Density Conditions on Gabor Frames

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To Mom, Dad, and Nate

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SUMMARY

Frames in a Hilbert space are a set of elements similar to bases, for they are complete, but frames are not necessarily minimal sets. With frames, elements have unconditionally convergent expansions just as with bases. In this paper, we discuss the basic idea of frames as well as some of their properties and advantages. We also discuss Riesz bases and their relationship to frames. We then move our focus to Gabor frames and their properties, mainly concerned with a density condition for Gabor frames. Our goal is to show that in order for a Gabor system to be a frame, it must possess finite upper Beurling density and a lower Beurling density of at least 1. As a consequence, we show that a system consisting of translations of a finite set of functions cannot be a frame. Not original to the author, these results are based on the paper by Christensen, Deng, and Heil.

CHAPTER I

INTRODUCTION

In a Hilbert space, the idea of a basis is important, for bases yield many powerful tools critical to the study of Hilbert spaces. One such tool is the fact that any element in the space has an unconditionally convergent representation in terms of a basis. While a basis provides important information, the criteria to be a basis is high, generating a problem with finding bases or retrieving bases from overcomplete sets. In recent studies, many have found that this overcompleteness we work to remove potentially provides benefits rather than hindrance. This is the idea of frames. Frames provide some of the same tools given by bases, such as an unconditionally convergent representation, yet frames do not require linear independence or uniqueness of vectors. The overcompleteness can also prove useful, for it allows frames to be less sensitive to perturbations or manipulations.

In this paper, we will give a general overview of frames. The first formal introduction of frames was by Duffin and Schaeffer. The addition of the papers from Young and from Daubechies, Grossmann, and Meyer helped form the fundamentals that have allowed frame theory to expand to what it is today. We will only discuss some of the basics from this theory and then will expand to other ideas such as Riesz bases, Gabor systems, and Gabor frames. In the Gabor systems and Gabor frames, we generate the systems by taking a set of modulations and translations of a finite set of functions. This is also the idea of wavelets, except we take translations and dilations of a mother wavelet to generate the wavelet system. Under the proper conditions, Gabor frames and wavelets give us the ability to reconstruct our space in terms of a finite set of functions.

Allowing us to represent functions in terms of frame expansions, our interest is in the density conditions on Gabor frames. We will show that Gabor frames must have finite Beurling density, with an even stronger condition that its lower density must at least be 1. In addition, these density results render the consequence that on $\mathbb{R}^d$, systems generated by translations of a finite set of functions cannot be frames. These ideas and results presented are mainly those from the paper "Density of Gabor Frames" by Christensen, Deng, and Heil, who credit much of their inspirations to Ramanathan and Steger's paper "Incompleteness of Sparse Coherent States."

In Chapter II we begin with definitions and theorems that will be used throughout the paper. In Chapter III we introduce basic definitions and ideas for frames and Riesz bases and then follow with the more specific topic of Gabor frames in Chapter IV. For Chapter V we discuss Beurling density, giving some examples as well as a lemma that relates an idea of separation to that of Beurling density. In Chapter VI we present two major theorems, the Homogeneous Approximation Property (HAP) and the Comparison Theorem, and follow with the statement and proof of the main theorem on the density of Gabor frames. Finally, in Chapter VII we show that frames of translates on $\mathbb{R}^d$ cannot exist.

CHAPTER II

BACKGROUND, NOTATION, AND OTHER TOOLS

In this section we briefly state some common theorems and definitions typically used in real or time-frequency analysis. For a more complete source, see [8].

We denote the Euclidean norm on $\mathbb{R}^d$ by $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$. The dot product is $x \cdot y = x_1 y_1 + \cdots + x_d y_d$.

For $h > 0$, define $Q_h(x) = \Pi_{i=1}^d [x_i - \frac{h}{2}, x_i + \frac{h}{2}]$, so that $Q_h(x)$ is the cube centered at $x \in \mathbb{R}^d$ with sidelengths h.

H will always denote a separable Hilbert space. $L^2(\mathbb{R}^d)$ is the Hilbert space of functions such that $\|f\|_2 = (\int_{\mathbb{R}^d} |f(x)|^2)^{1/2} < \infty$. The L^2-inner product is $\langle f, g \rangle = \int f(x)\overline{g(x)}dx$.

A sequence $\{x_n\}_{n \in I} \in H$ is a basis for H if for all $x \in H$ there exist unique scalars $c_n(x)$ such that $x = \sum_{n \in I} c_n(x) x_n$, where I is some indexing set. A basis is unconditional if $\sum_{n \in I} c_n(x) x_n$ converges unconditionally, i.e., it converges for any ordering of the index set. A basis is bounded if $0 < \inf \|x_n\| \leq \sup \|x_n\| < \infty$.

For H and K Hilbert spaces, the norm of a linear operator $L : H \longrightarrow K$ is defined by $\|L\| = \sup_{\|x\|_H = 1} \|Lx\|_K$. L is a topological isomorphism if it is a continous bijection of H onto K.

For a closed subspace $V \subset L^2(\mathbb{R}^d)$, the distance between any function $f \in L^2(\mathbb{R}^d)$ and V is

$$\mathit{distance}(f, V) = \inf_{u \in V} \|f - u\|_2 = \|f - P_V f\|_2,$$

where P_V is the orthogonal projection operator onto V.

Theorem 2.1 *Let $x, y \in H$ and $T : H \longrightarrow H$ be a positive operator.*

a. $|\langle x, y \rangle| \leq \|x\| \, \|y\|$ *(Cauchy-Schwarz Inequality).*

b. $|\langle Tx, y \rangle|^2 \leq \langle Tx, x \rangle \langle Ty, y \rangle$ *(Generalized Cauchy-Schwarz).*

Definition 2.2 *The* <u>*Fourier transform*</u> *of a function* $f \in L^1(\mathbb{R}^d)$ *is defined as*

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\omega}\, \mathrm{d}x.$$

We note that the Fourier Transform extends to all of $L^2(\mathbb{R}^d)$. To see this, take $f \in L^2(\mathbb{R}^d)$. Since $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, there exists a sequence $\{f_n\} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $f_n \longrightarrow f$ in $L^2(\mathbb{R}^d)$. Since $f_n(x) \in L^1(\mathbb{R}^d)$, $\hat{f}_n(\omega) = \int_{\mathbb{R}^d} f_n(x) e^{2\pi i\omega\cdot x}\, \mathrm{d}x$ exists. Now $\|\hat{f}_n - \hat{f}_m\|_2 = \|f_n - f_m\|_2 \to 0$, so $\{\hat{f}_n\}$ is Cauchy in $L^2(\mathbb{R}^d)$ and converges to some $g \in L^2(\mathbb{R}^d)$. Define $\hat{f} = g$.

Theorem 2.3 (Plancherel's theorem) *If* $f \in L^2(\mathbb{R}^d)$, *then*

$$\|f\|_2 = \|\hat{f}\|_2.$$

Definition 2.4 *For a fixed function* $g \neq 0$, *the* <u>*Short-Time Fourier Transform*</u> *of a function* f *with respect to* g *is defined as*

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t)\, \overline{g(t - x)} e^{-2\pi i t\cdot\omega} \mathrm{d}t \qquad \text{for } x, \omega \in \mathbb{R}^d.$$

For the context of this paper, f and g will be in $L^2(\mathbb{R}^d)$, so $V_g f$ will be well-defined on $\mathbb{R}^{2d}$.

Definition 2.5 *Let* $\{x_n\}_{n\in I}$ *be a sequence in a Hilbert space* H. *We say that a series* $\sum x_n$ *is* <u>*unconditionally convergent*</u> *if* $\sum x_n$ *converges for any rearrangement of the indices.*

Lemma 2.6 *Let* H *be a finite-dimensional Hilbert space, and* $B = \{b_1, \ldots, b_n\}$ *a basis for* H *with biorthogonal basis* $\tilde{B} = \{\tilde{b}_1, \ldots, \tilde{b}_n\}$. *If* $T : H \longrightarrow H$ *is linear, then* $trace(T) = \sum_{i=1}^{n} \langle Tb_i, \tilde{b}_i\rangle$.

Proof: We can write x in terms of the basis B and biorthogonal basis $\tilde{B}$ as

$$x = \sum_{i=1}^{n} \langle x, \tilde{b}_i\rangle b_i.$$

For the vector Tx we have

$$Tx = \sum_{i=1}^{n} \langle Tx, \tilde{b}_i\rangle\, b_i.$$

We can write the coordinates of the vectors x and Tx with respect to the basis B as

$$\begin{bmatrix} \langle x, \tilde{b}_1 \rangle \\ \vdots \\ \langle x, \tilde{b}_n \rangle \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \langle Tx, \tilde{b}_1 \rangle \\ \vdots \\ \langle Tx, \tilde{b}_n \rangle \end{bmatrix}.$$

By definition $[T]_B$ is the matrix such that

$$\forall x \in H, \quad [T]_B \, [x]_B = [Tx]_B.$$

If $x = b_1$ then

$$[T]_B \, [b_1]_B = [Tb_1]_B = \begin{bmatrix} \langle Tb_1, \tilde{b}_1 \rangle \\ \vdots \\ \langle Tb_1, \tilde{b}_n \rangle \end{bmatrix}.$$

Since $[b_1]_B = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$, $[T]_B \, [b_1]_B$ is the first column of $[T]_B$. Continuing this substitution for $x = b_1, \ldots, b_n$, we see that

$$\begin{aligned} [T]_B &= \Big[[Tb_1]_B \;\; [Tb_2]_B \;\; \ldots \;\; [Tb_n]_B] \Big]_B \\ &= \begin{bmatrix} \langle Tb_1, \tilde{b}_1 \rangle & \ldots & \langle Tb_n, \tilde{b}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle Tb_1, \tilde{b}_n \rangle & \ldots & \langle Tb_n, \tilde{b}_n \rangle \end{bmatrix}. \end{aligned}$$

Since $trace(T)$ is the sum of the diagonal entries of $[T]_B$, we have

$$tr(T) = \sum_{i=1}^{n} \langle Tb_i, \tilde{b}_i \rangle. \qquad \square$$

CHAPTER III

RIESZ BASES AND FRAMES

As mentioned in the introduction, frames were first introduced in [7] and then more fully emerged in [16], [6], and [5]. For more information on frames and Riesz bases, see [3] as well as those articles previously mentioned. For this chapter, we give some basic ideas of frames, beginning with the definition.

