta P}{\Delta L}\right)_g$, $\left(\frac{\Delta P}{\Delta L}\right)_\ell$, Sc_A, Sc_B, R_A^+, and R_B^+, on the Nusselt number are comparatively small. The almost-linear curves on the log-log plot in Figure 15 through Figure 20 indicate that Nu is a power function of these variables.

$$Nu = a \left(\frac{\Delta P}{\Delta L}\right)_g^b \left(\frac{\Delta P}{\Delta L}\right)_\ell^c Sc_A^{\,d} \; Sc_B^{\,e} \; R_A^{+f} \; R_B^{+g} \qquad (3.3.4)$$

The exponents in this equation, b, c, d, e, f, and g, are slopes of the curves in these figures. It is found by inspection that these exponents are almost independent of any variable except ω. Furthermore, it is also found that when the liquid flow rate is relatively small, ω is small, and Nu is independent of $\left(\frac{\Delta P}{\Delta L}\right)_g$, Sc_B and R_B^+. When the gas flow rate is relatively small, ω is large, and the Nusselt number becomes independent of Sc_A and R_A^+. For example, when $\omega = 5$,

$$Nu = 5.4 \times 10^{-7} H^{1.1} \left(\frac{\Delta P}{\Delta L}\right)_g^{-0.31} \left(\frac{\Delta P}{\Delta L}\right)_\ell^{0.94} Sc_B^{\,0.25} \; R_B^{+\,0.91} \qquad (3.3.5)$$

When $\omega = 0.05$ and $H = 5$,

$$Nu = 0.0136 \left(\frac{\Delta P}{\Delta L}\right)_f^{0.26} Sc_A^{0.66} R_A^{+0.78} \qquad (3.3.6)$$

These results are in good agreement with the data reported by Wales (45) at low gas flow rates. However, Wales' data show that the overall mass-transfer coefficient decreases when the gas flow rate increases. The calculations of this study on the same system (desorption of CO_2 from water into air) indicate that the mass-transfer coefficient increases slightly when the gas flow rate is increased. Bollinger (9) has reported that the mass-transfer coefficient for this system is independent of the gas flow rate. Jepsen (25) also concludes that mass-transfer coefficients in two-phase flow will increase when the flow rate of either phase is increased. Therefore, it must be concluded that the effect of the gas phase flow rate on the mass-transfer coefficients is uncertain at the present time.

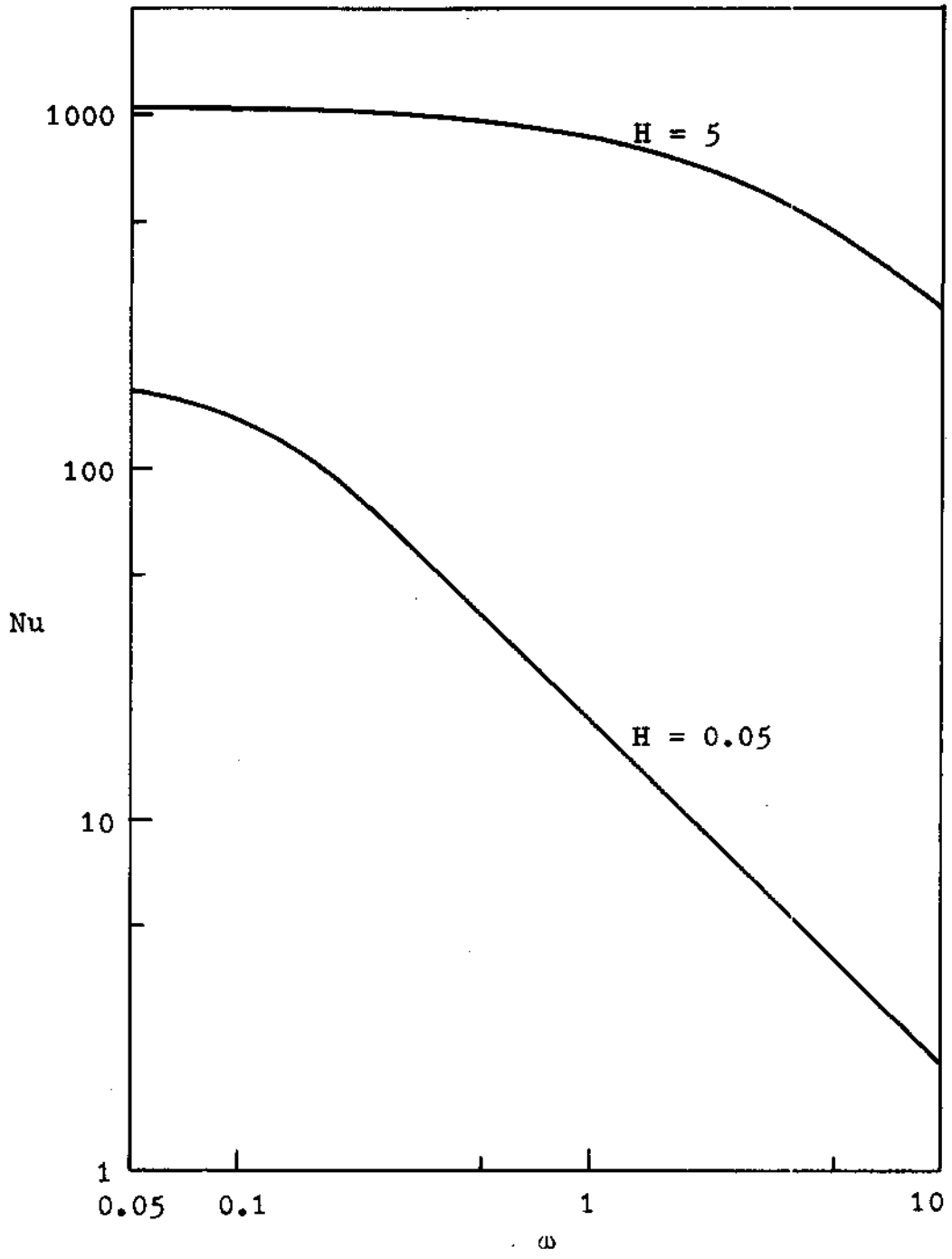

Figure 13. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of ω and H at $\left(\frac{\Delta P}{\Delta L}\right)_g = 2\times10^9$, $\left(\frac{\Delta P}{\Delta L}\right)_l = 10^8$, $Sc_A = 1$, $Sc_B = 700$, $R_A^+ = 4000$ and $R_B^+ = 2000$.

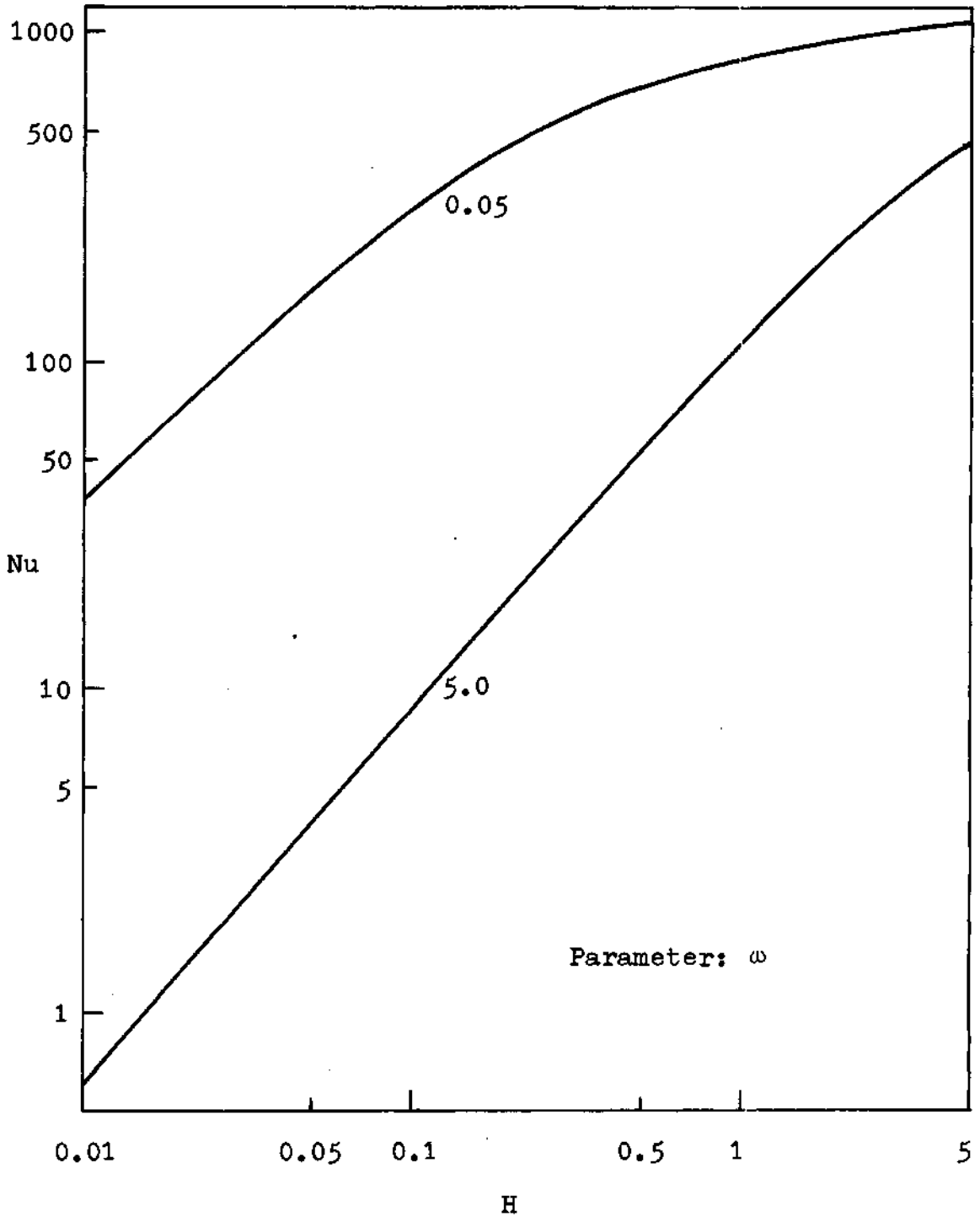

Figure 14. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of H and ω at $Sc_A = 1$, $Sc_B = 700$, $\left(\frac{\Delta P}{\Delta L}\right)_g = 2 \times 10^9$, $\left(\frac{\Delta P}{\Delta L}\right)_l = 10^8$, $R_A^+ = 4000$ and $R_B^+ = 2000$.

