

# TUTTE PATHS AND EVEN COVERS

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# TUTTE PATHS AND EVEN COVERS

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To my parents

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## SUMMARY

A Tutte path of a graph  $G$  is a path  $P$  of  $G$  such that every component of  $G - P$  has at most three attachments on  $P$ . Tutte paths are well studied in the literature due to their applications towards the Hamiltonian cycle problem. We prove the existence of Tutte paths in which the number of components is bounded for circuit graphs, a natural family of planar graphs which generalizes 3-connected planar graphs. As a consequence, we obtain a sharp lower bound for the circumference of essentially 4-connected planar graphs, answering a conjecture of Fabrici, Harant, Mohr, and Schmidt.

The Traveling Salesperson Problem (TSP) is a foundational problem in the optimization literature and generalizes the Hamiltonian cycle problem. Motivated by the TSP, we investigate even covers of subcubic graphs, i.e., finding a small number of cycles that cover the majority of the vertices (a graph is subcubic if its maximum degree is 3). As an application, we will show that if  $G$  is a 2-connected subcubic graph with  $n$  vertices and  $n_2$  vertices of degree 2, then  $G$  has a TSP walk of length at most  $\frac{5n+n_2}{4} - 1$ , establishing a conjecture of Dvořák, Král', and Mohar. There are an infinite family of subcubic (respectively, cubic) graphs whose minimum TSP walk have length  $\frac{5n+n_2}{4} - 1$  (respectively,  $\frac{5n}{4} - 2$ ). As this walk can be found in quadratic time, this provides a state-of-the-art  $\frac{5}{4}$ -approximation algorithm for the TSP on 2-connected cubic graphs, improving the prior best guarantee of  $\frac{9}{7}$ .

# CHAPTER 1

## INTRODUCTION

### 1.1 Background and History

A *Hamiltonian cycle* in a graph is a cycle that passes through every vertex of the graph. The study of Hamiltonian cycles has its origin with the 19th Century mathematician, Sir William Rowan Hamilton. Hamilton studied an algebra which characterized Hamiltonian cycles in the graph of the regular dodecahedron [16, 32, 33]. This algebra had a geometric interpretation of walking along the faces of the icosahedron, which Hamilton called the the icosian calculus. Hamilton branded this as a game, selling it to a game designer. This later proved to be a financial failure for the game maker [9]. While the name of such cycles traces back to Hamilton, it should be noted that Karper investigated the Hamiltonian cycle problem for more general polyhedra a few years prior to Hamilton [44].

Regardless of its origins, the study of Hamiltonian cycles has had profound impacts on mathematics, computer science, and optimization. The Four Color Theorem [4, 42] ( also [53]) states that every planar graph can be face 4-colorable. For quite a long time, the Four Color Theorem was the Four Color Conjecture. The question was first asked as early as the 1850s [9, 31]. The conjecture, at first receiving little attention, eventually developed a notoriety for its difficulty, accumulating a large list of ideas and techniques towards possible proof strategies [51, 54]. Many of these ideas, while failing their initial goal to prove the Four Color Theorem, matured into deep and rich combinatorial theories. It is well known now that if a plane graph is Hamiltonian, then it has a face 4-coloring. This connection between cycles and face-coloring can be traced as one of the early justifications for the serious academic study of Hamiltonian cycles, see for example [66]. Remarkably, even today, every proof of the Four Color Theorem has required computer assistance.

After giving a false proof of the Four Color Theorem, Tait [59] conjectured that every 3-connected planar cubic graph contains a Hamiltonian cycle. If true, such a claim would also imply the Four Color Theorem. Tutte [65] disproved this conjecture by giving a counterexample. Tutte's construction relied on a gadget obtained from modifying the graph of a pentagonal prism, now called a Tutte fragment in the literature. There has been additional work on finding more counterexamples to Tait's conjecture, see [2] or [34].

Strengthening the connectivity assumption in Tait's conjecture is sufficient for the existence of Hamiltonian cycles. Whitney [66] proved that 4-connected planar triangulations are Hamiltonian and Tutte [64] further generalized this by showing all 4-connected planar graphs are Hamiltonian. Thomassen [63] later showed that 4-connected planar graphs are Hamiltonian-connected, i.e., there exists a Hamiltonian path between any two specified vertices (a path is *Hamiltonian* if it contains every vertex of the graph). A small correction to Thomassen's proof was made by Chiba and Nishizeki [14] and a shorter alternative proof was given by Ozeki [52]. A slight improvement to Thomassen's result was given by both Sanders [55] and Thomas and Yu [60]. Algorithmic versions of these results have also been studied [8, 56].

We have seen that 4-connected planar graphs are Hamiltonian, and 3-connected planar graphs may not be. This leads to the natural question of understanding the relationship between connectivity and circumference of planar graphs. (The *circumference* of a graph is the length of a longest cycle in that graph.) For 2-connectivity, planar  $n$ -vertex graphs of size at least  $n \geq 5$  may have a circumference as low as 4, with the infinite family of complete bipartite graphs  $K_{2,n-2}$  exhibiting this bound. Chen and Yu [13] showed that the circumference of a 3-connected planar  $n$ -vertex graph is  $\Theta(n^{\log_3 2})$ . This bound too is best possible with Chen and Yu [13] giving a family of iterated planar triangulations that exhibit this bound.

For any positive integer  $k$ , a graph is *essentially  $k$ -connected* if it is  $(k - 1)$ -connected and, for any  $S \subseteq V(G)$  with  $|S| < k$ ,  $G - S$  is connected or has exactly two components one

of which is trivial (A component is *trivial* if it consists of a single vertex). A graph is *cubic* if all of its vertices have degree 3. Grünbaum and Malkevitch [30] observed that essentially 4-connected cubic planar  $n$ -vertex graphs have circumference at least  $3n/4$ , with Zhang [71] improving their bound by an additive constant of one. Jackson and Wormald [35] proved that the circumference of an essentially 4-connected  $n$ -vertex graph is at least  $(2n + 4)/5$ . Using a set of discharging rule, Fabrici, Harant, Mohr, and Schmidt [22] improved this to  $5(n + 2)/8$ . Fabrici, Harant, Mohr, and Schmidt [21] also showed that essentially 4-connected triangulations have circumference at least  $2(n + 4)/3$  and conjectured this bound could be extended to all essentially 4-connected planar graphs. The author and Yu [69] and independently Kessler and Schmidt [43] (using completely different methods) proved the following.

**Theorem 1.1.1.** *Let  $n \geq 6$  be an integer and let  $G$  be an essentially 4-connected planar  $n$ -vertex graph. Then the circumference of  $G$  is at least  $\lceil (2n + 6)/3 \rceil$ .*

This bound is best possible. Take a 4-connected triangulation  $T$  on  $k$  vertices, and inside each face of  $T$  add a new vertex and three edges from the new vertex to the three vertices in the boundary of that face. The resulting graph, say  $G$ , has  $n := 3k - 4$  vertices. Now take an arbitrary cycle  $C$  in  $G$ . For each  $x \in V(C)$  with degree three in  $G$ , deleting  $x$  from  $C$  and adding the edge between the two neighbors of  $x$  in  $C$ , we obtain a cycle in  $T$ , say  $D$ . Then  $|D| \leq k$ ; which implies  $|C| \leq 2k$ . Hence, the circumference of  $G$  is at most  $2k = 2(n + 4)/3 = \lceil (2n + 6)/3 \rceil$ .

The proof of Theorem 1.1.1 builds on the ideas and techniques from the proofs of Tutte [64] and Thomassen [63], finding a cycle  $C$  in a 2-connected graph  $G$  such that every component of  $G - C$  has at most three neighbors on  $C$ . A cycle or path with such property is *Tutte*. Note the assumption that  $G$  is 4-connected implies that  $C$  must be a Hamiltonian cycle. If we instead assume that  $G$  is essentially 4-connected, then each component of  $G - C$  would be a single vertex. Thus if we bound the number of components of  $G - C$ , then we also obtain a lower bound for the length of  $C$ . This was the main idea employed

in [69] to obtain the sharp lower bound for the circumference of essentially 4-connected graphs.

Another closely related problem to the Hamiltonian cycle problem is the famous Traveling Salesperson Problem. The *Travelling Salesperson Problem* (TSP) asks for a spanning cycle of minimum length in an edge-weighted complete graph. The problem has remained a cornerstone in the fields of combinatorics, computer science, and optimization. The TSP is NP-hard, as it generalizes the Hamiltonian cycle problem, one of Karp's original examples in his seminal paper on NP-completeness [37]. In fact, it is not possible to approximate the TSP in polynomial time within any constant factor of the optimum unless  $P = NP$  [70]. Thus it seems unlikely for there to exist an efficient algorithm to solve this problem. Regardless, the importance of the problem in practical matters cannot be understated, and has continued to receive significant attention.

An important special case of the TSP which admits a constant factor approximation is the *metric TSP* in which the edge weights form a metric, a natural assumption for many applications. A further specialization of the metric TSP is the *graphic TSP* in which the edge weights form the distance function in some underlying connected graph  $G$  on the same vertex set. This is equivalent to finding a spanning closed walk, a *TSP walk*, in  $G$  with the minimum number of edges. We denote this minimum length by  $\text{tsp}(G)$ .

The graphic TSP still contains the Hamiltonian cycle problem, and is thus NP-hard to solve exactly. A classic result of Christofides [15] and independently Serdyukov [7, 58] gives a  $\frac{3}{2}$ -approximation for the metric TSP. For many years, this had remained the best approximation ratio for any nontrivial special case of the metric TSP. The first improvement was made in 2005 by Gamarnik, Lewenstein, and Sviridenko [23] who gave a  $(\frac{3}{2} - \frac{5}{389})$ -approximation algorithm for the special case of the graphic TSP on 3-connected cubic graphs. Following this, Gharan, Saberi, and Singh [27] gave a  $(\frac{3}{2} - \epsilon)$ -approximation algorithm for the general graphic TSP. Then Mömke and Svensson [47] gave an approach for a 1.461-approximation algorithm for the graphic TSP, which Mucha [48] later improved

to a  $\frac{13}{9}$ -approximation by a better analysis of their techniques. Later, Sebő and Vygen [57] presented a new algorithm for an improved  $\frac{7}{5}$ -approximation for the graphic TSP. For the metric TSP, the  $\frac{3}{2}$  ratio was only very recently improved by Karlin, Klein, and Gharan [36] to  $(\frac{3}{2} - \varepsilon)$  for some small constant  $\varepsilon > 10^{-36}$ . While both the metric and graphic TSP allow for constant factor approximation, they remain APX-hard, i.e., unless  $P = NP$ , there does not exist polynomial-time approximation schemes. In particular, it is known that the metric and graphic TSPs are NP-hard to approximate within a  $\frac{123}{122}$  and  $\frac{185}{184}$ -factor of the optimum respectively [38, 46].

A special case of graphic TSP extensively studied is when the input graph is planar. Arora et al. [5] showed that planar graphs with arbitrary positive edge weights have a polynomial time approximation scheme for the TSP. Klein [45] later improved this, showing for any fixed  $\varepsilon > 0$ , there exists a linear-time algorithm that finds a TSP walk within  $(1 + \varepsilon)$  of the optimum. In particular, under the assumption of planarity, the TSP is no longer APX-hard. From the structural viewpoint, Kawarabayashi and Ozeki [41] showed that 3-connected planar graphs have TSP tours of length at most  $4(n-1)/3$ , with this bound being tight.

Another special case of the graphic TSP, namely on subcubic graphs, has received significant attention (a graph is *subcubic* if all of its vertices have degree at most 3). Subcubic and cubic graphs are simple classes of graphs which retain the inapproximability of the metric TSP. Even when restricted to subcubic and cubic graphs, it remains NP-hard to approximate within a  $\frac{685}{684}$  and  $\frac{1153}{1152}$ -factor respectively [39]. Furthermore, subcubic graphs are known to exhibit the worst-case behavior in a well-known conjecture from the 80's (see [28]), which asserts that the subtour elimination linear program relaxation for the metric TSP has an integrality gap of  $\frac{4}{3}$ . This  $\frac{4}{3}$ -integrality gap can be asymptotically realized by a family of subcubic graphs, see for example [6].

Note that a polynomial-time constructive proof of the  $\frac{4}{3}$ -integrality gap would yield a  $\frac{4}{3}$ -approximation algorithm for the TSP. Motivated by this, Aggarwal, Garg, and Gupta [1]

gave a  $\frac{4}{3}$ -approximation for 3-connected cubic graphs. This approximation guarantee was then extended to 2-connected cubic graphs by Boyd et al. [10], and to 2-connected subcubic graphs by Mömke and Svensson [47]. The  $\frac{4}{3}$  ratio was slightly improved for cubic graphs to  $(\frac{4}{3} - \frac{1}{61326})$  by Correa, Larreé, and Soto [18] and independently to  $(\frac{4}{3} - \frac{1}{8754})$  by Zuylen [72], which was further improved to 1.3 by Candráková and Lukot'ka [12], and later to  $\frac{9}{7}$  by Dvořák, Král', and Mohar [20].

Let  $G$  be a simple 2-connected subcubic graph. We write  $n(G)$  to denote the number of vertices in  $G$ , and  $n_2(G)$  to denote the number of degree 2 vertices in  $G$ . Dvořák, Král', and Mohar [20] showed that  $G$  has a TSP walk of length at most  $\frac{9n(G)+2n_2(G)}{7} - 1$ . They also constructed infinitely many subcubic (respectively, cubic) graphs whose minimum TSP walks have lengths  $\frac{5n(G)+n_2(G)}{4} - 1$  (respectively,  $\frac{5n(G)}{4} - 2$ ), and conjectured that  $\frac{5n(G)+n_2(G)}{4} - 1$  is the correct bound. The author, Yoo, and Yu proved the following.

**Theorem 1.1.2.** [67] *Let  $G$  be a 2-connected simple subcubic graph. Then  $\text{tsp}(G) \leq \frac{5n(G)+n_2(G)}{4} - 1$ . Moreover, a TSP walk of length at most  $\frac{5n(G)+n_2(G)}{4} - 1$  can be found in  $O(n(G)^2)$  time.*

Note this provides a  $\frac{5}{4}$ -approximation algorithm for the graphic TSP on simple cubic graphs. We remark that our algorithm is purely combinatorial and deterministic. We also characterize the extremal examples of Theorem 1.1.2; that is, the 2-connected simple subcubic graphs  $G$  such that  $\text{tsp}(G) = \frac{5n(G)+n_2(G)}{4} - 1$  (see Theorem 3.4.5). As pointed out by Dvořák et al. [20], Theorem 1.1.2 is false for non-simple graphs. This can be seen from the graph obtained from three internally disjoint paths between two vertices, each of length  $2k + 1$ , by the addition of parallel edges so that it becomes cubic.

## 1.2 Preliminaries

In order to prove Theorem 1.1.1, the strategy employed is to find a cycle  $C$  in a 2-connected planar graph  $G$  such that every component of  $G - C$  has at most three neighbors on  $C$  and

the number of components of  $G - C$  is as small as possible. We now introduce the related concepts and terminologies. For general references, we refer to [19] for graph theory and [70] for approximation algorithms.

Let  $G$  be a graph and  $H \subseteq G$ . An  $H$ -bridge of  $G$  is the subgraph of  $G$  induced by an edge in  $E(G) \setminus E(H)$  with both incident vertices on  $H$  or induced by the edges of  $G$  that are incident with one or two vertices in a single component of  $G - H$ . We use  $\beta_G(H)$  to denote the number of  $H$ -bridges in  $G$  with at least 3 vertices. For any  $H$ -bridge  $B$  of  $G$ , a vertex in  $V(B \cap H)$  is called an *attachment* of  $B$  on  $H$ . We say that  $H$  is a *Tutte subgraph* of  $G$  if every  $H$ -bridge of  $G$  has at most three attachments on  $H$ . Moreover, for any subgraph  $F \subseteq G$ ,  $H$  is said to be an  $F$ -*Tutte* subgraph of  $G$  if  $H$  is a Tutte subgraph of  $G$  and every  $H$ -bridge of  $G$  containing an edge of  $F$  has at most two attachments on  $H$ . A *Tutte cycle* (respectively, *Tutte path*) is a Tutte subgraph that is a cycle (respectively, path).

For any positive integer  $k$  and any graph  $G$ , a  $k$ -separation in  $G$  is a pair  $(G_1, G_2)$  of subgraphs of  $G$  such that  $|V(G_1 \cap G_2)| = k$ ,  $G = G_1 \cup G_2$ ,  $E(G_1) \cap E(G_2) = \emptyset$ , and  $G_i \not\subseteq G_{3-i}$  for  $i = 1, 2$ . A  $k$ -cut in  $G$  is a set  $S \subseteq V(G)$  with  $|S| = k$  such that there exists a separation  $(G_1, G_2)$  in  $G$  with  $S = V(G_1 \cap G_2)$  and  $G_i - G_{3-i} \neq \emptyset$  for  $i = 1, 2$ .

To state our technical result, we need further notation. Given a plane graph  $G$  and a cycle  $C$  in  $G$ , we say that  $(G, C)$  is a *circuit graph* if  $G$  is 2-connected,  $C$  is the outer cycle of  $G$  (i.e.  $C$  bounds the infinite face of  $G$ ), and, for any 2-cut  $T$  in  $G$ , each component of  $G - T$  must contain a vertex of  $C$ . Note as  $G$  is embedded into the plane,  $C$  has a clockwise orientation and a counterclockwise orientation. For any distinct elements  $x, y \in V(C) \cup E(C)$ , we use  $xCy$  to denote the subpath of  $C$  in clockwise order from  $x$  to  $y$  such that  $x, y \notin E(xCy)$ , and we say that  $xCy$  is *good* if  $G$  has no 2-separation  $(G_1, G_2)$  with  $V(G_1 \cap G_2) = \{s, t\}$  such that  $x, s, t, y$  occur on  $xCy$  in order,  $sCt \subseteq G_2$ , and  $|G_2| \geq 3$ .

Moreover, let

$$\tau_{Gxy} = \begin{cases} 1, & xCy \text{ is not good,} \\ 1, & |\{x, y\} \cap E(C)| = 1 \text{ and } x \text{ and } y \text{ are incident,} \\ 1/2, & |\{x, y\} \cap E(C)| = 1 \text{ and } |xCy| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

We remark now these  $\tau$  parameters are a rescaling of the  $\tau$  parameters present in Wigal and Yu [69], see Corollary 2.2.9. If there is no danger of confusion, we may drop the reference to  $G$ . In nonrigorous terms, these  $\tau$  parameters for particular choices of  $x$  and  $y$  will function as a measurement from how far a circuit graph  $(G, C)$  is from being 3-connected. Our main technical theorem for Tutte paths is the following.

**Theorem 1.2.1.** *Let  $(G, C)$  be a circuit graph and let  $u, v \in V(C)$  be distinct and  $e \in E(C)$ , such that  $u, e, v$  occur on  $C$  in clockwise order. Then  $G$  has a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that  $e \in E(P)$  and*

$$\beta(P) \leq (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev} \tag{1.1}$$

Theorem 1.2.1 is a strengthening of the technical theorem presented in [69]. In particular, the number of nontrivial bridges is now bounded in terms of the vertices in the path  $P$ , rather than the number of vertices of the graph  $G$ . In Corollary 2.2.9, we recover the original technical result in terms of bounding the number of nontrivial bridges in terms of the vertex count. As an application, in Theorem 2.3.1 we obtain sharp circumference bounds for essentially 4-connected graphs.

In Section 2.1, we consider some special cases for Theorem 1.2.1. In Section 2.2, we prove Theorem 1.2.1. In Section 2.3, we apply Theorem 1.2.1 providing sharp circumference bounds for essentially 4-connected planar graphs.

To prove our result on TSP walks in subcubic graphs, i.e., Theorem 1.1.2, we follow [20] in considering spanning subgraphs  $F$  of  $G$  in which every vertex has even degree. We call such a spanning subgraph  $F$  an *even cover*, and note that when  $G$  is a subcubic,  $F$  consists of vertex-disjoint cycles and isolated vertices. We let  $c(F)$  denote the number of cycles in  $F$  and  $i(F)$  denote the number of isolated vertices in  $F$ . The *excess* of  $F$  is defined to be

$$\text{exc}(F) = 2c(F) + i(F).$$

For a graph  $G$ , let  $\mathcal{E}(G)$  denote the set of even covers of  $G$ , and define the *excess* of  $G$  as

$$\text{exc}(G) = \min_{F \in \mathcal{E}(G)} \text{exc}(F).$$

As an example, consider the graph  $\Theta$  which consists of three internally disjoint paths between two vertices, each path consisting of  $k$  vertices of degree 2. Every even cover may contain at most one cycle. As a consequence, the even cover consisting of a cycle and  $k$  isolated vertices obtains the minimum excess. Thus for  $k \geq 1$ ,

$$\text{exc}(\Theta) = 2 + k \leq \frac{(3k + 2) + 3k}{4} + 1 = \frac{n(\Theta) + n_2(\Theta)}{4} + 1,$$

with equality when  $k = 1$  (in which case  $\Theta \cong K_{2,3}$ ).

It is observed in [20] that if  $G$  is a subcubic graph, then there is a linear relation between  $\text{tsp}(G)$  and  $\text{exc}(G)$ :

**Proposition 1.2.2** (Dvořák et al. [20]). *Let  $G$  be a subcubic graph. Then*

$$\text{tsp}(G) = \text{exc}(G) - 2 + n(G). \tag{1.2}$$

*Moreover, an even cover  $F \in \mathcal{E}(G)$  can be converted into a TSP walk in  $G$  of length  $\text{exc}(F) - 2 + n(G)$  in linear time.*

Thus, to prove Theorem 1.1.2, it suffices to show that

$$\text{exc}(G) \leq \frac{n(G) + n_2(G)}{4} + 1. \quad (1.3)$$

and that an even cover  $F$  of  $G$  satisfying this bound can be found in quadratic time.

To prove Theorem 1.1.2, we attack the problem in the following way. A natural strategy is given an edge  $e \in E(G)$ , to find an even cover  $F$  such that  $e \in E(F)$  such that  $\text{exc}(F) \leq \frac{n(G) + n_2(G)}{4} + 1$  as in (1.3). This is not always possible, but in this case, one can find an even cover  $F'$  such that  $e \notin E(F')$  and  $\text{exc}(F')$  satisfies (1.3). In addition, if for every even cover  $F$  such that  $e \notin F$  violates (1.3), a similar situation holds. To prove this behavior indeed occurs for even covers, an understanding of the structural properties of the graphs for when (1.3) fails is necessary.

In Section 3.1, we develop our key definitions and state our accompanying technical theorem (Theorem 3.1.4) from which Theorem 1.1.2 will follow. In Section 3.2, we provide some technical lemmas on the structure of the extremal graphs for Theorem 3.1.4, which we call  $\theta$ -chains. We complete the proof of Theorem 3.1.4 in Section 3.3. In Section 3.4, we characterize extremal graphs for Theorem 3.1.4. In Section 3.5, we give a quadratic-time algorithm that finds an even cover  $F$  in simple 2-connected subcubic graphs  $G$  with  $\text{exc}(F) \leq \frac{n(G) + n_2(G)}{4} + 1$ .

We end this section with notation. For a positive integer  $k$ , let  $[k] = \{1, \dots, k\}$ . If  $G$  and  $H$  are graphs, we write  $G \cup H$  (respectively,  $G \cap H$ ) to denote the union (intersection) of  $G$  and  $H$ . Let  $G$  be a graph. If  $S$  is a set of vertices or a set of edges, we let  $G - S$  denote the subgraph of  $G$  obtained by deleting elements of  $S$  as well as edges incident with a vertex in  $S$ . When  $S = \{s\}$  is a singleton, we simply write  $G - s$ . If  $H$  is a subgraph of  $G$ , we let  $G - H := G - V(G \cap H)$ . For a collection of 2-element subsets of  $V(G)$ , we write  $G + S$  for the graph with vertex set  $V(G)$  and edge set  $E(G) \cup S$ . However, for  $x, y \in V(G)$  we use  $G + xy$  to denote the graph obtained from  $G$  by adding a (possibly

parallel) edge between  $x$  and  $y$ . For a subgraph  $H \subseteq G$  and a set  $S \subseteq V(G)$ , we let  $H + S$  denote the subgraph of  $G$  such that  $V(H + S) = V(H) \cup S$  and  $E(H + S) = E(H)$ . For  $S \subseteq V(G)$ , we use  $N(S)$  to denote the neighborhood of  $S$  in  $G$ . If  $S = \{s\}$  is a singleton, we simply write  $N(s)$ . When  $|N(S)| \in \{1, 2\}$ , *suppressing*  $S$  means deleting  $S$  and adding a (possibly loop or parallel) edge between the vertices of  $N(S)$ . When  $S = \{s\}$  is a singleton, *suppressing*  $s$  means suppressing  $\{s\}$ .

## CHAPTER 2

### TUTTE PATHS

In this chapter, we provide results on bounding the number of nontrivial bridges of Tutte paths in circuit graphs.

#### 2.1 Special Cases

In this section, we prove some technical lemmas and special cases to Theorem 1.2.1. The following lemma is concerned with the base cases of the induction.

**Lemma 2.1.1.** *Let  $(G, C)$  be a circuit graph and let  $u, v \in V(C)$  be distinct and  $e \in E(C)$ , such that  $u, e, v$  occur on  $C$  in clockwise order. If  $e = uv$  or  $|G| = 3$  then Theorem 1.2.1 holds. In particular,  $G$  has a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that  $e \in E(P)$ , and  $\beta(P) = 1$  if  $e = uv$  and  $\beta(P) = 0$  if  $e \neq uv$  and  $|G| = 3$ .*

*Proof.* As  $G$  is 2-connected, we have  $|G| \geq 3$ . First suppose  $e = uv$ . Then  $vCu$  is not good because of the 2-separation  $(uCv, G - uv)$ ; so  $\tau_{vu} = 1$ . Moreover, since  $u, v$  are both incident with  $e$ ,  $\tau_{ue} = \tau_{ev} = 1$ . Hence,  $P := uv$  gives the desired  $C$ -Tutte path as  $\beta(P) = 1 = (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev}$ .

