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A THEORY OF GENERALIZED MOUFANG LOOPS

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## CHAPTER I

## INTRODUCTION

The purpose of the present research is to investigate the properties of loops which belong to a class of loops which we shall call generalized Moufang loops. In 1934 Ruth Moufang [6] investigated the properties of the multiplicative systems of alternative division rings. By deleting the zero element from such a system, one obtains a loop, and Moufang showed that the identity

$$(1.1) \quad gx \cdot yg = (g \cdot xy)g$$

holds for any three elements  $g$ ,  $x$ , and  $y$  belonging to such a loop. Any loop in which (1.1) holds is called a Moufang loop. In 1959 J. Marshall Osborn [7], while studying a certain class of weak inverse property loops, discovered that in any loop  $G$  of the type he was considering, the identity

$$(1.2) \quad gx \cdot (y\theta_g \cdot g) = (g \cdot xy)g$$

held where  $g$ ,  $x$ , and  $y$  are in  $G$  and where  $\theta_g$  is an automorphism of  $G$  depending only on  $g$ . We shall call any loop satisfying (1.2) a *generalized Moufang loop*.

In the first chapter we shall consider the origin of generalized Moufang loops from the study of weak inverse property loops; the material is expository in nature and, except where it is otherwise specified, appears in a paper by Osborn [7]. In the second chapter, we shall discuss some of the basic properties common to any generalized Moufang loop; give an example to show that the class of generalized Moufang loops is strictly larger than the class of loops previously considered by Osborn; and investigate the properties of generalized Moufang loops which also have the weak inverse property. In the third chapter we shall further investigate the properties of the loops which Osborn had considered previously, and in the fourth chapter we shall consider some of the theory of holomorphs of generalized Moufang loops.

We shall write all loops which appear in this paper multiplicatively, and unless specified otherwise, we shall use 1 to denote the identity of a loop. The symbol  $\rho$  will be used to denote the mapping which takes an element  $x$  of a loop onto its right inverse, and we shall denote the image of  $x$  under the mapping  $\rho$  by  $x^\rho$ . The symbol  $\lambda$  will denote the inverse of the mapping  $\rho$ , and the image of  $x$  under  $\lambda$  will be denoted by  $x^\lambda$ ; that is,  $x^\lambda$  is the left inverse of  $x$ . To each element  $y$  in a loop we shall use  $R(y)$ ,  $L(y)$  to denote the permutations defined by the equations:

$$(1.3) \quad xR(y) = xy \quad \text{and}$$

$$(1.4) \quad xL(y) = yx$$

The inverses of  $R(y)$  and  $L(y)$  will be denoted by  $R^{-1}(y)$  and  $L^{-1}(y)$ , respectively.

We now consider the definition of a weak inverse property loop.

Theorem 1.1. Let  $G$  be a loop. Then the following four statements are equivalent.

(1.5) If  $x$ ,  $y$ , and  $z$  are any three elements in  $G$  and if  $xy \cdot z = 1$ , then we have  $x \cdot yz = 1$ .

(1.6) For any two elements  $x$  and  $y$  in  $G$ , we have  $y(xy)^\rho = x^\rho$ .

(1.7) For any two elements  $x$  and  $y$  in  $G$ , we have  $(xy)^\lambda x = y^\lambda$ .

(1.8) If  $x$ ,  $y$ , and  $z$  are any three elements in  $G$  and if  $x \cdot yz = 1$ , then we have  $xy \cdot z = 1$ .

Proof: Assume that (1.5) holds. Let  $x$  and  $y$  be any two elements of  $G$ .

By definition we have  $xy \cdot (xy)^\rho = 1$ . Hence we obtain  $x \cdot y(xy)^\rho = 1$  by

(1.5). Hence we have  $x \cdot y(xy)^\rho = xx^\rho$ , and it follows that  $y(xy)^\rho = x^\rho$  since a loop is cancellative. Thus we have established (1.6).

Now assume that (1.6) holds. Replacing  $y$  by  $x$  and  $x$  by  $(xy)^\lambda$  in (1.6), we may write that  $x((xy)^\lambda x)^\rho = xy$ . Hence we see that  $((xy)^\lambda x)^\rho = y$ ; that is,  $(xy)^\lambda x = y^\lambda$ , and (1.7) is established.

Now assume that (1.7) holds. Let  $x$ ,  $y$ , and  $z$  be in  $G$  such that

$x \circ yz = 1$ . Then we have  $x = (yz)^\lambda$ . Now we also have  $(yz)^\lambda y = z^\lambda$ . Hence we see that  $(yz)^\lambda y \cdot z = 1$ ; that is,  $xy \cdot z = 1$ , and we have proved (1.8).

We have now proven that (1.5) implies (1.8). In like manner one can show that (1.8) implies (1.7), that (1.7) implies (1.6), and that (1.6) implies (1.5). Hence, we have proven the equivalence of all of the statements.  $\square$

Definition 1.1. A loop is called a *weak inverse property loop* if and only if it satisfies any, and hence all, of the conditions of Theorem 1.1.

Special cases of weak inverse property loops are inverse property loops and cross inverse property loops. Inverse property loops are loops in which corresponding to each element  $x$  there is an element  $x^{-1}$  such that  $x^{-1}x = xx^{-1} = 1$ , and, for any  $y$ ,  $x^{-1}(xy) = (yx)x^{-1} = y$ . R. H. Bruck [2] has shown that in such loops  $(xy)^{-1} = y^{-1}x^{-1}$ , and hence  $y(xy)^{-1} = y(y^{-1}x^{-1}) = x^{-1}$  which shows that such loops have the weak inverse property. Cross inverse property loops are loops in which corresponding to each element  $x$  there is an element  $x^\rho$  such that  $xx^\rho = 1$ , and  $xy \cdot x^\rho = 1$  for all  $y$  in the loop. In such loops we have the identity  $y(xy)^\rho = (xy \cdot yL^{-1}(xy))(xy)^\rho = yL^{-1}(xy)$ . Thus we see that  $xy \cdot y(xy)^\rho = y$ , but we also have that  $xy \cdot x^\rho = y$ , and hence  $y(xy)^\rho = x^\rho$  which implies that these loops are weak inverse property loops. Bruck [2] has shown that Moufang loops are inverse property loops; hence Moufang loops are also weak inverse property loops. We shall show later in this chapter (Example 1.1) that the class of all weak inverse property loops is strictly larger than the class consisting of all inverse property loops

and cross inverse property loops.

The following theorem provides a useful representation for  $R^{-1}(y)$  and  $L^{-1}(y)$  in any weak inverse property loop.

Theorem 1.2. Let  $y$  be an element of a weak inverse property loop  $G$ .

Then we have the following representations for  $R^{-1}(y)$  and  $L^{-1}(y)$ :

$$(1.9) \quad R^{-1}(y) = \rho L(y) \lambda$$

$$(1.10) \quad L^{-1}(y) = \lambda R(y) \rho$$

Proof: Let  $y$  be in  $G$ . Then, for any  $x$  in  $G$ , we have the identity  $y(xy)^{\rho} = x^{\rho}$  by (1.6). This implies that  $\rho = R(y)\rho L(y)$ . Hence we have  $R^{-1}(y) = \rho L(y)\lambda$ , and we have established (1.9). Taking the inverse of both sides of (1.9), we obtain the identity  $R(y) = \rho L^{-1}(y)\lambda$ . Hence we have  $L^{-1}(y) = \lambda R(y)\rho$ , and we have established (1.10). ■

Bruck [2] has shown that the mapping  $\rho^2$  is an automorphism of inverse property loops, and R. Artzy [1] has shown that  $\rho^2$  is an automorphism of cross inverse property loops; we now show that this result holds in weak inverse property loops.

Theorem 1.3. In a weak inverse property loop, the mapping  $\rho^2$  and its inverse  $\lambda^2$  are automorphisms.

Proof: Let  $G$  be any weak inverse property loop and suppose that  $x$  and  $y$  are in  $G$ . By (1.6) we have the identity  $y(xy)^{\rho} = x^{\rho}$ , and hence

$(y(xy)^\rho)^\rho = x^{\rho^2}$ . Thus  $(xy)^\rho \cdot x^{\rho^2} = (xy)^\rho \cdot (y(xy)^\rho)^\rho = y^\rho$ , where we have used (1.6) again at the last step. It follows that  $y^{\rho^2} = ((xy)^\rho x^{\rho^2})^\rho$ , from which we obtain that  $x^{\rho^2} \cdot y^{\rho^2} = x^{\rho^2} ((xy)^\rho x^{\rho^2}) = (xy)^{\rho^2}$  by (1.6) again. Hence  $\rho^2$  is an automorphism of  $G$ , and its inverse  $\lambda^2$  is also an automorphism of  $G$ . ■

In any loop  $G$  the set  $N_\lambda = \{a \in G: a \cdot xy = ax \cdot y, \text{ for all } x \text{ and } y \text{ in } G\}$  is a subloop of  $G$  called the *left nucleus* of  $G$ .<sup>\*</sup> Similarly, it can be shown that the sets  $N_\mu = \{a \in G: xa \cdot y = x \cdot ay \text{ for all } x, y \in G\}$  and  $N_\rho = \{a \in G: xy \cdot a = x \cdot ya \text{ for all } x, y \in G\}$  form subloops of  $G$  called the *middle nucleus* and the *right nucleus*, respectively.<sup>\*</sup> The intersection  $N_\lambda \cap N_\mu \cap N_\rho$  is also a subloop called the *nucleus*. Since these sets are subloops in  $G$ , it is immediately evident from their definitions that they are groups.

In general the left, middle, and right nuclei are distinct from one another; Bruck [2] has given examples of loops in which they are distinct. However, Bruck [2] and Artzy [1] have shown that these three nuclei do coincide with each other in inverse property and cross inverse property loops. We shall show that the nuclei coincide in weak inverse property loops, but to do so we need the concept of autotopism. Associated with any loop  $G$  is a group called the autotopism group of  $G$ . This group consists of all ordered triples of permutations on the elements of  $G$  which have the property that for any such triple  $(U, V, W)$ , the identity  $xU \cdot yV = (xy)W$  holds for all  $x$  and  $y$  in  $G$ . It is easily verified that

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<sup>\*</sup> See Bruck [2]. In that paper he uses the term *associator* for the term *nucleus*.

this set forms a group under the operation given by  $(U,V,W)(U',V',W') = (UU',VV',WW')$ , where  $UU'$  denotes the ordinary composition of  $U$  with  $U'$  and likewise for  $VV'$  and  $WW'$ . We now prove the following.

Lemma 1.1. Let  $(U,V,W)$  be an autotopism of a weak inverse property loop  $G$ . Then the triples  $(V,\lambda W\rho,\lambda U\rho)$  and  $(\rho W\lambda,U,\rho V\lambda)$  are autotopisms of  $G$ .

Proof: From (1.5) of Theorem 1.1, we have the identity  $yV \cdot [xU \cdot yV]^\rho = [xU]^\rho$  for all  $x$  and  $y$  in  $G$ . But since  $xU \cdot yV = (xy)W$ , we have  $yV \cdot [(xy)W]^\rho = [xU]^\rho$ . Since this holds for all  $x$  and  $y$  in  $G$ , it holds when  $x = (yz)^\lambda$ . Thus we have  $yV \cdot [(yz)^\lambda y]^\rho = [(yz)^\lambda U]^\rho$ . But by (1.7) of Theorem 1.1, we see that  $(yz)^\lambda y = z^\lambda$ . Hence we have  $yV \cdot [z^\lambda W]^\rho = [(yz)^\lambda U]^\rho$ . It follows that  $(V,\lambda W\rho,\lambda U\rho)$  is an autotopism of  $G$ .

Now by (1.7) of Theorem 1.1 we have the identity  $(xU \cdot yV)^\lambda xU = (yV)^\lambda$  for all  $x, y$  in  $G$ . Again this implies that  $((xy)W)^\lambda xU = (yV)^\lambda$ . Letting  $y = (zx)^\rho$ , we obtain the identity  $((x(zx)^\rho)W)^\lambda \cdot xU = ((zx)^\rho V)^\lambda$ , which implies that  $(z^\rho W)^\lambda \cdot xU = ((zx)^\rho V)^\lambda$ . Hence  $(\rho W\lambda,U,\rho V\lambda)$  is an autotopism of  $G$ . ■

We are now ready to prove that the left, middle, and right nuclei of a weak inverse property loop coincide.

Theorem 1.4. In a weak inverse property loop  $G$ , the left, middle, and right nuclei coincide with each other and with the nucleus.

Proof: Let  $a$  be in the left nucleus  $N_\lambda$  of  $G$ . Then we have the identity  $ax \cdot y = a \cdot xy$  for all  $x$  and  $y$  in  $G$ . But this implies that the triple  $(L(a), I, L(a))$  is an autotopism of  $G$ , where we have used  $I$  to denote the identity permutation. Conversely, if the triple  $(L(a), I, L(a))$  is an autotopism of  $G$ , then  $a$  is in the left nucleus of  $G$ . We can also prove that  $a$  is in the right nucleus  $N_\rho$  of  $G$  if and only if the triple  $(I, R(a), R(a))$  is an autotopism of  $G$ .

Thus suppose  $a$  is in  $N_\rho$ . Then  $(I, R(a), R(a))$  is an autotopism of  $G$ , and by Lemma 1.1  $(\rho R(a)\lambda, I, \rho R(a)\lambda)$  is an autotopism of  $G$ . Since  $\rho^2$  is an automorphism of  $G$ ,  $(\lambda^2, \lambda^2, \lambda^2)$  and  $(\rho^2, \rho^2, \rho^2)$  are autotopisms of  $G$ , and it follows that  $(\lambda^2, \lambda^2, \lambda^2)(\rho R(a)\lambda, I, \rho R(a)\lambda)^{-1}(\rho^2, \rho^2, \rho^2) = (\lambda R^{-1}(a)\rho, I, \lambda R^{-1}(a)\rho)$  is an autotopism of  $G$ . But by Theorem 1.2 we have  $\lambda R^{-1}(a)\rho = L(a)$ , hence it is true that  $(\lambda R^{-1}(a)\rho, I, \lambda R^{-1}(a)\rho) = (L(a), I, L(a))$  which shows that  $a$  is in  $N_\lambda$ . A similar argument shows that if  $a$  is in  $N_\lambda$ , then  $a$  is in  $N_\rho$ . Hence we have  $N_\lambda = N_\rho$ .

Now suppose that  $a$  is in  $N_\rho$ . Then  $(I, R(a), R(a))$  is an autotopism of  $G$ , and we obtain that  $(R(a), L^{-1}(a), I)$  is an autotopism of  $G$  by applying Lemma 1.1 and Theorem 1.2 in succession. Hence we have  $xa \cdot zL^{-1}(a) = xz$  for all  $x$  and  $z$  in  $G$ . Letting  $z = ay$ , we obtain that  $xa \cdot y = x \cdot ay$  for all  $x$  and  $y$  in  $G$ ; that is,  $a$  is in  $N_\mu$ . A similar argument shows that if  $a$  is in  $N_\mu$ , then  $a$  is in  $N_\rho$ . It follows from all of the above that the four nuclei of  $G$  coincide. ■

The above theorem allows us to give another characterization of weak inverse property loops.

Theorem 1.5. A loop  $G$  is a weak inverse property loop if and only if  $xy \cdot z = x \cdot yz$  whenever  $xy \cdot z$  is in any one of the four nuclei of  $G$ .

Proof: Since  $1$  belongs to the nucleus of any loop, the hypotheses immediately imply that  $G$  is a weak inverse property loop by (1.5). Thus, let  $G$  be a weak inverse property loop, in which case the nuclei coincide. Suppose that  $x$ ,  $y$ , and  $z$  are in  $G$  and that  $xy \cdot z$  is in the nucleus. Let  $a = xy \cdot z$ . Then it follows that  $(xy \cdot z)a^{-1} = 1$ , and by two applications of (1.5) we obtain that  $x \cdot y(za^{-1}) = 1$ . Now  $a^{-1}$  is in the nucleus since the nucleus is a group. Hence we have  $1 = x \cdot y(za^{-1}) = x((yz)a^{-1})$ . By (1.8) we see that  $(x \cdot yz)a^{-1} = x \cdot (yz)a^{-1} = 1$ . Hence we have  $(xy \cdot z) = a = (x \cdot yz)$ . ■

The class of weak inverse property loops which Osborn studied and which satisfy (1.2) is the class of loops all of whose loop isotopes have the weak inverse property. Two algebraic systems are said to be *isotopic* if and only if there is a triple  $(U, V, W)$  of one-to-one mappings having domain  $S$  and range  $S'$  such that for any  $x$  and  $y$  in  $S$ ,  $xU \cdot yV = (xy)W$ . The triple  $(U, V, W)$  is called an *isotopism*. If  $G$  is a loop and if  $f$  and  $g$  in  $G$ , then we may define another binary operation  $\circ$  on  $G$  by the equation  $x \circ y = xR^{-1}(g) \cdot yL^{-1}(f)$ . It can be shown that the new system  $G_\circ$  is a loop with identity  $fg$  which is isotopic to the original loop; it is called a *principal isotope* of  $G$ . It can also be shown that if  $G$  is a loop, then any loop isotopic to  $G$  is isomorphic to a principal isotope of  $G$ .\*

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\* See Bruck [2]. For readers interested in the origin of the concept of isotopic loops, the paper "What is a loop?" by Bruck [4] has an introductory account.

We now turn our attention to the question of whether or not a particular loop isotope of a given weak inverse property loop also is a weak inverse property loop.

Theorem 1.6. Let  $f$  and  $g$  be any two elements of a weak inverse property loop  $G$ . Then the principal isotope  $G_\circ$  of  $G$  given by  $x \circ y = xR^{-1}(g) \cdot yL^{-1}(f)$  has the weak inverse property if and only if the triple of mappings

$$(1.12) \quad (L([fg]^\lambda)R^{-1}(f), L(f)R^{-1}(g), R^{-1}(g)L([fg]^\lambda))$$

is an autotopism of  $G$ .

Proof: First, we calculate the mapping  $\rho_\circ$ , the right inverse mapping in  $G_\circ$ . From the relation  $x \circ x^{\rho_\circ} = fg$ , we obtain that  $xR^{-1}(g) \cdot x^{\rho_\circ}L^{-1}(f) = fg$ . Now by (1.7) we have

$$(xR^{-1}(g) \cdot (x^{\rho_\circ})L^{-1}(f))^\lambda \cdot xR^{-1}(g) = [x^{\rho_\circ}L^{-1}(f)]^\lambda.$$

Hence,  $(fg)^\lambda \cdot xR^{-1}(g) = [x^{\rho_\circ}L^{-1}(f)]^\lambda$ , or

$$x^{\rho_\circ} = xR^{-1}(g)L([fg]^\lambda)\rho L(f) \quad \text{for all } x \text{ in } G.$$

It follows that

$$(1.13) \quad \rho_\circ = R^{-1}(g)L([fg]^\lambda)\rho L(f).$$

Notice that we have derived (1.13) without assuming that  $G_o$  is a weak inverse property loop.

Now assume that  $G_o$  is a weak inverse property loop. Then by (1.6), for any  $x, y$  in  $G$ , we have  $y_o(xoy)^{\rho_o} = x^{\rho_o}$ , or  $yR^{-1}(g) \cdot [xR^{-1}(g) \cdot yL^{-1}(f)]^{\rho_o} L^{-1}(f) = x^{\rho_o}$ . By letting  $x = ug$  and  $y = fv$ , we see that this identity implies that  $(fv)R^{-1}(g) \cdot (uv)^{\rho_o} L^{-1}(f) = (ug)^{\rho_o}$  for all  $u, v \in G$ . Since  $G$  is a weak inverse property loop, by (1.7),  $(ug)^{\rho_o \lambda} \cdot (fv)R^{-1}(g) = [(uv)^{\rho_o} L^{-1}(f)]^\lambda$  for all  $u$  and  $v$  in  $G$ . But this implies that the triple of permutations  $(R(g)\rho_o\lambda, L(f)R^{-1}(g), \rho_o L^{-1}(f)\lambda)$  is an autotopism of  $G$ . Using (1.13), we obtain that  $(L([fg]^\lambda)\rho L(f)\lambda, L(f)R^{-1}(g), R^{-1}(g)L([fg]^\lambda))$  is an autotopism of  $G$ . But  $\rho L(f)\lambda = R^{-1}(f)$  by Theorem 1.2. Hence (1.12) is an autotopism of  $G$ . A similar argument shows that if  $G$  is a weak inverse property loop and if  $f$  and  $g$  are in  $G$  such that (1.12) is an autotopism of  $G$ , then the principal isotope  $G_o$  of  $G$  given by  $xoy = xR^{-1}(g) \cdot yL^{-1}(f)$  is a weak inverse property loop. ■

Definition 1.2. A weak inverse property loop  $G$  is said to be an *Osborn loop* if and only if every loop isotope of  $G$  has the weak inverse property.

Bruck [2] has shown that all of the loop isotopes of Moufang loops are Moufang loops. Since any Moufang loop is also a weak inverse property loop, we obtain that all Moufang loops are Osborn loops. We shall show that all Osborn loops are generalized Moufang loops, and we shall also give an example of an Osborn loop which is not a Moufang loop.

Lemma 1.2. Let  $G$  be a weak inverse property loop and suppose that the principal isotope  $G_o$  of  $G$  given by  $xoy = x \cdot yL^{-1}(f)$  is a weak inverse property loop for any fixed  $f$  in  $G$ . Then we have

$$(1.14) \quad L(f)R(f) = R^{-1}(f^0)L(f);$$

the triple of permutations

$$(1.15) \quad (L(f), R^{-1}(f^0), L(f)R(f))$$

is an autotopism; and

$$(1.16) \quad L(fx) = R(f^0)L(x)L(f)R(f) = R(f^0)L(x)R^{-1}(f^0)L(f),$$

for all  $x$  in  $G$ .

Proof: From (1.12), the triple  $(L(f^\lambda)R^{-1}(f), L(f), L(f^\lambda))$  is an autotopism of  $G$  from which we obtain that the triple  $(L(f), \lambda L(f^\lambda)_\rho, \lambda L(f^\lambda)R^{-1}(f)_\rho)$  is an autotopism of  $G$  by Lemma 1.1. Now, for any  $x$  in  $G$  and for any automorphism  $\alpha$  of  $G$ , it is clear that  $L(x)\alpha = \alpha L(x\alpha)$ ,  $R(x)\alpha = \alpha R(x\alpha)$ ,  $L^{-1}(x)\alpha = \alpha L^{-1}(x\alpha)$ , and  $R^{-1}(x)\alpha = \alpha R^{-1}(x\alpha)$ . Hence we have  $\lambda L(f^\lambda)_\rho = \lambda L(f^\lambda)_\rho \lambda^2 = \rho L(f^0)\lambda = R^{-1}(f^0)$  by Theorem 1.2 and Theorem 1.3. Similarly, we have  $\lambda L(f^\lambda)R^{-1}(f)_\rho = \lambda L(f^\lambda)_\rho \lambda R^{-1}(f)_\rho = R^{-1}(f^0)L(f)$ . Thus the triple  $(L(f), \lambda L(f^\lambda)_\rho, \lambda L(f^\lambda)R^{-1}(f)_\rho) = (L(f), R^{-1}(f^0), R^{-1}(f^0)L(f))$  is an autotopism of  $G$ . It follows that  $fx \cdot 1R^{-1}(f^0) = xR^{-1}(f^0)L(f)$  for all  $x$  in  $G$ . But, since  $1R^{-1}(f^0) \cdot f^0 = 1$ , we have  $1R^{-1}(f^0) = f$ . Hence we

see that  $fx \cdot f = xR^{-1}(f^0)L(f)$  for all  $x$  in  $G$ , and it follows that  $L(f)R(f) = R^{-1}(f^0)L(f)$ . Thus we have established (1.14), and using (1.14) in the autotopism  $(L(f), R^{-1}(f^0), R^{-1}(f^0)L(f))$ , we have proved that (1.15) is an autotopism of  $G$ . Using the autotopism (1.15), we have the identity  $fx \cdot yR^{-1}(f^0) = (xy)L(f)R(f)$  for all  $x$  and  $y$  in  $G$ . Hence  $yR^{-1}(f^0)L(fx) = yL(x)L(f)R(f)$  for all  $y$  in  $G$ . It follows that  $R^{-1}(f^0)L(fx) = L(x)L(f)R(f)$ , or  $L(fx) = R(f^0)L(x)L(f)R(f)$  which, together with (1.14), establishes (1.16).  $\square$

Theorem 1.7. Let  $G$  be an Osborn loop. Then, corresponding to each element  $g$  in  $G$ , the permutation  $L(g)L(g^\lambda)$ , which we denote by  $\theta_g$ , is an automorphism of  $G$ . Furthermore,

$$(1.17) \quad \theta_g = L(g)L(g^\lambda) = R^{-1}(g^0)R^{-1}(g) = L(g)R(g)L^{-1}(g)R^{-1}(g),$$

and, for any two elements  $x, z$  in  $G$ .

$$(1.18) \quad gx \cdot (z\theta_g \cdot g) = (g \cdot xz)g.$$

Proof: We set  $f = 1$  in (1.12) and take the inverse of the result to obtain that  $(L^{-1}(g^\lambda), R(g), L^{-1}(g^\lambda)R(g))$  is an autotopism of  $G$ . Hence we see that  $1L^{-1}(g^\lambda) \cdot xR(g) = xL^{-1}(g^\lambda)R(g)$  for any  $x$  in  $G$ . It follows that  $g \cdot xR(g) = xL^{-1}(g^\lambda)R(g)$ . Hence we have  $R(g)L(g) = L^{-1}(g^\lambda)R(g)$ . Thus the autotopism  $(L^{-1}(g^\lambda), R(g), L^{-1}(g^\lambda)R(g))$  can be written in the form  $(L^{-1}(g^\lambda), R(g), R(g)L(g))$ . By (1.15),  $(L(g), R^{-1}(g^0), L(g)R(g))$  is an

autotopism of  $G$ , where we have substituted  $f$  for  $g$ . Hence, the triple  $(L(g), R^{-1}(g^{\rho}), L(g)R(g)(L^{-1}(g^{\lambda}), R(g), R(g)L(g))^{-1} = (L(g)L(g^{\lambda}), R^{-1}(g^{\rho})R^{-1}(g), L(g)R(g)L^{-1}(g)R^{-1}(g))$  is an autotopism of  $G$ . But  $1L(g)L(g^{\lambda}) = g^{\lambda} \cdot g = 1$ ;  $1R^{-1}(g^{\rho})R^{-1}(g) = gR^{-1}(g) = 1$ , and  $1L(g)R(g)L^{-1}(g)R^{-1}(g) = (gg)L^{-1}(g)R^{-1}(g) = 1$ . Now if  $(U,V,W)$  is any autotopism of  $G$  such that  $1U = 1V = 1W = 1$ , then  $xU \cdot 1 = xW$  for any  $x$  in  $G$ , which implies that  $U = W$ . Similarly, we can prove that  $V = W = U$ . Applying this to the specific case at hand, we have  $L(g)L(g^{\lambda}) = R^{-1}(g^{\rho})R^{-1}(g) = L(g)R(g)L^{-1}(g)R^{-1}(g)$ , and we have established (1.17). Letting  $\theta_g$  denote this mapping, we have proved that  $(\theta_g, \theta_g, \theta_g)$  is an autotopism of  $G$ . Hence  $x\theta_g \cdot y\theta_g = (xy)\theta_g$  for any  $x, y$  in  $G$ . It follows that  $\theta_g$  is an automorphism of  $G$ . From (1.17) we obtain that  $R^{-1}(g^{\rho}) = \theta_g R(g)$ . Hence we may write the autotopism (1.15) as  $(L(g), \theta_g R(g), L(g)R(g))$ , where we have replaced  $f$  by  $g$ . But this says that, for any  $x$  and  $z$  in  $G$ .

$$gx \cdot (z\theta_g \cdot g) = (g \cdot xz)_g$$

and we have established (1.18). ■

It follows immediately from Theorem 1.7 that Osborn loops are generalized Moufang loops.

