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AN INVESTIGATION OF TWO MEASURES OF
LINEAR SYSTEM MEMORY

A THESIS

Presented to

The Faculty of the Graduate Division

by

William Elbert Jackson, III

In Partial Fulfillment

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Master of Science in Industrial Engineering

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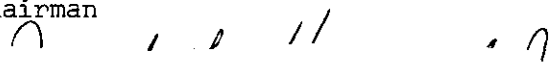
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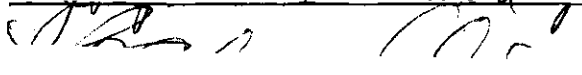
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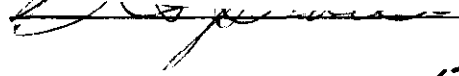
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SUMMARY

The primary objective of this investigation is to make an evaluation of two measures of linear system memory length to determine which, if either, is the better measure of memory length. The two measures with which the investigation is concerned are defined as follows:

$$|M| = \frac{\int_0^{\infty} t|h(t)|dt}{\int_0^{\infty} |h(t)|dt}$$

and

$$M_2 = \frac{\int_0^{\infty} th^2(t)dt}{\int_0^{\infty} h^2(t)dt} .$$

In these expressions $h(t)$ is the system impulse response function, and $|h(t)|$ indicates the absolute value of this function.

A brief introduction to the concept of linear system memory is given. To make the evaluation of $|M|$ and M_2 , certain criteria and relationships are established. Formulas for $|M|$ and M_2 with respect to certain systems are then used to determine whether these criteria and relationships are met. Comparison and evaluation of this information is done with graphs and tables, which are included. On the basis of the investigation certain conclusions and recommendations are made.

Considering linear systems as a whole, it is concluded that the measure M_2 is the better measure of memory length. This conclusion is reached primarily because the measure $|M|$ is very difficult to evaluate analytically for nonmonotone systems.

It is recommended, however, that for the case of comparison of monotone systems that the measure $|M|$ be used. The ease of derivation and the simple form of the formulas for this measure make this recommendation important.

It is also recommended that for the case of monotone systems in cascade that $|M|$ be used as an approximation to M_2 . This recommendation is possible due to certain characteristics of the impulse response function for systems in cascade.

CHAPTER I

INTRODUCTION

General

In recent years, a new scientific discipline known as system theory has emerged. It has been the aim of this discipline to provide the specialists of many widely-varying and dissimilar fields with a common language and a set of analytical tools. One inevitable outcome of this effort has been the interest in (if not the necessity of) classifying (or at least comparing) systems on the basis of some common characteristic(s). To a large extent, this has been done. Systems have been classified or compared according to whether they are linear or non-linear, stationary or nonstationary, stable or unstable, memory type or memoryless. Most of these classifications have been successful. One basis for classification, however, which has received comparably little attention is that associated with the notion of system memory length.

The concepts of system memory and memory length are probably best explained by viewing a system as a "box" (see Figure 1) with input

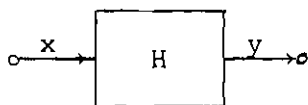


Figure 1. System Representation

(excitation) x and output (response) y with system input-output relation

$$y = Hx$$

where H serves both to identify the system and its operator. With x and y as functions of time t , this relationship can be represented as the continuous time system

$$y(t) = Hx(t). \quad (1)$$

The system H is said to be memory type if the output $y(t)$ depends not only on $x(t)$ but on the input at least at some past times. If the output $y(t)$ depends on at most $x(t)$, H is called a memoryless system. If the output $y(t)$ is completely determined by the input on the interval $[t-\ell, t](\ell \geq 0)$, the system is said to have memory of length ℓ . In this sense, a memoryless system has memory of length zero. If ℓ is finite and nonzero, the system is said to have a finite memory.*

This definition of system memory and memory length is entirely general and can be applied to any system which can be visualized or put into the form represented in Figure 1. The class of memory type systems is a very important one. In this class are found those systems which are most capable of "dealing" with their environment. This is evidenced by one outstanding member of this class, the human system itself.

*In this definition it is important to assume that the input x was applied at time $t = -\infty$.

Although the above definition of memory length is the one generally accepted, it is not the one which might be most useful. In this investigation a slightly different conception of memory length is used. This conception will be made clear in the following sections. However, for the present and for what follows, the above definition is quite sufficient.

It should be evident that if, among memory-type systems, some measure of memory length could be "reasonably" defined, a basis for classifying (or comparing) these systems would exist. To do this it would be necessary to state the nature of the relationship between the input x and the output y of Equation (1) in some more exact way. For all memory type systems this would probably be a difficult task. For linear memory type systems, however, this input-output relation can be simply and precisely stated. It is with this class of systems (memory type linear systems) that this investigation is concerned.

Linear System Memory

It is the purpose of this section to make the meaning of linear system memory clear and to introduce terms and definitions which will allow a precise statement of the nature of this investigation.

Simply stated, a system is linear if it satisfies the principle of superposition. That is, a system H is linear if and only if

$$H(\alpha x_1 + \beta x_2) = \alpha Hx_1 + \beta Hx_2,$$

where α, β are arbitrary constants and x_1, x_2 are any two inputs. The

class of physical systems which can be modeled with a linear input-output relation assumes a place of fundamental importance in all scientific work. Probably the most basic and most important representation of these systems is the transfer function representation illustrated schematically in Figure 2.

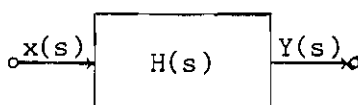


Figure 2. Transfer Function Representation of a System

Here $X(s)$ is the Laplace Transform of the input, $Y(s)$ is the Laplace Transform of the output, and $H(s)$ the system transfer function. The input-output relation for such a representation takes the form

$$Y(s) = H(s)X(s).^*$$
(2)

It will be tacitly assumed in this representation that the system is time-invariant. This amounts to saying that the system parameters are constant.

The relation given in Equation (2) establishes an important way of thinking about linear systems. One recognizes Equation (2) as the relationship which expresses the output y as the convolution of the

*A certain basic familiarity with this representation is assumed. A background for this discussion will be found in any of the references in the Bibliography dealing with linear systems.

function h and the input x . That is, the inverse transform of Equation (2) is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau^* . \quad (3)$$

The function h is the inverse transform of the system transfer function $H(s)$ and is called the system impulse response or weighting function. The relationship in Equation (3) expresses the important fact that for linear systems a knowledge of the system weighting function enables one to find the response of the system to any specific input x . Hence in a sense the linear system is completely characterized by its weighting function or equivalently by its transfer function, $H(s)$.

The concepts of memory and memory length are in general physical concepts. Hence, in order to relate these concepts to a transfer system, some assumptions about the system transfer function should be made which will insure that it is possible physically. This can be done by assuming that the transfer function is both physically realizable and stable. This amounts to making two assumptions, one about the system impulse response function and the other about the system transfer function.

A (linear) system is said to be physically realizable if its impulse response function is real for positive time and zero for negative time. This is another way of saying that the system does not react to inputs it has not received. For actual real world systems this is

* This relationship is sometimes expressed in terms of the convolution operator $*$ as $y = x*h$.

the case, and hence from a physical standpoint such an assumption is justified.

The other assumption relating to the stability of a system is of equal importance and is stated as follows: A (linear) system is stable if its transfer function has no poles in the right-half s-plane or on the imaginary axis. It is otherwise unstable.*

As a consequence of this definition it follows that the system impulse response function must approach zero at $t \rightarrow \infty$. This fact is shown easily using the final-value theorem,

$$\lim_{t \rightarrow \infty} h(t) = \lim_{s \rightarrow 0} sH(s).$$

Since the definition of stability has excluded the possibility of a pole at the origin, the value of this limit must be zero, i.e.

$$\lim_{t \rightarrow \infty} h(t) = 0.$$

From these two definitions it follows that the relationship expressed in Equation (3) for time-invariant, physically realizable, stable systems with input x applied at time 0 is given by

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau. \quad (4)$$

*This definition of stability is somewhat more restrictive than is generally used. In many cases systems with poles on the imaginary axis are classified as stable. In this investigation, however, the meaning of stability will be the one given here.

This relationship states that the output $y(t)$ of a linear system is a weighted combination of the inputs on the interval $[0,t]$. For a stable system this weighting is one which decreases with time; or, put another way, for a stable system past inputs assume less and less importance as time passes. This concept is illustrated in Figure 3 using the system weighting function.

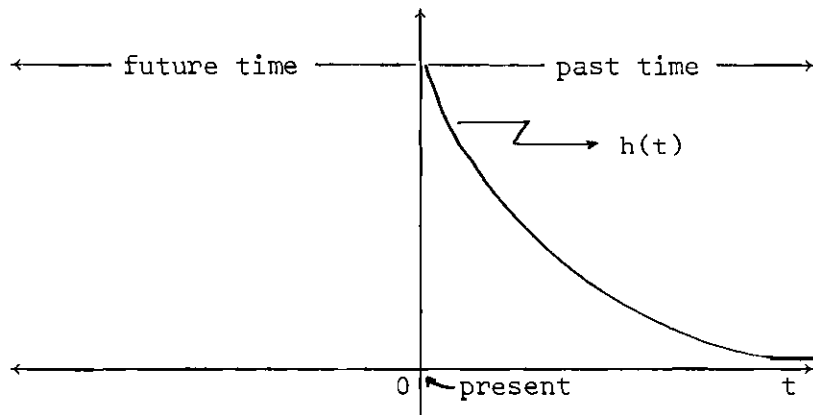


Figure 3. The Past, Present and Future for $h(t)$

The above statements make clear the concept of memory for linear systems. This meaning is essentially given by the convolution integral in Equation (4). The output $y(t)$ of the system depends on the inputs to the system over the "past" interval $[0,t]$ in a way which is completely determined by the system weighting function h . The assumption of a stable system insures that the system forgets. All memory type systems exhibit this characteristic, and in general, it is this property which gives meaning to the notion of memory length and the measures of memory length defined in the next section. The concept of memory length

at this point should be fairly clear. It should also be evident that any measure of it must be a function of the system weighting function h .*

Proposed Measures and Memory Length

The two measures of memory length with which this investigation is concerned are defined as follows:

$$|M| = \frac{\int_0^{\infty} t |h(t)| dt}{\int_0^{\infty} |h(t)| dt} \tag{5}$$

$$M_2 = \frac{\int_0^{\infty} t h^2(t) dt}{\int_0^{\infty} h^2(t) dt}$$

With these measures defined, a meaning of memory length can be clearly stated. Obviously $|M|$ and M_2 will, in general, give some measure of the length of the period beyond which the weighting function h does not differ in any significant fashion from zero. In this investigation, this period or time interval will be what is meant by system memory length. Intuitively such an interval conception of memory length is reasonable, for in this interval are "contained" those past inputs which most influence the present output. For many systems, however,

*For a more extended introduction to the memory concept for linear systems, one is referred to Brown and Nilsson [1].

this conception will differ from the one defined previously in terms of the parameter ℓ . For systems for which $\ell = +\infty$ both $|M|$ and M_2 , as well as the interval they measure, will in general take on finite values. For other systems for which ℓ is finite, $|M|$ and M_2 are finite and in some cases, have the same value as ℓ . One notes that both $|M|$ and M_2 might be looked at as measures of central tendency in the weighting function. Both measures have been appropriately modified to be valid for weighting functions which take on both positive and negative values. What might not be clear, however, is which, if either, of these measures is a better measure of memory length. The determination of this fact is one of the subjects of this investigation.

Present Investigation

Objective and Purpose

Equations (5) define two measures of linear system memory length. It is obvious that any attempt to classify or compare systems must be done on the basis of one or the other of these measures. It would be incorrect to compare two systems using $|M|$ for one system and M_2 for the other. Some attempt should be made to evaluate $|M|$ and M_2 on the basis of how well each measures memory length.

The primary objective of this investigation is to make such an evaluation of $|M|$ and M_2 . Secondary objectives include the derivation of some formulas useful for evaluating $|M|$ and M_2 in relation to specific systems and pointing out some memory characteristics which these formulas bring to attention. The purpose of the investigation is to furnish some results which will be useful in classifying (or comparing) linear systems.

Method of Investigation

To make the evaluation of $|M|$ and M_2 , certain criteria and relationships are established. The formulas for $|M|$ and M_2 with respect to certain systems (composed in general of cascade and parallel system combinations) are then used to determine whether these criteria and relationships are met. For some of the basic systems comparisons with results found in the literature are also made. All systems are represented by their transfer function or in some cases simply by an equivalent block diagram. These transfer functions are those which generally arise from systems described by linear constant coefficient differential equations. Other types of systems (or models of systems) might have been considered, but these allow for the necessary evaluation and also provide useful formulas. For certain of the more complicated nondimensional forms of $|M|$ and M_2 computer evaluations were made.

Form of Results

The results of the evaluation appear primarily in graphical form. Any formulas presented are stated in as concise a form as possible. The results of computer evaluation appear in table form.

Scope and Limitations

The general procedure used to obtain the formula results is applicable to the type systems considered in this investigation. For higher order systems, however, the derivations become increasingly complex; and in this sense, the procedure is limited. The results obtained can be generalized to all systems whose impulse response functions have a similar form. The formula results are of course restricted for use with systems for which they are derived.

Assumptions

In this investigation a certain familiarity with linear systems and their analysis is assumed. No attempt has been made at giving an exposition of linear systems beyond that given in the introduction. Concepts and notation particular to this thesis, however, are presented in Chapter III.

CHAPTER II

LITERATURE SURVEY

Apparently very little attention has been given to the notion of system memory length. The only literature found to be relevant to this investigation concerns itself with measures of system delay and measures of system performance. As one might suspect, system memory is intimately associated with system performance. In general a system with long memory will experience long delays and have a poor performance, while a system with short memory will have short delays and good performance. An introduction to the literature and concepts associated with measures of system delay and performance is thus in order.

The literature dealing with measures of system delay or system time delay is not as extensive as that related to performance measures and is reported most fully here. Part of this literature is quite significant and is reported in detail.

The literature related to performance measures is more extensive. For this reason only certain parts of the more important works relevant to this investigation are reported. Parts of this literature are also reported in detail.

System Delay

Experimentally the delay of a system is defined as the time which the system takes to respond to 50 per cent of its final value after the application of a unit step input $U(t)$. System delay is important in its

relation to system response or rise time. Rise time is usually defined as the time it takes a system to respond from 10 to 90 per cent of its final value after the application of the step input $U(t)$. Response time is an important characterization of a system, particularly in its relation to the system's frequency response. What is needed, however, are definitions of delay time and response time which are useful in computational work. The experimental definitions given above are useful in the laboratory but are very difficult to use in any theoretical investigation.

Elmore [3] did the basic work of formulating definitions of delay time and rise time in relation to the system impulse response function. This work was done, however, with particular regard to wideband amplifiers and considering only nonnegative impulse response functions.* Brown [2] carried this work a step further by considering impulse response functions which can assume both positive and negative values. His work is more general and is not restricted to any particular type of system. For this reason it is his work which is mainly reported here.

Figure 4 shows how the delay time and the response time of a system are related to its impulse response function. For monotone systems, Brown suggests (with Elmore's work in mind, no doubt) the following measures for these quantities, respectively:

*Such systems are called monotone systems since their response to a step input is monotone increasing.

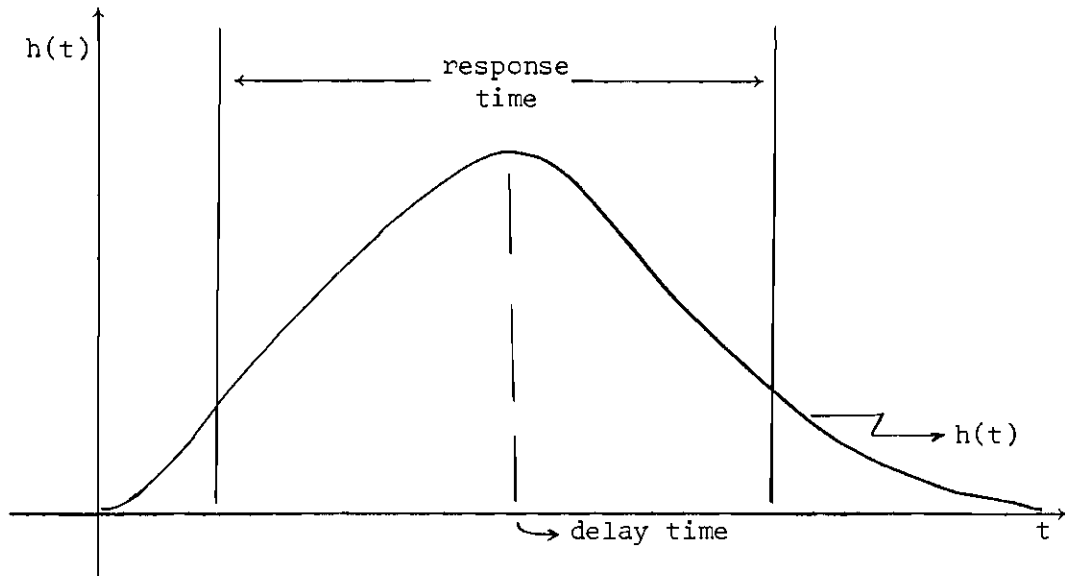


Figure 4. Response Time and Delay

$$\bar{t} \equiv \frac{\int_{-\infty}^{\infty} th(t)dt}{\int_{-\infty}^{\infty} h(t)dt}$$

(6)

$$\sigma^2 \equiv \frac{\int_{-\infty}^{\infty} (t-\bar{t})^2 h(t)dt}{\int_{-\infty}^{\infty} h(t)dt}$$

These analytical definitions correspond fairly closely with the experimental definitions for most work. In the event that \bar{t} corresponds to the median of the distribution of the impulse response function, the experimental and analytical definitions agree exactly. This follows easily since the derivative of the unit step response is $h(t)$, and 50 per cent of the "mass" of h lies on either side of the median point.

The definition of system response time is not of particular

importance to this investigation; it is only reported here to make clear the use of delay time. However, for the case of monotone systems, \bar{t} corresponds to the definition of memory length $|M|$ given in Equations (5); and some important results are given for this case. The following two results are given by Brown for delay time:

- (a)* If $H(s)$ is the system transfer function for a monotone system, then

$$\bar{t} = - \frac{H'(0)}{H(0)} ,$$

where

$$H'(s) = \frac{dH(s)}{ds} .$$

- (b)* If n monotone systems are cascaded and each individual delay is given as $\bar{t}_1, \dots, \bar{t}_n$, then the delay \bar{t} for the over-all system is given by

$$\bar{t} = \sum_{i=1}^n \bar{t}_i .$$

Result (a) follows easily from the fact that

*Corresponding results were also given for response time. Only the results for delay time are reported here.

$$H(s) = \int_{-\infty}^{\infty} e^{-st} h(t) dt$$

and

$$H'(s) = -\int_{-\infty}^{\infty} t e^{-st} h(t) dt.$$

Result (b) is proved by induction using result (a). Result (b) is clearly true for the case $n = 1$. For $n > 1$ let H_1, \dots, H_n be the transfer functions for the individual systems; $H = \prod_{i=1}^n H_i$; and $G = \prod_{i=1}^{n-1} H_i$. Assume that the result is true for $n-1$, that is

$$\sum_{i=1}^{n-1} \bar{t}_i = \frac{-G'(0)}{G(0)}.$$

Now

$$\bar{t} = \frac{-H'(0)}{H(0)} \quad \text{and} \quad H = GH_n.$$

It follows that $H' = G'H_n + GH_n'$ and hence

$$\begin{aligned} \bar{t} &= -\frac{G'(0)H_n(0) + G(0)H_n'(0)}{G(0)H_n(0)} \\ &= -\left[\frac{G'(0)}{G(0)} + \frac{H_n'(0)}{H_n(0)} \right] \\ &= \sum_{i=1}^{n-1} \bar{t}_i + \bar{t}_n = \sum_{i=1}^n \bar{t}_i, \end{aligned}$$

as was to be shown.