Definition 3.1 *A collection of elements $\{x_n\}_{n\in I}$ in a Hilbert Space H, where I is a countable index set, is called a frame for H if there exist positive numbers A and B such that for all $x \in H$,*

$$A\|x\|^2 \leq \sum_{n\in I} |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

The constants A and B are called frame bounds, where A is a lower frame bound and B is an upper frame bound.

Some frames have special properties and are given specific names. Whenever A and B can be chosen to be equal, we call it a tight frame or an A-tight frame. Specifically, if $A = B = 1$, then it is a 1-tight frame or a Parseval frame. Also, if a frame fails to be a frame whenever any element is removed, then it is called an exact frame.

Lemma 3.2 *If $\{x_n\}$ is a frame, then it is also complete in H.*

Proof. Suppose there exists an $x \in H$ such that $\langle x, x_n\rangle = 0$ for all n. Since $\{x_n\}$ is a frame, we have that $A\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 = 0$. Since A is strictly positive, x must be 0. □

Lemma 3.3 *Every finite spanning set in a finite-dimensional Hilbert space is a frame.*

Proof. Let H be a finite-dimensional Hilbert space with dimension n, and suppose that the set $\{x_i\}_{i\in I}$ spans H, where $I = \{1, \ldots, m\}$ and $m \geq n$. Without loss of generality, assume that the first n vectors are linearly independent. Form an $n \times n$ matrix V from $\{x_1, \ldots, x_n\}$ by setting

$$V = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}.$$

Now, let c denote the vector

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

If we let $w = Vc$, then $w = Vc = \sum_{i=1}^{n} c_i v_i$. But if $w = Vc$, then $c = V^{-1}w$. Denote V^{-1} by U, where

$$U = \begin{bmatrix} \overline{u}_{1,1} & \ldots & \overline{u}_{1,n} \\ \vdots & \ddots & \vdots \\ \overline{u}_{n,1} & \ldots & \overline{u}_{1,1} \end{bmatrix}.$$

We then have that

$$c = Uw = \begin{bmatrix} \overline{u}_1 & \ldots & \overline{u}_n \end{bmatrix} \begin{bmatrix} w \end{bmatrix} = \begin{bmatrix} \langle w, u_1 \rangle \\ \vdots \\ \langle w, u_n \rangle \end{bmatrix},$$

and we have that $w = \sum_{i=1}^{n} \langle w, u_i \rangle v_i$.

Now computing the norm, we see that

$$\begin{aligned} \|w\|^2 = |\langle w, w \rangle| &= |\langle \sum_{i \in I} \langle w, u_i \rangle v_i, w \rangle| \\ &\leq \sum_{i \in I} |\langle w, u_i \rangle| \, |\langle v_i, w \rangle| \\ &\leq \Big(\sum_{i \in I} |\langle w, u_i \rangle|^2 \Big)^{1/2} \Big(\sum_{i \in I} |\langle w, v_i \rangle|^2 \Big)^{1/2} \\ &\leq \Big(\sum_{i \in I} \|u_i\|^2 \Big)^{1/2} \|w\| \Big(\sum_{i \in I} |\langle w, v_i \rangle|^2 \Big)^{1/2}. \end{aligned} \tag{1}$$

This implies that

$$\|w\|^4 \leq \sum_{i \in I} \|u_i\|^2 \|w\|^2 \sum_{i \in I} |\langle w, v_i \rangle|^2,$$

hence

$$\Big(\sum_{i \in I} \|u_i\|^2 \Big)^{-1} \|w\|^2 \leq \Big(\sum_{i \in I} |\langle w, v_i \rangle|^2 \Big).$$

Thus $A = (\sum_{i\in I} \|u_i\|^2)^{-1}$ is a positive number and hence is a lower frame bound for $\{x_i\}$.

To find an upper bound, use Cauchy-Schwarz on equation (1) to obtain

$$\sum_{i\in I} |\langle w, v_i\rangle|^2 \leq \sum_{i\in I} \|v_i\|^2 \|w\|^2.$$

Then $B = \sum_{i\in I} \|v_i\|^2$ is an upper frame bound for $\{x_i\}$. □

We now give the definition of a Riesz basis and follow with a theorem showing the equivalence of Riesz bases, bounded unconditional bases, and exact frames.

Definition 3.4 *Let $\{x_n\}_{n\in I}$ be a basis for H. $\{x_n\}_{n\in I}$ is a Riesz basis if there exists a topological isomorphism $T : H \longrightarrow H$ such that for all n, $Tx_n = y_n$, where y_n is some orthonormal basis for H.*

Lemma 3.5 *Every basis in a finite-dimensional Hilbert space is a Riesz basis.*

Proof. Let H be a finite-dimensional Hilbert space, and $\{x_i\}_{i=1}^n$ be a basis for H. Since we have a basis, we have a biorthogonal basis $\{\tilde{x}_i\}_{i=1}^n$. Also, since $\{x_i\}$ is a basis, by Lemma (3.3) it is also a frame. We claim that $\{x_i\}_{i=1}^n$ is an exact frame. For a fixed i, $\langle x_i, \tilde{x}_i\rangle = 1$ and $\langle x_j, \tilde{x}_i\rangle = 0$ for $j \neq i$. This implies that $\tilde{x}_i$ is orthogonal to the closed linear span of $\{x_j\}_{j\neq i}$. This shows that $\{x_j\}_{j\neq i}$ is not complete, hence, it is not a frame. We have shown that $\{x_i\}_{i=1}^n$ is an exact frame, or equivalently, a Riesz basis. □

Lemma 3.6 *If $\{x_n\}_{n\in I}$ is a frame for H with upper frame bound B, then for any $x \in H$,*

$$\Big\|\sum_{n\in I} \langle x, x_n\rangle x_n\Big\|^2 \leq B \sum_{n\in I} |\langle x, x_n\rangle|^2.$$

Proof: Without loss of generality, assume that $I = \mathbb{N}$. Let F be a finite subset of $\mathbb{N}$. Then we have

$$\begin{aligned}
\left\|\sum_{n\in F}\langle x, x_n\rangle\right\|^2 &= \sup_{\|y\|=1}\left|\left\langle\sum_{n\in F}\langle x, x_n\rangle x_n,\, y\right\rangle\right|^2 \\
&= \sup_{\|y\|=1}\left|\sum_{n\in F}\langle x, x_n\rangle\langle x_n, y\rangle\right|^2 \\
&= \sup_{\|y\|=1}\left|\left\langle\{\langle x, x_n\rangle\}_{n\in I}, \{\langle y, x_n\rangle\}_{n\in N}\right\rangle_{\ell^2}\right|^2 \\
&\leq \sup_{\|y\|=1}\|\langle x, x_n\rangle\|^2\,\|\langle y, x_n\rangle\|^2 \qquad \text{(by Cauchy-Schwarz)} \\
&= \sup_{\|y\|=1}\sum_{n\in F}|\langle x, x_n\rangle|^2\sum_{n\in F}|\langle x_n, y\rangle|^2 \\
&\leq B\sup_{\|y\|=1}\sum_{n\in F}|\langle x, x_n\rangle|^2\,\|y\|^2 \qquad \text{(by definition of frame)} \\
&= B\sum_{n\in F}|\langle x, x_n\rangle|^2.
\end{aligned}$$

But $\sum|\langle x, x_n\rangle|^2$ is an unconditionally convergent series of real numbers, so it follows that $\sum_{n\in F}\langle x, x_n\rangle x_n$ converges unconditionally in H. In particular, taking $F = \{1, \ldots, N\}$ and letting N approach infinity, we obtain $\left\|\sum_n\langle x, x_n\rangle x_n\right\|^2 \leq B\sum_n\|\langle x, x_n\rangle\|^2$. □

An important operator associated with a frame is the frame operator $S : H \longrightarrow H$, defined by

$$Sx = \sum_{n\in I}\langle x, x_n\rangle x_n.$$

Note that Sx converges unconditionally by the previous lemma.

Let us show that for any frame $\{x_n\}$ in H, the frame operator S is a bounded, linear, and positive definite operator from H to H. S is linear since

$$\begin{aligned}
S(x+y) &= \sum_{n\in I}\langle x+y, x_n\rangle x_n \\
&= \sum_{n\in I}\Big(\langle x, x_n\rangle + \langle y, x_n\rangle\Big) x_n \\
&= \sum_{n\in I}\langle x, x_n\rangle x_n + \sum_{n\in I}\langle y, x_n\rangle x_n \\
&= Sx + Sy
\end{aligned}$$

and

$$\begin{aligned} S\alpha x &= \sum_{n\in I} \langle \alpha x, x_n \rangle x_n \\ &= \sum_{n\in I} \alpha \langle x, x_n \rangle x_n \\ &= \alpha \sum_{n\in I} \langle x, x_n \rangle x_n \\ &= \alpha S x. \end{aligned}$$

We will show S is continuous by showing that S is the composition of two continuous operators, the analysis and synthesis operator. The analysis operator, denoted C, is the operator from H to ℓ^2 defined by

$$Cx = \{\langle x, x_n \rangle\}_{n\in I}, \qquad x \in H.$$

C is bounded since

$$\|Cx\|^2 = \sum_{n\in I} |\langle x, x_n \rangle|^2 \leq B\|x\|^2,$$

where B is the upper frame bound for $\{x_n\}$. In particular, $\|C\| \leq B^{1/2}$.

The synthesis operator, denoted D, is the operator from ℓ^2 to H defined by

$$Dc = \sum_{n\in I} c_n x_n.$$

To show that D is bounded, we first need to know that $\sum_{n\in I} c_n x_n$ converges unconditionally. To see this, choose an ordering of I so that $I = \{n_1, n_2, ...\}$ and let S_L denote the partial sum up to L, so $S_L = \sum_{k=1}^{L} c_{n_k} x_{n_k}$. Since $c \in \ell^2$, $\sum_{k\in\mathbb{N}} |c_{n_k}|^2$ is finite. For $\epsilon \geq 0$ there is an $N_0 \geq 0$ such that if $k \geq N_0$, then $\sum_{k\geq N_0} |c_{n_k}|^2 \leq \epsilon$. Without loss of generality, let

$L \geq M \geq N_0$. Then we have

$$\begin{aligned}
\|S_L - S_M\|^2 &= \Big| \sum_{k=M+1}^{L} c_{n_k} x_{n_k} \Big|^2 \\
&= \sup_{\|x\|=1} \Big| \Big\langle \sum_{k=M+1}^{L} c_{n_k} x_{n_k}, x \Big\rangle \Big|^2 \\
&= \sup_{\|x\|=1} \Big| \sum_{k=M+1}^{L} \langle c_{n_k} x_{n_k}, x \rangle \Big|^2 \\
&\leq \sup_{\|x\|=1} \Big(\sum_{k=M+1}^{L} |c_{n_k}|^2 \Big) \Big(\sum_{n=M+1}^{L} |\langle x_{n_k}, x \rangle|^2 \Big) \\
&\leq \sup_{\|x\|=1} B\|x\|^2 \sum_{k=M+1}^{L} |c_{n_k}|^2 \leq B\epsilon.
\end{aligned}$$

This implies that the series is Cauchy with respect to any ordering of $\mathbb{N}$, hence the series defining Dc converges unconditionally.