Corollary 1.1. If  $G$  is an Osborn loop which is also an inverse property, cross inverse property, or commutative loop, then  $G$  is a Moufang loop.

Proof: If  $G$  has the inverse property, then  $x\theta_g = g^{-1}(gx) = x$  for all  $x$ ,  $g$  in  $G$ . Hence  $\theta_g$  is the identity mapping for all  $g$  in  $G$ ; this implies that the loop  $G$  is Moufang by (1.18). If  $G$  is commutative, then we have  $L(g) = R(g)$ . Hence we see that  $\theta_g = L(g)R(g)L^{-1}(g)R^{-1}(g) = R(g)R(g)R^{-1}(g)R^{-1}(g) = 1$ , and again  $G$  is Moufang. If  $G$  has the cross inverse property, then we have  $xL(g^\lambda)R(g) = (g^\lambda x)g = x$ . Thus we obtain that  $L(g^\lambda) = R^{-1}(g)$ . Hence we have  $L(g)L(g^\lambda) = L(g)R^{-1}(g) = L(g)R(g)L^{-1}(g)R^{-1}(g)$  by (1.17). It follows that  $R(g) = L(g)$ ; that is,  $G$  is commutative, and hence  $G$  is Moufang. ■

We now concern ourselves with a structure result for Osborn loops. Associated with any loop  $G$ , there is a permutation group which is generated by all permutations of  $G$  of the form  $R(x)$  or  $L(x)$ , where  $x$  is in  $G$ . An inner mapping  $U$  of  $G$  is an element of the permutation group for which  $1U = 1$ . Inner mappings are the loop theory analog of inner automorphisms in group theory. A subgroup  $N$  of a group  $G$  is said to be normal in  $G$  if and only if, for every inner automorphism  $\alpha$  of  $G$ , it is true that  $N\alpha \subseteq N$ . A subloop  $N$  of a loop  $G$  is said to be normal in  $G$  if and only if it is true that  $NU \subseteq N$  for every inner mapping  $U$  of  $G$ . Bruck [2] has shown that the elementary theory of normal subgroups, factor groups, and homomorphisms carries over entirely to the loop theory case with the exception of Lagrange's theorem. We now show that the nucleus  $N$  of an Osborn loop is normal.

Lemma 1.3. Let  $G$  be a loop in which the four nuclei coincide and in which, corresponding to each element  $x$  in  $G$ , there are autotopisms  $(U, V, W)$

and  $(U',V',W')$  such that  $U = L(x)$ ,  $U' = R(x)$ . Then the nucleus  $N$  of  $G$  is normal in  $G$ .

Proof: We show first that to every inner mapping  $U$  of  $G$ , there corresponds a  $u$  in  $G$  such that  $(xy)U \cdot u = xU \cdot (yU \cdot u)$  for all  $x, y$  in  $G$ .

Now  $U$  has the form  $U = U_1 U_2 \cdots U_n$ , where  $U_i = L(x)$ ,  $L^{-1}(x)$ ,  $R(x)$ , or  $R^{-1}(x)$  since  $U$  is an inner mapping. By the hypotheses and by the fact that the autotopisms of a loop  $G$  form a group, we know that, for each  $i = 1, 2, \dots, n$ , there exists mappings  $V_i$  and  $W_i$  such that  $(U_i, V_i, W_i)$  is an autotopism of  $G$ . Let  $V = V_1 V_2 \cdots V_n$  and  $W = W_1 W_2 \cdots W_n$ . Then  $(U, V, W)$  is an autotopism of  $G$ . Applying this autotopism to the pair  $(1, y)$ , we have  $1U \cdot yV = yW$ , or  $yV = yW$  for all  $y$  in  $G$ . Now set  $u = 1V$ . Then  $xU \cdot u = xW$  for all  $x$  in  $G$ . It follows that  $xU \cdot (yU \cdot u) = xU \cdot (yW) = xU \cdot yV = (xy)W = (xy)U \cdot u$ .

Now suppose that  $n$  belongs to  $N$  and that  $U$  is an inner mapping of  $G$ . Then  $V = U^{-1}$  is an inner mapping of  $G$ . Let  $a = nU$  and let  $v$  be the element in  $G$  associated with  $V$  such that  $xV \cdot (yV \cdot v) = (xy)V \cdot v$  for all  $x, y$  in  $G$ . Then we see that  $(ax)V \cdot v = aV \cdot (xV \cdot v) = n(xV \cdot v) = (n \cdot xV)v$ , or  $(ax)V = n \cdot xV$  for every  $x$  in  $G$ . Hence we have  $(ax \cdot y)V \cdot v = (ax)V \cdot (yV \cdot v) = (n \cdot xV)(yV \cdot v) = n(xV \cdot (yV \cdot v)) = n((xy)V \cdot v) = (n \cdot (xy)V)v = (a \cdot xy)V \cdot v$ , or  $ax \cdot y = a \cdot xy$  for all  $x$  and  $y$  in  $G$ . Hence  $a$  is in  $N$ . It follows that  $N$  is normal in  $G$ . ■

Bruck [3] gives essentially the same proof as above; however, his theorems are stated for Moufang loops.

Theorem 1.8. Let  $G$  be an Osborn loop. Let  $N$  be the nucleus of  $G$ . Then  $N$  is a normal subloop of  $G$ , and the factor loop  $G/N$  is a Moufang loop.

Proof: To show that  $N$  is normal, it is sufficient to show that for each  $f$  in  $G$ , the permutations  $L(f)$  and  $R(f)$  occur as the first permutations of some autotopism of  $G$ . By (1.15),  $L(f)$  occurs as the first permutation of some autotopism of  $G$ . In (1.12) we let  $g = f^0$  obtaining that  $(R^{-1}(f), L(f)R^{-1}(f^0), R^{-1}(f^0))$  is an autotopism of  $G$ , and hence  $(R(f), R(f^0)L^{-1}(f), R(f^0))$  is an autotopism of  $G$ . It now follows that  $N$  is normal in  $G$  by the preceding lemma.

Now let  $a$  and  $b$  denote the autotopisms of  $G$  obtained from (1.12) by substituting  $g = 1$  and  $f = 1$ , respectively. Let  $c$  denote the autotopism (1.12). Then  $abc^{-1} = (L(f^\lambda)R^{-1}(f), L(f), L(f^\lambda))(L(g^\lambda), R^{-1}(g), R^{-1}(g)L(g^\lambda))(R(f)L^{-1}([fg]^\lambda), R(g)L^{-1}(f), L^{-1}([fg]^\lambda)R(g)) = (L(f^\lambda)R^{-1}(f)L(g^\lambda)R(f)L^{-1}([fg]^\lambda), I, L(f^\lambda)R^{-1}(g)L(g^\lambda)L^{-1}([fg]^\lambda)R(g))$  is an autotopism of  $G$ . Applying  $abc^{-1}$  to the pair  $(x, g)$ , we have  $xL(f^\lambda)R^{-1}(f)L(g^\lambda)R(f)L^{-1}([fg]^\lambda)R(g) = xR(g)L(f^\lambda)R^{-1}(g)L(g^\lambda)L^{-1}([fg]^\lambda)R(g)$ , or  $xL(f^\lambda)R^{-1}(f)L(g^\lambda)R(f) = xR(g)L(f^\lambda)R^{-1}(g)L(g^\lambda)$ ; hence we have  $L(f^\lambda)R^{-1}(f)L(g^\lambda)R(f) = R(g)L(f^\lambda)R^{-1}(g)L(g^\lambda)$ . But from (1.16), with  $f, x$  replaced by  $g^\lambda, f^\lambda$ , respectively, we have  $L(g^\lambda f^\lambda) = R(g)L(f^\lambda)R^{-1}(g)L(g^\lambda)$ . Hence the autotopism  $abc^{-1}$  takes the form

$$(L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda), I, L(f^\lambda)R^{-1}(g)L(g^\lambda)L^{-1}([fg]^\lambda)R(g)).$$

For the moment, let this autotopism be denoted by  $(U, I, W)$ . Then we see that  $xU \cdot y = (xy)W$  for all  $x, y$  in  $G$ . Hence we have  $xU \cdot 1 = xW$  for all

$x$  in  $G$ , which implies that  $U = W$ . Thus we see that  $xU \cdot y = (xy)U$  for all  $x, y$  in  $G$ , and it follows that  $1U \cdot y = yU$ . Letting  $u = 1U$ , we have  $U = L(u)$ . Hence  $(L(u), I, L(u))$  is an autotopism of  $G$ ; that is, for all  $x, y \in G$ , we have  $ux \cdot y = u \cdot xy$ . Thus we see that  $u$  is in  $N$ . Now let  $\phi$  be the natural homomorphism from  $G$  onto  $G/N$ . Recalling that  $U = L(g^\lambda f^\lambda) L^{-1}([fg]^\lambda)$  and that  $u = 1U$ , we have  $[fg]^\lambda u = g^\lambda f^\lambda$ . Hence  $[f\phi \cdot g\phi]^\lambda \cdot u\phi = (g\phi)^\lambda (f\phi)^\lambda$ . But  $u\phi$  is the identity of  $G/N$  since  $u$  belongs to  $N$ . Thus we have  $(xy)^\lambda = y^\lambda x^\lambda$  for all  $x, y$  in  $G/N$ . It is clear that  $G/N$  is an Osborn loop; hence we have  $[xy]^\lambda x = y^\lambda$  since an Osborn loop is a weak inverse property loop. It follows that  $(y^\lambda x^\lambda) x = y^\lambda$ . Similarly, it can be shown that  $x^\lambda = x^\rho$  and  $x^\lambda (xy^\lambda) = y^\lambda$ . But this implies that  $G/N$  is an inverse property loop; hence  $G/N$  is Moufang by Corollary 1.1. ■

In Chapter III we shall strengthen the above theorem. We now turn our attention to the automorphism  $\theta_x$ , which was introduced in Theorem 1.7, and we construct an example of an Osborn loop which is not a Moufang loop.

Theorem 1.9. Let  $G$  be an Osborn loop. Let  $x$  be in  $G$  and let  $b$  be in the nucleus of  $G$ . Let  $a$  be such that  $x\theta_x = xa$ . Then we have the following identities:

$$(1.19) \quad \theta_b = I$$

$$(1.20) \quad (bx)^\lambda = x^\lambda b^{-1}, \text{ and } (bx)^\rho = x^\rho b^{-1}$$

$$(1.21) \quad \theta_{bx} = \theta_x, \text{ and } \theta_{xb} = \theta_x$$

$$(1.22) \quad \theta_x = \theta_x \lambda^i = \theta_x \rho^i \text{ for all } i \geq 0$$

$$(1.23) \quad x^{\lambda^i} \theta_x = x^{\lambda^{i+2}}, \text{ and } x^{\rho^i} \theta_x = x^{\rho^{i-2}} \text{ for all } i \geq 0$$

$$(1.24) \quad (xx)^\lambda = (xx)^\rho = x^\lambda \cdot x^\rho$$

$$(1.25) \quad x^{\lambda^i} a = x^{\lambda^{i+2}}; \quad ax^{\lambda^i} = x^{\lambda^{i-2}}; \quad x^{\rho^i} a = x^{\rho^{i-2}}; \quad ax^{\rho^i} = x^{\rho^{i+2}}$$

for all  $i \geq 0$ .

Furthermore,  $a$  is in the nucleus of  $G$ .

Proof: Suppose  $b$  is in the nucleus  $N$  and suppose that  $x$  is in  $G$ . Then we have  $x\theta_b = b^{-1}(bx) = x$ . Hence (1.19) holds. Furthermore, we have  $x^\lambda b^{-1} \cdot bx = x^\lambda \cdot b^{-1} bx = x^\lambda x = 1$ . It follows that  $x^\lambda b^{-1} = (bx)^\lambda$  for all  $x$  in  $G$ . Similarly, we may prove that  $x^\rho b^{-1} = (bx)^\rho$  for all  $x$  in  $G$ . Thus we have established (1.20). Now if  $y$  is in  $G$ , then  $y\theta_{bx} = (bx)^\lambda (bx \cdot y) = x^\lambda \cdot b^{-1} \cdot (b \cdot xy) = x^\lambda \cdot b^{-1} (b \cdot xy) = x^\lambda \cdot (xy)\theta_b = x^\lambda \cdot xy = y\theta_x$ . Since the nucleus is normal in  $G$ , there is some  $c$  in the nucleus such that  $xb = cx$ . Thus we have  $\theta_{xb} = \theta_{cx} = \theta_x$ . Hence (1.21) holds. Furthermore, we see that  $y\theta_x \cdot x = xx^\rho \cdot (y\theta_x \cdot x) = (x \cdot x^\rho y)x$  for all  $x$  and  $y$  in  $G$  by the generalized Moufang identity. Hence we have  $y\theta_x = yL(x^\rho)L(x) = y\theta_{x^\rho}$  for all  $x$  and  $y$  in  $G$ . It follows that  $\theta_x = \theta_{x^\rho}$  for all  $x$  in  $G$ . Replacing  $x$  by  $x^\lambda$ , we have  $\theta_{x^\lambda} = \theta_x$  for all  $x$  in  $G$ . It follows by induction that  $\theta_x = \theta_{x^{\lambda^i}} = \theta_{x^{\rho^i}}$  for all  $i \geq 0$ . Thus (1.22) holds. In particular we see that

$x^{\lambda^i} \theta_x = x^{\lambda^i} \theta_{x^{\lambda^{i+1}}} = x^{\lambda^{i+1}} (x^{\lambda^{i+1}} \cdot x^{\lambda^{i+2}}) = x^{\lambda^{i+2}}$  for all  $i \geq 0$ . A similar argument shows that  $x^{\rho^i} \theta_x = x^{\rho^{i-2}}$ . Hence (1.23) holds. Now it is also true that  $x^\lambda \cdot xx = x \theta_x = x^{\lambda^2}$ . Hence we have  $(x^\lambda \cdot xx)^\lambda = x^{\lambda^3}$ , and  $(xx)^\lambda = (x^\lambda \cdot xx)^\lambda x^\lambda = x^{\lambda^3} \cdot x^\lambda$ , by (1.7), or  $x^\lambda \cdot x^\rho = (x^{\lambda^3} \cdot x^\lambda)^{\rho^2} = (xx)^{\lambda \rho^2} = (xx)^\rho$  since  $\rho^2$  is an automorphism of  $G$ . A similar argument shows that  $x^{\rho^2} = x \theta_x^{-1} = xx \cdot x^\rho$  and hence that  $(xx \cdot x^\rho)^\rho = x^{\rho^3}$ . Thus we have  $x^\rho \cdot x^{\rho^3} = (xx)^\rho$  by (1.6). Therefore we see that  $x^\lambda \cdot x^\rho = (x^\rho \cdot x^{\rho^3})^{\lambda^2} = (xx)^{\rho \lambda^2} = (xx)^\lambda$ . Thus we have  $(xx)^\lambda = x^\lambda x^\rho = (xx)^\rho$ , and we have established (1.24). Now from  $xa = x \theta_x = x \theta_{x^\rho} = x(x^\rho x)$  we obtain that  $a = x^\rho x$ . Let  $\phi$  be the natural homomorphism from  $G$  onto  $G/N$ . Then we have  $a\phi = x^\rho \phi \cdot x\phi = (x\phi)^\rho \cdot (x\phi)$ . But since  $G/N$  is a Moufang loop, we have  $(x\phi)^\rho \cdot (x\phi) = (x\phi)^\lambda \cdot (x\phi) = 1\phi$ . Hence  $a$  is in  $N$ . Now  $1 = x^{\lambda^2} \cdot x^\lambda = xa \cdot x^\lambda = x \cdot ax^\lambda$  implies  $ax^\lambda = x^\rho$ . It follows by (1.24) that  $x^\lambda x^\lambda = (x^\lambda x^\lambda)^{\rho^2} = x^\rho x^\rho = x^\rho \cdot ax^\lambda = x^\rho a \cdot x^\lambda$ , and hence  $x^\rho a = x^\lambda$ . Thus we have  $x^\rho a = x^\lambda = (x^{\lambda^2})^\rho = (x \theta_x)^\rho = x^\rho \theta_x = x^\rho \theta_{x^\rho}$ . Now if  $i$  is even, we have  $x^{\lambda^i} a = x^{\lambda^i} a^{\lambda^i} = (xa)^{\lambda^i} = x^{\lambda^{i+2}}$ . Similarly, if  $i$  is odd, we have  $x^{\lambda^i} a = (x^\rho)^{\lambda^{i+1}} a = (x^\rho a)^{\lambda^{i+1}} = (x^\lambda)^{\lambda^{i+1}} = x^{\lambda^{i+2}}$ . Thus we have  $x^{\lambda^i} a = x^{\lambda^{i+2}}$  for all  $i \geq 0$ . A similar proof shows that  $x^{\rho^i} a = x^{\rho^{i-2}}$  for all  $i \geq 0$ .

Now we have  $1 = x^\lambda x = x^\rho a \cdot x = x^\rho \cdot ax$ , since  $a$  is in  $N$ . Thus we see that  $ax = x^{\rho^2}$ . If  $i$  is even, then we have  $ax^{\rho^i} = (ax)^{\rho^i} = x^{\rho^{i+2}}$ . If  $i$  is odd, then we have  $ax^{\rho^i} = a(x^\lambda)^{\rho^{i+1}} = (ax^\lambda)^{\rho^{i+1}} = (x^\rho)^{\rho^{i+1}} = x^{\rho^{i+2}}$ . Hence we obtain that  $ax^{\rho^i} = x^{\rho^{i+2}}$  for all  $i \geq 0$ . A similar argument shows that  $ax^{\lambda^i} = x^{\lambda^{i-2}}$ , and we have established (1.25). ■

Our next goal is to exhibit an Osborn loop. The loop we give is the one which Osborn [7] constructs. We shall establish properties for this loop which he establishes, but we shall use a different method. We shall use a result of Eric Wilson [8], a result which establishes a sufficient condition for a weak inverse property loop to be isomorphic to all of its loop isotopes.

Lemma 1.5. Let  $G$  be any loop and let  $f, g$  be in  $G$ . Then  $G$  is isomorphic to the principal isotope  $G_o$  given by  $xoy = xR^{-1}(g) \cdot yL^{-1}(f)$  if and only if there exists an autotopism  $(U, V, W)$  of  $G$  such that  $lU = f$  and  $lV = g$ .

Proof: Suppose that there exists an autotopism  $(U, V, W)$  of  $G$  such that  $lU = f$  and  $lV = g$ . Then if  $x$  belongs to  $G$ , we have  $lU \cdot xV = xW$ ; that is,  $f \cdot xV = xW$ . Hence we have  $xW = xVL(f)$  for all  $x$  in  $G$ . It follows that  $W = VL(f)$ . Similarly, it can be shown that  $W = UR(g)$ . Hence the autotopism  $(U, V, W)$  takes the form  $(WR^{-1}(g), WL^{-1}(f), W)$ . It follows that  $xWoyW = xWR^{-1}(g) \cdot yWL^{-1}(f) = xU \cdot yV = (xy)W$ . Hence  $W$  is an isomorphism from  $G$  onto  $G_o$ .

Now assume that  $G$  is isomorphic to  $G_o$ . Then there is a one-to-one mapping  $W$  from  $G$  onto itself such that  $xWoyW = (xy)W$ . Hence we have  $xWR^{-1}(g) \cdot yWL^{-1}(f) = (xy)W$ . It follows that  $(WR^{-1}(g), WL^{-1}(f), W)$  is an autotopism of  $G$ . Furthermore, we see that  $lWR^{-1}(g) = (fg)R^{-1}(g) = f$ , and  $lWL^{-1}(f) = (fg)L^{-1}(f) = g$ . ■

Theorem 1.10 (Wilson's Condition). Let  $G$  be a loop in which the identity

$$(1.26) \quad c(cx)^{\rho} = (cy)[c \cdot xy]^{\rho}$$

holds for all  $c, x$ , and  $y$  in  $G$ . Then  $G$  is a weak inverse property loop which is isomorphic to all of its loop isotopes.

Proof: With  $c = 1$  the identity (1.26) reads  $x^{\rho} = y(xy)^{\rho}$  for all  $x, y$  in  $G$ ; that is,  $G$  is a weak inverse property loop. Now, in (1.26) we replace  $x$  by  $(yz)^{\lambda}$  and obtain that  $c[c(yz)^{\lambda}]^{\rho} = (cy)[c \cdot (yz)^{\lambda}y]^{\rho}$ . Applying (1.7), we obtain  $c[c(yz)^{\lambda}]^{\rho} = (cy)[cz^{\lambda}]^{\rho}$  for all  $c, y, z$  in  $G$ . Hence the triple  $(L(c), \lambda L(c)\rho, \lambda L(c)\rho L(c))$  is an autotopism of  $G$  for all  $c$  in  $G$ . By Lemma 1.1 the triple  $(L(c)\rho L(c)^{\lambda}, L(c), L(c))$  is also an autotopism of  $G$  for all  $c$  in  $G$ . By Theorem 1.2 we have  $\rho L(c)\lambda = R^{-1}(c)$ , and  $(L(c)R^{-1}(c), L(c), L(c))$  is an autotopism of  $G$ . Thus we have shown that  $A(c) = (L(c), \lambda L(c)\rho, \lambda L(c)\rho L(c))$  and  $B(c) = (L(c)R^{-1}(c), L(c), L(c))$  are autotopisms of  $G$  for each  $c$  in  $G$ . Now consider the autotopism  $B(g^{\lambda}L(f)^{-1}\rho)A(f)$  for each  $f, g$  in  $G$ . Denoting the first and second permutations of this autotopism by  $U$  and  $V$ , respectively, we have

$$U = L(g^{\lambda}L^{-1}(f)\rho)R^{-1}(g^{\lambda}L^{-1}(f)\rho)L(f), \quad \text{and}$$

$$V = L(g^{\lambda}L^{-1}(f)\rho)\lambda L(f)\rho,$$

and it follows that  $1U = (g^{\lambda}L^{-1}(f)\rho)R^{-1}(g^{\lambda}L^{-1}(f)\rho)L(f) = f$  and

$1V = g^{\lambda} L^{-1}(f) \rho \lambda L(f) \rho = g$ . It follows that  $G$  is isomorphic to all of its isotopes by Lemma 1.5, since  $f$  and  $g$  were chosen arbitrarily in  $G$ .

We now give Osborn's [7] loop  $H$ .

Example 1.1. Let  $H$  be the set of all ordered pairs of integers. We introduce a binary operation on  $H$  defined by the following equations:

$$(1.27) \quad [2i, k][2j, m] = [2i+2j, k+m]$$

$$[2i+1, k][2j, m] = [2i+2j+1, k+m+j]$$

$$[2i, k][2j+1, m] = [2i+2j+1, m-k]$$

$$[2i+1, k][2j+1, m] = [2i+2j+2, m-k-j].$$

By routine checking, one can verify that  $H$  is a loop. We first identify the nucleus of this loop.

Lemma 1.6. Given any  $u, v$ , and  $w$  in  $H$ , we have  $u \cdot vw = uv \cdot w$  if and only if at least one of these elements is "even"; that is, the first integer of the ordered pair is an even integer. Otherwise, we have  $(u \cdot vw) = (uv \cdot w)a$ , where  $a = [0, 1]$ .

Proof: Let  $u = [2i+1, k]$ ,  $v = [2j+1, m]$ , and  $w = [2r+1, p]$ . Then we see that  $u \cdot vw = [2i+1, k][2j+2r+2, p-m-r] = [2i+2j+2r+3, p-m+k+j+1]$ , and

$uv \cdot w = [2(i+j+r+1)+1, p-m+k+j]$ . Hence, if all of  $u$ ,  $v$  and  $w$  are "odd," then we have  $(uv \cdot w)a = u \cdot vw$ , and no odd elements are in the nucleus. By direct checking one may verify that, if any of  $u, v, w$  are even, then it is true that  $uv \cdot w = u \cdot vw$ . ■

Corollary 1.2. The nucleus of  $H$  consists of precisely the even elements. In particular,  $a = [0,1]$  is in the nucleus.

Corollary 1.3. The loop  $H$  is a weak inverse property loop.

Proof: Suppose that  $xy \cdot z = [0,0]$ . Then at least one of the elements  $x$ ,  $y$ , or  $z$  must be even. Hence we see that  $x \cdot yz = [0,0]$ .

We now show, using Wilson's condition, that  $H$  is isomorphic to all of its loop isotopes; this will imply that all of its loop isotopes are weak inverse property loops; that is,  $H$  is an Osborn loop.

Theorem 1.11. The loop  $H$  is isomorphic to all of its loop isotopes.

Proof: We show that  $H$  satisfies Wilson's condition. Thus let  $c, x$ , and  $y$  be in  $H$ .

Case I. Let  $c$  be even. Then  $c$  is in the nucleus, and it follows that  $(cy)[c \cdot xy]^p = (cy)[(xy)^p c^{-1}] = c[y(xy)^p c^{-1}] = cx^p c^{-1} = c(cx)^p$  by Theorem 1.9. Hence (1.26) is satisfied in this case.

Case II. Let  $y$  be even. Then we have  $cy \cdot [c \cdot xy]^\rho = cy \cdot [cx \cdot y]^\rho = cy \cdot [y^{-1}(cx)^\rho] = c(cx)^\rho$ , and (1.26) is verified in this case.

Case III. Let  $c$  and  $y$  be odd and let  $x$  be even. Then  $cy \cdot [c \cdot xy]^\rho = (cy)[cx \cdot y]^\rho$ . Now the product  $cx \cdot y$  is even. Hence  $(cy)[cx \cdot y]^\rho = c \cdot y[cy \cdot x]^\rho = c(cx)^\rho$  by (1.6). It follows that (1.26) is true in this case.