Thus for cascaded monotone systems the individual system delays add; or for this investigation, for cascaded monotone systems the individual system values of $|M|$ add. This is a significant result. Elmore also proves it and shows that if the over-all normalized system transfer function has the form

$$H(s) = \frac{\sum_{i=1}^n a_i s^i + 1}{\sum_{i=1}^m b_i s^i + 1},$$

then

$$\bar{t} = b_1 - a_1.$$

Brown extends the definitions given in Equations (6) to consider the response time for arbitrary systems. It is evident that \bar{t} and σ^2 as defined in Equations (6) will not do for the case where $h(t)$ takes on both positive and negative values. The two extensions which Brown makes use $|h|$ and $|h|^2$ (or $|h|$ and h^2 for real-valued h) in place of h given in (6). This makes the definition of delay time the same as the definition given in Equations (5) for memory length. Brown, however, immediately discards $|h|$ as being too difficult to work with. For arbitrary systems then response time and delay time are defined as

$$\alpha^2 \equiv \frac{\int_{-\infty}^{\infty} (t-\bar{t})^2 |h(t)|^2 dt}{\int_{-\infty}^{\infty} |h(t)|^2 dt}$$

$$\bar{t} \equiv \frac{\int_{-\infty}^{\infty} t |h(t)|^2 dt}{\int_{-\infty}^{\infty} |h(t)|^2 dt} .$$

Brown's main concern is with system response time. He admits that there does not seem to exist any results in the arbitrary case similar to the results (a) and (b) for monotone systems. He assumes $\bar{t} = 0$ and continues his analysis with α^2 . This analysis is not relevant to this investigation and is not reported here.

Due to the close association between the memory length and time delay of a system, it will be of interest to briefly cite several other definitions of time delay for linear systems. These definitions, however, have only a passing interest since in general they depend not only on characteristics of the system, but also on characteristics of the input signal, an undesirable property of any measure which might be used to classify or compare systems. For a linear system with system function given by

$$H(j\omega) = A(\omega)e^{-j\theta(\omega)}$$

Papoulis [4] defines the group and phase delay, respectively, as follows:

$$t_{gr}(\omega) = \frac{d\theta(\omega)}{d\omega}$$

$$t_{\text{ph}}(\omega) = \frac{\theta(\omega)}{\omega} .$$

These terms are used to describe the delay of various parts of a signal as it passes through a linear system. To be meaningful, however, they depend upon a linear phase characteristic and a constant amplitude of the system in the frequency range of the input signal.

Simpson [5] defines another way of measuring the time delay of a system which minimizes the mean-square error between input and output signals. This distortion error is defined as

$$D(\tau) = \lim_{T \rightarrow \infty} \left[\frac{1}{2T} \int_{-T}^T [y(t) - x(t-\tau)]^2 dt \right],$$

where $x(t)$ is the system input and $y(t)$ the system output. By squaring the integrand and identifying appropriate correlation functions, an expression is reached which can be minimized with respect to τ . The value of τ which makes D minimum is then taken as the average time delay of the system. In this case, knowledge of system impulse response and input autocorrelation functions is required.

Performance Measures

Performance measures of the past, present, and future have been conveniently summarized by Schultz and Rideout [6]. These measures are in general designed to "measure" how well the actual response of a given system meets some desired response. Measures of performance have been defined in many ways, but those which are relevant to this investigation

are those which Schultz and Rideout have called performance measures of the past.

These early performance measures were in general defined for linear transfer systems and were based on integrals of the error function

$$e(t) = x(t) - y(t) ,$$

where $x(t)$ is the input and $y(t)$ the output of the system. Schultz and Rideout summarized these measures in the form

$$I = \int_0^{\infty} F[e(t),t]dt. \quad (7)$$

The only difference between one performance measure and another was a difference in the form of the function F . Some of the most studied forms were $|e|$, $t|e|$, e^2 , and te^2 . A system was said to perform in an optimal manner or was said to be the "best" among many if it minimized the particular integral employed. Much of the literature deals with studies which were made comparing a number of these performance measures on systems which were subject to step inputs. Values of the integrals I were in most cases made on an analog computer, and good or bad measures of performance were judged on the basis of various criteria.

It is to be noted that the performance measures summarized in Equation (7) are very closely related to the measures of memory length $|M|$ and M_2 . In fact, for a system subject to a pulse input the forms $t|e|$ and te^2 simply become $t|h|$ and th^2 . Thus, except for a normalizing

constant, $|M|$ and M_2 are included as past performance measures. This investigation, however, will continue to refer to $|M|$ and M_2 as measures of system memory length.

All the investigations of past performance measures encountered in making this literature survey used step inputs, except for one which used a square wave input. These investigations were in general concerned only with the minimum points of the integral forms and whether or not any particular form was selective.* Credit or discredit was also given certain forms for various reasons. From the point of view of this investigation the minimum points and the credit or discredit of forms similar to $|M|$ and M_2 are of interest. In the literature reported here this information has been included along with certain other graphical and analytical results. Considerable duplication among investigations was also found. For this reason what is surveyed here has been restricted to only a few of the outstanding papers on past performance measures.

Hall [7] was one of the first to use a performance measure of the form given in Equation (7). His work was done using the integral of square error (ISE),

$$I = \int_0^{\infty} e^2(t)dt .$$

This investigation is cited here principally because it contained the

* A particular integral form is selective if graphically its curve takes on a sharp minimum. See for instance the minimum for $t|e|$ in Figure 5.

only reported analytical results. For a second-order system with transfer function

$$\frac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}$$

the value of the ISE was found to be

$$I = \frac{1}{\omega_o} \frac{1 + 4\zeta^2}{4\zeta},$$

where ω_o is the system natural frequency and ζ the system damping ratio. For a fixed value of ω_o , I was found to have a minimum for $\zeta = 0.5$. Analytical results were also given for a third-order system, but these results are complex and will not be reported here.

A paper of particular importance was by Graham and Lathrop [8]. This investigation was primarily concerned with performance measures as criteria for developing standard forms.* A number of performance measures of the form given in Equation (7); namely those associated with e , te , $|e|$, e^2 , $t|e|$, te^2 , t^2e^2 , and $t^2|e|$; were investigated with respect to various linear systems. No analytical work was reported, and graphical presentations of results were obtained by analog methods.

Of particular interest was the fact that the integral $\int_0^{\infty} t|e|dt$, known as the time-multiplied absolute-value of error (ITAE) criterion, was found to have exceptional merit as a performance measure. This was

* A standard form is a set of values for a system's parameters or coefficients which define it in some optimal fashion.

largely due to the fact that it was particularly selective and easy to evaluate by analog means. A set of standard forms for systems through the eighth order was generated using this criterion.

Important also was the fact that the integrals associated with te^2 , t^2e^2 , and $t^2|e|$ were considered to be difficult to handle either analytically or with the analog computer and were discarded in the investigation.

One of the experiments carried out by Graham and Lathrop involved the second-order system

$$\frac{1}{s^2 + 2\zeta s + 1}$$

and the criteria $\int_0^{\infty} |e| dt$ and the ITAE. The system was subject to a step input and the criteria evaluated and graphically presented for various values of the damping ratio ζ . Figure 5 is a reproduction of these results. Of particular importance here is the fact that the criteria minimum occurred for $\zeta \approx 0.7$.

Still another result related to this second-order system involved solution time or what is sometimes called settling time.* Solution time is also a measure of system performance, and the relationship between its value and those of other performance measures is important. Five per cent solution time for the second-order system was calculated for various values of the damping ratio. These values were found to be

*Solution time or settling time is defined as the time taken by the output of a system to reach and remain within a certain percentage of its final value.

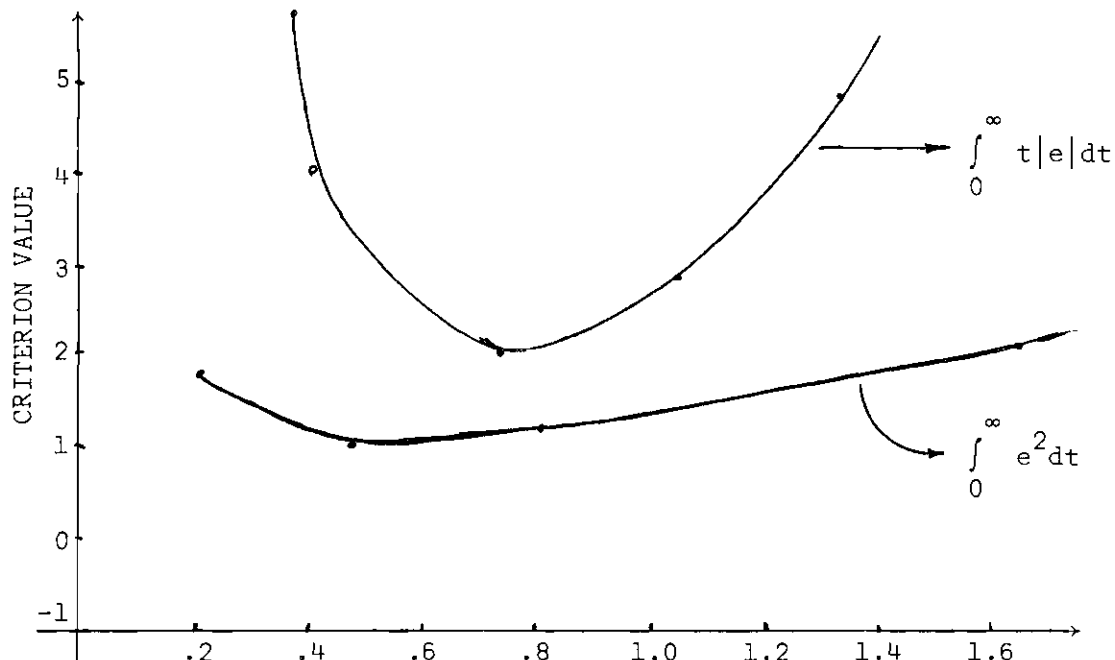


Figure 5. Criteria for the Step-Function Responses of Second-Order Systems

generally large for large and small values of the damping ratio. Its minimum value also occurred for $\zeta \approx 0.7$.

Another investigation of interest was done by Fickeisen and Stout [9]. This investigation was concerned with the performance measures

$$\int_0^{\infty} |e| dt \quad \text{and} \quad \int_0^{\infty} e^2 dt .$$

Analog methods were used to evaluate these measures with respect to a second-order system subject to a square wave input. These results were graphically presented for various values of the system damping ratio. A minimum for $\int_0^{\infty} |e| dt$ was found to occur for a value of the damping

ratio near 0.6 in contrast to the value of approximately 0.7 obtained by Graham and Lathrop using a step input. The minimum for $\int_0^{\infty} e^2 dt$ occurred near 0.5 in agreement with Hall's analytical result.

Results were also obtained for the third-order system

$$\frac{\omega_c}{s + \omega_c} \cdot \frac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}.$$

For this particular system it was possible to plot contours of constant values of the performance measures in relation to the second-order system damping ratio and the frequency ratio ω_c/ω_o . The diagram for the mean-square error criterion is partially reproduced in Figure 6.

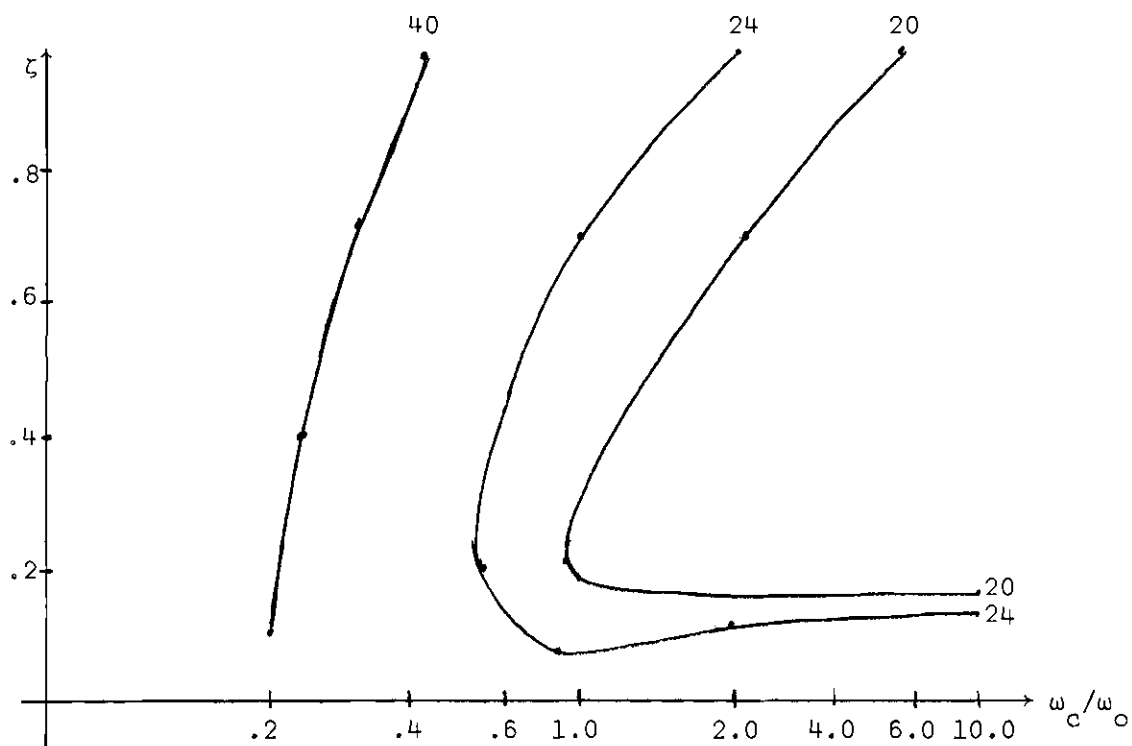


Figure 6. Mean Square Error Contours for a Third-Order Servomechanism as Functions of ζ and ω_c/ω_o

As will be shown later in Chapter V, a diagram similar to this one can also be constructed for M_2 . The diagram of Figure 6 serves as a point of comparison.

CHAPTER III

PRELIMINARY CONSIDERATIONS

Introduction

The purpose of this chapter is to introduce certain criteria, concepts, and notation necessary for the investigation of Chapter IV and V. Also included are certain derivations of $|M|$ and M_2 which are of preliminary importance and which would not appropriately fit in the remainder of this investigation. No attempt has been made to make all these topics seem logically related. In the main they are not. Of principal importance are the criteria and other relationships which form the basis for a large part of this investigation.

Criteria and Other Relationships

Suppose μ were any measure of system memory length. Then there are certain basic criteria which might be reasonably required of the measure. These criteria can be stated as follows:

- (i) $\mu = 0$, for all memoryless systems ($l=0$).
- (ii) $\mu > 0$, for all memory type systems ($l>0$).
- (iii) $\mu = \tau$, for a pure delay system with delay time equal to τ .
- (iv) $[\mu]$ (the dimension of μ) = time.

In addition to these basic criteria, there are certain other relationships which are of interest. For systems combined in cascade combination the additive property would intuitively seem to apply.

- (v) $\mu = \sum \mu_i$, where μ_i is the measure value for system i and μ the over-all system value.

For systems in parallel combination a relationship worth investigating is

- (vi) $\mu_{\min} \leq \mu \leq \mu_{\max}$, where μ_{\max} , μ_{\min} are the maximum and minimum measure values, respectively, taken over all component systems, and μ again the over-all system value.

An immediate observation is that $|M|$ and M_2 , as defined by Equations (5), satisfy criterion (iv) by definition. Also the results reported in Chapter II showed that, as indicated by $|M|$, memory length for monotone systems cascaded seems to add in a regular fashion. In this investigation it should be kept in mind that verification of $|M|$ and M_2 with regard to relationships (v) and (vi) will be done more to reveal information concerning memory length than for the purposes of evaluation. Whether or not relationships (v) and (vi) hold true depends both on $|M|$, M_2 , and memory length itself. In the remainder of this investigation, criteria (i) - (iii), as well as the relationships (v) and (vi), will be checked for $|M|$ and M_2 whenever appropriate and possible (of course, such checking for $|M|$ and relationship (v) is not necessary for monotone systems). Criterion (iii) is checked in this chapter, leaving only criteria (i) and (ii) for Chapters IV and V. In general criteria (i) and (ii) will not be cited for each system investigated, but will be cited only when and if a violation or near violation of either or both criteria occurs.

Percentage Points

Another consideration which will be useful in making an evaluation of $|M|$ and M_2 , at least for monotone systems, is contained in the idea of percentage points. If a monotone system were initially at rest and were then subject to a unit step input, one possible measure of memory length would be the time the output of the system took to reach a certain fixed percentage of its final value. This is a reasonable measure since certainly this time depends on that period beyond which the weighting function does not differ significantly from zero. For the same system it thus follows that it would be interesting to determine what percentage of the final value of the output of the system has been reached when say $|M|$ or M_2 time units have passed.* It is these percentages which will be referred to in the remainder of this investigation as the percentage points of $|M|$ or M_2 . On first thought it would seem desirable to find that these percentage points remained fairly constant from system to system as well as for all variations in the parameters of any particular system. If this were not the case, some explanation should be made. For a number of simple monotone systems these percentage points have been calculated and compared. In addition to disclosing the actual value of these points, these calculations also show an interesting consistency.

To further define these points and to introduce some notation, let $A(t)$ be the response of a monotone system to a unit step input.

* Recall that in the literature reported in Chapter II, Brown [2] defined $|M|$ as a measure of system delay or a measure of the 50 per cent point of a system.

Then the percentage points of $|M|$ and M_2 are respectively defined as

$$\% |M| = \frac{A(|M|)}{A(\infty)} \times 100$$

$$\% M_2 = \frac{A(M_2)}{A(\infty)} \times 100 ,$$

where

$$A(\infty) = \lim_{t \rightarrow \infty} A(t) \neq 0 .$$

Nondimensional Time

The use of nondimensional time will be extremely helpful in this investigation for comparing and presenting various time-related quantities, including percentage points. This concept is not particularly complicated; but due to its almost universal application to the type of systems considered in this investigation, some mention of it is made here. Its usefulness stems primarily from the fact that it reduces the number of parameters necessary to characterize a system by one. It follows that it therefore also reduces the number of parameters necessary to express $|M|$ or M_2 by one. That is, if $|M|$ and M_2 for any particular system are expressed as functions of n parameters; normally time constants, natural frequencies, and damping ratios;

$$\left. \begin{array}{l} |M| \\ \\ M_2 \end{array} \right\} = F(a_1, \dots, a_n)$$

then it is always possible to reduce the dimension of the problem by one, expressing $|M|$ and M_2 in terms of $n-1$ parameters,

$$\left. \begin{array}{l} |M|/T \\ M_2/T \end{array} \right\} = F(a'_1, \dots, a'_{n-1})$$

or

$$\left. \begin{array}{l} |M|\omega_n \\ M_2\omega_n \end{array} \right\} = F(a'_1, \dots, a'_{n-1}) .$$

These forms of $|M|$ and M_2 along with the parameters a'_1, \dots, a'_{n-1} are also dimensionless. As will be demonstrated in Chapter IV and V, the problem in $n-1$ dimensions makes possible comparisons and evaluations of $|M|$ and M_2 which could not otherwise have been made. Obviously, however, if n is large, a reduction in the dimension of the problem by one is of no great advantage. The greatest advantage is obtained when $n=1, 2$, and 3 .

System Evaluations

At this point a check of $|M|$ and M_2 will be made with respect to two systems which do not appropriately belong in either Chapters IV or V. The first system is the pure delay system of criterion (iii) given above, and the second a "perfect memory" system. The memory length of each of these systems is obvious, and thus what the values of $|M|$ and M_2 are in these cases is important.