By a calculation similar to the above we see that

$$\|Dc\|^2 = \Big\| \sum_{n \in I} c_n x_n \Big\|^2 \leq B\|c\|^2,$$

showing that D is bounded, and in fact $\|D\| \leq B^{1/2}$.

Since D and C are both bounded, then

$$DCx = D(\{\langle x, x_n \rangle\}_{n \in I}) = \sum_{n \in I} \langle x, x_n \rangle x_n = Sx,$$

and the frame operator S must be bounded as well.

While the boundedness of D and C help us to prove the boundedness of S, it is also important to note that D and C are adjoints of each other. We need this fact in order to prove that the frame operator is positive definite. To see that D and C are adjoints, we find that for all $x \in H$ and $c \in \ell^2$,

$$\begin{aligned}
\langle Dc, x \rangle &= \Big\langle \sum_{n \in I} c_n x_n, x \Big\rangle \\
&= \sum_{n \in I} c_n \langle x_n, x \rangle \\
&= \Big\langle c, \{\langle x, x_n \rangle\}_{n \in I} \Big\rangle_{\ell^2} \\
&= \langle c, Cx \rangle,
\end{aligned}$$

thus $C = D^*$. Now we can show that S is positive definite. First note that for $x \in H$,

$$\begin{aligned}
\langle Sx, x \rangle &= \left\langle \sum_{n \in I} \langle x, x_n \rangle x_n, x \right\rangle \\
&= \sum_{n \in I} \langle x, x_n \rangle \langle x_n, x \rangle \\
&= \sum_{n \in I} |\langle x, x_n \rangle|^2.
\end{aligned}$$

Then if $\langle Sx, x \rangle = 0$, this implies that

$$A\|x\|^2 \le \sum_{n \in I} |\langle x, x_n \rangle|^2 = \langle Sx, x \rangle = 0.$$

Since A is strictly positive, x must be 0, and it follows that S is positive definite.

We also see from this short computation that S is invertible. We have that

$$A\|x\|^2 \le \langle Sx, x \rangle \le \|Sx\| \|x\|.$$

If $Sx = 0$, then $\|Sx\| = 0$ which implies that $A\|x\| \le 0$. Since A is strictly positive, x must be 0, and S is injective. To see that S is surjective, first we will show that Range(S) is a closed subspace. Suppose $x_j \longrightarrow x \in H$, and $Sx_j \longrightarrow z \in H$. Since S is continuous, the Closed Graph Theorem yields $z = Sx$, so the range(S) is closed.

Now, suppose there is a $g \in H$ such that $g \perp$ range(S). Then for all $f \in H$, $\langle Sf, g \rangle = 0$, so $\langle Sg, g \rangle = 0$. It follows that $g = 0$, and Range(S) is dense in H. Now we have that $\text{Range}(S) = \overline{\text{Range}(S)} = H$, hence S is surjective.

We summarize the above discussion in the following theorem:

Theorem 3.7 *The frame operator $S : H \longrightarrow H$ has the following properties:*

a. S is linear.

b. S is bounded.

c. S is positive definite.

d. S is bijective, thus invertible.

Theorem 3.8 *If $\{x_n\}_{n\in I}$ is a frame for H, then $\{S^{-1}x_n\}_{n\in I}$ is also a frame, called the* *dual frame. The dual frame has lower frame bound $\frac{1}{B}$ and upper frame bound $\frac{1}{A}$, where A and B are lower and upper frame bounds for $\{x_n\}_{n\in I}$ respectively.*

Proof. We can see S^{-1} is positive since $0 \leq A\|S^{-1}x\|_2^2 \leq \langle S(S^{-1}x), (S^{-1}x)\rangle = \langle x, S^{-1}x\rangle \leq \|x\|\,\|S^{-1}x\|$. This implies that $\|S^{-1}x\| \leq \frac{1}{A}\|x\|$. Then since $\langle Sx, x\rangle = \sum_{n\in I}|\langle x, x_n\rangle|^2$ for every x, we have

$$\begin{aligned}
\sum_{n\in I}|\langle x, S^{-1}x_n\rangle|^2 &= \sum_{n\in I}|\langle S^{-1}x, x_n\rangle|^2 \\
&= \langle S(S^{-1}x), S^{-1}x\rangle \\
&= \langle x, S^{-1}x\rangle \leq \|x\|\,\|S^{-1}x\| \\
&\leq \frac{1}{A}\|x\|^2,
\end{aligned}$$

and $\frac{1}{A}$ is an upper frame bound. To find the lower frame bound, we compute

$$\begin{aligned}
\|x\|^4 &= (\langle x, x\rangle)^2 \\
&= |\langle S^{-1}(Sx), x\rangle|^2 \\
&\leq \langle S^{-1}Sx, Sx\rangle\langle S^{-1}x, x\rangle \quad \text{(by generalized Cauchy-Schwarz)} \\
&= \langle x, Sx\rangle\,\langle x, S^{-1}x\rangle \\
&\leq B\|x\|^2\langle S(S^{-1}x), S^{-1}x\rangle \\
&= B\|x\|^2\sum_{n\in I}\left|\langle x, S^{-1}x_n\rangle\right|^2.
\end{aligned}$$

Dividing both sides by $B\,\|x\|^2$, we have that $\frac{1}{B}$ is a lower frame bound. □

We end this chapter with the following theorem on relationships between frames and Riesz bases.

Theorem 3.9 *Let $\{x_n\}_{n\in I}$ be a sequence in H. The following are equivalent:*

a. $\{x_n\}$ is a Riesz basis.

b. $\{x_n\}$ is a bounded unconditional basis.

c. $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal.

d. $\{x_n\}$ *is an exact frame.*

Proof. *a.* $\Rightarrow$ *b.* Suppose $\{x_n\}$ is a Riesz basis for H. Then there exists an orthonormal basis $\{e_n\}_{n\in I} \in H$ and a topological isomorphism $T : H \longrightarrow H$ such that $x_n = Te_n$ for all n. Since $\{e_n\}$ is an orthonormal basis, it is a bounded unconditional basis. Bounded unconditional bases are preserved under topological isomorphism (for a proof, see [11]), and it follows that $\{x_n\}$ is a bounded unconditional basis.

b. $\Rightarrow$ *c.* Suppose $\{x_n\}$ is a bounded unconditional basis, with biorthogonal basis $\{y_n\}$. For any $x \in H$, we can write $x = \sum_{n\in I} \langle x, x_n\rangle\, y_n$. We can also write x in terms of the frame operator as $x = S^{-1}Sx = S^{-1}\sum_{n\in I}\langle x, x_n\rangle\, x_n = \sum_{n\in I}\langle x, x_n\rangle\, S^{-1}x_n$. The uniqueness of basis expansions implies that $S^{-1}x_n = y_n$, and so $\{S^{-1}x_n\}$ is biorthogonal to $\{x_n\}$.

c. $\Rightarrow$ *d.* Suppose $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal. For some fixed m, take the sequence $\{x_n\}_{n\neq m}$. Since $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal, $\langle x_n, S^{-1}x_m\rangle = 0$ for $n \neq m$. But since $S^{-1}x_m \neq 0$, this implies that $\{x_n\}_{n\neq m}$ is not complete. By Lemma 3.2, $\{x_n\}_{n\neq m}$ is not a frame, so $\{x_n\}$ is exact.

d. $\Rightarrow$ *a.* Suppose $\{x_n\}$ is an exact frame. We know that S^{-1} exists, and so $S^{-\frac{1}{2}}$ also exists. We claim that $\{S^{-\frac{1}{2}}x_n\}$ is an orthonormal basis. First note that $\{x_n\}$ being an exact frame implies that it is biorthogonal to $S^{-1}x_n$, and we appeal to [12] for a proof. Then we have $\langle S^{-\frac{1}{2}}x_m, S^{-\frac{1}{2}}x_n\rangle = \langle x_m, S^{-1}x_n\rangle = \delta_{mn}$, and so $\{S^{-\frac{1}{2}}x_n\}$ is orthonormal. We know that since $\{x_n\}$ is a frame, it is complete. We also know that since $\{x_n\}$ is exact, it is also minimal. Since $S^{-\frac{1}{2}}$ is a topological isomorphism, it preserves minimality and completeness. It follows that $\{S^{-\frac{1}{2}}x_n\}$ is an orthonormal basis, and since $S^{-\frac{1}{2}}$ is a topological isomorphism, $\{x_n\}$ is a Riesz basis. $\square$

CHAPTER IV

GABOR SYSTEMS AND GABOR FRAMES

In this chapter we give an abbreviated description of Gabor systems and Gabor frames. The study of Gabor systems is quite extensive due to their highly effective applications. Although we only briefly discuss these systems, one can see [10] for a more detailed examination.

On $L^2(\mathbb{R}^d)$ we have the translation and modulation operators. For $a, b \in \mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$, let T_a denote the translation operator $T_a(g) = g(x-a)$, and M_b denote the modulation operator $M_b(g) = e^{2\pi i b\cdot x} g(x)$. These two operators complement each other, and their composition is called a time-frequency shift. While translation yields a shift in time, modulation yields a shift in frequency. We can see this by taking the Fourier transform of $M_b g$:

$$\begin{aligned}
(M_b g)^\wedge(\omega) &= \int_{\mathbb{R}^d} g(x)\, e^{2\pi i b\cdot x}\, e^{2\pi i w\cdot x}\, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} g(x)\, e^{2\pi i(\omega - b)\cdot x}\, \mathrm{d}x \\
&= \hat{g}(w-b) \\
&= (T_b g)^\wedge(\omega).
\end{aligned}$$

Since the modulation and translation operators act as shifts in the time or frequency domains, we will define a time-frequency shift as a composition of these two operators. A time-frequency shift is written as $M_b T_a g(x) = e^{2\pi i b\cdot x} g(x-a)$. We also find that $T_a M_b g(x) = e^{-2\pi i a\cdot x} M_b T_a g(x)$, for modulation and translation almost commute in such a way that switching the order of operation merely results in an added phase factor. We see this with the following computation:

$$
\begin{aligned}
T_a M_b g(x) &= T_a(M_b g)(x) \\
&= M_b g(x-a) \\
&= e^{2\pi i b\cdot(x-a)} g(x-a) \\
&= e^{-2\pi i a\cdot x} e^{2\pi i b\cdot x} g(x-a) \\
&= e^{-2\pi i a\cdot x} M_b T_a g(x).
\end{aligned}
$$

Having defined time-frequency shifts, we note that we can write the Short-Time Fourier Transform $V_g f$ as an inner product of f against time-frequency shifts of g, for

$$
\begin{aligned}
V_g f(x,\omega) &= \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i t\cdot\omega} \mathrm{d}t \\
&= \int_{\mathbb{R}^d} f(t)\, \overline{M_\omega T_x g}\, \mathrm{d}t \\
&= \langle f, M_\omega T_x g \rangle.
\end{aligned}
$$

Also, the continuity of M_b and T_a allow the Short-Time Fourier Transform to be uniformly continuous. We see this in the following lemma.