Case IV. Let  $c$ ,  $y$ , and  $x$  be odd. Then  $[c \cdot xy]^\rho$  is odd, and we have  $(cy) \cdot [c \cdot xy]^\rho = c \cdot y[(cx \cdot y)a]^\rho a^{-1} = c \cdot y[a^{-1}(cx \cdot y)^\rho]a^{-1} = c \cdot y[(cx \cdot y)^\rho]$  by Lemma 1.6 and the fact that  $a^{-1}z = za$  for any odd element  $z$  in  $H$ . Now we have  $c \cdot y[(cx \cdot y)^\rho] = c(cx)^\rho$  by (1.6). Hence we have  $(cy)[c \cdot xy]^\rho = c(cx)^\rho$ .

We have shown that Wilson's condition holds in all cases. It follows that  $H$  is isomorphic to all of its loop isotopes and that  $H$  is an Osborn loop. ■

By direct checking, one can verify that, if  $x$  is odd, then  $x^\lambda \neq x^\rho$ . Hence  $H$  cannot be a Moufang loop. Now since  $H$  is an Osborn loop, this implies that  $H$  cannot be an inverse property or a cross inverse property loop by Corollary 1.1; hence  $H$  is also an example of a weak inverse property loop which is not an inverse property or cross inverse property loop.

In group theory one says that the group  $G$  is generated by the set  $S$  if the intersection of every subgroup of  $G$  which contains the set  $S$  is  $G$  itself. If the set  $S$  is finite, one says that the group is finitely generated. If  $n$  is the greatest lower bound on the cardinality of

all generating sets for  $G$ , one says that  $G$  has  $n$  generators. These ideas carry over to loop theory. We shall show that the loop  $H$ , constructed above, has one generator and that, if  $H'$  is an Osborn loop with one generator, then  $H'$  is a homomorphic image of  $H$ ; that is,  $H$  is the free Osborn loop on one generator. It is possible to give a direct proof of this by using induction. However, such a proof is quite tedious. We shall instead appeal to the theory of universal algebras to show the existence of such a loop and prove that the loop  $H$  constructed above is isomorphic to the free Osborn loop on one generator.\*

Theorem 1.12. The loop  $H$ , which was constructed above, is the free Osborn loop on one generator.

Proof: Let  $K$  be the free Osborn loop on one generator and suppose that  $K$  is generated by the element  $x$ . Now it is easy to see that  $[1,0]^0[0,1] = [-1,-1][1,0] = [0,1]$  and that  $[1,0]$  generates  $H$ . It follows that there is a homomorphism  $\phi$  from  $K$  onto  $H$  such that  $x\phi = [1,0]$ . Now  $x^\lambda \cdot xx = x\alpha$  for some  $\alpha$  in  $K$ . Hence we have  $(x\phi) \cdot (\alpha\phi) = (x\alpha)\phi = (x^\lambda \cdot xx)\phi = [1,0][0,1] = x\phi \cdot [0,1]$ , which implies that  $\alpha\phi = [0,1]$ . Now  $\alpha$  is in the nucleus  $N$  of  $K$  by Theorem 1.9. Let  $A$  be the subloop of  $K$  generated by  $\alpha$ . Then  $A$  is a group since  $\alpha$  is in  $N$ . We shall show that  $A$  is normal in  $K$ . By Theorem 1.9 we have  $ax \cdot a = a \cdot xa = ax^{\lambda^2} = x$ . Hence we see that  $ax = xa^{-1}$  and  $xa = a^{-1}x$ . It follows immediately that  $xA = Ax$ . Now let  $S = \{y \in K: yA = Ay\}$ . If  $r$  and  $s$  are in  $S$ , then we have

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\* See Cohn [5] for an account of the theory of Universal Algebras.

$(rs)A = r \cdot sA = r \cdot As = rA \cdot s = Ar \cdot s = A \cdot rs$ . Hence  $rs$  is in  $S$ . A similar argument shows that, if  $p$  and  $q$  are such that  $rp = s$  and  $qr = s$ , then we have  $pA = Ap$  and  $qA = Aq$ . It follows that  $S$  is a loop. But  $x$  belongs to  $S$ . Hence we see that  $S = K$  since  $K$  is generated by  $x$ . Thus we have  $KA = AK$ . Bruck [2] has shown that the inner mapping group of a loop  $G$  is generated by the mappings  $R(x)R(y)R^{-1}(xy)$  and  $R(y)L(x)R^{-1}(xy)$  and their inverses for all  $x$  and  $y$  in  $G$ . Now, if  $b$  is in  $A$ , then we have  $(bx \cdot y)R^{-1}(xy) = (b \cdot xy)R^{-1}(xy) = b$  and  $(x \cdot by)R^{-1}(xy) = (xb \cdot y)R^{-1}(xy) = (cx \cdot y)R^{-1}(xy) = (c \cdot xy)R^{-1}(xy) = c$  for some  $c$  in  $A$  since  $S = K$ . Similarly, one can show the same results for  $R(xy)R^{-1}(y)R^{-1}(x)$  and  $R(xy)L^{-1}(x)R^{-1}(y)$ . It follows that if  $I$  is the inner mapping group of  $K$ , then  $AI \subseteq A$ . Hence  $A$  is normal in  $K$ .

We have shown that  $\alpha\phi = [0,1]$ . Hence  $\phi(A)$  is the group generated by  $[0,1]$  in  $H$ . It is clear that  $\phi(A)h = h\phi(A)$  for all  $h$  in  $H$ ; hence  $\phi(A)$  is normal in  $H$  since  $\phi(A)$  is contained in the nucleus of  $H$ . Now  $\phi(a)$  is the infinite cyclic group; hence the restriction of  $\phi$  to  $A$ , denoted by  $\phi|_A$ , is an isomorphism from  $A$  onto  $\phi(A)$ .

Now consider  $K/A$ . We shall show that  $K/A$  is a group. Let  $S = \{pA: (pA)^{\lambda^2} = pA \text{ for } p \text{ in } K\}$ . Then, if  $p, q$  are in  $K$ , we have  $(pAqA)^{\lambda^2} = (pA)^{\lambda^2}(qA)^{\lambda^2} = pAqA$ . Hence,  $pAqA$  is in  $S$ . Furthermore,  $(pA)L^{-1}(qA)$  and  $(pA)R^{-1}(qA)$  are in  $S$  by similar arguments. It follows that  $S$  is a loop. Now  $xA$  is in  $S$  since  $(xA)^{\rho}(xA) = (x^{\rho}x)A = aA = A$ , or  $(xA)^{\rho} = (xA)^{\lambda}$ . Hence we have  $S = K/A$  since  $xA$  generates  $K/A$ . It follows that  $\lambda^2$  is the identity automorphism in  $K/A$ . Since the automorphism  $\theta_{xA}$  agrees with  $\lambda^2$  on the generating set  $\{xA\}$ , we have proved that  $\theta_{xA}$  is the identity automorphism in  $K/A$ .

Now let  $y_A$  be in  $K/A$ . We have  $y_A(x_A \cdot y_A)^{-1} = (x_A)^{-1}$  by (1.6); hence we see that  $(y_A)\theta_{x_A y_A}^{-1} = (y_A)\theta_{x_A}$ , or  $y_A = (y_A)\theta_{x_A} \theta_{x_A y_A}$ . By using (1.7), one can show that  $x_A = (x_A)\theta_{x_A y_A} \theta_{y_A}$ . Now we have  $\theta_{x_A y_A} \theta_{y_A} = L(x_A y_A)L([x_A y_A]^{-1})\theta_{y_A} = \theta_{y_A} L([x_A y_A]\theta_{y_A})L(([x_A y_A]\theta_{y_A})^{-1})$  since  $\theta_{y_A}$  is an automorphism. Hence we have  $\theta_{y_A y_A} \theta_{y_A} = \theta_{y_A} \theta_{(x_A y_A)\theta_{y_A}}$ . But we see that  $(x_A y_A)\theta_{y_A} = (x_A y_A)L(y_A)L([y_A]^{-1}) = ((xy)L(y)L(y^\lambda))_A = [(xy)\theta_y]_A$ . Let  $\eta$  be the natural mapping from  $K$  onto  $K/N$ , where  $N$  is the nucleus of  $K$ . Then we have  $((xy)\theta_y)_\eta = ((xy)L(y)L(y^\lambda))_\eta = ((xy)\eta)L(y\eta)L((y\eta)^{-1}) = (xy)\eta$  since  $K/N$  is an inverse property loop. It follows that  $(xy)\theta_y = xy \cdot q$  where  $q$  is in  $N$ . Hence we have  $(y_A y_A)\theta_{y_A} = (x_A y_A)q_A$  where  $q_A$  is in the nucleus of  $K/A$ . By Theorem 1.9 we have  $\theta_{x_A y_A} = \theta_{(x_A y_A)q_A} = \theta_{(x_A y_A)\theta_{y_A}}$ . Thus we see that  $x_A = (x_A)\theta_{y_A} \theta_{x_A y_A}$ . Hence we also have  $x_A y_A = ((x_A)\theta_{y_A} \cdot (y_A)\theta_{x_A})\theta_{x_A y_A}$ , or  $(x_A y_A)\theta_{x_A y_A}^{-1} = (x_A)\theta_{y_A} \cdot (y_A)\theta_{x_A}$ . But since  $(x_A y_A)\theta_{x_A y_A}^{-1} = (x_A y_A)^{\circ 2} = x_A y_A$  by Theorem 1.9 and the fact that inverses are unique in  $K/A$ , we have  $x_A y_A = (x_A)\theta_{y_A} \cdot y_A$  since  $\theta_{x_A}$  is the identity mapping in  $K/A$ . It follows that  $(x_A)\theta_{y_A} = x_A$ . Furthermore, since  $\theta_{y_A}$  is an automorphism and since  $\theta_{y_A}$  coincides with the identity mapping on the generating set  $\{x_A\}$ ,  $\theta_{y_A}$  is the identity mapping. It follows that  $K/A$  is a Moufang loop since  $y_A$  was chosen arbitrarily in  $K/A$ . Bruck [3] has shown that Moufang loops are diassociative; that is, that every subloop of a Moufang loop which is generated by one or two elements is a group. Since  $K/A$  has one generator,  $K/A$  is a group.

Now let  $\zeta$  be the natural homomorphism from  $H$  onto  $H/\phi(A)$ . Then  $\phi\eta$  maps  $K$  onto an infinite cyclic group, and  $A$  is contained in the kernel of  $\phi\eta$ . Now, for each  $z_A$  in  $K/A$ , define  $(z_A)\bar{\phi} = z\phi \cdot \phi(A)$ . Note that  $\bar{\phi}$  is a well-defined mapping since, if  $y_A = z_A$ , then  $z = ya^j$ , for some

integer  $j$ . Hence we have  $z\phi \cdot \phi(A) = y\phi(a\phi)^j \cdot \phi(A) = y\phi \cdot \phi(A)$ . Now, for any  $yA$  and  $zA$  in  $K/A$ , we have  $(yAzA)\bar{\phi} = (yz)A\bar{\phi} = (yz)\phi \cdot \phi A = (y\phi \cdot \phi A)(z\phi \cdot \phi(A)) = (yA)\bar{\phi} (zA)\bar{\phi}$ . Hence  $\bar{\phi}$  is a homomorphism from  $K/A$  into  $H/\phi(A)$ . It is clear that  $\bar{\phi}$  is onto. But  $K/A$  is a cyclic group, and  $H/\phi(A)$  is the infinite cyclic group. Hence  $\bar{\phi}$  is an isomorphism. Now let  $z\phi = [0,0]$ . Then we have  $(zA)\bar{\phi} = z\phi \cdot \phi(A) = 0$ , but, since  $\bar{\phi}$  is an isomorphism, this implies that  $zA$  is the identity in  $K/A$ . Hence we see that  $z$  is in  $A$ . Now  $\phi$  is an isomorphism when restricted to  $A$ ; hence  $z$  must be the identity of  $K$ . It follows that  $\phi$  is an isomorphism of  $K$  onto  $H$ . ■

## CHAPTER II

## GENERALIZED MOUFANG LOOPS

In this chapter we concern ourselves with the basic properties of generalized Moufang loops and their relationship to weak inverse property loops. We shall show that, although they have many properties in common with Osborn loops, they form a strictly larger class of loops. We now define formally the concept of generalized Moufang loop which was defined informally in Chapter I.

Definition 2.1. Let  $G$  be a loop. We say that  $G$  is a *generalized Moufang loop* or *M-loop* if and only if, corresponding to each element  $x$  in  $G$ , there is an automorphism  $\theta_x$  of  $G$  such that, for any  $y$  and  $z$  in  $G$ ,

$$(2.1) \quad xz (y\theta_x \cdot x) = (x \cdot zy)x.$$

By Theorem 1.7 we see immediately that Osborn loops are generalized Moufang loops. Since Moufang loops are special cases of Osborn loops, we see that Moufang loops are also special cases of M-loops. We begin our investigation of M-loops by deriving some of the properties of the automorphisms  $\theta_x$ .

Lemma 2.1. Let  $G$  be an M-loop. Then, for every  $x$  in  $G$ , we have the following identities:

$$(2.2) \quad xy \cdot x = x \cdot (y\theta_x \cdot x) \quad \text{for all } y \text{ in } G$$

$$(2.3) \quad \theta_x = L(x)L(x^\lambda) = R^{-1}(x^\rho)R^{-1}(x) = L(x)R(x)L^{-1}(x)R(x)$$

$$(2.4) \quad \theta_x = \theta_{x^\lambda} = \theta_{x^\rho}$$

$$(2.5) \quad x\theta_x = x^{\lambda^2}$$

Proof: If  $y$  is in  $G$ , then (2.2) is just a special case of (2.1) with  $z = 1$ . Now by (2.2) we have  $yL(x)R(x) = y\theta_x R(x)L(x)$ . Since  $y$  is any element in  $G$ , we have  $\theta_x = L(x)R(x)L^{-1}(x)R^{-1}(x)$ , and we have established part of (2.3). Note that  $y\theta_x \cdot x = xx^\rho \cdot (y\theta_x \cdot x) = (x \cdot x^\rho y)x$  by (2.1).

Hence we have  $y\theta_x = x \cdot x^\rho y = yL(x^\rho)L(x)$ . It follows that  $\theta_x = L(x^\rho)L(x)$

for all  $x$  in  $G$ . Replacing  $x$  by  $x^\lambda$ , we see that  $\theta_{x^\lambda} = L(x)L(x^\lambda)$  for all

$x$  in  $G$ . Now consider  $y\theta_{x^\lambda}\theta_x^{-1} = (x^\lambda \cdot xy)\theta_x^{-1} = (x^\lambda)\theta_x^{-1} \cdot (xy)\theta_x^{-1}$ . We

have  $x^\lambda\theta_x^{-1} = x^\lambda R(x)L(x)R^{-1}(x)L^{-1}(x) = [x(x^\lambda x)]R^{-1}(x)L^{-1}(x) =$

$1L^{-1}(x) = x^\rho$ , and we also have  $(xy)\theta_x^{-1} = (xy)L^{-1}(x)L^{-1}(x^\rho) = yL^{-1}(x^\rho)$ .

Hence we have  $y\theta_{x^\lambda}\theta_x^{-1} = x^\lambda\theta_x^{-1} \cdot (xy)\theta_x^{-1} = x^\rho \cdot yL^{-1}(x^\rho) = y$ . It follows

that  $\theta_{x^\lambda}\theta_x^{-1} = I$ , or  $\theta_x = \theta_{x^\lambda} = L(x)L(x^\lambda)$ .

We have now shown that  $\theta_x = L(x)L(x^\lambda) = L(x)R(x)L^{-1}(x)R^{-1}(x)$  and

that  $\theta_{x^\lambda} = \theta_x$ . From  $\theta_{x^\lambda} = \theta_x$  for all  $x$  in  $G$ , it immediately follows,

replacing  $x$  by  $x^\rho$ , that  $\theta_x = \theta_{x^\rho}$  for all  $x$  in  $G$ , and we have established

(2.4). To establish (2.3) we have only to show that  $\theta_x = R^{-1}(x^\rho)R^{-1}(x)$ .

Now we have  $xz \cdot yx = (x \cdot z(y\theta_x^{-1}))x = x((z \cdot y\theta_x^{-1})\theta_x \cdot x) = x \cdot ((z\theta_x \cdot y) \cdot x)$ ,

for all  $y$  and  $z$  in  $G$ , where we have used (2.1), (2.2) and the fact that

$\theta_x$  is an automorphism. In this identity choose  $y = x^\lambda$ . Then we have

$xz = x((z\theta_x^\lambda)x^\lambda \cdot x)$ , or  $z = (z\theta_x^\lambda)x^\lambda \cdot x$ . Since  $z$  is arbitrary in  $G$ , we have  $\theta_x^{-1} = R(x^\lambda)R(x)$ , or  $\theta_x = R^{-1}(x)R^{-1}(x^\lambda)$ . Hence we have  $\theta_{x^\rho} = R^{-1}(x^\rho)R^{-1}(x)$  for all  $x$  in  $G$ . But we also have  $\theta_x = \theta_{x^\rho} = R^{-1}(x^\rho)R^{-1}(x)$ , and we have verified (2.3). Finally, we have  $x\theta_{x^\lambda} = xL(x^\lambda)L(x^{\lambda^2}) = x^{\lambda^2} = x\theta_x$ , and we have verified (2.5). ■

The above lemma raises the question of the existence of a loop  $G$  with the properties that, corresponding to each element  $x$  in  $G$ , there is an automorphism  $\theta_x$  of  $G$ , given by (2.3), and that  $G$  is not an  $M$ -loop. We now show that there do exist such loops.

Example 2.1. Let  $G_2$  be the free group on two generators; that is,  $G_2$  is a group with two generators and every group with two generators is a homomorphic image of  $G_2^*$ . Let  $\Lambda$  be a cyclic group of order 2. Let  $K$  be the cartesian product  $G_2 \times \Lambda$ . We introduce a binary operation on  $K$  as follows. If  $e$  generates  $\Lambda$ , we define

$$(2.6) \quad [x, e^i][y, e^j] = [xy, e^{i+j+1}],$$

if the set  $\{x, y\}$  generates  $G_2$ . Otherwise, we define

$$(2.7) \quad [x, e^i][y, e^j] = [xy, e^{i+j}].$$

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\*The existence of  $G_2$  is shown in any book on universal algebras. For example, see [5].

It is clear that the binary operation is well defined and that  $[1, e^0]$  is an identity for the system. In Lemmas 2.2, 2.3, and Theorem 2.1,  $K$  refers to the above example.

Lemma 2.2. The system  $K$  is an inverse property loop.

Proof: We have already shown that  $K$  has an identity. Suppose that  $[a, e^i]$  and  $[b, e^j]$  are in  $K$ .

Case I. The set  $\{a, b\}$  generates  $G_2$ . Then the set  $\{a, a^{-1}b\}$  generates  $G_2$  since the group generated by the set  $\{a, a^{-1}b\}$  obviously must contain the set  $\{a, a(a^{-1}b)\} = \{a, b\}$ . Similarly, the set  $\{a, ba^{-1}\}$  generates  $G_2$ . It follows that  $[a, e^i][a^{-1}b, e^{j-i-1}] = [b, e^j]$  and that  $[ba^{-1}, e^{j-i-1}][a, e^i] = [b, e^j]$ . Furthermore, if  $[x, e^k]$  is such that  $[a, e^i][x, e^k] = [b, e^j]$ , then  $ax = b$ . Hence we have  $x = a^{-1}b$ . Then, since  $\{a, a^{-1}b\}$  generates  $G_2$ , we must have  $e^{i+k+1} = e^j$ . Hence we have  $i + k + 1 = j \pmod{2}$ . It follows that  $k = j - 1 - i \pmod{2}$ . Hence we have  $[x, e^k] = [a^{-1}b, e^{j-i-1+2n}]$ , where  $n$  is an arbitrary integer. But, since  $\Lambda$  has order 2, we have  $e^{j-i-1+2n} = e^{j-i-1}$  for any integer  $n$ . It follows that  $[a^{-1}b, e^{j-i-1+2n}]$  actually represents a unique element in  $K$ . Hence the solution  $[a^{-1}b, e^{j-i-1}]$  to the equation  $[a, e^i][x, e^k] = [b, e^j]$  is unique. Similarly, it is clear that the solution  $[ba^{-1}, e^{j-i-1}]$  to the equation  $[x, e^k][a, e^i] = [b, e^j]$  is unique.

Case II. The set  $\{a, b\}$  does not generate  $G_2$ . The argument for this case parallels the one above and is omitted.

We have shown that  $K$  is a loop. Now suppose that  $[a, e^i]$  is in  $K$ . Then we have  $[a, e^i][a^{-1}, e^{-i}] = [1, e^0]$  and  $[a^{-1}, e^{-i}][a, e^i] = [1, e^0]$ . Hence we see that  $x^\lambda = x^\rho$  for all  $x$  in  $K$ . Now, if  $\{a, b\}$  does not generate  $G_2$ , then neither do the sets  $\{a^{-1}, ab\}$  and  $\{ba, a^{-1}\}$  generate  $G_2$ . Hence we have  $[a^{-1}, e^{-i}]( [a, e^i][b, e^j] ) = [b, e^j] = ([b, e^j][a, e^i])[a^{-1}, e^{-i}]$ . If  $\{a, b\}$  does generate  $G_2$ , then the sets  $\{a^{-1}, ab\}$  and  $\{a^{-1}, ba\}$  also generate  $G_2$  since the respective groups generated by these sets contain the set  $\{a, b\}$ . It follows that  $[a^{-1}, e^{-i}]( [a, e^i][b, e^j] ) = [a^{-1}, e^{-i}][ab, e^{j+1}] = [b, e^{j+2}] = [b, e^j]$  and that  $( [b, e^j][a, e^i] )[a^{-1}, e^{-i}] = [ba, e^{i+j+1}][a^{-1}, e^{-i}] = [b, e^{j+2}] = [b, e^j]$ . Hence  $K$  is an inverse property loop. ■

Lemma 2.3. Corresponding to each  $\bar{x} = [x, e^i]$  in  $K$ , there exists an automorphism  $\theta_{\bar{x}}$  given by  $\theta_{\bar{x}} = L(\bar{x})L(\bar{x}^\lambda) = R^{-1}(\bar{x}^\rho)R^{-1}(\bar{x}) = L(\bar{x})R(\bar{x})L^{-1}(\bar{x})R^{-1}(\bar{x}) = I$ , where  $I$  is the identity mapping.

Proof: Since  $K$  is an inverse property loop, we have  $\bar{y}L(\bar{x})L(\bar{x}^\lambda) = \bar{x}^\lambda(\bar{x}\bar{y}) = \bar{x}^{-1}(\bar{x}\bar{y}) = \bar{y}$ , for every  $\bar{y} = [y, e^j]$ . Hence we have  $L(\bar{x})L(\bar{x}^\lambda) = I$ , and  $L(\bar{x})L(\bar{x}^\lambda)$  is an automorphism of  $K$ . Similarly, it is clear that  $R^{-1}(\bar{x}^\rho)R^{-1}(\bar{x}) = I$  is an automorphism.

Now suppose that the set  $\{x, y\}$  generates  $G_2$ . Then we have  $\bar{x}\bar{y}\bar{x} = ([x, e^i][y, e^j])[x, e^i] = [xy, e^{i+j+1}][x, e^i] = [xyx, e^{2i+2j+2}]$ , since the set  $\{x, yx\}$  generates  $G_2$  if the set  $\{x, y\}$  does. But we also have  $\bar{x}\bar{y}\bar{x} = [x, e^i]( [y, e^j][x, e^i] ) = [x, e^i][yx, e^{i+j+1}] = [xyx, e^{2i+j+2}] = \bar{x}\bar{y}\bar{x}$ .

A similar calculation shows that, if the set  $\{x, y\}$  does not generate  $G_2$ , then we still have  $\bar{x}\bar{y}\bar{x} = \bar{x}\bar{y}\bar{x}$ . It follows that  $\bar{x}\bar{y}\bar{x} =$

$\bar{x}\bar{y}\bar{x}$ , for any  $\bar{x}$  and  $\bar{y}$  in  $K$ . Hence we have  $L(\bar{x})R(\bar{x}) = R(\bar{x})L(\bar{x})$ , or  $L(\bar{x})R(\bar{x})L^{-1}(\bar{x})R^{-1}(\bar{x}) = I = L(\bar{x})L(\bar{x}^{-1}) = R^{-1}(\bar{x}^0)R^{-1}(\bar{x})$ . ■

Theorem 2.1. The loop  $K$  is not an  $M$ -loop.

Proof: We show first that  $K$  has two generators. Suppose the set  $\{x,y\}$  generates  $G_2$ . We shall prove that neither of the sets  $\{x^3,y\}$  and  $\{x^2,xy\}$  generates  $G_2$ .

Suppose that the set  $\{x^3,y\}$  generates  $G_2$ . Consider the direct product  $\bar{\Lambda}$  of the cyclic group of order three with the integers. This group is a homomorphic image of  $G_2$ . Let  $\eta$  denote this homomorphism. We show that the set  $\{(x\eta)^3,y\eta\}$  generates  $\bar{\Lambda}$  if the above assumption holds. Suppose that  $a$  is in  $\bar{\Lambda}$  and that  $\Lambda'$  is a subgroup of  $\bar{\Lambda}$  such that  $\{(x\eta)^3,y\eta\} \subseteq \Lambda'$ . Then we have  $\{x^3,y\} \subseteq \eta^{-1}(\Lambda')$ . But  $\eta^{-1}(\Lambda')$  is a subgroup of  $G_2$ . Hence we have  $\eta^{-1}(\Lambda') = G_2$  since we are assuming that the set  $\{x^3,y\}$  generates  $G_2$ . Thus we see that  $\eta^{-1}(a) \subseteq \eta^{-1}(\Lambda')$ . Hence we have  $a$  in  $\Lambda'$  since the mapping  $\eta$  is single valued. Since  $a$  was arbitrarily chosen in  $\bar{\Lambda}$ , it follows that  $\Lambda' = \bar{\Lambda}$ . Thus we have shown that any subgroup  $\Lambda'$  of  $\bar{\Lambda}$  containing the set  $\{(x\eta)^3,y\eta\}$  must be all of  $\bar{\Lambda}$ . Hence  $\{(x\eta)^3,y\eta\}$  generates  $\bar{\Lambda}$ . This is clearly false. Hence  $\{x^3,y\}$  does not generate  $G_2$ . A similar argument shows that the set  $\{x^2,y\}$  does not generate  $G_2$ . Since the set  $\{x,xy\}$  generates  $G_2$ , we know now that the set  $\{x^2,xy\}$  does not generate  $G_2$ .

Now let  $\bar{x} = [x,e^0]$  and  $\bar{y} = [y,e^0]$ , where the set  $\{x,y\}$  generates  $G_2$ . Then we have  $(\bar{x}^3\bar{y})^{-1}(\bar{x}^2\bar{x}\bar{y}) = ([x^3,e^0][y,e^0])^{-1} = ([x^2,e^0][xy,e]) = [y^{-1}x^{-3},e^0][x^3y,e] = [1,e]$ . Since  $[1,e]$  is not the identity of  $K$ , this

calculation shows that the loop  $K$  is not a group and that the subloop of  $K$  generated by  $\{\bar{x}, \bar{y}, \bar{e}\}$ , where  $\bar{e} = [1, e]$ , is the same as the one generated by the set  $\{\bar{x}, \bar{y}\}$ .