Now assume  $e \neq uv$  and  $|G| = 3$ . Further assume by symmetry that  $u$  is not incident with  $e$ . Then  $\tau_{vu} = 0$ ,  $\tau_{ue} = 1/2$ , and  $\tau_{ev} = 1$ . Hence,  $P := C - uv$  gives the desired  $C$ -Tutte path as  $\beta(P) = 0 = (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev}$ . □

The following two lemmas are concerned with the existence of particular 2-cuts.

**Lemma 2.1.2.** *Suppose  $n \geq 4$  is an integer and Theorem 1.2.1 holds for graphs on at most  $n - 1$  vertices. Let  $(G, C)$  be a circuit graph on  $n$  vertices,  $u, v \in V(C)$  be distinct, and  $e \in E(C)$  such that  $u, e, v$  occur on  $C$  in clockwise order. If  $G$  has a 2-separation  $(G_1, G_2)$*

such that  $\{u, v\} \subseteq V(G_1)$ ,  $\{u, v\} \not\subseteq V(G_2)$ ,  $e \in E(G_2)$ , and  $|G_2| \geq 3$ , then  $G$  has a  $C$ -Tutte Path  $P$  between  $u$  and  $v$  such that  $e \in E(P)$  and  $\beta(P) \leq (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev}$ .

*Proof.* Let  $V(G_1 \cap G_2) = \{x, y\}$  with  $x \in V(eCv)$  and  $y \in V(uCe)$ . See Figure 2.1. Let  $G'_i := G_i + xy$  for  $i \in \{1, 2\}$  such that  $G'_1$  is a plane graph with outer cycle  $C_1 := xCy + yx$  and  $G'_2$  is a plane graph with outer cycle  $C_2 := yCx + xy$ , i.e., both  $(G'_1, C_1)$  and  $(G'_2, C_2)$  are circuit graphs. Let  $e_1 := xy$ . Since  $\{u, v\} \not\subseteq V(G_2)$ , we may assume by symmetry that  $u \neq y$ .

By assumption,  $G'_1$  has a  $C_1$ -Tutte path between  $u$  and  $v$  such that  $e_1 \in E(P_1)$  and

$$\beta_{G'_1}(P_1) \leq (|P_1| - 6)/2 + \tau_{G'_1vu} + \tau_{G'_1ue_1} + \tau_{G'_1e_1v},$$

and  $G'_2$  has a  $C_2$ -Tutte path  $P_2$  between  $x$  and  $y$  such that  $e \in E(P_2)$  and

$$\beta_{G'_2}(P_2) \leq (|P_2| - 6)/2 + \tau_{G'_2xy} + \tau_{G'_2ye} + \tau_{G'_2ex}.$$

Let  $P := (P_1 - e_1) \cup P_2$ , and note  $P$  is a  $C$ -Tutte path in  $G$  between  $u$  and  $v$  such that  $e \in E(P)$ . Moreover, as  $|P_1| + |P_2| = |P| + 2$ ,  $\tau_{G'_1vu} = \tau_{Gvu}$ , and  $\tau_{G'_2xy} = 0$ , we have,

$$\begin{aligned} \beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|P| - 6)/2 - 2 + \tau_{Gvu} + \tau_{G'_1ue_1} + \tau_{G'_1e_1v} + \tau_{G'_2ye} + \tau_{G'_2ex}. \end{aligned}$$

We claim that  $\tau_{G'_1e_1v} + \tau_{G'_2ex} \leq \tau_{Gev} + 1$ . This is clear if  $\tau_{Gev} = 1$ . If  $\tau_{Gev} = 1/2$  then  $|eCv| = 2$ , and, hence,  $|e_1C_1v| = |xCv| = 2$  or  $|eC_2x| = |eCx| = 2$ ; so  $\tau_{G'_1e_1v} = 1/2$  or  $\tau_{G'_2ex} = 1/2$ , and the claim holds as well. Now suppose  $\tau_{Gev} = 0$ . Then  $|eCv| \geq 3$  and  $eCv$  is good. So  $|e_1C_1v| \geq 3$  and  $e_1C_1v$  is good in  $G'_1$ , or  $|eC_2x| \geq 3$  and  $|eC_2x|$  is good in  $G'_2$ , or  $|e_1C_1v| = |eC_2x| = 2$ . Hence,  $\tau_{G'_1e_1v} = 0$ , or  $\tau_{G'_2ex} = 0$ , or  $\tau_{G'_1e_1v} = \tau_{G'_2ex} = 1/2$ . In either case, the claim holds.

By similar argument,  $\tau_{G'_1ue_1} + \tau_{G'_2ye} \leq \tau_{Gue} + 1$ . So  $\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} +$

$\tau_{Gue} + \tau_{Gev}$ .

□

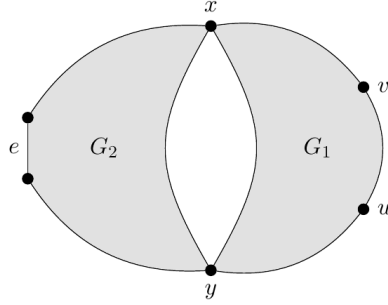


Figure 2.1: The separation  $(G_1, G_2)$  of  $G$ .

**Lemma 2.1.3.** *Suppose  $n \geq 4$  is an integer and Theorem 1.2.1 holds for graphs on at most  $n - 1$  vertices. Let  $(G, C)$  be a circuit graph on  $n$  vertices,  $u, v \in V(C)$  be distinct, and  $e = xy \in E(C)$ , such that  $u, x, y, v$  occur on  $C$  in clockwise order.*

*If  $\{u, x\}$  or  $\{v, y\}$  is a 2-cut in  $G$  then  $G$  has a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that  $e \in E(P)$  and  $\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$ .*

*Proof.* Suppose  $\{u, x\}$  or  $\{v, y\}$  is a 2-cut in  $G$ , say  $\{u, x\}$  by symmetry. Then  $G$  has a 2-separation  $(G_1, G_2)$  such that  $xCu \subseteq G_1$ ,  $uCx \subseteq G_2$ , and  $|G_2| \geq 3$ . We choose  $(G_1, G_2)$  so that  $G_2$  is maximal. See Figure 2.2. Note then  $ux \notin E(G_1)$  and  $\tau_{Gue} = 1$ .

*Case 1.*  $G_1$  is 2-connected.

Let  $C_1$  denote the outer cycle of  $G_1$ . As  $(G, C)$  is a circuit graph,  $(G_1, C_1)$  is a circuit graph as well. By assumption,  $G_1$  has a  $C_1$ -Tutte path  $P$  between  $u$  and  $v$  such that  $e \in E(P)$  and

$$\beta_{G_1}(P) \leq (|P| - 6)/2 + \tau_{G_1vu} + \tau_{G_1ue} + \tau_{G_1ev}.$$

Note that  $\tau_{G_1vu} = \tau_{Gvu}$ ,  $\tau_{G_1ue} = 0$  (as  $ux \notin E(G_1)$ ), and  $\tau_{G_1ev} = \tau_{Gev}$ . So

$$\beta_G(P) = \beta_{G_1}(P) + 1 \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$$

and thus  $P$  is the desired path.

Case 2.  $G_1$  is not 2-connected.

Let  $G'_1 := G_1 + ux$  be the plane graph with outer cycle  $C_1 := xCu + ux$ , and let  $G'_2 := G_2 + xu$  be the plane graph with outer cycle  $C_2 := uCx + xu$ . As  $(G, C)$  is a circuit graph, we see that both  $(G'_1, C_1)$  and  $(G'_2, C_2)$  are circuit graphs. Note  $\tau_{G'_1vu} = \tau_{Gvu}$ ,  $\tau_{G'_1ue} = 1/2$ , and  $\tau_{G'_1ev} = \tau_{Gev}$ . By assumption,  $G'_1$  has a  $C_1$ -Tutte path  $P_1$  between  $u$  and  $v$  such that  $e \in E(P_1)$  and

$$\begin{aligned}\beta_{G'_1}(P_1) &\leq (|P_1| - 6)/2 + \tau_{G'_1vu} + \tau_{G'_1ue} + \tau_{G'_1ev} \\ &= (|P_1| - 6)/2 + \tau_{Gvu} + (\tau_{Gue} - 1/2) + \tau_{Gev}.\end{aligned}$$

As  $G_1$  is not 2-connected,  $ux \in E(P_1)$ .

Choose  $e' \in E(uC_2x)$  such that  $\tau_{G'_2e'x} = 1/2$  and  $\tau_{G'_2ue'} \leq 1$ . Note  $\tau_{G'_2xu} = 0$ . By assumption,  $G'_2$  has a  $C_2$ -Tutte path  $P_2$  between  $x$  and  $u$  such that  $e' \in E(P_2)$  and

$$\beta_{G'_2}(P_2) \leq (|P_2| - 6)/2 + \tau_{G'_2xu} + \tau_{G'_2ue'} + \tau_{G'_2e'x} \leq (|P_2| - 6)/2 + 3/2$$

Now  $P := (P_1 - ux) \cup P_2$  is a  $C$ -Tutte path in  $G$  between  $u$  and  $v$  such that  $e \in E(P)$ .

Moreover,

$$\begin{aligned}\beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|P_1| - 6)/2 + \tau_{Gvu} + (\tau_{Gue} - 1/2) + \tau_{Gev} + (|P_2| - 6)/2 + 3/2 \\ &< (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.\end{aligned}$$

So  $P$  is the desired path. □

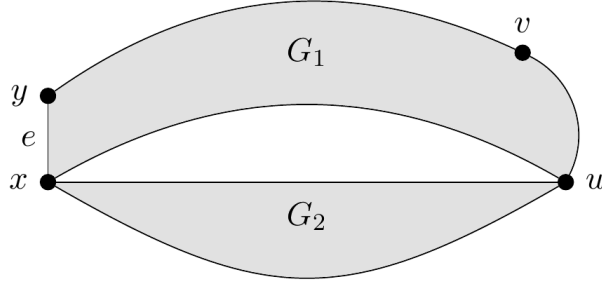


Figure 2.2: The other separation  $(G_1, G_2)$  of  $G$ .

The following lemma is included for later convenience where we want to find a Tutte path that includes three designated vertices. A vertex count analog to this lemma first appeared in [68].

**Lemma 2.1.4.** *Suppose  $n \geq 3$  is an integer and Theorem 1.2.1 holds for graphs on at most  $n - 1$  vertices. Let  $(G, C)$  be a circuit graph on  $n$  vertices and  $u, v, z \in V(C)$  be distinct such that  $u, z, v$  occur on  $C$  in clockwise order. Then  $G$  has a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that  $z \in V(P)$  and  $\beta(P) \leq (|P| - 3)/2 + \tau_{vu}$ .*

*Proof.* First suppose  $n = 3$  and choose  $e \in E(G)$  such that  $e \neq uv$ . Then by Lemma 2.1.1, then  $G$  has a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that  $\beta(P) = 0$ . In particular,  $P$  is Hamiltonian and thus  $z \in V(P)$ . We proceed with induction on  $n$ .

Now suppose  $n > 3$  and there is no 2-cut separating  $z$  from  $\{u, v\}$ . Choose an edge  $e$  such that  $\tau_{Gue} \leq 1/2$ . By induction there exists a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that  $e \in V(P)$  and

$$\begin{aligned} \beta_G(P) &\leq (|P| - 6)/2 + \tau_{Gue} + \tau_{Gev} + \tau_{Gvu} \\ &\leq (|P| - 3)/2 + \tau_{Gvu}. \end{aligned}$$

As there is no 2-cut separating  $z$  from  $\{u, v\}$ , we have  $z \in V(P)$ .

Now let  $(G_1, G_2)$  be a 2-separation in  $G$  with  $\{u, v\} \subseteq V(G_1)$  and  $z \in V(G_2)$ . Let  $V(G_1 \cap G_2) = \{x, y\}$  such that  $u, x, z, y, v$  appear on  $C$  in clockwise order. Let  $G'_1 :=$

$G_1 + xy$  and  $G'_2 := G_2 + xy$  be plane graphs with respective outer cycles  $C_1 := yCx + xy$  and  $C_2 := xCy + xy$ . Then  $(G'_i, C_i)$  are circuit graphs for  $i \in [2]$ .

As  $|G'_1| < n$ , Theorem 1.2.1 holds for  $G'_1$  (by assumption), thus  $G'_1$  has a  $C_1$ -Tutte path  $P_1$  between  $u$  and  $v$  such that  $e := xy \in E(P_1)$  and

$$\begin{aligned}\beta_{G'_1}(P_1) &\leq (|P_1| - 6)/2 + \tau_{G'_1 ue} + \tau_{G'_1 ev} + \tau_{G'_1 vu} \\ &\leq (|P_1| - 2)/2 + \tau_{G'_1 vu}.\end{aligned}$$

Note that  $\tau_{G'_2 yx} = 0$ . Thus, by induction,  $G'_2$  has a  $C_2$ -Tutte path  $P_2$  between  $x$  and  $y$  such that  $z \in V(P_2)$  and

$$\beta_{G'_2}(P_2) \leq (|P_2| - 3)/2.$$

Let  $P := (P_1 - xy) \cup P_2$ . As  $|P_1| + |P_2| = |P| + 2$ , we have

$$\begin{aligned}\beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|P_1| - 2)/2 + \tau_{G'_1 vu} + (|P_2| - 3)/2 \\ &= (|P| - 3)/3 + \tau_{Gvu}.\end{aligned}$$

As  $z \in V(P)$ , we have constructed the desired Tutte path. □

## 2.2 Proof of Theorem 1.2.1

We apply induction on  $n = |G|$ . By Lemma 2.1.1 and by symmetry, we may assume that  $u$  is not incident with  $e$  and  $|G| = n \geq 4$ , and the assertion holds for graphs on at most  $n - 1$  vertices. Let  $e = v'v''$  such that  $u, v', v'', v$  occur on  $C$  in clockwise order. We may conclude the following by Lemma 2.1.3.

**Claim 2.2.1.** *Neither  $\{u, v'\}$  nor  $\{v', v''\}$  is a cut in  $G$ .*

Moreover, by Lemma 2.1.2, we may assume that  $G$  has no 2-cut  $T \neq \{u, v\}$  separating  $e$  from  $\{u, v\}$ . Thus, by planarity,  $uCe$  is contained in a block of  $G - eCv$ , which is denoted by  $H$ . (See Figure 2.3) Note that  $H \cong K_2$  or  $H$  is 2-connected.

**Claim 2.2.2.**  *$H$  is 2-connected.*

*Proof.* For, suppose that  $H \cong K_2$ . Note that  $v'$  must have degree 2 in  $G$  and  $G - v'$  is 2-connected; for otherwise, there exists a vertex  $z \in V(v''Cv)$  such that  $\{v, z\}$  is a 2-cut in  $G$  separating  $e$  from  $\{u, v\}$ , contradicting Lemma 2.1.2. Let  $C' := v''Cu + uv''$  be the outer cycle of  $G' := (G - v') + uv''$ , and let  $e' := uv''$ . Note that  $(G', C')$  is a circuit graph,  $\tau_{G'ue'} = 1 = \tau_{Gue} + 1/2$ ,  $\tau_{G'e'v} = \tau_{Gev}$ , and  $\tau_{G'vu} = \tau_{Gvu}$ . Hence, by the induction hypothesis,  $G'$  has a  $C'$ -Tutte path  $P'$  between  $u$  and  $v$  such that  $e' \in E(P')$  and

$$\beta_{G'}(P') \leq (|P'| - 6)/2 + \tau_{G'vu} + \tau_{G'ue'} + \tau_{G'e'v}$$

Now  $P := (P' - e') \cup uv'v''$  is a  $C$ -Tutte path in  $G$  between  $u$  and  $v$  such that  $e \in E(P)$  and  $\beta_G(P) = \beta_{G'}(P') \leq (|P'| - 6)/2 + \tau_{G'vu} + \tau_{G'ue'} + \tau_{G'e'v} = (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$ . ■

By Claim 2.2.2, let  $D$  denote the outer cycle of  $H$ . Our strategy is to use the induction hypothesis to find a path in  $H$  and extend it to the desired path in  $G$  along  $eCv$ . Let  $w \in V(vCu)$  such that  $wDv' = wCv'$  and, subject to this,  $wCv'$  is maximal. By Lemma 2.1.4, we have the following claim.

**Claim 2.2.3.**  *$H$  contains a  $D$ -Tutte path  $P_H$  between  $u$  and  $v'$  such that  $w \in V(P_H)$  and*

$$\beta_H(P_H) \leq (|P_H| - 6)/2 + \tau_{Gue} + 3/2.$$

We wish to extend  $P_H$  along  $eCv$  to the desired path  $P$  in  $G$ . Thus we need a useful description of the structure of the part of  $G$  that lies between  $H$  and  $eCv$ . See Figure 2.3.

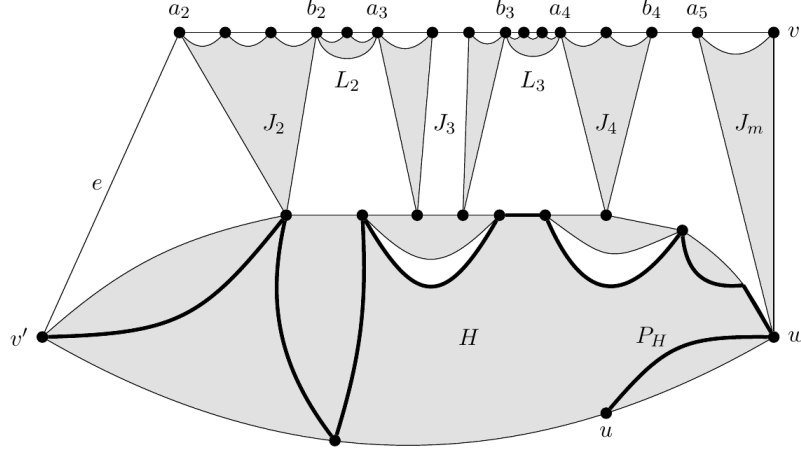


Figure 2.3: The subgraph  $H$  of  $G$  and the bridges between  $eCv$  and  $H$

Let  $\mathcal{B}$  be the set of  $(H \cup eCv)$ -bridges of  $G$ . Then  $G = H \cup eCv \cup (\bigcup_{B \in \mathcal{B}} B)$ . As  $H$  is a block of  $G - eCv$ ,  $|B \cap H| \leq 1$  for all  $B \in \mathcal{B}$ .

For  $B_1, B_2 \in \mathcal{B}$  with  $|B_1 \cap H| = 1 = |B_2 \cap H|$ , we denote by  $B_1 \sim B_2$  if  $V(B_1 \cap H) = V(B_2 \cap H) \subseteq V(P_H)$ , or if there exists a  $P_H$ -bridge  $B$  of  $H$  such that  $V(B_1 \cap H) \cup V(B_2 \cap H) \subseteq V(B - P_H)$ . Clearly,  $\sim$  is an equivalence relation on  $\mathcal{B}$ . Let  $\mathcal{B}_i$ ,  $i \in [m]$ , be the equivalence classes of  $\mathcal{B}$  with respect to  $\sim$ , such that  $H \cap (\bigcup_{B \in \mathcal{B}_i} B)$ ,  $i \in [m]$ , occur on  $D$  in order from  $v'$  to  $w$ , with  $v' \in V(B)$  for all  $B \in \mathcal{B}_1$  and  $w \in V(B')$  for all  $B' \in \mathcal{B}_m$ . Let  $a_i, b_i \in V(eCv)$  such that

- (a)  $a_i \in V(B)$  for some  $B \in \mathcal{B}_i$  and  $b_i \in V(B')$  for some  $B' \in \mathcal{B}_i$  (possibly  $B = B'$ ),
- (b)  $v'', a_i, b_i, v$  occur on  $eCv$  in order, and
- (c) subject to (a) and (b),  $a_i C b_i$  is maximal.

Note that  $v'' = a_1$  and  $b_m = v$ . Let  $J_i$  denote the union of  $a_i C b_i$ , all members of  $\mathcal{B}_i$ , those  $(H \cup eCv)$ -bridges of  $G$  whose attachments are all contained in  $a_i C b_i$ , and the  $P_H$ -bridge of  $H$  containing  $B \cap H$  for all  $B \in \mathcal{B}_i$ .

For  $1 < i < m$ , let  $L_i$  denote the union of  $b_i C a_{i+1}$  and those  $(eCv \cup H)$ -bridges of  $G$  whose attachments are all contained in  $b_i C a_{i+1}$ . Note that  $|J_i \cap P_H| \in \{1, 2\}$  for all  $1 \leq i \leq m$ . Let

- $\mathcal{J}_1$  be the set of all  $J_i$ ,  $1 < i < m$ , such that  $|J_i \cap P_H| = 1$  and  $a_i \neq b_i$ ,
- $\mathcal{J}_2$  be the set of all  $J_i$ ,  $1 < i < m$ , such that  $|J_i \cap P_H| = 2$ , and
- $\mathcal{L}$  be the set of all  $L_i$ ,  $1 < i < m$ .

By Lemma 2.1.2, we have  $|J_1| = 2$ . Thus we have the following claim.

**Claim 2.2.4.**  $J_1$  has a path  $P_1 = J_1$  such that  $\beta_{J_1}(P_1) = 0 = (|J_1| - 1)/2 - 1/2$ .

**Claim 2.2.5.** For  $J_i \in \mathcal{J}_1$ ,  $J_i$  has an  $a_i$ - $b_i$  path  $P_i$  such that  $P_i \cup (J_i \cap P_H)$  is an  $a_i C b_i$ -Tutte subgraph of  $J_i$  and

$$\beta_{J_i}(P_i \cup (J_i \cap P_H)) \leq \begin{cases} (|P_i| - 1)/2 - 1, & \text{if } eCv \text{ is good and } |a_i C b_i| \geq 3, \\ (|P_i| - 1)/2 - 1/2, & \text{if } eCv \text{ is good and } |a_i C b_i| = 2, \\ (|P_i| - 1)/2, & \text{otherwise.} \end{cases}$$

*Proof.* Let  $V(J_i \cap P_H) = \{x\}$ . Consider the plane graph  $J'_i := J_i + a_i x$  with outer cycle  $C_i$  consisting of  $a_i C b_i$ , the edge  $e_i := x a_i$ , and the path in the outer walk of  $J_i$  between  $b_i$  and  $x$  not containing  $a_i$ . Then  $(J'_i, C_i)$  is a circuit graph. Note that  $\tau_{J'_i x e_i} = 1$  and  $\tau_{J'_i b_i x} = 0$ .

Hence, by the induction hypothesis,  $J'_i$  has a  $C_i$ -Tutte path  $P'_i$  between  $x$  and  $b_i$  such that  $e_i \in E(P'_i)$  and  $\beta_{J'_i}(P'_i) \leq (|P'_i| - 6)/2 + \tau_{J'_i e_i b_i} + 1$ . Note that  $\tau_{J'_i e_i b_i} \leq 1$ . If  $eCv$  is good, then  $\tau_{J'_i e_i b_i} \leq 1/2$  (as  $a_i \neq b_i$ ), and  $\tau_{J'_i e_i b_i} = 0$  if  $|a_i C b_i| \geq 3$ . Let  $P_i := P'_i - x$ . As  $|P'_i| = |P_i| + 1$ ,  $P_i$  gives the desired path. ■

**Claim 2.2.6.** For  $J_i \in \mathcal{J}_2$ ,  $J_i$  has an  $a_i$ - $b_i$  path  $P_i$  such that  $P_i \cup (J_i \cap P_H)$  is an  $a_i C b_i$ -Tutte subgraph of  $J'_i$  and

$$\beta_{J_i}(P_i \cup (J_i \cap P_H)) \leq \begin{cases} (|P_i| - 1)/2, & \text{if } eCv \text{ is good and } a_i \neq b_i, \\ (|P_i| - 1)/2 + 1, & \text{otherwise.} \end{cases}$$

*Proof.* If  $a_i = b_i$ , we let  $P_i = a_i$  which gives the desired path. Now suppose  $a_i \neq b_i$ . Let  $V(J_i \cap P_H) = \{x, y\}$ , let  $J'_i$  be the block of  $J_i - \{x, y\}$  containing  $a_i C b_i$ , and let  $C_i$  be the outer cycle of  $J'_i$  with  $a_i C_i b_i = a_i C b_i$ . We may further assume that  $v', y, x, w$  occur on  $D$  in clockwise order.

By planarity there exists a vertex  $z \in V(b_i C_i a_i) \setminus \{a_i, b_i\}$  such that  $b_i C_i z - z$  contains no neighbor of  $y$  and  $z C_i a_i - z$  contains no neighbor of  $x$ . By Lemma 2.1.4,  $J'_i$  contains a Tutte path  $P_i$  between  $a_i$  and  $b_i$  such that  $z \in V(P_i)$  and

$$\beta_{J'_i}(P_i) \leq (|P_i| - 6)/2 + \tau_{J'_i a_i b_i} + 3/2.$$

As  $\beta_{J_i}(P_i) \leq \beta_{J'_i}(P_i) + 1$ , and  $\tau_{J'_i a_i b_i} = 0$  if  $eCv$  is good,  $P_i$  is the desired Tutte path.  $\blacksquare$

**Claim 2.2.7.**  $J_m$  has a path  $P_m$  between  $a_m$  and  $b_m = v$  such that  $P_m + w$  is an  $a_m C w$ -Tutte subgraph of  $J_m$  and

$$\beta_{J_m}(P_m + w) \leq \begin{cases} (|P_m| - 1)/2 - 1 + \tau_{Gvu}, & \text{if } eCv \text{ is good and } |a_m C b_m| \geq 3, \\ (|P_m| - 1)/2 - 1/2 + \tau_{Gvu}, & \text{if } eCv \text{ is good and } |a_m C b_m| = 2, \\ (|P_m| - 1)/2 + \tau_{Gvu}, & \text{otherwise.} \end{cases}$$

*Proof.* First, suppose  $a_m = b_m$ . Then  $P_m = a_m$  gives the desired path.

