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THE ELASTIC CATENARY

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THE ELASTIC CATENARY

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CHAPTER I

THE PROBLEM AND SOME PREVIOUS DISCUSSIONS OF IT

The classical problem of determining the equilibrium configuration of a perfectly flexible inelastic string hanging free under the influence of gravity has the catenary as its solution. The pertinent mathematical model is particularly simple and may be analyzed as an isoperimetric problem of the calculus of variations[1]. When the assumption of inelasticity is abandoned (and thus the string is allowed to elongate), determination of the equilibrium configuration is substantially more difficult. The mathematical model is no longer an isoperimetric problem, and recourse to the methods of vectorial mechanics is appropriate.

The object of this study is to present a rationally satisfying solution of the problem of the elastic catenary in which the steps from the physical problem to the mathematical model are made in a logical manner. To achieve this object, the following procedure is used. First, the physical problem is stated. Then simplifying assumptions are made, their purpose being to facilitate the construction of a tractable mathematical model while retaining, as nearly as practicable, the essential features of the actual

1. See Appendix I.

problem. A mathematical model is constructed on the basis of the simplifying assumptions, and this model is analyzed mathematically. Obviously, the solution thus produced represents the solution to an idealized physical problem; and the relative agreement of the approximation with the real solution (if one exists) is a measure of the adequacy of the model used.

The problem of the elastic catenary may be stated as follows.

> A perfectly flexible elastic string (or other suitable continuum), when unstretched, has length $2L_o$, uniform cross-sectional area a_o, constant density ρ, and modulus of elasticity E. The string is suspended between two points of support whose projections on a horizontal plane are a distance 2L apart. The points of support may differ in altitude by an amount M; and the distance $\sqrt{(2L)^2 + M^2}$ between them may be greater than, equal to, or less than the unstretched length $2L_o$ of the string. The horizontal reaction at each support is designated by the letter H. The string is allowed to reach an equilibrium configuration under the influence of gravity. Determine the equation representing the equilibrium configuration.

Five assumptions about the physical problem are made. First, it is assumed that the only stress acting on a cross section normal to the axis of the string is axially directed and uniformly distributed over the cross section. Second, the string is assumed to be linearly elastic; that is, Hooke's law is assumed to apply and the ratio of stress to strain to be the modulus of elasticity E (a constant of the material). Third, it is assumed that the density of the string in its equilibrium state is the same as its original constant density. This assumption implies incompressibility of the material; for some materials it is probably a poor approximation to the physical state. Fourth, the string is assumed to be perfectly flexible; that is, it can transmit no bending moments along its length. Finally, it is assumed that the curve representing the suspended string has a horizontal tangent either at one of the supports or at a point between them. When the points of support are at equal heights, this assumption is fulfilled; but when the supports are at unequal heights, a horizontal tangent may not exist. In the case of an elastic catenary with no horizontal tangent, however, there is an easy extension of the analysis which leads to a method of complete solution; this method is presented in Appendix III.

Several authors have given solutions for the elastic catenary. In 1860, Clebsch (1)* gave solutions for both the inelastic and elastic catenary subject to an arbitrary force field. Using the Hamilton-Jacobi procedure, he reduced the solution of the problems to the discovery of a solution to a partial differential equation. But his solution for the elastic case is so brief that it is almost undecipherable, a fact previously noted by Todhunter and Pearson (9), who say, "The statement is so brief that it is difficult to follow the reasoning of this last section."

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* Numbers in parentheses after names refer to entries in the bibliography.

Furthermore, Clebsch neglected the change of cross section which accompanies the stretching, thus limiting the applicability of his results to cases in which no appreciable "necking" occurs. In 1903 Maclaurin (6) produced a solution for the elastic catenary which accounted for stiffness in the string. However, he also neglected the change of cross section; and in addition his work involved the truncation of the expansions of functions in infinite series. In 1915 Young (10) corrected several errors of manipulation in Maclaurin's paper. More recently (1927) a solution appeared in a book by MacMillan (7). But the change of cross section was again neglected, and the equation for the deflection is not stated explicitly in terms of the natural parameters H, L, and L_o of the problem. Thus, in each of the previous solutions mentioned, some simplifying approximation was made in order to obtain a solution, and a rational basis for the approximation was not given.

CHAPTER II

MATHEMATICAL ANALYSIS OF THE ELASTIC CATENARY

To construct a mathematical model for the elastic catenary, suppose that there exists an equilibrium configuration of the suspended string. Introduce an x-y cartesian coordinate system, measuring x positively to the right and y positively upward from the point of minimum height of the equilibrium curve. Measure arc length s along the curve positively in a counterclockwise direction from the position $x = 0$, $y = 0$. For each particular catenary, there will exist non-negative numbers X_o and X_1 such that $-X_o \leq x \leq X_1$ and $X_o + X_1 = 2L$, the horizontal distance between supports. Assume that the arc length s is a one-to-one differentiable function of x with domain $[-X_o, X_1]$; then the extent of the interval of arc length in the stretched string is given by $s([-X_o, X_1])$. Assume that the graph of the equilibrium configuration of the string is given by $y = y(x)$, where y is a differentiable function of x with domain $[-X_o, X_1]$. Let $a = a(s)$ denote the cross-sectional area of the stretched string. Assume that a is a continuous function of s with domain $s([-X_o, X_1])$ and that $a(s) > 0$ for all s in the domain of a. Note that since the domain of a is closed, the assumption that a is continuous and pos-