First, consider a pure delay system defined as follows. For input $x(t)$ and output $y(t)$

$$\begin{aligned} y(t) &= \int_0^t x(z)\delta(t-\tau-z)dz \\ &= x(t-\tau), \end{aligned}$$

where $t \geq \tau > 0$. $\delta(t)$ is the unit pulse function and is defined such that

$$\int_{-\infty}^{\infty} \delta(t-\tau)dt = 1.0$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau).$$

Thus, it is easy to see that

$$|M| = \frac{\int_0^{\infty} t\delta(t-\tau)dt}{\int_0^{\infty} \delta(t-\tau)dt} = \tau.$$

With some consideration for what is meant by $\delta^2(t)$, which will not be taken up here,* it is also easy to show that

*One possible way to define $\delta^2(t)$ is by $\lim_{T \rightarrow 0} 1/T^2 e^{-2t/T}$. Using this definition it can be shown (easily) that $M_2 = \tau$.

$$M_2 = \tau.$$

Thus criterion (iii) is satisfied by both $|M|$ and M_2 . Also in this case note that the value for $|M|$ and M_2 equals the memory length of the system as defined in Chapter I in terms of the parameter ℓ , $\ell=\tau$ in this case.

The "perfect memory" system is defined by its weighting function

$$h(t) = \begin{cases} 1, & 0 \leq t \leq L \\ 0, & \text{otherwise} \end{cases}$$

or its transfer function

$$H(s) = \frac{1 - e^{-Ls}}{s}.$$

It is obvious in this case that memory length equals L . $|M|$ and M_2 in this case, however, are both $L/2$.

Absolutely Symmetric Weighting Functions

Both of the previous systems had weighting functions which were symmetric, and in both cases $|M|$ and M_2 were equal and equal to the point of symmetry. This result can be extended and generalized to include those systems whose weighting functions are absolutely symmetric.

A function $h(t)$ is said to be absolutely symmetric about a point a if

$$|h(a-t)| = |h(a+t)|$$

for all $t > 0$. For absolutely symmetric weighting functions symmetric about some point $a > 0$, $|M| = M_2 = a$. To show this, note that

$$\int_0^{\infty} (a-t)|h(t)|dt = 0 = a \int_0^{\infty} |h(t)|dt - \int_0^{\infty} t|h(t)|dt$$

or

$$\int_0^{\infty} t|h(t)|dt = a \int_0^{\infty} |h(t)|dt$$

and thus $|M| = a$. A similar argument can also be given for $h^2(t)$, showing that $M_2 = a$. This result will be used later in Chapter V.

Notation

To conclude this chapter a further note on notation will be made. On occasion when reference to both $|M|$ and M_2 is to be made on an equal basis, the symbol M will be used to stand for $|M|$ and M_2 . This notation will be used mainly in graphs and other places where space is limited. Obvious extensions of this in the forms $\% M$ and $\omega_n M$ will also be occasionally used.

CHAPTER IV

FIRST AND SECOND-ORDER SYSTEMS

Introduction

In this chapter $|M|$ and M_2 are studied in relation to first- and second-order systems. A consideration of these systems is important for three reasons. First, the values for $|M|$ and M_2 in this case are simple; and thus some evaluation of these measures can be easily obtained. Secondly, many systems are by nature first- or second-order systems. The values of $|M|$ and M_2 for these systems are thus important. And thirdly, first- and second-order systems form the indivisible building blocks of higher-order systems. For this reason, these systems are properly considered before the higher-order systems in the next chapter.

The results given here for the first-order system are given primarily for the second and third of the above reasons. Only a small amount of evaluation is included.

The results for the second-order system on the other hand are given primarily for the first and second reasons. Evaluations are obtained by comparison with the results reported in the literature cited in Chapter II. An example second-order system is also included.

In keeping with a general pattern, the first- and second-order systems considered in this chapter are defined by their transfer functions and equivalently by their weighting function. To further identify the systems, their pole locations are also specified.

First-Order Systems

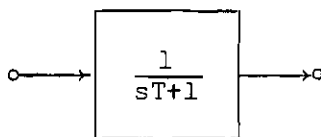


Figure 7. First-Order System

The first-order system is one of the simplest systems considered in this investigation. It is defined by its transfer function

$$H(s) = \frac{1}{sT + 1}$$

or equivalently by its weighting function

$$h(t) = \begin{cases} 1/T e^{-t/T}, & t \geq 0 \\ 0 & , t < 0 \end{cases}$$

The system is strictly monotone ($h(t) > 0$) for all values of the parameter $T > 0$ and is characterized by a single pole located on the real axis at $-1/T$.

Examples of first-order systems include all those which can be described by an ordinary first-order differential equation. The mechanical spring-dashpot system or the simple RC electrical network are common examples. Generally these systems are simply represented in

the block diagram form of Figure 7, which is often referred to as a time-constant block.

Since the weighting function in this case is simple and non-negative, the derivation of $|M|$ and M_2 is particularly easy, giving

$$|M| = \frac{\frac{1}{T} \int_0^{\infty} t e^{-t/T} dt}{\frac{1}{T} \int_0^{\infty} e^{-t/T} dt} = \frac{H'(0)}{H(0)} = T$$

and

$$M_2 = \frac{\frac{1}{T^2} \int_0^{\infty} t^2 e^{-2t/T} dt}{\frac{1}{T^2} \int_0^{\infty} e^{-2t/T} dt} = \frac{T}{2} .$$

The nondimensional forms of the system weighting function, unit step response, and $|M|$ and M_2 are

$$h(\alpha) = \begin{cases} e^{-\alpha}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

$$C(\alpha) = \begin{cases} 1 - e^{-\alpha}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

$$\frac{|M|}{T} = 1.0$$

$$\frac{M_2}{T} = 0.5$$

where α is the nondimensional time unit equal to t/T . A graphical summary of these forms is given in Figure 8.

The percentage points for the first-order system are found using the step response $C(\alpha)$. These points are

$$C(|M|/T) = 1 - e^{-1} = 0.632 \text{ or } \% |M| = 63.2$$

$$C(M_2/T) = 1 - e^{-0.5} = 0.394 \text{ or } \% M_2 = 39.4$$

For this system these points remain constant for all values of the parameter T .

One can see that the values T and $T/2$ obtained for $|M|$ and M_2 in this case give reasonable results. Their relative positions with respect to the weighting function are indicated in Figure 8. For large values of T , the large values of $|M|$ and M_2 correctly reflect the long memory of the system indicated by the relatively flat shape of the weighting function. As $T \rightarrow 0$, $h(t) \rightarrow \delta(t)$; and the system approaches a memoryless condition. Again the values of $|M|$ and M_2 give the right indication.

The results for the first-order system are summarized:

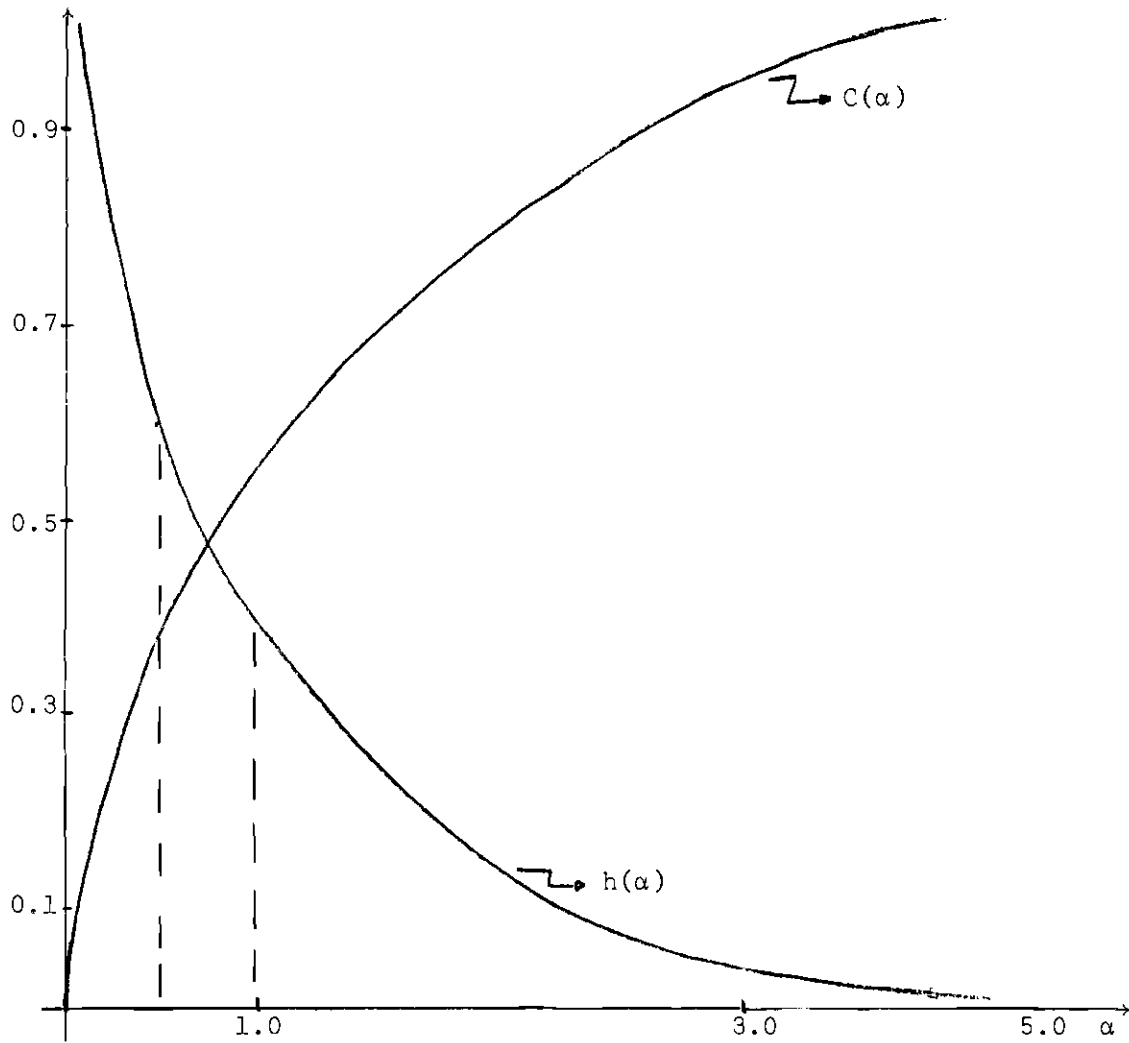


Figure 8. $h(\alpha)$ and $C(\alpha)$: First-Order System

$$h(\alpha) = \begin{cases} e^{-\alpha}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

$$C(\alpha) = \begin{cases} 1 - e^{-\alpha}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

$$\frac{|M|}{T} = 1.0$$

$$\frac{M_2}{T} = 0.5$$

$$\begin{aligned} |M| &= T \\ M_2 &= T/2 \end{aligned} \quad T > 0$$

$$\begin{aligned} \% |M| &= 63.2 \\ \% M_2 &= 39.4 \end{aligned}$$

Second-Order Systems

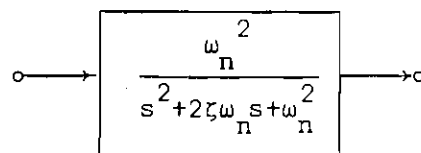


Figure 9. Second-Order System

The analysis of $|M|$ and M_2 for second-order systems will be carried out using the second-order system defined by the transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (\text{Figure 9})$$

In the complex s -plane the system is characterized by two poles, the location of which is specified by the system natural frequency ω_n and the system damping ratio ζ . These relationships are shown in Figure 10.

In general the response of the second-order system to a step input, unlike the response of the first-order system, can be either

oscillatory or monotone. Which mode of behavior persists depends on the value of the damping ratio. If the value of the damping ratio is less than one, the response will be oscillatory. If it is greater than or equal to one, the response will be monotone. For each behavior pattern the form of the system weighting function is different. The forms of this function for $\zeta < 1.0$ and $\zeta \geq 1.0$ are

I. ($0 < \zeta < 1.0$)

$$h(t) = \begin{cases} \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t} \sin \omega_r t, & t \geq 0 \\ 0 & t < 0 \end{cases}$$

II. ($\zeta > 1.0$)

$$h(t) = \begin{cases} \frac{\omega_n}{2\sqrt{\zeta^2-1}} (e^{-s_1 t} - e^{-s_2 t}), & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where ω_r is the system damped natural frequency $\omega_n \sqrt{1-\zeta^2}$, $s_1 = \omega_n (\zeta - \sqrt{\zeta^2-1})$, and $s_2 = \omega_n (\zeta + \sqrt{\zeta^2-1})$. The general form of $h(t)$ along with the corresponding step response of the system for various values of ζ is illustrated in Figure 11.

From these results it is obvious that $|M|$ and M_2 will have to be calculated for both the case $\zeta < 1.0$ and $\zeta > 1.0$. These calculations are straightforward though a bit tedious and are presented in the

Appendix. The results, however, are given in Table 1.

Table 1. $|M|$ and M_2 : Second-Order System

$ M $	M_2
$\frac{2\zeta}{\omega_n} + \frac{2\pi}{\omega_r} \left[\frac{r}{1-r^2} \right]$	$\frac{\zeta}{\omega_n} + \frac{1}{2\zeta\omega_n}, 0 < \zeta < 1.0$
$\frac{2\zeta}{\omega_n}$	$\frac{\zeta}{\omega_n} + \frac{1}{2\zeta\omega_n}, \zeta > 1.0$

where $r = e^{-\pi\phi}$, $\omega_r = \omega_n \sqrt{1-\zeta^2}$, $\phi = \frac{1}{\sqrt{1-\zeta^2}}$.

It is desirable in this case to give values for $|M|$ and M_2 which are in a usable form. For this reason Table 2 has been generated, giving values of $|M|$ (in nondimensional time) for the case of $\zeta < 1.0$. Included in this table are also the values of the angle β (see Figure 10). The nondimensional forms of both $|M|$ and M_2 are plotted in Figure 12.

Percentage points for the second-order system are defined only for the case where $\zeta > 1.0$ (monotone system). These points, however, will not be calculated here but will be taken up in the next chapter, where the second-order system (for $\zeta > 1.0$) will be viewed as two cascaded first-order systems.

Table 2. Values for $\omega_n |M|$: Second-Order System

β	$\omega_n M $	$\zeta = \cos\beta$	β	$\omega_n M $	$\zeta = \cos\beta$
0.05	1.998	0.999	0.80	1.809	0.697
0.10	1.990	0.995	0.85	1.852	0.660
0.15	1.998	0.989	0.90	1.911	0.622
0.20	1.960	0.980	0.95	1.990	0.582
0.25	1.938	0.969	1.00	2.092	0.540
0.30	1.912	0.955	1.05	2.223	0.498
0.35	1.882	0.939	1.10	2.393	0.454
0.40	1.852	0.921	1.15	2.612	0.408
0.45	1.823	0.900	1.20	2.901	0.362
0.50	1.797	0.878	1.25	3.242	0.315
0.55	1.777	0.853	1.30	3.838	0.268
0.60	1.763	0.825	1.35	4.647	0.219
0.65	1.759	0.796	1.40	5.945	0.170
0.70	1.764	0.765	1.45	8.342	0.121
0.75	1.780	0.732	1.50	14.162	0.171