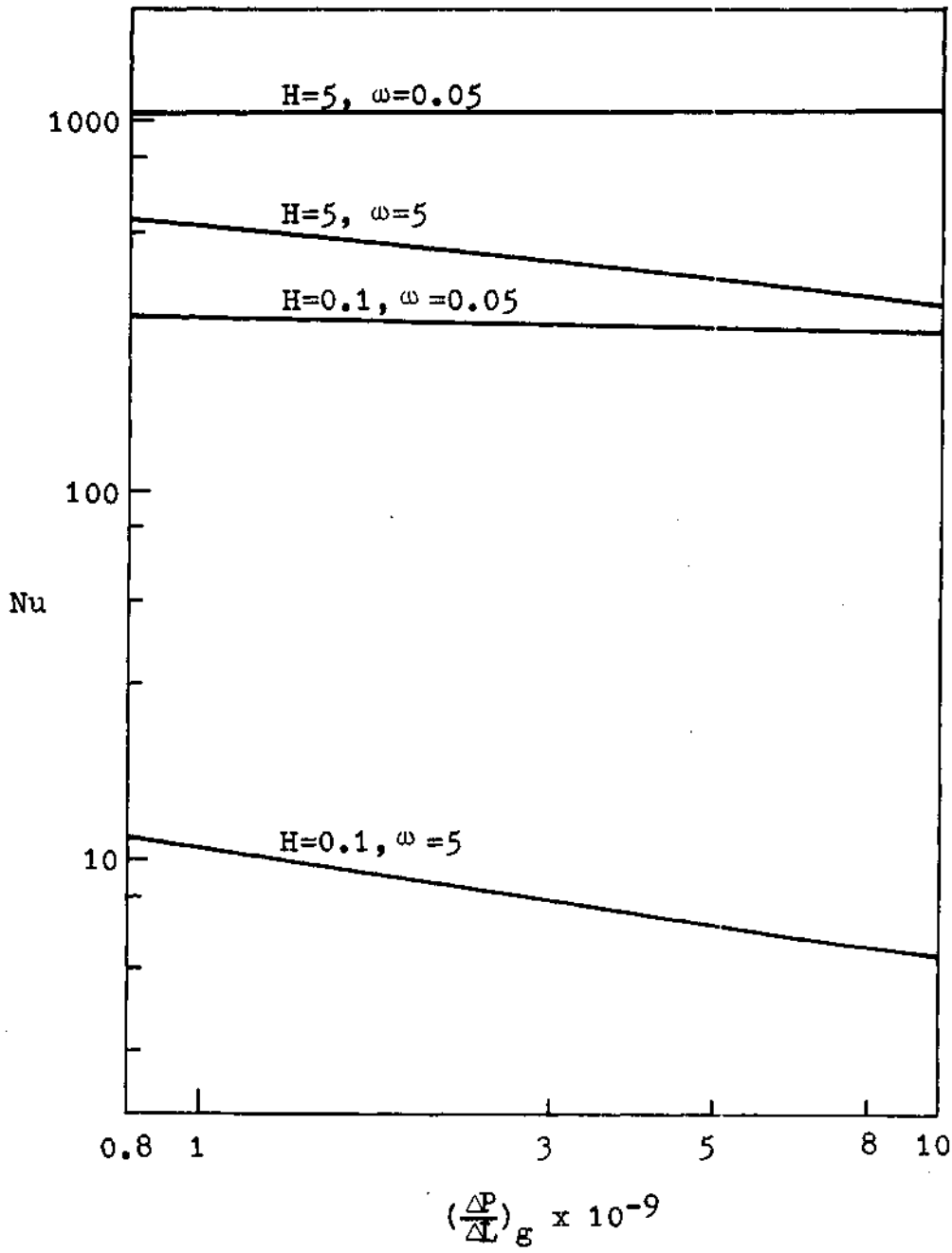

Figure 15. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of H, ω and $(\frac{\Delta P}{\Delta L})_g$ at $(\frac{\Delta P}{\Delta L})_l = 10^8$, $R_A^+ = 4000$, $Sc_A = 1$, $R_B^+ = 2000$ and $Sc_B = 700$.

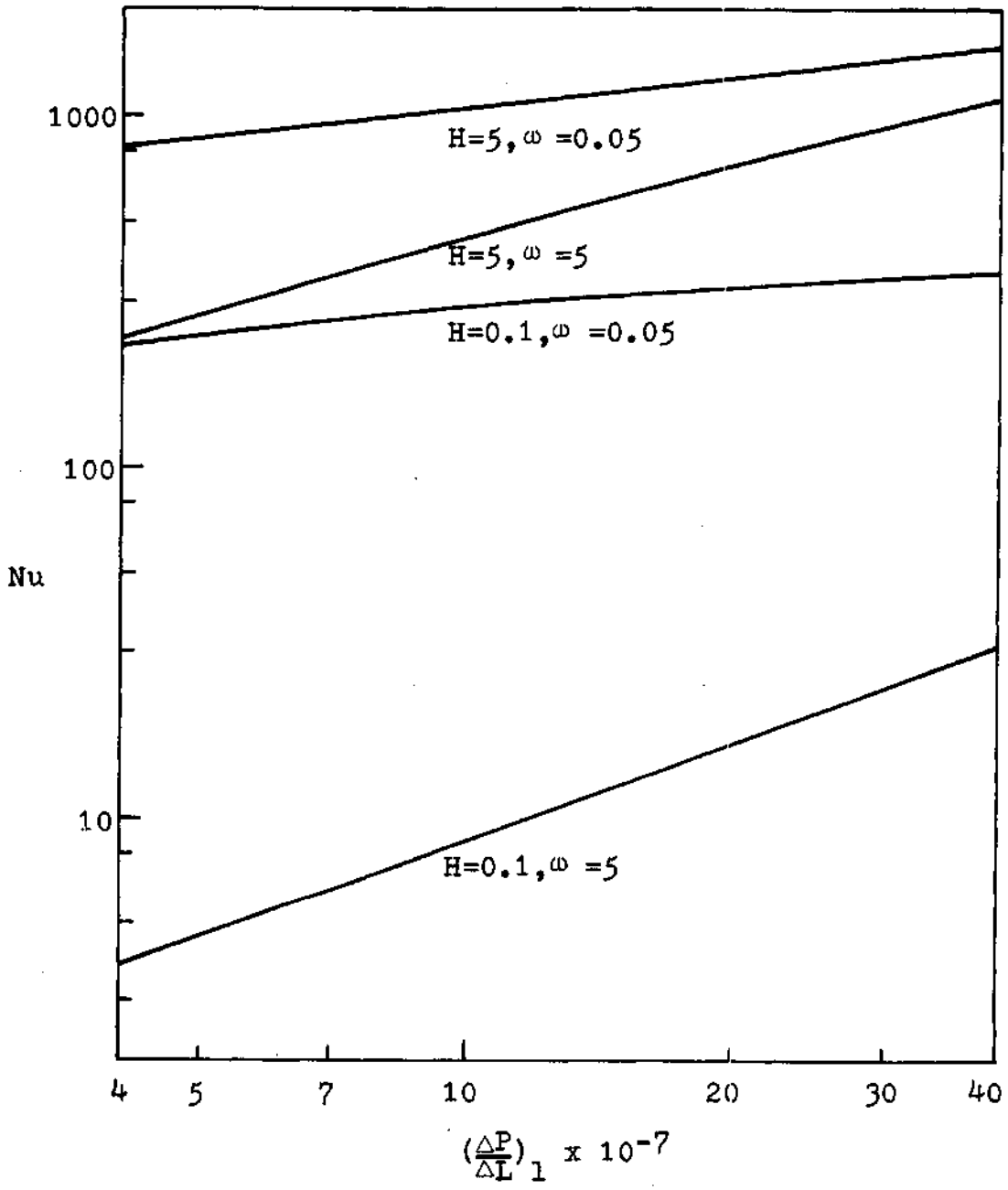

Figure 16. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of $(\frac{\Delta P}{\Delta L})_1$, H and ω at $Sc_A = 1$, $Sc_B = 700$, $(\frac{\Delta P}{\Delta L})_g = 2x10^9$, $R_A^+ = 4000$ and $R_B^+ = 2000$.

