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PROBABILISTIC FRACTURE MECHANICS AND OPTIMUM FRACTURE CONTROL ANALYTICAL PROCEDURES FOR A REUSABLE SOLID ROCKET MOTOR CASE

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NOTATIONS

A	a specified crack depth
C	constant in Paris's equation for crack growth
H	height of casing
K	stress intensity factor
K_C	critical stress intensity factor
K_N	stress intensity factor after N uses
N	number of uses of the motor case
p	Pressure
R_o	outer radius of the casing
a	surface crack depth
a_c	critical crack depth
a_N	crack depth after N uses
c_o	half the length of a surface crack
c_1	payload cost per pound
c_2	cost of total payload
c_3	cost of articles and accessories at proof test
c_i	component cost
c_{ii}	component cost
c_{iii}	component cost
c_{iv}	component cost
c_T	total cost
f	probability density function
n	constant in Paris's equation
n	constant in Collipriest's equation
p	proof load factor
t	thickness of the case
t_N	thickness of the case after N uses
x	random variable representing crack depth
z	standard normal variable
γ,η	shape parameter
Δt	thickness decreased during grit blasting
ΔK	stress intensity range
ϵ,λ	maximum initial crack depth, scale parameter

a_o	minimum crack depth
ϕ	shape parameter
$\bar{\gamma}$	density of the material of the casing
y	yield stress

Abstract

A methodology for the reliability analysis of a reusable solid rocket motor case has been discussed in this paper. The analysis is based on probabilistic fracture mechanics and probability distribution for initial flaw sizes. The developed reliability analysis can be used select the structural design variables of the solid rocket motor case on the basis of minimum expected cost and specified reliability bounds during the projected design life of the case. Effects of failure prevention plans such as non-destructive inspection and the material erosion between missions can also be considered in the developed procedure for selection of design variables. The reliability-based procedure that has been discussed in this paper can easily be modified to consider other similar structures of reusable space vehicle systems with different failure prevention plans.

INTRODUCTION

Structural components of a solid rocket motor case are considered to be fracture critical whenever the game plan is to recover and reuse the motor case for a designated number of missions. Proof tests, conducted on the case between missions, are also significant to rendering the structural components fracture critical. Proof load levels may significantly affect the design life of the structure. A fracture control plan is, therefore, necessary and are considered in the design of the case.

In particular, this paper is concerned with the fracture control of the most critical membrane areas of the case. All discussions and methodologies presented in this paper can, however, be used whenever similar fracture critical structures of a reusable space vehicle system are designed. Some modification might be necessary in particular structures. Significant loads are applied to the motor case during flight and water recovery operation of each mission. The applied stresses from all other events during the mission are assumed in this analysis to be not significant enough to result in cyclic or time dependent crack growth. If the test or analysis indicate the possibility of other critical loading events they can be included in the fracture control plan by extending the reported analysis. Before each mission, the case is also subjected a proof test. The loads applied during the proof tests can result in significant amount of crack growth. Grit blasting is assumed to be used between each mission. This reduces the effective depth of cracks and the thickness of the membrane by a selected amount. While the effective depth of crack is reduced, the refurbishment grit blasting operation has the effect of increasing the applied stresses. This necessitates a larger initial thickness of the membranes than that would be required otherwise. Therefore, any design of the membrane of the case must arrive at an initial wall thickness t, the thickness t that will be decreased between each mission and the proof load factor p. For example, a large value of initial wall thickness results in increased reliability, but results in the need for increased propellant, increased cost of operation and reduced pay load capability. On the other hand, a small initial wall thickness increases the probability of failure and the resulting loss of the reusable space vehicle system and the pay load. Therefore, there is a need for optimizing the initial wall thickness. Similar arguments can be presented to explain the need for selecting the other design variables such as Δt and p by optimizing the desired objective function of cost and weight.

In general, these design variables depend on the probability distribution for the initial flaw sizes present in the membrane, applied stresses during the use of the vehicle, crack growth characteristics of the material, fracture control plans, specified reliability bounds, weight and cost considerations. The paper describes a reliability-based procedure that can be used to select the design variables of a solid rocket motor case in a reusable space vehicle system by using probabilistic fracture mechanics and cost or weight considerations.

Method of Approach

It is assumed that careful Non Destructive Inspection (NDI) techniques can detect initial cracks greater than the surface length of $2c_o$ and depth of a_o with 100% success. Sometimes, it is assumed that cracks corresponding to surface length 2c = 0.1 inch can be identified 100% of the time.[1] If the corresponding maximum depth is 0.05 inch there is no possibility of existence any initial cracks of depth larger than 0.05 inch. Such an initial crack depth distribution is assumed to be analytically represented by Johnson S_b distribution.[2] Reasons for this assumption can be explained as follows. One of the requirements of any assumed distribution is that the minimum and maximum crack depths be bounded within finite limits. Depending on the thickness and the available techniques of non destructive inspection techniques, there is a finite maximum depth of possible crack. It is not infinity as is provided by distributions such as normal distribution, gamma or log-normal distributions. The minimum value of depth of crack can be assumed to be zero or a small number . Such a distribution can be obtained as the transformation of the usual normal variate. One such transformation is the following.

$$z = \gamma + \eta \ \ln \frac{x - \varepsilon}{\lambda + \varepsilon - x} \qquad \varepsilon \leq x \leq \varepsilon + \lambda \qquad (1)$$

In this equation, z is the standard normal variable and x is the variable of interest i.e., the crack depth. The four available parameters are γ, η, ε, & λ. The minimum and maximum crack depths fix ε & λ respectively. The parameters can be called shape parameters and can be determined from percentiles of the observed data.

The density function for the probabilistic model is written as follows

$$f_{ao}(a_o) = \frac{\eta}{2\pi} \frac{\lambda}{(a_o - \varepsilon)\ (\lambda - a_o + \varepsilon)} \exp \left\{ -\tfrac{1}{2} \left[\gamma + \eta \ \ln \left(\frac{a_o - \varepsilon}{\lambda - a_o + \varepsilon} \right) \right]^2 \right\} \qquad (2)$$

$$\varepsilon \leq a_o \leq \varepsilon + \lambda \ , \quad \eta > 0$$

$$-\infty \leq \gamma \leq \infty \quad , \quad \lambda > 0$$

$$-\infty \leq \varepsilon \leq \infty$$

This emperical distribution is called Johnson S_b distribution. It should be noted that it is possible to obtain other emperical distributions to represent the crack depths.