We now show that the set  $\{\bar{x}, \bar{y}, \bar{e}\}$  generates  $K$ ; hence it will follow that the set  $\{\bar{x}, \bar{y}\}$  generates  $K$ . Let  $E$  be the subloop of  $K$  generated by the element  $\bar{e}$ . It is clear from (2.6) and (2.7) that  $E$  is contained in the nucleus of  $K$  and that  $EK = KE$ . If we define a mapping  $\phi$  from  $K$  onto  $G_2$  by setting  $[z, e^j]\phi = z$ , it is immediately evident that  $\phi$  is a homomorphism and that  $E$  is the kernel of  $\phi$ . Bruck [2] has shown that the kernel of a homomorphism is normal in any loop  $G$  and that any loop homomorphic image  $G'$  is isomorphic to the factor loop of  $G$  over the kernel of the homomorphism. Hence the factor loop  $K/E$  is isomorphic to  $G_2$ , and it follows that the set  $\{\bar{x}E, \bar{y}E\}$  generates  $K/E$ . Now suppose  $\bar{z}$  is in  $K$ . Let  $K'$  be the subloop of  $K$  generated by the set  $\{\bar{x}, \bar{y}\}$ . Suppose  $\bar{z}$  is not in  $K'$ . Then there exists an element  $\bar{z}'$  in  $K'$  such that  $\bar{z}E = \bar{z}'E$  since the image of  $K'$  in  $K/E$  is  $K/E$  because the image of  $K'$  in  $K/E$  contains the set  $\{\bar{x}E, \bar{y}E\}$  which generates  $K/E$ . It follows that  $\bar{z} = \bar{z}'e^j$ , for some integer  $j$ , and that  $\bar{z}$  is in the subloop generated by  $\{\bar{x}, \bar{y}, \bar{e}\}$ . Since  $\bar{z}$  was arbitrarily chosen in  $K$ , we have proved that the set  $\{\bar{x}, \bar{y}, \bar{e}\}$  generates  $K$ . But this then implies that the set  $\{\bar{x}, \bar{y}\}$  generates  $K$ . Thus  $K$  is generated by a set consisting of two elements.

Now suppose that  $K$  is an  $M$ -loop. Since the automorphisms associated with each  $x$  in  $K$  all coincide with the identity mapping, the identity (2.1) reduces to (1.1), and  $K$  would be a Moufang loop. Moufang [6] and Bruck [3] have shown that any Moufang loop which is generated by a set consisting of two generators is a group. The system  $K$  is not a

group; hence  $K$  cannot be an  $M$ -loop. ■

Despite the above example, it is still possible to give a simple characterization of  $M$ -loops by using autotopisms; we now proceed to do this.

Theorem 2.2. Let  $G$  be a loop. Then the following statements are equivalent:

- (i) The loop  $G$  is an  $M$ -loop.
- (ii) For every  $f$  in  $G$ , the triple

$$(2.8) \quad (L(f), R^{-1}(f^0), L(f)R(f))$$

is an autotopism of  $G$ .

- (iii) For every  $f$  in  $G$ , the triple

$$(2.9) \quad (L(f), R^{-1}(f^0), R^{-1}(f^0)L(f))$$

is an autotopism of  $G$ .

Proof: Suppose that  $G$  is an  $M$ -loop. Then we have  $fx \cdot (y\theta_f \cdot f) = (f \cdot xy)f$  for every  $f, x,$  and  $y$  in  $G$ .

It follows that the triple  $(L(f), \theta_f R(f), L(f)R(f))$  is an autotopism of  $G$ . By Lemma 2.1 we have  $\theta_f R(f) = R^{-1}(f^0)R^{-1}(f)R(f) = R^{-1}(f^0)$ . Hence the triple (2.8) is an autotopism of  $G$ .

Now suppose that  $G$  is a loop in which the triple (2.8) is an autotopism of  $G$  for every  $f$  in  $G$ . Applying this autotopism to the pair  $(1, x)$ , we obtain that  $f \cdot xR^{-1}(f^0) = xL(f)R(f)$ ; that is,  $xR^{-1}(f^0)L(f) = L(f)R(f)$  for all  $x$  in  $G$ . Hence we have  $L(f)R(f) = R^{-1}(f^0)L(f)$ , and we have proved that the triple (2.9) is an autotopism of  $G$  for every  $f$  in  $G$ . In like manner, if the triple (2.9) is an autotopism of  $G$  for every  $f$  in  $G$ , then we may show that (2.8) is an autotopism of  $G$  for every  $f$  in  $G$  by applying the autotopism (2.9) to the pair  $(x, 1)$ .

Now suppose that  $G$  is a loop in which (2.9), and hence (2.8), is an autotopism of  $G$  for every  $f$  in  $G$ . Taking the inverse of (2.9) and substituting  $f^\lambda$  for  $f$ , we obtain that  $(L^{-1}(f^\lambda), R(f), L^{-1}(f^\lambda)R(f))$  is an autotopism of  $G$ . Now applying this autotopism to the pair  $(1, x)$ , we obtain  $1L^{-1}(f^\lambda) \cdot xf = xL^{-1}(f)R(f)$ ; that is,  $f \cdot xf = xL^{-1}(f^\lambda)R(f)$  since  $1L^{-1}(f^\lambda) = f$ . Since  $x$  was chosen arbitrarily in  $G$ , we have proved that  $R(f)L(f) = L^{-1}(f^\lambda)R(f)$ . It follows that the triple  $(L^{-1}(f^\lambda), R(f), R(f)L(f))$  is an autotopism of  $G$ . Thus the triple  $\alpha = (L(f), R^{-1}(f^0), L(f)R(f))[(L^{-1}(f^\lambda), R(f), R(f)L(f))^{-1}] = (L(f)L(f^\lambda), R^{-1}(f^\lambda)R^{-1}(f), L(f)R(f)L^{-1}(f)R^{-1}(f))$  is an autotopism of  $G$ . Now we have  $1L(f)L(f^\lambda) = f^\lambda f = 1$ ,  $1R^{-1}(f^0)R^{-1}(f) = fR^{-1}(f) = 1$ , and  $1L(f)R(f)L^{-1}(f)R^{-1}(f) = ffL^{-1}(f)R^{-1}(f) = 1$ . Hence each of the permutations in the autotopism  $\alpha$  leaves the identity element of  $G$  fixed. Applying the autotopism  $\alpha$  to the pair  $(x, 1)$ , we obtain that  $xL(f)L(f^\lambda) \cdot 1R^{-1}(f^\lambda)R^{-1}(f) = xL(f)R(f)L^{-1}(f)R^{-1}(f)$ ; that is,  $L(f)L(f^0) = L(f)R(f)L^{-1}(f)R^{-1}(f)$ .

Similarly, it can be shown that  $R^{-1}(f^0)R^{-1}(f) = L(f)R(f)L^{-1}(f)R^{-1}(f)$ . Thus, letting  $\theta_f = L(f)L(f^\lambda)$ , we obtain that

$(\theta_f, \theta_f, \theta_f)$  is an autotopism of  $G$ , hence  $\theta_f$  is an automorphism of  $G$ . Now (2.8) can be written  $(L(f), \theta_f R(f), L(f)R(f))$ . Thus

$$fx \cdot (y\theta_f \cdot f) = (f \cdot xy)f,$$

for every  $f, x, y$  in  $G$ , and we have shown that  $G$  is an M-loop. ■

Corollary 2.1. Let  $G$  be an M-loop. Then, for any  $f$  in  $G$ , we have

$$(2.10) \quad L(f)R(f) = R^{-1}(f^\theta)L(f);$$

the triple

$$(2.11) \quad (L^{-1}(f^\lambda), R(f), R(f)L(f))$$

is an autotopism of  $G$ ;

$$(2.12) \quad R(f)L(f) = L^{-1}(f^\lambda)R(f).$$

These facts are contained in the proof of Theorem 2.2, and a separate proof here is not needed.

Corollary 2.2. Let  $G$  be a loop. Then the following statements are equivalent:

- (i) The loop  $G$  is an M-loop.
- (ii) The triple

$$(2.13) \quad (L(f^\lambda), R^{-1}(f), L^{-1}(f)R^{-1}(f))$$

is an autotopism of  $G$ , for every  $f$  in  $G$ .

Proof: If  $G$  is an  $M$ -loop, then the inverse of (2.13) is an autotopism by (2.11), and hence (2.13) itself must be an autotopism of  $G$ .

Now suppose that (2.13) is an autotopism of  $G$  for every  $f$  in  $G$ . Then  $(L^{-1}(f^\lambda), R(f), R(f)L(f))$  is an autotopism of  $G$  for every  $f$  in  $G$ . Applying this autotopism to the pair  $(x, 1)$ , we obtain that  $xL^{-1}(f^\lambda) \cdot 1R(f) = xR(f)L(f)$ , or  $xL^{-1}(f^\lambda)R(f) = xR(f)L(f)$ . Since  $x$  is arbitrary in  $G$ , we have proved that  $L^{-1}(f^\lambda)R(f) = R(f)L(f)$ . It follows that the triple  $(L^{-1}(f^\lambda), R(f), L^{-1}(f^\lambda)R(f))$  is an autotopism of  $G$ . Taking the inverse of this autotopism and replacing  $f$  by  $f^0$ , we obtain (2.9). Hence,  $G$  is an  $M$ -loop by Theorem 2.2. ■

Corollary 2.3. Let  $G$  be a loop. Then the following statements are equivalent:

(i) If  $f, x$  are in  $G$ , then

$$(2.14) \quad L(fx) = R(f^0)L(x)L(f)R(f).$$

(ii) The loop  $G$  is an  $M$ -loop.

(iii) If  $f, x$  are in  $G$ , then

$$(2.15) \quad L(fx) = R(f^0)L(x)R^{-1}(f^0)L(f).$$

Proof: Suppose that (2.14) holds for every  $f$  and  $x$  in  $G$ . Then we have  $fx \cdot y = (f(x \cdot yf^{\rho})) \cdot f$  for every  $y$  in  $G$ . Substituting  $yR^{-1}(f^{\rho})$  for  $y$  in this identity, we have  $fx \cdot yR^{-1}(f^{\rho}) = (f \cdot xy) \cdot f$ . This identity holds for all  $y$  and  $x$  in  $G$ . Hence the triple  $(L(f), R^{-1}(f^{\rho}), L(f)R(f))$  is an autotopism of  $G$ . It follows that  $G$  is an  $M$ -loop by Theorem 2.2.

Now suppose that  $G$  is an  $M$ -loop. Then by Theorem 2.2 the triple  $(L(f), R^{-1}(f^{\rho}), R^{-1}(f^{\rho})L(f))$  is an autotopism of  $G$  for every  $f$  in  $G$ . Hence we have  $fx \cdot yR^{-1}(f^{\rho}) = (xy)R^{-1}(f^{\rho})L(f)$  for all  $x$  and  $y$  in  $G$ . Replacing  $y$  by  $yf^{\rho}$  in this identity, we obtain that  $fx \cdot y = (x \cdot yf^{\rho})R^{-1}(f^{\rho})L(f)$  for all  $y$  in  $G$ . Hence we have  $L(fx) = R(f^{\rho})L(x)R^{-1}(f^{\rho})L(f)$ , and we have established (2.15).

Now suppose that (2.15) holds for all  $f$  and  $x$  in  $G$ . Then we have the identity  $fx \cdot y = (x \cdot yf^{\rho})R^{-1}(f^{\rho})L(f)$  for all  $y$  in  $G$ . Replacing  $y$  by  $yR^{-1}(f^{\rho})$  in this identity, we have the identity  $fx \cdot yR^{-1}(f^{\rho}) = (xy)R^{-1}(f^{\rho})L(f)$  for all  $x$  and  $y$  in  $G$ . It follows that the triple  $(L(f), R^{-1}(f^{\rho}), R^{-1}(f^{\rho})L(f))$  is an autotopism of  $G$  for each  $f$  in  $G$ . Hence by Theorem 2.2  $G$  is an  $M$ -loop. Then by Corollary 2.1 we obtain that (2.15) holding for all  $f, x$  in  $G$  implies that (2.14) holds for all  $f, x$  in  $G$ . ■

Corollary 2.4. Let  $G$  be an  $M$ -loop and let  $G'$  be a loop homomorphic image of  $G$ . Then  $G'$  is an  $M$ -loop.

Proof: Let  $\phi$  be a homomorphism from  $G$  onto  $G'$  and let  $f', x'$ , and  $y'$  be elements in  $G'$ . Since the range of  $\phi$  is all of  $G'$ , we may find elements  $f, x$ , and  $y$  in  $G$  so that  $f\phi = f'$ ,  $x\phi = x'$ , and  $y\phi = y'$ .

Now since  $G$  is an  $M$ -loop, we have  $fx \cdot y = (f(x \cdot yf^{\rho})) \cdot f$  by (2.14). Hence we have  $f'x' \cdot y' = (f'(x' \cdot y'(f')^{\rho})) \cdot f'$  since  $\phi$  is a homomorphism. Since  $y'$  may be any element in  $G'$ , it follows that  $L(f'x') = R(f'^{\rho})L(x')L(f')R(f')$  for all  $f'$  and  $x'$  in  $G'$ . Thus  $G'$  is an  $M$ -loop by Corollary 2.2. ■

It can be seen from the above results that  $M$ -loops have many properties in common with Osborn loops. Another property which they have in common is one that they share with any weak inverse property loop; namely, the four nuclei of an  $M$ -loop coincide. Before proving this, however, we prove the following.

Lemma 2.4. Let  $G$  be an  $M$ -loop and suppose that  $a$  is in the left, middle, or right nucleus of  $G$ . Then we have  $a\theta_x = a$  and  $x\theta_a = x$  for any  $x$  in  $G$ .

Proof: We use Lemma 2.1 frequently in what follows. Suppose that  $a$  is in the left nucleus of  $G$ . Then  $a\theta_x^{-1} = ax \cdot x^{\rho} = a \cdot xx^{\rho} = a$  for all  $x$  in  $G$ . Hence we have  $a\theta_x = a$ . We have  $x\theta_a = x\theta_{a^{-1}} = a \cdot a^{-1}x = aa^{-1}x = x$ .

If  $a$  is in the middle nucleus of  $G$ , then  $a\theta_x = (xa \cdot x)L^{-1}(x)R^{-1}(x) = (x \cdot ax)L^{-1}(x)R^{-1}(x) = a$ . Furthermore, we have  $x\theta_a = a^{-1}ax = a^{-1}a \cdot x = x$ .

Now suppose that  $a$  is in the right nucleus of  $G$ . Then we have  $a\theta_x = x^{\lambda} \cdot xa = x^{\lambda}x \cdot a = a$ . Furthermore, we have  $x\theta_a = (ax \cdot a)L^{-1}(a)R^{-1}(a) = (a \cdot xa)L^{-1}(a)R^{-1}(a) = x$ . ■

We are now ready to prove the coincidence of the four nuclei.

Theorem 2.3. The four nuclei of an M-loop  $G$  coincide.

Proof: Let  $N_\lambda$ ,  $N_\mu$ , and  $N_\rho$  denote the left, middle, and right nuclei of  $G$ , respectively. Suppose  $a$  is in  $N_\lambda$ . Let  $x, y$  be in  $G$ . Then we have  $x \cdot ya = (a \cdot xL^{-1}(a)) \cdot (ya) = (a \cdot xL^{-1}(a))(y\theta_a \cdot a) = (a(xL^{-1}(a) \cdot y))a = ((a \cdot xL^{-1}(a))y)a = xy \cdot a$ . It follows that  $a$  belongs to  $N_\rho$  and that  $N_\lambda \subseteq N_\rho$ .

Now suppose that  $a$  is in  $N_\rho$ . Let  $x, y$  be in  $G$ . Then we have  $x \cdot ay = (y \cdot xL^{-1}(y)) \cdot ay = (y \cdot xL^{-1}(y))(a\theta_y \cdot y) = (y \cdot (xL^{-1}(y) \cdot a))y = ((y \cdot xL^{-1}(y)) \cdot a)y = xa \cdot y$ . It follows that  $a$  is in  $N_\mu$  and that  $N_\lambda \subseteq N_\rho \subseteq N_\mu$ .

Now suppose that  $a$  is in  $N_\mu$ . Let  $x, y$  be in  $G$ . Then we have  $y(a \cdot xy) = ya \cdot xy = (y \cdot (a \cdot x\theta_y^{-1}))y = y[(a \cdot x\theta_y^{-1})\theta_y \cdot y] = y[(a\theta_y \cdot x)y] = y(ax \cdot y)$ , where we have made one application of (2.2). It follows that  $a \cdot xy = ax \cdot y$  for all  $x, y$  in  $G$  and that  $N_\lambda \subseteq N_\rho \subseteq N_\mu \subseteq N_\lambda$ , or  $N_\lambda = N_\rho = N_\mu$ . Hence we have  $N = N_\lambda \cap N_\rho \cap N_\mu = N_\lambda = N_\rho = N_\mu$ . ■

Yet another property shared in common by M-loops and Osborn loops is the following.

Theorem 2.4. If an M-loop  $G$  is an inverse property, cross inverse property, or commutative loop, the  $G$  is a Moufang loop.

The proof of this theorem is identical to the proof of Corollary 1.1 and is omitted.

Despite the common properties which Osborn loops share with M-loops, there is one property which all Osborn loops have but which M-loops need not have; it is the weak inverse property. We shall give an example of an M-loop which is not a weak inverse property loop, but we shall need the following lemma.

Lemma 2.5. If  $G$  is a weak inverse property M-loop and if  $x$  is in  $G$ , then we have  $(xx)^\lambda = (xx)^\rho$ ; that is, squares have unique inverses in  $G$ .

Proof: By (2.3) and (2.5) of Lemma 2.1, we have  $x^\lambda(xx) = x\theta_x = x^{\lambda^2}$ . Hence we have  $1 = x^{\lambda^2} \circ x^\lambda = x^\lambda(xx) \circ x^\lambda$ , and, since  $G$  is a weak inverse property loop, we also have  $x^\lambda \circ (xx)x^\lambda = 1$ . But by (2.2) of Lemma 2.1, we have  $1 = x^\lambda(xx) \circ x^\lambda = x^\lambda \circ ((xx)\theta_x \circ x^\lambda)$ . It follows that  $x^\lambda \circ (xx)x^\lambda = 1 = x^\lambda \circ ((xx)\theta_x \circ x^\lambda)$ , or  $(xx) = (xx)\theta_x$ . Hence we have  $xx = (xx)\theta_x = x\theta_x \circ x\theta_x = x^{\lambda^2} \circ x^{\lambda^2} = (xx)^{\lambda^2}$  since  $\lambda^2$  is an automorphism of  $G$  by Theorem 1.3. Hence we have  $(xx)^\lambda = (xx)^\rho$ . ■

Example 3.2. Let  $M$  be the cross product of the even integers with the integers. We introduce a binary operation on  $M$  by defining  $[2i, m][2j, n] = [2(i+j), (m+n) - ij(2j-1)]$ .

Theorem 2.5. The system  $M$  is an M-loop which is not a weak inverse property loop.

Proof: We show first that the system  $M$  is a loop. It is clear that

$[0,0][2i,m] = [2i,m][0,0] = [2i,m]$ ; hence  $[0,0]$  is an identity for  $M$ .

Now let  $[2i,m]$  and  $[2j,n]$  be given. Then we have

$$[2i,m][2j-2i,n-m+i(j-i)(2(j-i)-1)] = [2j,n], \text{ and}$$

$[2j-2i,n-m+i(j-i)(2(j-i)-1)]$  is in  $M$ . It is also clear that if  $[x,y]$  is in  $M$  and if  $[2i,m][x,y] = [2j,n]$ , then we have  $x = 2j - 2i$ , and  $y = n-m+i(j-i)(2(j-i)-1)$ . It follows that  $M$  is a loop.

By Corollary 2.3, we need only show that in  $M, L(fx) =$

$R(f^0)L(x)L(f)R(f)$ , for every  $f$  and  $x$  in  $M$ , in order to show that  $M$  is an  $M$ -loop. Thus let  $f = [2i,k]$ ,  $x = [2j,q]$ , and  $y = [2r,p]$ . Then we

have  $fx = [2i+2j,(k+q)-ij(2j-1)]$ , and  $yL(fx) = fx \cdot y = [2(i+j+r),$

$(k+q+p)-2ij^2+ij-2ir^2-2jr^2+ir+jr]$ . Furthermore, we have  $f^0 =$

$[2(-i),2i^3+i^2-k]$ , and  $yR(f^0)L(x)L(f)R(f) = [2(i+j+r),$

$p+k+q+2i^3+i^2-2i^2r-ir-2jr^2+4ijr-2i^2j+rj-ij-2ij^2-2ir^2-2i^3-4ijr+4i^2j+4i^2r+ij+ir-i^2-2i^2j-2i^2r+ij+ir] = [2(i+j+r),p+q+r-2ij^2+ij-2ir^2-2jr^2+jr+ir] =$

$fx \cdot y$ . It follows that  $L(fx) = R(f^0)L(x)L(f)R(f)$ . Hence  $M$  is an  $M$ -loop.

Now let  $x = [2(1),0]$ . Then we have  $xx = [2(2),-1]$ , and

$[2(2),-1][2(-2),21] = [0,20+4(-4-1)] = [0,0]$ ; hence we see that  $(xx)^0 =$

$[2(-2),21]$ . But we also have  $[2(-2),-11][2(2),-1] = [0,-12+4(3)] =$

$[0,0]$ ; hence we have  $(xx)^\lambda = [2(-2),-11] \neq [2(-2),21] = (xx)^0$ . Hence

squares do not have unique inverses in  $M$ , and in view of Lemma 2.5,  $M$

is not a weak inverse property loop. ■

Theorem 2.6. The nucleus  $N$  of the  $M$ -loop  $M$  of Example 2.3 consists of all elements of the form  $[0,m]$ , where  $m$  is an integer. Furthermore,  $N$  is normal in  $M$ , and the factor loop  $M/N$  is the infinite cyclic group.

Proof: Let  $[2i,k]$ ,  $[2j,q]$ , and  $[2r,p]$  be any elements in  $M$  such that  $([2i,k][2j,q])[2r,p] = [2i,k]([2j,q][2r,p])$ . Then we have  $[2i+2j+2r,k+q+p-ij(2j-1)-r(i+j)(2r-1)] = [2(i+j+r),k+q+p-rj(2r-1)-i(r+j)(2(r+j)-1)]$ . This implies that  $-2ij^2+ij-2ir^2+ir-2jr^2+jr = -2jr^2+jr-2ir^2-4ijr-2ij^2+ir+ij$ . It follows that  $4ijr = 0$ , and hence we have either  $i = 0$ ,  $j = 0$  or  $r = 0$ . Hence if  $[x,y]$  is in  $N$ , then  $x = 0$ ; that is,  $N \subseteq \{[0,m]: m \text{ is an integer}\}$ . Conversely, for any  $i,j,k,q$ , and  $m$ , we have  $[0,m]([2i,k][2j,q]) = [0,m][2i+2j,k+q-ij(2j-1)] = [2i+2j,k+q+m-ij(2j-1)] = [2i,k+m][2j,q] = ([0,m][2i,k])[2j,q]$ . Hence  $[0,m]$  is in the nucleus of  $M$ . It follows that  $N = \{[0,m]: m \text{ is an integer}\}$ . Now, for each  $[x,y]$  in  $M$ , define  $[x,y]\phi = x$ . Then  $\phi$  is a well-defined mapping from  $M$  onto the set of even integers. It is clear that  $([x,y][u,v])\phi = [x,y]\phi \cdot [u,v]\phi$ . Hence  $\phi$  is a homomorphism from  $M$  onto the set of even integers with kernel  $N$ . It follows that  $N$  is normal in  $M$  and that the factor loop  $M/N$  is an infinite cyclic group since  $M/N$  is isomorphic to the even integers. ■

The above theorem shows that  $M$ -loops can have a structure that is quite close to the structure of Osborn loops even though they may not be Osborn loops.

We now show that if an  $M$ -loop also has the weak inverse property, then it must be an Osborn loop. The proof is broken down into a sequence of six lemmas and a theorem.

Lemma 2.6. Let  $G$  be an  $M$ -loop which is also a weak inverse property loop. Let  $x$  be any element of  $G$ . Then the following triples are automorphisms of  $G$ :

$$(2.16) \quad (L(x^\lambda)R^{-1}(x), L(x), L(x^\lambda))$$

$$(2.17) \quad (R(x), L(x^\lambda)R(x^\rho), R(x^\rho))$$

$$(2.18) \quad (R(x), R(x^\rho)L^{-1}(x), R(x^\rho))$$

$$(2.19) \quad (R^{-1}(x), R(x)L(x), R^{-1}(x^\lambda))$$

$$(2.20) \quad (R^{-1}(x), L^{-1}(x^\lambda)R(x), R^{-1}(x^\lambda))$$

Proof: Since  $G$  is an  $M$ -loop, the triple  $(L(x), R^{-1}(x^\rho), R^{-1}(x^\rho)L(x))$  is an autotopism of  $G$  by Theorem 2.2. Hence the triple

$$(2.21) \quad (\rho R^{-1}(x^\rho)L(x)\lambda, L(x), \rho R^{-1}(x^\rho)\lambda)$$

is an autotopism of  $G$  by Lemma 1.1.

Since  $\lambda^2$  is an automorphism of  $G$ , we have  $\rho R(x^\rho)\lambda = \rho R(x^\rho)\lambda^2\rho = \rho\lambda^2R([x^\rho]^{\lambda^2})\rho = \lambda R(x^\lambda)\rho$ . Hence we see that  $\rho R^{-1}(x^\rho)\lambda = \lambda R^{-1}(x^\lambda)\rho$ . We have also  $\lambda R^{-1}(x^\lambda)\rho = L(x^\lambda)$  by Theorem 1.2. It follows that  $\rho R^{-1}(x^\rho)L(x)\lambda = \rho R^{-1}(x^\rho)\lambda\rho L(x)\lambda = \lambda R^{-1}(x^\lambda)\rho\rho L(x)\lambda = L(x^\lambda)R^{-1}(x)$ , where again we have used Theorem 1.2. Hence the autotopism (2.21) can be written in the form  $(L(x^\lambda)R^{-1}(x), L(x), L(x^\lambda))$ , and we have established that (2.16) is an autotopism of  $G$ .

Now we apply Lemma 1.1 to the autotopism (2.8) to obtain that the triple

$$(2.22) \quad (R^{-1}(x^\rho), \lambda L(x)R(x)\rho, \lambda L(x)\rho)$$

is an autotopism of  $G$ .