Lemma 4.1 *For $f, g \in L^2(\mathbb{R}^d)$, $V_g f$ is uniformly continuous on $\mathbb{R}^{2d}$.*

Proof. First we show that the translation and modulation operators are continuous. For continuity of translation, fix $\epsilon > 0$, and let a really simple function g be defined as $g = \sum_{i=1}^{N} c_i \chi_{Q_i}$, where Q_i is a cube of the form $\Pi_{j=1}^{d}[a_j, b_j]$, Pick a really simple function g such that $\|f-g\|_2 < \frac{\epsilon}{3}$ and $\|T_x g - g\|_2 < \frac{\epsilon}{3}$. Then

$$
\begin{aligned}
\|T_x f - f\|_2 &= \|T_x f - T_x g + T_x g - g + g - f\|_2 \\
&\leq \|T_x f - T_x g\|_2 + \|T_x g - g\|_2 + \|g - f\|_2 \\
&\leq \|T_x\| \|f - g\|_2 + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}
$$

To see that modulation is continuous, we use Plancherel's theorem and find that

$$
\lim_{\omega \to 0} \|M_\omega f - f\|_2 = \lim_{\omega \to 0} \|T_\omega \hat{f} - \hat{f}\|_2 = 0. \quad \square
$$

Having defined time-frequency shifts, we can now state the definition of a Gabor frame.

Definition 4.2 *For $g \in L^2(\mathbb{R}^d)$, and $a, b \in \Lambda$, where Λ is some indexing set contained in $\mathbb{R}^{2d}$, the set of time-frequency shifts of g, denoted*

$$G(g, \Lambda) = \{e^{2\pi i b\cdot x} g(x-a)\}_{a,b\in\Lambda},$$

is called a Gabor system. If the Gabor system is also a frame, then it is called a Gabor frame.

Example 4.3 *$G(\chi_{Q_1(0)}, \mathbb{Z}^{2d})$ is a Gabor frame for $L^2(\mathbb{R}^d)$. More precisely, it is an orthonormal (hence Riesz) basis for $L^2(\mathbb{R}^d)$.*

Proof. We will show that $G(\chi_{Q_1(0)}, \mathbb{Z}^{2d})$ is an orthonormal basis, for then by Theorem 3.9, $G(\chi_{Q_1(0)}, \mathbb{Z}^{2d})$ is also a frame. Let $a, b, c, d \in \mathbb{Z}^d$, and look at M_bT_af, $M_dT_cf \in G(\chi_{Q_1(0)}, \mathbb{Z}^d)$. If $a \neq c$, the two functions have disjoint support, hence $\langle M_bT_af, M_dT_cf\rangle = 0$. If $a = c$ and $b \neq d$, then we have

$$\begin{aligned}\langle M_bT_af, M_dT_cf\rangle &= \int_{\chi_{Q_1(a)}} e^{2\pi i b\cdot x}\, e^{-2\pi i d\cdot x}\, \mathrm{d}x \\ &= \int_{a_n-1/2}^{a_n+1/2} \cdots \int_{a_1-1/2}^{a_1+1/2} e^{2\pi i(b-d)\cdot x}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n.\end{aligned}$$

For $k = 1, \ldots, d$,

$$\begin{aligned}\int_{a_k-1/2}^{a_k+1/2} e^{2\pi i(b_k-d_k)\cdot x_k}\, \mathrm{d}x_k &= \frac{e^{2\pi i(b_k-d_k)\cdot(a_k+1/2)} - e^{-2\pi i(b_k-d_k)\cdot(a_k-1/2)}}{2\pi i(b_k-d_k)} \\ &= \frac{e^{\pi i(b_k-d_k)}\, 2i\sin(2\pi(b_k-d_k)a_k)}{2\pi(b_k-d_k)} \\ &= 0 \qquad \text{since } a_k \in \mathbb{Z},\end{aligned}$$

thus $\int_{a-1/2}^{a+1/2} e^{2\pi i(b-d)\cdot x}\, \mathrm{d}x = 0$. If $a = c$ and $b = d$, then

$$\langle M_bT_af, M_dT_cf\rangle = \int_{\chi_{Q_1(a)}} |e^{2\pi i b\cdot x}|^2\, \mathrm{d}x = \int_{Q_1(a)} 1\, \mathrm{d}x = 1.$$

To see that $G(\chi_{Q_1(0)}, \mathbb{Z}^d)$ is a basis, suppose

$$\begin{aligned}0 = \langle g, T_aM_bf\rangle &= \langle T_{-a}f, M_b\chi_{Q_1(a)}\rangle \\ &= \int_{\mathbb{R}^d} T_{-a}f(x)\chi_{Q_1(a)}(x)e^{-2\pi i b\cdot x}\mathrm{d}x \\ &= \langle T_{-a}f \cdot \chi_{Q_1(a)}, e^{2\pi i b\cdot x}\rangle\end{aligned}$$

We know that the exponentials are an orthonormal basis for $L^2(Q_1(a))$. This implies that $fT_{-a} \cdot \chi_{Q_1(a)} = 0$. This is true for all $a \in \mathbb{Z}^d$, thus $f = 0$, and so $G(\chi_{Q_1(0)}, \mathbb{Z}^d)$ must be a basis. $\square$

We note that in general, the dual frame of a Gabor frame is not a Gabor frame. However, if $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, then the dual frame of a Gabor frame is indeed a Gabor frame. To see this, we must first show that the frame operator S commutes with the translation and modulation operators T_a and M_b. With translation we have

$$\begin{aligned} S(T_{aj}f) &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle T_{aj}f, T_{ak}M_{bn}g\rangle T_{ak}M_{bn}g \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{ak-aj}M_{bn}g\rangle T_{ak}M_{bn}g \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{ak}M_{bn}g\rangle T_{ak+aj}M_{bn}g \\ &= T_{aj}\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{ak}M_{bn}g\rangle T_{ak}M_{bn}g \\ &= T_{aj}Sf. \end{aligned}$$

For modulation we have

$$\begin{aligned} S(M_{bl}f) &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle M_{bl}f, T_{ak}M_{bn}g\rangle T_{ak}M_{bn}g \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, M_{-bl}T_{ak}M_{bn}g\rangle T_{ak}M_{bn}g \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}e^{2\pi i(bl-ak)}\langle f, T_{ak}M_{b(n-l)}g\rangle T_{ak}M_{bn}g \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}e^{2\pi i(bl-ak)}\langle f, T_{ak}M_{bn}g\rangle T_{ak}M_{bn+bl}g \\ &= \sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}e^{2\pi i(bl-ak)}e^{-2\pi i(bl-ak)}\langle f, T_{ak}M_{bn}g\rangle M_{bl}T_{ak}M_{bn}g \\ &= M_{bl}\sum_{n\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\langle f, T_{ak}M_{bn}g\rangle T_{ak}M_{bn}g \\ &= M_{bl}Sf. \end{aligned}$$

Suppose $g = S^{-1}f$. Using the fact that S commutes with T_{ak}, we see that

$$ST_{ak}g = ST_{ak}S^{-1}f = T_{ak}SS^{-1}f = T_{ak}f.$$

Applying S^{-1} to both sides, we have $S^{-1}T_{ak}f = T_{ak}S^{-1}f$. The same argument is used to show that S^{-1} commutes with M_{bn}. Now when we compute the dual frame, we have

$S^{-1}(T_{ak}M_{bn}g) = T_{ak}M_{bn}(S^{-1}g)$, and it follows that the dual frame is also a Gabor frame, $G(S^{-1}g, a, b)$.

CHAPTER V

DENSITY

Suppose we have a sequence of points of $\mathbb{R}^d$, and we are concerned with how spread out or how tightly packed together the points may be to each other. If this is the case, then we need to find information about how dense the sequence is. The following definitions can help us understand and learn about density. In this chapter we discuss the well-established idea of Beurling density, and we appeal to [4] for additional details.

Definition 5.1 *Let $E = \{e_i\}_{i \in I}$ be an arbitrary sequence of points in $\mathbb{R}^d$. Then E is δ-uniformly separated if $\delta = \inf_{i \neq j} |e_i - e_j| > 0$, where δ is a positive number called the separation constant. E is relatively uniformly separated if it is formed by taking a finite union of E_k sequences that are each δ_k-uniformly separated.*

Definition 5.2 *Let E be an arbitrary sequence of points in $\mathbb{R}^d$. For $h > 0$ and $Q_h(x)$ a cube in $\mathbb{R}^d$ with side length h and center x, we define*

$$v^+(h) = \sup_{x \in \mathbb{R}^d} \#(E \cap Q_h(x)) \quad \textit{and} \quad v^-(h) = \inf_{x \in \mathbb{R}^d} \#(E \cap Q_h(x)).$$

For any cube with sidelength h, $v^+(h)$ tells us the largest number of points of E that can lie inside the cube, so in a sense it is measuring the largest "mass" of E within the cube. Similarly, $v^-(h)$ tells us the smallest number of points that we may fit in a cube, so $v^-(h)$ gives us a measure of the smallest mass. Since the cubes have sidelength h, h^d is the volume of $Q_h(x)$. A seemingly natural way to define density follows from our accounts of mass and volume.

Definition 5.3 *The upper Beurling density of E is*

$$D^+(E) = \limsup_{h \to \infty} \frac{v^+(h)}{h^d}.$$

The lower Beurling density of E is

$$D^-(E) = \liminf_{h\to\infty} \frac{v^-(h)}{h^d}.$$

If $D^+(E) = D^-(E)$, we say that E has uniform Beurling density.