This probability distribution for initial crack depth changes after each mission, each proof test and each time the material as removed from the wall thickness. The change in distribution after each mission and each proof test is due to the crack growth resulting from the applied stresses. This crack growth also depends on the present length of the crack, applied stress and the material that are responsible for the crack growth. In this analysis,the applied stresses and material properties are assumed to be known deterministically. If the initial crack length were also known deterministically the crack length after each use can be determined from equations such as Paris' equation[3], Foreman's equation[4] or Collipriests equations[5]. Because initial crack lengths are not known deterministically, crack length after each use of the vehicle is again another probablistic distribution that has to be estimated.

The cumulative density function (CDF) for crack length after N uses is denoted by $F(a_N)$. This represents the probability that $a_N \leq A$ after N uses. Each use is defined as one flight, one proof test and a material removal. Crack growth due to time related effects such as stress corrosion have been neglected.

If $F(a_N)$ is known, the probability distribution for the stress intensity factor K_N can be obtained from the knowledge of the applied stresses. The probability distribution $F(K_N)$ for stress intensity factor can be used to estimate the probability failure P_f which is the probability of stress intensity factor K_N greater than or equal to the critical stress intensity factor during the projected design life of the structure. The critical stress intensity factor is denoted by K_c. In this analysis, stresses and the material properties are assumed to be known deterministically. However, the applied stress changes after each use due to material removal. Therefore, the probability of failure can be expressed as the probability of $a_N \geq a_c$. In this expression a_c is the critical crack depth that can be obtained from the critical stress intensity factor and the applied stress corresponding to that particular mission. This relationship between the stress intensity and the applied stress is discussed in the next section.

Stress Intensity Factor

For the analysis of the stress intensity factor in the membrane, an infinite plate model with elliptical surface flaws that are oriented perpendicular to the applied stress has been assumed. The relationship between the stress intensity factor, the applied tensile stress and crack depth is given by[1]

$$K = \sqrt{\frac{1.2\ \pi\ \sigma^2 a}{Q\left(\frac{a}{c}\right)}} \qquad (3)$$

where

$$Q\ \left(\frac{a}{c}\right) = \emptyset^2 - 0.212\left(\frac{\sigma}{\sigma_y}\right)^2 \qquad (4)$$

In this equation, is the yield stress and $\emptyset$ is a function of the ratio of crack depth to crack length (a/c). Variation $\emptyset^2$ with (a/c) is given in reference 1.

Because the crack depth a is a random variable the stress intensity factor K is also a random variable. In general, both crack depth a and crack length 2c are random variables and there is a need for a joint distribution for a and c. In this analysis, only the crack depth is considered as the random variable. It is also assumed that the probability distribution for crack depth a is known initially and is given by a Johnson S_b distribution[2]. The density function for the distribution is given in equation (1). This probability distribution for crack depth changes with use. The next step will be to determine the change and the new probability distribution after each flight and proof test.

Probability Distributions for Crack Depth After Use

The following symbols are used to properly account for the changes in probability distributions.

$f(a_o)$: Probability density function for the initial crack depth

$F(a_o)$: Cumulative distribution function for initial crack depth

$F(a_{op})$: Cumulative distribution function for initial crack depth after the first proof test

$F(a_N)$: Cumulative distribution function after N flights and (N+1) tests

$F(a_{Np})$: Cumulative distribution function after N flights and N proof tests.

$F(a_N)$: Cumulative distribution function after material removal from the wall thickness.

Then,

$$\frac{da}{dN} = c\ (\Delta K)^n \tag{5}$$

where C and n are empirical constants. Alternately, the rate of crack growth can be assumed to be given by Foreman's equation[4] or Collipriest's equation[5], if they are found to represent the situation more accurately. For example, Collipriest's equation can be written as follows:

$$\frac{da}{dN} = D \exp\left[\bar{n}\ \frac{\ln K_c - \ln\Delta K}{z}\ \tanh^{-1}\left\{\frac{\ln\Delta K - \frac{1}{2}\,(\ln K_c\ (1-R) + \ln\Delta K)}{\frac{1}{2}\,(\ln K_c(1-R) - \ln K)}\right\}\right.$$
$$\left. + \ln\left\{c \exp\left(\bar{n}\left(\frac{\ln K_c + \ln K_o}{z}\right)\right)\right.\right] \tag{6}$$

where n is an empirical constant. By integrating either of the selected equations (5) or (6) crack depth after N+1 uses can be determined if the crack depth after N uses and N proof tests are known deterministically, i.e.,

$$a_{N+1} = a_{N+1} \left\{ a_{NP} \right\} \tag{7}$$

Similarly, crack depth after the proof test can be determined from equation (5) or(6) if the crack depth before proof test is known deterministically, i.e.,

$$a_{NP} = a_{NP} \left\{ a_N \right\} \tag{8}$$

These functions represented by equations (7) or (8) can be determined analytically or in the form of quadratures from equation (5) or (6). From equation (7), a_{N+1} can be obtained for every known value of a_{NP}. Similarly, a_{NP} can be obtained for every known value of a_N from equation (8). However, both a_{NP} and a_N are random variables in the present analysis. In this case, equation (7) can be used to obtain the probability distribution for a_{N+1} if the probability distribution for a_{NP} is known by using the principle of transformation of random variables. It should be noted that all equations similar to (7) or (8) involving crack depths are increasing functions. This property is useful in transforming the random variables.

For example, the probability density function for a_{N+1} can be written as follows

$$f(a_{N+1}) = f\left[a_{N+1}\left\{a_{Np}\right\}\right]\left|\frac{da_{Np}}{da_{N+1}}\right| \tag{9a}$$

similarly

$$f(a_{Np}) = f\left[a_{Np}\ \left\{a_{N}\right\}\right]\left|\frac{da_{N}}{da_{Np}}\right| \tag{9b}$$

Equations (7) and (8) can be written for every value of N from zero to the projected number of uses.

Details of obtaining these equations for the membrane of the solid rocket motor case, with the expression for stress intensity given by equation (2) and Paris' equations for crack growth, is discussed in the Appendix I.