Now assume  $a_m \neq b_m$  and consider the plane graph  $J'_m := J_m + a_m w$  with outer cycle  $C_m := a_m C w + a_m w$ . Then  $(J'_m, C_m)$  is a circuit graph. Let  $e_m := a_m w$ . Note that  $\tau_{J'_m b_m w} \leq \tau_{Gvu}$  and  $\tau_{J'_m w e_m} = 1$ .

Hence, by induction hypothesis,  $J'_m$  has a  $C_m$ -Tutte path between  $b_m$  and  $w$  such that  $e_m \in E(P'_m)$  and

$$\begin{aligned} \beta_{J'_m}(P'_m) &\leq (|P'_m| - 6)/2 + \tau_{Gvu} + 1 + \tau_{J'_m e_m b_m} \\ &= (|P'_m| - 2)/2 - 1 + \tau_{Gvu} + \tau_{J'_m e_m b_m}. \end{aligned}$$

Note that  $\tau_{J'_m e_m b_m} \leq 1/2$  (when  $eCv$  is good) and  $\tau_{J'_m e_m b_m} \leq 1$  (when  $eCv$  is not good). Furthermore if  $eCv$  is good and  $|a_m C b_m| \geq 3$ , then  $\tau_{J'_m e_m b_m} = 0$ . Hence,  $P_m = P'_m - w$  gives the desired path. ■

Next, we consider those  $(eCv \cup H)$ -bridges of  $G$  with all attachments contained in  $eCv$ .

**Claim 2.2.8.**  $L_i$  contains a  $b_i C a_{i+1}$ -Tutte path  $Q_i$  from  $b_i$  to  $a_{i+1}$  such that

$$\beta_{L_i}(Q_i) \leq \begin{cases} (|Q_i| - 1)/2 - 1, & \text{if } |b_i C a_{i+1}| \geq 3, \\ (|Q_i| - 1)/2 - 1/2, & \text{if } |b_i C a_{i+1}| = 2, \\ (|Q_i| - 1)/2, & \text{otherwise.} \end{cases}$$

In particular,  $\beta_{L_i}(Q_i) \leq \max\{0, (|Q_i| - 1)/2 - 1\}$ .

*Proof.* If  $|b_i C a_{i+1}| \leq 2$  then let  $Q_i := b_i C a_{i+1}$ ; we see that  $\beta_{L_i}(Q_i) = 0$ . So suppose  $|b_i C a_{i+1}| \geq 3$ . Then consider the plane graph  $L'_i := L_i + b_i a_{i+1}$  with outer cycle  $D_i := b_i C a_{i+1} + a_{i+1} b_i$ . Note that  $(L'_i, D_i)$  is a circuit graph. Choose an edge  $e'_i \in E(b_i C a_{i+1})$  so that  $\tau_{L'_i b_i e'_i} = 1/2$ . Note that  $\tau_{L'_i a_{i+1} b_i} = 0$  and  $\tau_{L'_i e'_i a_{i+1}} \leq 1$ . Then by induction hypothesis,  $L'_i$  contains a  $D_i$ -Tutte path  $Q_i$  between  $b_i$  and  $a_{i+1}$  such that  $e'_i \in E(Q_i)$  and  $\beta_{L'_i}(Q_i) \leq (|Q_i| - 6)/2 + 3/2 = (|Q_i| - 1)/2 - 1$ . ■

We now form the path  $P$  by taking the union of  $P_H, P_i$  for  $i \in [m]$ , and  $Q_i$  for  $i \in [m - 1]$ . Clearly,  $P$  is between  $u$  and  $v$  and contains  $e$ .

It is easy to see that if  $B$  is a  $P$ -bridge of  $G$  then  $B$  is a  $P_H$ -bridge of  $H$ , or a  $(P_i \cup (J_i \cap P_H))$ -bridge of  $J_i$ , for some  $J_i \in \mathcal{J}_1 \cup \mathcal{J}_2$  or a  $Q_i$ -bridge of some  $L_i$ . Thus,  $P$  is a  $C$ -Tutte path in  $G$  between  $u$  and  $v$  containing  $e$ .

If we extend  $P_H$  from  $v'$  to  $v$  through  $J_1, L_1, J_2, L_2, \dots, J_{m-1}, L_{m-1}, J_m$  in order, we see that

- $P_1$  and  $P_H$  double count  $v'$ ;

- $P_m$  and  $P_H \cup (P_1 \cup Q_1) \cup \dots \cup (P_{m-1} \cup Q_{m-1})$  double count  $a_m$ ;
- for  $1 < i < m$ ,  $P_i$  and  $P_H \cup (P_1 \cup Q_1) \cup \dots \cup (P_{i-1} \cup Q_{i-1})$  double count  $a_i$ .
- for  $1 < i < m$ ,  $Q_i$  and  $P_H \cup (P_1 \cup Q_1) \cup \dots \cup (P_{i-1} \cup Q_{i-1}) \cup P_i$  double count  $b_i$ .

Note that for each  $J_i \in \mathcal{J}_2$ , the  $P_H$ -bridge of  $H$  contained in  $J_i$  does not contribute to the count of  $\beta_G(P)$ . We calculate  $\beta_G(P)$  as follows.

$$\begin{aligned} \beta_G(P) &= \beta_H(P_H) + \beta_{J_1}(P_1) + \beta_{J_m}(P_m + w) + \sum_{J_i \in \mathcal{J}_1} \beta_{J_i}(P_i \cup (J_i \cap P_H)) + \\ &\quad \sum_{J_i \in \mathcal{J}_2} (\beta_{J_i}(P_i \cup (J_i \cap P_H)) - 1) + \sum_{i=1}^{m-1} \beta_{L_i}(Q_i). \end{aligned}$$

Suppose  $eCv$  is not good. Then  $\tau_{Gev} = 1$ . Thus, by Claims 2.2.3, 2.2.4, 2.2.5, 2.2.6, 2.2.7, and 2.2.8, we have

$$\begin{aligned} \beta_G(P) &\leq ((|P_H| - 6)/2 + \tau_{Gue} + 3/2) + ((|P_1| - 1)/2 - 1/2) + ((|P_m| - 1)/2 + \tau_{Gvu}) \\ &\quad + \sum_{J_i \in \mathcal{J}_1 \cup \mathcal{J}_2} (|P_i| - 1)/2 + \sum_{L_i \in \mathcal{L}} \max\{0, (|Q_i| - 1)/2 - 1\} \\ &\leq (|P| - 6)/2 + 1 + \tau_{Gvu} + \tau_{Gue} \\ &= (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}. \end{aligned}$$

So suppose  $eCv$  is good. Let  $\mathcal{J}'_1 := \{J_i \in \mathcal{J}_1 \text{ or } i = m : |a_i C b_i| \geq 3\}$ ,  $\mathcal{J}''_1 := \{J_i \in \mathcal{J}_1 \text{ or } i = m : |a_i C b_i| = 2\}$ ,  $\mathcal{J}'_2 := \{J_i \in \mathcal{J}_2 : a_i \neq b_i\}$ ,  $\mathcal{J}''_2 := \{J_i \in \mathcal{J}_2 : a_i = b_i\}$ ,  $\mathcal{L}' := \{L_i \in \mathcal{L} : |b_i C a_{i+1}| \geq 3\}$ , and  $\mathcal{L}'' := \{L_i \in \mathcal{L} : |b_i C a_{i+1}| = 2\}$ . Note then by Claims 2.2.3, 2.2.4, 2.2.5, 2.2.6, 2.2.7, and 2.2.8 we have,

$$\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + 1 - |\mathcal{J}'_1| - |\mathcal{J}''_1|/2 - |\mathcal{J}'_2| - |\mathcal{L}'| - |\mathcal{L}''|/2.$$

We may assume  $|\mathcal{J}'_1| = |\mathcal{J}'_2| = |\mathcal{L}'| = 0$  as otherwise (1.1) holds. If  $|eCv| = 2$ , then  $\tau_{Gev} = 1/2$  and if  $|eCv| \geq 3$ , then  $\tau_{Gev} = 0$ . As  $|\mathcal{J}''_1| + |\mathcal{L}''| = |eCv| - 1$  (follows from

$|\mathcal{J}'_1| = |\mathcal{J}'_2| = |\mathcal{L}'| = 0$ ), (1.1) holds and Theorem 1.2.1 follows.  $\square$

We now apply Theorem 1.2.1 to recover the original technical theorem of Wigal and Yu [69]. A challenge to their theorem was accounting for the vertices in the bridges of the path. Care had to be taken to avoid double counting these vertices, as the path may need to be extended through a bridge in a previous iteration of the induction. To handle this, a contraction strategy is employed.

**Corollary 2.2.9.** [69] *Let  $n \geq 3$  be an integer, let  $(G, C)$  be a circuit graph on  $n$  vertices, let  $u, v \in V(C)$  be distinct, and let  $e \in E(C)$ , such that  $u, e, v$  occur on  $C$  in clockwise order. Then  $G$  has a  $C$ -Tutte path between  $u$  and  $v$  such that  $e \in E(P)$  and*

$$\beta(P) \leq (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.$$

*Proof.* First suppose  $e = uv$ . Letting  $P := uv$ , we have that  $\tau_{Gvu} = 1, \tau_{Gue} = 1$ , and  $\tau_{Gev} = 1$ . As  $\beta_G(P) = 1$  and  $n \geq 3$ , the inequality holds. Now suppose  $e \neq uv$  and  $|G| = 3$ . By symmetry, we may assume  $u$  is not incident with  $e$ . Let  $P := C - uv$ . As  $\beta_G(P) = 0, \tau_{Gvu} = 0, \tau_{Gue} = 1/2$ , and  $\tau_{Gev} = 1$ , the inequality holds. Thus we may assume  $e \neq uv$  and  $|G| > 3$  and we proceed with induction on  $n$ .

**Claim 2.2.10.** *If  $(G_1, G_2)$  is a 2-separation in  $G$  such that  $u, v \in V(G_1), e \in E(G_1)$ , and  $V(G_1 \cap G_2) \subseteq V(C)$ , then  $|G_2| = 3$ .*

*Proof.* For otherwise, let  $(G_1, G_2)$  be a 2-separation in  $G$  such that  $u, v \in V(G_1), e \in E(G_1), V(G_1 \cap G_2) \subseteq V(C)$ , and  $|G_2| > 3$ . Let  $V(G_1 \cap G_2) = \{x, y\}$  such that  $yCx \subseteq G_1$  and  $xCy \subseteq G_2$ . Let  $G'_1 := G_1 + \{t, tx, ty\}$ , where  $t$  is a new vertex and  $C_1 := yCxt$ . Then  $(G'_1, C_1)$  is a circuit graph. We apply induction on  $G'_1$  to find a Tutte path  $P_1$  in  $G'_1$  between  $u$  and  $v$  such that  $e \in E(P_1)$  and

$$\beta_{G'_1}(P_1) \leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3$$

If  $t \notin V(P_1)$ , then let  $P := P_1$ . As  $|G'_1| \leq n$ ,  $\tau_{G'_1vu} = \tau_{Gvu}$ ,  $\tau_{G'_1ue} = \tau_{Gue}$ , and  $\tau_{G'_1ev} = \tau_{Gev}$ ,  $P$  is the desired Tutte path.

So suppose  $t \in V(P_1)$ . Let  $C_2 := xCy + yx$  and  $G'_2 := G_2 + xy$ . Note then  $(G'_2, C_2)$  is a circuit graph. As  $|xCy| \geq 3$ , choose edge  $e' \in E(xCy)$  such that  $\tau_{G'_2xe'} \leq 1/2$ . As  $\tau_{G'_2yx} = 0$ , by induction,  $G'_2$  has a  $C_2$ -Tutte path  $P_2$  between  $x$  and  $y$  such that  $e' \in E(P_2)$  and

$$\begin{aligned} \beta_{G'_2}(P_2) &\leq (|G'_2| - 6)/3 + (2\tau_{G'_2yx} + 2\tau_{G'_2xe'} + 2\tau_{G'_2e'y})/3 \\ &\leq (|G'_2| - 3)/3 \end{aligned}$$

Let  $P := (P_1 - t) \cup P_2$ . As  $|G'_1| + |G'_2| = n + 3$  we have

$$\begin{aligned} \beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3 + (|G'_2| - 3)/3 \\ &= (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3. \end{aligned}$$

■

**Claim 2.2.11.** *If there is a 3-separation  $(G_1, G_2)$  in  $G$  such that  $C \subseteq G_1$ , then  $|G_2| = 4$ .*

*Proof.* Assume otherwise, and let  $(G_1, G_2)$  be such a separation that minimizes  $|V(G_2)| + |E(G_2)|$  subject to  $|G_2| \geq 5$ . Let  $V(G_1 \cap G_2) = \{x, y, z\}$ .

Let  $G'_1$  be the graph with  $V(G'_1) = V(G_1) \cup \{t\}$  where  $t$  is a new vertex in the face of  $G_1$  containing  $G_2$ , and  $E(G'_1) = E(G_1) \cup \{xt, yt, zt\}$ . Note that  $(G'_1, C)$  is a circuit graph; so by induction there exists a  $C$ -Tutte path  $P_1$  between  $u$  and  $v$  such that  $e \in E(P_1)$  and

$$\beta_{G'_1}(P_1) \leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3.$$

If  $t \notin V(P_1)$ , we let  $P := P_1$ . As  $\tau_{G'_1vu} = \tau_{Gvu}$ ,  $\tau_{G'_1ue} = \tau_{Gue}$ ,  $\tau_{G'_1ev} = \tau_{Gev}$ , and

$|G'_1| < n$ , we have that  $P$  is the desired Tutte path.

Thus we may assume  $t \in V(P_1)$ . Without loss of generality we may assume  $xt, yt \in E(P_1)$ .

Suppose  $z \in V(P_1)$ . Let  $G'_2 := G_2 + \{xz, yz\}$  and  $C_2$  be the outer cycle of  $G'_2$  such that  $xzy \subseteq C_2$ . Then  $(G'_2, C_2)$  is a circuit graph. We may assume without loss of generality that  $x, z, y$  appear on  $C_2$  in clockwise order. By induction, as  $xC_2z$  and  $yC_2x$  are both good and  $|yC_2x| \geq 3$ , there exists a Tutte path  $P_2$  in  $G'_2$  between  $x$  and  $z$  such that  $yz \in E(P_2)$  and

$$\beta_{G'_2}(P_2) \leq (|G'_2| - 4)/3.$$

Let  $P := (P_1 - t) \cup (P_2 - z)$ . As  $|G'_1| + |G'_2| = |G| + 4$ , we have that

$$\begin{aligned} \beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3 + (|G'_2| - 4)/3 \\ &= (n - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3. \end{aligned}$$

Now assume  $z \notin V(P_1)$ . Note by our choice of the minimality of  $|V(G_2)| + |E(G_2)|$ ,  $xz, yz \notin E(G_2)$ . Let  $G'_2$  be the block of  $G_2 + xy$  containing  $x$  and  $y$  and  $C_2$  be its outer cycle. If  $z \in V(G'_2)$  we let  $z' := z$ , otherwise we let  $z' \in V(G_2)$  be the cut vertex of  $G_2 + xy$  separating  $z$  from  $x$  and  $y$ . We may assume  $x, z', y$  appear on  $C_2$  in clockwise order.

First suppose  $z' \neq z$ . By Lemma 2.1.4,  $G'_2$  has a  $C_2$ -Tutte path  $P_2$  between  $x$  and  $y$  such that  $z' \in V(P_2)$  and

$$\beta_{G'_2}(P_2) \leq (|P_2| - 3)/2.$$

Note  $\beta_{G'_2}(P_2) + |P_2| = |G'_2|$  as otherwise we could find a 3-cut contradicting the choice of

$(G_1, G_2)$ , the minimality of  $|V(G_2)| + |E(G_2)|$ . Thus

$$|G'_2| = |P_2| + \beta_{G'_2}(P_2) \leq (3|P_2| - 3)/2.$$

Thus  $(2|G'_2| + 3)/3 \leq |P_2|$  and  $\beta_{G'_2}(P_2) = |G'_2| - |P_2| \leq (|G'_2| - 3)/3$ . Let  $P = (P_1 - t) \cup P_2$ .

As  $|G'_1| + |G'_2| = |G| + 3$  (as  $z \neq z'$ ), we have

$$\begin{aligned} \beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3 + (|G'_2| - 3)/3 \\ &= (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3. \end{aligned}$$

Now suppose  $z = z'$ . As  $xz, yz \notin E(G'_2)$ , there is a choice of  $e' \in E(C_2)$  such that  $e'$  contains  $z'$  and  $\tau_{G'_2e'y} = 0$ . By Theorem 1.2.1,  $G'_2$  has a  $C_2$ -Tutte path  $P_2$  between  $x$  and  $y$  such that  $e' \in E(P_2)$  and

$$\beta_{G'_2}(P_2) \leq (|P_2| - 4)/2.$$

Let  $B$  be the  $P_1$ -bridge containing  $z$ . We claim that  $B - (P_1 \cap B) - z$  has at most one component. First suppose  $|P_1 \cap B| = 2$  and  $B - (P_1 \cap B) - z \neq \emptyset$ . As  $(G'_1, C)$  is a circuit graph and  $(P_1 \cap B)$  is a 2-cut, each component of  $G'_1 - (P_1 \cap B)$  contains a vertex of  $C$ . As  $C$  is a cycle, this implies  $B - (P_1 \cap B) - z$  has at most one component. Now suppose  $|P_1 \cap B| = 3$  and  $B - (P_1 \cap B) - z$  has at least two components. This would imply the existence of a  $K_{3,3}$  minor, directly contradicting the planarity of  $G'_1$ .

Again, by our choice of  $(G_1, G_2)$ , we have  $\beta_{G'_2}(P_2) + |P_2| = |G'_2|$  and  $|G'_2| \leq (3|P_2| - 4)/2$ . In particular  $(2|G'_2| + 4)/3 \leq |P_2|$  and  $\beta_{G'_2}(P_2) = |G'_2| - |P_2| \leq (|G'_2| - 4)/3$ . Letting  $P = (P_1 - t) \cup P_2$ , as  $B - (P_1 \cap B) - z$  has at most one component,  $\beta_G(P) =$

$\beta_{G'_1}(P_1) + \beta_{G'_2}(P_2)$ . As  $|G'_1| + |G'_2| = |G| + 4$ , we have

$$\begin{aligned}\beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3 + (|G'_2| - 4)/3 \\ &= (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.\end{aligned}$$

■

Let  $(G, C)$  be a circuit graph on  $n \geq 3$  vertices. By Theorem 1.2.1, there exists a  $C$ -Tutte path  $P$  between  $u$  and  $v$  such that

$$\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.$$

By Claims 2.2.10 and 2.2.11 we have  $n = |P| + \beta_G(P)$ . It follows,

$$n = |P| + \beta_G(P) \leq (3|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.$$

Solving for  $|P|$  we have

$$(2n + 6)/3 - 2(\tau_{Gvu} + \tau_{Gue} + \tau_{Gev})/3 \leq |P|.$$

Thus we have

$$\beta_G(P) = n - |P| \leq (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.$$

□

### 2.3 Essentially 4-connected Planar Graphs

We now apply Corollary 2.2.9 to obtain sharp circumference bounds for essentially 4-connected graphs. We follow the argument as presented in [69]. Note that we need  $n \geq 6$  in Theorem 2.3.1, however, when  $n \geq 5$  the graph  $G$  is Hamiltonian.

**Theorem 2.3.1.** [69] *Let  $n \geq 6$  be an integer and let  $G$  be any essentially 4-connected  $n$ -vertex planar graph. Then the circumference of  $G$  is at least  $\lceil (2n + 6)/3 \rceil$ .*

*Proof.* First suppose  $G$  is 4-connected. Fix a planar drawing of  $G$  and let  $T$  be the outer cycle of  $G$ . Let  $uv, e \in E(T)$  be distinct. By applying Corollary 2.2.9,  $G$  has a  $T$ -Tutte path  $P$  between  $u$  and  $v$  such that  $e \in E(P)$ . As  $G$  is 4-connected,  $\beta_G(P) = 0$ , so  $P$  is in fact a Hamiltonian path. Hence,  $P + uv$  is a Hamiltonian cycle in  $G$  and has length  $n$ , which is at least  $(2n + 6)/3$  (as  $n \geq 6$ ).

Hence, we may assume  $G$  is not 4-connected. Then, since  $G$  is essentially 4-connected, there exists  $x \in V(G)$  such that  $x$  has degree 3 in  $G$ . So let  $N_G(x) = \{u, v, w\}$  and let  $H := G - x$  and assume that  $H$  is a plane graph with  $u, v, w$  on the outer cycle  $C$  of  $H$  in counter clockwise order. Note that  $(H, C)$  is a circuit graph.

Suppose two of  $|uCw|, |wCv|, |vCu|$  is at least 3. Without loss of generality, we may assume that  $|uCw| \geq 3$  and  $|wCv| \geq 3$ . Let  $e \in E(uCw)$  be incident with  $w$ . Then  $\tau_{Hvu} = 0, \tau_{Hue} \leq 1/2$ , and  $\tau_{Hev} = 0$ . Hence by Corollary 2.2.9,  $H$  has a  $C$ -Tutte path between  $u$  and  $v$  such that  $e \in E(P)$  and  $\beta_H(P) \leq (n - 7)/2 + 1/3 = (n - 6)/3$ . Thus,  $Q := P \cup uCw$  is a Tutte cycle in  $G$  such that  $\beta_G(Q) \leq (n - 6)/3$ . Since  $G$  is essentially 4-connected, every  $Q$ -bridges is a  $K_{1,3}$ . Hence  $|Q| \geq n - (n - 6)/3 = (2n + 6)/3$ .

So we may assume that  $|wCv| = |vCu| = 2$ . Consider the plane graph  $K := H - wv$  whose outer cycle  $D$  contains  $vCw$ . Since  $G$  is essentially 4-connected,  $K$  is 2-connected; so  $(K, D)$  is a circuit graph. We can choose  $e \in E(wDv)$  incident with  $w$ . Now  $\tau_{Kvu} = 0, \tau_{Kue} \leq 1/2$ , and  $\tau_{Kev} \leq 1/2$ .

If  $\tau_{Kue} = 0$  or  $\tau_{Kev} = 0$ , then by Corollary 2.2.9,  $K$  has a  $D$ -Tutte path  $P$  between  $u$

and  $v$  such that  $e \in E(P)$  and  $\beta_K(P) \leq (n-7)/3 + 1/3 = (n-6)/3$ . Thus  $Q := P \cup uxv$  is a cycle in  $G$  with  $|Q| \geq n - (n-6)/3 = (2n+6)/3$ .

So assume  $\tau_{Kue} = \tau_{Kev} = 1/2$  and, hence,  $|wDv| = 3$  and  $|uDw| = 2$ . Since  $n \geq 6$  and  $G$  is essentially 4-connected, one of  $\{v, w\}$  has a neighbor inside  $D$ , say  $w$  by symmetry. Now consider the plane graph  $J := H - uw$ , which is 2-connected as  $G$  is essentially 4-connected. Let  $F$  denote the outer cycle of  $J$ , which contains  $\{u, v, w\}$ . Clearly,  $(J, F)$  is a circuit graph. Choose  $f \in E(uFw)$  incident with  $w$ . Then  $\tau_{Juf} \leq 1/2$ , and  $\tau_{Jfv} = 0$ , and  $\tau_{Jvu} = 0$ . Hence, by Corollary 2.2.9,  $J$  has an  $F$ -Tutte path between  $u$  and  $v$  such that  $f \in E(P)$  and  $\beta_J(P) \leq (n-7)/3 + 1/3 = (n-6)/3$ . Thus  $Q := P \cup uxv$  is a cycle in  $G$  with  $|Q| \geq n - (n-6)/3 = (2n+6)/3$ .

□

## CHAPTER 3

### EVEN COVERS

In this chapter, we provide results on even covers in subcubic graphs and apply them to the travelling salesperson problem. This chapter is joint work with Youngho Yoo and Xingxing Yu [67].

#### 3.1 Subcubic Chains

In order to help with induction, we consider even covers which contain or avoid a specified edge. Let  $G$  be a graph and let  $e \in E(G)$ . We write  $\mathcal{E}(G, e)$  to denote the set of even covers of  $G$  containing  $e$ , and  $\widehat{\mathcal{E}}(G, e)$  to denote the set of even covers of  $G$  not containing  $e$ . Define

$$\text{exc}(G, e) := \min_{F \in \mathcal{E}(G, e)} \text{exc}(F) - 2$$

$$\widehat{\text{exc}}(G, e) := \min_{F \in \widehat{\mathcal{E}}(G, e)} \text{exc}(F)$$

Clearly, we have  $\text{exc}(G) = \min\{\text{exc}(G, e) + 2, \widehat{\text{exc}}(G, e)\}$  for any edge  $e \in E(G)$ . The “ $-2$ ” in the definition of  $\text{exc}(G, e)$  leads to a natural interpretation of the quantities  $\delta(G, e)$  and  $\widehat{\delta}(G, e)$  defined below, and also results in simpler calculations as it accounts for the fact that the cycle  $C$  of  $F$  containing  $e$  will often only be used as a path  $C - e$  as part of a larger cycle (see Propositions 3.1.1 and 3.1.2).