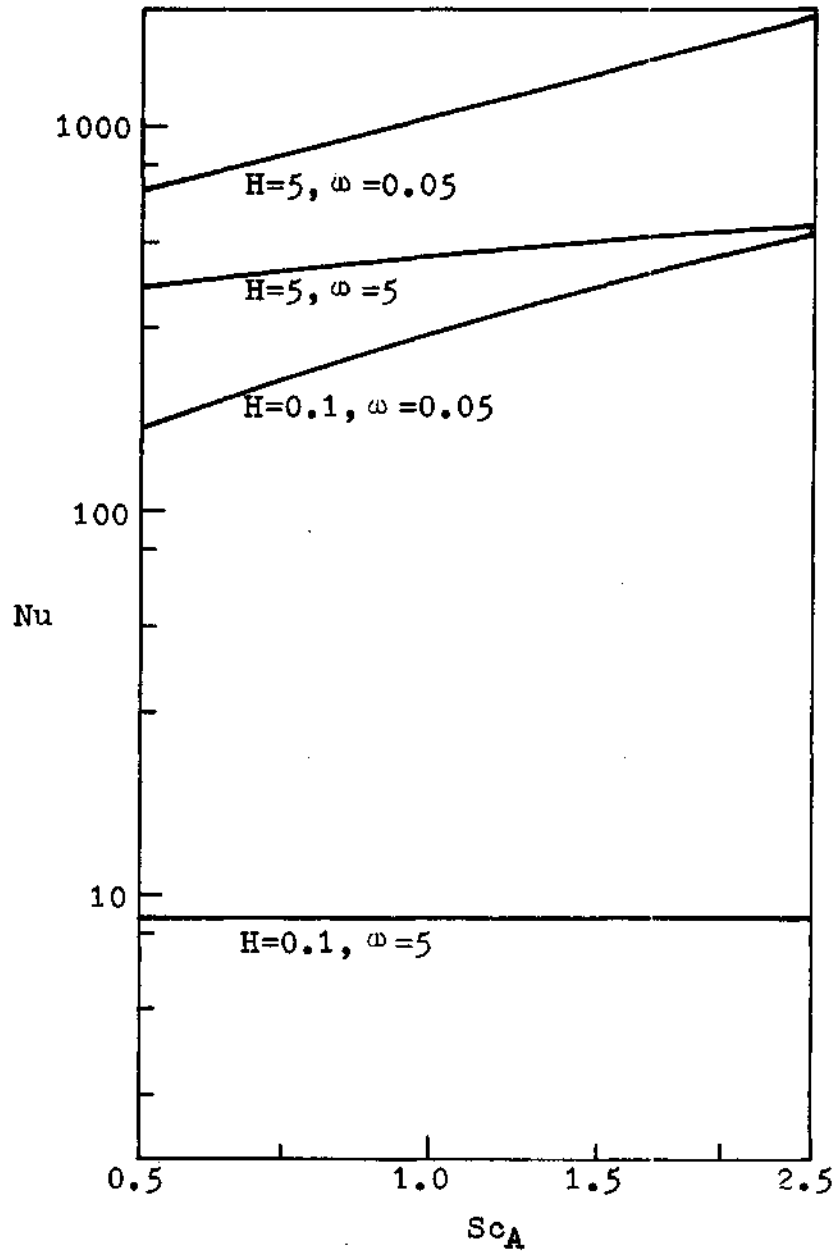

Figure 17. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of Sc_A, H and ω at $(\frac{\Delta P}{\Delta L})_g = 2x10^9$, $(\frac{\Delta P}{\Delta L})_l = 10^8$, $R_A^+ = 4000$, $R_B^+ = 2000$ and $Sc_B = 700$.

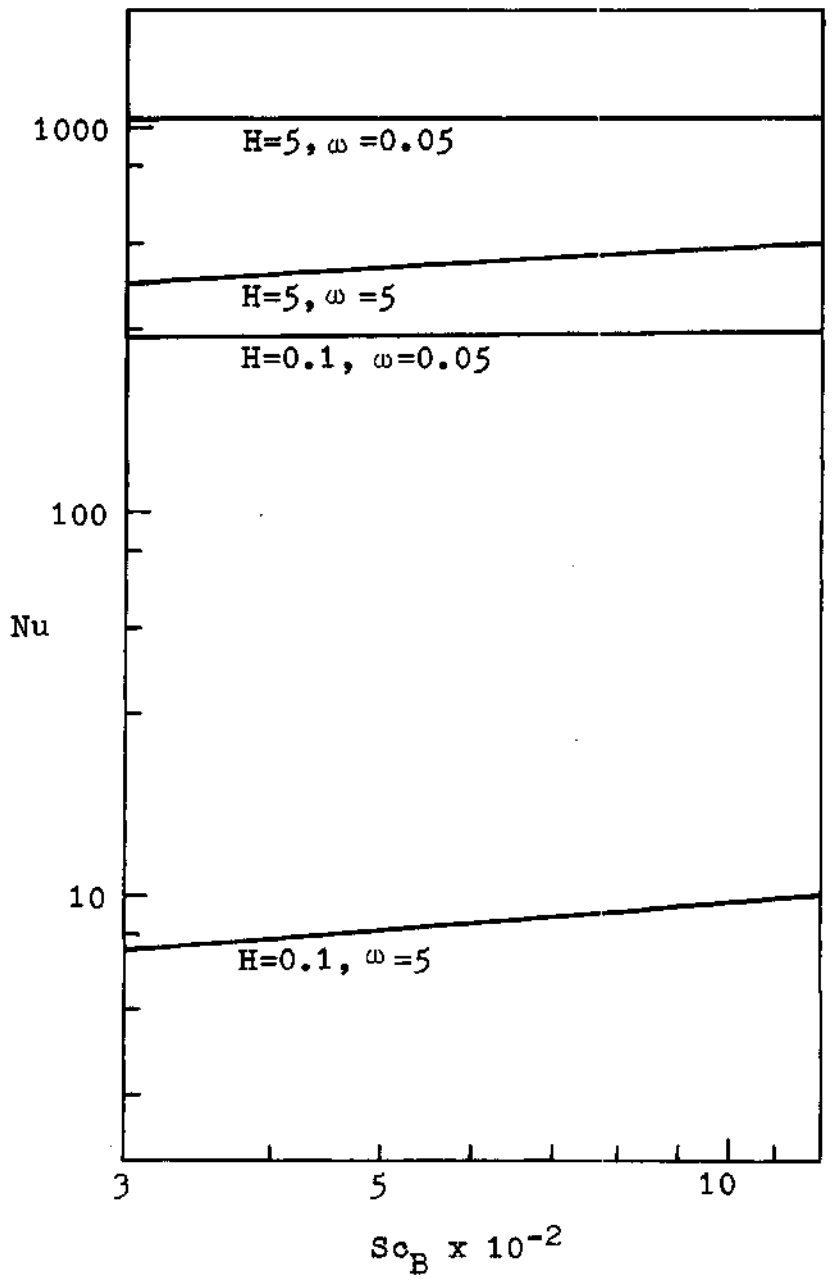

Figure 18. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of Sc_B, H and ω at $(\frac{\Delta P}{\Delta L})_g = 2x10^9$, $(\frac{\Delta P}{\Delta L})_l = 10^8$, $R_A^+ = 4000$, $R_B^+ = 2000$ and $Sc_A = 1$.

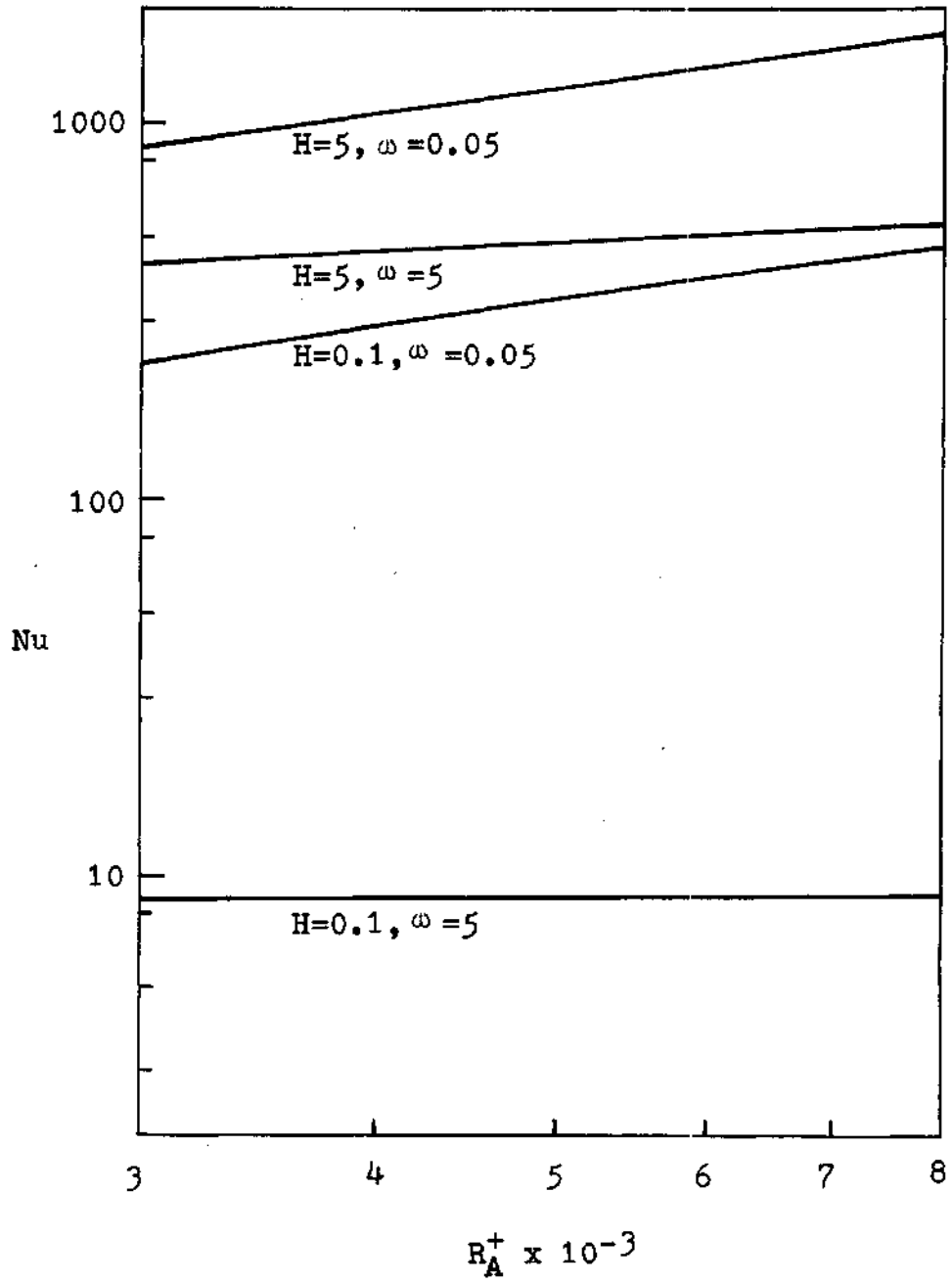

Figure 19. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of R_A^+, H and ω at $\left(\frac{\Delta P}{\Delta L}\right)_g = 2 \times 10^9$, $\left(\frac{\Delta P}{\Delta L}\right)_l = 10^8$, $R_B^+ = 2000$, $Sc_A = 1$ and $Sc_B = 700$.