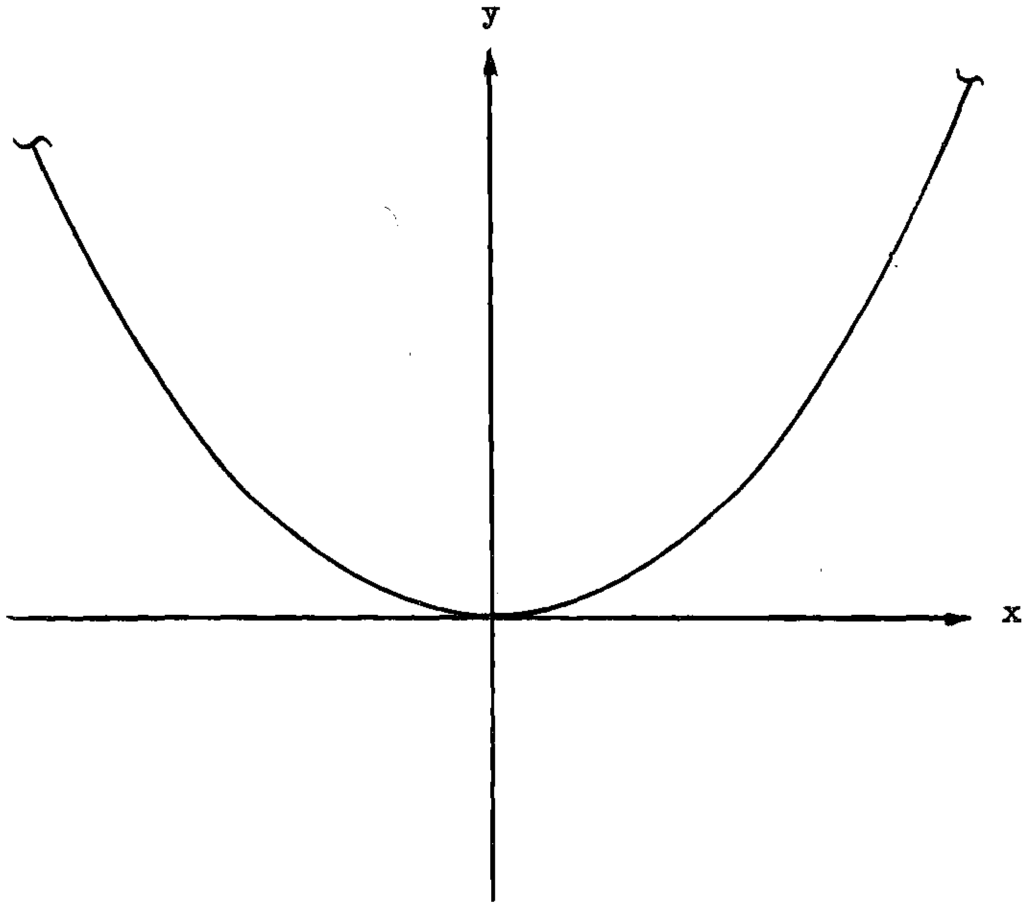

Figure 1. Coordinate System for the Inelastic Catenary

itive throughout its domain implies the existence of a positive constant c such that $a(s) > c$ for all s in the domain of a. Let $T = T(s)$ denote the axial tension in the string at arc length position s, and assume that T is a differentiable function of s with domain $s([-X_o, X_1])$. Let $\theta = \theta(s)$ denote the angle of inclination of the tangent to the graph of y at arc length position s, and assume that θ is a differentiable function of s with domain $s([-X_o, X_1])$.

First, a relation between cross-sectional area and tension is developed. Let s be an arbitrary but fixed point in the arc length interval $s((0, X_1))$ (note that $s > 0$). Define a sequence $\{[a_k, b_k]\}_{k=1}^{\infty}$ of nested arc length intervals centered about the point s with norm $\frac{1}{k}$: $a_k = s - \frac{1}{2k}$, $b_k = s + \frac{1}{2k}$. Then as $k \to \infty$, $[a_k, b_k] \to s$; and there exists a positive integer K such that for all $k > K$, $[a_k, b_k] \subset (0, s(X_1))$ — that is, the interval $[a_k, b_k]$ does not extend beyond the arc length interval $s((0, X_1))$. Let $\Delta s_k = b_k - a_k$, and denote by Δs_k^o the length which the segment $[a_k, b_k]$ had in the unstretched state.

For each finite positive value of $k > K$, consider the segment of the stretched string contained in the interval $[a_k, b_k]$. By the assumption of incompressibility, the mass of this segment is the same as the mass $\Delta s_k^o \, a_o \rho$ of the segment in the unstretched state. Hence,

$$\Delta s_k^\circ \, a_\circ \varrho = \int_{a_k}^{b_k} \varrho \, a(s) \, ds \qquad \text{for each } k > K.$$

By a mean value theorem of integral calculus, there is a value ξ_k, where $a_k < \xi_k < b_k$, such that

$$\varrho \, a(\xi_k) = \frac{1}{b_k - a_k} \int_{a_k}^{b_k} \varrho \, a(s) \, ds.$$

Thus

$$\Delta s_k^\circ \, a_\circ \varrho = \Delta s_k \varrho \, a(\xi_k); \qquad \text{or}$$

$$\frac{\Delta s_k}{\Delta s_k^\circ} = \frac{a_\circ}{a(\xi_k)} \; . \tag{1}$$

Note that $\xi_k \rightarrow s$ as $k \rightarrow \infty$ by the nested interval theorem.

By definition the modulus of elasticity E of the string is

$$E = \lim_{k \rightarrow \infty} \left[\frac{T(s)}{a(s)} \Big/ \frac{\Delta s_k - \Delta s_k^\circ}{\Delta s_k^\circ} \right] .$$

It is assumed that E is finite, positive, and a constant independent of s for $0 < s < s(X_1)$. Since for fixed s the quotient $\frac{T(s)}{a(s)}$ is independent of k, the definition of E as a limit and the assumption that E is finite and positive imply that the quantity

$$\frac{\Delta s_k - \Delta s_k^o}{\Delta s_k^o}$$

has a finite non-zero limit as $k \rightarrow \infty$. Since

$$\frac{\Delta s_k - \Delta s_k^o}{\Delta s_k^o} = \frac{\Delta s_k}{\Delta s_k^o} - 1 ,$$

the quantity $\frac{\Delta s_k}{\Delta s_k^o}$ also approaches a limit with increasing k. Hence, from equation (1),

$$\lim_{k \rightarrow \infty} \frac{\Delta s_k}{\Delta s_k^o} = \frac{a_o}{a(s)} .$$

Then from the definition of E

$$E = \frac{\frac{T(s)}{a(s)}}{\lim_{k \rightarrow \infty} \frac{\Delta s_k}{\Delta s_k^o} - 1} = \frac{\frac{T(s)}{a(s)}}{\frac{a_o}{a(s)} - 1} ,$$

from which it follows that

$$a(s) = a_o - \frac{T(s)}{E} . \qquad (2)$$

This relation between cross-sectional area and tension is one of those which distinguish the elastic catenary from the inelastic case.

Now some general properties of all catenaries having a horizontal tangent are developed. Since $s = s(x)$ is a one-to-one differentiable function of x, we may consider any function $F(s) = F[s(x)]$ as a function of x and vice versa. For the sake of a simpler notation, the same symbol $F(s) = F(x)$ will be used to denote the value of such a function F at $s = s(x)$.

At any point x such that $0 < s(x) \leq s(X_1)$,

$$\frac{dy}{dx} = \tan [\theta(s)]$$

and

$$H = T(s) \cos [\theta(s)] .$$

If $W(s)$ denotes the weight of the string in the arc length interval $[0,s]$, then

$$W(s) = T \sin [\theta(s)] .$$

Hence

$$\frac{W(s)}{H} = \tan [\theta(s)] .$$

But

$$W(s) = \int_0^s a(u) \rho g \, du ;$$

so

$$\frac{dW}{ds} = a(s)\rho g \quad . \tag{3}$$

Recall that $\frac{dy}{ds} = \sin\theta$ and $\frac{dx}{ds} = \cos\theta$. From the previous equations for H and W(s),

$$\frac{dW}{ds} = \frac{dT}{ds}\sin\theta + T\cos\theta\frac{d\theta}{ds} \ ;$$

and

$$\frac{dH}{ds} = \frac{dT}{ds}\cos\theta - T\sin\theta\frac{d\theta}{ds} = 0,$$

since H is a constant. Hence

$$\frac{dT}{ds} = T\tan\theta\frac{d\theta}{ds} \quad ,$$

and

$$\frac{dW}{ds} = T\tan\theta\sin\theta\frac{d\theta}{ds} + T\cos\theta\frac{d\theta}{ds} = T\frac{d\theta}{ds}\,\frac{\sin^2\theta + \cos^2\theta}{\cos\theta}$$

$$= T\,\frac{\frac{d\theta}{ds}}{\cos\theta} = \frac{\frac{dT}{ds}}{\tan\theta\cos\theta} \quad ,$$

since $T\frac{d\theta}{ds} = \frac{\frac{dT}{ds}}{\tan\theta}$, $\theta \neq 0$. But

$$\frac{dT}{dy} = \frac{dT}{ds}\frac{ds}{dx}\frac{dx}{dy} = \frac{dT}{ds}\cdot\frac{1}{\frac{dx}{ds}}\cdot\frac{1}{\frac{dy}{dx}} = \frac{\frac{dT}{ds}}{\cos\theta\tan\theta} \quad .$$