The next step is to obtain a tool for change of probability distribution due to the material removal from the wall thickness.

Material Removal and the Change of Probability Distribution

Due to material removal after each use, the effective crack depth is reduced by Δt . Thus, new crack depth is

$$\tilde{a}_N = a_N - \Delta t \tag{10}$$

It is assumed that Δt is a constant. Thus, by using the principles of transformation of random variables[2], the probability density function for a_N can be written as follows.

$$p(\bar{a}_N) = f\ (\bar{a}_N + \Delta t) \tag{11}$$

In this equation, $p\ (\bar{a}_n)$ represents the density function for a_N and f represents the functional form of the probability density function for a_N.

Probability of Failure

By following the method discussed in the preceeding two sections probability density function for crack depth can be obtained after every flight, proof test and material removal. From the density function, cumulative probabilities can be obtained by integration. Integration after the transformation of variables as discussed in equations (9), (10), and (11) needs the determination of appropriate limits of integration consistent with the transformation of variables. This is also discussed in the Appendix I. If $F(a_N)$ represents the cumulative density function after N flights & N proof tests the probability of failure is given by the probability of $a \geq a_{cN}$. The quanity of a_{cN} corresponds to K_c and the applied stress at the N^{th} use.

It is to be noted that the probability of failure changes with different selections of the initial wall thickness t, increased loading due to proof test, the material removed Δt and the number of designated number of missions. The increased loading due to proof tests is denoted by a factor p. A cost function or a weight function can be formulated from this knowledge of probability of failure and other related unit-cost or weight. Such a cost or weight function depends on t, p, and number of missions N. It is possible to select these design variables by minimizing the cost or weight function subject to appropriate realiability bounds. The effect of non destructive inspection (NDI) is indirectly related to initial flaw distribution. Additional NDI effects such as the rejection of structures are not considered in the analysis. However, they can be included as units related to the probability of failure. A numerical example is illustrated in the next section to illustrate the developments of the paper.

Numerical Example

For the numerical example, it is assumed that the Johnson S_b distribution for the initial crack depth is such that the minimum crack depth is zero and the maximum crack depth is 0.1 inch. Paris's equation for crack growth is assumed with

$$c = 0.847 \times 10^{-18}$$

$$n = 3.0$$

The variation of ϕ^2 with (a/c) as shown in figure 1 is approximated by a quadratic relation.

The primary objective of reusing the solid rocket motor case is to reduce the cost of operation of the reusable space vehicle system in which it is used. However, as the number of uses is increased, the probability of failure increases because of the propagation of the crack depth. On the otherhand, smaller number of uses increases reliability and also the cost is distributed over a smaller number of uses. This means the casing has to be replaced after a fewer number of uses.

A larger initial thickness would increase the weight of the casing and costs more in terms of payload. But the probability failure is less if the thickness is more. The proof test factor p and the material erosion Δt are kept constant in this example. However, they also can be varied and their effect on total cost can be considered in the most general case. The total cost function c_T, therefore, comprises the following component costs.

i) Initial cost of the casing, c_i,
ii) Expected cost of flight failure c_{ii},
iii) Expected cost of proof test failure c_{iii} and

iv) cost due to multiple usage, c_{iv}.

The initial cost c_i is given by the product of the weight of the casing and the cost per pound of the system, i.e.,

$$c_i = \pi \, (2R_o t_N - t_N^2) \, H \, \gamma \, c_1 \tag{12}$$

where R_o = outer radius of the casing
t_N = thickness of the casing at the N^{th} cycle
H = height of the casing
γ = density of the material
c_1 = payload cost per pound

The expected cost of flight failure is the product of the probability of flight failure and the entire payload cost, i.e.

$$c_{ii} = P_N \cdot c_2 \tag{13}$$

where P_N is the probability of failure at the N^{th} flight c_2 is the total cost of the payload. Similarly the cost of proof test failure is

$$c_{iii} = P_{Np}\, c_3 \tag{14}$$

where p_{np} is the probability of failure at the N^{th} proof test and c_3 is cost of articles and accessories of proof test. Finally, the cost due to multiple usage is given as follows:

$$c_{iv} = c_3 \Big/ (N)^{0.3} \tag{15}$$

Thus, substituting all the components, the total cost function c_T is given by the following equation

$$c_T = c_i + c_{ii} + c_{iii} + c_{iv} \tag{16}$$

The following numerical values are used[1,6] in evaluating equation (16).

$\bar{\gamma}$ = 0.3 lbs/cubic inch

H = 816 inches

R_o = 72.5 inches

C_1 = \$1624 per lbs.

C_2 = \$250 x 10^6

C_3 = \$2x10^6

Results

The initial thickness to is varied from 0.535" to 0.435" in steps of 0.005". Also 1% of the initial thickness is eroded after each flight. The total cost function is calculated for various initial thicknesses and use cycles by means of a digital computer. Figure 2 illustrates the variation of the cost function with t_o and N. If it is obvious that as the number of uses increases, the minimum occurs at a higher initial thickness. For example, for 18 missions the minimum cost occurs at an initial thickness of 0.48 inch. The initial thickness to give minimum cost for 20 mission cycles increases to 0.497 inch, for 22 missions the thickness required is 0.512".

Figure 3 delineates the variation of reliability with initial thickness, after 20 missions cycles. The reliability corresponding to the minimum cost for 20 uses is 99.3%. If this reliability is not adequate enough, then a higher initial thickness should be used even though the total cost will be higher than the minimum.

General Procedure

Based on the preceding example, a general procedure can be delienated in the following steps.

1. Obtain the parameters of the Johnson S_b distribution[2] for the initial flaw size.

2. Obtain the stress in the membrane from the known geometry of the case and wall thickness.

$$\sigma = p \frac{PR_o}{t}$$

   In the equation p is the proof stress factor. During flight, p is replaced by a value of 1. Pressure P is the MEOP pressure on the case and R_o is the radius of the case.

3. Obtain the new CDF and density function for the crack depth after the proof test.

4. Obtain the new CDF for the crack depth during the flight following the proof test.

5. Estimate the probability of failure.

6. Compute the cost function parameters.

7. Obtain the new CDF after the material removal.

8. Repeat steps 2 to 7 for the new thickness and the next mission until the total number of missions are complete.

9. Change t and N and repeat the calculations as necessary.

10. Select the design variables for the minimum value of the objective function subject to reliability constraints.

A computer program has been written to carry out these steps (see Appendix II).