Example 5.4 *The lattice $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ has uniform Beurling density $\frac{1}{\alpha\beta}$.*

First we note that $D(\alpha\mathbb{Z} \times \beta\mathbb{Z}) = D(\alpha\mathbb{Z})D(\beta\mathbb{Z})$. Looking at the one-dimensional lattice $\alpha\mathbb{Z}$, we find $D(\alpha\mathbb{Z}) = \frac{1}{\alpha}$. For $h > \alpha$, each $Q_h(h\alpha n)$ contains at most $([h/\alpha] + 1)$ points of $\alpha\mathbb{Z}$, where $[h/\alpha] = \max_{r\in\mathbb{Z}}\{r : r \leq h/\alpha\}$. Also, each $Q_h(h\alpha n)$ contains at least $([\frac{h}{\alpha}])$ points of $\alpha\mathbb{Z}$. Computing the densities, we have

$$D^-(\alpha\mathbb{Z}) = \liminf_{h\to\infty} \frac{[h/\alpha]}{h} = \frac{1}{\alpha},$$

$$D^+(\alpha\mathbb{Z}) = \limsup_{h\to\infty} \frac{[h/\alpha]+1}{h} = \frac{1}{\alpha},$$

and it follows that $D(\alpha\mathbb{Z}) = \frac{1}{\alpha}$. In a similar fashion we find that $D(\beta\mathbb{Z}) = \frac{1}{\beta}$. The proof for the higher dimensional case is similar.

We note that for any set E we have $0 \leq D^-(E) \leq D^+(E) \leq \infty$. Also, if E is a disjoint union of sequences $E_1, E_2, ..., E_k$, then $\#(E \cap Q_h(x)) = \sum_{j=1}^k \#(E_j \cap Q_h(x))$, hence

$$\begin{aligned}
D^+(E) &= \limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}^d}(\sum_{j=1}^k \#(E_j \cap Q_h(x)))}{h^d} \\
&\leq \limsup_{h\to\infty} \frac{\sum_{j=1}^k(\sup_{x\in\mathbb{R}^d} \#(E_j \cap Q_h(x)))}{h^d} \\
&= \sum_{j=1}^k \limsup_{h\to\infty} \left(\frac{v^+(h)}{h^d}\right) \\
&= \sum_{j=1}^k D^+(E).
\end{aligned}$$

With a similar computation for the lower side, we have

$$\sum_{j=1}^k D^-(E_j) \leq D^-(E) \leq D^+(E) \leq \sum_{j=1}^k D^+(E_j). \qquad (2)$$

Some of the inequalities of equation (2) may be strict. We give the following example.

Example 5.5 *Let $E = \mathbb{Z} \in \mathbb{R}$. Define $E_1 = \{n \in \mathbb{Z} : n < 0\}$ and $E_2 = \{n \in \mathbb{Z} : n \geq 0\}$. Then $E = E_1 \cup E_2$ and we have that*

$$\sum_{j=1}^{2} D^-(E_j) < D^-(E) = D^+(E) < \sum_{j=1}^{2} D^+(E_j)$$

Computing the densities of E_1 and E_2, we have that

$$D^-(E_1) = \liminf_{h\to\infty} \frac{\min_{x\in\mathbb{R}} \#(E_1 \cap Q_h(x))}{h} = 0,$$

$$D^-(E_2) = \liminf_{h\to\infty} \frac{\min_{x\in\mathbb{R}} \#(E_2 \cap Q_h(x))}{h} = 0,$$

$$D^+(E_1) = \limsup_{h\to\infty} \frac{\max_{x\in\mathbb{R}} \#(E_1 \cap Q_h(x))}{h} = \limsup_{h\to\infty} \frac{h+1}{h} = 1,$$

and

$$D^+(E_2) = \limsup_{h\to\infty} \frac{\max_{x\in\mathbb{R}} \#(E_2 \cap Q_h(x))}{h} = \limsup_{h\to\infty} \frac{h+1}{h} = 1.$$

Now computing the density for the integers, we have that

$$D^-(E) = \liminf_{h\to\infty} \frac{\min_{x\in\mathbb{R}} \#(E \cap Q_h(x))}{h} = \limsup_{h\to\infty} \frac{h}{h} = 1,$$

and

$$D^+(E) = \limsup_{h\to\infty} \frac{\max_{x\in\mathbb{R}} \#(E \cap Q_h(x))}{h} = \limsup_{h\to\infty} \frac{h+1}{h} = 1.$$

We see that

$$0 = D^-(E_1) + D^-(E_2) < D^-(E) = D^+(E) < D^+(E_1) + D^+(E_2) = 2.$$

The following lemma shows a relationship between separation and density, giving insight about sequences whose upper Beurling density is finite.

Lemma 5.6 *Let $E \subseteq \mathbb{R}^d$ be any arbitrary sequence of points. Then the following statements are equivalent:*

a. $D^+(E)$ is finite.

b. E is relatively uniformly separated.

c. For every $h > 0$, there exists an integer $N_h > 0$ such that

$$N_h = \sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn)) < \infty.$$

d. For some $h > 0$, there exists an integer $N_h > 0$ such that

$$N_h = \sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn)) < \infty.$$

Proof. *a.* $\Rightarrow$ *d.* Fix h, and suppose that $D^+(E)$ is finite. This implies that $\limsup_{h\to\infty} \frac{v^+(h)}{h^d} < \infty$. Then for some h, there exists an integer M_h such that for all $h > M_h$,

$$\frac{v^+(h)}{h^d} - D^+(E) < \epsilon.$$

$D^+(E)$ is finite, which implies that $v^+(h)$ is also finite. But $\sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn)) \leq v^+(h) = \sup_{x\in\mathbb{R}^d} \#(E \cap Q_h(x))$, implying that $\sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn)) < \infty$, and it follows that there is an N_h such that $N_h = \sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn))$.

d. $\Rightarrow$ *c.* Suppose there is some h_0 and some integer N_{h_0} such that $N_{h_0} = \sup_{n\in\mathbb{Z}^d} \#(E \cap Q_{h_0}(h_0 n)) < \infty$. For $h < h_0$, $\sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn)) \leq \sup_{n\in\mathbb{Z}^d} \#(E \cap Q_{h_0}(h_0 n))$. For $h > h_0$, each $Q_h(hn)$ is contained in the union of finitely many $Q_{h_0}(h_0 n)$, say at most K of them, each containing at most N_{h_0} points of E; hence $\sup_{n\in\mathbb{Z}^d} \#(E \cap Q_h(hn)) \leq KN_{h_0} < \infty$.

c. $\Rightarrow$ *b.* Suppose that there exists an $h > 0$ and $N_h > 0$ such that for any cube $Q_h(hn)$, the cube has at most N_h points of E. Now let $v_1, v_2, ..., v_{2^d}$ be the vertices of the unit cube $[0,1]^d$. Then define $B_k = \bigcup_{n\in Z_k} Q_h(hn)$ and $Z_k = (2\mathbb{Z})^d + v_k$. See Figure 1 for a graphical interpretation. Essentially, Z_k is partitioning $\mathbb{R}^d$ into a grid whose points are determined by the v_k vertices. The B_k form groups of cubes that correspond to each vertex, hence $\bigcup_{k=1}^{2d} B_k$ partitions $\mathbb{R}^d$. This implies that $\mathbb{R}^d$ is the disjoint union of the B_k's. Also, if $n, m \in Z_k$, where $n \neq m$, then

$$distance(Q_h(hn), Q_h(hm)) = \inf\{|x - y| : x \in Q_h(hn) \text{ and } y \in Q_h(hm)\} \geq h.$$

We can take the sequences $\{e_i : e_i \in B_k\}$ and regroup them into N_h uniformly separated sequences. To do this, take one e_i from each $Q_h(hn)$ to make a sequence. Pick another e_i from each $Q_h(hn)$ different from the previous one picked. Continue this way until all elements are chosen. Since each $Q_h(hn)$ has at most N_h elements, we can form no more than N_h sequences. Also, since the distance between each cube is greater than or equal to h, and for each of the the new sequences all of the elements are chosen from different cubes,

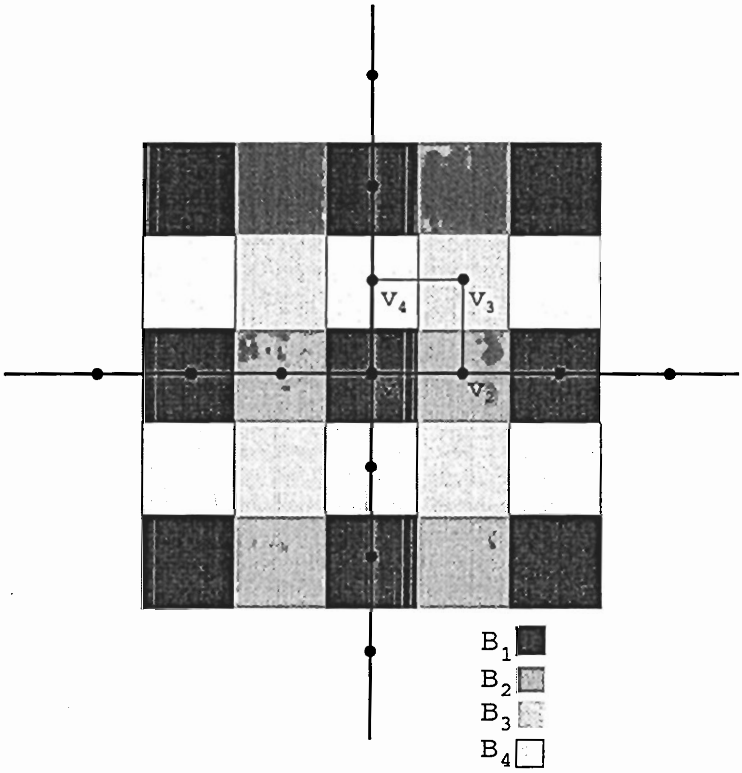

Figure 1: Formation of B_k in $\mathbb{R}^2$.

each sequence is uniformly separated. With this grouping, we have formed N_h uniformly separated sequences for each B_k. Since there are 2^d sets of B_k that partition $\mathbb{R}^d$, our entire set E can be broken up into $2^d N_h$ uniformly separated sequences.

b. $\Rightarrow$ *a.* Suppose that E is relatively uniformly separated. We can partition I into disjoint sets $I_1, ..., I_r$ such that each $E_k = \{e_i\}_{i \in I_k}$ is δ_k-uniformly separated. Let $\delta = \min\{\delta_1/2, ..., \delta_r/2\}$. Choosing δ this way makes certain that any cube Q with sidelength δ contains at most one element of E_k and thus at most r elements of E. Then for any positive number h, a cube with sidelength $h\delta$ contains at most $r(h+1)^d$ elements of E. This implies that for each $h, v^+(h\delta) \leq r(h+1)^d$. Then we have:

$$D^+(E) = \limsup_{h \to \infty} \frac{v^+(h\delta)}{(h\delta)^d} \leq \limsup_{h \to \infty} \frac{r(h+1)^d}{(h\delta)^d} = \frac{r}{\delta^d} < \infty. \quad \square$$

CHAPTER VI

DENSITY OF GABOR FRAMES

Our goal in this chapter is to prove a result that defines density conditions on Gabor frames. The results following are not unique to this paper, but are based on those found in the paper by Christensen, Deng, and Heil (see [4]). The paper [15] by Ramanathan and Steger highly influenced [4], yielding either [4] or [15] as good references for this chapter.