Hence

$$\frac{dW}{ds} = \frac{dT}{dy} .$$

Recall that $a(s) = a_o - \frac{T(s)}{E}$. Then from (3)

$$\frac{dT}{dy} = \rho g \left[a_o - \frac{T(s)}{E} \right] ; \quad \text{and}$$

$$\frac{dT}{dx} = \frac{dT}{dy}\frac{dy}{dx} = \rho g (a_o - \frac{T(s)}{E}) \frac{\sqrt{T^2 - H^2}}{H} ,$$

since $\frac{dy}{dx}$ is the slope of the equilibrium curve, to which the tension T(s) is parallel. Thus,

$$\frac{1}{(a_o - \frac{T}{E}) \sqrt{T^2 - H^2}} \frac{dT}{dx} = \frac{\rho g}{H} .$$

But $T = H \sec \theta$; therefore

$$\frac{H(\sec \theta)(\tan \theta) \frac{d\theta}{dx}}{(a_o - \frac{H}{E} \sec \theta) \sqrt{H^2(\sec^2\theta - 1)}} = \frac{\rho g}{H} .$$

Multiply the left member of the above equation by $\frac{\cos \theta}{\cos \theta}$. Then

$$\frac{1}{- \frac{H}{E} + a_o \cos \theta} \frac{d\theta}{dx} = \frac{\rho g}{H} .$$

Now

$$a(s) = a_o - \frac{T(s)}{E} \quad ,$$

$$T(s) = H(s) \sec \theta, \quad \text{and}$$

$$a(s) = a_o - \frac{H}{E} \sec \theta.$$

Since $a(s) > 0$, $a_o > \frac{H}{E} \sec \theta$, and since $\sec \theta \geqslant 1$, then $a_o > \frac{H}{E}$. Using this fact and integrating with respect to x gives

$$\frac{1}{\sqrt{a_o^2 - (\frac{H}{E})^2}} \ln \frac{(a_o - \frac{H}{E}) + \sqrt{a_o^2 - (\frac{H}{E})^2} \tan \frac{\theta}{2}}{(a_o - \frac{H}{E}) - \sqrt{a_o^2 - (\frac{H}{E})^2} \tan \frac{\theta}{2}} = \frac{\rho g x}{H} .$$

Note: That the denominator of the argument of the logarithm is positive may be verified as follows.

If the string does not break, the cross-sectional area $a(s)$ remains positive. Equation (2) then implies that $T < a_o E$. But

$$T = \frac{H}{\cos \theta} ;$$

so

$$\frac{H}{\cos \theta} < a_o E,$$

or, since $\cos \theta > 0$,

$$\cos \theta > \frac{H}{a_o E} .$$

Therefore

$$1 - \cos\theta < 1 - \frac{H}{a_oE} ;$$

$$1 + \cos\theta > 1 + \frac{H}{a_oE} ;$$

and $$\tan\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} < \sqrt{\frac{1 - \frac{H}{a_oE}}{1 + \frac{H}{a_oE}}} = \sqrt{\frac{a_o - \frac{H}{E}}{a_o + \frac{H}{E}}}$$

Finally,

$$(a_o - \frac{H}{E}) - \sqrt{a_o^2 - (\frac{H}{E})^2}\ \tan\frac{\theta}{2} > (a_o - \frac{H}{E}) - \sqrt{a_o^2 - (\frac{H}{E})^2}\sqrt{\frac{a_o - \frac{H}{E}}{a_o + \frac{H}{E}}}$$

$$= 0,$$

as was to be shown.

Now let $a_o - \frac{H}{E} = u$ and $\sqrt{a_o^2 - (\frac{H}{E})^2} = v$. Then

$$\ln\frac{u + v\tan\frac{\theta}{2}}{u - v\tan\frac{\theta}{2}} = \frac{v\rho gx}{H} , \quad \text{and}$$

$$u + v\tan\frac{\theta}{2} = e^{\frac{v\rho gx}{H}}(u - v\tan\frac{\theta}{2}).$$

Thus

$$v(\tan\frac{\theta}{2})(1 + e^{\frac{v\rho gx}{H}}) = u(e^{\frac{v\rho gx}{H}} - 1), \quad \text{and}$$

$$\tan\frac{\theta}{2} = \frac{u}{v}\tanh\frac{v\rho gx}{2H} .$$

Then

$$\theta = 2 \operatorname{Tan}^{-1}\left[\frac{a_o - \frac{H}{E}}{\sqrt{a_o^2 - (\frac{H}{E})^2}} \tanh \frac{\rho g}{2H} \sqrt{a_o^2 - (\frac{H}{E})^2}\; x\right].$$

But $\frac{dy}{dx} = \tan \theta$. Hence y is expressed as a function of x by the relation

$$y(x) = \int_0^x \tan\left\{2 \operatorname{Tan}^{-1}\left[\frac{a_o - \frac{H}{E}}{\sqrt{a_o^2 - (\frac{H}{E})^2}} \tanh\left(\frac{\rho g}{2H} \sqrt{a_o^2 - (\frac{H}{E})^2}\, t\right)\right]\right\} dt, \qquad (4)$$

where $0 \leqslant x \leqslant X_1$. Note that y(x) as given above is an even function of x.

Let $\alpha = \frac{a_o - \frac{H}{E}}{\sqrt{a_o^2 - (\frac{H}{E})^2}}$ and $\beta = \sqrt{a_o^2 - (\frac{H}{E})^2}\; \frac{\rho g}{2H}$. Now

$$\tan \theta = \tan 2(\tfrac{\theta}{2}) = \frac{\tan \frac{\theta}{2} + \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}.$$

But $\frac{\theta}{2} = \operatorname{Tan}^{-1}(\alpha \tanh \beta x)$. Hence

$$\tan \frac{\theta}{2} = \tan [\operatorname{Tan}^{-1}(\alpha \tanh \beta x)] = \alpha \tanh \beta x.$$

Thus

$$\tan \theta = \frac{2 \alpha \tanh \beta x}{1 - \alpha^2 \tanh^2 \beta x},$$

and

$$y = \int_0^x \tan [\theta(t)]\, dt = \int_0^x \frac{2 \alpha \tanh \beta t \, dt}{1 - \alpha^2 \tanh^2 \beta t}.$$

Let $\xi = \alpha^2 \tanh^2 \beta t$; then $d\xi = 2 \alpha^2 \beta \tanh \beta t \operatorname{sech}^2 \beta t \, dt$.