Conclusions and Recommendations

This paper has demonstrated that the reliability analysis based on probabilistic fracture mechanics can be used to optimize the selection of the design variables of a solid rocket motor case. In particular, basic design variables such as the thickness and projected design life as well as the fracture control variables such as the proof factor and material erosion can be included in the analysis. Accuracy in estimation of the initial flaw size distribution is reflected in the assessment of the risks involved in the design. By knowing the risks involved in the design, weight and cost can be reduced from those obtained by the conventional deterministic analysis and use of arbitrary safety margins.

This report is only a first step in the development of procedures based probabilistic fracture mechanics. Additional work that is necessary can be listed as follows:

1. A more accurate analysis can be obtained by considering the joint distribution for the crack dpeth and crack length along the surface.

2. Accurate methods of estimation of the probability distribution for the initial flaw size distribution should be developed.

3. In particular, effects of water impact and time dependent crack growth, stress corrosion, should be considered. This is particularly important if the missions are spaced over years.

4. Uncertainties in external loads and material properties should be considered.

5. Accuracy of the different models for crack growth (in the point of view of probabilistic fracture mechanics) should be evaluated.

6. Alternate fracture control plans and more accurate stress intensity measures based on cylindrical geometry can be considered.

7. Cost of NDI efforts in relation to the cost that will be incurred by additional safety factor should be evaluated in the point of view of improved reliability.

8. Thermal effects should be considered.

References

1. Kapp, J. R., "Fracture Control Plan for Space Shuttle Solid Rocket Motor Case," TWR 10689 Preliminary Report 1975.

2. Hahn, H., and Shapiro, "Statistical Models In Engineering" John Wiley and Sons, Inc., New York p. 198, 1967

3. Paris, P. C., and Erdogen, F., "A Critical Analysis of Crack Propagation Laws," Journal of Basic Engineering Trans ASME Series D No. 85, p. 528 1963

4. Foreman, R. G., Kearney, V. E., Engle, R. M., "Numerical Analysis of Crack Propagation In Cyclic Loaded Structures," Journal of Basic Engineering Trans ASME Series D, 88, p. 459, 1967.

5. Creager, M., "MSFC Crack Growth Analysis Computer Program" NASA CR - 143923, 1975.

6. Thomas, J. M., and Hanagud, S. "Reliability-Based Econometrics of Aerospace Structural Systems" NASA TN D-7646, June, 1974.

APPENDIX I

This appendix describes estimation of the new CDF of crack depth after use from a knowledge of the old CDF and probability density before use.

Crack Growth Rate

The rate at which the crack depth increases is given by Paris' equation as follows.

$$\frac{da}{dN} = c(\Delta K)^n = 0.847 \times 10^{-16} (\Delta K)^n$$

For subsequent convenience in algebra, the value of n is taken to be 3.0. The suggested value from current state of art is 2.48 and c is equal to 0.867×10^{18}. By substituting for

$$\frac{da}{dN} = 0.847 \left[c_4 \left\{ \frac{a}{c_5 + c_2 \frac{a}{c} + c_3 \left(\frac{a}{c}\right)^2} \right\}^{\frac{1}{2}} \right]^3 \qquad \text{(A1)}$$

Simplifying this further,

$$\frac{da}{dN} = c_6 \left\{ \frac{a}{c_5 + c_2 \left(\frac{a}{c}\right) + c_3 \left(\frac{a}{c}\right)^2} \right\}^{1.5} \qquad \text{(A2)}$$

where

$$c_6 = 0.847 \times c_4^{3} \times 10^{-18} \qquad \text{(A3)}$$

Separating the variables a and N in $\frac{da}{dN}$, it follows that

$$dN = \frac{1}{c_6} \left\{ \frac{c_5 + c_2(\frac{a}{c}) + c_3(\frac{a}{c})^2}{a} \right\}^{1.5} da \qquad \text{(A4)}$$

Integrating both sides between state (1) and state (2) the following equation is obtained

$$[N]^2 = \frac{1}{c_6} \int_{a_1}^{a_2} \left\{ \frac{c_5 + c_2(\frac{a}{c}) + c_3(\frac{a}{c})^2}{a} \right\}^{1.5} da \qquad \text{(A5)}$$

In order to evaluate the integral on the right hand side, it is found necessary to expand the numerator of the integrant binomially.

Now consider the numerator of the integrand with $C_5 = 1$. Neglecting terms of higher order than $(a/_{c})^3$, it follows that

$$\left\{1 + c_2 \left(\frac{a}{c}\right) + c_3 \left(\frac{a}{c}\right)^2\right\}^{1.5}$$

$$= 1.0 + 1.5\, c_2 \left(\frac{a}{c}\right)$$

$$+\left[1.5c_3 + 1.5\ (0.25)\right] \left(\frac{a}{c}\right)^2$$

$$+ \left[0.75c_2c_3 - 0.25(0.5)^2 c_2{}^3\right] \left(\frac{a}{c}\right)^3 \qquad \text{(A6)}$$

Letting

$$P_1 = \frac{1}{c}\ 1.5\ c_2 \qquad \text{(A7)}$$

$$P_2 = \frac{1}{c^2} \left\{1.5\ c_3 + 0.375\ c_2{}^2\right\} \qquad \text{(A8)}$$

and

$$P_3 = \frac{1}{c^3} \left\{0.75c_2c_3 - (0.25)^2\ c_2{}^3\right\} \qquad \text{(A9)}$$

Then, it follows that

$$\left\{1 + c_2 \left(\frac{a}{c}\right) + c_3 \left(\frac{a}{c}\right)^2\right\} = 1.0 + P_1a + P_2a^2 + P_3a^3 \qquad \text{(A10)}$$

Substituting in the integral the following result is obtained

$$\left[N\right]_{N_1}^{N_2} = \frac{1}{c_6}\left[-\frac{1}{0.5}(a)^{-0.5} + \frac{P_1}{0.5}(a)^{0.5} + \frac{P_2}{1.5}(a)^{1.5} + \frac{P_2}{1.5}a^{2.5}\right]_{a_1}^{a_2} \tag{A11}$$