Theorem 6.1 *For $k = 1, \dots r$, choose a nonzero function $g_k \in L^2(\mathbb{R}^d)$ and an arbitrary sequence $\Lambda_k \subset \mathbb{R}^{2d}$. Let Λ be the disjoint union of $\Lambda_1, \dots, \Lambda_r$.*

a. If $\cup_{k=1}^{r} G(g_k, \Lambda_k)$ possesses an upper frame bound, then $D^+(\Lambda) < \infty$.

b. If $\cup_{k=1}^{r} G(g_k, \Lambda_k)$ is a frame for $L^2(\mathbb{R}^d)$, then $D^-(\Lambda) \geq 1$.

Before we can prove this theorem, we must fill our mathematical tool box with other theorems and lemmas. We begin with a special case of Theorem 6.1.a.

Theorem 6.2 *Choose a nonzero $g \in L^2(\mathbb{R}^d)$ and a sequence $\Lambda \subseteq \mathbb{R}^{2d}$. If $G(g, \Lambda)$ possesses an upper frame bound, then Λ is relatively uniformly separated.*

Proof. The proof is by contrapositive with aid from the Short-Time Fourier Transform. Suppose not, so that Λ is not relatively uniformly separated. Choose an $f \in L^2(\mathbb{R}^d)$ such that $\|f\|_2 = 1$. Also, choose $N > 0$. Since Λ is not relatively uniformly separated, then by Lemma 5.6, there exists some cube $Q_h(p, q)$ in $\mathbb{R}^{2d}$ that contains at least N elements of Λ. We will now utilize the short-time Fourier transform. Since $V_g(f)$ is continuous and nonzero, it must be bounded away from zero on some cube in $\mathbb{R}^{2d}$, say $Q_h(c, d)$. Then let $\mu = \inf_{(x,y) \in Q_h(c,d)} |V_g f(x, y)| > 0$. Now for any point $(a, b) \in Q_h(p, q)$, the point

$(a-(p-c), b-(q-d))$ is in $Q_h(c,d)$. We then observe that

$$\begin{aligned}
&\sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |\langle M_{q-d}T_{p-c}f, M_bT_ag\rangle|^2 \\
&= \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |\langle f, T_{-p+c}M_{-q+d}M_bT_ag\rangle|^2 \\
&= \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |\langle f, M_{b-q+d}T_{a-p+c}g\rangle|^2 \\
&= \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |V_gf(a-p+c, b-q+d)|^2.
\end{aligned}$$

Since each element of this sum is bounded below by μ^2, and there are at least N elements being summed, it follows that this pseudo-Plancherel sum is greater than or equal to $N\mu^2$. Since $\|M_{q-d}T_{p-c}f\|_2 = 1$, we have the following:

$$N\mu^2 = N\mu^2\|M_{q-d}T_{p-c}f\|_2^2 \leq \sum_{(a,b)\in\Lambda\cap Q_h(p,q)} |\langle M_{q-d}T_{p-c}f, M_bT_ag\rangle|^2.$$

We see that the sum is bounded below by a number that can be made arbitrarily large, thus the sequence set $G(g,\Lambda)$ cannot possess an upper frame bound. □

We now state a lemma that aids in the proof for the Homogeneous Approximation Property. A proof of this lemma can be found in [4], in which the Bargman transform is the key element (see [13].

Lemma 6.3 *Let $h > 0$ be fixed and let $\phi(x)$ denote the Gaussian function $\phi(x) = e^{-(\pi/2)x^2}$. There exists a constant K such that for all $f \in L^2(\mathbb{R}^d)$ and $(p,q) \in \mathbb{R}^{2d}$,*

$$|\langle \phi, M_qT_pf\rangle|^2 \leq K \int\int_{Q_h(p,q)} |\langle \phi, M_yT_xf\rangle|^2 dxdy.$$

To introduce notation, let $g_1, ..., g_r \in L^2(\mathbb{R}^d)$ and $\Lambda_1, ..., \Lambda_r \subset \mathbb{R}^{2d}$ such that $\cup_k G(g_k, \Lambda_k)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Denote the dual frame of $\cup_k G(g_k, \Lambda_k)$ by $\{\tilde{g}_{k,a,b}\}_{(a,b)\in\Lambda_k, k=1,...,r}$. Also, for $h > 0$ and $(p,q) \in \mathbb{R}^{2d}$, denote a finite dimensional subspace of $L^2(\mathbb{R}^d)$ by

$$W(h,p,q) = \text{span}\{\tilde{g}_{k,a,b} : (a,b) \in Q_h(p,q) \cap \Lambda_k, k = 1, ..., r\}. \quad (3)$$

Theorem 6.4 (Homogeneous Approximation Property) *Let $g_1, ..., g_r \in L^2(\mathbb{R}^d)$ and $\Lambda_1, ..., \Lambda_r \subset \mathbb{R}^{2d}$ be such that $\cup_k G(g_k, \Lambda_k)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Then for each $f \in L^2(R^d)$ and for all $\epsilon > 0$, there exists an $R > 0$ such that for all $(p, q) \in \mathbb{R}^{2d}$,*

$$distance(M_q T_p f, W(R, p, q)) < \epsilon. \tag{4}$$

Proof. Suppose we have $g_1, ..., g_r \in L^2(\mathbb{R}^d)$ and $\Lambda_1, ..., \Lambda_r \subset \mathbb{R}^{2d}$ such that $\cup_k G(g_k, \Lambda_k)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Since $\cup_k G(g_k, \Lambda_k)$ is a frame, by Theorem 6.2, each Λ_k is relatively uniformly separated. Now Λ_k can be divided into a finite union of uniformly separated subsequences $\Lambda_{k_1}, ..., \Lambda_{k_r}$ where each Λ_{k_i} is δ_{k_i}-uniformly separated. Now define $\delta = \max\{\frac{\delta_1}{2}, .., \frac{\delta_r}{2}\}$.

Let H be the subset of $L^2(\mathbb{R}^d)$ such that a function f is in H if and only if f satisfies equation (4).

Claim 6.5 *H is closed under finite linear combinations and L^2-limits.*

To see that the claim holds, let $f_1, ...f_n \in H$. Choose $\epsilon > 0$. Then for each f_i, there exists an R_i such that for all $(p, q) \in \mathbb{R}^{2d}$, $\|M_q T_p f - g_{R_i}\| < \epsilon$, where $g_{R_i} \in W(R_i, p, q)$. Define $R = \max\{R_1, \ldots, R_n\}$, and let $g_R = g_{R_1} + g_{R_2} + \cdots + g_{R_n}$ so that $g_R \in W(R, p, q)$. Now looking at the distance we have

$$\begin{aligned}
&\|M_q T_p(f_1 + \cdots + f_n) - g_r\|_2 \\
&= \|M_q T_p f_1 + \cdots + M_q T_p f_n - g_{R_1} - g_{R_2} - \cdots - g_{R_n}\|_2 \\
&\leq \|M_q T_p f_1 - g_{R_1}\|_2 + \|M_q T_p f_2 - g_{R_2}\|_2 + \cdots + \|M_q T_p f_n - g_{R_n}\|_2 \\
&\leq n\epsilon.
\end{aligned}$$

Also, if $f \in H$ and c is a scalar then

$$\|M_q T_p cf - cg_R\|_2 = \|cM_q T_p f - cg_R\|_2 = |c|\|M_q T_p f - cg_R\|_2 \leq |c|\epsilon,$$

thus H is closed under finite linear combinations. To see that H is closed under L^2 limits, suppose that f is in the closure of H. Fix $\epsilon > 0$. There exists a $g \in H$ such that

$$\|f - g\|_2 < \epsilon.$$

Since g is in H, we know that for any point (p, q), there exists an R such that

$$distance(M_q T_p g, W(R, p, q)) < \epsilon.$$

Since the closest point to $M_q T_p g$ is the orthogonal projection of $M_q T_p g$ onto W, we know that

$$\|M_q T_p g - P_W(M_q T_p g)\|_2 < \epsilon.$$

To see that f is in H, let R be the same R that works for g. Then

$$\begin{aligned}
&\|M_q T_p f - P_W(M_q T_p f)\|_2 \\
&\leq \|M_q T_p f - M_q T_p g\|_2 + \|M_q T_p g - P_w(M_q T_p g)\|_2 + \|P_W(M_q T_p g) - P_W(M_q T_p f)\|_2 \\
&= \|M_q T_p(f - g)\|_2 + \|M_q T_p g - P_w(M_q T_p g)\|_2 + \|P_W M_q T_p(g - f)\|_2 \\
&\leq \|M_q T_p\| \|(f - g)\|_2 + \|M_q T_p g - P_w(M_q T_p g)\|_2 + \|P_W\| \|M_q T_p\| \|g - f\|_2 \\
&= \|(f - g)\|_2 + \|M_q T_p g - P_w(M_q T_p g)\|_2 + \|g - f\|_2 \\
&\leq 3\epsilon,
\end{aligned}$$

and so H is closed under L^2-limits. This finishes the claim.

Now H is closed under finite linear combinations and L^2-limits, and for the Gaussian function $\phi(x) = e^{-(\pi/2)x^2}$, the set of time-frequency shifts $\{M_b T_a \phi\}_{(a,b) \in \mathbb{R}^{2d}}$ is complete in $L^2(\mathbb{R}^d)$. For a proof of this statement, we refer to [10]. Once we show that all time-frequency shifts of ϕ are in H, then we can conclude that H must be all of $L^2(\mathbb{R}^d)$.