When $t = 0$, $\xi = 0$; when $t = x$, $\xi = \alpha^2 \tanh^2 \beta x$. So

$$y = \int_0^{\alpha^2\tanh^2\beta x} \frac{2\alpha \tanh\beta t \, d\xi}{2\alpha^2 \beta \tanh\beta t (\operatorname{sech}^2\beta t)(1-\xi)}$$

$$= \frac{\alpha}{\beta} \int_0^{\alpha^2\tanh^2\beta x} \frac{d\xi}{(1-\xi)(\alpha^2-\xi)} .$$

The fact that $\operatorname{sech}^2\beta t = \frac{1}{\alpha^2}(\alpha^2 - \xi)$ has been used. But

$$\frac{1}{(1-\xi)(\alpha^2-\xi)} = \frac{1}{1-\alpha^2}\left(\frac{1}{\alpha^2-\xi} - \frac{1}{1-\xi}\right);$$

therefore

$$y = \frac{\alpha}{\beta(1-\alpha^2)} \int_0^{\alpha^2\tanh^2\beta x} \left(\frac{1}{\alpha^2-\xi} - \frac{1}{1-\xi}\right) d\xi .$$

Since $a_0 > \frac{H}{E} > 0$, it follows from the definition of α that

$$0 < \alpha < 1,$$

and hence that

$$\alpha^2 < 1.$$

Then

$$y = \frac{\alpha}{\beta(1-\alpha^2)} \left[-\ln|\alpha^2-\xi| + \ln|1-\xi| \right]_0^{\alpha^2\tanh^2\beta x}$$

$$= \frac{\alpha}{\beta(1-\alpha^2)} \left[\ln \frac{1-\alpha^2\tanh^2\beta x}{\alpha^2(1-\tanh^2\beta x)} - \ln \frac{1}{\alpha^2} \right]$$

$$= \frac{\alpha}{\beta(1-\alpha^2)} \left[\ln \frac{1-\alpha^2\tanh^2\beta x}{\operatorname{sech}^2\beta x} \right]; \qquad (5)$$

or

$$y = \frac{\alpha}{\beta(1-\alpha^2)} \left[\ln (1-\alpha^2\tanh^2\beta x) - 2\ln \operatorname{sech}\beta x \right]; \qquad (6)$$

or

$$y = \frac{\alpha}{\beta(1-\alpha^2)} \left[\ln (1+\alpha\tanh\beta x) + \ln (1-\alpha\tanh\beta x) - 2\ln \operatorname{sech}\beta x \right]. \qquad (7)$$

CHAPTER III

THE ELASTIC CATENARY WITH SUPPORTS OF EQUAL HEIGHT

Consider now an elastic catenary whose end supports are of equal height. Elementary statical considerations lead to the conclusions that the equilibrium curve of the string is symmetrical about the y-axis, that $X_o = X_1 = L$, and that the vertical reactions at the end supports are equal. The conclusions of symmetry are borne out by the mathematical model, since the function derived in the preceding chapter for the equilibrium curve is an even function.

Various formulations of the problem of the elastic catenary with horizontal supports may now be considered. In the preceding chapter, an explicit formula (equation (6)) for the equilibrium curve was produced in which the parameters α and β depend only upon the quantities ρ, g, a_o, H and E. Because of the coordinate system introduced in the mathematical analysis of this problem, however, information in addition to this formula is required to completely determine the equilibrium curve of the string. For example, knowledge of either the maximum value of y or of the distance between supports would permit a graph of the curve to be drawn. The quantities ρ, g, a_o and E are constants which are assumed to be given in <u>any</u> formulation of the problem; the remaining parameters to be considered are H, L and L_o. These three parameters are not independent; it is shown below that, given any two parameters, the third may be calculated from those given. Hence, given any two of the quantities H, L and L_o, a complete solution of the problem of the elastic catenary with supports of equal height is established.

Recall that T(x) is the axial tension in the string. Let V(x) be the vertical component of T(x); the horizontal component is H. Then

$$\sqrt{T^2 - H^2} = V$$

and

$$\tan \theta = \frac{\sqrt{T^2 - H^2}}{H} .$$

But it has already been shown that $\tan \frac{\theta}{2} = \alpha \tanh \beta x$; so

$$\frac{\sqrt{T^2 - H^2}}{T + H} = \alpha \tanh \beta x, \quad \text{and}$$

$$\frac{V}{H + \sqrt{V^2 + H^2}} = \alpha \tanh \beta x.$$

The total weight of the string is $2\rho g a_o L_o$; and at the supports, $V = \rho g a_o L_o$. Hence

$$\frac{\rho g a_o L_o}{H + \sqrt{H^2 + (\rho g a_o L_o)^2}} + \frac{a_o - \frac{H}{E}}{\sqrt{a_o^2 - (\frac{H}{E})^2}} \tanh \frac{\rho g L}{2H} \sqrt{a_o^2 - (\frac{H}{E})^2} . \quad (8)$$

This transcendental equation relates the three quantities H, L and L_o; given any two, we may find the third.[2] Thus, the problem of the elastic catenary with supports of equal height is completely solved for the three cases where H and L, H and L_o, or L and L_o are given. For, if L and H are given, α and β are determined, the distance between supports is known, and the maximum ordinate y(L) may be calculated directly. In the other two cases, H or L may be found from equation (8), thus reducing the other cases to the first case.

2. See Appendix II.

CHAPTER IV

THE ELASTIC CATENARY WITH SUPPORTS OF UNEQUAL HEIGHTS

The remaining problem is that of the elastic catenary whose end supports are not at the same height. Suppose that the left support is at the lower height, and let M denote the vertical distance between the two supports. As before, 2L is the horizontal distance between supports, and the origin of the coordinate system is taken at the lowest point of the deflected string where the tangent line is horizontal. Note that this analysis is not applicable to catenaries which have no point of horizontal tangency. It is obvious that such catenaries can be realized physically. It is assumed here, however, that a horizontal tangent is realized on the catenary under consideration. Then the solution for y given in equation (5) is applicable for this case also, since no assumptions about the heights of the supports were used in deriving it.

For this case, the values of y at the end supports are obviously different; numbers x_1 and x_2, where $x_2 > x_1 \geq 0$, must now be determined such that

$$y(x_2) - y(-x_1) = M$$

and

$$x_1 + x_2 = 2L.$$

By use of equation (5), the first condition becomes

$$\frac{\alpha}{\beta(1-\alpha^2)}\left[\ \ln\left(\frac{1-\alpha^2\tanh^2\beta x_2}{\operatorname{sech}^2\beta x_2}\right) - \ln\left(\frac{1-\alpha^2\tanh^2(-\beta x_1)}{\operatorname{sech}^2(-\beta x_1)}\right)\right]=M,$$

which implies that

$$\ln\left[\frac{(1-\alpha^2\tanh^2\beta x_2)\ \operatorname{sech}^2\beta x_1}{(1-\alpha^2\tanh^2\beta x_1)\ \operatorname{sech}^2\beta x_2}\right] = \frac{\beta(1-\alpha^2)}{\alpha}\ M. \qquad (9)$$

But $x_1 = 2L - x_2$; hence

$$\ln\left[\frac{(1-\alpha^2\tanh^2\beta x_2)\ \operatorname{sech}^2\beta(2L-x_2)}{(1-\alpha^2\tanh^2\beta(2L-x_2))\ \operatorname{sech}^2\beta x_2}\right] = \frac{\beta(1-\alpha^2)}{\alpha}\ M. \qquad (10)$$

Given values for H and L, this transcendental equation is to be solved for x_2, and then x_1 may be calculated. When x_1 and x_2 have been determined, the problem is completely solved, since the distances between the origin and the supports are then known.