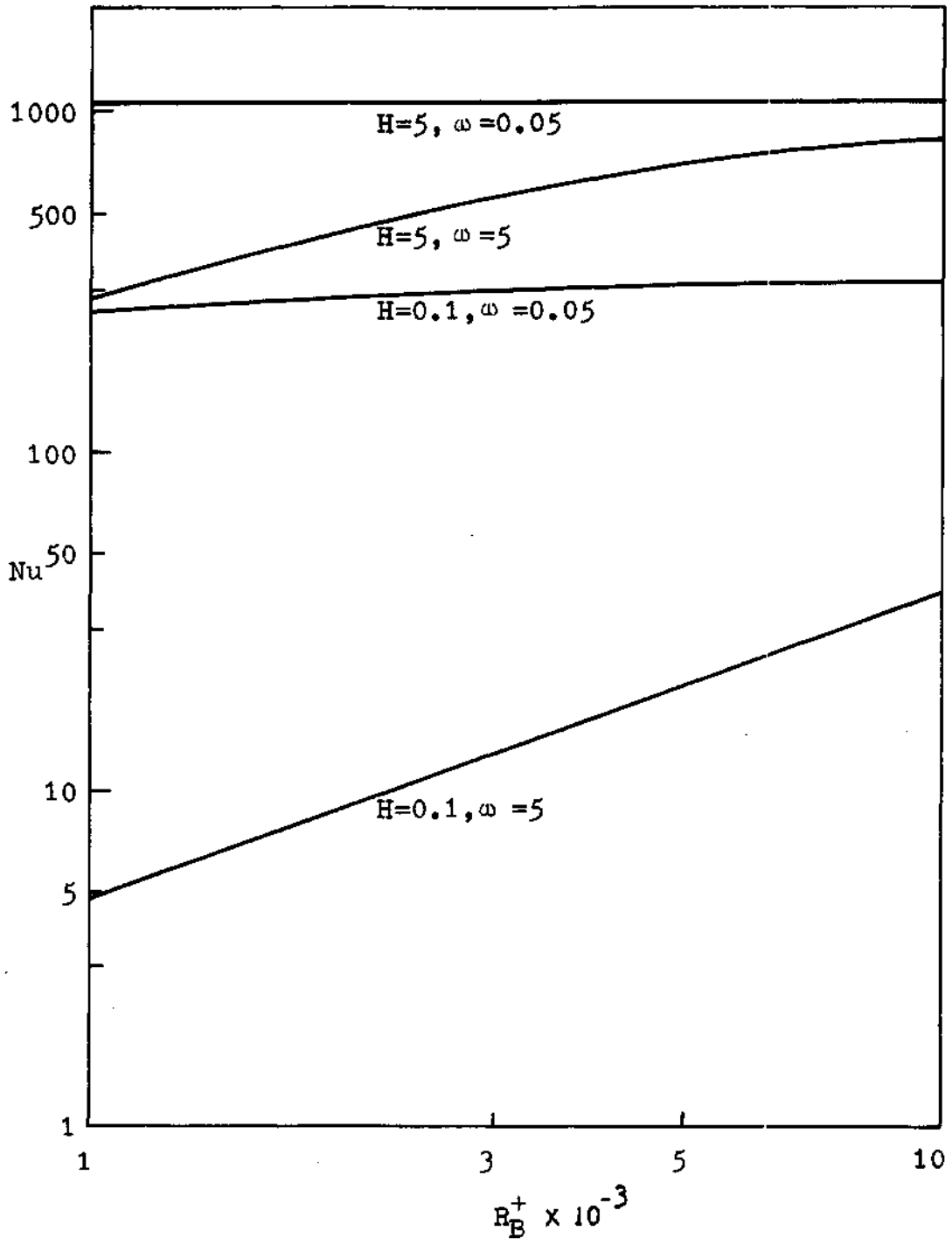

Figure 20. Nusselt Number for Turbulent Two-Phase Annular Flow as a Function of R_B^+, H and ω at $Sc_A = 1$, $Sc_B = 700$, $\left(\frac{\Delta P}{\Delta L}\right)_g = 2 \times 10^9$, $\left(\frac{\Delta P}{\Delta L}\right)_l = 10^8$ and $R_A^+ = 4000$.

CHAPTER IV

CONCLUSIONS

1. Flatter velocity profiles result in greater gas-phase controlled mass-transfer coefficients for laminar annular flow.

2. Less convex velocity profile and thinner film thicknesses give a greater liquid-phase controlled mass-transfer coefficient for laminar annular flow.

3. Nusselt numbers for interphase mass transfer for laminar annular flow are constant for a certain range of values of H. Within this range, the additivity principle of resistances is valid. The Nusselt numbers are smaller for values of H outside this range. They are also inversely proportional to Φ when Φ is large, but are independent of Φ when Φ is small.

4. The effects of entrained liquid droplets and interfacial wave motions on mass transfer in turbulent two-phase annular flow can be correlated as a function of molecular diffusivity.

5. The Nusselt number of gas-liquid two-phase annular flow in a horizontal pipe can be expressed as

$$Nu = a \left(\frac{\Delta P}{\Delta L}\right)_g^b \left(\frac{\Delta P}{\Delta L}\right)_\ell^c Sc_A^d \; Sc_B^e \; R_A^{+f} \; R_B^{+g}$$

Where a is a function of ω and H, and the exponents b, c, d, e, g, and g vary significantly only with ω.

APPENDIX

COMPUTER PROGRAMS

The calculations in this study were made on a Burroughs B-5500 Computer operated by the Rich Electronic Computer Center of the Georgia Institute of Technology. The Programming language was B-5500 ALGOL (43).

The first three programs following here calculate Nusselt numbers for the three cases of mass transfer in laminar flow discussed in Chapter II. The fourth program calculates Nusselt numbers for the case of mass transfer in turbulent flow discussed in Chapter III. Statements of input data follow immediately after declaration statements in each program. Eigenvalues are calculated by trial and error to four significant figures. Outputs are presented following each program respectively.

```
      BURROUGHS B-5500 ALGOL COMPILER LEVEL   6       FRIDAY,   5/ 1/70,

%COMPILE BIN0295/GAS    *ALGOL           ,065800006    *0295 WU D C

%PROCESS=       00000020;IO=    00000010.
%DATA.
%
%
%
%  THIS PROGRAM CALCULATES LOCAL NUSSELT NUMBER FOR GAS PHASE-CONTROLLED
%  MASS TRANSFER FOR LAMINAR ANNULAR FLOW, NU = NU(E1), AS SHOWN IN
%  SECTION 2 OF CHAPTER II.
%
BEGIN %

FILE OUT LONE 6(3,15); %
FORMAT FM1("WHEN E1 =",F5.2," THEN"),FM2(//" I   LAMBDA   COEFFICIENT"),

 FM3(I2,F9.3,E14.3),FM4(//X8,"ETA",X10,"NU"), FM5("   INFINITY",E12.3),
 FM6(E11.3,E12.3); %

REAL E1,ER,DE,EL,RN,FE,R1,E,NU,FL,FR; INTEGER X,Y,I,J; %
REAL ARRAY A[0:100],R,S[0:40],C,T[0:11]; %
DEFINE D7 = RN←DE*2; NU; IF Y = 1 THEN GO TO # , %
       D8 = ER←FE; Y←1; IF EL×ER < 0 THEN GO TO # , %
       D9 = EL←FE; Y←0; IF EL×ER > 0 THEN GO TO # ; %
PROCEDURE NU; BEGIN A[1]←-E1×RN/4; FE←1+A[1]; FOR J←2 STEP 1 UNTIL 80 DO
 BEGIN A[J]←RN×(A[J-2]-E1×A[J-1])/(4×J×J); FE←FE+A[J] END END;
LABEL M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,M12,M13,M14,M15;
 E1←1.05; WRITE(LONE,FM1,E1); X←I←R1←FL←0; WRITE(LONE,FM2); A[0]←1;
M1: R1←R1+1;RN←R1×R1; NU;   IF X = 1 THEN GO TO M2; FR←FE;X←1; %
 IF FL×FR < 0.0 THEN GO TO M3;GO TO M1; M2: FL←FE;X←0;%
 IF FL×FR > 0.0 THEN GO TO M1; M3: DE←R1-1.1;EL←0.0;Y←0;%
 M4: DE←DE+0.1; D7 M5;D8 M6;GO TO M4;M5: D9 M4;M6: DE←DE-0.11; Y←EL←0;
 M7: DE←DE+0.01; D7 M8;D8 M9;GO TO M7;M8: D9 M7;M9: DE←DE-0.011; Y←EL←0;
 M10: DE←DE+0.001; D7 M11; D8 M12; GO TO M10; M11: D9 M10; %
 M12: DE←DE-0.0011; Y←EL←0; M13: DE←DE+0.0001; D7 M14; D8 M15;GO TO M13;
 M14: D9 M13; M15: I←I+1;T[I]←(E1-0.5)×RN; FOR Y←1 STEP 1 UNTIL 39 DO
 BEGIN EL←1; FOR J←1 STEP 1 UNTIL 80 DO EL←EL+A[J]×(Y/40)↑(2×J);
 S[Y]←Y×(E1-(Y/40)↑2)×EL/40; R[Y]←S[Y]×EL END; RN←FE←EL←ER←0;
 FOR Y←1 STEP 2 UNTIL 39 DO BEGIN RN←RN+S[Y]; FE←FE+R[Y] END;
 FOR Y←2 STEP 2 UNTIL 38 DO BEGIN EL←EL+S[Y]; ER←ER+R[Y] END; %
 RN←4×RN+2×EL; C[I]←RN/(4×FE+2×ER); WRITE(LONE,FM3,I,DE,C[I]);
 C[I]←C[I]×RN; IF I < 6 THEN GO TO M1; WRITE(LONE,FM4); %
 FOR E←0.0001,0.0003162,0.001,0.003162,0.01,0.03162,0.1,0.3162 DO
 BEGIN EL←ER←0; FOR I←1 STEP 1 UNTIL 6 DO BEGIN ER←ER+C[I]/EXP(E×T[I]);
 EL←EL+C[I]×T[I]/EXP(E×T[I]) END;NU←EL/ER;WRITE(LONE,FM6,E,NU) END;
 WRITE(LONE,FM5,T[1]) END.
```