Fix any $(s, t) \in \mathbb{R}^{2d}$, and look at any $(p, q) \in \mathbb{R}^{2d}$. The time-frequency shifts of ϕ belong to L^2, so the function $M_q T_p(M_t T_s \phi)$ has a frame expansion

$$M_q T_p(M_t T_s \phi) = \sum_{k=1}^{r} \sum_{(a,b) \in \Lambda_k} \langle M_q T_p(M_t T_s \phi), M_b T_a g_k \rangle \, \tilde{g}_{k,a,b}.$$

Let $P_{W(R,p,q)}$ denote the orthogonal projection onto $W(R, p, q)$. We want to find how close we are to this subspace. Let $S = (distance(M_q T_p(M_t T_s \phi), W(R, p, q)))^2$. Then

$$
\begin{aligned}
S &= (distance(M_qT_p(M_tT_s\phi), W(R,p,q)))^2 \\
&\leq \|M_qT_p(M_tT_s\phi)\| - P_{W(R,p,q)}M_qT_p(M_tT_s\phi)\|_2^2 \\
&\leq \Big\| \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k} \langle M_qT_p(M_tT_s\phi), M_bT_ag_k\rangle \tilde{g}_{k,a,b} \\
&\quad - \sum_{k=1}^{r} \sum_{(a,b)\in(\Lambda_k\cap Q_R(p,q))} \langle M_qT_p(M_tT_s\phi), M_bT_ag_k\rangle \tilde{g}_{k,a,b} \Big\|_2^2 \\
&= \Big\| \sum_{k=1}^{r} \sum_{(a,b)\in(\Lambda_k\setminus Q_r(p,q))} \langle M_qT_p(M_tT_s\phi), M_bT_ag_k\rangle \tilde{g}_{k,a,b} \Big\|_2^2 . \qquad (5)
\end{aligned}
$$

By Lemma 3.6,

$$
\begin{aligned}
(5) &\leq A^{-1} \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus Q_r(p,q)} |\langle M_qT_p(M_tT_s\phi), M_bT_ag_k\rangle|^2 \\
&= A^{-1} \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus Q_r(p,q)} |\langle \phi, M_{b-q-t}T_{a-p-s}g_k\rangle|^2 .
\end{aligned}
$$

Now since $M_{b-q-t}T_{a-p-s}g_k \in L^2(\mathbb{R}^d)$ and $(a-p-s, b-q-t) \in \mathbb{R}^{2d}$, by Lemma 6.3 there exists a constant K such that

$$
\begin{aligned}
|\langle \phi, M_{b-q-t}T_{a-p-s}g_k\rangle|^2 &\leq K \iint_{Q_\delta(a-p-s,b-q-t)} |\langle \phi, M_yT_xg_k\rangle|^2 \, dx\, dy \qquad \text{(by Lemma 6.3)} \\
&= K \iint_{Q_\delta(p+s-a,q-t+b)} |\langle M_{-y}T_{-x}\phi, g_k\rangle|^2 \, dx\, dy \\
&= K \iint_{Q_\delta(p+s-a,q-t-b)} |V_\phi g_k(x,y)|^2 \, dx\, dy.
\end{aligned}
$$

We then have that

$$
\begin{aligned}
S &\leq A^{-1}K \sum_{k=1}^{r} \sum_{(a,b)\in\Lambda_k\setminus Q_R(p,q)} \iint_{Q_\delta(p+s-a,q+t-b)} |V_\phi g_k(x,y)|^2 \, dx\, dy \\
&\leq A^{-1}K \sum_{k=1}^{r} \iint_{\mathbb{R}^{2d}\setminus Q_{R-\delta}(s,t)} |V_\phi g_k(x,y)|^2 \, dx\, dy. \qquad (6)
\end{aligned}
$$

But each $V_\phi g_k$ is in $L^2(\mathbb{R}^{2d})$, so $|V_\phi g_k(x,y)| < \infty$. Now we choose R sufficiently large enough so that equation (6) can be made arbitrarily small for any $(p,q) \in \mathbb{R}^{2d}$. It follows that all time-frequency shifts of ϕ belong to H. □

Corollary 6.6 *Let $g_1, ..., g_r \in L^2(\mathbb{R}^d)$ and $\Lambda_1, ..., \Lambda_r \subset \mathbb{R}^{2d}$ be such that $\cup_k G(g_k, \Lambda_k)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Then for each $f \in L^2(R^d)$ and for all $\epsilon > 0$, there exists an $R > 0$ such that $\forall (p,q) \in \mathbb{R}^{2d}$, $\forall h > 0$, $\forall (x,y) \in Q_h(p,q)$,*

$$distance(M_y T_x f, W(R+h,p,q)) < \epsilon. \tag{7}$$

Proof. If $(x,y) \in Q_h(p,q)$, then $W(R,x,y) \subset W(R+h,p,q)$; furthermore,

$$distance(M_y T_x f, W(h+R,p,q)) \leq \; distance(M_y T_x f, W(R,p,q)). \quad \square$$

The Homogeneous Approximation Property aids in the proof of the next theorem. The Comparison Theorem gives insight on the relationship between Gabor frame and Riesz basis density. We will show that Riesz bases will be less dense than Gabor frames.

Theorem 6.7 (The Comparison Theorem) *Let $g_1, ..., g_r \in L^2(\mathbb{R}^d)$ and $\Lambda_1, ..., \Lambda_r \subset \mathbb{R}^{2d}$ be such that $\cup_{k=1}^r G(g_k, \Lambda_k)$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds A and B. Let $f_1, ..., f_s \in L^2(\mathbb{R}^d)$ and $\Delta_1, ..., \Delta_s \subset \mathbb{R}^{2d}$ be such that $\cup_{j=1}^s G(f_j, \Delta_j)$ is a Riesz basis for $L^2(\mathbb{R}^d)$. For Λ the disjoint union of $\Lambda_1, ..., \Lambda_r$ and Δ the disjoint union of $\Delta_1, ..., \Delta_s$,*

$$D^-(\Delta) \leq D^-(\Lambda) \text{ and } D^+(\Delta) \leq D^+(\Lambda).$$

Proof. Using analogous notation as before, let $\tilde{f}_{j,a,b}$ denote the dual frame of $\cup_{j=1}^s G(f_j, \Delta_j)$ and let

$$V(h,p,q) = \text{span}\{M_b T_a f_j : (a,b) \in Q_h(p,q) \cap \Delta_j, j = 1, ..., s\}.$$

Each Δ_j is relatively uniformly separated, so by Lemma 5.6, $V(h,p,q)$ is a finite dimensional subspace. Also, since frame elements are uniformly bounded, there is a constant C such that $\|\tilde{f}_{j,a,b}\| \leq C \; \forall \; j, a$, and b.

Now choose $\epsilon > 0$. Since $f_j \in L^2(\mathbb{R}^d)$, then by Strong HAP, there exists an R_j such that for all $h > 0$, for all points $(p,q) \in \mathbb{R}^{2d}$, and for all $(x,y) \in Q_h(p,q)$,

$$distance(M_y T_x f_j, W(h+R_j,p,q)) < \frac{\epsilon}{C}. \tag{8}$$

Let $R = \max\{R_1 \ldots R_j\}$. Using the new R, we now have that

$$distance(M_y T_x f_j, W(h+R,p,q)) < \frac{\epsilon}{C}. \tag{9}$$

Let $h > 0$ and fix $(p,q) \in \mathbb{R}^{2d}$. Define $T : V(h,p,q) \longrightarrow V(h,p,q)$ by

$$T = P_{V(h,p,q)} P_{W(h+R,p,q)} P_{V(h,p,q)}.$$

For simpler notation let $P_V = P_{V(h,p,q)}$ and $P_W = P_{W(h+R,p,q)}$ so that $T = P_V P_W P_V$. Since $\cup_{j=1}^{s} G(f_j, \Delta_j)$ is a Riesz basis, hence an exact frame, it is biorthogonal to its dual frame, $\{\tilde{f}_{j,a,b}\}$. We would like to use this biorthogonal property to compute the trace of T. Now $\{\tilde{f}_{j,a,b}\}$ does not necessarily lie in V, however, $\{P_V \tilde{f}_{j,a,b}\}$ is also biorthogonal to $\cup_{j=1}^{s} G(f_j, \Delta_j)$. To see this, we compute

$$\begin{aligned}
\langle M_b T_a f_j, P_V \tilde{f}_{l,c,d} \rangle &= \langle P_V M_b T_a f_j, \tilde{f}_{l,c,d} \rangle \\
&= \langle M_b T_a f_j, \tilde{f}_{l,c,d} \rangle \\
&= \delta_{jl,ac,bd}.
\end{aligned}$$

By Lemma (2.6) we can now compute the trace of T:

$$\text{trace}(T) = \sum_{j=1}^{s} \sum_{(a,b) \in Q_h(p,q) \cap \Delta_j} \langle T(M_b T_a f_j), P_V \tilde{f}_{j,a,b} \rangle.$$

Then we have

$$\begin{aligned}
\langle T(M_b T_a f_j), \tilde{f}_{j,a,b} \rangle &= \langle P_V P_W P_V (M_b T_a f_j), \tilde{f}_{j,a,b} \rangle \\
&= \langle P_W (M_b T_a f_j), P_V \tilde{f}_{j,a,b} \rangle \\
&= \langle (P_W - I)(M_b T_a f_j), P_V \tilde{f}_{j,a,b} \rangle + \langle M_b T_a f_j, P_V \tilde{f}_{j,a,b} \rangle \\
&= \langle (P_W - I)(M_b T_a f_j), P_V \tilde{f}_{j,a,b} \rangle + \langle P_V M_b T_a f_j, \tilde{f}_{j,a,b} \rangle \\
&= \langle (P_W - I)(M_b T_a f_j), P_V \tilde{f}_{j,a,b} \rangle + \langle M_b T_a f_j, \tilde{f}_{j,a,b} \rangle \qquad (10) \\
&= \langle (P_W - I)(M_b T_a f_j), P_V \tilde{f}_{j,a,b} \rangle + 1. \qquad (11)
\end{aligned}$$

We note that equation (11) follows from equation (10) since $\cup_{j=1}^{s} G(f_j, \Delta_j)$ is biorthogonal to $\{\tilde{f}_{j,a,b}\}$. Also, from equation (11) we have that

$$|\langle (P_W - I) M_b T_a f_j, P_V \tilde{f}_{j,a,b} \rangle| \leq \|(P_W - I) M_b T_a, f_j\|_2 \|P_V \tilde{f}_{j,a,b}\|_2 \leq \frac{\epsilon}{C} \times C = \epsilon,$$

and so

$$\langle (P_W - I)M_bT_af_j, P_V\tilde{f}_{j,a,b}\rangle \geq -\epsilon.$$

Combining all the equations we have

$$\begin{aligned} trace(T) &\geq \sum_{j=1}^{s} \sum_{(a,b)\in Q_h(p,q)\cap\Delta_j} (1-\epsilon) \\ &= (1-\epsilon)\sum_{j=1}^{s} \#(Q_h(p,q)\cap\Delta_j) \\ &= (1-\epsilon)\#(Q_h(p,q)\cap\Delta), \end{aligned}$$

where the last equality holds since Δ is the disjoint union of $\Delta_1, ..., \Delta_s$.