Since  $\rho^2$  is an automorphism of  $G$ , it follows that  $\lambda L(x)\rho = \lambda L(x)\rho^2\lambda = \lambda\rho^2L(x^{\rho^2})\lambda = \rho L(x^{\rho^2})\lambda = R^{-1}(x^{\rho^2})$ , where we have used Theorem 1.2 at the last step. Hence we see that  $\lambda L(x)R(x)\rho = \lambda L(x)\rho\lambda R(x)\rho = \rho L(x^{\rho^2})\lambda R(x)\rho = R^{-1}(x^{\rho^2})L^{-1}(x)$ , where we have again used Theorem 1.2. Thus  $(R^{-1}(x^\rho), R^{-1}(x^{\rho^2})L^{-1}(x), R^{-1}(x^{\rho^2}))$  is an autotopism of  $G$ ; hence  $(R(x^\rho), L(x)R(x^{\rho^2}), R(x^{\rho^2}))$  is an autotopism of  $G$  for any  $x$  in  $G$ . Substituting  $x^\lambda$  for  $x$ , we obtain that  $(R(x), L(x^\lambda)R(x^\rho), R(x^\rho))$  is an autotopism of  $G$  for all  $x$  in  $G$ , and we have established that (2.17) is an autotopism of  $G$ . Applying (2.17) to the pair  $(1, y)$ , we obtain that  $x \cdot yL(x^\lambda)R(x^\rho) = yR(x^\rho)$ . Hence we have  $yL(x^\lambda)R(x^\rho)L(x) = yR(x^\rho)$  for all  $y$  in  $G$ . Thus we have  $L(x^\lambda)R(x^\rho)L(x) = R(x^\rho)$ , or  $L(x^\lambda)R(x^\rho) = R(x^\rho)L^{-1}(x)$ . It follows that the autotopism (2.17) can be written as  $(R(x), R(x^\rho)L^{-1}(x), R(x^\rho))$ , and we have established (2.18).

Now we take the inverse of (2.9) and substitute  $x^\lambda$  for  $x$  to obtain that  $(L^{-1}(x^\lambda), R(x), L^{-1}(x^\lambda)R(x))$  is an autotopism of  $G$ . Since (2.16) is an autotopism of  $G$ , it follows that  $(L^{-1}(x^\lambda), R(x), L^{-1}(x^\lambda)R(x))(L(x^\lambda)R^{-1}(x), L(x), L(x^\lambda)) = (R^{-1}(x), R(x)L(x), L^{-1}(x^\lambda)R(x)L(x^\lambda))$  is an autotopism of  $G$ . By Corollary 2.1, with  $f$  replaced by  $x^\lambda$  in (2.10), we have  $R^{-1}(x)L(x^\lambda) = L(x^\lambda)R(x^\lambda)$ . Hence we have  $L^{-1}(x^\lambda)R(x) = [R^{-1}(x)L(x^\lambda)]^{-1} = R^{-1}(x^\lambda)L^{-1}(x^\lambda)$ . It follows that  $L^{-1}(x^\lambda)R(x)L(x^\lambda) = R^{-1}(x^\lambda)L^{-1}(x^\lambda)L(x^\lambda) = R^{-1}(x^\lambda)$  and that  $(R^{-1}(x), R(x)L(x), R^{-1}(x^\lambda))$  is an autotopism of  $G$ ; hence we have established that (2.19) is an autotopism

of  $G$ . Now by (2.12) of Corollary 2.1, we have  $R(x)L(x) = L^{-1}(x^\lambda)R(x)$ . Hence  $(R^{-1}(x), L^{-1}(x^\lambda)R(x), R^{-1}(x^\lambda))$  is an autotopism of  $G$ , and we have established that (2.20) is an autotopism of  $G$ . ■

Lemma 2.7. Let  $G$  be a weak inverse property loop. Then  $G$  is an M-loop if and only if, for every  $f$  and  $x$  in  $G$ ,

$$(2.23) \quad L(fx) = L(x)R^{-1}(x^0)L(f)R(x^0).$$

Proof: Suppose that  $G$  is an M-loop. Then by (2.18),  $(R(x), R(x^0)L^{-1}(x), R(x^0))$  is an autotopism of  $G$  for every  $x$  in  $G$ . It follows that we have the identity  $fx \cdot yR(x^0)L^{-1}(x) = (fy)R(x^0)$  for every  $f$  and  $y$  in  $G$ . Substituting  $yL(x)R^{-1}(x^0)$  for  $y$  in the above identity, we obtain that  $fx \cdot y = (f \cdot yL(x)R^{-1}(x^0))x^0$ , or  $yL(fx) = yL(x)R^{-1}(x^0)L(f)R(x^0)$ . Since  $y$  is any element in  $G$ , we have (2.23).

Now suppose that (2.23) holds for every  $f$  and  $x$  in  $G$ . Then we have  $fx \cdot y = (f \cdot (xy)R^{-1}(x^0))x^0$  for every  $y$  in  $G$ . Substituting  $yR(x^0)L^{-1}(x)$  for  $y$  in the above identity, we obtain the identity  $fx \cdot yR(x^0)L^{-1}(f) = (fy)x^0$ . It follows that  $(R(x), R(x^0)L^{-1}(x), R(x^0))$  is an autotopism of  $G$  for every  $x$  in  $G$ . Hence  $(R(x^0), R(x^{0^2})L^{-1}(x^0), R(x^{0^2}))$  is an autotopism of  $G$  for every  $x$  in  $G$ . By Lemma 1.1,  $(\rho R^{-1}(x^{0^2})\lambda, R^{-1}(x^0), \rho L(x^0)R^{-1}(x^{0^2})\lambda)$  is an autotopism of  $G$ . Since  $\lambda^2$  is an automorphism of  $G$ , we have  $\rho R(x^{0^2})\lambda = \rho R(x^{0^2})\lambda^2\rho = \rho\lambda^2R([x^{0^2}]^{\lambda^2})\rho = \lambda R(x)\rho = L^{-1}(x)$ , where we have used Theorem 1.2. Hence the autotopism above can be written  $(L(x), R^{-1}(x^0), R^{-1}(x^0)L(x))$ . It follows that  $G$  is an M-loop by Theorem 2.2. ■

Lemma 2.8. Let  $G$  be any loop and suppose that  $(U, I, W)$  is an autotopism of  $G$ , where  $I$  is the identity mapping. Then the element  $u = 1U$  is in the left nucleus of  $G$ .

Proof: Applying the autotopism  $(U, I, W)$  to the pair  $(x, 1)$ , we obtain that  $xU \cdot 1 = xW$ , or  $U = W$ . Now applying the autotopism  $(U, I, U)$  to the pair  $(1, x)$ , we obtain that  $u \cdot x = xU$ , or  $L(u) = U$ . Hence  $(L(u), I, L(u))$  is an autotopism of  $G$ . It follows that for any  $x$  and  $y$  in  $G$ ,  $ux \cdot y = u \cdot xy$ . Hence the element  $u$  is in the left nucleus of  $G$ . ■

Lemma 2.9. Let  $G$  be a weak inverse property  $M$ -loop. Then we have  $x\theta_x L^{-1}(x) = x^\rho x$ , and  $x^\rho x$  is in the nucleus  $N$  of  $G$  for every  $x$  in  $G$ .

Proof: Let  $x$  be in  $G$ . Then we have  $x\theta_x L^{-1}(x) = x\theta_{x^\rho} L^{-1}(x) = x(x^\rho x)L^{-1}(x) = x^\rho x$  by Lemma 2.1. We now show that  $x^\rho x$  is in  $N$ . By Lemma 2.6 the triples  $(L(x^\lambda)R^{-1}(x), L(x), L(x^\lambda))$  and  $(R(x), L(x^\lambda)R(x^\rho), R(x^\rho))$  are autotopisms of  $G$ . By Theorem 2.2,  $(L(x), R^{-1}(x^\rho), R^{-1}(x^\rho)L(x))$  is an autotopism of  $G$ . Furthermore,  $(\theta_x^{-1}, \theta_x^{-1}, \theta_x^{-1})$  is an autotopism of  $G$  since  $\theta_x$  is an automorphism of  $G$ . Thus the product  $(L(x^\lambda)R^{-1}(x), L(x), L(x^\lambda))(R(x), L(x^\lambda)R(x^\rho), R(x^\rho))(L(x), R^{-1}(x^\rho), R^{-1}(x^\rho)L(x))(\theta_x^{-1}, \theta_x^{-1}, \theta_x^{-1}) = (L(x^\lambda)L(x)\theta_x^{-1}, L(x)L(x^\lambda)\theta_x^{-1}, L(x^\lambda)L(x)\theta_x^{-1}) = (L(x^\lambda)L(x)\theta_x^{-1}, I, L(x^\lambda)L(x)\theta_x^{-1})$  is an autotopism of  $G$ . Now let  $u = 1L(x^\lambda)L(x)\theta_x^{-1}$ . Then by Lemma 2.1 we see that  $u = (xx^\lambda)\theta_x^{-1} = x^{\rho^2}x^\rho$ . By Lemma 2.8,  $x^{\rho^2}x^\rho$  is in the left nucleus of  $G$ . In view of Theorem 2.3 we may conclude that  $x^\rho x$  is in the nucleus of  $G$  for every  $x$  in  $G$ . ■

Lemma 2.10. Let  $G$  be a weak inverse property M-loop and let  $x$  be in  $G$ . Then, letting  $a = x^\rho x$ , we have  $xa = x^{\lambda^2}$ ,  $ax^\lambda = x^\rho$ ,  $x^\rho a = x^\lambda$ ,  $ax = x^{\rho^2}$ ,  $xa^{-1} = ax$ ,  $a^{-1}x^\lambda = x^\lambda a$ ,  $a^{-1}x^\rho = x^\rho a$ .

Proof: We have  $xa = x(x^\rho x) = xL(x^\rho)L(x) = x\theta_{x^\rho} = x\theta_x = x^{\lambda^2}$  by Lemma 2.1. Now we also have  $1 = x^{\lambda^2} \cdot x^\lambda = xa \cdot x^\lambda$ , and, since  $G$  is a weak inverse property loop, we see that  $x \cdot ax^\lambda = 1$ . Thus we have  $ax^\lambda = x^\rho$ , since  $xx^\rho = 1 = x \cdot ax^\lambda$ . Since squares of elements in  $G$  have unique inverses by Lemma 2.5 and since  $\rho^2$  is an automorphism of  $G$ , we have  $x^\lambda x^\lambda = (x^\lambda x^\lambda)^{\rho^2} = x^\rho x^\rho = x^\rho \cdot ax^\lambda$ . By Lemma 2.9,  $a$  is in the nucleus of  $G$ . Hence  $x^\lambda x^\lambda = x^\rho \cdot ax^\lambda = x^\rho a \cdot x^\lambda$ . It follows that  $x^\rho a = x^\lambda$ . Now we have  $1 = x^\lambda x = x^\rho a \cdot x = x^\rho \cdot ax$ , which implies that  $ax = x^{\rho^2}$ . Furthermore, we have  $x^\rho \cdot xa^{-1} = x^\rho x \cdot a^{-1} = a \cdot a^{-1} = 1$ , which implies that  $xa^{-1} = x^{\rho^2} = ax$ . From  $ax^\lambda = x^\rho$ , we obtain that  $ax^\lambda a = x^\rho a = x^\lambda$ . Hence we have  $x^\lambda a = a^{-1}x^\lambda$  and  $x^\lambda a^{-1} = ax^\lambda$ . From  $x^\rho a = x^\lambda$ , it follows that  $ax^\rho a = ax^\lambda = x^\rho$ . Hence we have  $ax^\rho = x^\rho a^{-1}$  and  $x^\rho a = a^{-1}x^\rho$ . ■\*

Lemma 2.11. Let  $G$  be a weak inverse property M-loop and let  $N$  be the nucleus of  $G$ . Then  $N$  is normal in  $G$ , and the factor loop  $G/N$  is a Moufang loop.

Proof: If  $x$  is in  $G$ , then by Theorem 2.2 and Lemma 2.6,  $(L(x), R^{-1}(x^\rho), L(x)R(x))$  and  $(R(x), L(x^\lambda)R(x^\rho), R(x^\rho))$  are autotopisms of  $G$ . It follows

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\*The proof of this lemma appears in Osborn's [7] paper; however, there he assumes that  $G$  is an Osborn loop.

that  $N$  is normal in  $G$  by Lemma 1.3.

We now show that for any  $x$  and  $y$  in  $G$ , the element  $e = y\theta_x L^{-1}(y)$  is in  $N$ . In (2.16) of Lemma 2.6, we replace  $x$  by  $x^\rho$  to obtain that  $(L(x)R^{-1}(x^\rho), L(x^\rho), L(x))$  is an autotopism which we will denote by  $\alpha$ . In (2.9) of Theorem 2.2, we replace  $f$  by  $y^\lambda$  to obtain that  $(L(y^\lambda), R^{-1}(y), R^{-1}(y)L(y^\lambda))$  is an autotopism of  $G$  whose inverse we will denote by  $\beta$ . From (2.20) of Lemma (2.6) we obtain that  $(R^{-1}(x), L^{-1}(x^\lambda)R(x), R^{-1}(x^\lambda))$  is an autotopism which we will denote by  $\gamma$ . Finally, we let  $\delta$  denote the autotopism  $(L([yx]^\lambda), R^{-1}(yx), L([yx]^\lambda)R([yx]^\lambda))$  which is obtained from (2.8) by substituting  $[yx]^\lambda$  for  $f$ . Then  $\delta\alpha\beta\gamma = (L([yx]^\lambda)L(x)R^{-1}(x^\rho)L^{-1}(y^\lambda)R^{-1}(x), R^{-1}(yx)L(x^\rho)R(y)L^{-1}(x^\lambda)R(x), L([yx]^\lambda)R([yx]^\lambda)L(x)L^{-1}(y^\lambda)R(y)R^{-1}(x^\lambda))$  is an autotopism of  $G$ . Now consider the second permutation of  $\delta\alpha\beta\gamma$ , which we shall call  $V$ . From Theorem 1.2 we have  $R^{-1}(yx) = \rho L(yx)\lambda$ . Hence by Lemma 2.7, we have  $R^{-1}(yx) = \rho L(yx)\lambda = \rho L(x)R^{-1}(x^\rho)L(y)R(x^\rho)\lambda = (\rho L(x)\lambda)(\rho R^{-1}(x^\rho)\lambda)(\rho L(y)\lambda)(\rho R(x^\rho)\lambda)$ . Since  $\lambda^2$  is an automorphism of  $G$ , we have  $\rho R(x^\rho)\lambda = \rho R(x^\rho)\lambda^2\rho = \rho\lambda^2 R([x^\rho]^\lambda)\rho = \lambda R(x^\lambda)\rho$ . Hence we have  $\rho R^{-1}(x^\rho)\lambda = \lambda R^{-1}(x^\lambda)\rho$  and  $(\rho L(x)\lambda)(\rho R^{-1}(x^\rho)\lambda)(\rho L(y)\lambda)(\rho R(x^\rho)\lambda) = (\rho L(x)\lambda)(\lambda R^{-1}(x^\lambda)\rho)(\rho L(y)\lambda)(\lambda R(x^\lambda)\rho) = R^{-1}(x)L(x^\lambda)R^{-1}(y)L^{-1}(x^\lambda)$  by Theorem 1.2. It follows that  $V = R^{-1}(x)L(x^\lambda)R^{-1}(y)L^{-1}(x^\lambda)L(x^\rho)R(y)L^{-1}(x^\lambda)R(x)$ .

Now if  $a = x^\rho x$ , then by the preceding lemma we have  $ax^\lambda = x^\rho$ , and  $a$  is in  $N$ . Hence we have  $zL(x^\rho) = zL(ax^\lambda) = ax^\lambda \cdot z = a \cdot x^\lambda z = zL(x^\lambda)L(a)$  for any  $z$  in  $G$ . Thus we see that  $L(x^\rho) = L(ax^\lambda) = L(x^\lambda)L(a)$ . Furthermore, we have  $zL(a)R(y) = az \cdot y = a \cdot zy = zR(y)L(a)$  for all  $y$  and  $z$  in  $G$ ; hence we have  $L(a)R(y) = R(y)L(a)$  for all  $y$  in  $G$ .

It follows that  $V = R^{-1}(x)L(x^\lambda)R^{-1}(y)L^{-1}(x^\lambda)L(x^\lambda)L(a)R(y)L^{-1}(x^\lambda)R(x) = R^{-1}(x)L(x^\lambda)R^{-1}(y)L(a)R(y)L^{-1}(x^\lambda)R(x) = R^{-1}(x)L(x^\lambda)R^{-1}(y)R(y)L(a)L^{-1}(x^\lambda)R(x) = R^{-1}(x)L(x^\lambda)L(a)L^{-1}(x^\lambda)R(x) = R^{-1}(x)L(x^\rho)L^{-1}(x^\lambda)R(x)$ . But since  $x^\lambda = x^\rho a$  by the preceding lemma, we have  $zL(x^\lambda) = zL(x^\rho a) = x^\rho a \cdot z = x^\rho \cdot az = zL(a)L(x^\rho)$  for any  $z$  in  $G$ .

Hence we may write  $L^{-1}(x^\lambda) = L^{-1}(x^\rho a) = (L(a)L(x^\rho))^{-1} = L^{-1}(x^\rho)L^{-1}(a)$ .

It follows that  $V = R^{-1}(x)L(x^\rho)L^{-1}(x^\rho)L^{-1}(a)R(x) = R^{-1}(x)L^{-1}(a)R(x) = (L(a)R(x))^{-1}R(x) = (R(x)L(a))^{-1}R(x) = L^{-1}(a)R^{-1}(x)R(x) = L^{-1}(a)$ .

It follows that  $\delta\alpha\beta\gamma = (L([yx]^\lambda)L(x)R^{-1}(x^\rho)L^{-1}(y^\lambda)R^{-1}(x), L^{-1}(a), L([yx]^\lambda)R([yx]^\lambda)L(x)L^{-1}(y^\lambda)R(y)R^{-1}(x^\lambda))$ . Since  $a$  is in  $N$ , we obtain that  $(L(a), I, L(a))$  is an autotopism of  $G$ . Since  $G$  is a weak inverse property loop,  $(\rho L(a)\lambda, L(a), \rho I\lambda) = (R^{-1}(a), L(a), I)$  is an autotopism of  $G$  by Lemma 1.1 and Theorem 1.2. Denoting this autotopism by  $\epsilon$ , we have  $\delta\alpha\beta\gamma\epsilon = (L([yx]^\lambda)L(x)R^{-1}(x^\rho)L^{-1}(y^\lambda)R^{-1}(x)R^{-1}(a), I, L([yx]^\lambda)R([yx]^\lambda)L(x)L^{-1}(y^\lambda)R(y)R^{-1}(x^\lambda))$  is an autotopism of  $G$ .

It follows that the element  $u = 1L([yx]^\lambda)L(x)R^{-1}(x^\rho)L^{-1}(y^\lambda)R^{-1}(x)R^{-1}(a)$  is in  $N$  by Lemma 2.8 and Theorem 2.3. Thus  $ua = 1L([yx]^\lambda)L(x)R^{-1}(x^\rho)L^{-1}(y^\lambda)R^{-1}(x)$  is in  $N$  since both  $u$  and  $a$  are in  $N$ . It follows that  $x[yx]^\lambda = (y^\lambda(ua \cdot x)) \cdot x^\rho = (y^\lambda ua \cdot x)x^\rho = (y^\lambda ua)\theta_x^{-1}$ . Now we have  $[yx]^\lambda = [yx]^\rho b$ , for some  $b$  in  $N$  by the preceding lemma. Hence we see that  $y^\rho b = x(yx)^\rho b = x(yx)^\lambda = (y^\lambda ua)\theta_x^{-1}$ . Thus we have  $(y^\rho b)\theta_x = y^\lambda \cdot ua$ , or  $(y\theta_x)^\rho \cdot b = (y\theta_x)^\rho \cdot b\theta_x = (y^\rho b)\theta_x = y^\lambda ua$ , or  $(y\theta_x)^\rho = y^\lambda uab^{-1}$ . Now we have  $y^\lambda = y^\rho c$  for some  $c$  in  $N$ ; hence we have  $y^\rho \theta_x = y^\rho \cdot cuab^{-1}$ . Since  $y$  may be any element in  $G$ , we have  $y\theta_x = ye$  for some  $e$  in  $N$ .

We now show that the factor loop  $G/N$  is a Moufang loop. By Corollary 2.4,  $G/N$  is an M-loop. Now suppose  $xN$  is in  $G/N$  and consider  $\theta_{xN}$ , an automorphism of  $G/N$ . For any  $yN$  in  $G/N$ , we have  $(yN)\theta_{xN} := (yN)L(xN)L([xN]^\lambda) = (xN)^\lambda[(xN)(yN)] = (x^\lambda \cdot xy)N = (y\theta_x)N = (y \cdot y\theta_x L^{-1}(y))N = yN \cdot (y\theta_x L^{-1}(y))N$ . But we have just shown that  $y\theta_x L^{-1}(y)$  is in  $N$ . Hence we have  $(yN)\theta_{xN} = yN$ . It follows that  $\theta_{xN}$  is the identity mapping in  $G/N$  for all  $xN$  in  $G/N$ . Thus it is obvious that  $G/N$  is a Moufang Loop. ■

Theorem 2.7. A weak inverse property M-loop  $G$  is an Osborn loop.

Proof: We shall show first that the triple  $(L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda), I, L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda))$  is an autotopism of  $G$  for all  $f$  and  $g$  in  $G$ . Let  $N$  be the nucleus of  $G$ . Let  $u = L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda)$ . Then we have  $[fg]^\lambda \cdot u = g^\lambda f^\lambda$ . Suppose  $\phi$  is the natural homomorphism of  $G$  onto  $G/N$ ; then  $(g\phi)^\lambda \cdot (f\phi)^\lambda = [f\phi \cdot g\phi]^\lambda \cdot u\phi$ . By the preceding lemma,  $G/N$  is a Moufang loop. Hence  $G/N$  is an inverse property loop, which implies that  $(g\phi)^\lambda \cdot (f\phi)^\lambda = (g\phi)^{-1} \cdot (f\phi)^{-1} = [f\phi \cdot g\phi]^{-1} = [f\phi \cdot g\phi]^\lambda$ . It follows that  $u\phi$  is the identity element in  $G/N$ . Thus we see that  $u$  is in  $N$ . Hence  $(L(u), I, L(u))$  is an autotopism of  $G$ . Let  $x$  be any element in  $G$ . Then we have  $xL(g^\lambda f^\lambda)L^{-1}([fg]^\lambda) = (g^\lambda f^\lambda \cdot x)L^{-1}([fg]^\lambda) = ([fg]^\lambda u \cdot x)L^{-1}([fg]^\lambda) = ([fg]^\lambda \cdot ux)L^{-1}([fg]^\lambda) = ux = xL(u)$ . It follows that  $L(u) = L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda)$  and that

$$(2.24) \quad (L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda), I, L(g^\lambda f^\lambda)L^{-1}([fg]^\lambda))$$

is an autotopism of  $G$  for every  $f$  and  $g$  in  $G$ .

By Corollary 2.3 we have  $L(g^\lambda f^\lambda) = R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)$ . Hence the autotopism (2.24) may be written in the form

$$(2.25) \quad (R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda), I,$$

$$R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda)).$$

Applying this autotopism to the pair  $(x, g)$ , we obtain that

$$xR(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda) \cdot g = (xg)R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda)$$

$$\text{or that } xR(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda)R(g) =$$

$$xR^2(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda). \text{ Since } x \text{ may be any element in } G, \text{ we}$$

$$\text{have } R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda)R(g) = R^2(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda),$$

$$\text{or } L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda)R(g) = R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda).$$

It follows that (2.25) may be written in the form

$$(2.26) \quad (R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda), I,$$

$$L(f^\lambda)L(g^\lambda)R(g^\lambda)L^{-1}([fg]^\lambda)R(g)).$$

By Corollary 2.3 and Lemma 2.7, we have  $R(g)L(f^\lambda)L(g^\lambda)R(g^\lambda) =$

$$L(g^\lambda f^\lambda) = L(f^\lambda)R^{-1}(f)L(g^\lambda)R(f). \text{ Using this identity on the first permu-}$$

tation of (2.26) and using on the last permutation of (2.26) the fact

$$\text{that } L(f^\lambda)R(g^\lambda) = R^{-1}(g)L(f^\lambda), \text{ which we obtain from (2.10) of Corollary}$$

2.1, we obtain that

$$(2.27) \quad (L(f^\lambda)R^{-1}(f)L(g^\lambda)R(f)L^{-1}([fg]^\lambda), I,$$

$$L(f^\lambda)R^{-1}(g)L(g^\lambda)L^{-1}([fg]^\lambda)R(g)).$$

Let the inverse of (2.27) be denoted by  $\delta$ . Let  $\gamma$  be the autotopism  $(L(g^\lambda), R^{-1}(g), R^{-1}(g)L(g^\lambda))$  which is obtained from (2.9) by substituting  $g^\lambda$  for  $f$ . Let  $\beta$  be the autotopism  $(L(f^\lambda)R^{-1}(f), L(f), L(f^\lambda))$  which may be derived from (2.16) by substituting  $f$  for  $x$ . Then the autotopism  $\delta\beta\gamma$  is  $(L([fg]^\lambda)R^{-1}(f)L^{-1}(g^\lambda)R(f)L^{-1}(f^\lambda)L(f^\lambda)R^{-1}(f)L(g^\lambda), L(f)R^{-1}(g), R^{-1}(g)L([fg]^\lambda)L^{-1}(g^\lambda)R(g)L^{-1}(f^\lambda)L(f^\lambda)R^{-1}(g)L(g^\lambda)) = (L([fg]^\lambda)R^{-1}(f), L(f)R^{-1}(g), R^{-1}(g)L([fg]^\lambda))$ . This autotopism is (1.12); hence  $G$  is an Osborn loop. ■

We conclude this chapter with a few remarks concerning the normality of the nucleus of  $M$ -loops. All known examples of  $M$ -loops have normal nuclei; but it is not known whether this must be the case. We have seen that for any Osborn loop  $G$ ,  $L(fx) = L(x)R^{-1}(x^0)L(f)R(x^0)$  for any  $f$  and  $x$  in  $G$ . This identity does not hold in Example 2.2; however, we have the following theorem.

Theorem 2.8. Let  $G$  be an  $M$ -loop and suppose that  $L(fx) = L(x)R^{-1}(x^0)L(f)R(x^0)$  for every  $f$  and  $x$  in  $G$ . Then the nucleus  $N$  is normal in  $G$ .