WHEN E1 = 1.05 THEN

I	LAMBDA	COEFFICIENT
1	2.624	1.485@+00
2	6.448	-8.250@-01
3	10.285	6.107@-01
4	14.122	-4.992@-01
5	17.958	4.292@-01
6	21.801	-3.815@-01

ETA	NU
1.000@-04	1.281@+01
3.162@-04	1.260@+01
1.000@-03	1.200@+01
3.162@-03	1.046@+01
1.000@-02	7.823@+00
3.162@-02	5.457@+00
1.000@-01	4.141@+00
3.162@-01	3.791@+00
INFINITY	3.786@+00

```
      BURROUGHS B-5500 ALGOL COMPILER LEVEL   6    THURSDAY,   5/ 7/70,

%COMPILE BIN1022/LIQUID *ALGOL          ,065800006   *1022 WU D C

%PROCESS=       00000020;IO=    00000010.
%DATA.
%
%
%
%  THIS PROGRAM CALCULATES LOCAL NUSSELT NUMBERS FOR LIQUID PHASE-
%  CONTROLLED MASS TRANSFER FOR LAMINAR ANNULAR FLOW, NU = NU(K,E3),
%  AS SHOWN IN SECTION 3 OF CHAPTER II.
%
BEGIN %

FILE OUT LINE 6(3,15); %
FORMAT FM1("WHEN K =",F5.2," AND E3 =",F5.2," THEN E2 =",E10.3," AND"),

 FM2(/X4,"PHI =",E11.4,X10,"FROM EQ.(2.3.36)"),FM6(//X7,"ETA",X15,"NU"),
 FM3(/X5, "XI =",E11.4,X10,"FROM EQ.(2.3.34)"), FM7(E10.3,E17.4),
 FM4(//" I     GAMMA    LAMBDA    COEFF * INTEG"), FM5(I2,2(F9.3),E15.3),
 FM8("  INFINITY",E17.4); %

REAL RN,K,FE,EL,ER,FL,FR,R1,DE,E2,E3,INT,E,NU,W,PI; %
INTEGER L,I,J,X,Y,Z,MR; %
REAL ARRAY B,F,V[0:101],T,TP,N,C[0:11],U,XN[0:11,0:101],AA,BB, %
 P[0:11,0:11]; %
DEFINE L11Z = FOR L+1 STEP 1 UNTIL Z DO # , %
       L111 = FOR L+1 STEP 1 UNTIL 100 DO # , %
       J111 = FOR J+1 STEP 1 UNTIL 100 DO # , %
       J11M = FOR J+1 STEP 1 UNTIL MR DO # , %
       J1IM = FOR J+I STEP 1 UNTIL MR DO # , %
       I11M = FOR I+1 STEP 1 UNTIL MR DO # , %
       D7 = RN*DE*2; WU; IF Y = 1 THEN GO TO # , %
       D8 = ER+FE; Y+1; IF EL*ER < 0 THEN GO TO # , %
       D9 = EL+FE; Y+0; IF EL*FR > 0 THEN GO TO # ; %
PROCEDURE      JACOBI(OPT,N,A,B) ; %
VALUE          OPT,N ; %
INTEGER        OPT,N ; %
ARRAY          A,B[0,0] ;%
%         THIS PROCEDURE REPLACES THE N-TH ORDER, REAL, SYMMETRIC
%    MATRIX A[,] WITH A DIAGONAL MATRIX SUCH THAT THE ELEMENTS A[I,I],
%    I = 1,2,3,...,N, CONTAIN THE EIGENVALUES OF A[,] ARRANGED IN DE-
%    CREASING ORDER OF MAGNITUDE. THE EIGENVECTOR ASSOCIATED WITH THE
%    I-TH EIGENVALUE IS STORED IN THE I-TH COLUMN OF THE MATRIX B[,].
%    THE OPERATION PERFORMED DEPENDS UPON THE OPTION PARAMETER OPT AS
%    FOLLOWS:
%
%         IF OPT = 1, BOTH EIGENVALUES AND EIGENVECTORS ARE PRODUCED.
%
%         IF OPT = 2, ONLY EIGENVALUES ARE PRODUCED.
%
BEGIN
INTEGER        I,K,L,M,KL,KM ; %

REAL           Q,R,S,T,V,W,Z,ML,MM ; %
ARRAY          MX[0:N] ; %
INTEGER ARRAY KX[0:N] ; %
DEFINE         IDO1 = FOR I ← 1 STEP 1 UNTIL N DO # , %
               IDO2 = FOR I ← 1 STEP 1 WHILE I ≤ N DO # , %
               IDO3 = FOR I ← 1 STEP 1 WHILE I < L DO # , %
               IDO4 = FOR I ← L+1 STEP 1 WHILE I < M DO # , %
```

```
      IDD5 = FOR I ← M+1 STEP 1 WHILE I ≤ N DO # ; %
      IDD6 = FOR I ← 1 STEP 1 WHILE I < N DO # ; %
      KDD0 = FOR K ← 1 STEP 1 UNTIL N DO # ; %
      KDD1 = FOR K ← I+1 STEP 1 UNTIL N DO # ; %
      KDD2 = FOR K ← I+1 STEP 1 WHILE K ≤ N DO # ; %

      IF OPT ≠ 2 AND OPT ≠ 3 THEN %
      IDD1 %
BEGIN %
      B[I,I] ← 1.0 ; %
      KDD1 B[I,K] ← B[K,I] ← 0 ; %
END ; %
      Q ← 0 ; L ← 1 ; %
      IDD2 %
BEGIN %
      MM ← 0 ; KM ← N ; %
      KDD2 IF (W ← ABS(A[I,K])) > MM THEN %
BEGIN %
      MM ← W ; %
      KM ← K ; %
END ; %
      MX[I] ← MM ; KX[I] ← KM ; %
      IF MM > Q THEN BEGIN Q ← MM ; L ← I END ; %
END ; %
      M ← KX[L] ; %
      R ← ABS(A[L,L]) + ABS(A[M,M]) ; %
      Q ← W + R ; %
      WHILE R ≠ Q DO %
BEGIN %
      Q ← A[L,M] + A[L,M] ; %
      R ← A[L,L] - A[M,M] ; %
      S ← 2×SQRT(R×R + Q×Q) ; %
      T ← SQRT(0.5 + ABS(R)/S) ; %
      S ← Q/(S×T) ; %
      IF R < 0 THEN BEGIN Z ← T ; T ← S ; S ← Z END ; %
      IF OPT ≠ 2 THEN %
      IDD2 %
BEGIN %
      Q ← B[I,L] ; R ← B[I,M] ; %
      B[I,L] ← T×Q + S×R ; %
      B[I,M] ← T×R - S×Q ; %
END ; %
      ML ← MM ← 0 ; KL ← KM ← N ; %
      IDD3 %
BEGIN %
      Q ← A[I,L] ; R ← A[I,M] ; %
      A[I,L] ← V ← T×Q + S×R ; %
      A[I,M] ← Z ← T×R - S×W ; %
      Z ← ABS(Z) ; V ← ABS(V) ; W ← MX[I] ; %
      IF Z > W OR V > W THEN %
      IF Z > V THEN BEGIN MX[I] ← Z ; KX[I] ← M END %
               ELSE BEGIN MX[I] ← V ; KX[I] ← L END %
END ; %
      IDD4 %
BEGIN %
      Q ← A[L,I] ; R ← A[I,M] ; %
      A[L,I] ← V ← T×Q + S×R ; %
      A[I,M] ← Z ← T×R - S×Q ; %
      IF ABS(V) > ML THEN BEGIN ML ← ABS(V) ; KL ← I END ; %
      IF ABS(Z) > MX[I] THEN BEGIN MX[I] ← ABS(Z) ; %
                                   KX[I] ← M END ; %
END ; %
```

```
        ID05 %
    BEGIN %
        Q ← A[L,I] ; R ← A[M,I] ; %
        A[L,I] ← V ← Y×Q + S×R ; %
        A[M,I] ← Z ← Y×R - S×Q ; %
        IF ABS(V) > ML THEN BEGIN ML ← ABS(V) ; KL ← I END ; %
        IF ABS(Z) > MM THEN BEGIN MM ← ABS(Z) ; KM ← I END ; %
    END ; %
        MX[L] ← ML ; KX[L] ← KL ; %
        MX[M] ← MM ; KX[M] ← KM ; %
        Q ← A[L,L] ; R ← A[M,M] ; Z ← A[L,M] ; %
        V ← T×S ; T ← T×T ; S ← S×S ; %
        A[L,M] ← Z×(T-S) + V×(R-Q) ; %
        V ← 2×(Z×V) ; %
        A[L,L] ← T×Q + S×R + V ; %
        A[M,M] ← T×R + S×Q - V ; %
        Q ← ML ; %
        ID06 IF (R ← MX[I]) > Q THEN BEGIN Q ← R ; L ← I END ; %
        M ← KX[L] ; %
        R ← ABS(A[L,L]) + ABS(A[M,M]) ; %
        Q ← Q+R ; %
    END ; %
        KD00 ID01 IF I ≠ K THEN A[I,K] ← 0 ; %
END JACOBI ; %


PROCEDURE WU; BEGIN FE←1; B[2]←-RN/2; FOR L←3 STEP 1 UNTIL 7 DO
 B[L]←((1-K)×(L-1)↑2×B[L-1]-RN×B[L-2]+RN×(1-K)×B[L-3])/(L×(L-1)); L11Z
 FE←FE+B[L] END; %
PROCEDURE DC; BEGIN EL←ER←0; FOR L←1 STEP 2 UNTIL 99 DO EL←EL+F[L]; FOR
 L←2 STEP 2 UNTIL 98 DO ER←ER+F[L]; INT←(F[0]+4×EL+2×ER+F[100])/300 END;
LABEL M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,M12,M13,M14,M15; %
 E3←1.0; K←0.9; MR←6; Z←90; PI←3.14159265359; B[0]←1; B[1]←V[0]←FL0]←0;
 X←I←R1←FL←0;E2←(1-K×K)/2-E3×(1-K×K+2×K×K×LN(K))/(2×(1-K×K)); W←1;
M1: R1←R1+1;RN←R1×R1; WU;   IF X = 1 THEN GO TO M2; FR←FE;X←1; %
 IF FL×FR < 0.0 THEN GO TO M3;GO TO M1; M2: FL←FE;X←0;%
 IF FL×FR > 0.0 THEN GO TO M1; M3: DE←R1-1.1;EL←0.0;Y←0;%
 M4: DE←DE+0.1; D7 M5;D8 M6;GO TO M4;M5: D9 M4;M6: DE←DE-0.11; Y←EL←0;
 M7: DE←DE+0.01; D7 M8;D8 M9;GO TO M7;M8: D9 M7;M9: DE←DE-0.011; Y←EL←0;
 M10: DE←DE+0.001; D7 M11; D8 M12; GO TO M10; M11: D9 M10; %
 M12: DE←DE-0.0011; Y←EL←0; M13: DE←DE+0.0001; D7 M14; D8 M15;GO TO M13;
 M14: D9 M13; M15: I←I+1; T[I]←RN; J111 BEGIN U[I,J]←1; L11Z %
 U[I,J]←U[I,J]+B[L]×(J/100)↑L END; IF I < MR THEN GO TO M1; %
 L111 V[L]←(1-(1-K)×L/100)×(2×(1-K)×L/100-((1-K)×L/100)↑2 %
 +E3×LN(1-(1-K)×L/100)); I11M J11M BEGIN L111 F[L]←V[L]×U[I,L]×U[J,L];
 DC; P[I,J]←P[J,I]←INT/E2 END; WRITE(LINE,FM1,K,E3,1/E2); F[0]←1; I11M
 BEGIN L111 F[L]←(1-(1-K)×L/100)×U[I,L]↑2; DC; N[I]←INT END; I11M J11M
 AA[I,J]←AA[J,I]←P[I,J]/SQRT(T[I]×T[J]×N[I]×N[J]); JACOBI(1,MR,AA,BB);
 FR←SQRT((1-K×K+E3×LN(K))/(E2×PI)); WRITE(LINE,FM2,FR); F[0]←0;
 L111 F[L]←SQRT(2×(1-K)×L/100-((1-K)×L/100)↑2 +E3×LN(1-(1-K)×L/100)) ;
 DC; FL←PI×SQRT(E2)/INT; WRITE(LINE,FM3,FL); WRITE(LINE,FM4); I11M BEGIN
 J11M BB[J,I]←BB[J,I]/SQRT(T[J]×N[J]); TP[I]←1/SQRT(AA[I,I]) END;
 I11M BEGIN L111 BEGIN XN[I,L]←0; J11M XN[I,L]←XN[I,L]+BB[J,I]×U[J,L]
 END; J111 F[J]←V[J]×XN[I,J]; DC; DE←INT↑2; J111 F[J]←F[J]×XN[I,J]; DC;
 C[I]←DE/INT; WRITE(LINE,FM5,I,SQRT(T[I]),TP[I],C[I]) END; %
 WRITE(LINE,FM6); FOR E←0.0001,0.003162,0.001,0.003162,0.01,0.03162,0.1
 DO BEGIN FL←FR←0; I11M BEGIN FL←FL+C[I]/(AA[I,I]×EXP(E/AA[I,I])); %
 FR←FR+C[I]/EXP(E/AA[I,I]) END; NU←FL×(1+K)/(FR×2×K); %
 WRITE(LINE,FM7,E,NU) END;NU←(1+K)/(AA[1,1]×2×K);  WRITE(LINE,FM8,NU)
 END. %
```