This method suffices for the case where H and L are given; but if either H and L_o or L and L_o are given, equation (10) does not yield an explicit solution for x_2. For these two cases, the concept of the <u>reduced</u> <u>catenary</u> is introduced. The reduced catenary is a catenary whose supports are of equal height and spaced a distance $2x_1$ or $2x_2$ apart. Thus, given a particular elastic catenary whose supports are of unequal heights, there correspond two reduced catenaries, one smaller than the other. Elementary statical considerations lead to the conclusion that the horizontal reactions for both reduced catenaries are identical with those of the catenary from which they are derived. The results of Chapter III are applicable to each reduced catenary. By use of equation (8), the vertical reactions at the supports of the reduced catenaries may be calculated as functions of H and x_1 or x_2. Thus, for the reduced catenary with supports spaced a distance $2x_1$ apart ($x_1 > 0$), the vertical reactions V_1 are given by

$$\frac{V_1}{H + \sqrt{H^2 + V_1^2}} = \sqrt{\frac{a_o - \frac{H}{E}}{a_o + \frac{H}{E}}} \tanh \frac{\rho g x_1}{2H} \sqrt{a_o^2 - (\frac{H}{E})^2} ;$$

and

$$\ln \left[\frac{1}{\frac{H}{V_1} + \sqrt{1 + (\frac{H}{V_1})^2}} \right] = \ln \sqrt{\frac{1 - \frac{H}{a_o E}}{1 + \frac{H}{a_o E}}}$$

$$+ \ln \left[\tanh \left(\frac{\rho g a_o x_1}{2H} \sqrt{1 - (\frac{H}{a_o E})^2} \right) \right].$$

Hence

$$-\sinh^{-1} (\frac{H}{V_1}) = -\tanh^{-1} (\frac{H}{a_o E}) + \ln \tanh (\frac{\rho g a_o x_1}{2H} \sqrt{1 - (\frac{H}{a_o E})^2}),$$

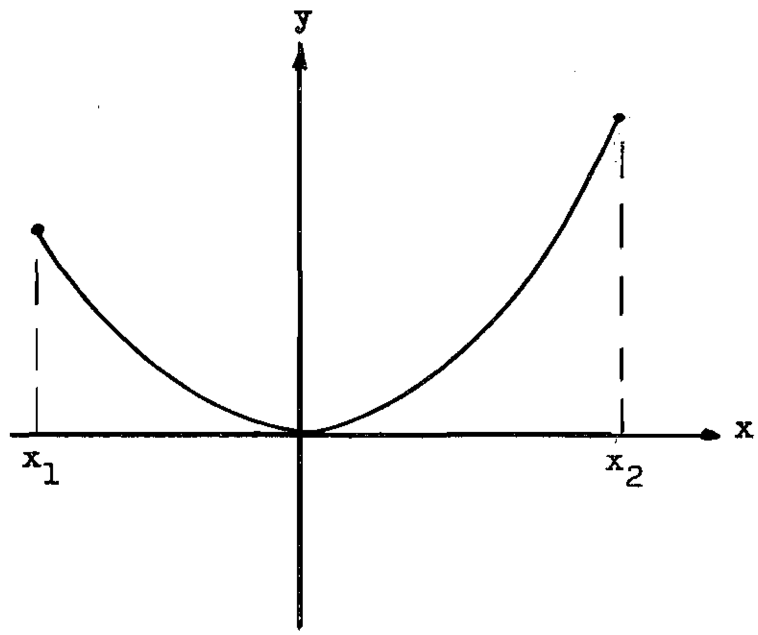

Figure 2(a). The Elastic Catenary with Supports of Unequal Heights.

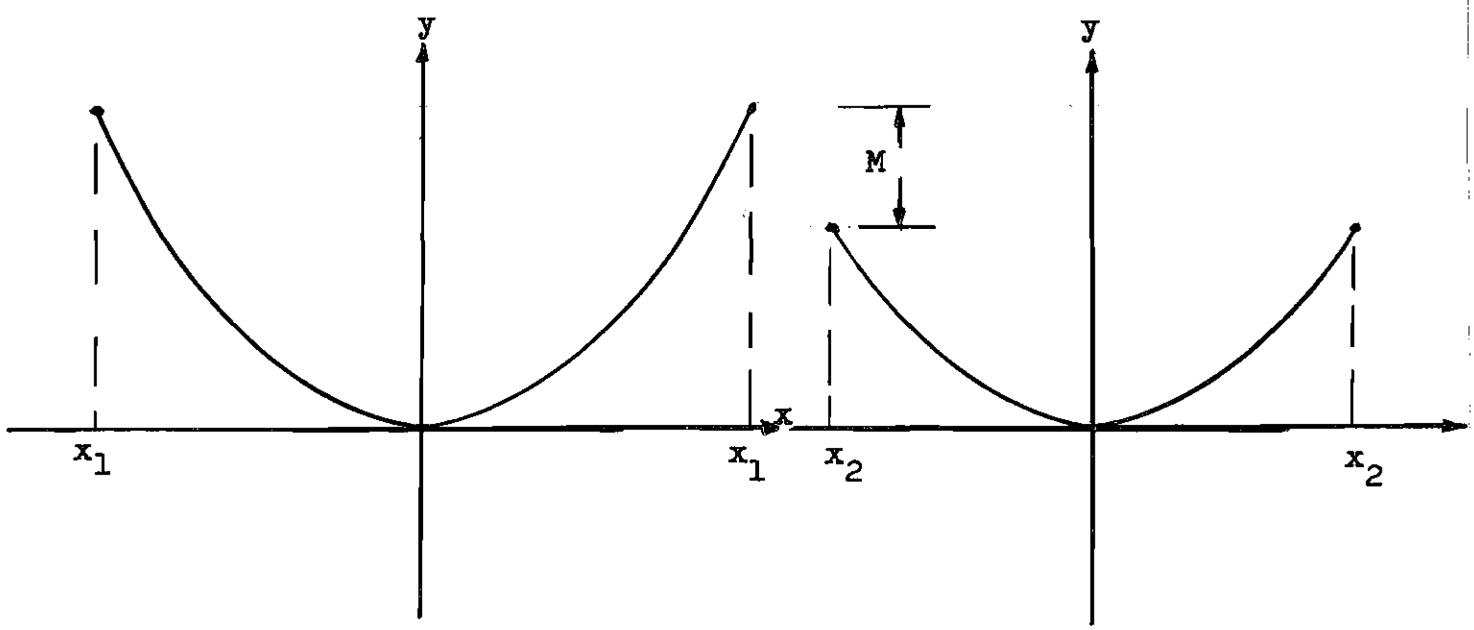

Figure 2(b). The Reduced Catenaries.

and

$$\frac{H}{V_1} = \sinh\left[\tanh^{-1}\left(\frac{H}{a_oE}\right) - \ln\tanh\frac{\rho g a_o x_1}{2H}\sqrt{1-\left(\frac{H}{a_oE}\right)^2}\right].$$

Thus

$$V_1 = H\,\mathrm{csch}\left[\tanh^{-1}\left(\frac{H}{a_oE}\right) - \ln\tanh\frac{\rho g a_o x_1}{2H}\sqrt{1-\left(\frac{H}{a_oE}\right)^2}\right]$$

if $x_1 > 0$. Note that $V_1 = 0$ if $x_1 = 0$.