Solution of a_1 as a function of a_2

Substituting the limits of integration in A(11)

$$c_6\,(N_2 - N_1) = -2a_2^{-0.5} + 2P_1(a_2)^{0.5} + \frac{2}{3}P_2a_2^{1.5}$$

$$+\frac{2}{5}P_3a_2^{2.5} + 2a_1^{-0.5} - 2P_1a_1^{0.5} - \frac{2}{3}P_2a_2^{1.5} - \frac{2P_3}{5}a_1^{2.5} \tag{A12}$$

Rearranging and neglecting terms of order higher than three, it reduces to the following equation

$$(a_1)^3 + p\,(a_1)^2 + q(a_1) + \gamma = 0 \tag{A13}$$

where

$$p = \frac{1.0}{\frac{8}{3}P_1P_2 - \frac{8}{5}P_3}\left(4P_1^2 - \frac{8P_2}{3}\right) \tag{A14}$$

$$q = \frac{-1.0}{\frac{8}{3} P_1 P_2 - \frac{8}{5} P_3} \left(8P_1 + c_1^{\,2}\right) \qquad \text{(A15)}$$

and

$$\gamma = \frac{4}{\frac{8}{3} P_1 P_2 - \frac{8}{5} P_3} \qquad \text{(A16)}$$

Now, the three roots of this cubic equation, $(a_1)^i$ are given as follows

$$\begin{aligned} a_1^{(1)} &= A + B - \frac{P}{3} \\ a_1^{(2)} &= \frac{A + B}{z} + \frac{A - B}{z} \sqrt{-3} - \frac{P}{3} \qquad \text{(A17)} \\ a_1^{(3)} &= \frac{A + B}{z} - \frac{A - B}{z} \sqrt{-3} - \frac{P}{3} \end{aligned}$$

where

$$\begin{aligned} A &= \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{\bar{a}^3}{27}}} \qquad \text{(A18)} \\ B &= \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{\bar{a}^3}{27}}} \\ \bar{a} &= \frac{1}{3}\left(3q - p^2\right), \quad b = \frac{1}{27}\left(2p^3 - qpq + 27\gamma\right) \end{aligned}$$

Transformation

Probability density of a_2 is given by

$$f_{a_2}(a_2) = \frac{da_1}{da_2} f_{a_1}(a_1) \tag{A19}$$

CDF of a_2 is then

$$\int_{\bar{a}1}^{a_2} f_{a_2}(a_2)\,da_2 = \int_{o}^{a_1(a_2)} f_{a_1}(a_1)\,da_1 \tag{A20}$$

$$\int_{o}^{a_1(a_2)} f_{a_1}(a_1)\,da_1 = \Big[F_{a_1}(a_1)\Big]^{a_1(a_2)} \tag{A21}$$

where $F_{a_1}(a_1)$ is the CDF of Johnson S_B distribution^2.

Now, it is needed to obtain a_1 as a function of a_2, No. of cycles etc. This can be done by solving the polynomial equation obtained previously in terms of a_1 and treating a_2, N_1 and N_2 as constants. The infinite degree polynomial equation is truncated at the 3rd degree for convenience.

Of the three roots only one will be the real root because of the physical nature of the problem, say $a_1(a_2)$

Then by substituting in the expression for the CDF of a_2

$$F_{a_2}(a_2) = \int_{o}^{\hat{a}_1(a_2)} f_{a_1}(a_1)\,da_1 \tag{A22}$$

or if the CDF of a_1 is known,

$$F_{a_2}(a_2) = \left[F_a(a_1)\right]_o^{a_1(a_2)} \tag{A23}$$

Thus, $F_{a_2}(a_2)$ is a function of the parameters of flaw distribution i.e., ϵ, λ, γ, η , the proof test factor p and the number of uses $(N_2 - N_1)$.

The effect of each of these parameters can be studied by calculating $F_{a_2}(a_2)$ for various cases, by means of a computer.

Parabolic Fit to $\phi^2(\frac{a}{c})$

Consider the range $0 \leq \phi^2 \leq 1$. In this range

it is attempted to fit a parabolic curve for

such as follows.

$$\phi^2 (\tfrac{a}{c}) = \bar{c}_1, + \bar{c}_2 (\tfrac{a}{c}) + \bar{c}_3 (\tfrac{a}{c})^2 \tag{A24}$$

In order to determine the three constants C_1, C_2 and C_3 three points are considered on the given curve.

$$\begin{aligned} \tfrac{a}{c} &= 0 & \phi^2 (\tfrac{a}{c}) &= 1.0 \\ \tfrac{a}{c} &= 0.5 & \phi^2 (\tfrac{a}{c}) &= 1.5 \\ \tfrac{a}{c} &= 1.0 & \phi^2 (\tfrac{a}{c}) &= 2.5 \end{aligned} \tag{A25}$$

Substituting the values for point (i),

Substituting the values for point (ii)

$$1.0 + \bar{c}_2 \ (0.5) + \bar{c}_3 \ (0.25) = 1.5 \tag{A27}$$

or

$$2 \ \bar{c}_2 + \bar{c}_3 = 2.0 \tag{A28}$$

Substituting the values for point (iii)

$$1.0 + \bar{c}_2 + \bar{c}_3 = 2.5 \tag{A29}$$

or

$$\bar{c}_2 + \bar{c}_3 = 1.5 \tag{A30}$$

Solving equations (2) and (3) simultaneously

$$c_2 = 0.5 \tag{A31}$$

and

$$c_3 \ 1.0 \tag{A32}$$

Thus the chosen parabolic fit is as follows

$$\emptyset^2 = 1.0 + 0.5 \frac{a}{c} + \frac{a^2}{c^2} \tag{A33}$$

Limits of Integration for the CDF of a_2

By hypothesis, the initial flaw a_1 has a Johnson S_b distribution[2]. Also, there is a functional relationship between the initial flaw size a_1 and the subsequent flaw size a_2 after N cycles. This relationship renders a_2 a random variable because a_1 is a random variable by hypothesis. Having known the range space of a_1 the range space of a_2 can be derived from the functional relationship between a_1 and a_2. Thus, if the lower limit of a_1 is zero, it follows from the functional relationship between a_1 and a_2 that the lower limit of a_2 is also zero. Next, if the upper limit of a_1 is a_1, the corresponding upper limit for a_2 can be obtained by solving the cubic relation between a_1 and a_2, as a function of the number of cycles $N^2 = N_2 - N_1$.