WHEN K = 0.90 AND E3 = 1.00 THEN E2 = 2.264@+01 AND

PHI = 7.8101@-01 FROM EQ.(2.3.36)

XI = 3.2870@+00 FROM EQ.(2.3.34)

I	GAMMA	LAMBDA	COEFF x INTEG
1	1.538	1.899	3.263@-02
2	4.702	5.191	4.285@-03
3	7.848	8.480	1.594@-03
4	10.991	11.794	7.420@-04
5	14.134	15.628	2.605@-04
6	17.276	26.780	6.681@-05

ETA	NU
1.000@-04	1.4822@+01
3.162@-04	1.4461@+01
1.000@-03	1.3496@+01
3.162@-03	1.1491@+01
1.000@-02	8.5291@+00
3.162@-02	5.6746@+00
1.000@-01	4.1179@+00
INFINITY	3.8047@+00

```
     BURROUGHS B-5500 ALGOL COMPILER LEVEL   6       FRIDAY,    5/ 1/70,
%COMPILE BIN0825/INTER  *ALGOL            .065800006    *0825 NU O C

%PROCESS=        00000020;IO=    00000010.
%DATA.
%
%
%
%  THIS PROGRAM CALCULATES LOCAL NUSSELT NUMBER FOR INTERPHASE MASS
%  TRANSFER FOR LAMINAR ANNULAR FLOW, NU = NU(K,H,PHI,E1,E3), AS SHOWN
%  IN SECTION 4 OF CHAPTER II.
%
BEGIN %

FILE OUT   LINE 6(3,15);%
FORMAT FM1(//"XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX"//),

 FM2("K =",F6.3,X3,"PHI =",E9.2,X3,"H =",E9.2,X3,"E1 =",F6.3,X3,"E3 =",
 F6.3), FM3(//" I     GAMMA    LAMBDA    COEFFICIENT"), FM6(E10.3,E12.3),
 FM4(I2,2(F9.3),E14.3), FM5(//X7,"ETA",X10,"NU"),FM7(" INFINITY",E13.5);
REAL E,F,G,DP,W,H,K,VS,KO,KH,E1,E2,E3,E4,M,S,R,DA,V,J0,J1,SS,SN,FN,HN,U,

 EF,TAU,DR,BN,EO,ET,ES,PHI; %
INTEGER    I,J,X,Y,Z,L,MR,XX;%
REAL ARRAY B[0:101],N,T,SD,JI,TP,CF,CP[0:11],JO,BX[0:11,0:41],
 AA,BB,P[0:11,0:11],A[0:41,0:41],O,Q,D[0:41]; %
DEFINE I1M = FOR I+1 STEP 1 UNTIL MR DO #, %
       J1M = FOR J+1 STEP 1 UNTIL MR DO #, %
       Y1M = FOR Y+1 STEP 1 UNTIL MR DO #, %
       J14 = FOR J+1 STEP 1 UNTIL 40 DO #, %
       Y14 = FOR Y+1 STEP 1 UNTIL 40 DO #, %
       DD1 = V+BN; WU; IF XX = 1 THEN GO TO #, %
       DD2 = EO+FN; XX+1; IF EO×EF < 0 THEN GO TO #, %
       DD3 = EF+FN; XX+0; IF EO×EF > 0 THEN GO TO #; %
PROCEDURE      JACOBI(OPT,N,A,B) ; %
VALUE          OPT,N ; %
INTEGER        OPT,N ; %
ARRAY          A,B[0,0] ;%
%        THIS PROCEDURE REPLACES THE N-TH ORDER, REAL, SYMMETRIC
%   MATRIX A[,] WITH A DIAGONAL MATRIX SUCH THAT THE ELEMENTS A[I,I],
%   I = 1,2,3,...,N, CONTAIN THE EIGENVALUES OF A[,] ARRANGED IN DE-
%   CREASING ORDER OF MAGNITUDE, THE EIGENVECTOR ASSOCIATED WITH THE
%   I-TH EIGENVALUE IS STORED IN THE I-TH COLUMN OF THE MATRIX B[,].
%   THE OPERATION PERFORMED DEPENDS UPON THE OPTION PARAMETER OPT AS
%   FOLLOWS:
%
%        IF OPT = 1, BOTH EIGENVALUES AND EIGENVECTORS ARE PRODUCED.
%
%        IF OPT = 2, ONLY EIGENVALUES ARE PRODUCED.
%
BEGIN
INTEGER        I,K,L,M,KL,KM ; %

REAL           Q,R,S,T,V,W,Z,ML,MM ; %
ARRAY          MX[0:N] ; %
INTEGER ARRAY  KX[0:N] ; %
DEFINE         ID01 = FOR I + 1 STEP 1 UNTIL N DO # , %
               ID02 = FOR I + 1 STEP 1 WHILE I ≤ N DO # , %
               ID03 = FOR I + 1 STEP 1 WHILE I < L DO # , %
               ID04 = FOR I + L+1 STEP 1 WHILE I < M DO # , %
               ID05 = FOR I + M+1 STEP 1 WHILE I ≤ N DO # , %
```

```
      ID06 = FOR I ← 1 STEP 1 WHILE I < N DO # ; %
      KD00 = FOR K ← 1 STEP 1 UNTIL N DO # ; %
      KD01 = FOR K ← I+1 STEP 1 UNTIL N DO # ; %
      KD02 = FOR K ← I+1 STEP 1 WHILE K ≤ N DO # ) %

      IF OPT ≠ 2 AND OPT ≠ 3 THEN %
      ID01 %
BEGIN %
      B[I,I] ← 1.0 ; %
      KD01 B[I,K] ← B[K,I] ← 0 ; %
END ; %
      Q ← 0 ; L ← 1 ; %
      ID02 %
BEGIN %
      MM ← 0 ; KM ← N ; %
      KD02 IF (W ← ABS(A[I,K])) > MM THEN %
BEGIN %
      MM ← W ; %
      KM ← K ; %
END ; %
      MX[I] ← MM ; KX[I] ← KM ; %
      IF MM > Q THEN BEGIN Q ← MM ; L ← I END ; %
END ; %
      M ← KX[L] ; %
      R ← ABS(A[L,L]) + ABS(A[M,M]) ; %
      Q ← Q + R ; %
      WHILE R ≠ Q DO %
BEGIN %
      Q ← A[L,M] + A[L,M] ; %
      R ← A[L,L] - A[M,M] ; %
      S ← 2×SQRT(R×R + Q×Q) ; %
      T ← SQRT(0.5 + ABS(R)/S) ; %
      S ← Q/(S×T) ; %
      IF R < 0 THEN BEGIN Z ← T ; T ← S ; S ← Z END ; %
      IF OPT ≠ 2 THEN %
      ID02 %
BEGIN %
      Q ← B[I,L] ; R ← B[I,M] ; %
      B[I,L] ← T×Q + S×R ; %
      B[I,M] ← T×R - S×Q ; %
END ; %
      ML ← MM ← 0 ; KL ← KM ← N ; %
      ID03 %
BEGIN %
      Q ← A[I,L] ; R ← A[I,M] ; %
      A[I,L] ← V ← T×Q + S×R ; %
      A[I,M] ← Z ← T×R - S×Q ; %
      Z ← ABS(Z) ; V ← ABS(V) ; W ← MX[I] ; %
      IF Z > W OR V > W THEN %
      IF Z > V THEN BEGIN MX[I] ← Z ; KX[I] ← M END %
               ELSE BEGIN MX[I] ← V ; KX[I] ← L END %
END ; %
      ID04 %
BEGIN %
      Q ← A[L,I] ; R ← A[I,M] ; %
      A[L,I] ← V ← T×Q + S×R ; %
      A[I,M] ← Z ← T×R - S×Q ; %
      IF ABS(V) > ML THEN BEGIN ML ← ABS(V) ; KL ← I END ; %
      IF ABS(Z) > MX[I] THEN BEGIN MX[I] ← ABS(Z) ; %
                                   KX[I] ← M END ; %
END ; %
      ID05 %
```

```
        BEGIN %
              Q ← A[L,I] ; R ← A[M,I] ; %
              A[L,I] ← V ← T×Q + S×R ; %
              A[M,I] ← Z ← T×R - S×Q ; %
              IF ABS(V) > ML THEN BEGIN ML ← ABS(V) ; KL ← I END ; %
              IF ABS(Z) > MM THEN BEGIN MM ← ABS(Z) ; KM ← I END ; %
        END ; %
              MX[L] ← ML ; KX[L] ← KL ; %
              MX[M] ← MM ; KX[M] ← KM ; %
              Q ← A[L,L] ; R ← A[M,M] ; Z ← A[L,M] ; %
              V ← T×S ; T ← T×T ; S ← S×S ; %
              A[L,M] ← Z×(T-S) + V×(R-Q) ; %
              V ← 2×(Z×V) ; %
              A[L,L] ← T×Q + S×R + V ; %
              A[M,M] ← T×R + S×Q - V ; %
              Q ← ML ; %
              IDO6 IF (R ← MX[I]) > Q THEN BEGIN Q ← R ; L ← I END ; %
              M ← KX[L] ; %
              R ← ABS(A[L,L]) + ABS(A[M,M]) ; %
              Q ← Q+R ; %
        END ; %
              KDOO IDO1 IF I ≠ K THEN A[I,K] ← 0 ; %
END JACOBI ; %


REAL PROCEDURE DER(F) ; %
 REAL ARRAY F[0] ;    BEGIN REAL A,B ; INTEGER I ; A←B←0 ; FOR I←1 STEP 2

 UNTIL 39 DO A←A+F[I] ; FOR I←2 STEP 2 UNTIL 38 DO B←B+F[I] ; %
 DER←(F[0]+4×A+2×B+F[40])/120 END ; %

PROCEDURE  WU ;    BEGIN          J14 BEGIN O[J]←COS(V×SIN(J×M/80)) ;
 O[J]←SIN(J×M/80)×SIN(V×SIN(J×M/80)) END ; J0←DER(0) ; J1←DER(0) ; SS←0 ;
 SN←1 ; B[2]←-(W×V×V)/2 ; FOR L←3 STEP 1 UNTIL Z DO %
 B[L]←((1-K)×(L-1)↑2×B[L-1]-W×V×V×B[L-2]+(1-K)×W×V×V×B[L-3])/(L×(L-1)) ;
 FOR L←2 STEP 1 UNTIL Z DO BEGIN SS←SS+L×B[L] ; SN←SN+B[L] END ;
 FN←SS×J0-K0×V×SN×J1 END ; %
LABEL  L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,L12 ; %
 WRITE (LINE,FM1) ; K←0.9 ; PHI←10 ; H←1 ; E1←1.05 ; E3←1.4 ; Z←90 ; MR←6 ;
 KH←H/(1+K) ; E2←(1-K×K)/2-E3×(1-K×K+2×K×K×LN(K))/(2×(1-K×K)) ; K0←PHI ;
 W←K0×K×H/(1+K) ; DP←1-E3/(2×K×K) ; E4←2/(2×E1-1) ; M←3.1415926535897932 ;