Proof: Let  $f, x$ , and  $y$  be in  $G$ . Then we have the identity  $fx \cdot y = (f \cdot (xy)R^{-1}(x^0))x^0$  by hypothesis. Substituting  $yR(x^0)L^{-1}(x)$  for  $y$  in

this identity, we obtain the identity  $fx \cdot yR(x^0)L^{-1}(x) = (fy) \cdot x^0$  for all  $f, y$  in  $G$ . Hence  $(R(x), R(x^0)L^{-1}(x), R(x^0))$  is an autotopism of  $G$ . Since  $(L(x), R^{-1}(x^0), L(x)R(x))$  is also an autotopism of  $G$  by Theorem 2.2; we have that  $N$  is normal in  $G$  by Lemma 1.3. ■

## CHAPTER III

## THE STRUCTURE OF OSBORN LOOPS

The main purpose of this chapter is to strengthen the result of Theorem 1.8. If we consider Osborn's loop  $H$  (Example 1.1), we see that it contains a normal subloop  $A$ , generated by the element  $a = [0,1]$ , such that the quotient loop  $H/A$  is Moufang. This subloop is a proper subloop of the nucleus. In this chapter we show that the nucleus of any Osborn loop  $G$  contains two subloops, one contained within the other, such that both subloops are normal in  $G$ ; the quotient loop of  $G$  over the larger subloop is Moufang; and the quotient loop over the smaller loop is an Osborn loop in which the left inverse of any element  $x$  is equal to that element's right inverse. We shall also show that these subloops are minimal in the sense that they are the smallest normal subloops which yield quotient loops having the properties mentioned above. We then investigate structure of these two subloops and present some results concerning their relationship to one another.

Definition 3.1. Let  $G$  be a loop and let  $A$  be the subloop of  $G$  generated by the set  $\{x^0x : x \in G\}$ . Then  $A$  is said to be the *inner canonical subloop* of  $G$ .

Definition 3.2. Let  $G$  be a loop and let  $E$  be the subloop of  $G$  generated by the set  $\{y\theta_x L^{-1}(y) : x, y \in G \text{ and } \theta_x = L(x)L(x^\lambda)\}$ . Then  $E$  is said to be the *outer canonical subloop* of  $G$ .

Note that  $x\theta_{x\rho}L^{-1}(x) = x^{\rho}x$  is in  $E$  for every  $x$  in  $G$ . Hence  $A$  is a subloop of  $E$ .

Lemma 3.1. Let  $G$  be an Osborn loop. Then the outer canonical subloop  $E$  of  $G$  is contained in the center of the nucleus of  $G$ .

Proof: Let  $N$  be the nucleus of  $G$  and consider the factor loop  $G/N$ . Let  $\phi$  be the natural homomorphism from  $G$  onto  $G/N$  and let  $x, y$  be in  $G$ . Then we have  $[y\theta_x L^{-1}(y)]\phi = [yL(x)L(x^\lambda)L^{-1}(y)]\phi = [(x^\lambda \cdot xy)L^{-1}(y)]\phi = [(x\phi)^\lambda \cdot (x\phi)(y\phi)]L^{-1}(y\phi)$ . Now  $G/N$  is a Moufang loop by Theorem 1.8. It follows that  $[(x\phi)^\lambda \cdot (x\phi)(y\phi)]L^{-1}(y\phi) = (y\phi)L^{-1}(y\phi) = 1$ , the identity of  $G/N$ . Hence  $y\theta_x L^{-1}(y)$  is in  $N$ ; that is,  $E$  is contained in the nucleus  $N$  of  $G$  since the generators of  $E$  are in  $N$ .

Now let  $b$  be in  $N$  and let  $e$  be a generator of  $E$  of the type specified in Definition 3.2. Then we have  $e = y\theta_x L^{-1}(y)$  for some  $x$  and  $y$  in  $G$ . The mapping  $L(y)R^{-1}(y)$  has the property that  $1L(y)R^{-1}(y) = 1$ . Hence  $L(y)R^{-1}(y)$  is an inner mapping of  $G$ , and the element  $f = bL(y)R^{-1}(y)$  is in the nucleus of  $G$  since  $N$  is normal in  $G$ . It follows that  $y \cdot eb = ye \cdot b = y\theta_x \cdot b = y\theta_x \cdot b\theta_x = (yb)\theta_x = (fy)\theta_x = f\theta_x \cdot y\theta_x = f \cdot ye = fy \cdot e = yb \cdot e = y \cdot be$ , where we have made use of Theorem 1.9 in asserting that  $b\theta_x = b$  and that  $f\theta_x = f$ . It follows that  $be = eb$ . Thus the generators of  $E$  commute with every element in  $N$ . Now suppose  $S = \{ge \in E: gb = bg \text{ for all } b \in N\}$ . Then we have  $gh \cdot b = g \cdot hb = g \cdot bh = gb \cdot h = bg \cdot f = b \cdot gh$  for all  $g, h$  in  $S$  and all  $b$  in  $N$ . Hence we see that  $gh$  is in  $S$ . Clearly  $e^{-1}$  is in  $S$  if  $e$  is in  $S$ . Hence  $S$  is a group. Since  $S$  contains a set which generates  $E$ , we have  $S = E$ . ■

Lemma 3.2. Let  $G$  be a loop and let  $M$  be a subloop of the nucleus of  $G$ . Suppose that  $S$  is a generating set for  $M$  such that  $Sg \subseteq gM$  and  $gS \subseteq Mg$  for all  $g$  in  $G$ . Then  $M$  is normal in  $G$ .

Proof: We show first that  $Mg = gM$  for all  $g$  in  $G$ . Let  $Q$  be the set  $\{a \in M: (ay)L^{-1}(y) \in M \text{ for all } y \in G\}$ . Note that  $S$  is contained in  $Q$ . Suppose that  $b, c$  are in  $Q$ . Then, for any  $y$  in  $G$ , we have  $(bc \cdot y)L^{-1}(y) = (b \cdot cy)L^{-1}(y) = (b \cdot [y \cdot (cy)L^{-1}(y)])L^{-1}(y) = [by \cdot ((cy)L^{-1}(y))]L^{-1}(y) = [(y \cdot ((by)L^{-1}(y))) \cdot ((cy)L^{-1}(y))]L^{-1}(y) = [y \cdot ((by)L^{-1}(y)) \cdot ((cy)L^{-1}(y))]L^{-1}(y) = [(by)L^{-1}(y)][(cy)L^{-1}(y)]$  and  $[(by)L^{-1}(y)][(cy)L^{-1}(y)]$  is in  $M$  since  $(by)L^{-1}(y)$  and  $(cy)L^{-1}(y)$  are in  $M$  and  $M$  is a subloop of the nucleus. It follows that  $bc$  is in  $Q$ . Now consider  $b^{-1}$  for any  $b$  in  $Q$ . We have  $(b^{-1}y) = y \cdot (b^{-1}y)L^{-1}(y)$  for any  $y$  in  $G$ . Hence we see that  $y = b \cdot b^{-1}y = b \cdot (y \cdot (b^{-1}y)L^{-1}(y)) = by \cdot (b^{-1}y)L^{-1}(y) = (y \cdot (by)L^{-1}(y)) \cdot (b^{-1}y)L^{-1}(y) = y[(by)L^{-1}(y) \cdot (b^{-1}y)L^{-1}(y)]$  since  $(by)L^{-1}(y)$  is in the nucleus of  $G$ . Hence we have  $(b^{-1}y)L^{-1}(y) = [(by)L^{-1}(y)]^{-1}$  and  $(b^{-1}y)L^{-1}(y)$  is in  $M$ . It follows that  $Q$  is a subgroup of  $M$ . But we have  $S \subseteq Q$  since  $Sg \subseteq gM$  for all  $g$  in  $G$ . Thus  $Q$  equals  $M$  since  $S$  generates  $M$ , and we see that  $Mg \subseteq gM$  for all  $g$  in  $G$ . In like manner, one can show that  $gM \subseteq Mg$  for all  $g$  in  $G$ ; thus we have  $gM = Mg$  for all  $g$  in  $G$ .

Bruck [2] has shown that the inner mapping group  $I$  of  $G$  is generated by all of the permutations of the form  $R(x,y) = R(x)R(y)R^{-1}(xy)$ ,  $T(x,y) = R(y)L(x)R^{-1}(xy)$ , where  $x, y$  are in  $G$ . Thus to show that  $M$  is normal, we need only show that  $M$  is mapped into itself by all such permutations and their inverses. Thus let  $a$  be in  $M$ . If  $x, y$  are in  $G$ , then

we have  $aR(x,y) = (ax \cdot y)R^{-1}(xy) = (a \cdot xy)R^{-1}(xy) = a$ , or  $aR(x,y) = a$ . Thus we have  $aR^{-1}(x,y) = a$ . It follows that  $(M)R(x,y) \subseteq M$  and  $(M)R^{-1}(x,y) \subseteq M$ . Furthermore, we have  $aT(x,y) = aR(y)L(x)R^{-1}(xy) = (x \cdot ay)R^{-1}(xy) = (xa \cdot y)R^{-1}(xy) = (((xa)R^{-1}(x) \cdot x)y)R^{-1}(xy)$ . Now we have shown that  $Mx = xM$  for all  $x$  in  $G$ ; hence we have  $(xa)R^{-1}(x)$  in  $M$ . It follows that  $(xa)R^{-1}(x)$  is in the nucleus and that  $aT(x,y) = (((xy)R^{-1}(x) \cdot x)y)R^{-1}(xy) = ((xa)R^{-1}(x) \cdot xy)R^{-1}(xy) = (xa)R^{-1}(x)$  and  $(xa)R^{-1}(x)$  is in  $M$ . Thus we see that  $MT(x,y) \subseteq M$ .

Now we also have  $aT^{-1}(x,y) = (a \cdot xy)L^{-1}(x)R^{-1}(y) = (ax \cdot y)L^{-1}(x)R^{-1}(y) = ((x \cdot (ax)L^{-1}(x))y)L^{-1}(x)R^{-1}(y) = (x \cdot ((ax)L^{-1}(x) \cdot y))L^{-1}(x)R^{-1}(y) = (ax)L^{-1}(x)$  since  $(ax)L^{-1}(x)$  is in  $M$  and the nucleus. It follows that  $MT^{-1}(x,y) \subseteq M$ . Hence  $M$  is normal in  $G$ . ■

The proof of the above lemma is contained implicitly in the proof of Theorem 1.12. We are now ready to prove that the subloops  $E$  and  $A$  are normal in  $G$ .

Theorem 3.1. Let  $G$  be an Osborn loop and let  $A$  be its inner canonical subloop of  $G$ . If  $a$  is in  $A$  and  $r$  is in  $G$ , then we have

$$(3.1) \quad a(rr) = (rr)a.$$

Furthermore,  $A$  is normal in  $G$ , and the factor loop  $G/A$  is an Osborn loop in which inverses are unique; that is, the left inverse of any element equals that element's right inverse.

Proof: Suppose that  $a$  is a generator of  $A$  of the type described in Definition 3.1 and that  $r$  is in  $G$ . Then by Lemma 2.10 and Theorem 1.9, we have  $sa = s\theta_s$  for some  $s$  in  $G$ . Now we see that  $(s \cdot rr)\theta_{s \cdot rr} = (s \cdot rr)^{\lambda^2} = s^{\lambda^2} \cdot rr = sa \cdot rr = s \cdot a(rr)$ , where we have used Theorem 1.9, and the fact that squares have unique inverses in the Osborn loop  $G$ . By definition of  $A$  we have  $(s \cdot rr)\theta_{s \cdot rr} = (s \cdot rr)d$  for some  $d$  in  $A$ . Using Theorem 1.9, we obtain that  $(s \cdot rr) = d(s \cdot rr)^{\lambda^2} = d \cdot (s \cdot rr)d$ . Hence we have  $d^{-1}(s \cdot rr) = (s \cdot rr)d$ . It follows that  $sa \cdot rr = (s \cdot rr)\theta_{s \cdot rr} = (s \cdot rr)d = d^{-1}(s \cdot rr) = d^{-1}s \cdot rr$ . Hence we have  $d^{-1}s = sa$ . But we also have  $asa = as^{\lambda^2} = s$  by Theorem 1.9. Hence we see that  $a^{-1}s = sa$ . It follows that  $d^{-1}s \cdot rr = sa \cdot rr = a^{-1}s \cdot rr$ , or  $d = a$ . Hence we have  $s \cdot a(rr) = sa \cdot rr = a^{-1}s \cdot rr = d^{-1}(s \cdot rr) = (s \cdot rr)d = s \cdot (rr)a$ , or  $a(rr) = (rr)a$ .

Now let  $S = \{g \in A: g(rr) = (rr)g \text{ for all } r \text{ in } G\}$ . We have shown that a generating set for  $A$  is  $S$ . If  $a, b$  are in  $S$ , then we have  $ab \cdot rr = a \cdot b(rr) = a \cdot (rr)b = a(rr) \cdot b = (rr)a \cdot b = (rr) \cdot ab$ , or  $ab$  is in  $S$ . Similarly, we have  $a^{-1}$  in  $S$  if  $a$  is in  $S$ .

It follows that  $S$  is a group and that  $S = A$ , since  $S$  contains a set which generates  $A$ . We have now established that (3.1) holds.

We can now prove that  $A$  is normal in  $G$ . Again let  $a$  be a generator of  $A$  of the type described in Definition 3.1; then we have  $a = x^{\rho}x$  for some  $x$  in  $G$ . Let  $y$  be in  $G$  and set  $b = y^{\rho}y$  and  $c = (xy)^{\rho}(xy)$ . Then we have  $(xy)c = (xy)\theta_{xy} = (xy)^{\lambda^2} = x^{\lambda^2}y^{\lambda^2} = xa \cdot yb = x \cdot ayb$  since  $\lambda^2$  is an automorphism of  $G$ . It follows that  $ay = ycb^{-1}$ . Since  $a(yy) = (yy)a$ , we have  $ay \cdot y = y \cdot ya$  or  $ycb^{-1} \cdot y = y \cdot ya$ . Thus we see that

$cb^{-1}y = ya$ . Now we have shown that if  $a$  is a generator of  $A$  of the type described above and if  $y$  is in  $G$ , then  $ay = y\mu$  and  $ya = \mu y$ , where  $\mu$  is in  $A$ . It follows that the set  $S = \{x^{\circ}x : x \in G\}$  has the property that  $Sg \subseteq gA$  and  $gS \subseteq Ag$  for all  $g$  in  $G$ . Thus  $A$  is normal in  $G$  by Lemma 3.2. Now consider the factor loop  $G/A$ . We have  $(xA)^{\lambda^2} = x^{\lambda^2}A = (x\theta_x)A = (x\theta_x)A = x(x^{\circ}x)A = xA$  for any  $xA$  in  $G/A$ . Hence we have  $(xA)^{\lambda} = (xA)^{\circ}$ . ■

We now show that  $A$  is minimal, in a certain sense.

Theorem 3.2. Let  $G$  be an Osborn loop with inner canonical subloop  $A$ . Let  $R$  be any normal subloop of  $G$  such that  $G/R$  is an Osborn loop in which inverses are unique. Then  $A$  is a subloop of  $R$ .

Proof: Let  $x$  be in  $G$ . We have  $(x^{\circ}x)R = (xR)^{\circ}(xR) = (xR)^{-1}(xR) = 1$ , the identity in  $G/R$ . Hence  $x^{\circ}x$  is in  $R$ . It follows that a generating set for  $A$  is contained in  $R$ ; hence  $A$  is a subloop of  $R$ . ■

We now proceed to show that  $E$  is normal in  $G$ .

Theorem 3.3. Let  $G$  be an Osborn loop. Then the outer canonical subloop  $E$  of  $G$  is normal in  $G$ , and the factor loop  $G/E$  is a Moufang loop.

Proof: Suppose that  $e$  is a generator of  $E$  of the type described in Definition 3.2. Then we have  $ye = y\theta_x$  for some  $x$  and  $y$  in  $G$ . Now let  $z$  be such that  $zy = x$ . Then we have  $(zy)\theta_x = (zy)\theta_{zy} = zy \cdot a$  for some  $a$  in  $A$ , the inner canonical subloop of  $G$ . Hence we have  $(zy)a = (zy)\theta_x = z\theta_x \cdot ye$ . Let  $f = z\theta_x L^{-1}(z)$ . Then  $f$  is in  $E$  and  $zf \cdot ye = zy \cdot a$ , or

$fye = ya$ . By Theorem 3.1,  $A$  is normal in  $G$ ; hence the element  $g = aL(y)R^{-1}(y)$  is in  $A$  since  $L(y)R^{-1}(y)$  is an inner mapping of  $G$ . It follows that  $fye = gy$ . Hence we have  $ye = f^{-1}gy$ , where  $f^{-1}g$  is in  $E$ .

Now let  $h$  be such that  $(yy)\theta_x = ye \cdot ye = yy \cdot h$ . Then  $h$  is in  $E$  and we see that  $ey = yhe^{-1}$ .

Now let  $r$  be any element in  $G$ . Let  $w$  be such that  $r = wy$ . Then we have  $re = wy \cdot e = w \cdot y\theta_x = (w\theta_x^{-1} \cdot y)\theta_x = h \cdot (w\theta_x^{-1} \cdot y)$ , and  $h$  is in  $E$  by the preceding part of the proof. Furthermore,  $w\theta_x$  can be written as  $kw$  for some  $k$  in  $E$ . Hence we have  $w = k \cdot w\theta_x^{-1}$ , or  $kw^{-1} = w\theta_x^{-1}$ . It follows that  $re = h(w\theta_x^{-1} \cdot y) = h(k^{-1}w \cdot y) = hk^{-1} \cdot wy = hk^{-1} \cdot r$ . We have shown that  $re = hk^{-1} \cdot r$ , where  $hk^{-1}$  is in  $E$ . Now consider  $(yr)\theta_x = yr \cdot \ell$  for some  $\ell$  in  $E$ . Then we have  $y\theta_x \cdot r\theta_x = yr \cdot \ell$ , or  $ye \cdot nm = yr \cdot \ell$ , where  $nm = r\theta_x$ . Thus we see that  $m$  is in  $E$  and  $enm = r\ell$ , or  $er = r\ell m^{-1}$  and  $\ell m^{-1}$  is in  $E$ . It follows that  $eg$  is in  $gE$  and  $ge$  is in  $Eg$  for all  $g$  in  $G$ . Thus we see that  $E$  is normal in  $G$  by Lemma 3.2.

Now consider the factor loop  $G/E$ . Let  $xE$  and  $yE$  be any two elements in  $G/E$ . Then we have  $(xE)^\lambda [(xE)(yE)] = (x^\lambda \cdot xy)E = (ye)E$  for some  $e$  in  $E$ . But this implies that  $(xE)^\lambda [(xE)(yE)] = (ye)E = yE$ . It follows that the automorphisms  $\theta_x$  of the factor loop  $G/E$  are identity mappings and that  $G/E$  is a Moufang loop. ■

Theorem 3.4. Let  $G$  be an Osborn loop and let  $R$  be a normal subloop of  $G$  such that  $G/R$  is a Moufang loop. Then  $R$  contains the subloop  $E$ .

Proof: Let  $e = y\theta_x L^{-1}(y)$  for some  $x, y$  in  $G$ . Then we have  $eR = (y\theta_x L^{-1}(y))R = ([x^\lambda \cdot xy]L^{-1}(y))R = [(xR)^\lambda \cdot (xR)(yR)]L^{-1}(yR) = 1R$  since  $G/R$  is a Moufang loop. It follows that  $e$  is in  $R$ . Since these  $e$ 's generate  $E$ , we have  $E \subseteq R$ . ■

We now turn our attention to the relationship of these two subloops to one another.

Lemma 3.3. Let  $G$  be an Osborn loop. Then the following identities hold for any  $x, y$  in  $G$ .

$$(3.2) \quad \theta_x \theta_y = \theta_y \theta_x$$

$$(3.3) \quad y\theta_{xy}^{-1} = y\theta_y^{-1}\theta_x$$

$$(3.4) \quad x\theta_{xy}^{-1} = x\theta_x^{-1}\theta_y$$

$$(3.5) \quad xy = x\theta_y \cdot y\theta_x$$

$$(3.6) \quad y\theta_{xx} = y$$

$$(3.7) \quad (yy)\theta_x = yy$$

Proof: For any  $z$  in  $G$ , we have  $z\theta_x \theta_y = zL(x)L(x^\lambda)\theta_y = (x^\lambda \cdot xz)\theta_y = (x\theta_y)^\lambda \cdot (x\theta_y \cdot z\theta_y) = (xf)^\lambda \cdot (xf \cdot z\theta_y)$  for some  $f$  in  $E$ , the outer canonical subloop of  $G$ . Thus we have  $z\theta_x \theta_y = z\theta_y \theta_{xf}$ . But, since  $f$  is in the

nucleus, we have  $\theta_{xf} = \theta_x$  by Theorem 1.9. Hence we see that  $z\theta_{xy} = z\theta_y$  for all  $z$  in  $G$ ; that is,  $\theta_{xy} = \theta_y$  and we have verified (3.2).

Now by Theorem 1.1, we have  $y(xy)^\rho = x^\rho$  since  $G$  is a weak inverse property loop. Thus we have  $y(xy)^\rho \cdot (xy)^{\rho^2} = x^\rho (xy)^{\rho^2} = x^\rho (x^{\rho^2} y^{\rho^2})$  since  $\rho^2$  is an automorphism of  $G$ . It follows that  $y\theta_{xy}^{-1} = y\theta_{(xy)^\rho}^{-1} = y(xy)^\rho \cdot (xy)^{\rho^2} = x^\rho (x^{\rho^2} y^{\rho^2}) = y^{\rho^2} \theta_{x^{\rho^2}} = y\theta_y^{-1} \theta_x$ , where we have used Theorem 1.7 and Theorem 1.9. Thus we have established (3.3).

Now we may also use Theorem 1.1 to conclude that  $(xy)^\lambda x = y^\lambda$ . Hence we have  $(xy)^\lambda \cdot (xy)^\lambda y = (xy)^\lambda y^\lambda = x^{\lambda^2} y^{\lambda^2} \cdot y^\lambda$ . It follows that  $x\theta_{xy} = x\theta_{(xy)^\lambda} = (xy)^\lambda \cdot (xy)^\lambda x = x^{\lambda^2} y^{\lambda^2} \cdot y^\lambda = x\theta_x \theta_y^{-1} = x\theta_x \theta_y^{-1}$ , where we have again used Theorem 1.7 and Theorem 1.9. Using (3.2), we obtain that  $x = x\theta_x \theta_y^{-1} \theta_y^{-1} = x\theta_{xy} \theta_x^{-1}$ , or  $x\theta_{xy}^{-1} = x\theta_x^{-1} \theta_y$ . Hence we have established (3.4).

Using all of the above identities, Theorem 1.9, and the fact that  $\rho^2$  is an automorphism, we obtain that  $(xy)^{\rho^2} = (xy)\theta_{xy}^{-1} = x\theta_{xy}^{-1} \cdot y\theta_{xy}^{-1} = x\theta_x^{-1} \theta_y \cdot y\theta_y^{-1} \theta_x = x^{\rho^2} \theta_y \cdot y^{\rho^2} \theta_x = (x\theta_y \cdot y\theta_x)^{\rho^2}$ . It follows that  $xy = x\theta_y \cdot y\theta_x$ , and we have established (3.5).

In order to calculate  $y\theta_{xx \cdot y}$ , we first calculate  $y\theta_{xx \cdot y}$ . If  $N$  is the nucleus of  $G$ , then in  $G/N$  we have  $(xx \cdot y)N = (xNxN) \cdot yN = xN \cdot (xNyN) = (x \cdot xy)N$  since the factor loop  $G/N$  is a Moufang loop. Thus, if  $n$  is such that  $(xx \cdot y) = (x \cdot xy)n$ , then we have  $(xx \cdot y)NnN = (x \cdot xy)nN = (xx \cdot y)N = (x \cdot xy)N$ . It follows that  $n$  is in  $N$ . Hence we see that  $\theta_{xx \cdot y} = \theta_{(x \cdot xy)n} = \theta_{x \cdot xy}$  by Theorem 1.9. Thus we have  $y\theta_{xx \cdot y} = y\theta_x \theta_x^{-1} \theta_{xx \cdot y} = [x^\lambda(xx)]\theta_x^{-1} \theta_{xx \cdot y} = [x^\lambda(xy)]\theta_x^{-1} \theta_{x \cdot xy} = [x^\lambda(xy)]\theta_{x \cdot xy} \theta_x^{-1}$ , where we have also used (3.2) and Theorem 1.7. Now consider  $[x^\lambda(xy)]\theta_{x \cdot xy} = x^\lambda \theta_{x \cdot xy} \cdot (xy)\theta_{x \cdot xy} = (x\theta_{x \cdot xy})^\lambda \cdot (xy)\theta_{x \cdot xy}$ . By (3.3) and (3.4) we have