Similarly for the other reduced catenary, the vertical reactions V_2 are given by

$$V_2 = H\,\mathrm{csch}\left[\tanh^{-1}\left(\frac{H}{a_oE}\right) - \ln\tanh\frac{\rho g a_o x_2}{2H}\sqrt{1-\left(\frac{H}{a_oE}\right)^2}\right],$$

since, in accordance with the conventions chosen, x_2 is always positive. But V_1 represents the actual vertical reaction at the support $x = -x_1$ of the catenary whose supports are of unequal height; and V_2 represents the vertical reaction at the support $x = x_2$. This conclusion is based on simple statical considerations, by use of the fact that the vertical reaction at the origin of the coordinate system is zero, since the tangent to the curve is horizontal there. Hence, the original weight of the unstretched string is equal to $V_1 + V_2$, or

$$V_1 + V_2 = 2\rho g a_o L_o$$

$$= H\left\{\operatorname{csch}\left[\tanh^{-1}\left(\frac{H}{a_oE}\right) - \ln\tanh\frac{\rho g a_o(2L-x_2)}{2H}\sqrt{1-\left(\frac{H}{a_oE}\right)^2}\right]\right.$$

$$\left. + \operatorname{csch}\left[\tanh^{-1}\left(\frac{H}{a_oE}\right) - \ln\tanh\frac{\rho g a_o x_2}{2H}\sqrt{1-\left(\frac{H}{a_oE}\right)^2}\right]\right\}, \quad (11a)$$

if $x_1 > 0$, and

$$V_1 + V_2 = V_2 = 2\rho g a_o L_o = H\operatorname{csch}\left[\tanh^{-1}\left(\frac{H}{a_oE}\right) - \ln\tanh\frac{\rho g a_o x_2}{2H}\sqrt{1-\left(\frac{H}{a_oE}\right)^2}\right], \quad (11b)$$

if $x_1 = 0$.

If either H and L_o or L and L_o are given, then equations (10) and (11) constitute a system of two equations to be solved for the two unknowns x_2 and the remaining parameter.

Questions of existence and uniqueness of solutions of equation (10) or equations (10) and (11) together, given pairs of the parameters H, L and L_o, are not discussed here. It is not to be expected that for all possible values of the constants, solutions for x_1 and x_2 exist, for the reason discussed earlier. In any particular case, trial-and-error methods may be used to calculate approximate solutions.

APPENDIX I

THE INELASTIC CATENARY

The problem of the inelastic catenary may be stated as follows:

> A perfectly flexible inelastic string of length L_o, uniform density ρ and cross-sectional area A is suspended between two supports spaced a horizontal distance L and a vertical distance M apart, where $L_o > \sqrt{L^2 + M^2}$. The string is allowed to reach an equilibrium configuration under the influence of gravity. Determine the equation representing the equilibrium configuration.

The equilibrium position of the string is characterized by the minimum of its gravitational potential; thus its center of gravity lies as low as possible (subject to the inextensibility of the string). This statement is an application of Torricelli's Law (a physical law), which is adopted as the model for this problem.

Introduce an x-y cartesian coordinate system, measuring x positively to the right from the left-hand support and y positively upward from the left-hand support, so that the coordinates of the right hand support are (L,M).

If the equilibrium position of the string is given by the function f, then the height of its center of gravity (relative to the coordinate system chosen above) is given by

$$\overline{y[f]} = \frac{\int_0^L f(x)\sqrt{1+(f'(x))^2}\,dx}{\int_0^L \sqrt{1+(f'(x))^2}\,dx} \quad .$$

Thus, to determine the actual equilibrium configuration, say $f_o(x)$ [if it exists], it is necessary to minimize $\overline{y}[f]$ subject to the side condition

$$\int_0^L \sqrt{1+(f'(x))^2}\,dx = L_o.$$

Assume that the solution function, if it exists, has a continuous second derivative on (0,L) and is continuous on [0,L]. Then the model for the inelastic catenary leads to the following isoperimetric problem in the calculus of variations.

> Find a function f of class C^2 on (0,L) and class C on [0,L] which minimizes
>
> $$\int_0^L f(x)\sqrt{1+(f'(x))^2}\,dx,$$
>
> and subject to the side condition
>
> $$\int_0^L \sqrt{1+(f'(x))^2}\,dx = L_o$$
>
> and such that
>
> $$f(0) = 0 \text{ and } f(L) = M.$$

A necessary condition that a function f be a solution of this problem is that it satisfy the Euler differential equation

$$\frac{\partial}{\partial f}\left[(f + \lambda)\sqrt{1 + (f')^2}\right] - \frac{d}{dx}\frac{\partial}{\partial f'}\left[(f + \lambda)\sqrt{1+(f')^2}\right]$$

$$= 0,$$

for a suitable choice of the parameter λ.
Thus

$$\frac{d}{dx}\left[\frac{(f + \lambda)\, f'}{\sqrt{1 + (f')^2}}\right] - \sqrt{1 + (f')^2} = 0.$$

The general integral for this differential equation is known, and is

$$f(x) = C \cosh \frac{x + D}{C} - \lambda \quad . \tag{12}$$

The three constants C, D and λ must be determined from the three equations

$$f(0) = 0, \quad f(L) = M, \quad \text{and} \int_0^L \sqrt{1 + (f'(x))^2} = L_o.$$

In terms of the solution (equation (12)), these conditions are

$$C \cosh \frac{D}{C} - \lambda = 0,$$

$$C \cosh \frac{L + D}{C} - \lambda = M \quad \text{and} \tag{13}$$

$$\int_0^L \cosh \frac{x + D}{C}\, dx = C \left\{ \sinh \frac{L + D}{C} - \sinh \frac{D}{C} \right\} = L_o.$$

The degenerate case of the catenary, where

$$L^2 + M^2 = L_o^2 ,$$

has been excluded in the statement of the problem, since it represents the trivial physical case. In mathematical terms, there are no admissible comparison curves except the straight line between (0,0) and (L,M).

Now let

$$\alpha = \frac{D}{C} \quad \text{and} \quad \beta = \frac{L + D}{C} .$$

Then, from equations (13),

$$M = C\,[\cosh\beta - \cosh\alpha\,] = 2C \sinh \frac{\beta - \alpha}{2} \sinh \frac{\beta + \alpha}{2}$$

and

$$L_o = C\,[\sinh\beta - \sinh\alpha\,] = 2C \sinh \frac{\beta - \alpha}{2} \cosh \frac{\beta + \alpha}{2} . \tag{14}$$

But $M < L_o$ implies $\frac{M}{L_o} < 1$; hence

$$\frac{\alpha + \beta}{2} = \tanh^{-1} \frac{M}{L_o} \triangleq \gamma \tag{15}$$

is uniquely determined. Now let $\delta = \frac{\beta - \alpha}{2}$. Substituting equation (15) into equation (14) and noting that

$$C = \frac{L}{\beta - \alpha} \quad \text{(so that } 2C = \frac{L}{\delta}\text{)}$$

yields the following transcendental equation for δ:

$$\frac{\delta L_o}{L \cosh \gamma} = \sinh \delta . \tag{16}$$

From equation (15),

$$\frac{L_o}{L \cosh \gamma} = \frac{L_o}{L} \sqrt{1 - \tanh^2 \gamma} = \frac{L_o}{L} \sqrt{1 - \left(\frac{M}{L_o}\right)^2}$$

$$= \frac{\sqrt{L_o^2 - M^2}}{L} > 1, \text{ since}$$

$$L_o^2 > L^2 + M^2 \text{ implies } L^2 < L_o^2 - M^2.$$

Hence

$$\frac{L_o}{L \cosh \gamma} > 1.$$

Thus, equation (16) has exactly two non-zero solutions, differing only in sign, say $\delta_1 > 0$ and $\delta_2 = -\delta_1$. In terms of the known solutions δ_1 and δ_2,

$$C = \pm \frac{M}{2\delta_1}, \quad D = -\frac{M}{2} \pm \gamma \frac{M}{2\delta_1}, \quad \text{and} \quad \lambda$$

can be calculated from equations(13).