To prove (1.3), it will be convenient to define the following parameters for a graph  $G$

and an edge  $e \in E(G)$ :

$$\delta(G, e) := \text{exc}(G, e) - \frac{n(G) + n_2(G)}{4},$$

$$\widehat{\delta}(G, e) := \widehat{\text{exc}}(G, e) - \frac{n(G) + n_2(G)}{4}.$$

Note that if every vertex of  $G$  has degree 2 or 3 (for instance, if  $G$  is subcubic and 2-connected), then  $\delta(G, e)$  and  $\widehat{\delta}(G, e)$  are always half-integral since  $n(G) + n_2(G) = (n(G) - n_2(G)) + 2n_2(G)$  where  $(n(G) - n_2(G))$  is the number of vertices of odd degree in  $G$ , which is always even.

A *subcubic chain*  $C$  is a simple connected subcubic graph, written as an alternating sequence  $C = xe_0B_1e_1B_2 \dots B_ke_ky$  for some nonnegative integer  $k$ , satisfying the following properties (see Figure 3.1):

- $\{e_0, \dots, e_k\}$  is the set of cut-edges of  $C$ ,
- $\{B_0, B_1, \dots, B_k, B_{k+1}\}$  is the set of connected components of  $C - \{e_0, \dots, e_k\}$ , where  $V(B_0) = \{x\}$  and  $V(B_{k+1}) = \{y\}$ ,
- $B_i$  is either a single vertex or 2-connected for all  $i \in [k]$ , and
- each  $e_i$  has one endpoint in  $B_i$  and one endpoint in  $B_{i+1}$  for all  $i = 0, \dots, k$ .

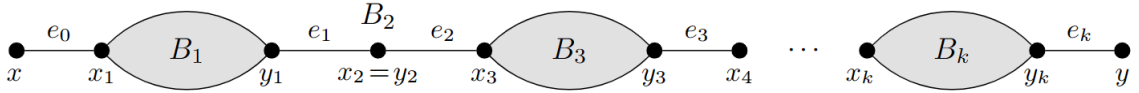


Figure 3.1: A subcubic chain

We say that  $C$  has *end points*  $x, y$  and has *end edges*  $e_0$  and  $e_k$ . A subcubic chain is *trivial* if  $k = 0$  (that is,  $C$  is an edge  $xy$ ), and *nontrivial* otherwise.

Let  $C = xe_0B_1e_1B_2 \dots B_ke_ky$  be a nontrivial subcubic chain. For  $i \in [k]$ , let  $x_i$  denote the endpoint of  $e_{i-1}$  in  $B_i$  and let  $y_i$  denote the endpoint of  $e_i$  in  $B_i$ . (Note that  $x_i \neq y_i$

when  $n(B_i) \neq 1$ , as  $C$  is subcubic.) We define  $\overline{B}_i = B_i + \overline{e}_i$  where  $\overline{e}_i = x_i y_i$ , and  $\overline{C} = C - \{x, y\} + e_C$  where  $e_C = x_1 y_k$ . We call each  $(\overline{B}_i, \overline{e}_i)$  a *chain-block* of  $C$ , and  $\overline{C}$  the *closure* of  $C$ . Note that the closure of a nontrivial subcubic chain  $C$  is a subcubic graph with no cut-vertex such that  $\overline{C} - e_C$  is simple. If  $C$  is a trivial subcubic chain, we define  $\text{exc}(\overline{C}, e_C) = \widehat{\text{exc}}(\overline{C}, e_C) = \delta(\overline{C}, e_C) = \widehat{\delta}(\overline{C}, e_C) = 0$ .

**Proposition 3.1.1.** *Let  $C = x e_0 B_1 e_1 B_2 \dots B_k e_k y$  be a subcubic chain, and let  $\{(\overline{B}_i, \overline{e}_i) : i \in [k]\}$  denote the chain-blocks of  $C$ . Then*

- $\text{exc}(\overline{C}, e_C) = \sum_{i=1}^k \text{exc}(\overline{B}_i, \overline{e}_i)$ ,
- $\widehat{\text{exc}}(\overline{C}, e_C) = \sum_{i=1}^k \widehat{\text{exc}}(\overline{B}_i, \overline{e}_i)$ ,
- $\delta(\overline{C}, e_C) = \sum_{i=1}^k \delta(\overline{B}_i, \overline{e}_i)$ , and
- $\widehat{\delta}(\overline{C}, e_C) = \sum_{i=1}^k \widehat{\delta}(\overline{B}_i, \overline{e}_i)$ .

*Proof.* If  $C$  is trivial then the proposition is true by definition (an empty sum is defined to be 0), so we may assume that  $C$  is nontrivial. Note that a cycle in  $\overline{C}$  contains  $e_C$  if and only if it contains all of  $e_1, \dots, e_{k-1}$ . This gives a natural bijective correspondence between even covers  $F \in \mathcal{E}(\overline{C}, e_C)$  and tuples of even covers  $(F_1, \dots, F_k)$  where  $F_i \in \mathcal{E}(\overline{B}_i, \overline{e}_i)$  for each  $i \in [k]$ . Indeed, this correspondence is obtained by “splitting” the cycle  $D$  of  $F$  containing  $e_C$  into  $k$  cycles,  $(D \cap \overline{B}_i) + \overline{e}_i$  for  $i \in [k]$ . With this correspondence, we have  $\text{exc}(F) = 2 + \sum_{i=1}^k (\text{exc}(F_i) - 2)$ . Hence,

$$\begin{aligned} \text{exc}(\overline{C}, e_C) &= \min_{F \in \mathcal{E}(\overline{C}, e_C)} \text{exc}(F) - 2 \\ &= \sum_{i=1}^k \min_{F_i \in \mathcal{E}(\overline{B}_i, \overline{e}_i)} (\text{exc}(F_i) - 2) \\ &= \sum_{i=1}^k \text{exc}(\overline{B}_i, \overline{e}_i). \end{aligned}$$

Since  $n(\overline{C}) = \sum_{i=1}^k n(\overline{B}_i)$  and  $n_2(\overline{C}) = \sum_{i=1}^k n_2(\overline{B}_i)$ , this also implies  $\delta(\overline{C}, e_C) = \sum_{i=1}^k \delta(\overline{B}_i, \overline{e}_i)$ .

Similarly, there is a natural bijective correspondence between even covers  $F \in \widehat{\mathcal{E}}(\overline{C}, e_C)$  and tuples  $(F_1, \dots, F_k)$  where  $F_i \in \widehat{\mathcal{E}}(\overline{B}_i, \overline{e}_i)$  for each  $i \in [k]$ . That is,  $F_i$  is the restriction of  $F$  on  $B_i$  for all  $i \in [k]$ . Moreover,  $\text{exc}(F) = \sum_{i=1}^k \text{exc}(F_i)$ . Hence,

$$\begin{aligned} \widehat{\text{exc}}(\overline{C}, e_C) &= \min_{F \in \widehat{\mathcal{E}}(\overline{C}, e_C)} \text{exc}(F) \\ &= \sum_{i=1}^k \min_{F_i \in \widehat{\mathcal{E}}(\overline{B}_i, \overline{e}_i)} \text{exc}(F_i) \\ &= \sum_{i=1}^k \widehat{\text{exc}}(\overline{B}_i, \overline{e}_i). \end{aligned}$$

This similarly gives  $\widehat{\delta}(\overline{C}, e_C) = \sum_{i=1}^k \widehat{\delta}(\overline{B}_i, \overline{e}_i)$ .  $\square$

The parameters  $\delta(\overline{C}, e_C)$  and  $\widehat{\delta}(\overline{C}, e_C)$  can be interpreted as the “difference” in the  $\delta$  or  $\widehat{\delta}$  of the overall graph  $G$  made by the presence of the subcubic chain  $C$  compared to a trivial chain (a single edge). This is formalized in the next proposition.

Let  $G$  be a graph containing a nontrivial subcubic chain  $C = xe_0B_1 \dots B_k e_k y$  such that  $C - \{x, y\}$  is a connected component of  $G - \{e_0, e_k\}$ . In this case, we say that  $C$  is a *subcubic chain of  $G$* . If  $C$  is a subcubic chain of  $G$ , we write  $G/C$  to denote the graph obtained by suppressing  $V(C) \setminus \{x, y\}$ , and write  $e_{G/C}$  to denote the resulting edge. We say that  $G/C$  is obtained from  $G$  by *suppressing  $C$* . A cycle in  $G$  containing the edge  $e_0$  (hence all of  $\{e_0, \dots, e_k\}$ ) is said to be a *cycle through  $C$* , and an *even cover through  $C$*  is an even cover of  $G$  containing a cycle through  $C$ .

**Proposition 3.1.2.** *Let  $C$  be a subcubic chain of a graph  $G$ , and let  $e$  be a cut-edge of  $C$ . Then  $\delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C)$  and  $\widehat{\delta}(G, e) = \widehat{\delta}(G/C, e_{G/C}) + \widehat{\delta}(\overline{C}, e_C)$ .*

*Proof.* Given an even cover  $F \in \mathcal{E}(G, e)$ ,  $e$  is contained in some cycle  $D$  in  $F$ . By splitting  $D$  into two cycles  $(D \cap G/C) + e_{G/C}$  and  $(D \cap C) + e_C$ , we obtain from  $F$  two even covers  $F' \in \mathcal{E}(G/C, e_{G/C})$  and  $F_C \in \mathcal{E}(\overline{C}, e_C)$  satisfying  $\text{exc}(F) = \text{exc}(F') + \text{exc}(F_C) - 2$ . This

bijjective correspondence gives

$$\begin{aligned}
\text{exc}(G, e) &= \min_{F \in \mathcal{E}(G, e)} \text{exc}(F) - 2 \\
&= \min_{F' \in \mathcal{E}(G/C, e_{G/C})} (\text{exc}(F') - 2) + \min_{F_C \in \mathcal{E}(\overline{C}, e_C)} (\text{exc}(F_C) - 2) \\
&= \text{exc}(G/C, e_{G/C}) + \text{exc}(\overline{C}, e_C).
\end{aligned}$$

Similarly, for any even cover  $F \in \widehat{\mathcal{E}}(G, e)$ , its restriction on  $G/C$  is in  $\widehat{\mathcal{E}}(G/C, e_{G/C})$  and its restriction on  $\overline{C}$  is in  $\widehat{\mathcal{E}}(\overline{C}, e_C)$ ; and we have  $\widehat{\text{exc}}(G, e) = \widehat{\text{exc}}(G/C, e_{G/C}) + \widehat{\text{exc}}(\overline{C}, e_C)$ .

Since  $n(G) = n(G/C) + n(\overline{C})$  and  $n_2(G) = n_2(G/C) + n_2(\overline{C})$ , the proposition follows from the definitions of  $\delta$  and  $\widehat{\delta}$ .  $\square$

We will show in Theorem 3.1.4 that  $\delta(G, e) + \widehat{\delta}(G, e) \leq 0$  for every 2-connected subcubic graph  $G$  and every edge  $e \in E(G)$  for which  $G - e$  is simple. If  $\delta(G, e) + \widehat{\delta}(G, e) = 0$ , then we say that  $(G, e)$  is *tight*. A subcubic chain  $C$  is *tight* if its closure  $(\overline{C}, e_C)$  is tight.

The next proposition states that a subcubic chain is tight if and only if all of its chain-blocks are tight.

**Proposition 3.1.3.** *Let  $C = xe_0B_1e_1B_2 \dots B_k e_k y$  be a subcubic chain, and assume  $\delta(\overline{B}_i, \overline{e}_i) + \widehat{\delta}(\overline{B}_i, \overline{e}_i) \leq 0$  for all  $i$ . Then  $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$ , with equality if and only if  $\delta(\overline{B}_i, \overline{e}_i) + \widehat{\delta}(\overline{B}_i, \overline{e}_i) = 0$  for all  $i \in [k]$ .*

*Proof.* Since  $\delta(\overline{B}_i, \overline{e}_i) + \widehat{\delta}(\overline{B}_i, \overline{e}_i) \leq 0$  for all  $i$ , we have by Proposition 3.1.1,

$$\delta(\overline{C}, e_C) = \sum_{i=1}^k \delta(\overline{B}_i, \overline{e}_i) \leq \sum_{i=1}^k (-\widehat{\delta}(\overline{B}_i, \overline{e}_i)) = -\widehat{\delta}(\overline{C}, e_C).$$

Hence,  $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$ , with equality if and only if  $\delta(\overline{B}_i, \overline{e}_i) + \widehat{\delta}(\overline{B}_i, \overline{e}_i) = 0$  for all  $i$ .  $\square$

We say that a subcubic chain  $C$  is *minimal* if it is tight and  $\delta(\overline{C}, e_C) = -\frac{1}{2}$ , and that  $C$

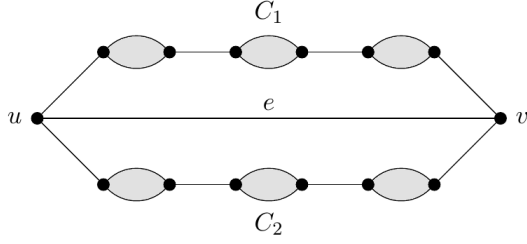


Figure 3.2: A rooted  $\theta$ -chain

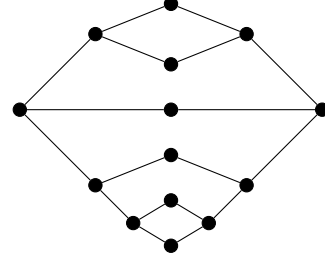


Figure 3.3: A minimal  $\theta$ -chain

is *near-minimal* if it is tight and  $\delta(\overline{C}, e_C) \in \{-\frac{1}{2}, -1\}$ . Two subcubic chains  $C_1$  and  $C_2$  are *balanced* if  $\delta(\overline{C}_1, e_{C_1}) = \delta(\overline{C}_2, e_{C_2})$ .

A  $\theta$ -chain is a graph  $G$  that is the union of three internally disjoint subcubic chains  $C_1, C_2, C_3$  with common endpoints. We call  $C_1, C_2, C_3$  the *chains of  $G$* . Note that the choices of the three chains  $C_1, C_2, C_3$  may not be unique (consider the graph obtained from two disjoint 4-cycles by adding two edges joining them so that the endpoints of the two edges are nonadjacent in each 4-cycle). A *rooted  $\theta$ -chain* is a pair  $(G, e)$  where  $G$  is a graph and  $e = uv \in E(G)$  such that  $G - e$  is the union of two internally disjoint subcubic chains  $C_1, C_2$  with common endpoints  $\{u, v\}$ . We call  $C_1, C_2$  the *chains of  $(G, e)$* . See Figure 3.2.

A (rooted)  $\theta$ -chain is *balanced* if all pairs of its chains are balanced, *tight* if the closures of its chains are all tight, and *(near) minimal* if all of its chains are (near) minimal. Note that a (near) minimal (rooted)  $\theta$ -chain is also balanced and tight by definition. See Figure 3.3.

We can now state our main result, which immediately implies (1.3). For inductive purposes, we allow the graph  $G$  to be a loop  $e$  on a single vertex and we also allow one edge of  $G - e$  to be parallel to  $e$ . In all cases however,  $G - e$  is a simple subcubic graph.

**Theorem 3.1.4.** *Let  $G$  be a 2-connected subcubic graph and let  $e = uv$  be an edge of  $G$  such that  $G - e$  is simple. Then the following statements hold:*

- (T1)  $\delta(G, e) \leq -\frac{1}{2}$ , with equality if and only if either  $G$  is a loop or  $(G, e)$  is a balanced tight rooted  $\theta$ -chain.

**(T2)** If  $G - e$  is 2-connected, then  $\widehat{\delta}(G, e) \leq \frac{3}{2}$ , with equality if and only if  $G - e$  is a minimal  $\theta$ -chain.

**(T3)** If  $\delta(G, e) = -1$ , then either

(a)  $G \cong K_4$ , or

(b)  $e$  has a parallel edge, and suppressing  $\{u, v\}$  to an edge  $e'$  results in a graph  $G'$  such that either  $G'$  is a loop or  $(G', e')$  is a near-minimal rooted  $\theta$ -chain, or

(c) there exists  $e' \in E(G)$  such that  $\{e, e'\}$  is a 2-edge-cut in  $G$ , and suppressing either subcubic chain  $C$  of  $G$  with end edges  $e, e'$  yields either a loop or a balanced tight rooted  $\theta$ -chain  $(G/C, e_{G/C})$ , or

(d)  $(G, e)$  is a rooted  $\theta$ -chain such that  $\min_{i \in [2]} \left( \delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_{3-i}, e_{C_{3-i}}) \right) = -\frac{1}{2}$ .

**(T4)**  $\delta(G, e) + \widehat{\delta}(G, e) \leq 0$ .

One immediate consequence of Theorem 3.1.4 is that if  $C$  is a subcubic chain, then  $\delta(\overline{C}, e_C) \leq -\frac{1}{2}$  unless  $C$  is trivial, in which case  $\delta(\overline{C}, e_C) = 0$  by definition. In particular,  $\delta(G, e) \leq -\frac{1}{2}$  for every nonempty 2-connected subcubic graph  $G$  and  $e \in E(G)$  such that  $G - e$  is simple. Hence, if  $C$  is a minimal subcubic chain, then by Proposition 3.1.1, it has exactly one chain-block  $(\overline{B}, \overline{e}_B)$ , and this chain-block satisfies  $\delta(\overline{B}, \overline{e}_B) = -\frac{1}{2}$ .

### 3.2 Properties of $\theta$ -chains

In this section, we derive useful properties of balanced, tight, or minimal  $\theta$ -chains assuming Theorem 3.1.4 for smaller graphs. We begin by proving statements **(T1)** and **(T3)** of Theorem 3.1.4, assuming Theorem 3.1.4 for smaller graphs, for the special case where  $(G, e)$  is a rooted  $\theta$ -chain (equivalently,  $G$  is simple and  $\{u, v\}$  forms a cut in  $G$ ). The proof is a relatively straightforward but illustrative demonstration of our techniques.

**Lemma 3.2.1.** *Let  $(G, e)$  be a simple rooted  $\theta$ -chain, and let  $C_1, C_2$  denote the two chains of  $(G, e)$ . Assume that Theorem 3.1.4 holds for graphs with fewer than  $n(G)$  vertices. Then*

(i)  $\delta(G, e) = -\frac{1}{2} + \min_{i \in [2]} \left( \delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_{3-i}, e_{C_{3-i}}) \right) \leq -\frac{1}{2}$ , with equality if and only if  $(G, e)$  is a balanced tight rooted  $\theta$ -chain,

(ii)  $\widehat{\delta}(G, e) \leq \frac{3}{2} + \delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2}) \leq \frac{1}{2}$ ,

(iii)  $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{1}{2}, \frac{1}{2})$  if and only if  $(G, e)$  is a minimal rooted  $\theta$ -chain, and

(iv) if  $\delta(G, e) = -1$  then  $\min_{i \in [2]} \left( \delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_{3-i}, e_{C_{3-i}}) \right) = -\frac{1}{2}$ .

*Proof.* An even cover  $F \in \mathcal{E}(G, e)$  corresponds to a pair  $(F_1, F_2)$  where  $F_i \in \mathcal{E}(\overline{C}_i)$  for each  $i \in [2]$  and  $F_i \in \mathcal{E}(\overline{C}_i, e_{C_i})$  for exactly one  $i \in [2]$ . This correspondence gives  $\text{exc}(F) = \text{exc}(F_1) + \text{exc}(F_2)$ . Since  $n(G) = n(\overline{C}_1) + n(\overline{C}_2) + 2$  and  $n_2(G) = n_2(\overline{C}_1) + n_2(\overline{C}_2)$ , we have

$$\begin{aligned} \text{exc}(G, e) &= \min_{F \in \mathcal{E}(G, e)} \text{exc}(F) - 2 \\ &= \min_{i \in [2]} \left( \min_{F_i \in \mathcal{E}(\overline{C}_i, e_{C_i})} (\text{exc}(F_i) - 2) + \min_{F_{3-i} \in \mathcal{E}(\overline{C}_{3-i}, e_{C_{3-i}})} \text{exc}(F_{3-i}) \right) \\ &= \min_{i \in [2]} (\text{exc}(\overline{C}_i, e_{C_i}) + \widehat{\text{exc}}(\overline{C}_{3-i}, e_{C_{3-i}})) \\ &= \min_{i \in [2]} \left( \frac{n(\overline{C}_i) + n_2(\overline{C}_i)}{4} + \delta(\overline{C}_i, e_{C_i}) + \frac{n(\overline{C}_{3-i}) + n_2(\overline{C}_{3-i})}{4} + \widehat{\delta}(\overline{C}_{3-i}, e_{C_{3-i}}) \right) \\ &= \min_{i \in [2]} \left( \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \delta(\overline{C}_i, e_{C_i}) + \delta(\overline{C}_{3-i}, e_{C_{3-i}}) \right). \end{aligned}$$

Therefore,

$$\delta(G, e) = -\frac{1}{2} + \min_{i \in [2]} \left( \delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_{3-i}, e_{C_{3-i}}) \right), \quad (3.1)$$

whence for  $i \in [2]$ ,

$$\delta(G, e) \leq -\frac{1}{2} + \delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_{3-i}, e_{C_{3-i}}). \quad (3.2)$$

By assumption, Theorem 3.1.4 holds for  $(\overline{C}_i, e_{C_i})$ ; so  $\delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_i, e_{C_i}) \leq 0$  for each  $i \in [2]$ . Adding the two inequalities of (3.2) gives

$$2\delta(G, e) \leq -1 + \sum_{i \in [2]} \left( \delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_i, e_{C_i}) \right) \leq -1.$$

Hence,

$$\delta(G, e) \leq -\frac{1}{2}. \quad (3.3)$$

Moreover,  $\delta(G, e) = -\frac{1}{2}$  if and only if all of the above inequalities are tight, which means  $(\overline{C}_1, e_{C_1})$  and  $(\overline{C}_2, e_{C_2})$  are tight, and

$$0 = \delta(\overline{C}_1, e_{C_1}) + \widehat{\delta}(\overline{C}_2, e_{C_2}) = \delta(\overline{C}_1, e_{C_1}) - \delta(\overline{C}_2, e_{C_2}).$$

In other words,  $C_1, C_2$  are balanced. Together with (3.1) and (3.3), this proves (i).

If  $F_i \in \mathcal{E}(\overline{C}_i, e_{C_i})$  for each  $i \in [2]$  then, by merging the cycles in  $F_i$  containing  $e_{C_i}$  for  $i \in [2]$ , we obtain an even cover  $F \in \widehat{\mathcal{E}}(G, e)$  with  $\text{exc}(F) = \text{exc}(F_1) + \text{exc}(F_2) - 2$ . So

$$\begin{aligned} \widehat{\text{exc}}(G, e) &\leq \min_{F \in \widehat{\mathcal{E}}(G, e)} \text{exc}(F) \\ &\leq \min_{F_1 \in \mathcal{E}(\overline{C}_1, e_{C_1})} \text{exc}(F_1) + \min_{F_2 \in \mathcal{E}(\overline{C}_2, e_{C_2})} (\text{exc}(F_2) - 2) \\ &= (\text{exc}(\overline{C}_1, e_{C_1}) + 2) + \text{exc}(\overline{C}_2, e_{C_2}) \\ &= \frac{n(\overline{C}_1) + n_2(\overline{C}_1)}{4} + \delta(\overline{C}_1, e_{C_1}) + \frac{n(\overline{C}_2) + n_2(\overline{C}_2)}{4} + \delta(\overline{C}_2, e_{C_2}) + 2 \\ &= \frac{n(G) + n_2(G)}{4} + \frac{3}{2} + \delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2}). \end{aligned}$$

Hence,

$$\widehat{\delta}(G, e) \leq \frac{3}{2} + \delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2}).$$

Since  $G$  is simple, each  $C_i$  is a nontrivial chain; so  $\delta(\overline{C}_i, e_{C_i}) \leq -\frac{1}{2}$  by the assumption that Theorem 3.1.4 holds for  $(\overline{C}_i, e_{C_i})$ . This gives  $\widehat{\delta}(G, e) \leq \frac{1}{2}$  and proves (ii).

To prove (iii), suppose  $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{1}{2}, \frac{1}{2})$ . Then  $\delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2}) = -1$  by (ii). Since  $\delta(\overline{C}_i, e_{C_i}) \leq -\frac{1}{2}$  for  $i \in [2]$  (by assumption),  $\delta(\overline{C}_i, e_{C_i}) = -\frac{1}{2}$  for each  $i \in [2]$ . Moreover, each  $(\overline{C}_i, e_{C_i})$  is tight (by (i)), so  $(G, e)$  is a minimal rooted  $\theta$ -chain.

Finally, note that (iv) follows from (i).  $\square$

The next lemma says that given a choice of adding an edge  $uv_1$  or  $uv_2$  to a 2-connected subcubic graph  $Z$ , the two quantities  $\delta(Z + uv_1, uv_1)$  and  $\delta(Z + uv_2, uv_2)$  cannot both be large.

**Lemma 3.2.2.** *Let  $Z$  be a 2-connected simple subcubic graph and let  $u, v_1, v_2$  be three distinct vertices of degree 2 in  $Z$ . Assume Theorem 3.1.4 holds for graphs with at most  $n(Z)$  vertices. Then  $\delta(Z + uv_1, uv_1) + \delta(Z + uv_2, uv_2) \leq -2$ .*

*Proof.* By the assumption that Theorem 3.1.4 holds for graphs with at most  $n(Z)$  vertices, we have  $\delta(Z + uv_i, uv_i) \leq -\frac{1}{2}$  for each  $i \in [2]$ , with equality if and only if  $(Z + uv_i, uv_i)$  is a balanced tight rooted  $\theta$ -chain. If both  $\delta(Z + uv_1, uv_1) \leq -1$  and  $\delta(Z + uv_2, uv_2) \leq -1$ , then there is nothing to prove. So we may assume by symmetry that  $\delta(Z + uv_1, uv_1) = -\frac{1}{2}$ ; thus  $(Z + uv_1, uv_1)$  is a balanced tight rooted  $\theta$ -chain. Note that it suffices to show that  $\delta(Z + uv_2, uv_2) \leq -\frac{3}{2}$ .