 VS←(E1-1)/((1/K↑2-1)/DP+2×(1/DP-1)×LN(K)) ; WRITE(LINE,FM2,K,K0,H,E1,E3) ;
 TAU←(1-K×K)×VS×E2×E4/(K↑4×H×DP) ; DR←K0×K/((1-K)×TAU) ; WRITE(LINE,FM3) ;
 B[0]←SD[0]←T[0]←J[0]←O[0]←1 ; X←I←XX←D[0]←B[1]←ET←HN←BN←EF←0 ;
L1 : HN←HN+0.1 ; V←HN ; WU ; IF X = 1 THEN GO TO L3 ; ES←FN ; X←1 ;
 IF ES×ET < 0.0 THEN GO TO L2 ; GO TO L1 ; %
L3 : ET←FN ; X←0 ; IF ES×ET > 0.0 THEN GO TO L1 ; L2 : BN←HN-0.11 ; XX←EF←0 ;
L4 : BN←BN+0.01 ; DD1 L6 ; DD2 L5 ; GO TO L4 ; L6 : DD3 L4 ; L5 : BN←BN-0.011 ;
 XX←EF←0 ; L7 : BN←BN+0.001 ; DD1 L8 ; DD2 L9 ; GO TO L7 ; L8 : DD3 L7 ; L9 :
 BN←BN-0.0011 ; XX←EF←0 ; L10 : BN←BN+0.0001 ; DD1 L11 ; DD2 L12 ; GO TO L10 ;
 L11 : DD3 L10 ; L12 : I←I+1 ; T[I]←V ; SD[I]←SS ; J0[I,0]←B[I,0]←1 ; J14 BEGIN
 Y14 A[J,Y]←COS(V×Y×SIN(J×M/80)/40) ; U←1 ; FOR Y←2 STEP 1 UNTIL Z DO
 U←U+B[Y]×(J/40)↑Y ; BX[I,J]←U END ; Y14 BEGIN J0[0,Y]←BX[0,Y]←1 ; J14
 O[J]←A[J,Y] ; J0[I,Y]←DER(0) END ; J1[I]←J1×K0 ; IF I < MR THEN GO TO L1 ;
 FOR I←0 STEP 1 UNTIL MR DO FOR J←I STEP 1 UNTIL MR DO BEGIN Y14 BEGIN
 DA←Y×E4×(E1-(Y/40)↑2)×SD[I]×J0[I,Y]×SD[J]×J0[J,Y]/40 ; %
 S←KH×(1-(1-K)×Y/40)×T[I]×J1[I]×BX[I,Y]×T[J]×J1[J]×BX[J,Y]/E2 ;
 D[Y]←DA+S×((1-K)×Y×(2-(1-K)×Y/40)/40+E3×LN(1-(1-K)×Y/40)) END ; S←R←0.0 ;
 P[I,J]←P[J,I]←2×DER(0) END ; %
 I1M BEGIN FOR Y←0 STEP 1 UNTIL 40 DO BEGIN S←T[I]×J1[I]×BX[I,Y] ;
 O[Y]←(1-(1-K)×Y/40)×S ; D[Y]←(SD[I]×J0[I,Y])↑2×Y/40+KH×(1-(1-K)×Y/40)×S×S
 END ; N[I]←DER(D)×2/CF[I]←2×H×DER(0)/(N[I]×(K+1)) END ; I1M FOR J←I STEP 1
```

```
UNTIL MR DO BEGIN AA[I,J]←(P[I,J]−P[0,I]×P[0,J]/(1+H))/SQRT(N[I]×N[J]);
AA[I,J]←AA[I,J]/(T[I]×T[J]);AA[J,I]←AA[I,J] END; JACOBI(1,MR,AA,BB);
I1M J1M BB[J,I]←BB[J,I]/(T[J]×SQRT(N[J]));ES←H/(1+H); D[0]←O[0]←Q[0]←0;
I1M BEGIN TP[I]←1/SQRT(AA[I,I]);%
F←0.0;J1M F←F+BB[J,I]×P[0,J];J1M BB[J,I]←−BB[J,I]×(1+H)/F  END;
I1M BEGIN Y14 BEGIN S←R←1.0;J1M BEGIN S←S+BB[J,I]×SD[J]×JO[J,Y];
R←R+BB[J,I]×T[J]×JI[J]×BX[J,Y] END;   O[Y]←Y×E4×(E1−(Y/40)*2)×S/40;
E←(1−(1−K)×Y/40)×((1−K)×Y×(2−(1−K)×Y/40)/40+E3×LN(1−(1−K)×Y/40))×R/E2;
D[Y]←E;Q[Y]←O[Y]×S+KH×E×R END; CP[I]←KH×DER(D)/DER(Q); %
N[I]←CP[I]×DER(O)×2;WRITE(LINE,FM4,I,T[I],TP[I],CP[I]) END; %
WRITE(LINE,FM5);%
FOR E←0.001,0.003162,0.01,0.03162,0.1,0.3162 DO BEGIN S←R←0; I1M BEGIN
S←S+N[I]/(AA[I,I]×EXP(E/AA[I,I]));R←R+N[I]/EXP(E/AA[I,I]) END;U←ES×S/R;
WRITE(LINE,FM6,E,U) END; ET←ES/AA[1,1]; WRITE(LINE,FM7,ET) END.
```

```
XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

K = 0.900    PHI = 1.00@+01    H = 1.00@+00    E1 = 1.050    E3 = 1.400

 I     GAMMA    LAMBDA    COEFFICIENT
 1     0.900     1.049     8.746@−02
 2     2.182     2.302    −5.698@−02
 3     3.410     3.417     2.294@−02
 4     4.013     4.154    −3.964@−03
 5     5.094     5.326    −1.763@−02
 6     6.371     8.607     1.265@−02

      ETA            NU
 1.000@−03      1.823@+00
 3.162@−03      1.771@+00
 1.000@−02      1.631@+00
 3.162@−02      1.345@+00
 1.000@−01      9.719@−01
 3.162@−01      6.888@−01
 INFINITY     5.49728@−01
```

```
      BURROUGHS B-5500 ALGOL COMPILER LEVEL   6     TUESDAY,   5/12/70,

%COMPILE BIN0509/TURB   *ALGOL          ,06S800006   *0509 WU D C

%PROCESS=       00000020;IO=    00000010.
%DATA.
%
%
%
%  THIS PROGRAM CALCULATES THE NUSSELT NUMBER FOR TURBULENT TWO-PHASE
%  ANNULAR FLOW AS A FUNCTION OF THE FOLLOWING EIGHT VARIABLES;
%     W   =  PARAMETER DEFINED BY EQ.(2.3.8)
%     H   =  PARAMETER DEFINED BY EQ.(2.4.23)
%    PG   =  PRESSURE GRADIENT WHICH WOULD EXIST IF GAS FLOWS ALONE
%    PL   =  PRESSURE GRADIENT WHICH WOULD EXIST IF LIQUID FLOWS ALONE
%   SC1   =  SCHMIDT NUMBER IN THE GAS PHASE
%   SC2   =  SCHMIDT NUMBER IN THE LIQUID PHASE
%   YR1   =  REDUCED RADIUS FOR GAS PHASE
%   YR2   =  REDUCED RADIUS FOR LIQUID PHASE
%
BEGIN %

FILE OUT  LINE 6(3,15);%
FORMAT FM1(X7,"H",X7,"W",X6,"PG",X6,"PL",X5,"YR1",X5,"YR2",X5,"SC1",X5,

 "SC2",X13," B",X9,"LAMBDA",X12," NU"), FM2(8(E8.1),3(E15.4)); %

REAL YC,TR,JL,K,JR,TK,HR,HL,W,FE,FK,FL,FR,ER,PG,PL,YR1,YR2,SC1,SC2,H,XX,
  B,CO,R1,NU;    INTEGER L,I,X; %
REAL ARRAY E,F,E1,F2,U1,U2,X1,X2[0:1002]; %
DEFINE L111 = FOR L←1 STEP 1 UNTIL 1000 DO # , %
       L110 = FOR L←1 STEP 1 UNTIL 100 DO # ,%
       L115 = FOR L←1 STEP 1 UNTIL 500 DO # , %
       DD1 = HL←R1×R1; WU; IF X = 1 THEN GO TO # ,%
       DD2 = FR←FE; X←1; IF FL×FR < 0 THEN GO TO # ,%
       DD3 = FL←FE; X←0; IF FL×FR > 0 THEN GO TO # ,%
       DIF = IF YC < 5 THEN E[L]←(YC/14.5)↑3 ELSE IF YC < 33 THEN
       E[L]←YC/5-0.959 ELSE # ; %
PROCEDURE WU; BEGIN I←TR←JL←0.0; L111 BEGIN %
 YC←(1-(1-K)×L/1000)*U2[L]×X2[L]; IF I = 0 THEN BEGIN JR←(JL+YC)×@-06;
 I←1; TK←YC; HR←TR+JR END ELSE BEGIN JR←(JL+4×TK+YC)/1.5@+06; HR←TR+JR;
 I←0; JL←YC; TR←HR END; X2[L+1]←X2[L-1]-HL×W×HR/(E2[L]×(1-(1-K)×L×@-03))
 END; I←TR←JL←0.0; L111 BEGIN YC←L×U1[L]×X1[L]; IF I = 0 THEN BEGIN
 JR←(JL+YC)×@-06; I←1; TK←YC; HR←TR+JR END ELSE BEGIN I←0; %
 JR←(JL+4×TK+YC)/1.5@+06; HR←TR+JR; JL←YC;TR←HR END; %
 X1[L+1]←X1[L-1]-HL×HR/(L×E1[L]) END; %
 FE←(X2[1001]-X2[999])×X1[1000]←FK×X2[1000]×(X1[999]-X1[1001]) END;
PROCEDURE DC; BEGIN FL←FR←0.0; FOR L←1 STEP 2 UNTIL 999 DO FL←FL+F[L];
FOR L←2 STEP 2 UNTIL 998 DO FR←FR+F[L]; ER←(F[0]+4×FL+2×FR+F[1000])/3000
 END; %
LABEL M1,M2,M3,M4,M5,M6,M7,M8,M9,M10,M11,M12; %
 WRITE(LINE,FM1); PG←2.0@+09; PL←1.0@+08; YR1←4000; YR2←2000; SC1←1;
 SC2←700; W←0.1; FOR H←0.01,0.05,0.1,0.5,1,5 DO BEGIN %
 XX←(PL/PG)↑0.5; K←1/(1+XX↑0.8)↑0.187;  B←7.40@-06×PL↑0.83×H↑0.21;
 CO←H/(1+H); F[0]←U1[1000]←E[0]←U2[0]←0; L115 BEGIN YC←YR2×(1-K)×L/1000;
 DIF E[L]←YC×(1-(1-K)×L×@-03)/2.5 END; L115 F[500+L]←E[500-L]; L111 BEGIN
 U2[L]←U2[L-1]+(1-(1-K)×(L-1)/1000)/(1+E[L-1]); E2[L]←E[L]×SC2/B +1;
 F[L]←U2[L]×(1-(1-K)×L/1000) END; DC; L111 U2[L]←U2[L]×(1+K)/(ER×2);
 L111 BEGIN YC←K×YR1×L/1000; DIF IF L < 200 THEN E[L]←YC×(1-K×L/1000)/2.5
 ELSE E[L]←YR1/14.4 END; F[1000]←2×U1[1000]; FK←-W×(1+K)/(H×K); L111 BEGIN
 E1[L]←E[1000-L]×SC1/B +1; U1[1000-L]←U1[1001-L]+(1001-L)/(1+E[L-1]);
 F[1000-L]←U1[1000-L]×(1000-L)/500 END; DC; L111 U1[L]←U1[L]/ER; %
```

```
      X1[0]+X1[1]+X2[0]+X2[1]+1.0) X+FL+R1+0) $
      M1: R1+R1+1) DD1 M2) DD2 M3) GO TO M1; M2: DD3 M1) M3: R1+R1-1.1)X+FL+0)
      M4: R1+R1+0.1;DD1 M5;DD2 M6)GO TO M4;M5: DD3 M4;M6: R1+R1-0.1)X+FL+0)
      M7: R1+R1+0.01;DD1 M8;DD2 M9;GO TO M7;M8: DD3 M7;M9: R1+R1-0.01)X+FL+0)
      M10: R1+R1+0.001;DD1 M11;DD2 M12;GO TO M10;M11: DD3 M10;M12: NU+HL*B*CO)
       WRITE(LINE,FM2,H,W,PG,PL,YR1,YR2,SC1,SC2,B,HL,NU) END END.
```