Hence, there are exactly <u>two</u> catenaries which connect the two supports and have the prescribed length; to the negative root of equation (16), $\delta_2 = -\delta_1$, corresponds a catenary opening downward, and to the positive root δ_1 corresponds a catenary opening upward. A second application of Torricell's Law gives as the solution the concave upward catenary

$$f(x) = C \cosh \frac{x + D}{C} - \lambda ,$$

where the constants are evaluated as shown above.

The above procedure only guarantees the necessity of equation (12) as the solution of the problem; but it can be shown that the solution (12) does indeed yield a minimum for the gravitational potential, subject to the inextensibility condition.

APPENDIX II

SOLUTIONS OF A CERTAIN TRANSCENDENTAL EQUATION

In Chapter III, physical considerations led to the transcendental equation (8):

$$\frac{\rho g a_o L_o}{H + \sqrt{H^2 + (\rho g a_o L_o)^2}} = \frac{a_o - \frac{H}{E}}{\sqrt{a_o^2 - (\frac{H}{E})^2}} \tanh \frac{\rho g L}{2H} \sqrt{a_o^2 - (\frac{H}{E})^2} .$$

Since $V = \rho g a_o L_o$, this equation becomes

$$\frac{V}{H + \sqrt{H^2 + V^2}} = \sqrt{\frac{1 - \frac{H}{a_o E}}{1 + \frac{H}{a_o E}}} \tanh\left(\frac{VL}{2HL_o}\sqrt{1 - (\frac{H}{a_o E})^2}\right). \qquad (17)$$

Given any two of the three parameters H, L and L_o ($0 < H < a_o E$, $0 < V < a_o E$) and with ρ, g, a_o and E fixed constants, it is desired to prove that a unique real solution for the third is guaranteed.

Suppose first that H and L_o are given. Then, equivalently, H and V are given ($0 < H < a_o E$, $0 < V < a_o E$) and

$$L = \frac{2HL_o}{V}\left(\sqrt{1 - (\frac{H}{a_o E})^2}\right)^{-1} \tanh^{-1} \left[\frac{V}{H + \sqrt{H^2 + V^2}} \sqrt{\frac{1 + \frac{H}{a_o E}}{1 - \frac{H}{a_o E}}}\right]. \qquad (18)$$

Thus L is uniquely determined.

Next, suppose that H and L are given. Then

$$\ln \left[\frac{V}{H + \sqrt{H^2+V^2}}\right] = \ln \sqrt{\frac{1 - \frac{H}{a_oE}}{1 + \frac{H}{a_oE}}} + \ln \tanh \left(\frac{\rho g L}{2H}\sqrt{a_o^2 - (\frac{H}{E})^2}\right).$$

Let $\mu = \frac{\rho g l}{2H} \sqrt{a_o^2 - (\frac{H}{E})^2}$, and note that μ is fixed when H and L are given. Hence

$$-\sinh^{-1}(\frac{H}{V}) = -\tanh^{-1} \frac{H}{a_oE} + \ln \tanh \mu$$

implies that

$$\frac{H}{V} = \sinh \left[\tanh^{-1} \frac{H}{a_oE} - \ln \tanh \mu \right]; \text{ or}$$

$$V = H \operatorname{csch} \left[\tanh^{-1} \frac{H}{a_oE} - \ln \tanh \mu \right]. \quad (19)$$

Thus $L_o = \frac{V}{\rho g a_o}$ is uniquely determined.

Finally, suppose that L and L_o are given. That at least one solution for H in the open interval $(0, a_oE)$ exists is shown by the following argument. Rearranging equation (17) yields

$$\frac{V}{H + \sqrt{H^2+V^2}} \sqrt{\frac{1 + \frac{H}{a_oE}}{1 - \frac{H}{a_oE}}} = \tanh \left(\frac{VL}{2HL_o} \sqrt{1 - (\frac{H}{a_oE})^2}\right). \quad (20)$$

This equation has exactly the same real solutions for H on $(0, a_oE)$ as does equation (17). When $H \to 0^+$, both sides of equation (20) approach 1. Considering the left hand side of equation (20) as a function of H on $(0, a_oE)$, we obtain

$$\frac{d}{dH}\left[\frac{V}{H+\sqrt{H^2+V^2}}\sqrt{\frac{1+\frac{H}{a_oE}}{1-\frac{H}{a_oE}}}\right] = \frac{-V}{H+\sqrt{H^2+V^2}}\,\frac{1}{\sqrt{H^2+V^2}}\sqrt{\frac{1+\frac{H}{a_oE}}{1-\frac{H}{a_oE}}}$$

$$+\ \frac{1}{a_oE}\ \frac{V}{H+\sqrt{H^2+V^2}}\sqrt{\frac{1-\frac{H}{a_oE}}{1+\frac{H}{a_oE}}}\ \frac{1}{(1-\frac{H}{a_oE})^2}\ ;$$

so that

$$\lim_{H\to 0^+}\left\{\frac{d}{dH}\left[\frac{V}{H+\sqrt{H^2+V^2}}\sqrt{\frac{1+\frac{H}{a_oE}}{1-\frac{H}{a_oE}}}\right]\right\} = \frac{1}{a_oE} - \frac{1}{V}\ .$$

But $\frac{1}{a_oE} - \frac{1}{V} < 0$, since $V < a_oE$. Considering the right hand side of equation (20) as a function of H on $(0, a_oE)$, we obtain

$$\frac{d}{dH}\left\{\tanh\left(\frac{VL}{2HL_o}\sqrt{1-(\frac{H}{a_oE})^2}\right)\right\} = \left[\frac{-VL\sqrt{1-(\frac{H}{a_oE})^2}}{2H^2L_o}\right.$$

$$\left. -\ \frac{VL}{2L_o(a_oE)^2\sqrt{1-(\frac{H}{a_oE})^2}}\right]\ \text{sech}^2\left[\frac{VL}{2HL_o}\sqrt{1-(\frac{H}{a_oE})^2}\right];$$

so that

$$\lim_{H\to 0^+}\left\{\frac{d}{dH}\left[\tanh\left(\frac{VL}{2HL_o}\sqrt{1-(\frac{H}{a_oE})^2}\right)\right]\right\} = 0.$$

Hence, the slope of the left member of equation (20) approaches a negative value as $H \to 0^+$, while the slope of the right member approaches zero as $H \to 0^+$. But as $H \to a_0E^-$, the left member of equation (20) approaches infinity, while the right member approaches 0. Hence the left and right members, considered as separate functions of H, must intersect at least once on $(0, a_0E)$; that is, equation (17) has at least one real solution on $(0, a_0E)$. The question of uniqueness is not answered here. The concerted efforts of the author, his advisor, and a large number of senior mathematics students have failed to produce either a proof of uniqueness or a counterexample (the use of the facilities of the Computer Center in the search for a counterexample is gratefully acknowledged). Note that for the case where L and L_0 are given, no explicit formula for H is determined, so that for any particular case, trial-and-error or approximation methods must be used to generate an approximate solution for H. It is clear that physical arguments about the determinacy of the actual problem are not sufficient to insure a unique solution for H, for the model used to idealize the physical problem is not perfect. A uniqueness theorem for equation (17) in all three cases considered above would have afforded a check on the consistency of the mathematical model, which must now only be conjectured.