Let  $C_1, C_2$  denote the two chains of  $(Z + uv_1, uv_1)$ . Let us assume without loss of generality that  $v_2 \in V(C_1)$ . Write  $C_1 = v_1 e_0 B_1 e_1 B_2 \dots B_k e_k u$  (where  $k \geq 1$ ) and write its chain-blocks  $(\overline{B}_i, \overline{e}_i)$  for all  $i \in [k]$ . Since  $C_1, C_2$  are balanced, we have  $\delta(\overline{C}_1, e_{C_1}) = \delta(\overline{C}_2, e_{C_2})$ , and since they are both tight, we have  $\delta(\overline{C}_i, e_{C_i}) + \widehat{\delta}(\overline{C}_i, e_{C_i}) = 0$  for  $i \in [2]$ .

So by Proposition 3.1.3 and the assumption that Theorem 3.1.4 holds for each  $(\overline{B}_i, \overline{e}_i)$ , we have

$$\delta(\overline{B}_i, \overline{e}_i) + \widehat{\delta}(\overline{B}_i, \overline{e}_i) = 0 \quad \text{for all } i \in [k]. \quad (3.4)$$

Let  $\ell \in [k]$  be the unique index such that  $v_2 \in B_\ell$ . (Note  $\ell$  is well defined as  $Z$  is subcubic and  $v_2$  has degree 2 in  $Z$ .) Let  $v'$  denote the vertex of  $B_\ell$  incident with  $e_{\ell-1}$ .

Then there is an even cover  $F \in \mathcal{E}(Z + uv_2, uv_2)$  obtained from a tuple  $(F', F_1, \dots, F_k)$  where  $F' \in \mathcal{E}(\overline{C}_2, e_{C_2})$ ,  $F_i \in \mathcal{E}(\overline{B}_i, \overline{e}_i)$  for each  $i \in [\ell - 1]$ ,  $F_\ell \in \mathcal{E}(B_\ell + v'v_2, v'v_2)$ , and  $F_j \in \widehat{\mathcal{E}}(\overline{B}_j, \overline{e}_j)$  for each  $j = \ell + 1, \dots, k$ . This gives  $\text{exc}(F) - 2 = (\text{exc}(F') - 2) + \sum_{i=1}^{\ell} (\text{exc}(F_i) - 2) + \sum_{j=\ell+1}^k \text{exc}(F_j)$ . Moreover, since  $n(B_\ell + v'v_2) = n(\overline{B}_\ell)$  and  $n_2(B_\ell + v'v_2) = n_2(\overline{B}_\ell)$ , we have

$$\begin{aligned} n(Z + uv_2) &= 2 + n(\overline{C}_2) + \sum_{i=1}^{\ell-1} n(\overline{B}_i) + n(B_\ell + v'v_2) + \sum_{j=\ell+1}^k n(\overline{B}_j), \\ n_2(Z + uv_2) &= n_2(\overline{C}_2) + \sum_{i=1}^{\ell-1} n_2(\overline{B}_i) + n_2(B_\ell + v'v_2) + \sum_{j=\ell+1}^k n_2(\overline{B}_j). \end{aligned}$$

This gives

$$\begin{aligned} \text{exc}(Z + uv_2, uv_2) &\leq \text{exc}(\overline{C}_2, e_{C_2}) + \sum_{i=1}^{\ell-1} \text{exc}(\overline{B}_i, \overline{e}_i) \\ &\quad + \text{exc}(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^k \widehat{\text{exc}}(\overline{B}_j, \overline{e}_j) \\ &= \frac{n(Z + uv_2) + n_2(Z + uv_2)}{4} - \frac{1}{2} + \delta(\overline{C}_2, e_{C_2}) + \sum_{i=1}^{\ell-1} \delta(\overline{B}_i, \overline{e}_i) \\ &\quad + \delta(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^k \widehat{\delta}(\overline{B}_j, \overline{e}_j), \end{aligned}$$

whence

$$\delta(Z + uv_2, uv_2) \leq -\frac{1}{2} + \delta(\overline{C}_2, e_{C_2}) + \sum_{i=1}^{\ell-1} \delta(\overline{B}_i, \overline{e}_i) + \delta(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^k \widehat{\delta}(\overline{B}_j, \overline{e}_j).$$

Note that  $\widehat{\text{exc}}(\overline{B}_\ell, \overline{e}_\ell) = \widehat{\text{exc}}(B_\ell + v'v_2, v'v_2)$  since both quantities are equal to the minimum excess of an even cover of  $B_\ell$ . This implies  $\widehat{\delta}(\overline{B}_\ell, \overline{e}_\ell) = \widehat{\delta}(B_\ell + v'v_2, v'v_2)$ . Using (3.4) and that  $\delta(B_\ell + v'v_2, v'v_2) + \widehat{\delta}(B_\ell + v'v_2, v'v_2) \leq 0$  as Theorem 3.1.4 holds for  $(B_\ell + v'v_2, v'v_2)$  (by assumption), we have

$$\begin{aligned}
\delta(Z + uv_2, uv_2) &\leq -\frac{1}{2} + \delta(\overline{C}_2, e_{C_2}) + \sum_{i=1}^{\ell-1} (-\widehat{\delta}(\overline{B}_i, \overline{e}_i)) \\
&\quad + (-\widehat{\delta}(B_\ell + v'v_2, v'v_2)) + \sum_{j=\ell+1}^k \widehat{\delta}(\overline{B}_j, \overline{e}_j) \\
&= -\frac{1}{2} + \delta(\overline{C}_2, e_{C_2}) + \sum_{i=1}^{\ell-1} (-\widehat{\delta}(\overline{B}_i, \overline{e}_i)) + (-\widehat{\delta}(\overline{B}_\ell, \overline{e}_\ell)) + \sum_{j=\ell+1}^k \widehat{\delta}(\overline{B}_j, \overline{e}_j) \\
&= -\frac{1}{2} + \delta(\overline{C}_2, e_{C_2}) + \sum_{j=1}^k \widehat{\delta}(\overline{B}_j, \overline{e}_j) - 2 \sum_{j=1}^{\ell} \widehat{\delta}(\overline{B}_j, \overline{e}_j) \\
&= -\frac{1}{2} + \delta(\overline{C}_2, e_{C_2}) + \widehat{\delta}(\overline{C}_1, e_{C_1}) - 2 \sum_{j=1}^{\ell} \widehat{\delta}(\overline{B}_j, \overline{e}_j) \\
&\hspace{20em} \text{(by Proposition 3.1.1)} \\
&= -\frac{1}{2} - 2 \sum_{j=1}^{\ell} \widehat{\delta}(\overline{B}_j, \overline{e}_j) \hspace{5em} \text{(as } C_1 \text{ and } C_2 \text{ are balanced and tight)} \\
&\leq -\frac{3}{2},
\end{aligned}$$

since  $-\widehat{\delta}(\overline{B}_j, \overline{e}_j) = \delta(\overline{B}_j, \overline{e}_j) \leq -1/2$  for all  $j \in [k]$  by (3.4) and the assumption that Theorem 3.1.4 holds for  $(\overline{B}_j, \overline{e}_j)$ .  $\square$

We can now prove the following lemma for  $\theta$ -chains.

**Lemma 3.2.3.** *Let  $G$  be a subcubic graph with  $e = uv \in E(G)$  such that  $G - e$  is simple and 2-connected. Assume that Theorem 3.1.4 holds for graphs with fewer than  $n(G)$  vertices. Let  $G_u$  be the graph obtained from  $G - e$  by suppressing  $u$  into an edge  $f_u$ , and assume that  $(G_u, f_u)$  is a rooted  $\theta$ -chain. Then*

- (i)  $\widehat{\delta}(G, e) \leq \frac{3}{2}$ , with equality if and only if  $G - e$  is a minimal  $\theta$ -chain whose three

minimal chains can be chosen to have common endpoints  $N(u) \setminus \{v\}$ ,

(ii)  $\delta(G, e) \leq -\frac{3}{2}$ , and

(iii)  $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{3}{2}, \frac{3}{2})$  if and only if  $G - e$  is a minimal  $\theta$ -chain and  $e$  joins two nonadjacent vertices of a 4-cycle in  $G - e$ .

*Proof.* Let  $N(u) \setminus \{v\} = \{x, y\}$ , the set of endpoints of  $f_u$ . Let  $C_1, C_2$  denote the two chains of  $(G_u, f_u)$  with common endpoints  $\{x, y\}$ , and let  $C_3$  denote the subcubic chain  $x(xu)u(uy)y$ . Note that  $n(G) = 2 + \sum_{i=1}^3 n(\overline{C}_i)$ ,  $n_2(G) = -2 + \sum_{i=1}^3 n_2(\overline{C}_i)$  (since the  $\overline{C}_i$ 's do not account for the edge  $e$ ), and  $\overline{C}_3$  is a loop. Let  $i_1, i_2, i_3$  be a permutation of  $[3]$  such that  $\delta(\overline{C}_{i_1}, e_{C_{i_1}}) \leq \delta(\overline{C}_{i_2}, e_{C_{i_2}}) \leq \delta(\overline{C}_{i_3}, e_{C_{i_3}})$ .

Consider a triple  $(F_1, F_2, F_3)$  such that  $F_{i_1} \in \mathcal{E}(\overline{C}_{i_1}, e_{C_{i_1}})$ ,  $F_{i_2} \in \mathcal{E}(\overline{C}_{i_2}, e_{C_{i_2}})$ , and  $F_{i_3} \in \widehat{\mathcal{E}}(\overline{C}_{i_3}, e_{C_{i_3}})$ . Let  $F \in \widehat{\mathcal{E}}(G, e)$  be obtained from  $F_1 \cup F_2 \cup F_3$  by merging the cycles in  $F_{i_1}, F_{i_2}$  through  $e_{C_{i_1}}, e_{C_{i_2}}$ . Then  $\text{exc}(F) - 2 = (\text{exc}(F_{i_1}) - 2) + (\text{exc}(F_{i_2}) - 2) + \text{exc}(F_{i_3})$ ; so

$$\begin{aligned} \widehat{\text{exc}}(G, e) - 2 &= \text{exc}(\overline{C}_{i_1}, e_{C_{i_1}}) + \text{exc}(\overline{C}_{i_2}, e_{C_{i_2}}) + \widehat{\text{exc}}(\overline{C}_{i_3}, e_{C_{i_3}}) \\ &= \frac{n(G) + n_2(G)}{4} + \delta(\overline{C}_{i_1}, e_{C_{i_1}}) + \delta(\overline{C}_{i_2}, e_{C_{i_2}}) + \widehat{\delta}(\overline{C}_{i_3}, e_{C_{i_3}}). \end{aligned}$$

Since Theorem 3.1.4 holds for  $(\overline{C}_i, e_{C_i})$  for each  $i \in [3]$  (by assumption), we have

$$\widehat{\delta}(\overline{C}_{i_3}, e_{C_{i_3}}) \leq -\delta(\overline{C}_{i_3}, e_{C_{i_3}}) \leq -\delta(\overline{C}_{i_2}, e_{C_{i_2}})$$

and  $\delta(\overline{C}_i, e_{C_i}) \leq -\frac{1}{2}$  for  $i \in [3]$ , which gives

$$\widehat{\text{exc}}(G, e) - 2 \leq \frac{n(G) + n_2(G)}{4} + \delta(\overline{C}_{i_1}, e_{C_{i_1}}) \leq \frac{n(G) + n_2(G)}{4} - \frac{1}{2}.$$

Therefore,  $\widehat{\text{exc}}(G, e) \leq \frac{n(G) + n_2(G)}{4} + \frac{3}{2}$ , and  $\widehat{\delta}(G, e) \leq \frac{3}{2}$ .

Suppose  $\widehat{\delta}(G, e) = \frac{3}{2}$ . Then the above inequalities hold with equality. Hence,  $-\frac{1}{2} =$

$\delta(\overline{C_{i_1}}, e_{C_{i_1}}) = \delta(\overline{C_{i_2}}, e_{C_{i_2}}) = \delta(\overline{C_{i_3}}, e_{C_{i_3}})$ . Since Theorem 3.1.4 holds for all  $(\overline{C_i}, e_{C_i})$  (by assumption),  $(\overline{C_i}, e_{C_i})$  is tight (hence minimal) for all  $i \in [3]$ . Therefore,  $G - e$  is a minimal  $\theta$ -chain with its three chains having common endpoints  $N(u) \setminus \{v\}$ .

Now suppose  $G - e$  is a minimal  $\theta$ -chain with the three minimal chains  $C_1, C_2, C_3$  with common endpoints  $N(u) \setminus \{v\}$ . Let  $F \in \widehat{\mathcal{E}}(G, e)$ . If  $F$  contains a cycle through two of  $C_1, C_2, C_3$ , then the above argument shows  $\text{exc}(F) = \frac{n(G) + n_2(G)}{4} + \frac{3}{2}$ . So we just need to show that if  $F$  does not contain a cycle through any of  $C_1, C_2, C_3$ , then  $\text{exc}(F) \geq \frac{n(G) + n_2(G)}{4} + \frac{3}{2}$ . Indeed, such  $F$  when restricted to  $(\overline{C_i}, e_{C_i})$  for  $i \in [3]$  gives a triple  $(F_1, F_2, F_3)$  such that  $F_i \in \widehat{\mathcal{E}}(\overline{C_i}, e_{C_i})$  for each  $i \in [3]$ , and  $\text{exc}(F) = 2 + \sum_{i=1}^3 \text{exc}(F_i)$  (since the two vertices of  $N(u) \setminus \{v\}$  are isolated in  $F$ ). So

$$\begin{aligned} \text{exc}(F) &\geq 2 + \sum_{i=1}^3 \widehat{\text{exc}}(\overline{C_i}, e_{C_i}) \\ &= 2 + \sum_{i=1}^3 \left( \frac{n(\overline{C_i}) + n_2(\overline{C_i})}{4} + \widehat{\delta}(\overline{C_i}, e_{C_i}) \right) \\ &= \frac{n(G) + n_2(G)}{4} + 2 + \sum_{i=1}^3 \widehat{\delta}(\overline{C_i}, e_{C_i}) \\ &= \frac{n(G) + n_2(G)}{4} + \frac{7}{2}. \end{aligned}$$

The last equality holds since  $\widehat{\delta}(\overline{C_i}, e_{C_i}) = \frac{1}{2}$  for each  $i \in [3]$ , completing the proof of (i).

We now prove (ii) and (iii). Let us assume without loss of generality that  $v \in V(C_1)$ , and write  $C_1 = xe_0B_1e_1B_2 \dots B_k e_k y$  with chain-blocks  $(\overline{B_i}, \overline{e_i})$ . Let  $\ell \in [k]$  denote the unique index such that  $v \in V(B_\ell)$ . By symmetry, we may assume that  $\sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \leq \sum_{j=\ell+1}^k \delta(\overline{B_j}, \overline{e_j})$ . Then, by the assumption that Theorem 3.1.4 holds for each  $(\overline{B_j}, \overline{e_j})$ , we have

$$\sum_{j=\ell+1}^k \widehat{\delta}(\overline{B_j}, \overline{e_j}) \leq \sum_{j=\ell+1}^k (-\delta(\overline{B_j}, \overline{e_j})) \leq - \left( \sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \right). \quad (3.5)$$

Consider the tuple of even covers  $(F_1, \dots, F_k, F^2)$ , where  $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$  for  $i \in [\ell-1]$ ,  $F_\ell \in \mathcal{E}(B_\ell + x'v, x'v)$  where  $x'$  is the endpoint of  $e_{\ell-1}$  in  $B_\ell$ ,  $F_j \in \widehat{\mathcal{E}}(\overline{B_j}, e_{B_j})$  for  $j =$

$\ell + 1, \dots, k$ , and  $F^2 \in \mathcal{E}(\overline{C_2}, e_{C_2})$ . This corresponds to an even cover  $F \in \mathcal{E}(G, e)$  containing a cycle through all of  $xe_0B_1 \dots B_{\ell-1}e_{\ell-1}, e, uy$ , and  $C_2$ , such that

$$\text{exc}(F) - 2 = \sum_{i=1}^{\ell-1} (\text{exc}(F_i) - 2) + (\text{exc}(F_\ell) - 2) + \sum_{j=\ell+1}^k \text{exc}(F_j) + (\text{exc}(F^2) - 2).$$

Since

$$\begin{aligned} n(G) &= \sum_{i=1}^{\ell-1} n(\overline{B_i}) + n(B_\ell + x'v) + \sum_{j=\ell+1}^k n(\overline{B_j}) + n(\overline{C_2}) + 3, \text{ and} \\ n_2(G) &= \sum_{i=1}^{\ell-1} n_2(\overline{B_i}) + n_2(B_\ell + x'v) + \sum_{j=\ell+1}^k n_2(\overline{B_j}) + n_2(\overline{C_2}) - 1, \end{aligned}$$

we have

$$\begin{aligned} \text{exc}(G, e) &\leq \sum_{i=1}^{\ell-1} \text{exc}(\overline{B_i}, \overline{e_i}) + \text{exc}(B_\ell + x'v, x'v) + \sum_{j=\ell+1}^k \widehat{\text{exc}}(\overline{B_j}, \overline{e_j}) + \text{exc}(\overline{C_2}, e_{C_2}) \\ &= \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \left( \sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \right) + \delta(B_\ell + x'v, x'v) \\ &\quad + \left( \sum_{j=\ell+1}^k \widehat{\delta}(\overline{B_j}, \overline{e_j}) \right) + \delta(\overline{C_2}, e_{C_2}) \\ &\leq \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \delta(B_\ell + x'v, x'v) + \delta(\overline{C_2}, e_{C_2}) \quad (\text{by (3.5)}) \\ &\leq \frac{n(G) + n_2(G)}{4} - \frac{3}{2}, \end{aligned}$$

where the last inequality follows as by our assumption Theorem 3.1.4 holds for  $(B_\ell + x'v, x'v)$  and  $(\overline{C_2}, e_{C_2})$ . Hence  $\delta(G, e) \leq -\frac{3}{2}$  and (ii) holds.

To prove (iii), suppose  $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{3}{2}, \frac{3}{2})$ . Then equality holds above, so we have  $\delta(B_\ell + x'v, x'v) = \delta(\overline{C_2}, e_{C_2}) = -\frac{1}{2}$ . Moreover,  $C_1$  and  $C_2$  are minimal chains (by (i)), which implies  $k = \ell = 1$  and  $\delta(\overline{B_\ell}, \overline{e_\ell}) = \delta(\overline{C_1}, e_{C_1}) = -\frac{1}{2}$  (by Proposition 3.1.1). So  $\delta(\overline{B_\ell}, \overline{e_\ell}) + \delta(B_\ell + x'v, x'v) = -1$ . Now  $B_\ell$  is a single vertex; otherwise, by applying Lemma 3.2.2 to  $B_\ell, x'$ , the other endpoint  $y'$  of  $\overline{e_\ell}$ , and  $v$ , we obtain  $\delta(\overline{B_\ell}, \overline{e_\ell}) + \delta(B_\ell +$

$x'v, x'v) = \delta(B_\ell + x'y', x'y') + \delta(B_\ell + x'v, x'v) \leq -2$ , a contradiction. Therefore, we have  $B_\ell = \{x'\} = \{v\}$ , and  $e$  joins two nonadjacent vertices of the 4-cycle  $xvyux$ .  $\square$

We conclude this section with a lemma bounding  $\widehat{\delta}(G, e)$ , which proves statement **(T2)** of Theorem 3.1.4, assuming Theorem 3.1.4 for smaller graphs.

**Lemma 3.2.4.** *Let  $G$  be a 2-connected subcubic graph with  $e = uv \in E(G)$  such that  $G - e$  is simple and 2-connected. Assume that Theorem 3.1.4 holds for graphs with fewer than  $n(G)$  vertices. Then  $\widehat{\delta}(G, e) \leq \frac{3}{2}$ , with equality if and only if  $(G_u, f_u)$  is a minimal rooted  $\theta$ -chain, where  $G_u$  is the graph obtained from  $G - e$  by suppressing  $u$  into an edge  $f_u$ .*

*Proof.* Since  $G - e$  is 2-connected, both  $u$  and  $v$  have degrees 3. Define  $G_u, f_u$  as stated in the lemma. We claim that

$$\widehat{\delta}(G, e) = \min\{\delta(G_u, f_u) + 2, \widehat{\delta}(G_u, f_u) + 1\}. \quad (3.6)$$

Indeed, there is a bijective correspondence between  $\widehat{\mathcal{E}}(G, e)$  and  $\mathcal{E}(G_u)$  obtained as follows. If  $F \in \widehat{\mathcal{E}}(G, e)$  contains a cycle through  $u$ , then we obtain  $F_u \in \mathcal{E}(G_u, f_u)$  by suppressing  $u$  in  $F$ , and we have  $\text{exc}(F) = \text{exc}(F_u)$ . Otherwise, if  $u$  is an isolated vertex in  $F$ , then we obtain  $F_u \in \widehat{\mathcal{E}}(G_u, f_u)$  by removing  $u$  from  $F$ , and we have  $\text{exc}(F) = \text{exc}(F_u) + 1$ . Since  $n(G) + n_2(G) = n(G_u) + n_2(G_u)$ , (3.6) follows from the definitions of  $\delta, \widehat{\delta}$ .

It follows from (3.6) that  $\widehat{\delta}(G, e) \leq \delta(G_u, f_u) + 2 \leq \frac{3}{2}$  by the assumption that Theorem 3.1.4 holds for  $(G_u, f_u)$ . Moreover,  $\widehat{\delta}(G, e) = \frac{3}{2}$  if and only if  $\delta(G_u, f_u) = -\frac{1}{2}$  and  $\widehat{\delta}(G_u, f_u) = \frac{1}{2}$ , which is equivalent to  $(G_u, f_u)$  being a minimal rooted  $\theta$ -chain by Lemma 3.2.1.  $\square$

### 3.3 Proof of Theorem 3.1.4

We proceed by induction on  $n(G)$ . Note that **(T4)** is implied by **(T1)** and **(T2)**: If  $\delta(G, e) \leq -1$  and  $\widehat{\delta}(G, e) \leq 1$ , then **(T4)** holds. Otherwise, we have  $\delta(G, e) = -\frac{1}{2}$  or  $\widehat{\delta}(G, e) = \frac{3}{2}$ . In

the former case, **(T4)** follows from **(T1)** and Lemma 3.2.1; in the latter case, **(T4)** follows from **(T2)** and Lemma 3.2.3. Also note that Lemmas 3.2.3 and 3.2.4 imply **(T2)**. Therefore, it suffices to prove **(T1)** and **(T3)**.

If  $G - \{u, v\}$  is disconnected, then **(T1)** and **(T3)** both hold by Lemma 3.2.1. So we may assume that  $G - \{u, v\}$  is connected. It now suffices to show that  $\delta(G, e) \leq -1$  and that if equality holds, then one of the outcomes of **(T3)** holds.

**Claim 3.3.1.** *We may assume that  $G$  is simple.*

*Proof.* Since  $G - e$  is simple, if  $G$  is not simple, then there is exactly one edge  $e^*$  parallel with  $e$ . Let  $G'$  be the graph obtained from  $G$  by suppressing  $\{u, v\}$  to an edge  $e'$ .

Then  $n(G) = n(G') + 2$  and  $n_2(G) = n_2(G')$ . By the inductive hypothesis, we have  $\delta(G', e') \leq -\frac{1}{2}$ . But every even cover  $F' \in \mathcal{E}(G', e')$  gives an even cover  $F \in \mathcal{E}(G, e)$  with the same excess, so

$$\begin{aligned} \delta(G, e) &= \min_{F \in \mathcal{E}(G, e)} \text{exc}(F) - 2 - \frac{n(G) + n_2(G)}{4} \\ &\leq \min_{F' \in \mathcal{E}(G', e')} \text{exc}(F') - 2 - \frac{n(G') + n_2(G') + 2}{4} \\ &= \delta(G', e') - \frac{1}{2} \\ &\leq -1. \end{aligned}$$

Now suppose  $\delta(G, e) = -1$ . Then both inequalities above are tight; in particular, we have  $\delta(G', e') = -\frac{1}{2}$ , and by the inductive hypothesis,  $G'$  is a loop or  $(G', e')$  is a balanced tight rooted  $\theta$ -chain. If  $G'$  is a loop then  $(G, e)$  satisfies (b) of **(T3)**. So assume that  $(G', e')$  is a balanced tight rooted  $\theta$ -chain, and let  $C_1, C_2$  denote the two chains of  $(G', e')$ .