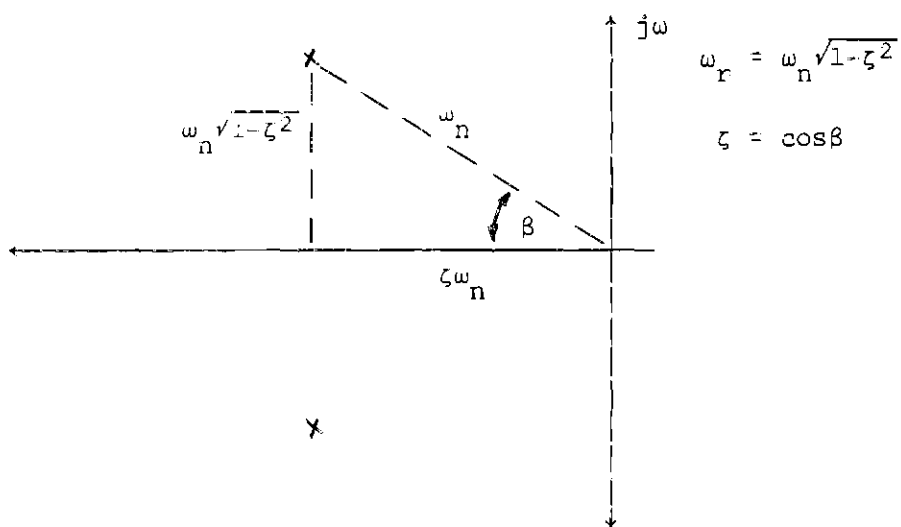


Figure 10. s-Plane Diagram of Second-Order System

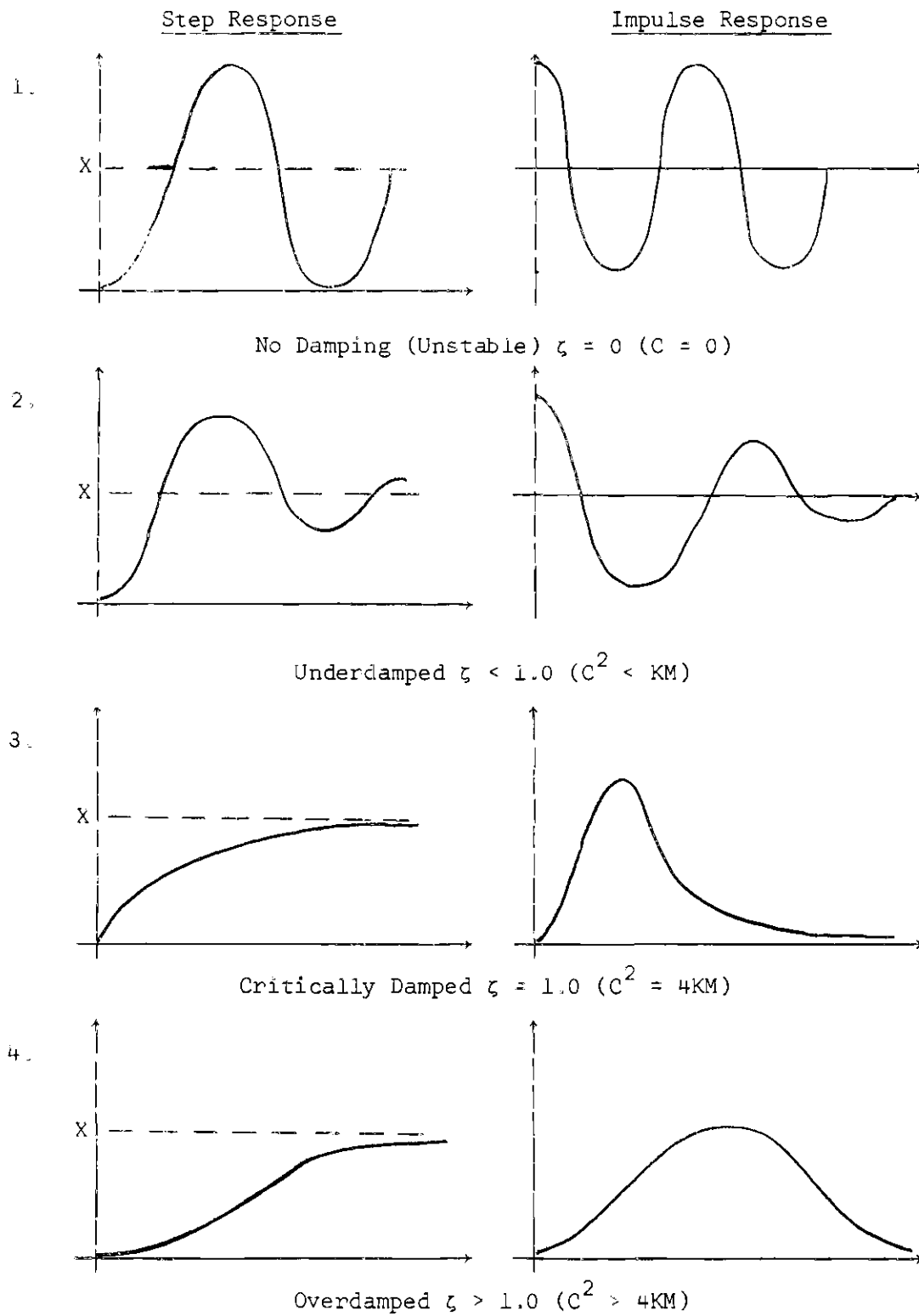


Figure 11. Impulse Response and Step Response of Second-Order System

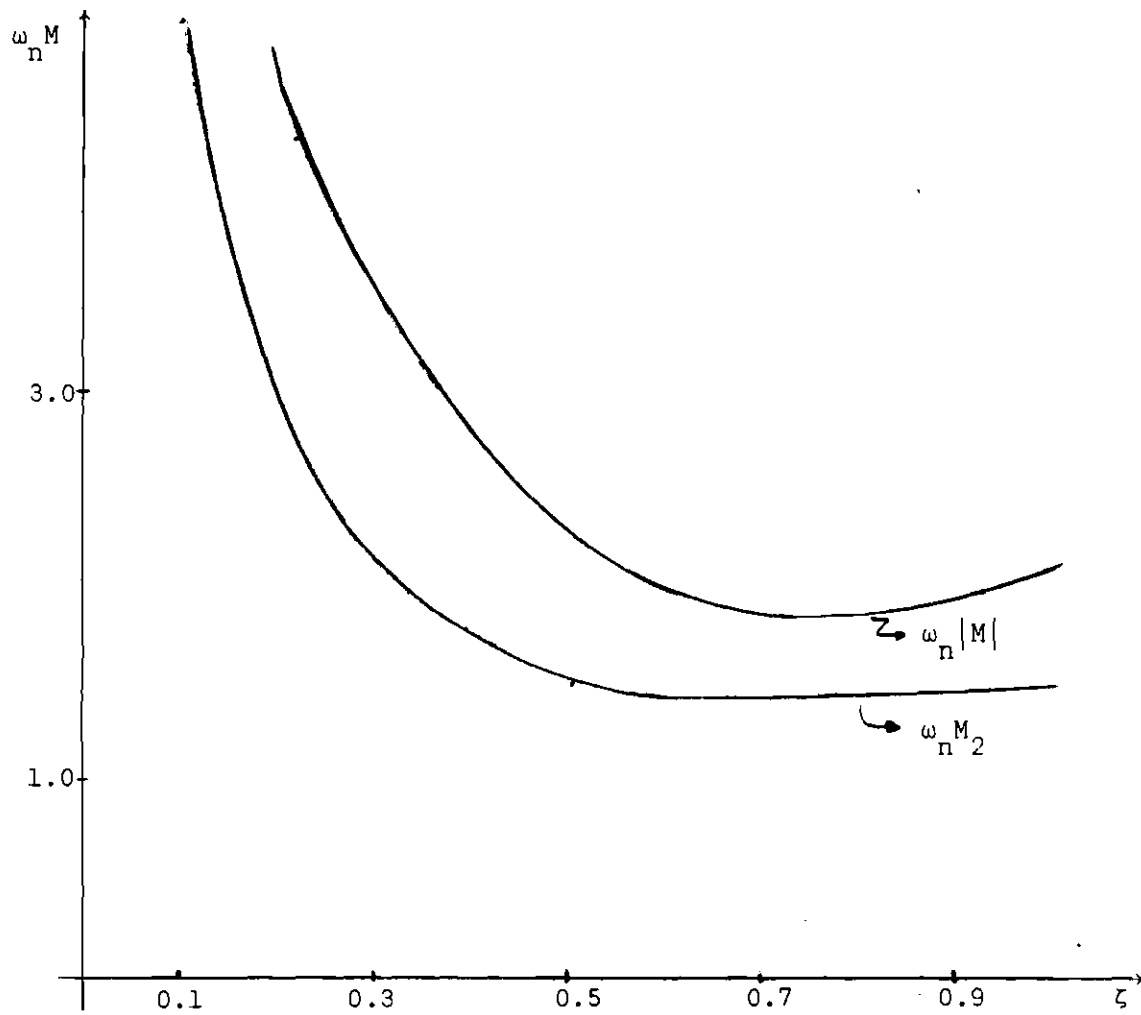


Figure 12. ω_n^M : Second-Order System

$$\omega_n^M = \begin{cases} 2\zeta & , \quad \zeta > 1.0 \\ 2\zeta + \frac{2\pi}{\sqrt{1-\zeta^2}} \left[\frac{r}{1-r^2} \right] & , \quad 0 < \zeta < 1.0 \end{cases}$$

$$\omega_n^M_2 = \zeta + \frac{1}{2\zeta}$$

$$\min(\omega_n^M) \approx 1.76 \text{ for } \zeta \approx 0.8$$

$$\min(\omega_n^M_2) = \sqrt{2} \text{ for } \zeta = \sqrt{2}/2$$

At this point a few general observations concerning the results obtained for $|M|$ and M_2 can be made. A comparison of Figure 12 and Figure 5 (Chapter II) shows, as was expected, a close correspondence between system performance and system memory length. In general for large and small values of the damping ratio performance was bad and memory length long. One look at Figure 11 shows the reason for such results. For small value of the damping ratio both the weighting function and the response of the system to a step input tend toward a sustained oscillatory behavior over an extended period. For large values of the damping ratio the step input response is slow or sluggish, while the weighting function shows a tendency to shift its mass toward the right or toward past inputs. Both cases indicate poor performance and long memory. $|M|$ and M_2 thus again seem to give reasonable results, and the correspondence with system performance should be clear.

Due to the close correspondence between performance and memory length, it might be suspected that the system with the shortest memory length would correspond with the system having the best performance. As the results in Chapter II indicated, both for performance measures and for solution time, this second-order system occurs for a damping ratio value of about 0.7. An easy calculation shows that the minimum value of M_2 occurs for $\zeta = \sqrt{2}/2 \approx 0.707$, a value which does not differ from 0.7 to any extent and what is more, is probably the exact statement of "about 0.7." The minimum point for $|M|$ cannot be calculated exactly very easily, but from Table 2, it seems to occur for $\zeta \approx 0.8$. This value does differ from 0.7; however, the broad minimums on all the

curves (see Figures 5 and 12) at this point makes this difference of only small significance.

The point $\zeta = \sqrt{2}/2$ has a further significance. For this value of the damping ratio the poles of the second-order system are located in the s-plane 45° from the real axis, see Figure 11. This follows since $\beta = \cos^{-1}(\sqrt{2}/2) = \pi/4$. This location of poles for a second-order system is a generally known optimum, and it is noteworthy that M_2 's minimum occurs at exactly this point. The minimum point of $|M|$ corresponds to a value of β equal to 37° and is not a generally well-known value. Also not to go unnoticed is the similarity between Hall's result for the ISE in Equation (8) and the result for M_2 given in Table 1.

From these results it seems that the minimum value of M_2 occurs very close to what is a generally accepted optimum performing system. If this system is taken to be the one having the minimum memory length, and it will be taken to be this system, then M_2 appears, at least for the second-order system, to do well as a measure of memory length. The measure $|M|$, however, does not seem to do badly. The single formula for M_2 for all values of the damping ratio is also another favorable point. The value of $|M|$ has two different forms, one of which is difficult to use without an accompanying table.

Example

At this point a specific physical example of a second-order system will be considered. The purpose of this example is not only to point out further differences and similarities between $|M|$ and M_2 but to illustrate their use in an actual physical situation.

A system commonly used to illustrate the behavior of the second-order system is the mechanical spring-mass-dashpot system of Figure 13. The input $f(t)$ and the output $x(t)$ of the system are governed by the second-order differential equation

$$M \frac{d^2 x(t)}{dt^2} + C \frac{dx(t)}{dt} + Kx(t) = Kf(t).$$

The transfer function corresponding to this equation is given by

$$H(s) = \frac{X(s)}{F(s)} = \frac{K/M}{s^2 + (C/M)s + K/M}$$

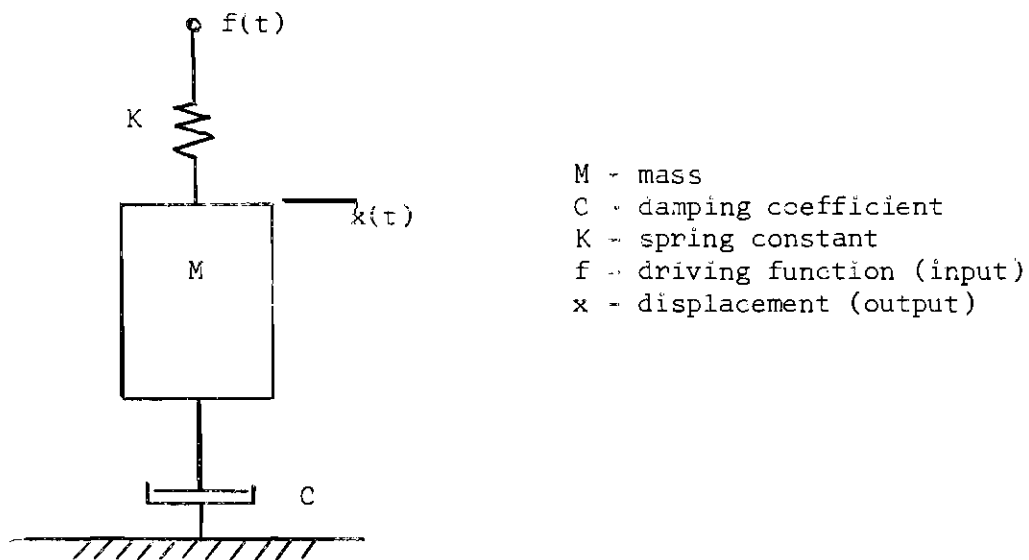


Figure 13. Spring-Mass-Dashpot System

or in standard form as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

$$\omega_n = \sqrt{K/M} \quad \text{and} \quad \zeta = \frac{C}{2\sqrt{MK}}.$$

The different responses of this system for various values of the parameters M , K , and C are also illustrated in Figure 11.

The values for $|M|$ and M_2 in this case are

$$M_2 = C/K + M/C$$

and

$$|M| = \begin{cases} C, K & , C^2 > 4KM \\ C/K + \frac{4\pi M}{\sqrt{4MK - C^2}} \left[\frac{r}{1 - r^2} \right] & , C^2 < 4KM \end{cases}$$

where

$$r = e^{-\pi\phi} \quad \text{and} \quad \phi = \frac{C}{\sqrt{4MK - C^2}}.$$

We can make a few general observations from these results. Both $|M|$ and M_2 increase without bound as the mass M becomes large (this may

not appear obvious for $|M|$, but it can be verified). This result is one we would intuitively expect, since a large mass would result in a more sluggish system with longer delay or memory length. On the other hand, an increasing value of the spring constant K might seem to imply a system with a steadily decreasing delay or memory length. But as $K \rightarrow \infty$, $M_2 \rightarrow M/C$ and $|M| \rightarrow 2M/C$, not zero.

As a numerical example and to illustrate the use of Table 2, suppose

$$M = 1 \text{ kilogram}$$

$$K = 625 \frac{\text{newtons}}{\text{meter}}$$

$$C = 30 \frac{\text{newton} \cdot \text{seconds}}{\text{meter}}$$

Then

$$\omega_n = \sqrt{K/M} = \sqrt{625} = 25 \text{ sec}^{-1}$$

$$\zeta = \frac{C}{2\sqrt{KM}} = \frac{30}{50} = 0.6$$

Thus

$$M_2 = \frac{\zeta}{\omega_n} + \frac{1}{2\zeta\omega_n} = 0.057 \text{ sec.}$$

Interpolating in Table 2 gives

$$\omega_n |M| = 1.954,$$

hence

$$|M| = 1.954/25 = 0.078 \text{ sec.}$$

CHAPTER V

CASCADE AND PARALLEL SYSTEM COMBINATIONS

Introduction

In this chapter $|M|$ and M_2 are studied in relation to cascade and parallel system combinations. The chapter is composed of two parts. The first part is concerned with monotone systems or systems whose weighting functions are non-negative. The second part is concerned with systems which are not monotone or nonmonotone systems. No attempt is made to investigate $|M|$ and M_2 in relation to these type systems in an entirely general way but only by representative monotone and nonmonotone systems. These representative systems are composed of cascade and parallel combinations of the first- and second-order systems of the previous chapter. The breakdown of the chapter into two parts is important since the investigation in relation to monotone and nonmonotone systems is quite different. To further facilitate the investigation care is taken to consider systems in a logical sequence which begins with the simpler and proceeds to the more complex systems. The primary basis for the investigation of each system is appropriate points from Chapter III (these being criterion (ii), relationships (v) and (vi), and the system percentage points). Other characteristics and observations related to $|M|$ and M_2 , however, are included when these are thought to contribute to the evaluation. Also for some systems the evaluation and discussion of $|M|$ and M_2 can proceed further

than it can for others. For this reason some sections are long and others more brief.

Monotone Systems

In this part of the chapter $|M|$ and M_2 are investigated in relation to cascade and parallel system combinations which are monotone. The particular systems which are investigated here are composed of cascade and parallel first-order system combinations. The investigation proceeds from the simplest to the more complex systems. A detailed discussion is in general given each system. A summary of important results is also included.

n Identical First-Order Systems Cascaded

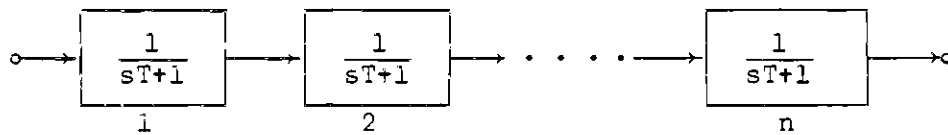


Figure 14. Identical First-Order Systems Cascaded

An order of complexity above that of a single first-order system is that of n identical cascaded such systems (Figure 14)*. The weighting function for the system is

*In this chapter systems will in general be identified through their block diagram and weighting function. The transfer function will not in general be given.

$$h(t) = \begin{cases} \frac{1/T^n}{(n-1)!} t^{n-1} e^{-t/T}, & t \geq 0 \\ 0 & , t < 0 \end{cases}$$

The system is characterized by an n -pole at $-1/T$ and a single positive parameter T .

Since each of the component first-order systems in this case is a monotone system, then from the results reported in Chapter II the value of $|M|$ for the over-all system must be the sum of its values for each individual system. Thus

$$|M| = nT.$$

The value of M_2 is obtained by a straightforward derivation and is given by

$$M_2 = \frac{\Gamma(2n)/(2/T)^{2n}}{\Gamma(2n-1)/(2/T)^{2n-1}} = T(n-1/2) = \left(\frac{T}{2}\right)(2n-1)$$

As with a single first-order system, the percentage points in this case do not depend on T . Their dependence is only on the number of components n . These values for $n=1, 2$, and 4 are given in Table 3. A correct conclusion is that

$$39.4 \leq \% M \leq 63.2$$

for all $n \geq 1$.

Table 3. % M: Identical First-Order Systems Cascaded

n	% M	% M ₂
1	63.2	39.4
2	59.5	44.2
4	56.6	46.8

It is evident from the value obtained for M_2 that its value for the over-all system is not the sum of its values for the individual component systems, in this case $T/2$. That is, relationship (v) (Chapter III) does not hold in general for M_2 . However, it is also evident that $|M|$ and M_2 differ by only a small amount for n sufficiently large.

A plot of the nondimensional form of the weighting function is shown in Figure 15. Summarized there also are relative relationships between $|M|$, M_2 , and the weighting function maximum.

$$\begin{array}{l}
 |M| = nT \\
 M_2 = T/2(2n-1) \\
 T > 0; n=1,2,\dots
 \end{array}$$

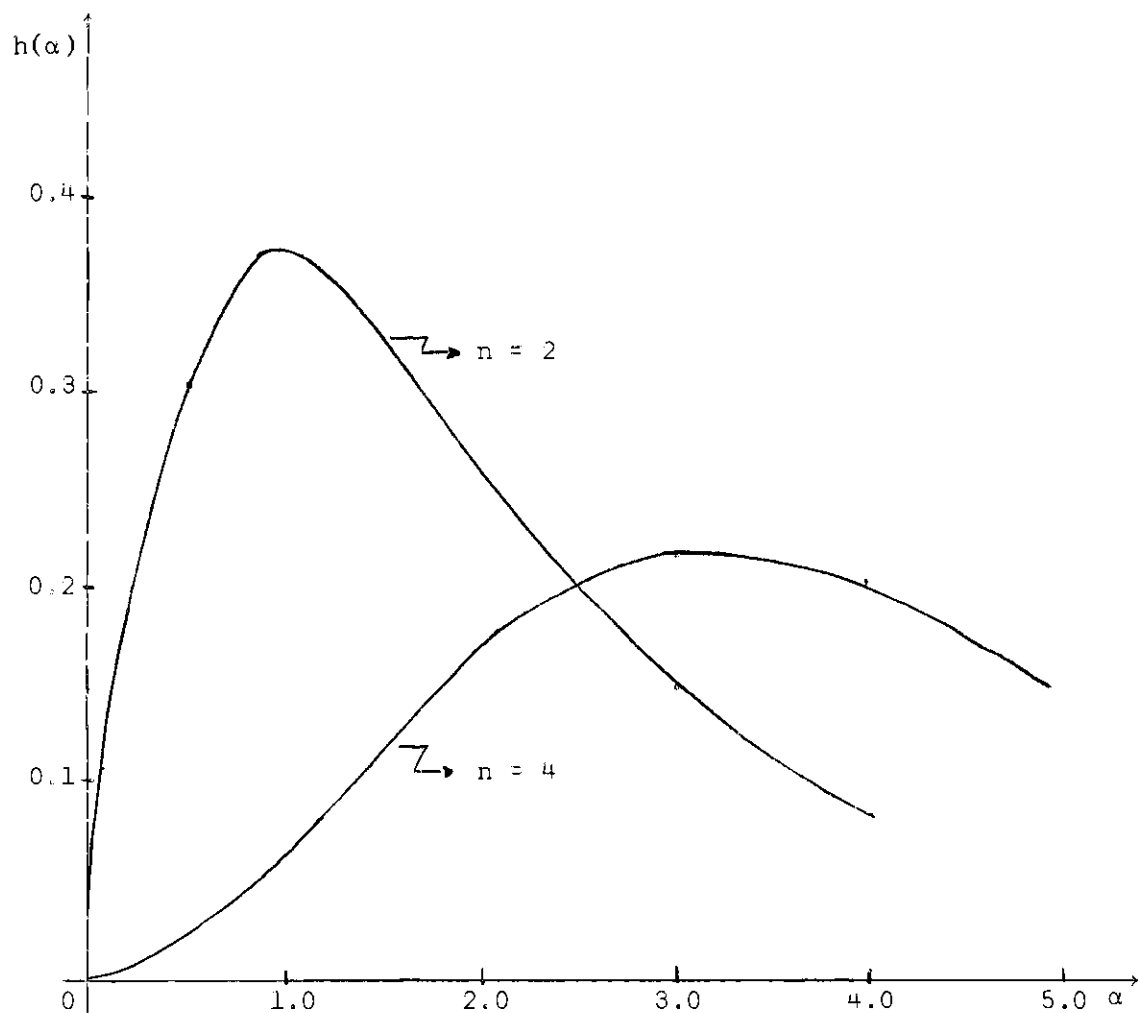


Figure 15. $h(\alpha)$: Identical First-Order Systems Cascaded

$$h(\alpha) = \begin{cases} \frac{\alpha^{n-1}}{(n-1)!} e^{-\alpha}, & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}$$

$$\frac{|M|}{T} = n \qquad \frac{M_2}{T} = n - \frac{1}{2}$$

$\max h(\alpha)$ occurs at $\alpha = n - 1$

$$\max h(t) < M_2 < |M|$$

Two First-Order Systems Cascaded

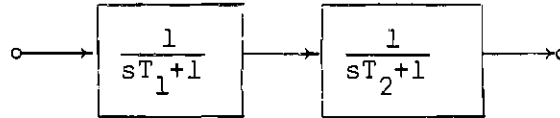


Figure 16. Two First-Order Systems Cascaded

The second-order system was first considered in Chapter IV. It was considered in the previous section for $n=2$. Here it (the monotone system) is viewed as two cascaded first-order systems for which T_1 and T_2 are in general not the same.