APPENDIX II

```
      PROGRAM MAIN (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C     ***************************************************************
C     THIS PROGRAM CALCULATES  THE RELIABILITY OF THE SRM CASING AT THE END OF
C     EACH PROOF TEST AND USE CYCLE GIVEN THE INITIAL FLAW DISTRIBUTION FOR
C     VARIOUS INITIAL THICKNESSES AND LAUNCH NUMBERS.
      DIMENSION F(30,1),Y(30,1),ZZ1(5)
      COMPLEX AA,BE,A1HAT
      READ(5,*)ALAMDA,EATA,GAMA
      EPS=0.0
      WRITE(6,100) ALAMDA,EATA,GAMA,EPS
      ENDFILE6
100   FORMAT(1H1,/,5X,"PARAMETERS OF THE JOHNSON SB DISTRIBUTION ARE
     1 LAMBDA =",F4.1,2X,"EATA =",F5.2,2X,"GAMA =",F5.2,2X,"EPSILON =",
     1F5.2)
      DO 5000 I=1,21
      AKP=1.05
C     "AKP" IS THE PROOF-TEST FACTOR
      WRITE(6,102)AKP
102   FORMAT(1H1,/,5X," PROOF - TEST FACTOR =",F10.3)
      STHIK=0.486*(1.-(I-11)/100.)
      WRITE(6,103)STHIK
103   FORMAT(/,5X," INITIAL THICKNESS OF SRM CASE =",F10.3)
      DO 4000 N=1,25
C     "N" IS THE NUMBER OF THE CYCLE CONSISTING OF ONE PROOF-TEST AND ONE USE.
      DO 3010 L=1,2
      IF(L.EQ.1) WRITE(6,104) N
      IF(L.EQ.2) WRITE(6,105) N
104   FORMAT(///,5X,"PROOF TEST NUMBER =",I13)
105   FORMAT(5X,"LAUNCH NUMBER OF SRM CASE =",I7)
C     ***************************************************************
C     THIS SECTION CALCULATES THE THICKNESS AT THE END OF EACH CYCLE.
      THIK=STHIK-(((STHIK/100.)*(N-1)))
      WRITE(6,106) THIK
106   FORMAT(5X,"THICKNESS OF SRM CASE=",F15.4)
C     ***************************************************************
C     THIS SECTION CALCULATES THE APPLIED STRESS  (SIGMP)
      SIGMP=AKP*950.7*72.5/THIK
      ACR=((93500/SIGMP)**2.0)/(1.2*3.143)
      IF(L.EQ.2) SIGMP=SIGMP/AKP
      WRITE(6,108)SIGMP
108   FORMAT(5X,"PROOF-STRESS =",F26.1)
C     ***************************************************************
C     THIS SECTION  CONSIDERS THE CUBIC APPROXIMATION.
      C=0.4
      C1=1.0
      C2=0.5
      C3=1.0
      C4=(SQRT(1.2*3.147))*SIGMP
      C6=0.847*(C4**3.0)
      C6=1.0E-18*C6
      P1=1.5*C2/C
      P2=((1.5*C3)+(1.5*0.25*C2*C2))/(C*C)
      P3=((1.5*0.5*C2*C3)-(0.0625*(C2**3.0)))/(C**3.0)
      A2=ACR
      Z1=2.0*P3*(A2**2.5)/5.0
      Z2=2.0*P2*(A2**1.5)/3.0
      Z3=2.0*P1*(SQRT(A2))
```

```
      Z4=-(2.0/SQRT(A2))-(C6*N)
      C7=Z1+Z2+Z3+Z4
      Z1=(4.0*P1*P1)-(8.0*P2/3.0)
      Z2=(8.0*P1*P2/3.0)-(8.0*P3/5.0)
      P=Z1/Z2
      Z1=-((8.0*P1)+(C7*C7))
      Q=Z1/Z2
      R=4.0/Z2
      ABAR=((3.0*Q)-(P*P))/3.0
      Z1=2.0*(P**3.0)
      Z2=-9*P*Q
      Z3=27*R
      B=(Z1+Z2+Z3)/27.0
      Z1=(B*B/4.0)+((ABAR/3.0)**3.0)
112   FORMAT(/," DISCREMINANT SQUARE =",E15.8)
      AA=((-B/2.0)+CSQRT(CMPLX(Z1,0.0)))
      BB=((-B/2.0)-CSQRT(CMPLX(Z1,0.0)))
      Q1=REAL(AA)
      Q2=AIMAG(AA)
      RR=(SQRT((Q1**2.0)+(Q2**2.0)))**(1.0/3.0)
      THET=ATAN(Q2/Q1)
      AR1=RR*COS(THET/3.0)
      AI1=RR*SIN(THET/3.0)
      AR2=RR*COS((THET+(2.*3.147))/3.0)
      AI2=RR*SIN((THET+(2.*3.147))/3.0)
      AR3=RR*COS((THET+(4.*3.147))/3.0)
      AI3=RR*SIN((THET+(4.*3.147))/3.0)