$x\theta_{x \cdot xy}^{-1} = x\theta_x^{-1}\theta_{xy}$  and  $(xy)\theta_{x \cdot xy}^{-1} = (xy)\theta_{xy}^{-1}\theta_x$ . Using these identities and the identity (3.2), we obtain that  $x\theta_{x \cdot xy} = x\theta_x\theta_{xy}^{-1}$  and  $(xy)\theta_{x \cdot xy} = (xy)\theta_{xy}\theta_x^{-1}$ . Hence we have  $[x^\lambda(xy)]\theta_{x \cdot xy} = (x\theta_{x \cdot xy})^\lambda \cdot (xy)\theta_{x \cdot xy} = (x\theta_x\theta_{xy}^{-1})^\lambda \cdot (xy)\theta_{xy}\theta_x^{-1} = (x\theta_x^{-1}\theta_{xy})^\lambda \cdot (xy)\theta_{xy}\theta_x^{-1} = (x\theta_x^{-1}\theta_{xy}\theta_x)^\lambda \cdot (xy)\theta_{xy}\theta_x^{-1} = (x\theta_y)^\lambda \cdot (xy)\theta_{xy}\theta_x^{-1}$ , where we have again made use of (3.2) and (3.4). It follows that  $[x^\lambda(xy)]\theta_{x \cdot xy}\theta_x^{-1} = [(x\theta_y)^\lambda \cdot (xy)\theta_{xy}\theta_x^{-1}]\theta_x^{-1}$ . Now let  $c$ ,  $a$ , and  $b$  be such that  $xa = x\theta_x$ ,  $yb = y\theta_y$ , and  $xy \cdot c = (xy)\theta_{xy}$ . Then  $a$ ,  $b$ , and  $c$  are in the nucleus by Theorem 1.9, and we have  $y\theta_{xx \cdot y} = [x^\lambda(xy)]\theta_{x \cdot xy}\theta_x^{-1} = [(x\theta_y)^\lambda \cdot (xy)\theta_{xy}\theta_x^{-1}]\theta_x^{-1} = [(x\theta_y)^\lambda \cdot ((xy)c)\theta_x^{-1}]\theta_x^{-1}$ . Also by Theorem 1.9, we have  $((xy)c)\theta_x^{-1} = (xy)\theta_x^{-1} \cdot c\theta_x^{-1} = (xy)\theta_x^{-1} \cdot c$  since  $c$  is in the nucleus. Thus we see that  $y\theta_{xx \cdot y} = [(x\theta_y)^\lambda \cdot ((xy)c)\theta_x^{-1}]\theta_x^{-1} = [(x\theta_y)^\lambda \cdot ((xy)\theta_x^{-1} \cdot c)]\theta_x^{-1} = [((x\theta_y)^\lambda \cdot (xy)\theta_x^{-1})c]\theta_x^{-1} = [(x\theta_y)^\lambda \cdot (xy)\theta_x^{-1}]\theta_x^{-1} \cdot c\theta_x^{-1} = [(x\theta_y)^\lambda \cdot (xy)\theta_x^{-1}]\theta_x^{-1} \cdot c$ . Now by (3.5) we have that  $xy = x\theta_y \cdot y\theta_x$ . Hence we see that  $y\theta_{xx \cdot y} = [(x\theta_y)^\lambda \cdot (xy)\theta_x^{-1}]\theta_x^{-1} \cdot c = [(x\theta_y)^\lambda \cdot (x\theta_y \cdot y\theta_x)\theta_x^{-1}]\theta_x^{-1} \cdot c = [(x\theta_y)^\lambda \cdot (x\theta_y\theta_x^{-1} \cdot y)]\theta_x^{-1} \cdot c = [(x\theta_y)^\lambda \cdot (x\theta_x^{-1}\theta_y \cdot y)]\theta_x \cdot c$ , where we have again made use of (3.2). Now from  $xa = x\theta_x$ , we obtain that  $x\theta_x^{-1} \cdot a = x$ , or  $x\theta_x^{-1} = xa^{-1}$ . Furthermore, by Theorem 1.9 we have  $xy \cdot c = (xy)\theta_{xy} = (xy)^{\lambda^2} = x^{\lambda^2}y^{\lambda^2} = x\theta_x \cdot y\theta_y = xa \cdot yb$ , from which we obtain that  $ayb = yc$ , or  $a^{-1}y = ybc^{-1}$ . It follows that

$$\begin{aligned}
y\theta_{xx \cdot y} &= [(x\theta_y)^\lambda \cdot (x\theta_x^{-1}\theta_y \cdot y)]\theta_x^{-1} \cdot c \\
&= [(x\theta_y)^\lambda \cdot (xa^{-1})\theta_y \cdot y]\theta_x^{-1} \cdot c = [(x\theta_y)^\lambda \cdot ((x\theta_y \cdot a^{-1})y)]\theta_x^{-1} \cdot c \\
&= [(x\theta_y)^\lambda \cdot (x\theta_y \cdot (a^{-1}y))]\theta_x^{-1} \cdot c = [(x\theta_y)^\lambda \cdot (x\theta_y \cdot ybc^{-1})]\theta_x^{-1} \cdot c \\
&= [(x\theta_y)^\lambda \cdot ((x\theta_y \cdot y)bc^{-1})]\theta_x^{-1} \cdot c = [((x\theta_y)^\lambda \cdot (x\theta_y \cdot y))bc^{-1}]\theta_x^{-1} \cdot c \\
&= ([ (x\theta_y)^\lambda \cdot (x\theta_y \cdot y) ]\theta_x^{-1} \cdot (bc^{-1})\theta_x^{-1})c = ([ (x\theta_y)^\lambda \cdot (x\theta_y \cdot y) ]\theta_x^{-1} \cdot bc^{-1})c \\
&= [(x\theta_y)^\lambda \cdot (x\theta_y \cdot y)]\theta_x^{-1} \cdot b,
\end{aligned}$$

where we have made use of Theorem 1.9 again. Now we have  $x\theta_y = xe$  for some  $e$  in the outer canonical subloop of  $G$ . Hence  $e$  is in the nucleus, and we see that  $\theta_x \theta_y = \theta_{xe} = \theta_x$  by Theorem 1.9. It follows that  $y\theta_{xx \cdot y} = [(x\theta_y)^\lambda \cdot (x\theta_y \cdot y)]\theta_x^{-1} \cdot b = y\theta_{x\theta_x} \theta_x^{-1} \cdot b = y\theta_{x\theta_x} \theta_x^{-1} \cdot b = yb$ . By (3.3) we have  $y\theta_{xx \cdot y}^{-1} = y\theta_y^{-1} \theta_{xx}$ , or  $y = y\theta_y^{-1} \theta_{xx} \theta_{xx \cdot y} = y\theta_{xx \cdot y} \theta_y^{-1} \theta_{xx}$ , where we have used (3.2). It follows that  $y\theta_{xx \cdot y} = y\theta_y \theta_{xx}^{-1} = (yb)\theta_{xx}^{-1} = y\theta_{xx}^{-1} \cdot b\theta_{xx}^{-1} = y\theta_{xx}^{-1} \cdot b$  by Theorem 1.9 and (3.2). Thus we see that  $y\theta_{xx}^{-1} \cdot b = y\theta_{xx \cdot y} = yb$ , or  $y = y\theta_{xx}^{-1}$ . Thus we have  $y\theta_{xx} = y$ , and we have established (3.6).

By (3.5) we have  $(yy)\theta_x \cdot x\theta_{yy} = yy \cdot x$ . But by (3.6) we also have  $x\theta_{yy} = x$ , where we have interchanged  $x$  and  $y$ . Thus we have  $yy \cdot x = (yy)\theta_x \cdot x\theta_{yy} = (yy)\theta_x \cdot x$ . It follows that  $(yy)\theta_x = yy$ , and we have established (3.7). ■

Corollary 3.1. Let  $G$  be an Osborn loop. Then the subloop of  $G$  generated by the set  $S = \{x^2 : x \in G\}$  is a Moufang loop.

Proof: Let  $S' = \{y \in G : y\theta_x = y \text{ for any } x \text{ in } G\}$ . If  $r$  and  $s$  are in  $S'$ , then we have  $(rs)\theta_x = r\theta_x \cdot s\theta_x = rs$  for all  $x$  in  $G$ ; that is,  $rs$  is in  $S'$ . Now let  $u$  and  $v$  be in  $G$  such that  $ru = s$  and  $vr = s$ . Then we have  $r \cdot u\theta_x = (ru)\theta_x = s\theta_x = s$ ; that is,  $u\theta_x = u$ . Similarly, we have  $v\theta_x = v$ . Hence  $u$  and  $v$  are in  $S'$ . It follows that  $S'$  is a subloop of  $G$ . By (3.7) we have  $y^2\theta_x = y^2$  for every  $y$  in  $G$ . It follows that  $S \subseteq S'$ . Hence  $S'$  contains the loop generated by  $S$ . Thus by Theorem 1.7 we see that  $xz \cdot yx = (xz) \cdot (y\theta_x \cdot x) = (x \cdot zy)x$  for any  $x, y$ , and  $z$  in the loop generated by  $S$ ; that is, the set  $S$  generates a Moufang loop. ■

Theorem 3.5. Let  $G$  be an Osborn loop. Let  $x, y$  be in  $G$  and let  $a, b, c$ , and  $e$  be in  $G$  such that  $xa = x\theta_x$ ,  $yb = y\theta_x$ ,  $(xy)c = (xy)\theta_{xy}$ , and  $ye = y\theta_x$ . Then we have the following identities:

$$(3.8) \quad eye = y$$

$$(3.9) \quad xe = x\theta_y$$

$$(3.10) \quad e^2 = abc^{-1}$$

Furthermore, the square of any element in the outer canonical subloop  $E$  is in the inner canonical subloop  $A$ .

Proof: Observe that  $e$  is in  $E$ , and hence  $e$  is in the nucleus of  $G$ . From (3.7) we have  $yy = (yy)_{\mathbf{x}}^{\theta} = ye^{\cdot}ye = y \cdot eye$ , which implies that  $eye = y$ . Thus we have established (3.8).

Now let  $f$  be such that  $xf = x\theta_y$ . Then by (3.5) we see that  $xy = x\theta_y \cdot y\theta_x = xf \cdot ye = x \cdot fye$ , or  $y = fye$ . Since we have shown that  $y = eye$ , we have  $f = e$ , and we have established (3.9).

Now we have  $(xy)_{\mathbf{x}}^{\theta} = xa \cdot ye = x \cdot aye$ . Since  $x \cdot ayb = xa \cdot yb = x^{\lambda^2} y^{\lambda^2} = (xy)^{\lambda^2} = (xy)_{\mathbf{xy}}^{\theta} = xy \cdot c = x \cdot yc$ , we see that  $ayb = yc$ , or  $ay = ycb^{-1}$ . Hence we have  $(xy)_{\mathbf{x}}^{\theta} = x \cdot aye = xy \cdot cb^{-1}e$ . By (3.8) we have  $cb^{-1}e(xy)cb^{-1}e = xy$ , where we have read  $xy$  for  $y$  and  $cb^{-1}e$  for  $e$ . It follows that  $(xy)e^{-1}bc^{-1} = cb^{-1}e(xy)$ . By Lemma 2.10 and the fact that the outer canonical subloop  $E$  is an Abelian group, we obtain that  $(xy)e^{-1}bc^{-1} = cb^{-1}e(xy) = b^{-1}ec(xy) = b^{-1}e(xy)c^{-1}$ , or  $(xy)e^{-1}b = b^{-1}e(xy)$ . By (3.9) we are permitted to read  $x$  for  $y$  in (3.8). Hence we have  $exe = x$ , or  $xe = e^{-1}x$ . We also have  $e^{-1}y = ye$  by (3.8). Thus we see that  $b^{-1}e(xy) = (xy)e^{-1}b = e^{-1}(xy)b = e^{-1}xb^{-1}y$ , where we have again used Lemma 2.10. It follows that  $b^{-1}ex = e^{-1}xb^{-1}$ , or  $e^2b^{-1}x = xb^{-1}$ . Now from  $a^{-1}xb^{-1}y = xa \cdot yb = x\theta_x \cdot y\theta_y = x^{\lambda^2} y^{\lambda^2} = (xy)^{\lambda^2} = (xy)_{\mathbf{xy}}^{\theta} = (xy)c = c^{-1}(xy) = c^{-1}x \cdot y$ , we obtain  $a^{-1}xb^{-1} = c^{-1}x$ , or  $xb^{-1} = c^{-1}ax$ . Thus we have  $e^2b^{-1}x = xb^{-1} = ac^{-1}x$ . It follows that  $e^2 = abc^{-1}$ , and we have established (3.10).

Result (3.10) shows that the square of any generating element of  $E$  of the type described in Definition 3.2 is in  $A$ . Since  $E$  is an Abelian group, it follows that the square of any element in  $E$  is in  $A$ . ■

Theorem 3.6. Let  $G$  be an Osborn loop which is generated by a set consisting of two elements. Then the inner canonical subloop  $A$  is generated by a set containing no more than three elements, and the outer canonical subloop  $E$  is generated by a set containing no more than four elements, three of which are in the inner canonical subloop  $A$ .

Proof: Let the set  $\{x,y\}$  generate  $G$  and let  $a,b$ , and  $c$  be such that  $xa = x\theta_x$ ,  $yb = y\theta_y$ , and  $xy^2c = (xy)\theta_{xy}$ . We shall show that the subloop  $S$  generated by the set  $\{a,b,c\}$  is normal in  $G$ , and we shall make free use of Lemma 2.10 and Theorem 1.9 in the calculations that follow. From  $(xy)c = (xy)\theta_{xy} = (xy)^{\lambda^2} = x^{\lambda^2}y^{\lambda^2} = xa \cdot yb = x \cdot ayb$ , we obtain that  $ay = ycb^{-1}$ . By Theorem 3.1, we have  $a \cdot yy = yy \cdot a$ . Hence we see that  $y \cdot ya = yy \cdot a = a \cdot yy = ay \cdot y = ycb^{-1} \cdot y$ , or  $ya = cb^{-1}y$ .

From  $c^{-1} \cdot xy = xy^2c = (xy)\theta_{xy} = xa \cdot yb = a^{-1}xb^{-1}y$ , we see that  $c^{-1}ax = xb^{-1}$ , or  $xb = ca^{-1}x$ . From Theorem 3.1, we obtain that  $bx \cdot x = b \cdot xx = xx \cdot b = x \cdot xb = x \cdot ca^{-1}x = xca^{-1} \cdot x$ . Hence we have  $bx = xca^{-1}$ .

From  $ay = ycb^{-1}$  we obtain that  $yc = ayb = ab^{-1}y$ , and from  $bx = xca^{-1}$  we obtain that  $ba^{-1}x = xc$ . Similarly we may prove that  $cx = xa^{-1}b$  and  $cy = yb^{-1}a$ .

Now let  $Q = \{d \in S: dx \in xS \text{ and } xd \in Sx\}$ . We have shown that  $a,b$ , and  $c$  are in  $Q$ . Now let  $r$  and  $s$  be in  $Q$  and let  $r'$  and  $s'$  be in  $S$  such that  $rx = xr'$  and  $sx = xs'$ . Then we have  $rs \cdot x = r \cdot sx = r \cdot xs' = rx \cdot s' = xr' \cdot s' = x \cdot r's'$ . We have just shown that  $(rs \cdot x)L^{-1}(x)$  is in  $S$ . Similarly, we can show that  $(x \cdot rs)R^{-1}(x)$  is in  $S$ . Hence we see that  $rs$  is in  $Q$ . Similarly, we may show that  $r^{-1}$  is in  $Q$ . It follows that  $Q$

is a loop and that  $Q = S$ . We have shown  $xS = Sx$ . We may also show that  $yS = Sy$ .

Now let  $Q'$  be the set  $\{z \in G: zS = Sz\}$ . Let  $r$  and  $s$  be in  $Q$ . Let  $d$  be any element in  $S$ . Then  $d \cdot rs = dr \cdot s = rd' \cdot s = r \cdot d's = r \cdot sd'' = rs \cdot d''$ , where  $d'$  and  $d''$  are in  $S$  such that  $dr = rd'$  and  $d's = sd''$ . Thus we see that  $d \cdot rs$  is in  $(rs)S$ . Since  $d$  was chosen arbitrarily in  $S$ , we have  $(rs)S \supseteq S(rs)$ . We may also show that  $(rs)S = S(rs)$ . Hence we have  $rs$  in  $Q'$ . Now let  $v$  be in  $G$  such that  $rv = s$ . Let  $d$  be in  $S$ . Then we have  $r \cdot vd = rv \cdot d = sd = d's = d' \cdot rv = d' \cdot r \cdot v = rd'' \cdot v = r \cdot d''v$ , where  $d'$  and  $d''$  are in  $S$  such that  $d's = sd$  and  $d'r = rd''$ . Thus we see that  $vd = d''v$  and that  $vS \subseteq Sv$  since  $d$  was arbitrarily chosen in  $S$ . We may also show that  $Sv \subseteq vS$ . Hence we see that  $Sv = vS$  and that  $v$  is in  $Q'$ . In like manner we may show that, if  $v'$  is such that  $v'r = s$ , then  $v$  is in  $Q'$ . It follows that  $Q'$  is a loop. Since we have shown above that  $x$  and  $y$  are in  $Q'$ , we have  $Q' = G$ . Now by Lemma 3.2,  $S$  is normal in  $G$  since we have shown that  $zS = Sz$  for every  $z$  in  $G$ .

Now consider the factor loop  $G/S$ . It is generated by set  $\{xS, yS\}$ . Now we have  $(xS)^{\lambda^2} = x^{\lambda^2}S = x\theta_x S = xa \cdot S = xS$ . Similarly, we see that  $(yS)^{\lambda^2} = yS$ . Since  $\lambda^2$  is an automorphism, it follows that  $\lambda^2$  is the identity mapping in  $G/S$ . Hence  $G/S$  is a loop in which inverses are unique. Now by Theorem 3.2, we have  $S \supseteq A$ . But it is clear that  $A \supseteq S$ . Thus we see that  $S = A$ .

If  $z$  is any element in  $G$ , we denote  $zS$  by  $\bar{z}$ . Thus  $\bar{x}$  and  $\bar{y}$  generate  $G/S$  since  $x$  and  $y$  generate  $G$ . We shall show that the outer canonical subloop of  $G/S$  is generated by the element  $\bar{e} = \bar{y}\theta_x^{-1}(\bar{y})$ . By Theorem 3.5, we have  $\bar{e}^2 = 1$ ,  $\bar{e}\bar{y} = \bar{y}\bar{e}$ , and  $\bar{e}\bar{x} = \bar{x}\bar{e}$ . Since  $\bar{e}$  is in

the nucleus of  $G/S$  and since the set  $\{\bar{x}, \bar{y}\}$  generates  $G/S$ , we see that the subloop  $E$  generated by  $\bar{e}$  is normal in  $G/S$  by Lemma 3.2 since the equations  $\bar{e}\bar{y} = \bar{y}\bar{e}$  and  $\bar{e}\bar{x} = \bar{x}\bar{e}$  together with the fact that  $\bar{e}$  is in the nucleus of  $G/S$  imply that  $\bar{e} G/S \subseteq G/S\bar{e}$  and that  $G/S\bar{e} \subseteq \bar{e} G/S$ . Now let  $Q = \{\bar{z} \in G/S: \bar{z}\theta_{\bar{y}}^{-1}(\bar{z}) \in E\}$ . Clearly, we have  $\bar{y}$  is in  $Q$ , and since  $x y = x e$  by (3.9), we have  $\bar{x}$  in  $Q$ . If  $\bar{r}, \bar{s}$  are in  $Q$ , then we have  $(\bar{r}\bar{s})\theta_{\bar{y}} = \bar{r}\bar{f} \cdot \bar{s}\bar{g} = \bar{r} \cdot \bar{f}\bar{s}\bar{g}$ , where  $\bar{f}$  and  $\bar{g}$  are in  $E$ . But since  $E$  is normal in  $G/S$ , we have  $\bar{f}\bar{s} = \bar{s}\bar{g}'$ , where  $\bar{g}'$  is in  $E$ . Thus  $(\bar{r}\bar{s})\theta_{\bar{y}} = \bar{r}\bar{s} \cdot \bar{g}\bar{g}'$  and  $\bar{g}\bar{g}'$  is in  $E$ . Hence we see that  $\bar{r}\bar{s}$  is in  $Q$ . Now let  $\bar{z}$  be such that  $\bar{r}\bar{z} = \bar{s}$ . Then we have  $\bar{r}\theta_{\bar{y}} \bar{z}\theta_{\bar{y}} = \bar{s}\theta_{\bar{y}}$ . Let  $\bar{f}'$  and  $\bar{h}$  be such that  $\bar{f}'\bar{r} = \bar{r}\theta_{\bar{y}}$  and  $\bar{z}\bar{h} = \bar{z}\theta_{\bar{y}}$ . Now we have  $\bar{f}'\bar{r} = \bar{r}\bar{f}$  for some  $\bar{f}$  in  $G/S$ . But, since  $r$  is in  $S$ , we have  $\bar{f}$  is in  $E$ . Hence we see that  $\bar{f}'$  is in  $E$  since  $E$  is normal in  $G/S$ . Now we have  $\bar{r}\bar{z} \cdot \bar{b} = \bar{f}'^{-1} \cdot \bar{s}\bar{g}$  since both  $\bar{h}$  and  $\bar{f}'$  are in the nucleus. Since  $\bar{f}'^{-1}$  is in  $E$  and since  $E$  is normal in  $G/S$ , we have  $\bar{f}'^{-1}\bar{s} = \bar{s}\bar{k}$ , where  $\bar{k}$  is in  $E$ . It follows that  $\bar{r}\bar{z} \cdot \bar{b} = \bar{s} \cdot \bar{k}\bar{y}$ , or  $\bar{h} = \bar{k}\bar{g}$  and  $\bar{h}$  is in  $E$ . Hence we have  $\bar{z}$  is in  $Q$ . In a similar manner one can show that if  $\bar{q}\bar{r} = \bar{s}$ , then  $\bar{q}$  is in  $Q$ . It follows that  $Q$  is a loop. Since  $Q$  contains  $\bar{x}$  and  $\bar{y}$ , we have  $Q = G/S$ . Similarly, we may also show that the set  $\{\bar{z} \in G/S: \bar{z}\theta_{\bar{x}}^{-1}(\bar{z}) \in E\} = G/S$ .

Now let  $\bar{q}$  be in  $G/S$  and let  $T_{\bar{q}} = \{\bar{z} \in G/S: \bar{z}\theta_{\bar{q}}^{-1}(\bar{z}) \in E\}$ . From  $\bar{y}\bar{q} = \bar{y}\theta_{\bar{q}} \cdot \bar{q}\theta_{\bar{y}}$ , we obtain that  $\bar{y}\bar{q} = \bar{y}\theta_{\bar{g}} \cdot \bar{q}'\bar{q}$ , where  $\bar{q}'\bar{q} = \bar{q}\theta_{\bar{y}}$ . Now we have  $\bar{q}'\bar{q} = \bar{q}\bar{q}''$  for some  $\bar{q}''$  in  $E$ . Since  $E$  is normal in  $G/S$ , we have  $\bar{q}''^{-1}$  in  $E$ . It follows that  $\bar{y}$  is in  $T_{\bar{q}}$ . Similarly, we can show that  $\bar{x}$  is in  $T_{\bar{q}}$ . By applying an argument to  $T_{\bar{q}}$  which is parallel to the one applied to  $Q$  above, we obtain that  $T_{\bar{q}} = G/S$ . Thus we have established that if  $\bar{z}$  and

$\bar{y}$  are in  $G/S$ , then  $\bar{z}\theta_q^{-1}L^{-1}(\bar{z})$  is in  $E$ ; that is  $E$  contains a set which generates  $\bar{E}$ . Hence we have  $\bar{E} \subseteq E$ , but it is clear that  $E \subseteq \bar{E}$ . Thus we see that  $E = \bar{E}$ .

Now let  $e$  be in  $G$  such that  $eS = \bar{e}$ . If  $f$  is an element of the outer canonical subloop  $E$  of  $G$ , then  $\bar{f} = fS$  is in  $\bar{E}$ . Hence we have  $\bar{f} = \bar{e}^i$  for some integer  $i$ . It follows that  $fe^{-i}$  is in the inner canonical subloop  $A$  or  $S$ . Hence  $f$  is in the subloop generated by  $\{e, a, b, c\}$ ; that is, the outer canonical subloop  $E$  of  $G$  is generated by the set  $\{e, a, b, c\}$ . ■

The above theorem is not true, in general, for  $M$ -loops, a fact which we now show.

Example 3.1. Let  $\bar{M}$  be the triple product of the integers with themselves. We introduce a binary operation on  $\bar{M}$  by the following equations.

$$(3.11) \quad [2i, k, m][2j, p, q] = [2i+2j, k+p-ij(2j-1), q+m-ij(2j-1)]$$

$$[2i+1, k, m][2j, p, q] = [2i+2j+1, k+p-ij(2j-1)-j^2+j, \\ n+m-ij(2j-1)-j^2]$$

$$[2i, k, m][2j+1, p, q] = [2i+2j+1, m+p-ij(2j+1), \\ q+k-ij(2j+1)]$$

$$[2i+1, k, m][2j+1, p, q] = [2i+2j+2, m+p-ij(2j+1)-j^2-j, \\ q+k-ij(2j+1)-j^2]$$

By direct computation it can be shown that  $\bar{M}$  has an identity  $[0,0,0]$  and that  $\bar{M}$  is an M-loop.

Theorem 3.7. The inner canonical subloop of  $\bar{M}$  consists of all elements of the form  $[0,n,m]$ , where  $n$  and  $m$  are integers.

Proof: Let  $x$  be in  $M$  and suppose that  $x$  is of the form  $[2i,k,m]$ . Then we have  $x^\rho = [2(-i), -k+2i^3+i^2, -m+2i^3+i^2]$ . Hence we have  $x^\rho x = [0,4i^3,4i^3]$ .

Now suppose that  $x$  is of the form  $[2i+1, k, m]$ . Then we have  $x^\rho = [2(-i-1)+1, -m+2i^3+4i^2+2i, -k+2i^3+4i^2+3i+1]$  and  $x^\rho x = [2(-i-1)+1, -m+2i^3+4i^2+2i, -k+2i^3+4i^2+3i+1][2i+1, k, m] = [0, 4i^3+6i^2+3i+1, 4i^3+6i^2+3i]$ .

It follows that the inner canonical subloop  $A$  is a subset of the set of elements of the form  $[0,n,m]$ . If  $x = [1,0,0]$ , we have  $x^\rho x = [0,1,0]$ , and, if  $x = [-1,0,0]$ , we have  $x^\rho x = [0,0,-1]$ . Since the set  $\{[0,1,0], [0,0,-1]\}$  generates the set of elements of the form  $[0,n,m]$  and is contained in  $A$ , we have  $A = \{[0,n,m]: n,m \text{ are integers}\}$ . ■

It is clear that the element  $[1,0,0]$  is a generator of  $M$ , but the inner canonical subloop of  $M$  is generated by no set consisting of less than two elements. One may verify by direct calculation that the direct product  $M \otimes M$  is an M-loop which has a generating set of two elements and whose inner canonical subloop has no generating set of less than four elements. We now show that Osborn's loop  $H$  is a homomorphic image of  $\bar{M}$ .

Theorem 3.8. Osborn's loop  $H$  (Example 1.1) is a homomorphic image of  $\bar{M}$ .

Proof: For any  $[i, j, k]$  in  $M$  define  $[i, j, k]\phi = [i, j, k]$ . Now let  $x$  and  $y$  be in  $\bar{M}$ .

Case I. The elements  $x$  and  $y$  are of the form  $x = [2i, k, m]$ ,  $y = [2j, \ell, n]$ . Then we have  $x\phi = [2i, k, m]$ ,  $y\phi = [2j, \ell, n]$ , and  $(xy)\phi = [2i+2j, k+\ell-(m+n)]$ . Furthermore, in Osborn's loop  $H$ ,  $x\phi \cdot y\phi = [2i, k, m][2j, \ell, n] = [2i+2j, k-m+\ell-n] = (xy)\phi$ .

Case II. The elements  $x$  and  $y$  are of the form  $x = [2i+1, k, m]$  and  $y = [2j, \ell, n]$ . Then we have  $x\phi = [2i+1, k, m]$ ,  $y\phi = [2j, \ell, n]$ , and  $(xy)\phi = [2j+2i+1, k+\ell-m-n+j] = [2i+1, k-m][2j, \ell, n] = x\phi \cdot y\phi$ .

Case III. The elements  $x$  and  $y$  are of the form  $[2i, k, m]$  and  $[2j+1, \ell, n]$ , respectively. Then we have  $(xy)\phi = ([2i+2j+1, \ell+m-n-k]) = [2i, k-m][2j+1, \ell, n] = x\phi \cdot y\phi$ .

Case IV. The elements  $x$  and  $y$  are of the form  $[2i+1, k, m]$  and  $[2j+1, \ell, n]$ , respectively. Then we have  $(xy)\phi = [2i+2j+2, \ell+m-n-k] = [2i+1, k-m][2j+1, \ell, n] = x\phi \cdot y\phi$ .

It follows that  $\phi$  is a homomorphism from  $M$  onto  $H$ . ■

We consider now the question of the coincidence of the inner and outer canonical subloops of an Osborn loop. It is not known whether or not the inner and outer canonical subloops must coincide, but one sufficient condition is known. Suppose that  $G$  is an Osborn loop in which  $x^2 = 1$  for all  $x$  in  $G$ . Then we have  $x^\lambda = x^\rho = x$  for all  $x$  in  $G$ , and hence inverses are unique in  $G$ . Furthermore, we have  $x(yx) = x(yx)^{-1} = y^{-1} = (xy)^{-1}x = (xy)x$  for all  $x$  and  $y$  in  $G$ . Hence we have  $y = yL(x)R(x)L^{-1}(x)R^{-1}(x)$  for all  $x$  and  $y$  in  $G$ . By Theorem 1.7 it follows that  $\theta_x$  is the identity mapping on  $G$  and hence that  $G$  is Moufang.