Then a pair of even covers  $F_1, F_2$  where  $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$  for each  $i \in [2]$  gives an even cover  $F \in \mathcal{E}(G, e)$  by combining the two cycles of  $F_i$  through  $e_{C_i}$  and adding the cycle

with edge set  $\{e, e^*\}$ , with

$$\text{exc}(F) - 2 = (\text{exc}(F_1) - 2) + (\text{exc}(F_2) - 2) + 2.$$

Since  $n(G) = n(\overline{C}_1) + n(\overline{C}_2) + 4$  and  $n_2(G) = n_2(\overline{C}_1) + n_2(\overline{C}_2)$ , we have

$$\begin{aligned} \text{exc}(G, e) &\leq \text{exc}(\overline{C}_1, e_{C_1}) + \text{exc}(\overline{C}_2, e_{C_2}) + 2 \\ &= \frac{n(G) + n_2(G)}{4} + 1 + \delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2}), \end{aligned}$$

so  $\delta(G, e) \leq 1 + \delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2})$ . Thus, we have  $\delta(\overline{C}_i, e_{C_i}) \in \{-\frac{1}{2}, -1\}$  for each  $i \in [2]$ ; in other words,  $(G', e')$  is a near-minimal rooted  $\theta$ -chain. So  $(G, e)$  satisfies (b) of **(T3)**. ■

**Claim 3.3.2.** *We may assume that  $e$  is not in any 2-edge-cut of  $G$ .*

*Proof.* Suppose there is an edge  $e'$  such that  $\{e, e'\}$  is a 2-edge-cut of  $G$ . Let  $C$  be a subcubic chain of  $G$  with end edges  $e, e'$ . By Proposition 3.1.2 and by the inductive hypothesis applied to  $(G/C, e_{G/C})$  and  $(\overline{C}, e_C)$ , we have

$$\delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C) \leq -1.$$

Moreover, if  $\delta(G, e) = -1$ , then  $\delta(G/C, e_{G/C}) = \delta(\overline{C}, e_C) = -\frac{1}{2}$ , so  $(G/C, e_{G/C})$  and  $(\overline{C}, e_C)$  are loops or balanced tight rooted  $\theta$ -chains and (c) of **(T3)** holds for  $(G, e)$ . ■

By Claim 3.3.2, let  $u_1, u_2$  denote the two neighbors of  $u$  distinct from  $v$ , and let  $v_1, v_2$  denote the two neighbors of  $v$  distinct from  $u$ . Moreover, there exist two disjoint paths  $P_1, P_2$  from  $\{u_1, u_2\}$  to  $\{v_1, v_2\}$  in  $G - \{u, v\}$ . We may assume without loss of generality that the set of endpoints of  $P_i$  is  $\{u_i, v_i\}$ ,  $i \in [2]$ .

Let  $S$  denote the set of all cut edges in  $G - \{u, v\}$ . Then each component of  $G - \{u, v\} - S$  is either an isolated vertex or 2-connected.

**Claim 3.3.3.** *For each  $i \in [2]$ , there is a unique component  $Z_i$  of  $G - \{u, v\} - S$ , such that there are three paths in  $G - \{u, v\}$  from  $Z_i$  to  $\{u_i, v_i, u_{3-i}\}$ , pairwise disjoint except possibly at their endpoints in  $Z_i$ , and there are three paths in  $G - \{u, v\}$  from  $Z_i$  to  $\{u_i, v_i, v_{3-i}\}$ , pairwise disjoint except possibly at their endpoints in  $Z_i$ . See Figures 3.4 and 3.5.*

*Proof.* By symmetry, it suffices to prove the claim for  $i = 1$ . First, we show that there is a unique component  $Z_1$  of  $G - \{u, v\} - S$  such that there are three paths in  $G - \{u, v\}$  from  $Z_1$  to  $\{u_1, v_1, u_2\}$ , pairwise disjoint except possibly at their endpoints in  $Z_1$ . Indeed, if there were two distinct such components  $Z, Z'$ , they are by definition separated by a cut-edge  $s \in S$  of  $G - \{u, v\}$ . But  $G - \{u, v\} - s$  has exactly two connected components, one of which contains at least two of  $\{u_1, v_1, u_2\}$ , so one of  $Z, Z'$  is separated from two vertices of  $\{u_1, v_1, u_2\}$  by a cut-edge, contradicting the assumptions on  $Z, Z'$ .

Similarly, there is a unique connected component  $Z'_1$  of  $G - \{u, v\} - S$  such that there are three paths in  $G - \{u, v\}$  from  $Z'_1$  to  $\{u_1, v_1, v_2\}$ , pairwise disjoint except possibly at their endpoints in  $Z'_1$ . We now show that  $Z_1 = Z'_1$ . Otherwise, there is a cut edge  $s$  of  $G - \{u, v\}$  separating  $Z_1$  from  $Z'_1$ . Then the two connected components of  $G - \{u, v\} - s$  each contain exactly one of  $\{u_1, v_1\}$  and exactly one of  $\{u_2, v_2\}$ . But this implies that  $\{e, s\}$  is a 2-edge-cut in  $G$ , contradicting Claim 3.3.2. ■

There are two cases to consider: either  $Z_1 \neq Z_2$  or  $Z_1 = Z_2$ . For  $i \in [2]$ , let  $u'_i$  (respectively,  $v'_i$ ) denote the vertex of  $Z_i$  that is the endpoint of a (possibly trivial) path in  $G - \{u, v\}$  from  $u_i$  (respectively,  $v_i$ ) to  $Z_i$  that is internally disjoint from  $Z_1 \cup Z_2$ . Note that  $u'_i$  and  $v'_i$  are uniquely determined. For  $i \in [2]$ , let  $U_i$  denote the unique (possibly trivial) subcubic chain of  $G - \{v, u_{3-i}\}$  with endpoints  $\{u, u'_i\}$ , and let  $V_i$  denote the unique subcubic chain of  $G - \{u, v_{3-i}\}$  with endpoints  $\{v, v'_i\}$ .

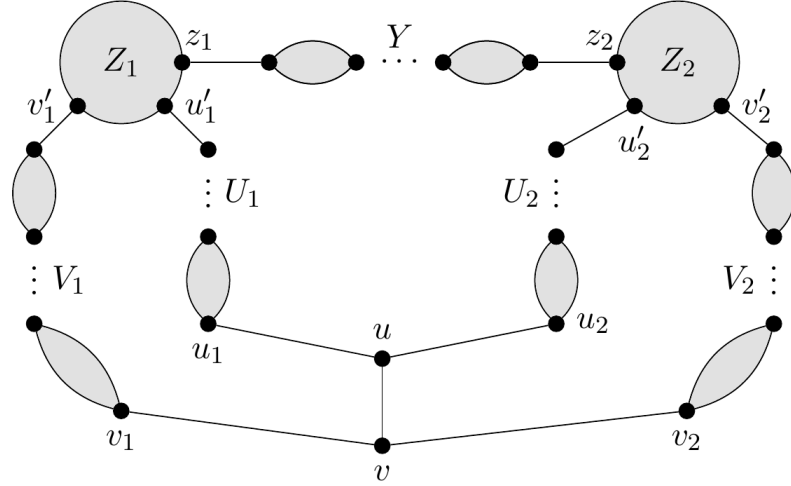


Figure 3.4:  $Z_1 \neq Z_2$

**Case 1:**  $Z_1 \neq Z_2$ .

There is a cut-edge separating  $Z_1$  and  $Z_2$  in  $G - \{u, v\}$  and there is a unique subcubic chain  $Y$  of  $G - \{u, v\}$  with an endpoint  $z_i \in Z_i$  for each  $i \in [2]$ , internally disjoint from  $Z_1 \cup Z_2$ . Then  $G$  is the union of  $U_1, U_2, V_1, V_2, Z_1, Z_2, Y$ , and the edge  $e = uv$ . We have, for  $i, j \in [2]$ ,

$$n(G) = n(\overline{U_1}) + n(\overline{U_2}) + n(\overline{V_1}) + n(\overline{V_2}) + n(Z_1 + u'_i z_1) + n(Z_2 + v'_j z_2) + n(\overline{Y}) + 2,$$

$$n_2(G) = n_2(\overline{U_1}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(\overline{V_2}) + n_2(Z_1 + u'_i z_1) + n_2(Z_2 + v'_j z_2) + n_2(\overline{Y}) - 2.$$

Suppose  $F \in \mathcal{E}(G, e)$  goes through  $U_1, Y$ , and  $V_2$ . Then there is a correspondence between  $F$  and the tuple  $(F_{U_1}, F_{Z_1}, F_Y, F_{Z_2}, F_{V_2}, F_{U_2}, F_{V_1})$ , where

- $F_{U_1} \in \mathcal{E}(\overline{U_1}, e_{U_1})$ ,  $F_{Z_1} \in \mathcal{E}(Z_1 + u'_1 z_1, u'_1 z_1)$ ,  $F_Y \in \mathcal{E}(\overline{Y}, e_Y)$ ,  $F_{Z_2} \in \mathcal{E}(Z_2 + v'_2 z_2, v'_2 z_2)$ ,  $F_{V_2} \in \mathcal{E}(\overline{V_2}, e_{V_2})$ , and
- $F_{U_2} \in \widehat{\mathcal{E}}(\overline{U_2}, e_{U_2})$ ,  $F_{V_1} \in \widehat{\mathcal{E}}(\overline{V_1}, e_{V_1})$ .

This gives

$$\begin{aligned}
\text{exc}(G, e) &\leq \text{exc}(\overline{U}_1, e_{U_1}) + \text{exc}(Z_1 + u'_1 z_1, u'_1 z_1) + \text{exc}(\overline{Y}, e_Y) + \text{exc}(Z_2 + v'_2 z_2, v'_2 z_2) \\
&\quad + \text{exc}(\overline{V}_2, e_{V_2}) + \widehat{\text{exc}}(\overline{U}_2, e_{U_2}) + \widehat{\text{exc}}(\overline{V}_1, e_{V_1}) \\
&= \frac{n(G) + n_2(G)}{4} + \delta(\overline{U}_1, e_{U_1}) + \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(\overline{Y}, e_Y) \\
&\quad + \delta(Z_2 + v'_2 z_2, v'_2 z_2) + \delta(\overline{V}_2, e_{V_2}) + \widehat{\delta}(\overline{U}_2, e_{U_2}) + \widehat{\delta}(\overline{V}_1, e_{V_1}),
\end{aligned}$$

hence

$$\begin{aligned}
\delta(G, e) &\leq \delta(\overline{U}_1, e_{U_1}) + \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v'_2 z_2, v'_2 z_2) + \delta(\overline{V}_2, e_{V_2}) \\
&\quad + \widehat{\delta}(\overline{U}_2, e_{U_2}) + \widehat{\delta}(\overline{V}_1, e_{V_1}).
\end{aligned} \tag{3.7}$$

Similarly, by considering an even cover in  $\mathcal{E}(G, e)$  through  $U_2, Y$ , and  $V_1$ , we obtain

$$\begin{aligned}
\delta(G, e) &\leq \delta(\overline{U}_2, e_{U_2}) + \delta(Z_2 + u'_2 z_2, u'_2 z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v'_1 z_1, v'_1 z_1) + \delta(\overline{V}_1, e_{V_1}) \\
&\quad + \widehat{\delta}(\overline{U}_1, e_{U_1}) + \widehat{\delta}(\overline{V}_2, e_{V_2}).
\end{aligned} \tag{3.8}$$

Now suppose  $\delta(G, e) \geq -1$ . Then

$$\begin{aligned}
-1 &\leq \delta(\overline{U}_1, e_{U_1}) + \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v'_2 z_2, v'_2 z_2) + \delta(\overline{V}_2, e_{V_2}) \\
&\quad + \widehat{\delta}(\overline{U}_2, e_{U_2}) + \widehat{\delta}(\overline{V}_1, e_{V_1}) \\
&\hspace{25em} \text{(by (3.7))} \\
&\leq -(\widehat{\delta}(\overline{U}_1, e_{U_1}) + \widehat{\delta}(\overline{V}_2, e_{V_2}) + \delta(\overline{U}_2, e_{U_2}) + \delta(\overline{V}_1, e_{V_1})) \\
&\hspace{25em} \text{(by inductive hypothesis)} \\
&\quad + \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v'_2 z_2, v'_2 z_2) \\
&= -(\widehat{\delta}(\overline{U}_1, e_{U_1}) + \widehat{\delta}(\overline{V}_2, e_{V_2}) + \delta(\overline{U}_2, e_{U_2}) + \delta(\overline{V}_1, e_{V_1})) \\
&\quad - (\delta(Z_2 + u'_2 z_2, u'_2 z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v'_1 z_1, v'_1 z_1)) \\
&\quad + (\delta(Z_2 + u'_2 z_2, u'_2 z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v'_1 z_1, v'_1 z_1)) \\
&\quad + \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v'_2 z_2, v'_2 z_2) \\
&\leq 1 + \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v'_2 z_2, v'_2 z_2) \\
&\hspace{25em} \text{(by (3.8))} \\
&\quad + \delta(Z_2 + u'_2 z_2, u'_2 z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v'_1 z_1, v'_1 z_1).
\end{aligned}$$

This gives

$$\begin{aligned}
-2 &\leq \delta(Z_1 + u'_1 z_1, u'_1 z_1) + \delta(Z_2 + v'_2 z_2, v'_2 z_2) + \delta(Z_2 + u'_2 z_2, u'_2 z_2) \\
&\quad + \delta(Z_1 + v'_1 z_1, v'_1 z_1) + 2\delta(\overline{Y}, e_Y) \leq -2,
\end{aligned}$$

since by the inductive hypothesis, the all terms are each at most  $-\frac{1}{2}$  except  $\delta(\overline{Y}, e_Y) = 0$  when  $Y$  is a trivial chain. Hence,  $\delta(G, e) = -1$ ,

$$\delta(Z_1 + u'_1 z_1, u'_1 z_1) = \delta(Z_2 + v'_2 z_2, v'_2 z_2) = \delta(Z_1 + u'_2 z_2, u'_2 z_2) = \delta(Z_1 + v'_1 z_1, v'_1 z_1) = -\frac{1}{2},$$

and  $\delta(\overline{Y}, e_Y) = 0$  (i.e.,  $Y$  is a trivial chain). By Lemma 3.2.2,  $Z_1$  and  $Z_2$  are single vertices.

So for  $i, j \in [2]$ ,  $\delta(Z_i + u'_j z_i, u'_j z_i) = \delta(Z_i + v'_j z_i, v'_j z_i) = -\frac{1}{2}$ . Hence, from (3.7) and (3.8),

and by the inductive hypothesis, we have

$$\delta(\overline{U}_i, e_{U_i}) = \delta(\overline{V}_j, e_{V_j}) = 0$$

for each  $i, j \in [2]$ , so  $U_i, V_j$  are all trivial chains as well. This proves that  $G \cong K_4$ , satisfying (a) of **(T3)**.

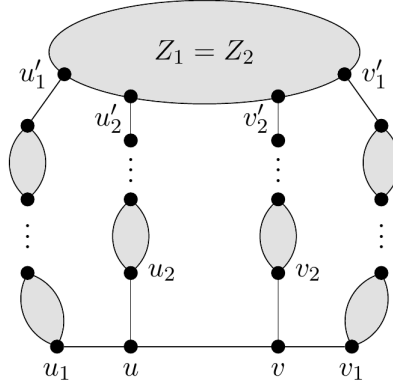


Figure 3.5:  $Z_1 = Z_2$

**Case 2:**  $Z_1 = Z_2$ .

Let  $Z := Z_1 = Z_2$ . Then  $u'_1, u'_2, v'_1, v'_2$  are distinct vertices (since  $G$  is subcubic and  $Z$  is 2-connected), and  $G$  is the union of  $U_1, U_2, V_1, V_2, Z$ , and the edge  $e$ . Note that

$$n(G) = n(\overline{U}_1) + n(\overline{U}_2) + n(\overline{V}_1) + n(\overline{V}_2) + n(Z + u'_i v'_j) + 2$$

$$n_2(G) = n_2(\overline{U}_1) + n_2(\overline{U}_2) + n_2(\overline{V}_1) + n_2(\overline{V}_2) + n_2(Z + u'_i v'_j) - 2.$$

For  $i, j \in [2]$ , let  $F \in \mathcal{E}(G, e)$  be an even cover through  $U_i$  and  $V_j$ . This corresponds to a tuple  $(F_{U_1}, F_{U_2}, F_{V_1}, F_{V_2}, F_Z)$  where

- $F_{U_i} \in \mathcal{E}(\overline{U}_i, e_{U_i})$ ,  $F_{V_j} \in \mathcal{E}(\overline{V}_j, e_{V_j})$ ,  $F_Z \in \mathcal{E}(Z + u'_i v'_j, u'_i v'_j)$ , and
- $F_{U_{3-i}} \in \widehat{\mathcal{E}}(\overline{U}_{3-i}, e_{U_{3-i}})$ ,  $F_{V_{3-j}} \in \widehat{\mathcal{E}}(\overline{V}_{3-j}, e_{V_{3-j}})$ ,

which gives

$$\begin{aligned}
\text{exc}(G, e) &\leq \text{exc}(\overline{U}_i, e_{U_i}) + \text{exc}(\overline{V}_j, e_{V_j}) + \text{exc}(Z + u'_i v'_j, u'_i v'_j) \\
&\quad + \widehat{\text{exc}}(\overline{U}_{3-i}, e_{U_{3-i}}) + \widehat{\text{exc}}(\overline{V}_{3-j}, e_{V_{3-j}}) \\
&= \frac{n(G) + n_2(G)}{4} + \delta(\overline{U}_i, e_{U_i}) + \delta(\overline{V}_j, e_{V_j}) + \delta(Z + u'_i v'_j, u'_i v'_j) \\
&\quad + \widehat{\delta}(\overline{U}_{3-i}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V}_{3-j}, e_{V_{3-j}}).
\end{aligned}$$

Hence, for all  $i, j \in [2]$ ,

$$\delta(G, e) \leq \delta(\overline{U}_i, e_{U_i}) + \delta(\overline{V}_j, e_{V_j}) + \delta(Z + u'_i v'_j, u'_i v'_j) + \widehat{\delta}(\overline{U}_{3-i}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V}_{3-j}, e_{V_{3-j}}) \tag{3.9}$$

We now show that  $\delta(G, e) \leq -\frac{3}{2}$ , which completes the proof of Theorem 3.1.4. Suppose to the contrary that  $\delta(G, e) \geq -1$ . Then by (3.9) and the inductive hypothesis,

$$\begin{aligned}
-1 &\leq \delta(\overline{U}_i, e_{U_i}) + \delta(\overline{V}_j, e_{V_j}) + \widehat{\delta}(\overline{U}_{3-i}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V}_{3-j}, e_{V_{3-j}}) + \delta(Z + u'_i v'_j, u'_i v'_j) \\
&\hspace{15em} \text{(by (3.9))} \\
&\leq -(\widehat{\delta}(\overline{U}_i, e_{U_i}) + \widehat{\delta}(\overline{V}_j, e_{V_j}) + \delta(\overline{U}_{3-i}, e_{U_{3-i}}) + \delta(\overline{V}_{3-j}, e_{V_{3-j}})) \\
&\hspace{15em} \text{(by (T4))} \\
&\quad + \delta(Z + u'_i v'_j, u'_i v'_j) \\
&= -(\widehat{\delta}(\overline{U}_i, e_{U_i}) + \widehat{\delta}(\overline{V}_j, e_{V_j}) + \delta(Z + u'_{3-i} v'_{3-j}, u'_{3-i} v'_{3-j}) + \delta(\overline{U}_{3-i}, e_{U_{3-i}}) \\
&\quad + \delta(\overline{V}_{3-j}, e_{V_{3-j}})) + \delta(Z + u'_i v'_j, u'_i v'_j) + \delta(Z + u'_{3-i} v'_{3-j}, u'_{3-i} v'_{3-j}) \\
&\leq 1 + \delta(Z + u'_i v'_j, u'_i v'_j) + \delta(Z + u'_{3-i} v'_{3-j}, u'_{3-i} v'_{3-j}). \hspace{5em} \text{(by (3.9))}
\end{aligned}$$

Hence for  $i, j \in [2]$ ,

$$-2 \leq \delta(Z + u'_i v'_j, u'_i v'_j) + \delta(Z + u'_{3-i} v'_{3-j}, u'_{3-i} v'_{3-j}) \tag{3.10}$$

On the other hand, applying Lemma 3.2.2 to  $u'_i, v'_1, v'_2$  and  $v'_j, u'_1, u'_2$ , we have for all  $i, j \in [2]$

$$\begin{aligned} \delta(Z + u'_i v'_1, u'_i v'_1) + \delta(Z + u'_i v'_2, u'_i v'_2) &\leq -2 \text{ and} \\ \delta(Z + u'_1 v'_j, u'_1 v'_j) + \delta(Z + u'_2 v'_j, u'_2 v'_j) &\leq -2. \end{aligned} \tag{3.11}$$

Now, setting  $i = j = 1$  and setting  $i = 1$  and  $j = 2$  in (3.10), we have

$$-4 \leq \delta(Z + u'_1 v'_1, u'_1 v'_1) + \delta(Z + u'_2 v'_2, u'_2 v'_2) + \delta(Z + u'_1 v'_2, u'_1 v'_2) + \delta(Z + u'_2 v'_1, u'_2 v'_1).$$

On the other hand, setting  $i = 1$  and  $i = 2$  in the first inequality of (3.11), we have

$$\delta(Z + u'_1 v'_1, u'_1 v'_1) + \delta(Z + u'_1 v'_2, u'_1 v'_2) + \delta(Z + u'_2 v'_1, u'_2 v'_1) + \delta(Z + u'_2 v'_2, u'_2 v'_2) \leq -4.$$

We thus have equality everywhere. In particular,  $\delta(G, e) = -1$  and we have equality in (3.10) and (3.11), which implies that for all  $i, j \in [2]$ ,

$$\delta(Z + u'_i v'_j, u'_i v'_j) = -1. \tag{3.12}$$

Since  $Z + u'_i v'_j$  has at least two vertices of degree 2 (namely  $u'_{3-i}$  and  $v'_{3-j}$ ), it is not isomorphic to  $K_4$ . Moreover, since  $Z$  is 2-connected,  $u'_i v'_j$  is not contained in any 2-edge-cut in  $Z + u'_i v'_j$ . So each  $(Z + u'_i v'_j, u'_i v'_j)$  satisfies (b) or (d) of **(T3)**.

We claim that  $u'_i v'_j \notin E(Z)$  for all  $i, j \in [2]$  (hence  $(Z + u'_i v'_j, u'_i v'_j)$  satisfies (d) of **(T3)**). For, suppose without loss of generality that  $u'_1 v'_1 \in E(Z)$ .

By the inductive hypothesis, (b) of **(T3)** holds for  $(Z + u'_1 v'_1, u'_1 v'_1)$ , so suppressing  $\{u'_1, v'_1\}$  in  $Z$  to an edge  $e'$  results in a graph  $Z'$  such that  $(Z', e')$  is a near-minimal rooted  $\theta$ -chain. Let  $C_1, C_2$  denote the two chains of  $(Z', e')$ . Assume without loss of generality that  $v'_2 \in V(C_1)$ . Since  $v'_2$  has degree 2 in  $Z$ , it is in the interior of  $C_1$ , and this implies that  $Z - \{u'_1, v'_2\}$  is connected and  $v'_2 u'_1 \notin E(Z)$ . Then  $(Z + u'_1 v'_2, u'_1 v'_2)$  satisfies (d) of **(T3)**,

which implies that  $Z - \{u'_1, v'_2\}$  is disconnected, a contradiction.

It follows that  $(Z + u'_i v'_j, u'_i v'_j)$  satisfy (d) of **(T3)** for all  $i, j \in [2]$ , so  $(Z + u'_i v'_j, u'_i v'_j)$  is a rooted  $\theta$ -chain for all  $i, j \in [2]$ . Consider the rooted  $\theta$ -chain  $(Z + u'_1 v'_1, u'_1 v'_1)$ . Since  $(Z + u'_1 v'_2, u'_1 v'_2)$  (respectively,  $(Z + u'_2 v'_1, u'_2 v'_1)$ ) is a rooted  $\theta$ -chain,  $\{v'_2\}$  (respectively,  $\{u'_2\}$ ) is a block in one of the chains of  $(Z + u'_1 v'_1, u'_1 v'_1)$ . Let  $C_1$  denote the subcubic chain of  $Z$  with end points  $\{u'_1, v'_1\}$  not containing  $v'_2$ , and let  $C_2$  denote the subcubic chain of  $Z$  with end points  $\{u'_1, v'_2\}$  not containing  $v'_1$ . Let  $D$  denote the subcubic chain of  $Z$  with end points  $\{v'_1, v'_2\}$  not containing  $u'_1$ .

Then for  $j \in [2]$ ,  $n(Z + u'_1 v'_j) = n(\overline{C_1}) + n(\overline{C_2}) + n(\overline{D}) + 3$  and  $n_2(Z + u'_1 v'_j) = n_2(\overline{C_1}) + n_2(\overline{C_2}) + n_2(\overline{D}) + 1$ . Thus for each  $j \in [2]$ , by forming an even cover in  $\mathcal{E}(Z + u'_1 v'_j, u'_1 v'_j)$  using even covers from  $\widehat{\mathcal{E}}(\overline{C_j}, e_{C_j})$ ,  $\mathcal{E}(\overline{D}, e_D)$ , and  $\mathcal{E}(\overline{C_{3-j}}, e_{C_{3-j}})$ , we obtain

$$\delta(Z + u'_1 v'_j, u'_1 v'_j) \leq -1 + \widehat{\delta}(\overline{C_j}, e_{C_j}) + \delta(\overline{D}, e_D) + \delta(\overline{C_{3-j}}, e_{C_{3-j}}).$$

Adding these two inequalities and using (3.12), we have

$$\begin{aligned} 0 &\leq \delta(\overline{C_1}, e_{C_1}) + \widehat{\delta}(\overline{C_1}, e_{C_1}) + 2\delta(\overline{D}, e_D) + \delta(\overline{C_2}, e_{C_2}) + \widehat{\delta}(\overline{C_2}, e_{C_2}) \\ &\leq 2\delta(\overline{D}, e_D) \end{aligned}$$

by **(T4)** applied to  $(\overline{C_i}, e_{C_i})$ . It follows that  $D$  is a trivial chain, and  $v'_1 v'_2 \in E(Z)$ .