      Q1=REAL(BB)
      Q2=AIMAG(BB)
      RR=(SQRT((Q1**2.0)+(Q2**2.0)))**(1.0/3.0)
      THET=ATAN(Q2/Q1)
      BR1=RR*COS(THET/3.0)
      BI1=RR*SIN(THET/3.0)
      BR2=RR*COS((THET+(2.*3.147))/3.0)
      BI2=RR*SIN((THET+(2.*3.147))/3.0)
      BR3=RR*COS((THET+(4.*3.147))/3.0)
      BI3=RR*SIN((THET+(4.*3.147))/3.0)
      A1HAT=AR1+BR1-(P/3.0)
      AA=REAL(A1HAT)
C     ************************************************************
C     THIS SECTION CONSIDERS THE QUADRATIC APPROXIMATION.
      Z1=(4.0*P1*P1)-(8.0*P2/3.0)
      Z2=-((8.0*P1)+(C7*C7))
      P=Z2/Z1
      Q=4.0/Z1
      A1=(-P+SQRT((P*P)-(4.0*Q)))/2.0
      WRITE(6,114)A1
114   FORMAT(5X,"UPPER LIMIT OF A1 =",E24.6)
      A1HA1=A1
      IF (A1HA1.GE.ALAMDA) Y(N1,1)=1.0
      IF (A1HA1.GE.ALAMDA) GO TO 3000
C     ************************************************************
C     THIS SECTION CALCULATES THE RELIABILITY
      N1=15
      DO 3000 K=2,N1
      A1=A1HA1*(K-1)/FLOAT(N1-1)
      Z1=EATA*ALAMDA/SQRT(2.0*3.147)
```

```
      Z2=1.0/(A1*(ALAMDA-A1))
      Z4=EATA*ALOG(A1/(ALAMDA-A1))
      Z3=EXP(-((GAMA+Z4)**2.0)/2.0)
      F(K,1)=Z1*Z2*Z3
 3000 CONTINUE
      F(1,1)=0.0
      CALL INTGRL(1,A1HA1,F,Y,N1)
 3005 WRITE(6,116)Y(N1,1)
 116  FORMAT(5X,"PROBABILITY OF NO FAILURE =",E14.5)
      ZZ1(L)=1.-Y(N1.1)
 3010 CONTINUE
 C    ***************************************************************************
 C    THIS SECTION CALCULATES THE TOTAL COST FUNCTION.
      C11=(1.0/FLOAT(N))*0.30
      C1=1624.
      C2=250.E+6
      C3=2.0E+6
      WT=3.143*816.*0.3*((145.*THIK)-(THIK*THIK))
      CTOT=(WT*C1)+(ZZ1(2)*C2)+(ZZ1(1)*C3)+(C3*C11)
      WRITE(6,118)CTOT
 118  FORMAT(5X,"********************* TOTAL COST FUNCTION =",F15.5)
 4000 CONTINUE
 5000 CONTINUE
 999  STOP
      END
```

```
      SUBROUTINE INTGRL (M,T,F,Y,N)
      DIMENSION A(30.30),B(30,30),C(30,30),F(30,1),Y(30,1)
      H=T/(720.*(N-1))
      DO 20 I=1,N
      DO 20 J=1,N
      B(I,J)=0.0
      A(I,J)=0.0
 20   CONTINUE
      DO 24 K=1,N
      DO 24 J=1,K
      B(K,J)=1.0
 24   CONTINUE
      A(2,1)=251.
      A(2,2)=646.
      A(2,3)=-264.
      A(2,4)=106.
      A(2,5)=-19.
      A(N,N-4)=-19.
      A(N,N-3)=106.
      A(N,N-2)=-264.
      A(N,N-1)=646.
      A(N,N)=251.
      A(N-1,N-4)=11.
      A(N-1,N-3)=-74.
      A(N-1,N-2)=456.
      A(N-1,N-1)=346.
      A(N-1,N)=-19.
      J=N-2
      DO 25 I=3,J
      A(I,I-2)=-19.
      A(I,I-1)=346.
      A(I,I)=456.
      A(I,I+1)=-74.
      A(I,I+2)=11.
 25   CONTINUE
      DO 30 I=1,N
      DO 30 J=1,N
      C(I,J)=0.0
      DO 30 K=1,N
      C(I,J)=C(I,J)+(B(I,K))*(A(K.J)*H)
 30   CONTINUE
      DO 35 I=1,N
      Y(I,1)=0.0
      DO 40 J=1,N
 40   Y(I,1)=Y(I,1)+((C(I,J))*(F(J,1)))
 35   CONTINUE
      IF (M.EQ.1) GO TO 45
      M=M-1
      DO 65 K=1,M
      DO 50 L=1,N
 50   F(L,1)=Y(L,1)
      DO 55 I=1,N
      Y(I,1)=0.0
      DO 60 J=1,N
 60   Y(I,1)=Y(I,1)+(C(I,J)*F(J,1))
 55   CONTINUE
 65   CONTINUE
```

```
 45   DO 80 I=1,N
 80   Y(I,1)=ABS(Y(I,1))
      RETURN
      END
```