## CHAPTER IV

## HOLOMORPHY THEORY

In this chapter we shall investigate the structure of some holomorphs of the loops we have been considering. The concept of a holomorph is familiar from group theory; we redefine it here for loops.

Definition 4.1. Let  $G$  be a loop and let  $A$  be any group of automorphisms of  $G$ . Then the *holomorph*  $A(G)$  of  $G$  by  $A$ , or the *A-holomorph* of  $G$ , is the set  $A \times G$  with a binary operation defined by

$$(\alpha, x)(\beta, y) = (\alpha\beta, x\beta \cdot y)$$

for all  $\alpha, \beta$  in  $A$  and all  $x, y$  in  $G$ .

Bruck [2] has shown that the holomorph of a loop is also a loop. We first investigate the holomorph of an M-loop.

Theorem 4.1. Let  $G$  be an M-loop and let  $A$  be a group of automorphisms of  $G$ . Then the following statements are equivalent:

(i) The triple

$$(4.1) \quad (L(f), R^{-1}(f\alpha^p), L(f)R(f\alpha))$$

is an autotopism of  $G$  for every  $f$  in  $G$  and every  $\alpha$  in  $A$ .

(ii) If  $f, x$  are in  $G$  and  $\alpha$  is in  $A$ , then we have

$$(4.2) \quad L(fx) = R(f\alpha^0)L(x)L(f)R(f\alpha).$$

(iii) The holomorph  $A(G)$  is an M-loop.

Proof: Suppose that (i) is true. Let  $f$  be any element in  $G$  and let  $\alpha$  be any automorphism of  $G$  which is in  $A$ . Then we have the identity  $fx \cdot yR^{-1}(f\alpha^0) = (f(xy))(f\alpha)$  for all  $x$  and  $y$  in  $G$ . Substituting  $y(f\alpha^0)$  for  $y$ , we obtain that  $fx \cdot y = (f(x \cdot y(f\alpha^0)))(f\alpha)$ , or  $yL(fx) = yR(f\alpha^0)L(x)L(f)R(f\alpha)$  for every  $y$  in  $G$ . The identity (4.2) now follows since  $y$  is any element in  $G$ .

Now suppose that (ii) is true. Let  $\bar{f} = (\gamma, f)$ ,  $\bar{x} = (\alpha, x)$ , and  $\bar{y} = (\beta, y)$  be in the holomorph  $A(G)$ . We shall show that  $\bar{y}L(\bar{f}\bar{x}) = \bar{y}R(\bar{f}^0)L(\bar{x})L(\bar{f})R(\bar{f})$ . Now we have  $\bar{f}\bar{x} = (\gamma, f)(\alpha, x) = (\gamma\alpha, f\alpha \cdot x)$  and  $\bar{f}\bar{x} \cdot \bar{y} = (\gamma\alpha\beta, (f\alpha\beta \cdot x\beta)y)$ . We also have  $\bar{f}^0 = (\gamma^{-1}, f^0\gamma^{-1})$  and  $\bar{y}R(\bar{f}^0)L(\bar{x})L(\bar{f})R(\bar{f}) = \bar{f}(\bar{x} \cdot \bar{y}\bar{f}^0) \cdot \bar{f} = (\gamma\alpha\beta, (f\alpha\beta \cdot (x\beta \cdot yf^0))f) = (\gamma\alpha\beta, yR(f^0)L(x\beta)L(f\alpha\beta)R(f)) = (\gamma\alpha\beta, yL(f\alpha\beta \cdot x\beta)) = (\gamma\alpha\beta, (f\alpha\beta \cdot x\beta)y) = \bar{f}\bar{x} \cdot \bar{y} = \bar{y}L(\bar{f}\bar{x})$ , where we have used (4.2) with  $f$  replaced by  $f\alpha\beta$ ,  $x$  replaced by  $x\beta$ , and  $\alpha$  replaced by  $(\alpha\beta)^{-1}$ . It follows that  $L(\bar{f}\bar{x}) = R(\bar{f}^0)L(\bar{x})L(\bar{f})R(\bar{f})$  and that  $A(G)$  is an M-loop by Corollary 2.3.

Now assume that (iii) is true. Then by Theorem 2.2 we have that the triple  $(L(\bar{f}), R^{-1}(\bar{f}^0), L(\bar{f})R(\bar{f}))$  is an autotopism of  $A(G)$  for every  $\bar{f} = (\gamma, f)$  in  $A(G)$ . Let  $\bar{x} = (\alpha, x)$  and  $\bar{y} = (\beta, y)$  be in  $A(G)$ . Then we have  $\bar{f}\bar{x} \cdot \bar{y}R^{-1}(\bar{f}^0) = (\bar{f} \cdot \bar{x}\bar{y})\bar{f}$ . We now calculate  $\bar{y}R^{-1}(\bar{f}^0)$ . Suppose

$\bar{y}R^{-1}(\bar{f}^0) = \bar{z} = (\delta, z)$  for some  $z$  in  $G$  and some  $\delta$  in  $A$ . Then we have  $\bar{y} = \bar{z}\bar{f}^0$ , or  $(\beta, y) = (\delta\gamma^{-1}, z\gamma^{-1}\cdot f^0\gamma^{-1})$ . It follows that  $\delta = \beta\gamma$  and that  $z = yR^{-1}(f^0\gamma^{-1})\gamma$ . Hence we have  $\bar{y}R^{-1}(\bar{f}^0) = (\beta\gamma, yR^{-1}(f^0\gamma^{-1})\gamma)$ . Thus from  $\bar{f}x\cdot\bar{y}R^{-1}(\bar{f}^0) = (\bar{f}\cdot\bar{x}\bar{y})\bar{f}$  we obtain that

$(\gamma\alpha, f\alpha\cdot x)(\beta\gamma, yR^{-1}(f^0\gamma^{-1})\gamma) = (\gamma\alpha\beta\gamma, (f\alpha\beta\gamma\cdot x\beta\gamma)\cdot yR^{-1}(f^0\gamma^{-1})\gamma) =$   
 $(\gamma\alpha\beta\gamma, (f\alpha\beta\gamma\cdot(x\beta\gamma\cdot y\gamma))f)$ . It follows that  $(f\alpha\beta\gamma\cdot x\beta\gamma)\cdot yR^{-1}(f^0\gamma^{-1})\gamma =$   
 $(f\alpha\beta\gamma\cdot(x\beta\gamma\cdot y\gamma))f$  for every  $f, x,$  and  $y$  in  $G$  and every  $\alpha, \beta,$  and  $\gamma$  in  $G$ . Since  $\gamma$  is an autotopism of  $G$ , we have  $(f\alpha\beta\cdot x\beta)\cdot yR^{-1}(f^0\gamma^{-1}) =$   
 $(f\alpha\beta\cdot(x\beta\cdot y))\cdot f\gamma^{-1}$  for all  $f, x,$  and  $y$  in  $G$  and all  $\alpha, \beta,$  and  $\gamma$  in  $A$ .

Substituting  $x\beta^{-1}$  for  $x$  and  $f\beta^{-1}\alpha^{-1}$  for  $f$ , we obtain that

$(fx)\cdot yR^{-1}(f^0\beta^{-1}\alpha^{-1}\gamma^{-1}) = (f\cdot xy)\cdot f\beta^{-1}\alpha^{-1}\gamma^{-1}$  for all  $f, x,$  and  $y$  in  $G$   
and all  $\alpha, \beta,$  and  $\gamma$  in  $A$ . Replacing  $\beta^{-1}\alpha^{-1}\gamma^{-1}$  by  $\alpha$ , we obtain that  
 $fx\cdot yR^{-1}(f^0\alpha) = (f\cdot xy)\cdot f\alpha$  for every  $f, x,$  and  $y$  in  $G$  and every  $\alpha$  in  $A$ .

Hence (i) is true. ■

We turn now to the problem of a holomorph of an Osborn loop.

Lemma 4.1. Let  $G$  be a weak inverse property loop and let  $A$  be an automorphism group of  $G$ . Then the holomorph  $A(G)$  is a weak inverse property loop.

Proof: Let  $(\alpha, x)$  and  $(\beta, y)$  be any two elements in  $A(G)$ . Then we have  $(\alpha, x)(\alpha^{-1}, x^0\alpha^{-1}) = (1, x\alpha^{-1}\cdot x^0\alpha^{-1}) = (1, 1)$ , where we have used 1 to denote the identity automorphism in  $A$ . It follows that  $(\alpha, x)^\rho = (\alpha^{-1}, x^0\alpha^{-1})$ . Hence we have  $(\beta, y)[(\alpha, x)(\beta, y)]^\rho = (\beta, y)(\alpha\beta, x\beta\cdot y)^\rho = (\beta, y)(\beta^{-1}\alpha^{-1}, (x\alpha^{-1}\cdot y\beta^{-1}\alpha^{-1})^\rho)$ . But we have  $y\beta^{-1}\alpha^{-1}\cdot(x\alpha^{-1}\cdot y\beta^{-1}\alpha^{-1})^\rho =$

$(x\alpha^{-1})^\rho = x^\rho\alpha^{-1}$  since  $G$  is a weak inverse property loop. Hence we see that  $(\beta,y)[(\alpha,x)(\beta,y)]^\rho = (\alpha^{-1}, y\beta^{-1}\alpha^{-1} \cdot (x\alpha^{-1} \cdot y\beta^{-1}\alpha^{-1})^\rho) = (\alpha^{-1}, x^\rho\alpha^{-1}) = (\alpha,x)^\rho$ . It follows that  $A(G)$  is a weak inverse property loop. ■

Bruck [2] has shown that any holomorph of an inverse property loop is also an inverse property loop.

Theorem 4.2. Let  $G$  be an Osborn loop and let  $A$  be any group of automorphisms of  $G$ . Then the holomorph  $A(G)$  is an Osborn loop if and only if  $A(G)$  is an M-loop.

Proof: Suppose that the holomorph  $A(G)$  is an M-loop. Since  $G$  is an Osborn loop, it must be a weak inverse property loop, and hence  $A(G)$  is a weak inverse property loop by Lemma 4.1. Since  $A(G)$  is an M-loop by assumption, it must also be an Osborn loop by Theorem 2.7.

Conversely, if  $A(G)$  is an Osborn loop, it is obvious that  $A(G)$  is also an M-loop. ■

Bruck [2] has proved that the inner mapping group  $I$  of a commutative Moufang loop  $G$  is an automorphism group of  $G$ . Furthermore, Bruck [2] establishes the existence of a commutative Moufang loop  $G$  whose  $I$ -holomorph  $I(G)$  is not Moufang. Since  $G$  is a Moufang loop, it must be an inverse property loop. It follows that  $I(G)$  cannot be an M-loop since all inverse property M-loops are Moufang loops by Theorem 2.4. Thus there are examples of M-loops with holomorphs which are not M-loops.

We now exhibit an Osborn loop with the property that any holomorph of it is an Osborn loop. This loop is the same loop which Osborn

[7] constructed and which we gave in the introduction as Example 1.1.

The proof of following theorem uses Theorem 1.10.

Theorem 4.3. Let  $H$  be the free Osborn loop on one generator. Let  $A$  be any automorphism group of  $H$ . Then the  $A$ -holomorph of  $H$  is isomorphic to all of its loop isotopes.

Proof: Let  $(\gamma, c)$ ,  $(\alpha, x)$ , and  $(\beta, y)$  be in  $A(H)$ . We shall show that

$(\gamma, c) [(\gamma, c)(\alpha, x)]^\rho = [(\gamma, c)(\beta, y)][(\gamma, c)((\alpha, x)(\beta, y))]^\rho$ . Now we have

$(\gamma, c)[(\gamma, c)(\alpha, x)]^\rho = (\gamma\alpha^{-1}\gamma^{-1}, (c \cdot (c\alpha \cdot x)^\rho)\alpha^{-1}\gamma^{-1})$  and

$[(\gamma, c)(\beta, y)][(\gamma, c)((\alpha, x)(\beta, y))]^\rho = (\gamma\alpha^{-1}\gamma^{-1}, (c \cdot y\beta^{-1})[c\alpha \cdot (x \cdot y\beta^{-1})]^\rho \alpha^{-1}\gamma^{-1})$ .

Hence we need only show that if  $x$ ,  $y$ , and  $c$  are in  $H$  and if  $\alpha$  is in  $A$ , we have the identity

$$c(c\alpha \cdot x)^\rho = (c \cdot y\beta^{-1})[c\alpha \cdot (x \cdot y\beta^{-1})]^\rho .$$

But this identity holds for all  $x$ ,  $y$ , and  $c$  in  $H$  and all  $\alpha, \beta$  in  $A$  if and only if the identity

$$(4.3) \quad c(c\alpha \cdot x)^\rho = (cy)[c\alpha \cdot (xy)]^\rho$$

holds for all  $x$ ,  $y$ , and  $c$  in  $H$  and all  $\alpha$  in  $A$ . We shall show that

(4.3) actually does hold in  $H$ .

Note that the nucleus of any loop is invariant under automorphisms. Since the nucleus of  $H$  coincides with the set of even elements, we have that if  $z$  is any element of  $H$  and if  $\delta$  is any automorphism of  $H$ ,

then  $z\delta$  is even if  $z$  is even, and  $z\delta$  is odd if  $z$  is odd.

Case I. Let  $c$  be even. Then we have  $(cy)[ca \cdot (xy)]^\rho = (cy)[(xy)^\rho \cdot c^{-1}a] = [(cy) \cdot (xy)^\rho] \cdot c^{-1}a = [cxy(xy)^\rho] \cdot c^{-1}a = cx^\rho \cdot c^{-1}a = c(x^\rho \cdot c^{-1}a) = c(ca \cdot x)^\rho$  where we have used Theorem 1.1 and Theorem 1.9.

Thus we have established (4.3) in this case.

Case II. Let  $y$  be even. Then we have  $(cy)[ca \cdot (xy)]^\rho = (cy)[(ca \cdot x)y]^\rho = c \cdot y[(ca \cdot x)y]^\rho = c(ca \cdot x)^\rho$ , and we have established (4.3) in this case.

Case III. Let  $c$  and  $y$  be odd and let  $x$  be even. Then we have  $(cy)[ca \cdot (xy)]^\rho = (cy)[(ca \cdot x)y]^\rho = cxy[(ca \cdot x)y]^\rho = c(ca \cdot x)^\rho$  since  $(ca \cdot x)y$  is even. Hence we have verified (4.3) in this case.

Case IV. Let  $c$ ,  $x$ , and  $y$  be odd. Let  $a = [0,1]$ . Then we have  $(cy)[ca \cdot (xy)]^\rho = (cy) \cdot [(ca \cdot x)y \cdot a]^\rho = (cy) \cdot [a^{-1}((ca \cdot x)y)^\rho]$  by Lemma 1.6 and Theorem 1.9. Now since  $((ca \cdot x)y)^\rho$  is an odd element, we have  $a^{-1}((ca \cdot x)y) = ((ca \cdot x)y)a$ . It follows that  $(cy)[ca \cdot (xy)]^\rho = (cy) \cdot [a^{-1}((ca \cdot x)y)^\rho] = (cy) \cdot [((ca \cdot x)y)^\rho a] = cxy[(ca \cdot x)y]^\rho = c(ca \cdot x)^\rho$ , where we have used Lemma 1.6 again and also the fact that  $A(H)$  is a weak inverse property loop. Thus we see that (4.3) holds in this case. It follows that  $A(H)$  is isomorphic to all of its loop isotopes and hence that all of its loop isotopes are weak inverse property loops. Hence  $A(H)$  is an Osborn loop. ■

The proof of the above theorem parallels closely that of Theorem 1.11. The theorem itself is not as strong as it seems, for, as we now show, the group of all automorphisms of  $H$  is an infinite cyclic group.

Theorem 4.4. The automorphism group of  $H$  is an infinite cyclic group.

Proof: Recall that  $x = [1,0]$  generates  $H$ . Consider the loop generated by  $x^\lambda$ . This loop must contain  $x$ , and hence the loop generated by  $x^\lambda$  is  $H$ . This implies the existence of an automorphism  $\alpha$  of  $H$  which maps  $x$  onto  $x^\lambda$ . We now show that this automorphism generates the automorphism group of  $H$ .

Let  $\beta$  be any automorphism of  $H$  and consider the factor loop  $H/A$ , where  $A$  is the normal subloop generated by the element  $a = [0,1]$ . It is clear from the definition of  $H$  that  $H/A$  is a cyclic group. Now from the relation  $x^{\lambda^2} = x\theta_x = xa$ , we obtain that  $(x\beta)^{\lambda^2} = x\beta \cdot a\beta$ , or  $[(x\beta)A]^{\lambda^2} = (x\beta)^{\lambda^2}A = x\beta A \cdot a\beta A$ . But we also have  $[(x\beta)A]^{\lambda^2} = x\beta A$  since  $H/A$  is a group. It follows that  $(x\beta)A = [(x\beta)A]^{\lambda^2} = x\beta A \cdot a\beta A$ ; that is,  $a\beta$  is in  $A$ . Thus we see that  $A\beta \subseteq A$ .

Now we define  $(yA)\phi = (y\beta)A$  for any  $yA$  in  $H/A$ . Let  $y$  and  $z$  be such that  $yA = zA$ . Let  $q$  be in  $(y\beta)A$ . Then  $q = y\beta \cdot r$  for some  $r$  in  $A$ . Hence we have  $q\beta^{-1} = y \cdot r\beta^{-1}$ . It follows that  $q\beta^{-1}$  is in  $yA = zA$  since  $A\beta \subseteq A$ . Thus we see that  $q\beta^{-1} = q's$  for some  $s$  in  $A$ , or  $q = z\beta \cdot s\beta$ , which is in  $(z\beta)A$ . Thus  $\phi$  is a well-defined function from  $H/A$  into  $H/A$ . If  $zA$  is any element in  $H/A$ , then  $(z\beta^{-1})A\phi = zA$ ; that is,  $\phi$  maps  $H/A$  onto  $H/A$ . Now we have  $(yA \cdot zA)\phi = ((yz)A)\phi = ((yz)\beta)A = (y\beta \cdot z\beta)A = (y\beta)A \cdot (z\beta)A = (yA)\phi \cdot (zA)\phi$ . Hence  $\phi$  is an endomorphism.

Since  $H/A$  is an infinite cyclic group, this implies that  $\phi$  is either the identity mapping or else  $\phi$  is the automorphism which takes every element onto its own inverse.

Suppose that  $\phi$  is the identity mapping. Then we have  $x^A = (xA)\phi = (x\beta)A$ ; that is,  $x\beta = x\alpha^i$  for some integer  $i$ . It follows that  $x\beta = x^{\lambda^{2i}} = x\alpha^{2i}$ . Since  $x$  generates  $H$ , we have proved that  $\beta = \alpha^{2i}$ . If  $\phi$  is not the identity mapping, then we have  $x^{\lambda}A = (xA)^{-1}$ ;  $(xA)\phi = (x\beta)A$ ; that is,  $x^{\lambda} = x\beta\alpha^i$  for some integer  $i$ . Hence we see that  $x\alpha = x^{\lambda} = x\beta\alpha^i = (x\beta)^{\lambda^{2i}} = (x^{\lambda^{2i}})\beta = x\alpha^{2i}\beta$ ; that is,  $\alpha = \alpha^{2i}\beta$  since  $x$  generates  $H/A$ . Hence we have  $\beta = \alpha^{1-2i}$ . It follows that the automorphism group of  $H$  is generated by  $\alpha$ . ■

We now consider some of the general properties of the holomorphs of  $M$ -loops.

Theorem 4.5. Let  $G$  be any loop and suppose that  $A$  is any automorphism group of  $G$ . Then the inner and outer canonical subloops of  $A(G)$  are isomorphic respectively to the inner and outer canonical subloops of  $G$ .

Proof: Let  $E$  be the outer canonical subloop of  $G$  and let  $\bar{E}$  be the outer canonical subloop of  $A(G)$ . If  $e$  is in  $E$ , we define  $e\phi = (1,e)$ , where  $1$  is the identity of  $A$ . It is clear that  $\phi$  is a one-to-one mapping from  $E$  onto a subset of  $A(G)$ .

If  $e$  is a generator of  $E$  of the type described in Definition 3.2, then we have  $re = r\theta_s$  for some  $r, s$  in  $G$ . Furthermore, we have

$$(1,r)\theta_{(1,s)} = (1,s)^{\lambda}((1,s)(1,r)) = (1,s)^{\lambda}(1,sr) = (1,re) = (1,r)(1,e).$$

Hence  $(\gamma, e)$  is a generator of  $\bar{E}$  of the type described in Definition 3.2.

Now suppose that  $(\gamma, g)$  is such a generator of  $\bar{E}$ . Then for some  $(\alpha, x)$  and  $(\beta, y)$  we have  $(\alpha, x)^\wedge((\alpha, x)(\beta, y)) = (\beta, y)(\gamma, g)$ , or  $(\beta, (x\beta)^\wedge \cdot (x\beta \cdot y)) = (\beta, y)(\gamma, g)$ . But this implies that  $\gamma = e$  and that  $g$  is a generator of  $E$  of the type described in Definition 3.2. Hence  $\phi$  is a one-to-one mapping from a set which generates  $E$  onto a generating set of  $\bar{E}$ . Now if  $e$  and  $g$  are any elements in  $R$ , we have  $e\phi \cdot g\phi = (\gamma, e)(\gamma, g) = (\gamma, eg) = (eg)\phi$ , which implies that  $\phi$  is an isomorphism from  $E$  into  $\bar{E}$ . But since  $E\phi$  contains a generating set for  $\bar{E}$ ,  $\phi$  must be an isomorphism from  $E$  onto  $\bar{E}$ . A similar argument shows that the same conclusion is valid for inner canonical subloops. ■

The above theorem shows that we cannot expect to construct examples of Osborn loops which have distinct inner and outer canonical subloops by taking holomorphs of Osborn loops whose inner and outer canonical subloops coincide. We conclude this chapter with a result which tells us something about the mappings  $\theta_x = L(x)L(x^\lambda)$  in holomorphs of M-loops.

Theorem 4.6. Let  $G$  be an M-loop and let  $A$  be an automorphism group of  $G$ . Then the mappings  $L(\bar{x})L(\bar{x}^\lambda)$  and  $R^{-1}(\bar{x}^\rho)R^{-1}(\bar{x})$  are automorphisms of the holomorph  $A(G)$  for every  $\bar{x}$  in  $A(G)$  if and only if  $\theta_{x\beta} = \theta_x$  for every  $x$  in  $G$  and every  $\beta$  in  $A$ . In this case we have  $L(\bar{x})L(\bar{x}^\lambda) = R^{-1}(\bar{x}^\rho)R^{-1}(\bar{x})$  for every  $\bar{x}$  in  $A(G)$ .

Proof: Let  $\bar{x} = (\alpha, x)$ ,  $\bar{y} = (\beta, y)$ , and  $\bar{z} = (\gamma, z)$  be elements in  $A(G)$  and let  $\theta_x^-$  denote the mappings  $L(\bar{x})L(\bar{x}^{-1})$  for all  $\bar{x}$  in  $A(G)$ . Then we have

$$\begin{aligned}\bar{y}\theta_x^- &= \bar{x}^{-1}(\bar{x}\bar{y}) = (\alpha^{-1}, x^{-1}\alpha^{-1})((\alpha, x)(\beta, y)) \\ &= (\beta, (x\beta)^{-1}(x\beta \cdot y)) = (\beta, y\theta_{x\beta}^-).\end{aligned}$$

The above calculation holds for all  $\bar{y}$  in  $A(G)$ ; hence we see that  $z\theta_x^- = (\gamma, z\theta_{xy}^-)$  and  $(\bar{y}\bar{z})\theta_x^- = (\beta\gamma, (y\gamma \cdot z)\theta_{x\beta\gamma}^-)$ . It follows that  $(\bar{y}\bar{z})\theta_x^- = \bar{y}\theta_x^- \cdot \bar{z}\theta_x^-$  if and only if  $(\beta\gamma, (y\gamma \cdot z)\theta_{x\beta\gamma}^-) = (\beta, y\theta_{x\beta}^-)(\gamma, z\theta_{xy}^-) = (\beta\gamma, y\gamma\theta_{x\beta\gamma}^- \cdot z\theta_{xy}^-)$ , where we have used the fact that  $\theta_s \gamma = \gamma\theta_{s\gamma}$  for all  $\gamma$  in  $A$  and all  $s$  in  $G$ . Hence  $\theta_x^-$  is an automorphism of  $A(G)$  if and only if the identity

$$(y\gamma)\theta_{x\beta\gamma}^- \cdot z\theta_{x\beta\gamma}^- = y\gamma\theta_{x\beta\gamma}^- \cdot z\theta_{xy}^-$$

holds for all  $x, y$ , and  $z$  in  $G$  and all  $\alpha, \beta$ , and  $\gamma$  in  $A$ ; that is,  $\theta_x^-$  is an automorphism of  $A(G)$  if and only if  $\theta_{x\beta\gamma}^- = \theta_{xy}^-$  for all  $x$  in  $G$  and all  $\beta, \gamma$  in  $A$ . Now  $\theta_{x\beta\gamma}^- = \theta_{xy}^-$  if and only if  $\gamma^{-1}\theta_{x\beta}^-\gamma^{-1} = \gamma^{-1}\theta_x^-\gamma$ , where we again use the fact that  $\theta_x \gamma = \gamma\theta_{x\gamma}$  for all  $x$  in  $G$  and all  $\gamma$  in  $A$ . Hence we have  $\theta_{x\beta\gamma}^- = \theta_{xy}^-$  for all  $x$  in  $G$  and all  $\beta, \gamma$  in  $A$  if and only if  $\theta_{x\beta}^- = \theta_x^-$  for all  $x$  in  $G$  and all  $\beta$  in  $A$ .

A similar argument shows that the mappings  $R^{-1}(\bar{x}^{-1})R^{-1}(\bar{x})$  are automorphisms for all  $\bar{x}$  in  $A(G)$  if and only if  $\theta_{x\beta}^- = \theta_x^-$  for all  $x$  in  $G$  and all  $\beta$  in  $A$ . Now suppose that these mappings are automorphisms. Let

$\bar{x} = (\alpha, x)$  and  $\bar{y} = (\beta, y)$  be any elements in  $A(G)$ . We calculated above that  $\bar{y}L(\bar{x})L(\bar{x}^\lambda) = (\beta, y\theta_{x\beta}) = (\beta, y\theta_x)$ . In like manner we may calculate that  $\bar{y}R^{-1}(\bar{x}^\rho)R^{-1}(\bar{x}) = (\beta, y\theta_{x\alpha^{-1}}) = (\beta, y\theta_x)$ . Hence we see that  $L(\bar{x})L(\bar{x}^\lambda) = R^{-1}(\bar{x}^\rho)R^{-1}(\bar{x})$ . ■

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## VITA

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