The weighting function for this system is given by

$$h(t) = \begin{cases} \frac{e^{-t/T_1} - e^{-t/T_2}}{T_1 - T_2}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad T_1 \neq T_2$$

and the values of $|M|$ and M_2 are

$$|M| = T_1 + T_2$$

$$M_2 = \frac{T_1}{2} + \frac{T_2}{2} + \frac{T_1 T_2}{T_1 + T_2}$$

for $T_1, T_2 > 0$. The value for $|M|$ follows from the theorem on cascaded systems, and the value for M_2 from a minor derivation.

Before going further it will be of considerable advantage to

express quantities in nondimensional form. To do this let $T_2 = kT_1$. In this case k is a dimensionless parameter relating the first-order system time constants. With this relationship values for $|M|$, M_2 , and $h(t)$ can be expressed in the nondimensional form

$$|M|/T_1 = k + 1$$

$$M_2/T_1 = \frac{k + 1}{2} + \frac{k}{k + 1}$$

and

$$h(\alpha) = \begin{cases} \frac{e^{-\alpha/k} - e^{-\alpha}}{k - 1}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

where $\alpha = t/T_1$ and $k \neq 1$. Note that these forms are functions of the relationship between T_1 and T_2 , not the actual values of these parameters. As will be seen, these forms are useful for comparative purposes.

Plots of $|M|/T_1$, M_2/T_1 , and $h(\alpha)$ are given in Figure 17 and Figure 18. The plot for $h(\alpha)$ appears for $k=2,4$, and 6; and included with it are the corresponding values for $|M|/T_1$ and M_2/T_1 . One particular observation should be made. In Figure 17 there is a general tendency for plots of $|M|/T_1$ and M_2/T_1 to separate for large k . An explanation for this is found in Figure 18 where for the large k values ($k=4$ and 6) the weighting function develops a long "tail." It is this

particular form of the weighting function which causes $|M|/T_1$ and M_2/T_1 (and hence $|M|$ and M_2) to take on quite different values. M_2 , having an affinity for the large values of the weighting function, tends to take on the smaller values, while $|M|$ takes on the larger value for increasing k . As will be brought out for a later system, this particular tendency can become quite marked. It is, for the most part, only a matter of opinion as to whether or not a particularly large or small measure of memory length is appropriate in any particular case, but the above tendency does show the difference in measure values which can exist for the same system or the same value of memory length. It also demonstrates the necessity of using the same measure to compare systems.

The percentage points for this system are plotted in Figure 19. These points are plotted on the unit interval $(0,1]$. It should be noted that the plots on this interval are the same as the ones that would be obtained for k on the interval $[1,+\infty)$. That is, the same value for $\% |M|$ and $\% M_2$ would be obtained for $k=k_0$ as for $k = 1/k_0$, $k_0 > 0$. This follows since it really does not make any difference whether the substitution $T_2 = kT_1$ is used or the substitution $T_1 = kT_2$ is used to obtain the nondimensional form. From these plots it follows that

$$59.5 \leq \% |M| \leq 63.2$$

$$39.4 \leq M_2 \leq 44.2$$

for any $T_1, T_2 > 0$.

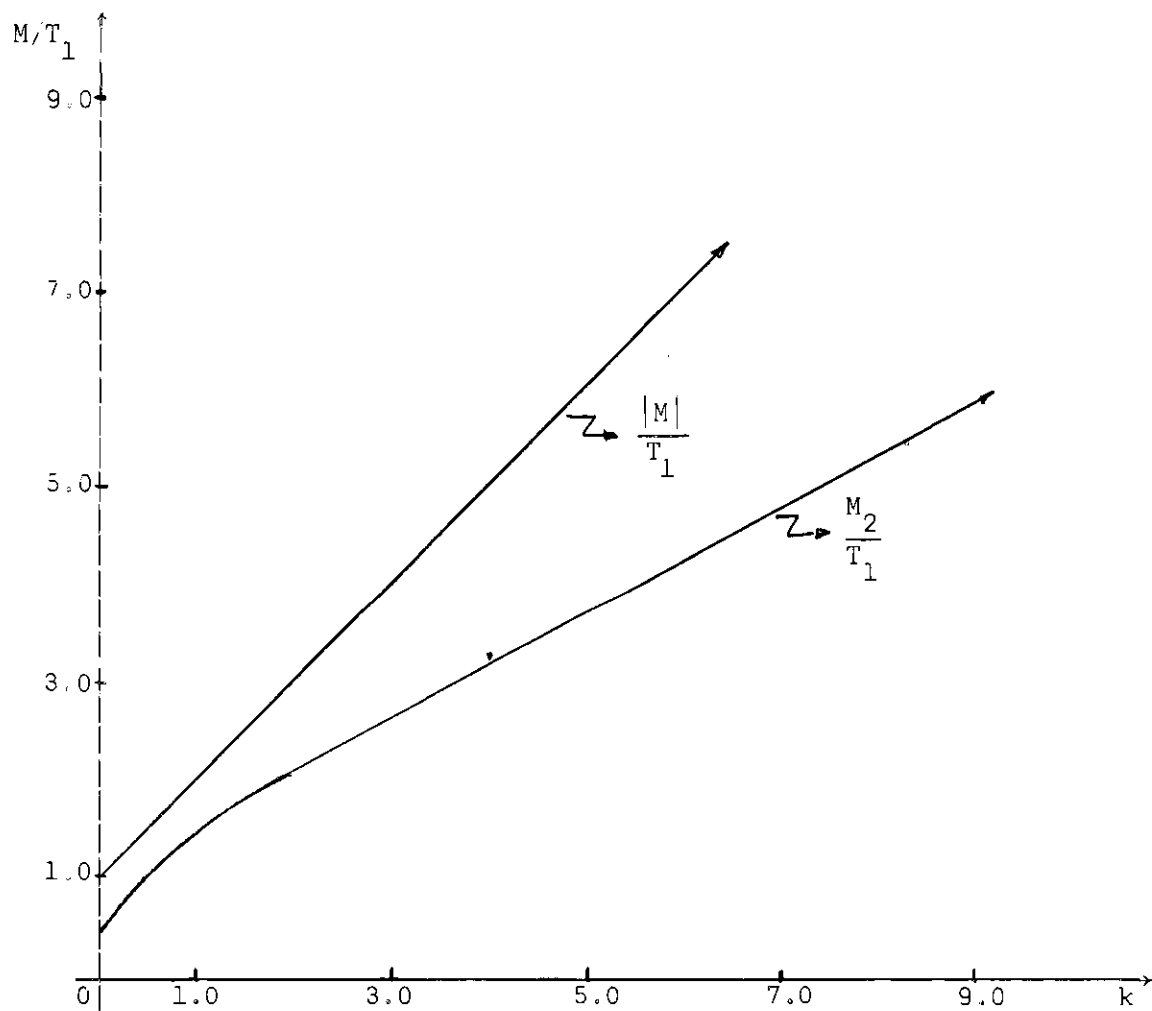


Figure 17. M/T_1 : Two First-Order Systems Cascaded

$$\frac{|M|}{T_1} = k + 1$$

$$\frac{M_2}{T_1} = \frac{k + 1}{2} + \frac{k}{k + 1}$$

$$\frac{|M|}{2T_1} < \frac{M_2}{T_1} < \frac{|M|}{T_1}$$

or

$$\frac{|M|}{2} < M_2 < |M|$$

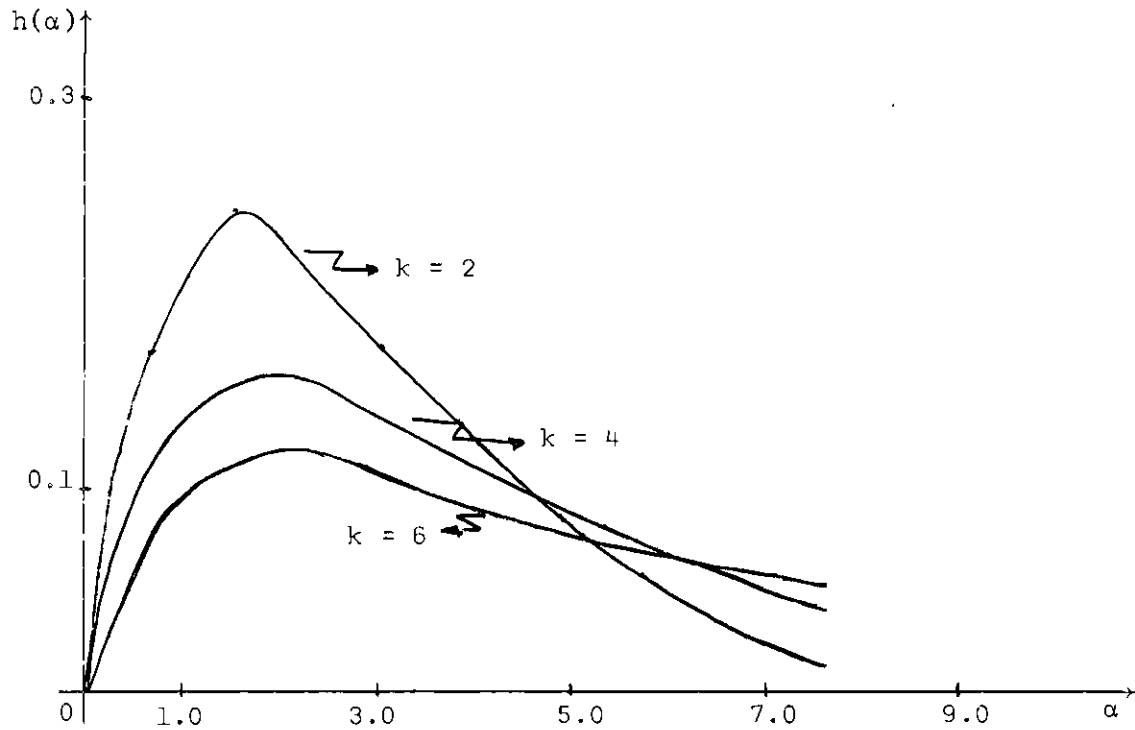


Figure 18. $h(\alpha)$: Two First-Order Systems Cascaded

$$h(\alpha) = \begin{cases} \frac{e^{-\alpha/k} - e^{-\alpha}}{k-1}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \quad k \neq 1$$

k	2	3	4
$ M /T_1$	3	5	7
M_2/T_1	2.15	3.30	4.35

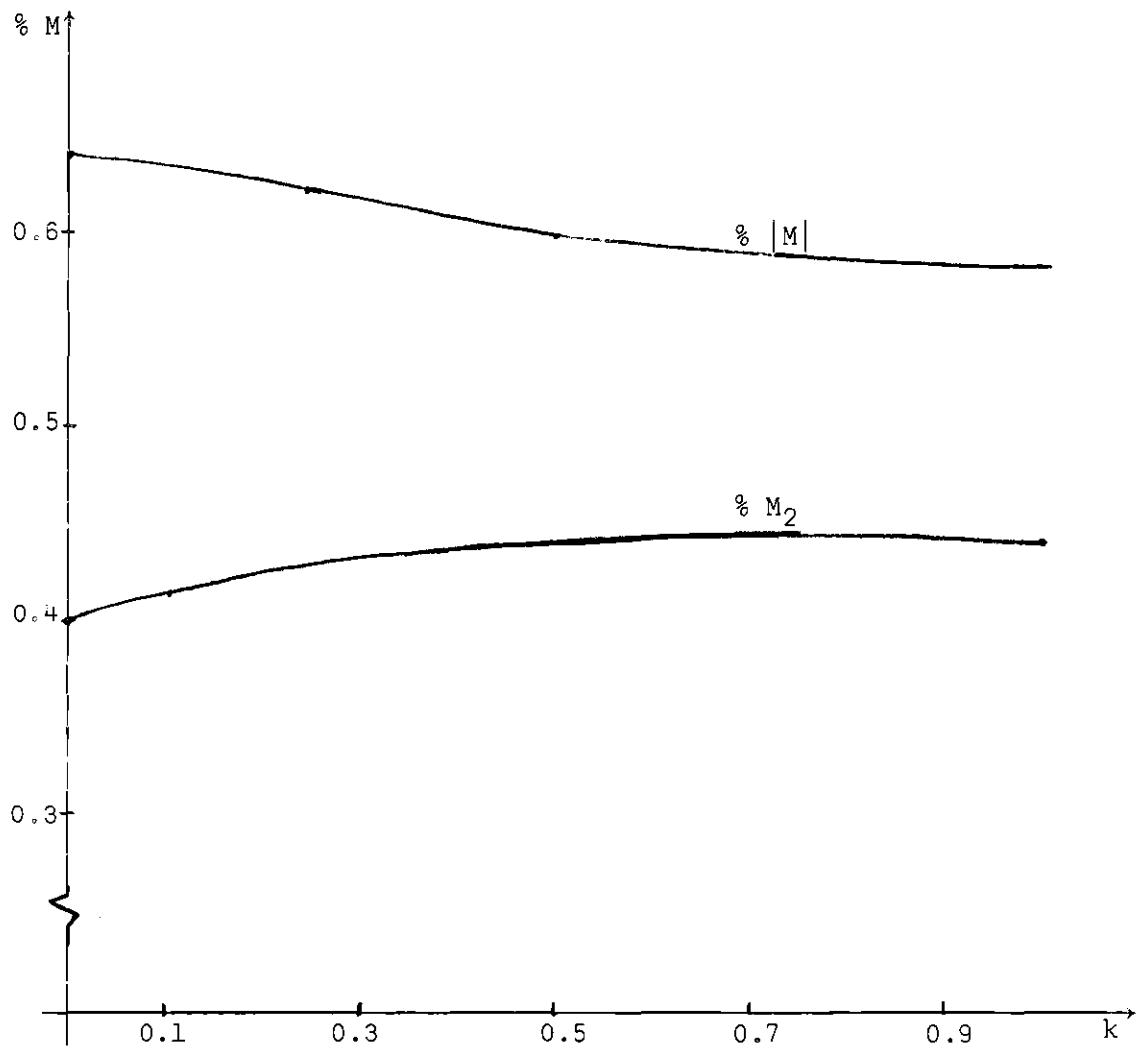


Figure 19. % M: Two First-Order Systems Cascaded

$$59.5 \leq \% |M| \leq 63.2$$

$$39.4 \leq \% M_2 \leq 44.2$$

Noteworthy here of course is the fact that for all variations in the weighting function there is only a small change in the percentage point values. This change for % $|M|$ is 3.7 and that for % M_2 is 4.8. As will again be seen, this small variation and these ranges of values hold rather generally, differing only slightly from one system to another.

$$\begin{aligned}
 |M| &= T_1 + T_2 \\
 M_2 &= \frac{T_1}{2} + \frac{T_2}{2} + \frac{T_1 T_2}{T_1 + T_2} \\
 T_1, T_2 &> 0
 \end{aligned}$$

n Identical First-Order Systems in Parallel

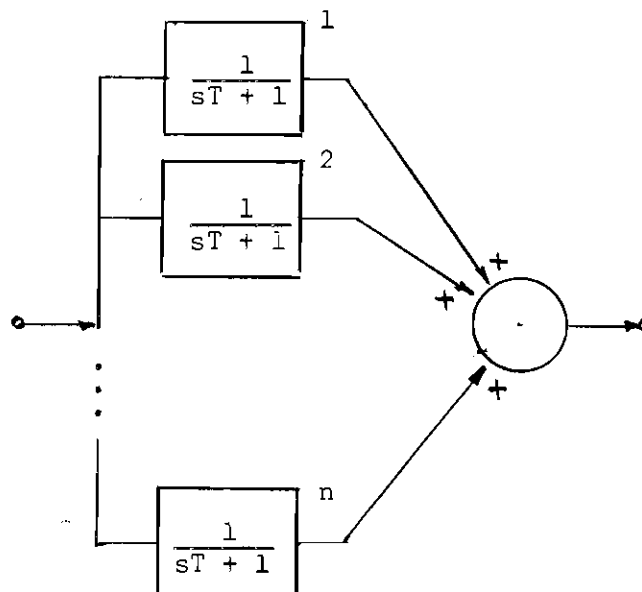


Figure 20. Identical First-Order Systems in Parallel

This system is the next logical system to consider in this investigation. The transfer function for the system is given by

$$H(s) = \sum_{l=1}^n \frac{1}{sT + 1} = \frac{n}{sT + 1},$$

which is immediately recognized as the transfer function of a first-order system with a gain of n . Since the system gain does not affect the value of memory length, the values for $|M|$ and M_2 in this case are the same as those for the first-order system.

This section is thus terminated by stating essentially the same results that were given in Chapter IV for the first-order systems.

$ M = T$	$\% M = 63.2$
$M_2 = \frac{T}{2}$	$\% M_2 = 39.4$
$T > 0$	$n=1,2,\dots$
$n = 1,2,\dots$	

Two First-Order Systems in Parallel

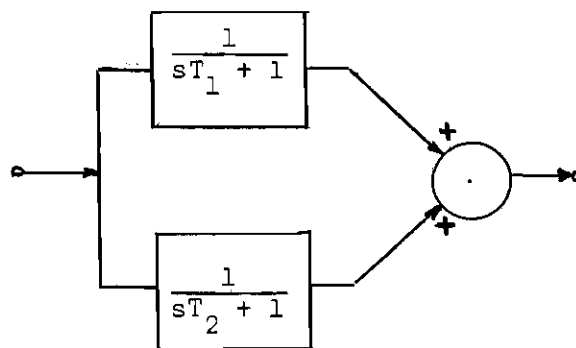


Figure 21. Two First-Order Systems in Parallel

This section is concerned with only two systems in parallel, one distinguished from the other in the value of its parameter T . Here certain especially important evaluative information is obtained.

The weighting function for this system, the sum of two exponential terms, is

$$h(t) = \begin{cases} 1/T_1 e^{-t/T_1} + 1/T_2 e^{-t/T_2}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The values for $|M|$ and M_2 are given by

$$|M| = \frac{T_1 + T_2}{2}$$

$$M_2 = \frac{T_1 T_2}{T_1 + T_2}$$

where $T_1, T_2 > 0$. Also by letting $T_2 = kT_1$, the nondimensional forms of these quantities and $h(t)$ are

$$|M|/T_1 = \frac{k + 1}{2}$$

$$M_2/T_1 = \frac{k}{k + 1}$$

and

$$h(\alpha) = \begin{cases} 1/k e^{-\alpha/k} + e^{-\alpha}, & \alpha \geq 0 \\ 0 & , \alpha < 0 \end{cases}$$

where $\alpha = t/T_1$.

The formulas for $|M|$ and M_2 in this case are of particular concern. Note that as $T_2 \rightarrow 0$, $M_2 \rightarrow 0$. In this particular situation M_2 would seem to indicate that the system was approaching a memoryless state or system. But as $T_2 \rightarrow 0$, $h(t) \rightarrow \delta(t) + 1/T_1 e^{-t/T_1}$, which is not memoryless. Also a look at Figure 23 for the case $k \rightarrow 0$ (corresponding to $T_2 \rightarrow 0$) demonstrates that the system does not become memoryless. Whereas T_2 cannot actually equal zero, it can, along with M_2 , be made arbitrarily close to zero. This would thus seem to be a very near violation of criterion (ii) given in Chapter III. The explanation for this behavior on the part of M_2 should be fairly obvious from Figure 23. The distorted form of the weighting function for small k and the general affinity of M_2 for large values is apparently sufficient to cause M_2 to approach zero as $T_2 \rightarrow 0$. In contrast note that $|M|$ does not behave in this fashion. As $T_2 \rightarrow 0$, $|M| \rightarrow T_1/2$, a value which intuitively might seem to be more reasonable measure of memory length or at least a value which is not arbitrarily close to zero. As will be pointed out for later parallel system combinations, this particular characteristic of M_2 seems to be universal, occurring whenever a similar distortion of the weighting function occurs.