By symmetry,  $u'_1 u'_2 \in E(Z)$ . Thus,  $\{u'_1 u'_2, v'_1 v'_2\}$  is a 2-edge-cut in  $Z$ . Let  $D_1, D_2$  denote the connected components of  $Z - \{u'_1 u'_2, v'_1 v'_2\}$  and (by relabeling  $u'_1, u'_2$  if necessary) assume  $u'_i, v'_i \in V(D_i)$  for  $i \in [2]$ .

Then for  $i, j \in [2]$ ,  $n(Z + u'_i v'_j, u'_i v'_j) = n(\overline{D_1}, e_{D_1}) + n(\overline{D_2}, e_{D_2}) + 4$  and  $n_2(Z + u'_i v'_j, u'_i v'_j) = n_2(\overline{D_1}, e_{D_1}) + n_2(\overline{D_2}, e_{D_2}) + 2$ . Thus, by forming an even cover in  $\mathcal{E}(Z +$

$u'_i v'_j, u'_i v'_j$ ) using even covers from  $\mathcal{E}(\overline{D}_k, e_{D_k})$  and  $\widehat{\mathcal{E}}(\overline{D}_{3-k}, e_{D_{3-k}})$  for  $k \in [2]$ , we get

$$\delta(Z + u'_i v'_j, u'_i v'_j) \leq -\frac{3}{2} + \delta(\overline{D}_k, e_{D_k}) + \widehat{\delta}(\overline{D}_{3-k}, e_{D_{3-k}})$$

Adding these two inequalities and using (3.12) and **(T4)**, we have

$$1 \leq \delta(\overline{D}_1, e_{D_1}) + \widehat{\delta}(\overline{D}_1, e_{D_1}) + \delta(\overline{D}_2, e_{D_2}) + \widehat{\delta}(\overline{D}_2, e_{D_2}) \leq 0,$$

a contradiction. This completes the proof of Theorem 3.1.4.

### 3.4 Extremal Examples

In this section, we give a structural characterization of the extremal examples of Theorem 1.1.2. Recall that for a subcubic graph  $G$  and any edge  $e \in E(G)$ , we have

$$\begin{aligned} \text{exc}(G) &= \min\{\text{exc}(G, e) + 2, \widehat{\text{exc}}(G, e)\} \\ &= \frac{n(G) + n_2(G)}{4} + \min\{\delta(G, e) + 2, \widehat{\delta}(G, e)\}. \end{aligned}$$

So if either  $\delta(G, e) \leq -\frac{3}{2}$  or  $\widehat{\delta}(G, e) \leq \frac{1}{2}$  for any edge  $e \in E(G)$ , then  $\text{exc}(G) \leq \frac{n(G) + n_2(G)}{4} + \frac{1}{2}$ . It follows that  $\text{exc}(G) = \frac{n(G) + n_2(G)}{4} + 1$  (equivalently,  $\text{tsp}(G) = \frac{5n(G) + n_2(G)}{4} - 1$ ) if and only if  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$  for all  $e \in E(G)$ .

**Proposition 3.4.1.** *Let  $G$  be a simple 2-connected subcubic graph and let  $e$  be an edge of  $G$ . Then  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$  if and only if either  $G \cong K_4$  or  $G$  is a minimal  $\theta$ -chain.*

*Proof.* Suppose  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ . Since  $\delta(G, e) = -1$ , one of the four outcomes of **(T3)** holds. If  $G \cong K_4$  then we are done. Since  $G$  is simple, (b) of **(T3)** cannot occur. Moreover, (d) of **(T3)** does not hold; otherwise,  $(G, e)$  is a simple rooted  $\theta$ -chain and, by Lemma 3.2.1 (ii),  $\widehat{\delta}(G, e) \leq \frac{3}{2} + \delta(\overline{C}_1, e_{C_1}) + \delta(\overline{C}_2, e_{C_2}) \leq 1/2$ , a contradiction.

Thus (c) of **(T3)** holds: there exists  $e' \in E(G)$  such that  $\{e, e'\}$  is a 2-edge-cut in  $G$  and suppressing either subcubic chain  $C$  of  $G$  with end edges  $e, e'$  yields a loop or a balanced tight rooted  $\theta$ -chain  $(G/C, e_{G/C})$ . Let  $C$  be a subcubic chain of  $G$  with end edges  $e, e'$ . Then by Proposition 3.1.2 and **(T4)**,

$$\begin{aligned} -1 &= \delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C) \\ &\leq -(\widehat{\delta}(G/C, e_{G/C}) + \widehat{\delta}(\overline{C}, e_C)) = -\widehat{\delta}(G, e) = -1. \end{aligned}$$

This implies that  $(\delta(G/C, e_{G/C}), \widehat{\delta}(G/C, e_{G/C})) = (\delta(\overline{C}, e_C), \widehat{\delta}(\overline{C}, e_C)) = (-\frac{1}{2}, \frac{1}{2})$ , and thus  $(\overline{C}, e_C)$  and  $(G/C, e_{G/C})$  are minimal rooted  $\theta$ -chains (by Lemma 3.2.1 (iii)). Therefore, by definition,  $G$  is a minimal  $\theta$ -chain (since it is the internally disjoint union of  $C$  and the two chains of  $(G/C, e_{G/C})$ , all of which are minimal).  $\square$

To give an alternate structural characterization of minimal (rooted)  $\theta$ -chains, we now describe an operation introduced in [20]. Let  $H$  be a graph and  $v \in V(H)$  be a vertex of degree 2. A  $\diamond$ -operation on  $H$  at  $v$  deletes  $v$  from  $H$ , adds a 4-cycle  $D$  disjoint from  $H - v$ , and adds a matching between the neighbors of  $v$  and two nonadjacent vertices in  $D$ . See Figure 3.6. We say that a graph is  $H$ -constructible if it can be obtained from  $H$  by repeated  $\diamond$ -operations.

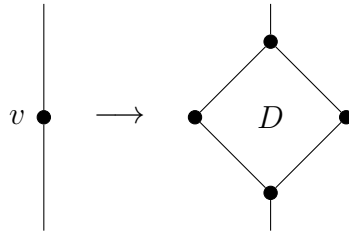


Figure 3.6: The  $\diamond$ -operation

It is observed in [20] that after each  $\diamond$ -operation, the excess of the new graph increases by 1 and the new quantity  $\frac{n(G)+n_2(G)}{4}$  also increases by 1. We will consider  $K_{2,3}$ -constructible graphs and  $K_4^-$ -constructible graphs, where  $K_4^-$  is the graph obtained from

the complete graph  $K_4$  by removing an edge. Note that  $\text{exc}(K_{2,3}) = \frac{n(K_{2,3})+n_2(K_{2,3})}{4} + 1$ ; thus, if  $G$  is  $K_{2,3}$ -constructible then  $\text{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$ .

**Proposition 3.4.2** (Dvořák et al. [20]). *Let  $G$  be a simple 2-connected subcubic graph. If  $G \cong K_4$  or  $G$  is  $K_{2,3}$ -constructible, then  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ .*

We show that the converse of Proposition 3.4.2 is also true, thereby giving a structural characterization of the extremal graphs for Theorem 1.1.2. First, we have an observation similar to Proposition 3.4.2. The *center* of  $K_4^-$  is the edge whose endpoints both have degree 3.

**Proposition 3.4.3.** *Let  $(G, e)$  be a simple minimal rooted  $\theta$ -chain. Then  $G$  is  $K_4^-$ -constructible, with the edge  $e$  corresponding to the center of  $K_4^-$ .*

*Proof.* By **(T1)** and Lemma 3.2.1 (iii),  $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{1}{2}, \frac{1}{2})$ . Let  $C_1$  and  $C_2$  be the chains of  $(G, e)$ . By the definition of a minimal rooted  $\theta$ -chain, for each  $i \in [2]$ , we have  $(\delta(\overline{C}_i, e_{C_i}), \widehat{\delta}(\overline{C}_i, e_{C_i})) = (-\frac{1}{2}, \frac{1}{2})$ , so  $(\overline{C}_i, e_{C_i})$  is either a loop or a minimal rooted  $\theta$ -chain by **(T1)** and Lemma 3.2.1. If  $(\overline{C}_i, e_{C_i})$  is not a loop, then by induction, it is  $K_4^-$ -constructible with  $e_{C_i}$  corresponding to the center of  $K_4^-$ . It follows that  $(G, e)$  is  $K_4^-$ -constructible with  $e$  corresponding to the center of  $K_4^-$ .  $\square$

**Proposition 3.4.4.** *Let  $G$  be a simple minimal  $\theta$ -chain. Then  $G$  is  $K_{2,3}$ -constructible.*

*Proof.* By definition, there exists a choice of three chains  $C_1, C_2, C_3$  of  $G$  with common endpoints such that  $G$  is the internally disjoint union  $C_1 \cup C_2 \cup C_3$ , and we have  $(\delta(\overline{C}_i, e_{C_i}), \widehat{\delta}(\overline{C}_i, e_{C_i})) = (-\frac{1}{2}, \frac{1}{2})$  for each  $i \in [3]$ . If  $G \cong K_{2,3}$ , then we are done. So we may assume without loss of generality that  $(\overline{C}_1, e_{C_1})$  is not a loop. Then it is a minimal rooted  $\theta$ -chain by Lemma 3.2.1, and by Proposition 3.4.3, it is  $K_4^-$ -constructible with the edge  $e_{C_1}$  corresponding to the center of  $K_4^-$ . On the other hand,  $(G/C_1, e_{G/C_1})$  is by definition a minimal rooted  $\theta$ -chain, so it is also  $K_4^-$ -constructible by Proposition 3.4.3, with  $e_{G/C}$  corresponding to the center of  $K_4^-$ . It follows that  $G$  is  $K_{2,3}$ -constructible.  $\square$

We thus have the following characterization of the extremal examples of Theorem 1.1.2.

**Theorem 3.4.5.** *Let  $G$  be a simple 2-connected subcubic graph. Then  $\text{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + 1$ , with equality if and only if either  $G \cong K_4$  or  $G$  is  $K_{2,3}$ -constructible.*

*Proof.* Let  $e \in E(G)$ . If  $\delta(G, e) \leq -\frac{3}{2}$  or  $\widehat{\delta}(G, e) \leq \frac{1}{2}$ , then  $\text{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$ . Otherwise, we have  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ , or equivalently,  $\text{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$ . Now if  $G \cong K_4$  or  $G$  is  $K_{2,3}$ -constructible, then  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$  by Proposition 3.4.2. Conversely, if  $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ , then by Propositions 3.4.1 and 3.4.4, either  $G \cong K_4$  or  $G$  is  $K_{2,3}$ -constructible.  $\square$

### 3.5 The Algorithm

We now provide an algorithm for finding a TSP walk of length at most  $\frac{5n(G)+n_2(G)}{4} - 1$  in any simple 2-connected subcubic graph  $G$ . This is achieved by following the proof of Theorem 3.1.4 to construct an even cover  $F$  of  $G$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$ . As noted by Dvořák et al. [20], modifying this even cover to our desired TSP walk takes linear time.

In the proof of Theorem 3.1.4, we often have a choice of routing a cycle through certain subcubic chains and not through others. For each such chain  $C$ , we “save”  $\delta(\overline{C}, e_C)$  by going through  $C$  and incur a “cost”  $\widehat{\delta}(\overline{C}, e_C)$  by not going through  $C$ . The key idea of Theorem 3.1.4 is that these costs and savings are (at worst) balanced, i.e.  $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$ . Of course, for a given subcubic graph  $G$  and an edge  $e$ , we cannot efficiently compute  $\delta(G, e)$  and  $\widehat{\delta}(G, e)$  exactly (unless P=NP). Instead, we compute “worst-case” estimates

$$(\Delta(G, e), \widehat{\Delta}(G, e)) \in \left\{ \left(-\frac{1}{2}, \frac{1}{2}\right), (-1, 1), \left(-\frac{3}{2}, \frac{3}{2}\right) \right\}$$

such that  $(\delta(G, e), \widehat{\delta}(G, e)) \leq (\Delta(G, e), \widehat{\Delta}(G, e))$  (coordinate-wise).

The natural approach would be to determine exactly when  $\delta(G, e) = -\frac{1}{2}$  or  $\widehat{\delta}(G, e) = \frac{3}{2}$  using our characterization of the extremal examples in Theorem 3.1.4, and then assigning  $(\Delta(G, e), \widehat{\Delta}(G, e))$  to be  $(-\frac{1}{2}, \frac{1}{2})$  or  $(-\frac{3}{2}, \frac{3}{2})$  respectively (and assign  $(-1, 1)$  in all other

cases). To check whether  $(G, e)$  is a minimal rooted  $\theta$ -chain (for example), we would need to first check that it is a rooted  $\theta$ -chain (which takes linear time) and then recursively check that each of its two chains are also minimal, taking quadratic time overall. This approach would result in a cubic algorithm to produce the desired even covers.

It turns out that a much simpler linear-time estimate is sufficient, and yields a quadratic-time algorithm to find the desired even covers. Indeed, by Lemma 3.2.1, if  $(G, e)$  is a rooted  $\theta$ -chain (regardless of whether it is tight or balanced), then we have  $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{1}{2}, \frac{1}{2})$ . And by Lemma 3.2.3, if  $G - e$  is simple and 2-connected and  $(G_u, f_u)$  is a rooted  $\theta$ -chain (where  $G_u$  is obtained from  $G - e$  by suppressing an endpoint  $u$  to an edge  $f_u$ ), then we have  $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{3}{2}, \frac{3}{2})$ .

We thus define an algorithm  $\text{Scan}(G, e)$  to estimate  $(\delta(G, e), \widehat{\delta}(G, e))$  as follows. If  $G$  is a loop or  $G - e$  is 2-connected,  $\text{Scan}(G, e)$  will assign

$$(\Delta(G, e), \widehat{\Delta}(G, e)) = \begin{cases} (-\frac{1}{2}, \frac{1}{2}) & \text{if } (G, e) \text{ is a loop or a rooted } \theta\text{-chain,} \\ (-\frac{3}{2}, \frac{3}{2}) & \text{if } (G_u, f_u) \text{ is a rooted } \theta\text{-chain,} \\ (-1, 1) & \text{otherwise.} \end{cases}$$

If  $G - e$  is not 2-connected (and it is not a loop), then  $(G, e)$  can be written as the closure  $(\overline{C}, e_C)$  of a subcubic chain  $C = xe_0B_1e_1 \cdots e_{k-1}B_k e_k y$  such that  $k \geq 2$  (if  $k = 1$ , then  $G - e = \overline{C} - e_C$  is 2-connected or an isolated vertex). In this case, our estimate on  $(G, e)$  will be the sum of the estimates of the chain-blocks  $(\overline{B}_i, e_{B_i})$  of  $C$ :

$$(\Delta(G, e), \widehat{\Delta}(G, e)) = \sum_{i=1}^k (\Delta(\overline{B}_i, e_{B_i}), \widehat{\Delta}(\overline{B}_i, e_{B_i})).$$

For the remainder of this section, given a 2-connected subcubic graph  $G$  and an edge  $e = uv \in E(G)$  such that  $G - e$  is simple and has no cut-vertex, we let  $u_1, u_2$  denote the two neighbors of  $u$  not equal to  $v$ , and denote by  $G_u$  the graph obtained by deleting  $e$  and suppressing  $u$  to an edge  $f_u = u_1u_2$ . Note that computing  $G_u$  and  $f_u$  takes constant time.

To resolve ambiguities in the choice of the vertex  $u$  in the edge  $e = uv$  (in the case where  $\widehat{\Delta}(G, e) = \frac{3}{2}$ ), we fix a linear ordering  $\leq$  of the vertices throughout, and assume that  $u \leq v$ .

**Proposition 3.5.1.** *Let  $G$  be a subcubic graph and let  $e = uv \in E(G)$  such that  $G - e$  is simple. Then  $\delta(G, e) \leq \Delta(G, e)$  and  $\widehat{\delta}(G, e) \leq \widehat{\Delta}(G, e)$ .*

*Proof.* First suppose  $G$  is a loop or  $G - e$  is 2-connected. If  $(G, e)$  is a loop or a rooted  $\theta$ -chain, then by Lemma 3.2.1,  $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{1}{2}, \frac{1}{2}) = (\Delta(G, e), \widehat{\Delta}(G, e))$ . If  $(G_u, f_u)$  is a rooted  $\theta$ -chain, then by Lemma 3.2.3,

$$(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{3}{2}, \frac{3}{2}) = (\Delta(G, e), \widehat{\Delta}(G, e)).$$

Otherwise, by Theorem 3.1.4, we have  $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-1, 1) = (\Delta(G, e), \widehat{\Delta}(G, e))$ .

Now suppose  $G - e$  is not 2-connected. Then we can write  $(G, e)$  as the closure  $(\overline{C}, e_C)$  of a subcubic chain  $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$  where  $k \geq 2$ . By Proposition 3.1.1 and by induction, we have

$$\begin{aligned} (\delta(\overline{C}, e_C), \widehat{\delta}(\overline{C}, e_C)) &= \sum_{i=1}^k (\delta(\overline{B}_i, e_{B_i}), \widehat{\delta}(\overline{B}_i, e_{B_i})) \\ &\leq \sum_{i=1}^k (\Delta(\overline{B}_i, e_{B_i}), \widehat{\Delta}(\overline{B}_i, e_{B_i})) \\ &= (\Delta(G, e), \widehat{\Delta}(G, e)). \end{aligned}$$

□

Checking whether  $(G, e)$  is a rooted  $\theta$ -chain is equivalent to checking whether  $G - \{u, v\}$  is disconnected, which can be done in linear time. More generally, we can determine the block structure of graphs with a depth first search (DFS) in  $O(n(G) + |E(G)|)$  time (e.g. [17]), which is  $O(n(G))$  when  $G$  is subcubic.

---

**Algorithm 1:**  $\text{Scan}(G, e)$ 

---

**Input** : A loop or a 2-connected subcubic graph  $G$  and  $e = uv \in E(G)$  such that  $G - e$  is simple

**Output:** A half integral vector  $(\Delta(G, e), \widehat{\Delta}(G, e)) \in \{(-\frac{1}{2}, \frac{1}{2}), (-1, 1), (-\frac{3}{2}, \frac{3}{2})\}$ .

```
1 if  $G - e$  has a cut-vertex then
2   |   Write  $(G, e)$  as the closure  $(\overline{C}, e_C)$  of a subcubic chain  $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ ;
3   |   return  $\sum_{i=1}^k \text{Scan}(\overline{B}_i, e_{B_i})$ ;
4 if  $G - \{u, v\}$  is disconnected or  $G$  is a loop then
5   |   return  $(-\frac{1}{2}, \frac{1}{2})$ ;
6 else if  $G_u - \{u_1, u_2\}$  is disconnected then
7   |   return  $(-\frac{3}{2}, \frac{3}{2})$ ;
8 else
9   |   return  $(-1, 1)$ ;
```

---

**Proposition 3.5.2.**  $\text{Scan}(G, e)$  can be computed in  $O(n(G))$  time.

*Proof.* If  $\text{Scan}(G, e)$  returns on lines 5, 7, or 9, then it performs at most three depth first searches, thus requiring  $O(n(G))$  time. Now suppose  $\text{Scan}(G, e)$  returns on line 3; that is,  $(G, e)$  is the closure of a subcubic chain  $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$  where  $k \geq 2$ . For all  $i \in [k]$ ,  $\overline{B}_i - e_{B_i}$  is either 2-connected or a single vertex, so  $\text{Scan}(\overline{B}_i, e_{B_i})$  will not execute line 2. Thus  $\text{Scan}(G, e)$  requires a depth first search on an input of size  $n(G)$  on line 1 and at most three depth first searches for each  $\overline{B}_i, i \in [k]$ . As  $\sum_{i=1}^k n(\overline{B}_i) < n(G)$ , we have that in all cases,  $\text{Scan}(G, e)$  requires  $O(n(G))$  time.  $\square$

We will define two algorithms  $\text{EC}(G, e)$  and  $\widehat{\text{EC}}(G, e)$  which will return an even cover  $F$  in  $\mathcal{E}(G, e)$  and  $\widehat{\mathcal{E}}(G, e)$  respectively such that  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$  and  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$  respectively. For convenience, we wrap these two algorithms in a main algorithm `ALGO` with preprocessing to handle the base case (where  $(G, e)$  is a loop) and the case where  $G - e$  is not 2-connected.

---

**Algorithm 2:**  $\text{Algo}(G, e, \text{flag})$ 

---

**Input** : A loop or a 2-connected subcubic graph  $G$  and  $e \in E(G)$  such that  $G - e$  is simple, and a binary input flag

**Output:**  $F \in \mathcal{E}(G, e)$  such that  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$  (if flag == true) or  $F \in \widehat{\mathcal{E}}(G, e)$  such that  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$  (if flag == false)

- 1 **if**  $G$  is a loop **then**
- 2     **if** flag == true **then**
- 3         **return**  $F = G$ ;
- 4     **else**
- 5         **return**  $F = G - e$ ;
- 6 **if**  $G - e$  is not 2-connected **then**
- 7     Write  $(G, e)$  as the closure  $(\overline{C}, e_C)$  of a subcubic chain  
        $C = xe_0B_1e_1B_2 \dots e_{k-1}B_ke_ky$ ;
- 8     Let  $F_i = \text{Algo}(\overline{B}_i, e_{B_i}, \text{flag})$  for all  $i \in [k]$ ;
- 9     **if** flag == true **then**
- 10         **return**  $F = \bigcup_{i=1}^k (F_i - e_{B_i}) + e + \{e_i : i \in [k-1]\}$ ;
- 11     **else**
- 12         **return**  $F = \bigcup_{i=1}^k F_i$ ;
- 13 Let  $(\Delta, \widehat{\Delta}) = \text{Scan}(G, e)$ ;
- 14 **if** flag == true **then**
- 15     **return**  $F = \text{EC}(G, e, \Delta)$ ;
- 16 **else**
- 17     **return**  $F = \widehat{\text{EC}}(G, e)$ ;

---

For the remainder of the section, we let  $f_{\text{Algo}} : \mathbb{N} \rightarrow \mathbb{N}$  denote a superadditive function (i.e.  $f_{\text{Algo}}(n_1) + f_{\text{Algo}}(n_2) \leq f_{\text{Algo}}(n_1 + n_2)$  for all  $n_1, n_2 \in \mathbb{N}$ ) such that  $\text{Algo}(G, e, \text{flag})$  takes at most  $f_{\text{Algo}}(n)$  steps on inputs of size at most  $n$ . We will show in the end that we can take  $f_{\text{Algo}}(n) = O(n^2)$ .

We now give the algorithm  $\widehat{\text{EC}}(G, e)$  used in line 17 of  $\text{Algo}(G, e, \text{flag})$ , which produces an even cover  $F \in \widehat{\mathcal{E}}(G, e)$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ . Recall that  $(G_u, f_u)$

is obtained from  $G$  and  $e = uv$  by deleting  $e$  and suppressing  $u$  to an edge  $f_u = u_1u_2$ .

---

**Algorithm 3:**  $\widehat{\text{EC}}(G, e)$

---

**Input** : A subcubic graph  $G$  and  $e = uv \in E(G)$  such that  $G - e$  is simple and 2-connected

**Output:** An even cover  $F \in \widehat{\mathcal{E}}(G, e)$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$  where

$$\widehat{\Delta}(G, e) = \text{Scan}(G, e)_2$$

1 Let  $F' = \text{Algo}(G_u, f_u, \text{true});$

2 **return**  $F = (F' - f_u) + \{u\} + \{u_1u, uu_2\};$

---

**Proposition 3.5.3.** *Suppose Algo is correct on inputs of size less than  $n$ . Then  $\widehat{\text{EC}}$  is correct and takes  $f_{\text{Algo}}(n - 1) + O(1)$  time for all inputs of size less than or equal to  $n$ .*

*Proof.* We clearly have  $F \in \widehat{\mathcal{E}}(G, e)$ . We claim that  $\Delta(G_u, f_u) + 2 \leq \widehat{\Delta}(G, e)$ . If  $\widehat{\Delta}(G, e) = \frac{3}{2}$ , there is nothing to prove (since  $\Delta \leq -\frac{1}{2}$ ). If  $\widehat{\Delta}(G, e) = 1$ , then  $(G_u, f_u)$  is not a rooted  $\theta$ -chain, so  $\Delta(G_u, f_u) \leq -1$ . Finally, suppose  $\widehat{\Delta}(G, e) = \frac{1}{2}$ . Then  $(G, e)$  is a rooted  $\theta$ -chain. This implies that  $(G_u, f_u)$  is the closure  $(\overline{C}, e_C)$  of a subcubic chain  $C$  with at least three blocks, so  $\Delta(G_u, f_u) = \Delta(\overline{C}, e_C) \leq -\frac{3}{2}$ . It follows that

$$\begin{aligned} \text{exc}(F) &= \text{exc}(F') \\ &\leq \frac{n(G) + n_2(G)}{4} + \Delta(G_u, f_u) + 2 \\ &\leq \frac{n(G) + n_2(G)}{4} + \widehat{\Delta}(G, e). \end{aligned}$$

For the time complexity, note that Algo is called only once on  $(G_u, f_u)$ , which takes  $f_{\text{Algo}}(n(G_u)) = f_{\text{Algo}}(n - 1)$  time. The remaining lines require constant time, thus  $\widehat{\text{EC}}$  runs in  $f_{\text{Algo}}(n - 1) + O(1)$  time.  $\square$

We now give the algorithm  $\text{EC}(G, e, \Delta)$  in line 15 of Algo, which produces an even cover  $F \in \mathcal{E}(G, e)$  such that  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$ . For clarity of presentation, we split the algorithm into three cases depending on the value  $\Delta$ . We first describe the case  $\Delta = -\frac{1}{2}$ .