H	W	PG	PL	YR1	YR2	SC1	SC2	B	LAMBDA	NU
1.0E-02	1.0E-01	2.0E+09	1.0E+08	4.0E+03	2.0E+03	1.0E+00	7.0E+02	1.2281E+01	2.3725E+02	2.8848E+01
5.0E-02	1.0E-01	2.0E+09	1.0E+08	4.0E+03	2.0E+03	1.0E+00	7.0E+02	1.7219E+01	1.7137E+02	1.4052E+02
1.0E-01	1.0E-01	2.0E+09	1.0E+08	4.0E+03	2.0E+03	1.0E+00	7.0E+02	1.9917E+01	1.3663E+02	2.4740E+02
5.0E-01	1.0E-01	2.0E+09	1.0E+08	4.0E+03	2.0E+03	1.0E+00	7.0E+02	2.7925E+01	6.7011E+01	6.2379E+02
1.0E+00	1.0E-01	2.0E+09	1.0E+08	4.0E+03	2.0E+03	1.0E+00	7.0E+02	3.2302E+01	4.8344E+01	7.8081E+02
5.0E+00	1.0E-01	2.0E+09	1.0E+08	4.0E+03	2.0E+03	1.0E+00	7.0E+02	4.5291E+01	2.7437E+01	1.0355E+03

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