Comment should also be made for the opposite case, namely the case for which $T_2 \rightarrow \infty$ or equivalently the case for which $k \rightarrow \infty$. Figure 22 shows that there is a marked difference in $|M|/T_1$ and M_2/T_1 or $|M|$ and M_2 in this situation. The explanation is the same as it was for two cascaded systems. As $k \rightarrow \infty$ the weighting function (see Figure 23) develops a long tail. The value of $|M|$, as before, increases without bound, while M_2 takes on the smaller values. This again brings out the distinctly different measures of memory length which $|M|$ and M_2 can have for the same system.

Attention is now turned to relationship (vi) of Chapter III. Since the system in Figure 21 is composed of two systems in parallel, it would be desirable to know the relation between $|M|$ and M_2 and the corresponding maximum and minimum memories of the components. For $|M|$ this is obvious. Assume $T_2 > T_1$, then certainly

$$T_1 < |M| < T_2$$

where T_1, T_2 are the values of $|M|$ for each of the component systems. For M_2 the result is also the same. To see this note that

$$M_2 = \frac{T_1 T_2}{T_1 + T_2} = \frac{T_2}{1 + T_2/T_1} < \frac{T_2}{2},$$

since $1 + T_2/T_1 > 2$. Also

$$T_1/2 < \frac{T_1}{1 + T_1/T_2} = M_2,$$

since $1 + T_1/T_2 < 2$. Thus

$$\frac{T_1}{2} < M_2 < \frac{T_2}{2}$$

where $T_1/2$, $T_2/2$, as it should be recalled, are the values of M_2 for the first-order system components. Of course a similar result holds for the case $T_1 > T_2$. Hence, relationship (vi) is verified in this case.

A plot of the percentage points for this system is given in Figure 24. Again it is sufficient to give these values for k on the unit interval $(0,1]$. Of particular note for these values is the fact that again there is only a small variation for all possible values of the system parameters T_1, T_2 . It is only for k small, where the weighting function is most distorted, that the percentage points change to any extent. Even when $k \rightarrow 0$, where $M_2 \rightarrow 0$, $\% M_2$ approaches the limiting value of 31.5.

To terminate this section attention will be called to the fact that the gains for each of the component first-order systems in Figure 21 were assumed to be the same, and in this case assumed to be one. If this is not the case, then the values obtained for $|M|$ and M_2 are not as simple as the ones given here. Considering unequal component system gains, however, does not add anything significant to what has already been done. For this reason no attempt is made to proceed along these lines either for this system or for any of the remaining parallel system combinations investigated.

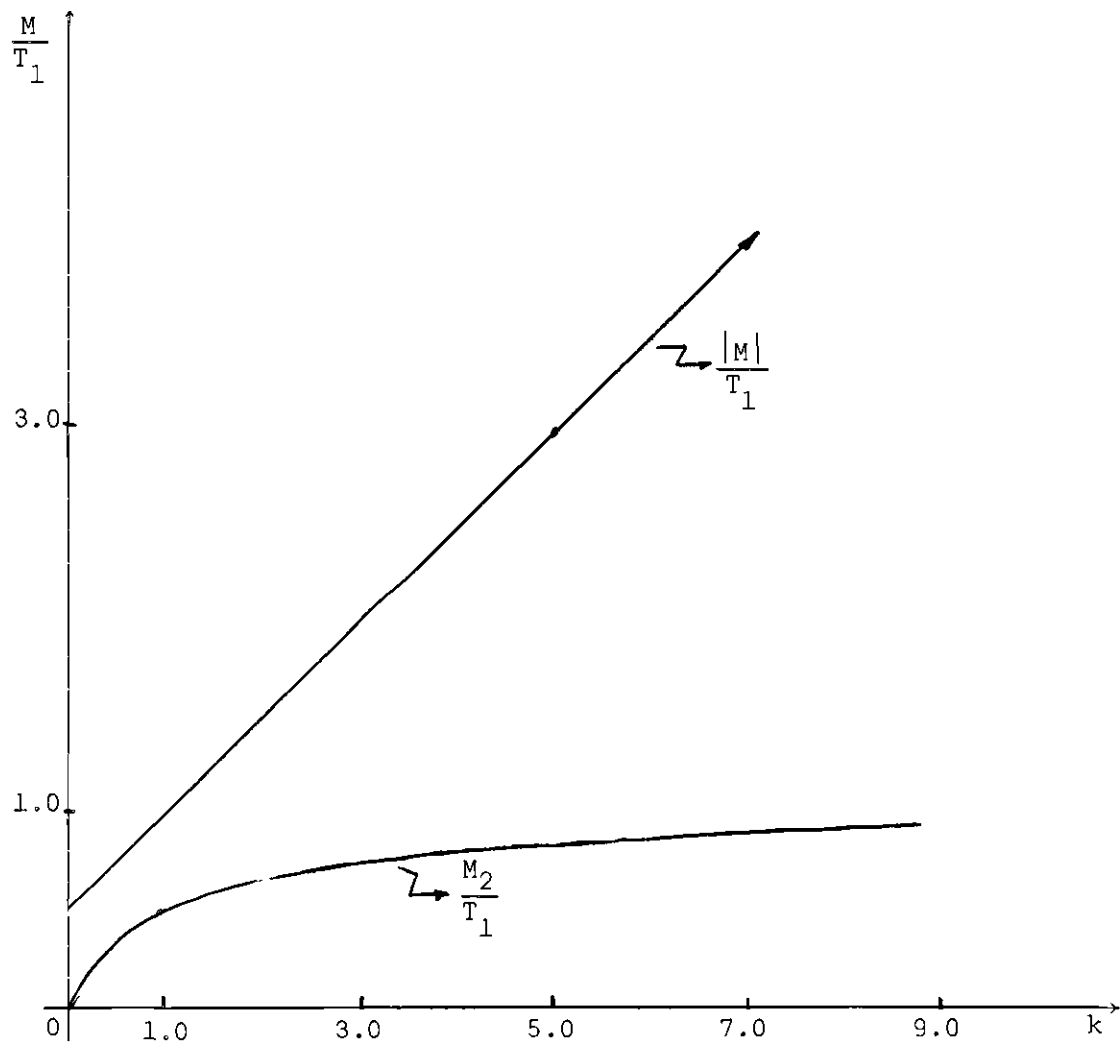


Figure 22. M/T_1 : Two First-Order Systems in Parallel

$$\frac{|M|}{T_1} = \frac{k + 1}{2}$$

$$\frac{M_2}{T_1} = \frac{k}{k + 1}$$

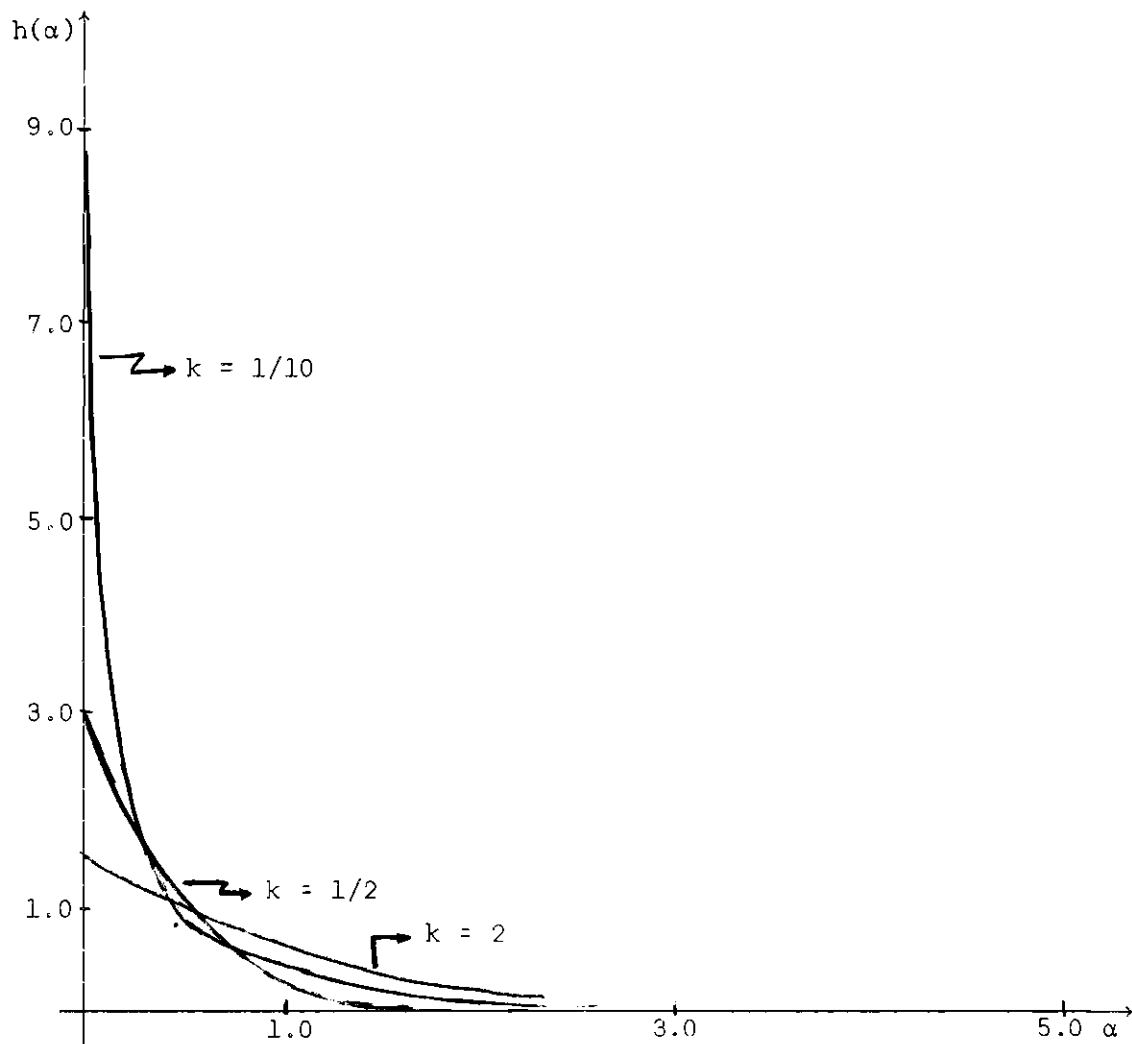


Figure 23. $h(\alpha)$: Two First-Order Systems in Parallel

$$h(\alpha) = \begin{cases} \frac{1}{k} e^{-\alpha/k} + e^{-\alpha}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases}$$

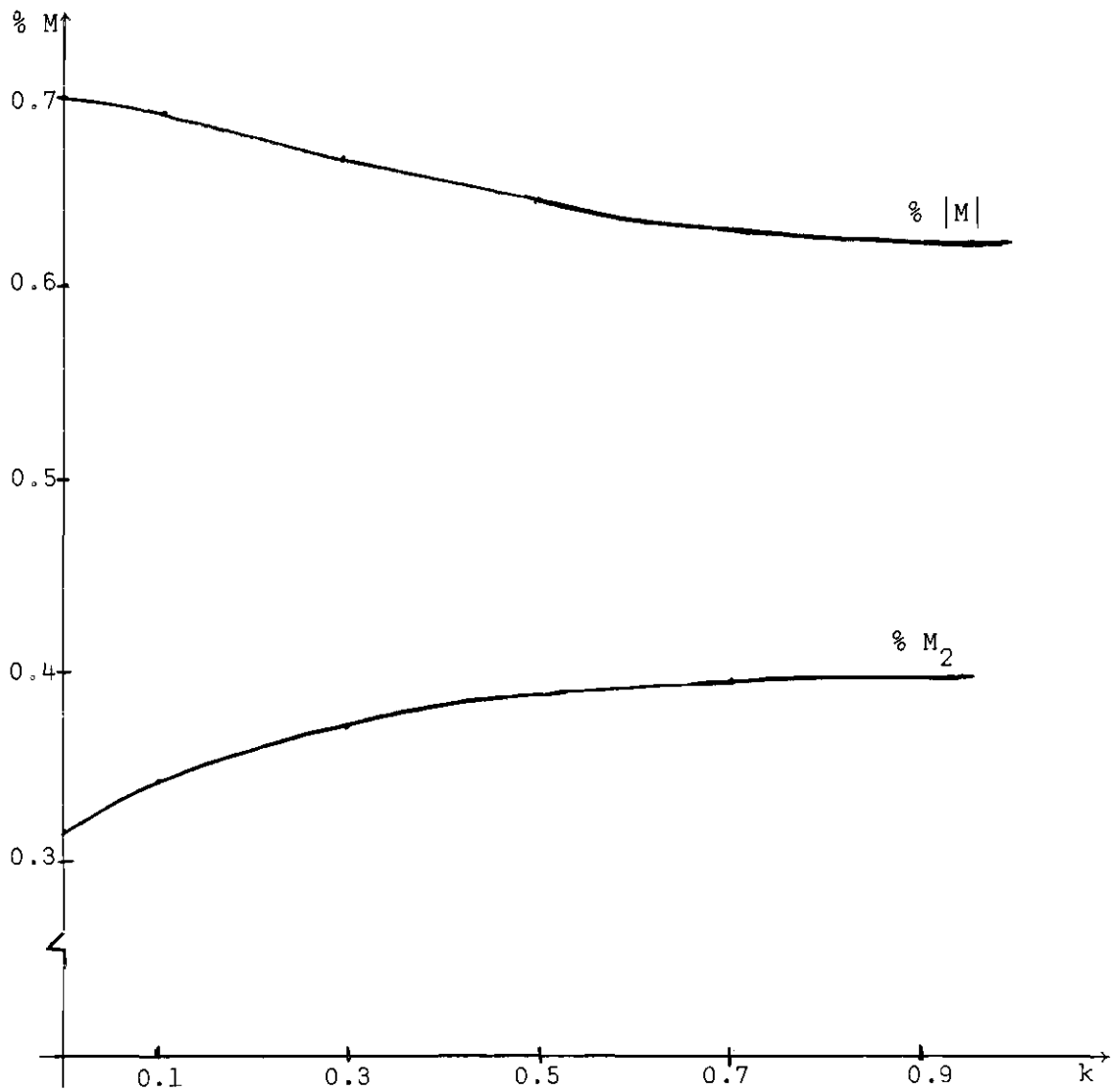


Figure 24. % M: Two First-Order Systems in Parallel

$$63.2 \leq \% |M| \leq 70$$

$$31.5 \leq \% M_2 \leq 39.4$$

$$\begin{aligned}
 |M| &= \frac{T_1 + T_2}{2} \\
 M_2 &= \frac{T_1 T_2}{T_1 + T_2} \\
 T_1, T_2 &> 0
 \end{aligned}$$

n First-Order Systems Cascaded

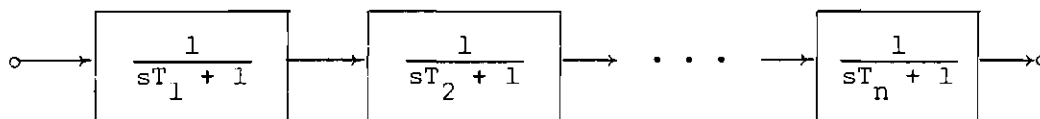


Figure 25. First-Order Systems Cascaded

In the first section of this chapter n identical cascaded systems were considered. In this section the more general case of n cascaded systems (Figure 25) is considered.

The weighting function for this system is given by

$$h(t) = \begin{cases} \sum_{i=1}^n A_i e^{-t/T_i}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where

$$A_1 = \begin{bmatrix} T_1^{n-2} \\ T_1 \end{bmatrix} \begin{bmatrix} n \\ \pi \\ 2 \end{bmatrix} \frac{1}{T_1 - T_i}$$

and

$$A_j = \left[T_j^{n-2} \right] \left[\prod_{\substack{i=1 \\ i \neq j}}^n \frac{1}{T_j - T_i} \right], \quad 1 < j \leq n.$$

The formulas for $|M|$ and M_2 are

$$|M| = \sum_{i=1}^n T_i$$

$$M_2 = \frac{\sum_{i=1}^n A_i^2 \left(\frac{T_i}{2}\right)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_i A_j \left(\frac{T_i T_j}{T_i + T_j}\right)^2}{\sum_{i=1}^n A_i^2 \left(\frac{T_i}{2}\right) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_i A_j \left(\frac{T_i T_j}{T_i + T_j}\right)}$$

The formula for $|M|$ of course follows from the theorem on cascaded monotone systems. The formula for M_2 looks somewhat formidable but can be easily arrived at using straightforward procedures.

Since the formula for M_2 in this case is complicated, it is difficult to make any evaluative or comparative remarks. One observation, however, will be made. Performing the division indicated in M_2 yields

$$M_2 = \sum_{i=1}^n \frac{T_i}{2} + R(T_1, \dots, T_n)$$

where R is a remaining term and a function of the parameter T_i . As will be seen in a later section dealing in a more general way with cascaded monotone systems, $R \rightarrow \sum_{i=1}^n T_i/2$ as $n \rightarrow \infty$. Thus $M_2 \rightarrow \sum_{i=1}^n T_i = |M|$ as $n \rightarrow \infty$. It is probably fortunate that this is the case, for it would

indeed be difficult to evaluate M_2 for n of any size. Certainly it is easier to approximate it by the sum $\sum_{i=1}^n T_i$.

$$|M| = \sum_{i=1}^n T_i$$

$$M_2 = \frac{\sum_{i=1}^n A_i^2 \left(\frac{T_i}{2}\right)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_i A_j \left(\frac{T_i T_j}{T_i + T_j}\right)^2}{\sum_{i=1}^n A_i^2 \left(\frac{T_i}{2}\right) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n A_i A_j \left(\frac{T_i T_j}{T_i + T_j}\right)}$$

$$A_1 = \begin{bmatrix} T_1^{n-2} \\ T_1 \end{bmatrix} \begin{bmatrix} n \\ \pi \\ 2 \end{bmatrix} \frac{1}{T_1 - T_i}$$

$$A_j = \begin{bmatrix} T_j^{n-2} \\ T_j \end{bmatrix} \begin{bmatrix} n \\ \pi \\ \sum_{\substack{i=1 \\ i \neq j}}^n \end{bmatrix} \frac{1}{T_j - T_i}, \quad 1 < j \leq n$$

$$T_i > 0$$

n First-Order Systems in Parallel

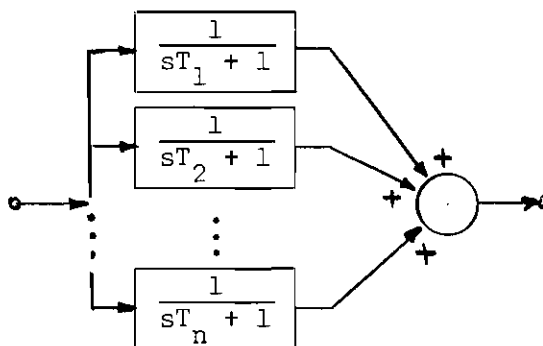


Figure 26. First-Order Systems in Parallel

In this section n systems in parallel (Figure 26) will be considered, the next logical extension of the system composed of n identical systems in parallel considered previously.