---

**Algorithm 4:**  $\text{EC}(G, e, -\frac{1}{2})$ 

---

**Input** : A subcubic graph  $G$  and  $e = uv \in E(G)$  such that  $G - e$  is simple and

2-connected, and  $\Delta(G, e) = -\frac{1}{2}$  (i.e.  $(G, e)$  is a rooted  $\theta$ -chain)

**Output:** An even cover  $F \in \mathcal{E}(G, e)$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$

- 1 Determine the subcubic chains  $C_1$  and  $C_2$  of  $(G, e)$  with a DFS;
  - 2 Let  $(\Delta(C_1), \widehat{\Delta}(C_1)) = \text{Scan}(\overline{C_1}, e_{C_1})$  and let  $(\Delta(C_2), \widehat{\Delta}(C_2)) = \text{Scan}(\overline{C_2}, e_{C_2})$ ;
  - 3 Relabel if necessary so that  $\Delta(C_1) + \widehat{\Delta}(C_2) \leq 0$ ;
  - 4 Let  $F_1 = \text{Algo}(\overline{C_1}, e_{C_1}, \text{true})$  and  $F_2 = \text{Algo}(\overline{C_2}, e_{C_2}, \text{false})$ ;
  - 5 Let  $v'$  be the neighbor of  $v$  in  $C_1$  and let  $u'$  be the neighbor of  $u$  in  $C_1$ ;
  - 6 **return**  $F = (F_1 - e_{C_1}) \cup F_2 + \{u, v\} + \{u'u, uv, vv'\}$ ;
- 

**Proposition 3.5.4.** *Suppose Algo is correct on inputs of size less than  $n = n(G)$ . Then  $\text{EC}(G, e, -\frac{1}{2})$  is correct and takes  $f_{\text{Algo}}(n-1) + O(n)$  time for all input graphs of size less than or equal to  $n$ .*

*Proof.* For correctness, first note that the relabeling step on line 3 is always possible as  $\Delta(C_i) = -\widehat{\Delta}(C_i)$  for  $i \in [2]$ . Since  $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 2$ ,  $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$ , and  $\text{exc}(F) = \text{exc}(F_1) + \text{exc}(F_2)$ , we have

$$\begin{aligned} \text{exc}(F) &= \text{exc}(F_1) + \text{exc}(F_2) \\ &\leq \frac{n(C_1) + n_2(C_1)}{4} + \Delta(C_1) + 2 + \frac{n(C_2) + n_2(C_2)}{4} + \widehat{\Delta}(C_2) \\ &\leq \frac{n(G) + n_2(G)}{4} + \frac{3}{2}. \end{aligned}$$

For the time complexity, line 1 requires  $O(n)$  time. By Proposition 3.5.2, line 2 requires  $O(n(\overline{C_1})) + O(n(\overline{C_2})) = O(n)$  time. By induction, line 4 takes  $f_{\text{Algo}}(n(\overline{C_1})) + f_{\text{Algo}}(n(\overline{C_2})) \leq f_{\text{Algo}}(n-1)$  time. Thus, in total,  $\text{EC}(G, e, -\frac{1}{2})$  takes  $f_{\text{Algo}}(n-1) + O(n)$  time of inputs of size  $n$ .  $\square$

Before we handle the analysis of  $\text{EC}(G, e, -1)$ , we first give an important subroutine which is an algorithmic version of Lemma 3.2.2.

---

**Algorithm 5:** Subroutine( $Z, u, v_1, v_2$ )

---

**Input** : A simple 2-connected subcubic graph  $Z$  and distinct vertices  $u, v_1, v_2$  of degree 2  
in  $Z$

**Output:**  $F \in \mathcal{E}(Z + uv_i, uv_i)$  for some  $i \in [2]$  with  $\text{exc}(F) \leq \frac{n(Z+uv_i)+n_2(Z+uv_i)}{4} + 1$

- 1 For each  $i \in [2]$ , let  $(\Delta_i, \widehat{\Delta}_i) = \text{Scan}(Z + uv_i, uv_i)$ ;
- 2 **if**  $\Delta_i \leq -1$  for some  $i \in [2]$  **then**
- 3     **return**  $F = \text{Algo}(Z + uv_i, uv_i, \text{true})$ ;
- 4 Let  $C_{i,1}, C_{i,2}$  denote the two subcubic chains of  $(Z + uv_i, uv_i)$ ,  $i \in [2]$ ;
- 5 Let  $(\Delta(C_{i,j}), \widehat{\Delta}(C_{i,j})) = \text{Scan}(\overline{C_{i,j}}, e_{C_{i,j}})$  for  $i, j \in [2]$ ;
- 6 Relabel if necessary so that  $\Delta(C_{1,1}) + \widehat{\Delta}(C_{1,2}) \leq -\frac{1}{2}$ ;
- 7 Let  $F_1 = \text{Algo}(\overline{C_{1,1}}, e_{C_{1,1}}, \text{true})$  and  $F_2 = \text{Algo}(\overline{C_{1,2}}, e_{C_{1,2}}, \text{false})$ ;
- 8 Let  $u'$  be the neighbor of  $u$  in  $C_{1,1}$  and  $v'$  be the neighbor of  $v_1$  in  $C_{1,1}$ ;
- 9 **return**  $F = (F_1 - e_{C_{1,1}}) \cup F_2 + \{u, v\} + \{u'u, uv_1, v_1v'\}$ ;

---

**Proposition 3.5.5.** *Suppose Algo is correct for all inputs of size less than or equal to  $n = n(Z)$ . Then Subroutine is correct and takes  $f_{\text{Algo}}(n) + O(n)$  time for all inputs of size less than or equal to  $n$ .*

*Proof.* We first analyze correctness. If we return on line 3, by correctness of Algo, we have  $\text{exc}(F) \leq \frac{n(Z+uv_i)+n_2(Z+uv_i)}{4} + 1$ . So assume  $\Delta_i = \Delta(Z + uv_i, uv_i) = -\frac{1}{2}$  for both  $i \in [2]$ . Thus both  $(Z + uv_i, uv_i)$  are rooted  $\theta$ -chains, which implies that  $v_{3-i}$  is a trivial block in one of the chains  $C_{i,1}$  and  $C_{i,2}$ . This then implies that  $\Delta(C_{i,1}) \neq \Delta(C_{i,2})$  for some  $i \in [2]$ . Thus the relabeling step on line 6 is always possible.

Now consider the even cover  $F$  returned on line 9. As  $n(Z+uv_1) = n(\overline{C_{1,1}}) + n(\overline{C_{1,2}}) +$

2,  $n_2(Z + uv_1) = n_2(\overline{C_{1,1}}) + n_2(\overline{C_{1,2}})$ , and  $\Delta(C_{1,1}) + \widehat{\Delta}(C_{1,2}) \leq -\frac{1}{2}$ , we have

$$\begin{aligned}
\text{exc}(F) &= \text{exc}(F_1) + \text{exc}(F_2) \\
&\leq \frac{n(\overline{C_{1,1}}) + n_2(\overline{C_{1,1}})}{4} + \Delta(\overline{C_{1,1}}) + 2 + \frac{n(\overline{C_{1,2}}) + n_2(\overline{C_{1,2}})}{4} + \widehat{\Delta}(\overline{C_{1,2}}) \\
&\leq \frac{n(Z + uv_1) + n_2(Z + uv_1)}{4} + \Delta(\overline{C_{1,1}}) + \widehat{\Delta}(\overline{C_{1,2}}) + \frac{3}{2} \\
&\leq \frac{n(Z + uv_1) + n_2(Z + uv_1)}{4} + 1.
\end{aligned}$$

For the time complexity, as  $n(\overline{C_{1,1}}) + n(\overline{C_{1,2}}) < n$ , lines 3 and 7 both take at most  $f_{\text{Algo}}(n)$  time. Furthermore, by Proposition 3.5.2, the remaining lines require  $O(n)$  time. Since we call exactly one of line 3 or 7,  $\text{Subroutine}(Z, u, v_1, v_2)$  takes  $f_{\text{Algo}}(n) + O(n)$  time. □

We are now ready to present  $\text{EC}(G, e, -1)$ .

---

**Algorithm 6:**  $\text{EC}(G, e, -1)$ 

---

**Input :** A subcubic graph  $G$  and  $e = uv \in E(G)$  such that  $G - e$  is simple and 2-connected, and  $\Delta(G, e) = -1$ .

**Output:**  $F \in \mathcal{E}(G, e)$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$

- 1 Let  $Z_1$  and  $Z_2$  be the blocks (or single vertices) of  $G - \{u, v\}$  as defined in Claim 3.3.3;
- 2 Define vertices  $u_i, u'_i, v_j, v'_j$  and subcubic chains  $U_i, V_j$  for  $i, j \in [2]$ , as in the proof of Theorem 3.1.4;
- 3 Let  $(\Delta(U_i), \widehat{\Delta}(U_i)) = \text{Scan}(\overline{U}_i, e_{U_i})$  and  $(\Delta(V_j), \widehat{\Delta}(V_j)) = \text{Scan}(\overline{V}_j, e_{V_j})$  for  $i, j \in [2]$ ;
- 4 **if**  $Z_1 \neq Z_2$  **then**
  - 5 Relabel vertices as necessary so that  $\Delta(U_1) + \Delta(V_2) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_1) \leq 0$ ;
  - 6 Let  $Z = Z_1 \cup Z_2 \cup Y$ , where  $Y$  is the subcubic chain from  $Z_1$  to  $Z_2$ ;
  - 7 Let  $F_{U_1} = \text{Algo}(\overline{U}_1, e_{U_1}, \text{true})$ ,  $F_{V_2} = \text{Algo}(\overline{V}_2, e_{V_2}, \text{true})$ ,  
 $F_{U_2} = \text{Algo}(\overline{U}_2, e_{U_2}, \text{false})$ ,  $F_{V_1} = \text{Algo}(\overline{V}_1, e_{V_1}, \text{false})$ , and  
 $F_Z = \text{Algo}(Z + u'_1 v'_2, u'_1 v'_2, \text{true})$ ;
  - 8 **return**  
 $F = (F_{U_1} - e_{U_1}) \cup (F_{V_2} - e_{V_2}) \cup F_{U_2} \cup F_{V_1} \cup (F_Z - u'_1 v'_2) + \{u, v\} + \{u_1 u, uv, vv_2\}$ ;
- 9 **else**
  - 10 Relabel vertices as necessary so that  $\Delta(U_1) + \Delta(V_i) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_{3-i}) \leq 0$  for  
 $i \in [2]$ ;
  - 11 Let  $F_Z = \text{Subroutine}(Z_1, u'_1, v'_1, v'_2)$ ;
  - 12 Relabel so that  $u'_1 v'_2 \in F_Z$ ;
  - 13 Let  $F_{U_1} = \text{Algo}(\overline{U}_1, e_{U_1}, \text{true})$ ,  $F_{V_2} = \text{Algo}(\overline{V}_2, e_{V_2}, \text{true})$ ,  
 $F_{U_2} = \text{Algo}(\overline{U}_2, e_{U_2}, \text{false})$ , and  $F_{V_1} = \text{Algo}(\overline{V}_1, e_{V_1}, \text{false})$ ;
  - 14 **return**  
 $F = (F_Z - u'_1 v'_2) \cup (F_{U_1} - e_{U_1}) \cup (F_{V_2} - e_{V_2}) \cup F_{U_2} \cup F_{V_1} + \{u, v\} + \{u_1 u, uv, vv_2\}$ ;

---

**Proposition 3.5.6.** *Suppose Algo is correct on all inputs of size less than  $n = n(G)$ . Then  $\widehat{\text{EC}}(G, e, -1)$  is correct and takes  $f_{\text{Algo}}(n - 1) + O(n)$  time for all inputs of size less than or equal to  $n$ .*

*Proof.* The proof of correctness follows the same structure of Section ???. The existence of  $Z_1$  and  $Z_2$  follows from Claim 3.3.3, and they can be determined from the block structure of  $G - \{u, v\}$  in linear time. As  $\Delta(U_i) = -\widehat{\Delta}(U_i)$  and  $\Delta(V_i) = -\widehat{\Delta}(V_i)$  for  $i \in [2]$ , the relabeling on lines 5 and 10 are always possible. Furthermore, regardless of whether  $Z_1 \neq Z_2$  or  $Z_1 = Z_2$ , we have

- $\text{exc}(F) - 2 = (\text{exc}(F_{U_1}) - 2) + (\text{exc}(F_{V_2}) - 2) + \text{exc}(F_{U_2}) + \text{exc}(F_{V_1}) + (\text{exc}(F_Z) - 2)$ ,
- $n(G) = n(\overline{U_1}) + n(\overline{V_2}) + n(\overline{U_2}) + n(\overline{V_1}) + n(Z + u'_1 v'_2) - 2$ , and
- $n_2(G) = n_2(\overline{U_1}) + n_2(\overline{V_2}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(Z + u'_1 v'_2) + 2$ .

By induction, we have  $\text{exc}(F_{U_1}) - 2 \leq \frac{n(\overline{U_1}) + n_2(\overline{U_1})}{4} + \Delta(U_1)$ ,  $\text{exc}(F_{V_2}) - 2 \leq \frac{n(\overline{V_2}) + n_2(\overline{V_2})}{4} + \Delta(V_2)$ ,  $\text{exc}(F_{U_2}) \leq \frac{n(\overline{U_2}) + n_2(\overline{U_2})}{4} + \widehat{\Delta}(U_2)$ , and  $\text{exc}(F_{V_1}) \leq \frac{n(\overline{V_1}) + n_2(\overline{V_1})}{4} + \widehat{\Delta}(V_1)$ . We argue now that in both cases we have

$$\text{exc}(F_Z) - 2 \leq \frac{n(Z + u'_1 v'_2) + n_2(Z + u'_1 v'_2)}{4} - 1. \quad (3.13)$$

If  $Z_1 = Z_2$ , this follows from Proposition 3.5.5. If  $Z_1 \neq Z_2$ , then  $(Z + u'_1 v'_2, u'_1 v'_2)$  is the closure of a subcubic chain with at least two blocks, namely  $Z_1$  and  $Z_2$ . By induction on its chain-blocks, we have

$$\begin{aligned} \text{exc}(F_Z) - 2 &\leq \frac{n(Z + u'_1 v'_2) + n_2(Z + u'_1 v'_2)}{4} + \Delta(Z + u'_1 v'_2, u'_1 v'_2) \\ &\leq \frac{n(Z + u'_1 v'_2) + n_2(Z + u'_1 v'_2)}{4} - 1 \end{aligned}$$

and (3.13) holds in both cases. Thus,

$$\begin{aligned} \text{exc}(F) - 2 &= (\text{exc}(F_{U_1}) - 2) + (\text{exc}(F_{V_2}) - 2) + \text{exc}(F_{U_2}) + \text{exc}(F_{V_1}) + (\text{exc}(F_Z) - 2) \\ &\leq \frac{n(G) + n_2(G)}{4} + \Delta(U_1) + \Delta(V_2) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_1) + \Delta(Z + u_1 v'_2, u'_1 v'_2) \\ &\leq \frac{n(G) + n_2(G)}{4} - 1. \end{aligned}$$

For the time complexity, note that we only call `Algo` and `Subroutine` on inputs whose sizes sum to less than  $n$ . As the remaining lines require  $O(n)$  time by Proposition 3.5.2, we have that the entire algorithm requires  $f_{\text{Algo}}(n-1) + O(n)$  time.  $\square$

We now present the final case for EC.

---

**Algorithm 7:**  $\text{EC}(G, e, -\frac{3}{2})$

---

**Input** : A subcubic graph  $G$  and  $e = uv \in E(G)$  with  $G - e$  is simple and 2-connected,

and  $\Delta(G, e) = -\frac{3}{2}$  (i.e.  $(G_u, f_u)$  is a rooted  $\theta$ -chain)

**Output:**  $F \in \mathcal{E}(G, e)$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$

- 1 Let  $C_1$  and  $C_2$  denote the chains of  $(G_u, f_u)$  with common endpoints  $f_u = \{u_1, u_2\}$  and  $v \in V(C_1)$ ;
  - 2 Let  $x_i \in V(C_2)$  be the neighbor of  $u_i$  for  $i \in [2]$ ;
  - 3 Write  $C_1 = u_1 e_0 B_1 \dots e_{k-1} B_k e_k u_2$ ;
  - 4 Let  $\ell \in [k]$  be the unique index such that  $v \in V(B_\ell)$ ;
  - 5 Let  $v'$  denote the endpoint of  $e_{\ell-1}$  in  $B_\ell$ , and let  $v''$  denote the endpoint of  $e_\ell$  in  $B_\ell$ ;
  - 6 Let  $D_1$  and  $D_2$  denote the chains of  $C_1$  with end points  $\{u_1, v'\}$  and  $\{v'', u_2\}$  respectively;
  - 7 For  $i \in [2]$ , let  $(\Delta(D_i), \widehat{\Delta}(D_i)) = \text{Scan}(\overline{D_i}, e_{D_i})$ ;
  - 8 Relabel if necessary so that  $\Delta(D_1) + \widehat{\Delta}(D_2) \leq 0$ ;
  - 9 Let  $F_2 = \text{Algo}(\overline{C_2}, e_{C_2}, \text{true})$ ,  $F_{D,1} = \text{Algo}(\overline{D_1}, e_{D_1}, \text{true})$ ,  
 $F_{D,2} = \text{Algo}(\overline{D_2}, e_{D_2}, \text{false})$ , and  $F_\ell = \text{Algo}(B_\ell + v'v, v'v, \text{true})$ ;
  - 10 **return**  $F = (F_2 - e_{C_2}) \cup (F_{D,1} - e_{D_1}) \cup F_{D,2} \cup (F_\ell - v'v) + \{u, u_1, u_2\} + \{e_0, e_{\ell-1}, u_1 x_1, uv, uu_2, u_2 x_2\}$ ;
- 

**Proposition 3.5.7.** *Suppose `Algo` is correct for all inputs of size less than  $n = n(G)$ . Then  $\text{EC}(G, e, -\frac{3}{2})$  is correct and takes  $f_{\text{Algo}}(n-1) + O(n)$  time for all inputs of size less than or equal to  $n$ .*

*Proof.* We first analyze the correctness of the returned even cover  $F$ . By induction, we have that  $\text{exc}(F_2) \leq \frac{n(\overline{C_2})+n_2(\overline{C_2})}{4} + \Delta(C_2) + 2$ ,  $\text{exc}(F_{D,1}) \leq \frac{n(\overline{D_1})+n_2(\overline{D_1})}{4} + \Delta(D_1) + 2$ ,  $\text{exc}(F_{D,2}) \leq \frac{n(\overline{D_2})+n_2(\overline{D_2})}{4} + \widehat{\Delta}(D_2)$ , and  $\text{exc}(F_\ell) \leq \frac{n(B_\ell+v'v)+n_2(B_\ell+v'v)}{4} + \frac{3}{2}$ . As  $\text{exc}(F) - 2 = (\text{exc}(F_2) - 2) + (\text{exc}(F_{D,1}) - 2) + \text{exc}(F_{D,2}) + (\text{exc}(F_\ell) - 2)$ ,  $n(G) = n(\overline{C_2}) + n(\overline{D_1}) +$

$n(\overline{D_2}) + n(B_\ell + v'v) + 3$ , and  $n_2(G) = n_2(\overline{C_2}) + n_2(\overline{D_1}) + n_2(\overline{D_2}) + n_2(B_\ell + v'v) - 1$ , we have

$$\begin{aligned} \text{exc}(F) - 2 &= (\text{exc}(F_2) - 2) + (\text{exc}(F_{D,1}) - 2) + \text{exc}(F_{D,2}) + (\text{exc}(F_\ell) - 2) \\ &\leq \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \Delta(C_2) + \Delta(D_1) + \widehat{\Delta}(D_2) + \Delta(B_\ell + v'v, v'v) \\ &\leq \frac{n(G) + n_2(G)}{4} - \frac{3}{2}, \end{aligned}$$

since  $\Delta(C_2), \Delta(B_\ell + v'v, v'v) \leq -\frac{1}{2}$  and  $\Delta(D_1) + \widehat{\Delta}(D_2) \leq 0$ . Thus  $\text{exc}(F)$  satisfies our desired bound.

For the time analysis, as we only call `Algo` on inputs whose sizes sum to less than  $n$ , line 9 takes at most  $f_{\text{Algo}}(n)$  time. Furthermore, by Proposition 3.5.2, the remaining lines require  $O(n)$  time. Thus,  $\text{EC}(G, e, -\frac{3}{2})$  takes  $f_{\text{Algo}}(n-1) + O(n)$  time.  $\square$

To summarize, we have the following.

**Corollary 3.5.8.** *Algo is correct and takes  $O(n^2)$  time.*

*Proof.* We show inductively that we can take  $f_{\text{Algo}}(n) = O(n^2)$ . First note that lines 1-5 take constant time. Line 6 takes linear time to check, and if executed, lines 7-12 take  $O(n) + \sum_{i=1}^k f_{\text{Algo}}(n(\overline{B_i})) \leq O(n) + \sum_{i=1}^k O(n(\overline{B_i})^2) = O(n^2)$ .

Line 13 take linear time by Proposition 3.5.2, and in lines 14-17, we execute exactly one of  $\text{EC}(G, e, \Delta)$  and  $\widehat{\text{EC}}(G, e)$ , which takes  $f_{\text{Algo}}(n-1) + O(n)$  time by Propositions 3.5.3, 3.5.4, 3.5.6, and 3.5.7. It follows that we can take  $f_{\text{Algo}}(n) = O(n^2)$ .  $\square$

**Corollary 3.5.9.** *Given a simple 2-connected subcubic graph  $G$ , we can find an even cover  $F$  of  $G$  with  $\text{exc}(F) \leq \frac{n(G) + n_2(G)}{4} + 1$  in quadratic time.*

*Proof.* Pick an arbitrary edge  $e \in E(G)$ . Run  $\text{Algo}(G, e, \text{true})$  and  $\text{Algo}(G, e, \text{false})$ . One of the returned even covers will have excess at most  $\frac{n(G) + n_2(G)}{4} + 1$ .  $\square$

Let us now complete the proof of Theorem 1.1.2, restated here for the reader's convenience.

*Proof.* By Corollary 3.5.9, we can find an even cover  $F$  of  $G$  with  $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$  in quadratic time. Then by Proposition 1.2.2, we can convert  $F$  to a TSP walk of length  $\text{exc}(F) - 2 + n(G) \leq \frac{5n(G)+n_2(G)}{4} - 1$  in linear time.  $\square$

If the input graph  $G$  is cubic (i.e.  $n_2(G) = 0$ ), then Theorem 1.1.2 finds a TSP walk of length at most  $\frac{5n(G)}{4} - 1$  in quadratic time. Since every TSP walk trivially has length at least  $n(G)$ , this gives a  $\frac{5}{4}$ -approximation algorithm for TSP walks in 2-connected cubic graphs. For general subcubic graphs, Theorem 1.1.2 finds a TSP walk of length at most  $\frac{3}{2}n(G)$  which trivially yields a  $\frac{3}{2}$ -approximation algorithm. The bound gets better for subcubic graphs with fewer vertices of degree 2; for example, if  $n_2(G) \leq \frac{1}{3}n(G)$ , then Theorem 1.1.2 yields a TSP walk of length at most  $\frac{4}{3}n(G)$ . We suspect that refining the ideas developed in this paper could lead to another  $\frac{4}{3}$ -approximation algorithm for subcubic graphs, matching the current best ratio by Mömke and Svensson [47], and possibly beyond.

## CHAPTER 4

### CONCLUSION

In Chapter 2, we developed a theory of Tutte paths where the number of possible bridges is bounded. In Chapter 3, we provided a  $5/4$ -approximation for the cubic TSP by means of finding efficient even covers. We will now discuss possible future directions for this work.

#### 4.1 Future Directions

There has been extensive work in extending Tutte path results to other surfaces. Thomas and Yu [60] showed that 4-connected projective-planar graphs are Hamiltonian and Kawarabayishi and Ozeki [40] later showed that such graphs are Hamiltonian-connected. Both of these results relied on Tutte path techniques. Both Gröbaum [29] and Nash-Williams [50] conjectured every 4-connected graph embedded in the torus is Hamiltonian. There has been partial results [3, 11, 61, 62] towards this conjecture, again relying on a Tutte path strategy. It is natural to try and extend our quantitative Tutte path result to these different settings.

A *2-walk* is a spanning walk that visits each vertex at most twice. As a relaxation of the Hamiltonian cycle problem, Gao and Richter [24] showed that every 3-connected planar graph has a 2-walk. Nakamoto, Oda, and Ota [49] asked if every 3-connected  $n$ -vertex planar graph has a 2-walk such that the number of vertices visited twice is  $n/3 + o(1)$ . Note if this were true, it would directly imply the result of Kawarabayishi and Ozeki [41] on the length of TSP walks in 3-connected planar graphs up to additive constant error. Later, Gao, Richter, and Yu [25] showed that every 3-connected planar graph has a 2-walk, such that any vertex visiting twice is contained in a 3-cut. This was developed by developing a system of distinct representatives (SDR) between a Tutte path and its bridges. It would be interesting if this SDR can be developed simultaneously while bounding the number of nontrivial bridges to be  $n/3 + o(1)$ . This could have potential applications in answering

Nakamoto, Oda, and Ota's question.

From an algorithm's perspective, our approximation algorithm for cubic TSP is unsophisticated as it simply provides a walk of length  $5n/4-1$  where  $n$  is the number of vertices. As every TSP walk has length at least  $n$ , this provides the  $5/4$ -approximation guarantee. To push the  $5/4$ -approximation guarantee further, a possible strategy is to develop efficient means to compute better lower bound guarantees on the optimal walk. This seems challenging, as this lower bound guarantee cannot be used to detect if a graph is Hamiltonian or not, as the Hamiltonian cycle problem remains NP-hard even when restricted to 3-connected cubic planar graphs [26].

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