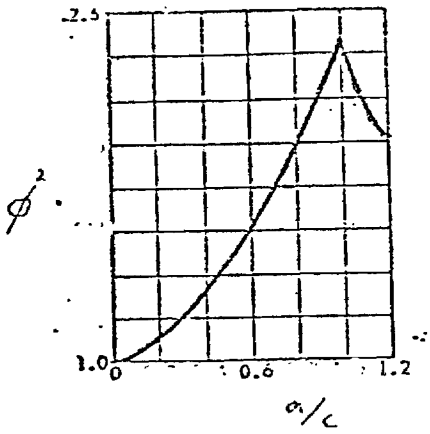

Figure No. 1. Variation of Shape Factor with a/c

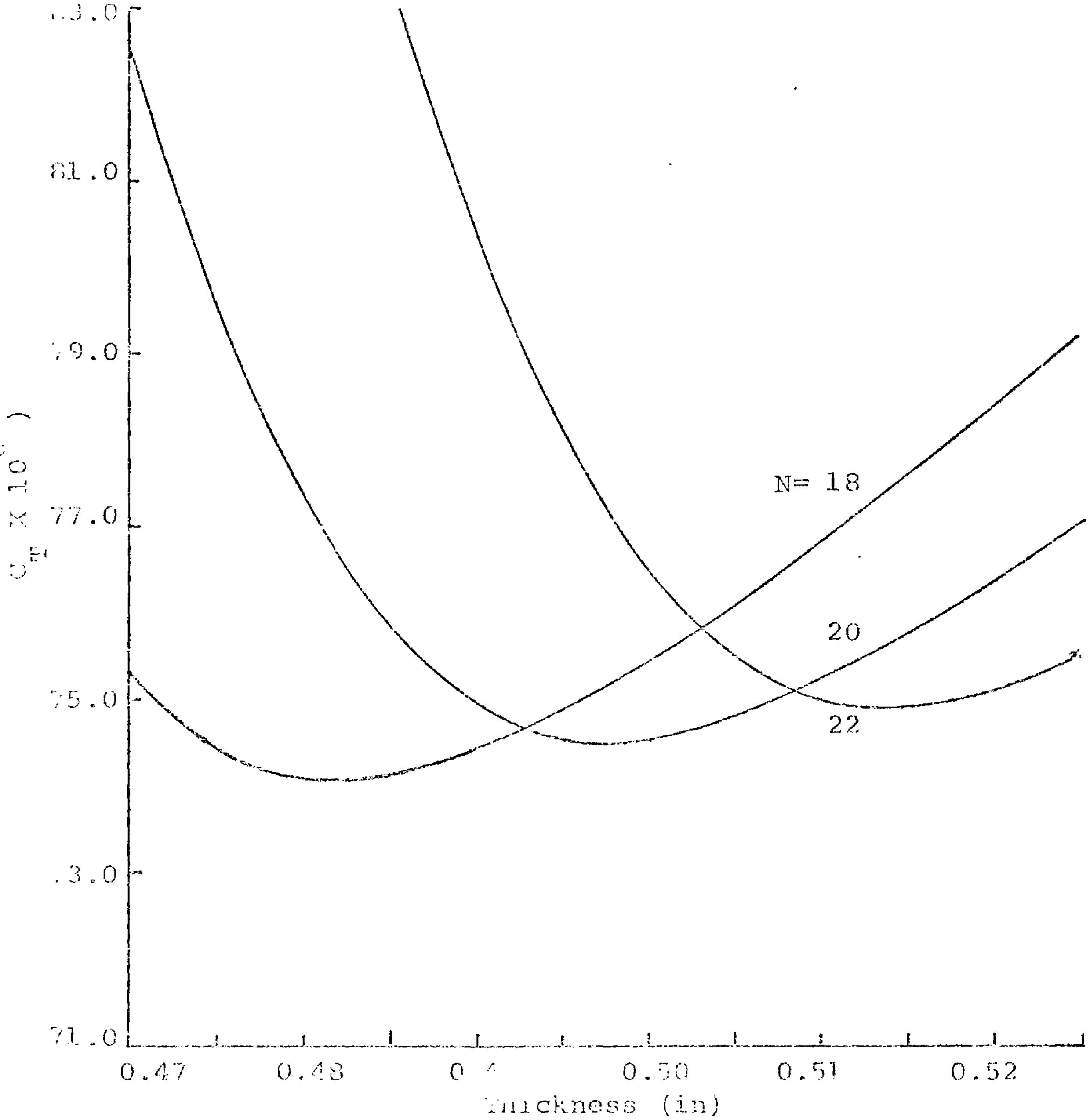

Figure 2. Optimization curves

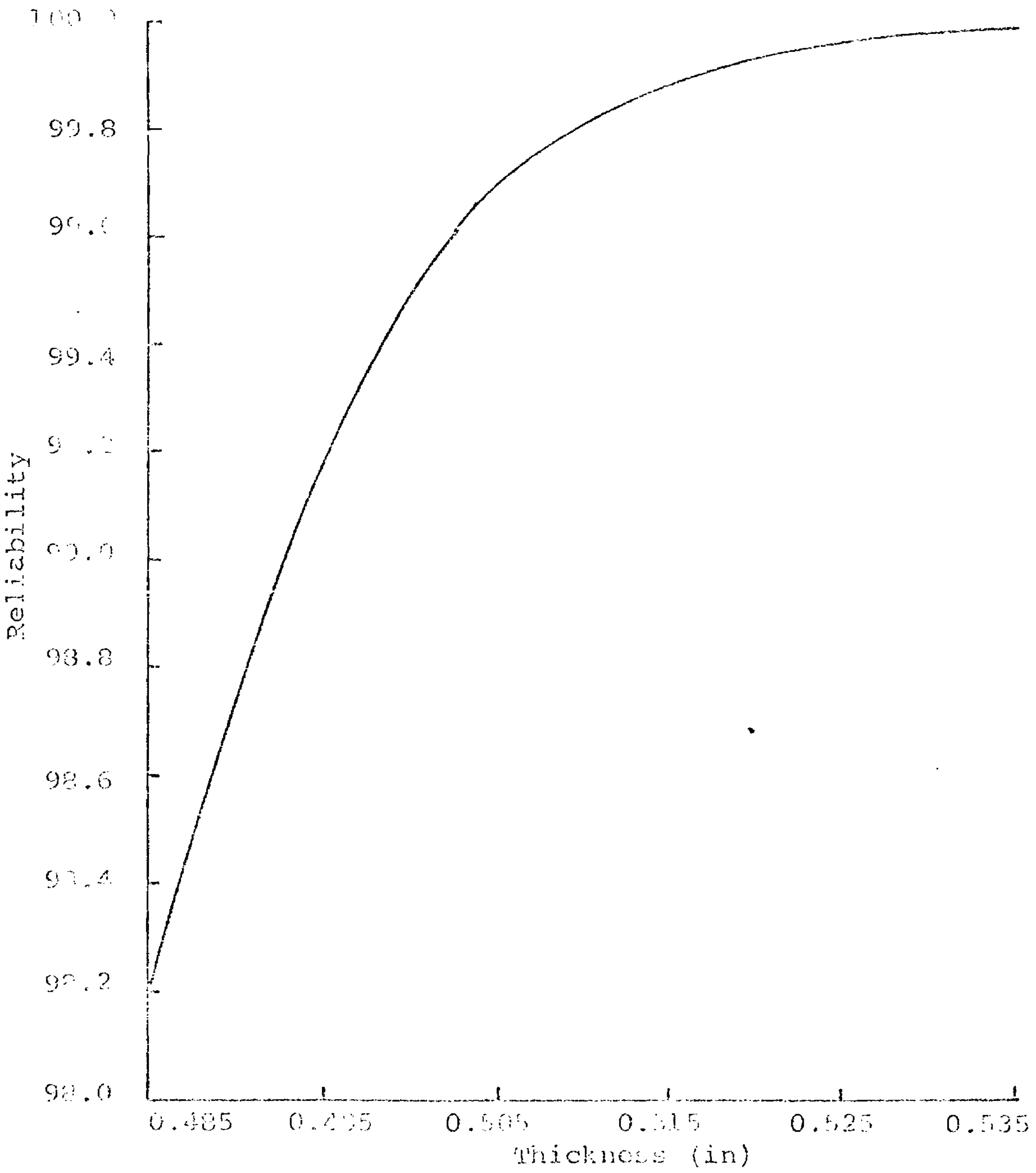

Figure 3. Reliability After 20 Launch Cycles