The weighting function for this system is given by

$$h(t) = \begin{cases} \sum_{i=1}^n 1/T_i e^{-t/T_i}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and the formulas for $|M|$ and M_2 as

$$|M| = \frac{\sum_{i=1}^n T_i}{n}$$

$$M_2 = \frac{n/2 + 4 \sum_{j=1}^{n-1} \sum_{i=j+1}^n \frac{T_i T_j}{(T_i + T_j)^2}}{\sum_{i=1}^n 1/T_i + 4 \sum_{j=1}^{n-1} \sum_{i=j+1}^n \frac{1}{T_i + T_j}}$$

The formula for M_2 again looks formidable. However, it provides certain useful information. The formula itself is obtained by using simple integration techniques and the fact that

$$\left(\sum_{i=1}^n x_i \right)^2 = \sum_{i=1}^n x_i^2 + 2 \sum_{j=1}^{n-1} \sum_{i=j+1}^n x_j x_i, \quad n \geq 2.$$

From this formula for M_2 it is obvious that $M_2 \rightarrow 0$ as $T_i \rightarrow 0$

for any i . This result was first mentioned in the case $n=2$. It is shown here to be true for any number of first-order systems in parallel combination. If any one of the component first-order systems approaches a memoryless state, then M_2 indicates that the same happens for the entire system. The reason for this indication by M_2 is the same as it was for the case for two parallel systems. The attraction of M_2 for large values and the distortion of the weighting function when some $T_i \rightarrow 0$ causes the result. Again criterion (ii) is nearly violated. Note again, too, that, as for the $n=2$ case, $|M|$ does not give this indication.

$$|M| = \frac{\sum_{i=1}^n T_i}{n}$$

$$M_2 = \frac{\frac{n/2 + 4}{\sum_{i=1}^n 1/T_i} + 4 \sum_{j=1}^{n-1} \sum_{i=j+1}^n \frac{T_i T_j}{(T_i + T_j)^2}}{\sum_{i=1}^n 1/T_i + 4 \sum_{j=1}^{n-1} \sum_{i=j+1}^n \frac{1}{T_i + T_j}} \quad *$$

* Though no discussion is made of the fact, it is true in this case that

$$\frac{T_{\min}}{2n} < M_2 < \frac{T_{\max}}{2} .$$

This fact can be easily shown. In addition no contradiction can be found for the statement

$$\frac{T_{\min}}{2} < M_2 < \frac{T_{\max}}{2} .$$

Though no proof of this fact could be made, it is strongly believed to be true for all $n \geq 2$ (it has already been shown true for the case $n=2$).

$$T_i > 0; \quad n=2,3,4,\dots$$

Monotone Systems Cascaded

In a number of the previous sections of this chapter $|M|$ and M_2 were studied in relation to systems whose components were cascaded first-order systems. At this point an important result concerning the relationship between $|M|$ and M_2 for the general case of monotone systems cascaded will be given. As one might suspect, there exists a close parallel between certain parts of this investigation, particularly certain parts of this section dealing with monotone systems, and some elements of probability theory. The definition of $|M|$ in many cases corresponds closely with the definition of expected value. In fact, $|M|$ and expected value have the same definition for those monotone systems whose weighting functions integrate to one. The definition of M_2 does not correspond to any definition of expected value, but it is still a measure of central tendency in the weighting function. Considering these statements, it should not be surprising that some of the tools associated with the area of probability theory might be useful in establishing certain general results.

The specific tool which will be used in this section is the central limit theorem. It is noted that the central limit theorem states that the distribution of the sum of n independent random variables is asymptotically normally distributed whatever the distribution of each of the individual random variables. Also finding the distribution of the sum of n independent random variables involves taking the

convolution of the distributions of each of the random variables in the sum. This same process of convolution is also used in finding the weighting function of n cascaded monotone systems.* If the weighting functions of the individual component systems in such a cascade should happen to also integrate to one, then the over-all weighting function must asymptotically approach a symmetric Gaussian error curve as the number of cascaded systems increases ($n \rightarrow \infty$)** . In fact, even if some or all of the individual weighting functions did not integrate to one, the over-all weighting function would still asymptotically approach a symmetric, bell-shaped curve. The fact that this over-all weighting function will always approach a symmetric curve for cascaded monotone systems is important. In Chapter III it was shown that for symmetric (in fact, absolutely symmetric) weighting functions $|M| = M_2 = a$, where a was the point of symmetry. If the over-all weighting function approaches symmetry, then certainly it can be said that $M_2 \rightarrow |M|$ at the same time. From this argument then the following statement can be made:

Let n monotone systems be cascaded. Then $M_2 \rightarrow |M|$ as $n \rightarrow \infty$, where $|M|$ and M_2 are evaluated on the over-all system.

The value of $|M|$ of course will be the sum of its values on each of the component systems.

One should recall that the above statement has already been used

* Recall that the convolution of n functions can be made by multiplying their Laplace Transforms together and then finding the inverse transform. This is exactly the way in which the over-all weighting function for n cascaded systems is obtained.

** It will be assumed here that all the conditions of the central limit theorem are satisfied.

to arrive at some of the earlier results for systems composed of first-order components cascaded. This statement could also have some practical value. For most monotone systems in cascade it is much easier to obtain the value for $|M|$ than it is to obtain the value for M_2 . If the number of systems in cascade was large, it would be much easier to use $|M|$ as an approximation to M_2 than it would be to try and evaluate M_2 should such an evaluation be called for. In any case, a method for approximating M_2 in certain situations is available.

It should also be pointed out that since $M_2 \rightarrow |M|$ as $n \rightarrow \infty$, then certainly $\% M_2 \rightarrow \% |M|$ at the same time. But as the number of cascaded systems increases, the resulting weighting function becomes more symmetric. It thus follows that both $\% |M|$ and $\% M_2$ approach 50 per cent. The entries in Table 3 at the beginning of this chapter demonstrate this result.

Summary of Important Results

At this point a brief summary of the results which have been obtained for $|M|$ and M_2 with respect to the previous sections will be given. These results are important and will form a major part of the basis from which the conclusions and recommendations of this investigation are made. These results are as follows:

1. For parallel system combinations it was found that $M_2 \rightarrow 0$ whenever any one of the component systems approached a memoryless state (that is, whenever the weighting function for any component system approached a unit impulse). This behavior of M_2 is a near violation of criterion (ii) (Chapter III) and is intuitively incorrect. The value

of $|M|$ for the same case on the other hand did not approach zero but what seemed like a reasonable positive value.

2. The general affinity of M_2 for large values in the system weighting function was found to cause large differences in the values of $|M|$ and M_2 for some systems. This affinity is also the cause of the previous result.

3. M_2 did not satisfy relationship (v). That is, it did not indicate that memory length added in a regular fashion for systems in cascade. $|M|$ did satisfy this relationship (this was actually shown in the literature and not in this investigation).

4. Formulas for M_2 in general tended to be complicated and hard to derive. Formulas for $|M|$ in general were simple and easy to derive.

5. Both $|M|$ and M_2 seemed to satisfy relationship (vi), thus indicating this as a characteristic of memory length.

$$6. \quad M_2 \leq |M| \quad \text{and} \quad 50 \leq \% |M| \leq 70$$

$$30 \leq \% M_2 \leq 50.$$

7. For n monotone systems cascaded, $M_2 \rightarrow |M|$ as $n \rightarrow \infty$.

Nonmonotone Systems

In the previous sections $|M|$ and M_2 were investigated in relation to monotone systems. In this part of the chapter investigation will proceed in relation to nonmonotone systems. In Chapter IV both $|M|$ and M_2 were considered in relation to an important member of this class, a second-order system (with damping ratio less than one). In the next two sections two third-order systems will be briefly considered. The first system consists of a first- and second-order system cascaded

and the second a first- and second-order system in parallel. In both these cases the poles of the second-order system component are complex; thus rendering the over-all system nonmonotone.

In making the transition of investigation from monotone to nonmonotone systems, some important changes will take place. In the following two sections the derivation and investigation of $|M|$ will not be made. The reason for this is that it is too difficult, with the exception of the second-order system investigated in Chapter IV, to derive $|M|$ for these systems. For monotone systems the absolute value in the definition of $|M|$ presents no difficulty. In the case of nonmonotone systems; however, the absolute value makes it necessary to locate the intervals on which the weighting function is positive and negative; and this means locating all the zeros of this function on the interval $[0, +\infty)$. For the second-order system of Chapter IV this was fairly easy. For nonmonotone systems in general this is a very difficult task. In the conclusions to this investigation this difficulty in deriving $|M|$ will be an important fact.

Since the derivation of $|M|$ will not be made, the evaluation of M_2 cannot be carried out by comparison with results obtained for $|M|$. One important point of evaluation for M_2 , however, can be made. In general the formulas for M_2 for nonmonotone systems are very involved. The analysis done in the next two sections, with little exception, will be to show how the value of M_2 can be obtained for a large number of nonmonotone systems, namely the type systems investigated in this thesis, with the use of tables. The limitations of this method will also be discussed.

First- and Second-Order Systems Cascaded

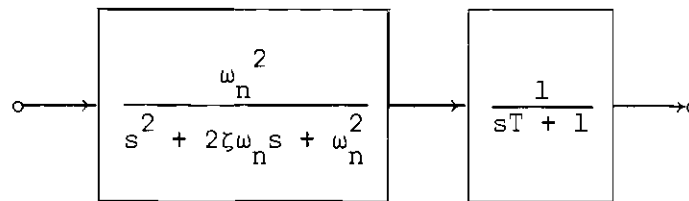


Figure 27. First- and Second-Order Systems Cascaded

In this section M_2 will be investigated in relation to the system illustrated in Figure 27. The weighting function for this system can be written as

$$h(t) = \begin{cases} \frac{\omega_n^2/T}{[(1/T - \zeta\omega_n)^2 + \omega_r^2]} \left[e^{-t/T} + \frac{\sqrt{(1/T - \zeta\omega_n)^2 + \omega_r^2}}{\omega_r} e^{-\zeta\omega_n t} \sin(\omega_r t - \phi) \right], & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where $\phi = \tan^{-1} \left[\frac{\omega}{1/T - \zeta\omega_n} \right]$ and $\omega_r = \omega_n \sqrt{1 - \zeta^2}$. Concern in this section of course will be for the system where $0 < \zeta < 1.0$ ($0 < \beta < \pi/2$)* and $T > 0$. The principal objective will simply be to introduce Table 4.

Since the value of M_2 for this system is quite involved, only its nondimensional form will be given. This form appears at the end of this

* Recall that $\zeta = \cos\beta$. In this section and the one which follows reference should be made to Figure 10 (Chapter IV) for the relationship among the second-order system parameters.

section. Its main usefulness is in the generation of Table 4. To obtain this form (a partial derivation of this form is made in the Appendix. Only an outline is given here), it is necessary to again introduce a nondimensional parameter k . The third-order system of Figure 27 is characterized by three parameters: a time constant T , a natural frequency ω_n , and a damping ratio ζ . In order to reduce this number to two, namely to k and ζ , the relation

$$k\zeta\omega_n = 1/T$$

is introduced. With this relation the formula for M_2 can be expressed in the nondimensional form $\omega_n M_2$ by simply factoring ω_n from the expression for M_2 . This form is a function of the two parameters k and ζ or k and β ; that is, $\omega_n M_2 = \omega_n M_2(k, \beta)$. In dealing with previous systems which could be characterized by two parameters, it was possible to express the nondimensional form of M_2 (and $|M|$) in terms of a single parameter k . The nondimensional form could then be graphically presented vs. k . In the present case, however, $\omega_n M_2$ must be graphically presented in contour diagram form vs. k and ζ . Such a contour diagram has been constructed and is given in Figure 29 for values of k up to 9 and for values of ζ up to 0.9. It is this diagram which is to be compared with the one in Figure 6 of Chapter II. Both diagrams have the same form. They differ in what is plotted and the values represented on the abscissa.

Corresponding to the contour diagram is a table of values for $\omega_n M_2$. These values are given in Table 4. Recall that in Chapter IV a

similar table (Table 2) was used to obtain values for $|M|$. How this can be done for higher order systems in general will be discussed in the next section. With Table 4 the values of M_2 can be determined for a large number of systems with relatively few calculations. An example showing how this is done for a specific system will be given.

Since the system in Figure 27 is composed of two systems in cascade, a word should be said about relationship (v). It is evident from the formula given below and the values of M_2 (see Chapter IV) for the component systems that this relationship does not hold in this case. This is generally true for nonmonotone systems composed of components in cascade. It shows that memory length for these systems is not a simple additive function of the memory lengths of the component systems.

$$\omega_n M_2 = \frac{N}{D}$$

$$N = N_1 + C_2 N_2 - C_3 N_3 + C_4 N_4 - C_5 N_5 - C_6 N_6$$

$$D = D_1 + C_2 D_2 - C_3 D_3 + C_4 D_4 - C_5 D_5 - C_6 D_6$$

where

$$N_1 = \frac{1}{4k^2 \cos^2 \beta} \qquad D_1 = \frac{1}{2k \cos \beta}$$

$$N_2 = \frac{\sin 2\theta_1}{A_1} \qquad D_2 = \frac{\sin \beta}{A_1}$$

$$N_3 = \frac{\cos 2\theta_1}{A_1}$$

$$D_3 = \frac{(k+1)\cos\beta}{A_1}$$

$$N_4 = \frac{1}{4\cos^2\beta}$$

$$D_4 = \frac{1}{2\cos\beta}$$

$$N_5 = \frac{\cos 2\beta}{4}$$

$$D_5 = \frac{\cos\beta}{2}$$

$$N_6 = \frac{\sin 2\beta}{4}$$

$$D_6 = \frac{\sin\beta}{2}$$

$$C_2 = 2B\cos\phi$$

$$\theta_1 = \tan^{-1} \left[\frac{\tan\beta}{k+1} \right]$$

$$C_3 = 2B\sin\phi$$

$$\phi = \tan^{-1} \left[\frac{\tan\beta}{k-1} \right], = \pi/2 \text{ if } k=1$$

$$C_4 = B^2/2$$

$$C_5 = (B^2/2)\cos 2\phi$$

$$A_1 = (k+1)^2\cos^2\beta + \sin^2\beta$$

$$C_6 = (B^2/2)\sin\phi$$

$$A_2 = (k-1)^2\cos^2\beta + \sin^2\beta$$

$$0 < \beta < \pi/2$$

$$B = \frac{\sqrt{A_2}}{\sin\beta}$$

$$k > 0$$

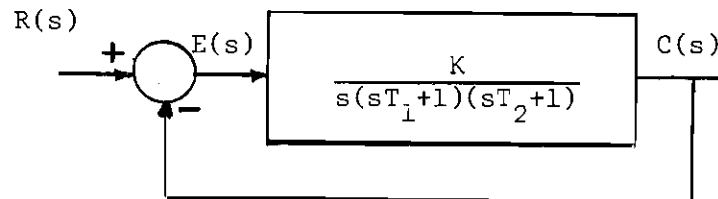
Example

Figure 28. Feedback Control System

Suppose it is desired to find M_2 for the above system with values for T_1, T_2 , and K as given below. The closed-loop transfer function for this system is given by

$$\frac{C(s)}{R(s)} = \frac{K}{s(sT_1+1)(sT_2+1) + K}$$

If

$$T_1 = 0.25 \text{ sec.}$$

$$T_2 = 0.10 \text{ sec.}$$

$$K = 2.25.$$

then

$$\frac{C(s)}{R(s)} = \frac{2.25}{s(0.25s+1)(0.10s+1) + 2.25}$$

$$= \frac{90}{s(s+4)(s+10) + 90}$$

$$= \frac{90}{(s_3+11.1)(s_1+i.45+2.47i)(s_2+1.45-2.47i)}$$

Thus

$$s_2 = -1.45 + 2.47i \quad \text{and}$$

$$\beta = \tan^{-1}\left(\frac{2.47}{1.45}\right) = \tan^{-1}(1.71) \approx 60^\circ \text{ or } \pi/3$$