Since $|\lambda| \leq \|T\| \leq 1$, for all eigenvalues λ of T, we also have that

$$\begin{aligned} \text{trace}(T) &\leq \text{rank}(T) \\ &\leq \dim(W(h+R,p,q)) \\ &\leq \sum_{k=1}^{s} \#(Q_{h+R}(p,q)\cap\Lambda_k) \\ &= \#(Q_{h+R}(p,q)\cap\Lambda). \end{aligned}$$

Hence we have that

$$(1-\epsilon)\#(Q_h(p,q)\cap\Delta) \leq \text{trace}(T) = \#(Q_{h+R}(p,q)\cap\Lambda).$$

As a result, we see that

$$(1-\epsilon)\frac{\#(Q_h(p,q)\cap\Delta)}{h^{2d}} \leq \frac{\#(Q_{h+R}(p,q)\cap\Lambda)}{(h+R)^{2d}}\frac{(h+R)^{2d}}{h^{2d}}.$$

Taking the limit as h goes to infinity, it follows that $(1-\epsilon)D^+(\Delta) \leq D^+(\Lambda)$ and $(1-\epsilon)D^-(\Delta) \leq D^-(\Lambda)$. But ϵ is arbitrary, hence the theorem is proved. □

Now that we have proved the HAP and the Comparison Theorem, we have all the powerful tools needed to prove the main result.

Proof of Theorem 6.1.a. Suppose that $\cup_k G(g_k, \Lambda_k)$ has an upper frame bound. Then by Theorem 6.2, each Λ_k is relatively uniformly separated. Since Λ is a finite union of relatively uniformly separated sequences, it is also relatively uniformly separated. By Lemma 5.6, $D+(\Lambda) < \infty$.

Proof of 6.1.b. Define $f = \chi_{Q_1(0)}$ and $\Delta = \mathbb{Z}^{2d}$. Computing the density for Δ, we have that

$$D^-(\Delta) = D^-(\mathbb{Z}^{2d}) = \liminf_{h\to\infty} \frac{\min \#(\mathbb{Z}^{2d} \cap Q_h(x))}{h^d} = \liminf_{h\to\infty} \frac{h^d}{h^d} = 1.$$

The Comparison Theorem then implies that $D^-(\Lambda) \geq D^-(\Delta) = 1$. □.

While we can conclude that $D^-(\Lambda) \geq 1$ with Theorem 6.1, we cannot conclude the stronger statement that $\sum_{k=1}^{s} D^-(\Lambda_k) \geq 1$. Consider the same $G(f, \Delta)$ defined by $f = \chi_{Q_1(0)}$ and $\Delta = \mathbb{Z}^{2d}$. Partition Δ by defining $\Delta_1 = \{n = (n_1, \ldots n_d) \in \mathbb{Z}^d : n_1 \geq 0\}$ and $\Delta_2 = \{n = (n_1, \ldots n_d) \in \mathbb{Z}^d : n_1 < 0\}$. Then $G(f, \Delta_1) \cup G(f, \Delta_2) = G(f, \Delta)$, and so $G(f, \Delta_1) \cup G(f, \Delta_2)$ is an orthonormal (hence Riesz) basis for $L^2(\mathbb{R}^d)$. However,

$$\sum_{k=1}^{2} D^-(\Delta_k) = 0 + 0 = 0.$$

Corollary 6.8 *Let $f_1, \ldots, f_s \in L^2(\mathbb{R}^d)$ and $\Delta_1, \ldots, \Delta_s \subset \mathbb{R}^{2d}$ such that $\cup_{k=1}^{s} G(f, \Delta_k)$ is a Riesz basis for $L^2(\mathbb{R}^d)$. Let Δ be the disjoint union of $\Delta_1, \ldots, \Delta_s$. Then Δ has uniform Beurling density $D(\Delta) = 1$.*

Proof. Let $g = \chi_{Q_1(0)}$ and $\Lambda = \mathbb{Z}^{2d}$. then $G(g, \Lambda)$ is an orthonormal basis (hence frame) for $L^2(\mathbb{R}^d)$. Now, as shown in the proof for Theorem 6.1.b, $D^-(\Lambda) = 1$. Also,

$$\Delta^+(\Lambda) = \limsup_{h\to\infty} \frac{\max_{x\in\mathbb{R}^d} \#(\Lambda \cap Q_h(x))}{h^d} = \limsup_{h\to\infty} \frac{(h+1)^d}{h^d} = 1.$$

Apply the Comparision Theorem to the frame $G(g, \Lambda)$ and the Riesz basis $\cup_{k=1}^{s} G(f_k, \Delta_k)$, and we have that

$$D^-(\Delta) \leq D^-(\Lambda) = 1 \text{ and } D^+(\Delta) \leq D^+(\Lambda) = 1.$$

Similarly, $G(g, \Lambda)$ is an orthonormal (hence Riesz) basis, and $\cup_{k=1}^{s} G(f_k, \Delta_k)$ is a Riesz basis, hence a frame. Apply the Comparison Theorem to the Riesz basis $G(g, \Lambda)$ and to the frame $\cup_{k=1}^{s} G(f_k, \Delta_k)$, and we have that

$$1 = D^-(\Lambda) \leq D^-(\Delta) \text{ and } 1 = D^+(\Lambda) \leq D^+(\Delta). \quad \square$$

CHAPTER VII

SOME RESULTS

As a consequence of the results found in the previous chapter, we find that frames solely composed of translations of a finite set of functions cannot exist. What follows is nearly a restatement of results originally found in [4]. It is still an open question as to whether there exists a Schauder basis for $L^2(\mathbb{R}^d)$ consisting of translations of a finite set of functions. See [14] for more details.

Theorem 7.1 *By Theorem 6.1.b, there are no frames for $L^2(\mathbb{R}^d)$ consisting of translations of a finite set of functions.*

Proof. For $g \in L^2(\mathbb{R}^d)$, let $T(g, F)$ be the set of translates of g by F, defined by $T(g, F) = \{T_a g : g \in L^2(\mathbb{R}^d) \text{ and } a \in \mathbb{F}\}$. Let $\cup_k T(g_k, F_k)$ be a collection of translates of finitely many functions g_k. Each $T(g_k, F_k)$ is a Gabor system of the form $G(g_k, F_k \times 0)$. Now suppose that $g_1, \ldots, g_r \in L^2(\mathbb{R}^d)$ and $F_1, \ldots F_r \subset \mathbb{R}^d$ are such that $\cup_k T(g_k, F_k)$ is a frame for $L^2(\mathbb{R}^d)$. Since $T(g_k, F_k) = G(g_k, F_k \times \{0\})$, Theorem 6.1.b. implies that $D^-(F \times \{0\}) \geq 1$, where F is the disjoint union of $F_1, \ldots, F_r$. This is a contradiction since $D^-(F \times \{0\}) = 0$. □

Remarking on the implications of Theorem 6.1.b, we see that if $\cup_k G(g_k, F_k)$ is a frame, then the disjoint union of points F must be distributed throughout all of $\mathbb{R}^{2d}$.

Theorem 7.2 *For $k = 1, \ldots r$, choose a nonzero function $g_k \in L^2(\mathbb{R}^d)$ and an arbitrary sequence $E_k \subset \mathbb{R}^{2d}$. Let E be the disjoint union of $E_1, \ldots, E_r$.*

a. If $\cup_{k=1}^r T(g_k, E_k)$ possesses an upper frame bound for $L^2(\mathbb{R}^d)$, then $D^+(E) < \infty$.

b. If $\cup_{k=1}^r T(g_k, E_k)$ possesses a lower frame bound for $L^2(\mathbb{R}^d)$, then $D^+(E) = \infty$.

Proof of part a. Suppose that $\cup_k T(g_k, E_k)$ has an upper frame bound. By Theorem 6.1.a, $D^+(E \times \{0\}) < \infty$. Now by Lemma 5.6, $E \times \{0\}$ is a relatively uniformly separated sequence of points in $\mathbb{R}^{2d}$. This implies that E is a relatively uniformly separated sequence of points in $\mathbb{R}^d$. Then by Lemma 5.6, $D^+(E) < \infty$.

Proof of part b. Suppose that $D^+(E) < \infty$. By Lemma 5.6, E is relatively uniformly separated, and so each E_k is also relatively uniformly separated. Therefore, each E_k is the union of δ_{kj}-separated subsequences Δ_{kj} for $j = 1, \ldots, s_k$. Define $\delta = \min\{\delta_{kj}/2\}$, and fix an $h < \delta$. Also, define $Q = Q_h(0)$. Since Δ_{kj} are δ_{kj} -separated, the cubes $\{Q + a\}_{a \in \Delta_{kj}}$ are disjoint. Now define B_{kj} such that it is the union of the disjoint cubes $\{Q + a\}$, so

$$B_{kj} = \cup_{a \in \Delta_{kj}} (Q + a).$$

With some computations we have that

$$\begin{aligned}
\sum_{k=1}^{r} \sum_{a \in E_k} |\langle \chi_Q, T_a g_k \rangle|^2 &= \sum_{k=1}^{r} \sum_{j=1}^{s_k} \sum_{a \in \Delta_{kj}} |\langle \chi_Q, \chi_Q T_a g_k \rangle|^2 \\
&\leq \sum_{k=1}^{r} \sum_{j=1}^{s_k} \sum_{a \in \Delta_{kj}} \|\chi_Q\|_2^2 \|\chi_Q T_a g_k\|_2^2 \\
&= \|\chi_Q\|_2^2 \sum_{k=1}^{r} \sum_{j=1}^{s_k} \int_{B_{kj}} |g_k(x)|^2 \mathrm{d}x.
\end{aligned}$$

Now, for j and k fixed and as $h \to \infty$, $\chi_{B_{kj}} |g_k(x)|^2$ converges to zero pointwise a.e. But for all h, $\chi_{B_{kj}} |g_k(x)|^2 \leq |g_k(x)|^2$, and so by the Lebesgue Dominated Convergence Theorem,

$$\lim_{h \to 0} \int_{B_{kj}} |g_k(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} \lim_{h \to 0} \chi_{B_{kj}} |g_k(x)|^2 = 0.$$

Thus we cannot find a lower frame bound for $\cup_k T(g_k, \Gamma_k)$. □

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