APPENDIX III

THE ELASTIC CATENARY WITH NO HORIZONTAL TANGENT

The problem of the elastic catenary with no horizontal tangent is stated as before:

> A perfectly flexible elastic string (or other suitable continuum), when unstretched, has length $2L_o$, uniform cross-sectional area a_o, constant density ρ, and modulus of elasticity E. The string is suspended between two points of support whose projections on a horizontal plane are a distance 2L apart. The points of support <u>may differ</u> in altitude by an amount M; and the distance $\sqrt{(2L)^2 + M^2}$ between them may be greater than, equal to, or less than the unstretched length $2L_o$ of the string. The horizontal reaction at each support is designated by the letter H. The string is allowed to reach an equilibrium configuration under the influence of gravity. Determine the equation representing the equilibrium configuration.

Four of the five assumptions made previously in constructing the mathematical model are retained; only the assumption that the catenary has a horizontal tangent is deleted. Without loss of generality the left hand support of the elastic catenary may be chosen as the lower of the two. Let V_1 be the vertical reaction and H the horizontal reaction at the lower support.

To given values of the parameters ρ, a_o, E and H there corresponds by use of the model of Chapter II a unique elastic catenary with a horizontal tangent (call this configuration the <u>constructed catenary</u>). Furthermore, the vertical support reaction $V = V[s(x)]$ of the constructed catenary is a strictly increasing continuous function of arc length s (measured from the point of horizontal tangency where $V(0) = 0$). Hence there exist a unique value s_1 such that $V[s_1] = V_1$ and a unique value x_1 such that $s(x_1) = s_1$, where $V[s(x_1)] = V_1$.

The following assertion will be proved:

> The portion of the constructed catenary for $x_1 \leq x \leq x_1+2L$ is exactly the curve for the given elastic catenary with no horizontal tangent.

Thus, the case of the elastic catenary with no horizontal tangent may be considered as a special case of the general analysis of Chapters II, III and IV.

Proof. Consider the portion of the constructed catenary for $0 \leq x \leq x_1$. Then, since the support reactions $V[s(x)] = V_1$ and H at $s = s_1$ coincide in magnitude with those of the lower support of the given elastic catenary, both curves have the same slope where these identical reactions occur (recall the assumption that the tension in the string acts axially). For the elastic catenary with no horizontal tangent, introduce a cartesian coordinate system by placing the left hand support of the given elastic catenary at the point $(x_1, y(x_1))$ of the constructed catenary and matching the slopes at this point. Now measure arc length s from $s = 0$ along the constructed catenary for $0 \leq s \leq s_1$ and along the given elastic catenary for $s \geq s_1$. Note that at this stage, arc length s is measured along the analytical curve given by equation (7) for $0 \leq s \leq s_1$; it is measured along the equilibrium curve for the given elastic catenary for $s \geq s_1$. It is this curve for $s \geq s_1$ which is to be determined.

The same analysis as before yields the relations

$$a(s) = a_0 - T(s)/E \text{ and}$$

$$H = T(s) \cos[\theta(s)], \text{ where } s \geq s_1.$$

If W(s) denotes the weight of the given elastic catenary in the arc length interval $[s_1, s]$, then

$$W(s) = T(s) \sin[\theta(s)] - T(s_1) \sin[\theta(s_1)].$$

Hence

$$\frac{W(s) + T(s_1) \sin[\theta(s_1)]}{H} = \tan \theta(s).$$

All further analysis of the model for this problem involves differentiation of the quantities T(s) and W(s); since W(s) differs only by a constant from its value in the work of Chapter II, the solutions coincide. Thus the assertion is proved.

APPENDIX IV

THEOREMS CITED IN THE TEXT

<u>Theorem 1</u> (A Mean Value Theorem of Integral Calculus). If f is a real-valued continuous function on the closed interval [a,b], there is some number C, where $a < C < b$, such that

$$\int_a^b f(x)\,dx = (b-a)\,f(C).$$

<u>Theorem 2</u> (Intermediate Value Theorem). If f is a real-valued continuous function on the closed interval $S=[a,b]$, then f assumes every value between its maximum, sup f[S], and its minimum, inf f[S].

<u>Theorem 3</u> (Nested Interval Theorem in E_1). If $\{F_n\}$ is a descending sequence of nonempty closed intervals in E_1, such that $d(F_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} F_n$ consists of a single point of E_1.

<u>Theorem 4</u> (Concerning the Isoperimetric Problem of the Calculus of Variations). Let $f = f(x,y,y')$ and $g = g(x,y,y')$ be continuous real-valued functions with continuous first- and second-order derivatives with respect to each variable,

and let L be a given number. If $y = y(x)$ is a real-valued continuous function and has continuous first- and second-order derivatives on $[x_1, x_2]$, with

$$y(x_1) = y_1, \quad y(x_2) = y_2, \qquad \text{and is such that}$$

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \quad \text{yields a minimum while}$$

$$C = \int_{x_1}^{x_2} g(x, y, y')\, dx = L,$$

then it is <u>necessary</u> that y satisfy the differential equation

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\frac{\partial h}{\partial y'} = 0, \quad \text{where}$$

$$h(x, y, y') = f(x, y, y') + \lambda\, g(x, y, y')$$

and λ is an arbitrary parameter.

BIBLIOGRAPHY

1. Clebsch, A., "Ueber die Gleichgewichtsfigur eines biegsamen Fadens," Crelles Journal für die reine und angewandte Mathematik, Vol. 57, 1860, pp. 93-110.

2. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. 1, New York, N.Y.: Interscience Publishers, Inc., 1963, p. 218.

3. Grüss, Gerhard, Variationsrechnung, Heidelberg, Quelle and Meyer, 1955, p. 2.

4. Grüss, Gerhard, Variationsrechnung, unpublished translation of parts of the German text by Professor G.B. Thomas, pp. 224-226.

5. McKibben, William P., Mathematical Models of Physical Phenomena, M. S. Thesis, Georgia Institute of Technology, June, 1963.

6. Maclaurin, R.C., "The Influence of Stiffness on the Form of a Suspended Wire or Tape," The Philosophical Magazine, Vol. VI, July-December, 1903, pp. 166-173.

7. MacMillan, W.D., Statics and the Dynamics of a Particle, New York, N.Y.: Dover Publications, Inc., pp. 160-172.

8. Sagan, Hans, Boundary and Eigenvalue Problems in Mathematical Physics, New York, N.Y.: John Wiley and Sons, Inc., 1961, p. 26.

9. Todhunter, I. and Pearson, K., A History of the Theory of Elasticity and of the Strength of Materials, Vol. II, New York, N.Y.: Dover Publications, Inc., 1960, pp. 107-108.

10. Young, A.E., "On the Form of a Suspended Wire or Tape including the Effect of Stiffness," The Philosophical Magazine, Vol. 29, January-June, 1915, pp. 96-127.