$$\therefore \zeta = \cos\beta \approx 0.50$$

$$\zeta\omega_n = 1.45$$

$$\omega_n = 2.9 \text{ sec}^{-1}$$

Also

$$-s_3 = 1/T = 11.1 \text{ sec}^{-1}$$

$$\therefore k = -s_3/\zeta\omega_n = 11.1/1.45 = 7.67$$

From Table 4 for $\zeta = 0.5$ and $k = 7.67$

$$\omega_n M_2 \approx 1.608$$

and

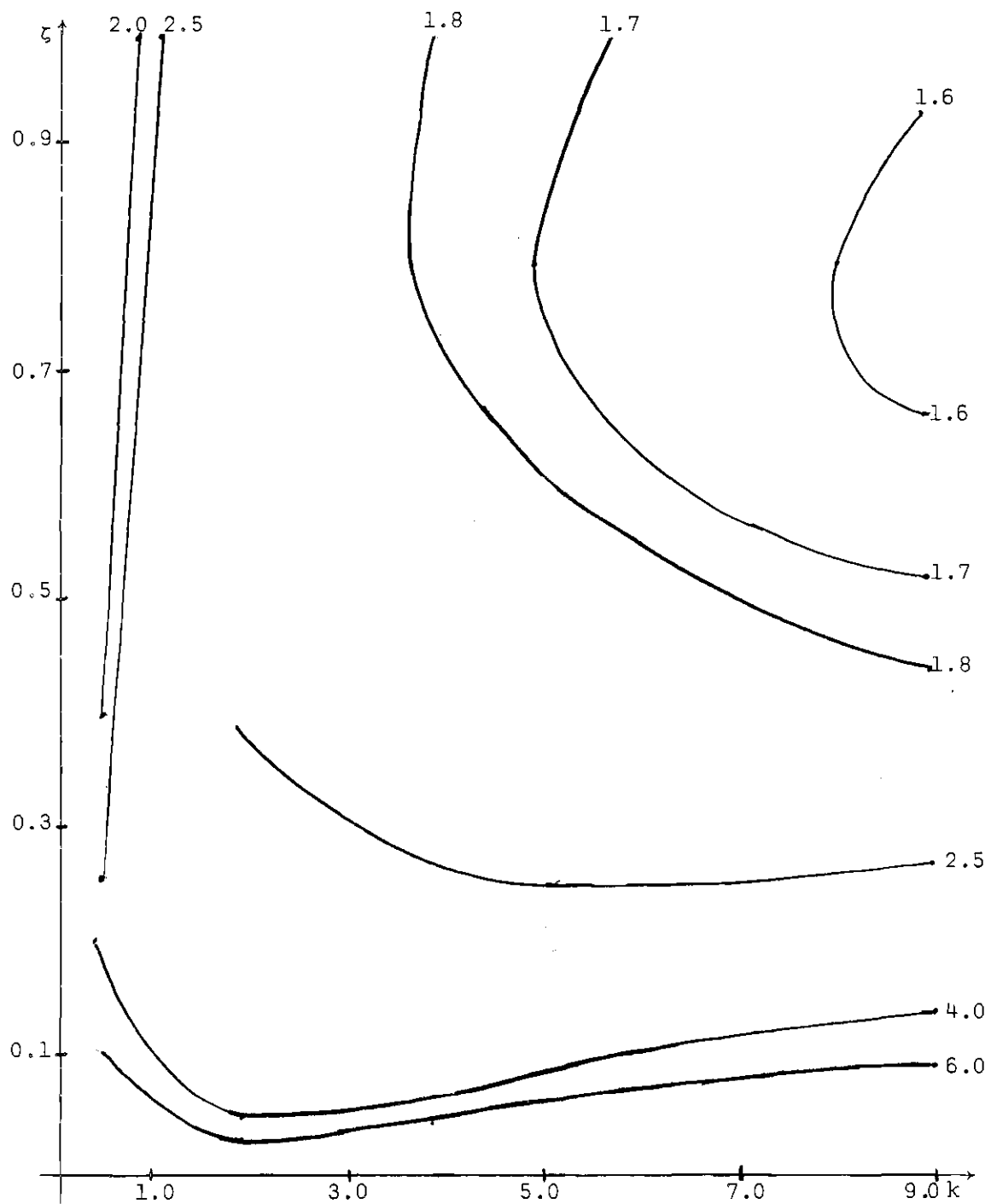


Figure 29. $\omega_n M_2$: First- and Second-Order Systems Cascaded

Table 4. ω_{n2} : First- and Second-Order Systems Cascaded

β	ζ	k																			
		0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0.05	0.999	0.857	2.500	2.205	2.042	1.938	1.866	1.814	1.775	1.744	1.719	1.698	1.681	1.667	1.654	1.644	1.634	1.626	1.619	1.612	1.606
0.10	0.995	0.856	2.500	2.205	2.041	1.937	1.866	1.813	1.774	1.743	1.718	1.697	1.680	1.665	1.653	1.642	1.633	1.625	1.617	1.611	1.605
0.15	0.989	0.856	2.500	2.204	2.041	1.936	1.865	1.812	1.772	1.741	1.716	1.695	1.678	1.663	1.651	1.640	1.631	1.622	1.615	1.608	1.602
0.20	0.980	0.856	2.500	2.204	2.040	1.935	1.863	1.810	1.770	1.739	1.713	1.693	1.675	1.661	1.649	1.637	1.628	1.619	1.612	1.605	1.599
0.25	0.969	0.856	2.500	2.204	2.039	1.934	1.861	1.808	1.768	1.736	1.710	1.689	1.672	1.657	1.644	1.633	1.624	1.615	1.608	1.601	1.595
0.30	0.955	0.856	2.500	2.204	2.038	1.933	1.859	1.806	1.765	1.733	1.707	1.686	1.668	1.653	1.640	1.629	1.620	1.611	1.603	1.597	1.590
0.35	2.939	0.856	2.500	2.204	2.038	1.932	1.858	1.803	1.762	1.730	1.704	1.682	1.664	1.649	1.636	1.625	1.615	1.606	1.598	1.591	1.585
0.40	0.921	0.856	2.501	2.205	2.038	1.931	1.856	1.801	1.759	1.727	1.700	1.678	1.660	1.644	1.631	1.620	1.610	1.601	1.593	1.586	1.580
0.45	0.900	0.857	2.503	2.206	2.039	1.930	1.855	1.800	1.757	1.724	1.697	1.674	1.656	1.640	1.627	1.615	1.605	1.596	1.588	1.580	1.574
0.50	0.878	0.858	2.506	2.209	2.040	1.931	1.855	1.798	1.755	1.721	1.694	1.671	1.652	1.636	1.622	1.610	1.600	1.591	1.583	1.575	1.569
0.55	0.853	0.860	2.510	2.213	2.043	1.933	1.856	1.799	1.755	1.720	1.692	1.669	1.649	1.633	1.619	1.607	1.596	1.586	1.578	1.570	1.564
0.60	0.825	0.863	2.514	2.218	2.048	1.937	1.858	1.800	1.755	1.720	1.691	1.668	1.648	1.631	1.617	1.604	1.593	1.583	1.574	1.567	1.560
0.65	0.796	0.867	2.520	2.224	2.054	1.992	1.863	1.804	1.758	1.722	1.693	1.668	1.648	1.631	1.616	1.603	1.591	1.581	1.572	1.564	1.557
0.70	0.765	0.873	2.528	2.233	2.063	1.951	1.870	1.810	1.764	1.727	1.697	1.672	1.651	1.633	1.617	1.604	1.592	1.582	1.573	1.564	1.557
0.75	0.732	0.882	2.537	2.245	2.075	1.962	1.881	1.820	1.773	1.735	1.704	1.678	1.656	1.638	1.622	1.608	1.596	1.585	1.576	1.567	1.559
0.80	0.697	0.895	2.548	2.259	2.091	1.978	1.896	1.834	1.786	1.747	1.715	1.689	1.666	1.647	1.631	1.616	1.604	1.592	1.582	1.574	1.565
0.85	0.660	0.913	2.561	2.277	2.111	1.999	1.916	1.854	1.804	1.765	1.732	1.705	1.682	1.662	1.645	1.680	1.616	1.605	1.594	1.585	1.577
0.90	0.622	0.940	2.577	2.299	2.137	2.025	1.943	1.880	1.830	1.789	1.756	1.728	1.704	1.683	1.665	1.650	1.636	1.624	1.613	1.603	1.594
0.95	0.582	0.978	2.595	2.325	2.168	2.059	1.978	1.915	1.864	1.823	1.788	1.759	1.735	1.713	1.695	1.678	1.664	1.651	1.640	1.629	1.620
1.00	0.540	1.033	2.617	2.356	2.206	2.102	2.023	1.960	1.909	1.867	1.832	1.802	1.777	1.755	1.735	1.718	1.703	1.690	1.678	1.667	1.657
1.05	0.498	1.112	2.644	2.393	2.253	2.155	2.079	2.018	1.968	1.926	1.891	1.860	1.834	1.811	1.791	1.773	1.757	1.743	1.730	1.719	1.708
1.10	0.454	1.226	2.677	2.436	2.309	2.220	2.150	2.093	2.044	2.003	1.968	1.937	1.910	1.887	1.866	1.847	1.831	1.816	1.802	1.790	1.779
1.15	0.408	1.391	2.721	2.488	2.376	2.300	2.239	2.187	2.143	2.104	2.069	2.039	2.012	1.988	1.967	1.948	1.931	1.915	1.901	1.888	1.876
1.20	0.362	1.632	2.781	2.552	2.458	2.398	2.350	2.308	2.270	2.235	2.204	2.175	2.149	2.126	2.105	2.085	2.068	2.052	2.037	2.023	2.011
1.25	0.315	1.984	2.872	2.635	2.558	2.520	2.491	2.463	2.436	2.409	2.383	2.359	2.335	2.314	2.294	2.275	2.257	2.241	2.226	2.213	2.200
1.30	0.268	2.507	3.017	2.753	2.690	2.676	2.670	2.664	2.654	2.641	2.625	2.609	2.592	2.575	2.558	2.541	2.526	2.511	2.496	2.483	2.470
1.35	0.219	3.307	3.267	2.942	2.881	2.889	2.912	2.934	2.949	2.958	2.961	2.960	2.955	2.998	2.939	2.928	2.918	2.906	2.895	2.884	2.873
1.40	0.170	4.596	3.737	3.289	3.205	3.225	3.276	3.331	3.380	3.421	3.454	3.478	3.496	3.508	3.516	3.520	3.521	3.520	3.517	3.513	3.508
1.45	0.120	6.926	4.731	4.039	3.880	3.883	3.955	4.040	4.126	4.207	4.280	4.394	4.400	4.447	4.486	4.520	4.597	4.570	4.588	4.603	4.614
1.50	0.070	12.384	7.417	6.137	5.780	5.723	5.781	5.884	6.004	6.126	6.246	6.359	6.465	6.563	6.653	6.736	6.812	6.880	6.943	7.000	7.051

$$M_2 = 1.608/2.9 = 0.55 \text{ sec}$$

which is considerably easier than evaluating the formula for $\omega_n M_2$.

First- and Second-Order System in Parallel

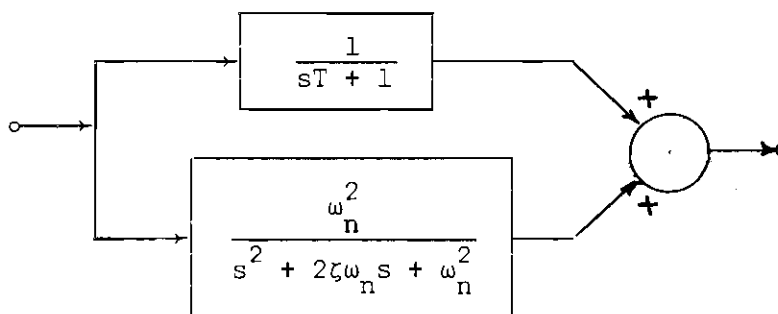


Figure 30. First- and Second-Order System in Parallel

This system is the parallel counterpart of the one in the previous section. The weighting function for this system is

$$h(t) = \begin{cases} 1/T e^{-t/T} + \frac{\omega_n^2}{\omega_r} e^{-\zeta\omega_n t} \sin\omega_r t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Again the relation $k\zeta\omega_n = 1/T$ can be used to obtain $\omega_n M_2$. This value appears at the end of this section. Also, as before, a contour diagram and a table of values for the nondimensional form are given. These are Figure 31 and Table 5, respectively.

In this section the discussion will be confined to two points. One point concerns relationship (vi) and the other concerns, as was previously mentioned, tables like 4 and 5 in general. Discussion of these two points will terminate this investigation.

To check relationship (vi) for this system the values in Table 5 can be used. Suppose that for the system in Figure 30

$$T/2 \leq \zeta/\omega_n + \frac{1}{2\zeta\omega_n}$$

or (using $k\zeta\omega_n = 1/T$)

$$1/2k\zeta \leq \zeta + 1/2\zeta$$

which implies

$$k \geq \frac{1}{2\zeta^2 + 1} .$$

What is to be determined in this case then is whether or not

$$T/2 \leq M_2 \leq \zeta/\omega_n + \frac{1}{2\zeta\omega_n}$$

or

$$1/2k\zeta \leq \omega_n M_2 \leq \zeta + 1/2\zeta$$

for all $k \geq 1/(2\zeta^2+1)$. The number $\omega_n M_2$ of course is in Table 5. In general this relation will be found to hold. However, for at least one

set of values for k and ζ it fails. Suppose ζ is small (β large), say $\zeta = 0.07$. Then $\omega_n M_2$ should satisfy

$$7.15/k \leq \omega_n M_2 \leq 7.22$$

for all $k \geq 1.0$. For $k = 1.0$ though $\omega_n M_2 = 6.881 \nmid 7.15$. If this relation is also checked for a few other small values of ζ with $k = 1.0$, it will also fail. Thus it seems that at least for these cases relationship (vi) fails to hold for M_2 . For most of the values in the table, as was just stated, the proper relationship does hold.

The other point to discuss concerns tables like 4 and 5. In Chapter III it was pointed out that a reduction in the number of parameters necessary to characterize a system could be made for all the type systems considered in this investigation. How this is done and how the corresponding nondimensional form of M_2 can be used has been demonstrated for two nonmonotone systems. How this might also be extended to systems with orders higher than the third is also not hard to see. The use of a few more nondimensional parameters like k would make it possible to reduce the number of characterizing parameters and develop tables like 4 and 5. Such tables, however, would obviously get very large as the order of the system to which they applied increased. In fact, if some way were not devised to generate such tables with some degree of efficiency, this method of evaluating M_2 would quickly run aground. One way such efficiency can be obtained is to choose the ranges of the parameters in the table to cover only the intervals which might be most used. Other ways to obtain efficiency also exist

and can be employed in a large number of cases.

$$\omega_n M_2 = \frac{N}{D} *$$

$$N = 1/4 + \frac{4(k^2+k)\cos^2\beta}{A^2} + \frac{1}{2\sin^2 2\beta} - \frac{\cos 2\beta}{8\sin^2\beta}$$

$$D = \frac{k\cos\beta}{2} + \frac{2k\cos\beta}{A} + \frac{1}{4\cos\beta}$$

$$A = (k^2+2k)\cos^2\beta + 1$$

$$k > 0 \quad \zeta = \cos\beta \quad 0 < \beta < \pi/2$$

* A partial derivation of this formula appears in the Appendix.

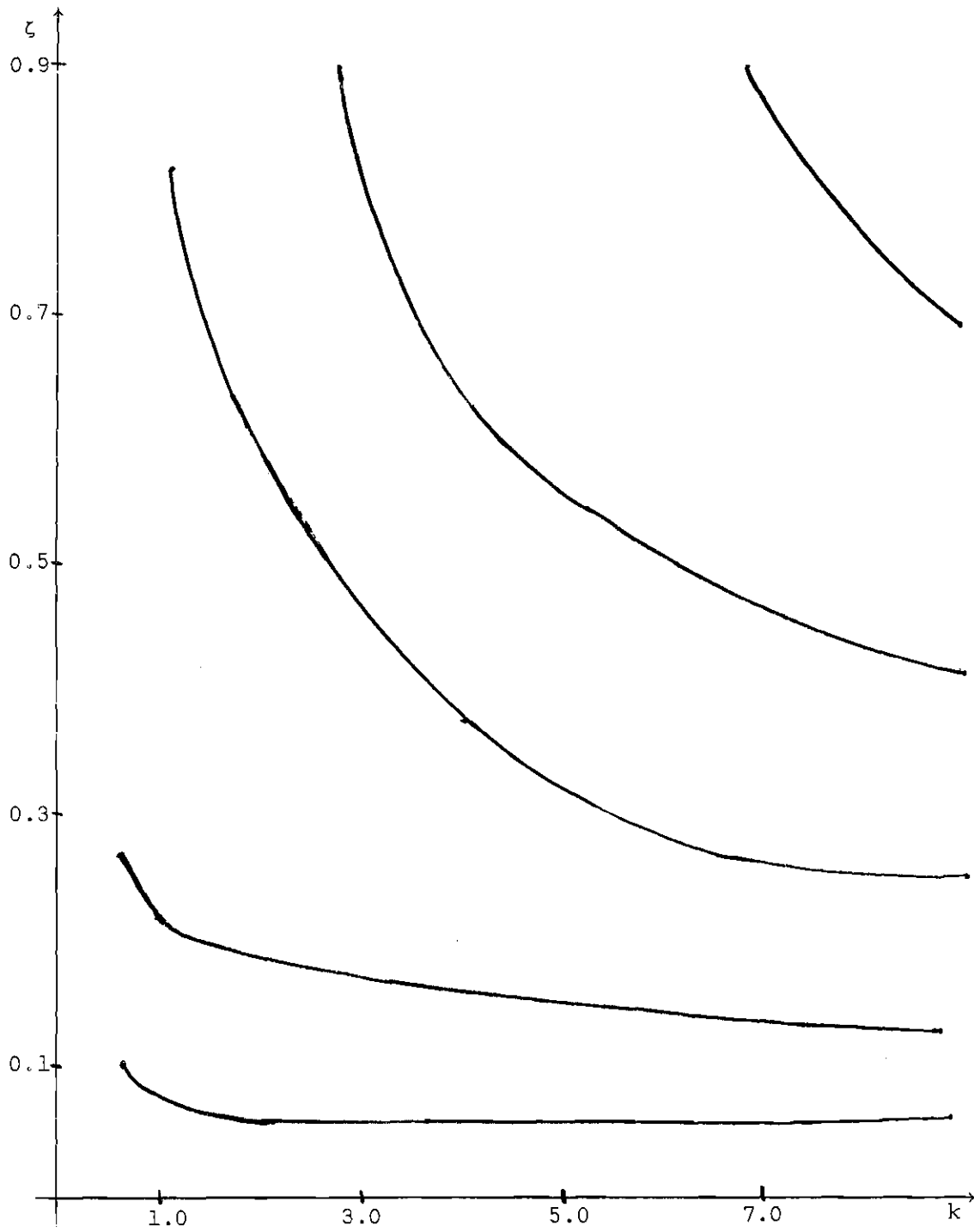


Figure 31. $\omega_n M_2$: First- and Second-Order Systems in Parallel

Table 5. $\omega_n M_2$: First- and Second-Order Systems in Parallel

β	ζ	k																			
		0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0
0.05	0.999	1.290	0.901	0.683	0.544	0.450	0.383	0.332	0.293	0.263	0.237	0.217	0.199	0.184	0.172	0.160	0.151	0.142	0.134	0.127	0.121
0.10	0.995	1.290	0.403	0.685	0.547	0.452	0.385	0.334	0.295	0.264	0.239	0.218	0.200	0.185	0.172	0.161	0.151	0.143	0.135	0.138	0.122
0.15	0.989	1.293	0.407	0.689	0.550	0.455	0.387	0.336	0.297	0.266	0.240	0.219	0.202	0.187	0.174	0.163	0.153	0.144	0.136	0.129	0.123
0.20	0.980	1.296	0.912	0.694	0.565	0.460	0.391	0.340	0.300	0.269	0.243	0.222	0.204	0.189	0.176	0.164	0.154	0.196	0.138	0.131	0.124
0.25	0.969	1.300	0.919	0.701	0.561	0.466	0.396	0.345	0.304	0.273	0.297	0.225	0.207	0.192	0.178	0.167	0.157	0.148	0.140	0.133	0.126
0.30	0.955	1.304	0.927	0.710	0.569	0.473	0.403	0.350	0.310	0.277	0.256	0.229	0.211	0.195	0.182	0.170	0.160	0.151	0.142	0.135	0.129
0.35	0.939	1.309	0.937	0.720	0.579	0.481	0.411	0.358	0.316	0.288	0.256	0.234	0.216	0.200	0.186	0.174	0.163	0.154	0.146	0.138	0.132
0.40	0.921	1.314	0.449	0.732	0.590	0.492	0.420	0.366	0.324	0.290	0.263	0.240	0.221	0.205	0.191	0.178	0.168	0.158	0.150	0.142	0.135
0.45	0.900	1.320	0.962	0.746	0.604	0.504	0.431	0.376	0.333	0.299	0.271	0.247	0.228	0.211	0.196	0.184	0.173	0.163	0.154	0.146	0.139
0.50	0.878	1.327	0.977	0.762	0.619	0.518	0.444	0.388	0.344	0.308	0.280	0.256	0.235	0.218	0.203	0.190	0.179	0.169	0.160	0.152	0.144
0.55	0.853	1.333	0.993	0.780	0.636	0.534	0.459	0.401	0.356	0.320	0.290	0.265	0.245	0.227	0.211	0.198	0.186	0.176	0.166	0.158	0.150
0.60	0.825	1.340	1.011	0.801	0.656	0.553	0.476	0.417	0.370	0.333	0.302	0.277	0.255	0.237	0.221	0.207	0.195	0.184	0.174	0.165	0.157
0.65	0.796	1.347	1.031	0.824	0.674	0.574	0.496	0.435	0.387	0.349	0.317	0.290	0.268	0.249	0.232	0.217	0.204	0.193	0.183	0.174	0.165
0.70	0.765	1.354	1.052	0.849	0.705	0.599	0.518	0.456	0.407	0.367	0.334	0.303	0.283	0.262	0.249	0.230	0.216	0.204	0.194	0.184	0.175
0.75	0.732	1.360	1.075	0.877	0.734	0.627	0.545	0.481	0.430	0.388	0.354	0.325	0.300	0.279	0.261	0.244	0.230	0.218	0.206	0.196	0.187
0.80	0.647	1.367	1.099	0.909	0.766	0.658	0.575	0.509	0.456	0.413	0.377	0.347	0.321	0.298	0.279	0.262	0.247	0.233	0.221	0.211	0.201
0.85	0.660	1.375	1.126	0.943	0.803	0.695	0.610	0.542	0.487	0.442	0.405	0.373	0.345	0.322	0.301	0.283	0.267	0.253	0.240	0.228	0.218
0.90	0.622	1.383	1.154	0.981	0.844	0.737	0.651	0.581	0.524	0.477	0.438	0.404	0.375	0.350	0.328	0.309	0.292	0.276	0.262	0.250	0.238
0.95	0.582	1.393	1.183	1.022	0.891	0.784	0.698	0.627	0.568	0.519	0.477	0.442	0.411	0.384	0.361	0.340	0.322	0.305	0.290	0.276	0.264
1.00	0.540	1.407	1.216	1.068	0.944	0.840	0.753	0.681	0.621	0.569	0.526	0.488	0.456	0.427	0.402	0.379	0.359	0.341	0.325	0.310	0.297
1.05	0.498	1.428	1.253	1.120	1.004	0.904	0.818	0.746	0.684	0.631	0.585	0.545	0.511	0.480	0.453	0.429	0.407	0.387	0.369	0.353	0.338
1.10	0.454	1.461	1.297	1.178	1.072	0.978	0.895	0.823	0.761	0.706	0.659	0.617	0.580	0.548	0.518	0.492	0.468	0.447	0.427	0.409	0.392
1.15	0.408	1.513	1.354	1.247	1.153	1.067	0.988	0.918	0.856	0.801	0.752	0.708	0.670	0.635	0.603	0.575	0.549	0.525	0.503	0.484	0.465
1.20	0.362	1.596	1.433	1.335	1.252	1.174	1.102	1.036	0.976	0.921	0.876	0.827	0.786	0.750	0.716	0.686	0.657	0.632	0.608	0.586	0.565
1.25	0.315	1.730	1.552	1.454	1.379	1.311	1.247	1.187	1.130	1.078	1.029	0.985	0.943	0.905	0.870	0.838	0.808	0.780	0.754	0.729	0.707
1.30	0.268	1.947	1.746	1.638	1.563	1.500	1.444	1.390	1.339	1.291	1.245	1.202	1.161	1.124	1.088	1.055	1.023	0.994	0.966	0.940	0.915
1.35	0.219	2.305	2.081	1.943	1.860	1.794	1.739	1.690	1.644	1.601	1.559	1.520	1.482	1.446	1.412	1.379	1.348	1.318	1.240	1.263	1.237
1.40	0.170	2.926	2.688	2.525	2.410	2.324	2.257	2.202	2.155	2.113	2.074	2.037	2.003	1.970	1.939	1.909	1.879	1.851	1.824	1.793	1.773
1.45	0.120	4.116	3.895	3.717	3.574	3.457	3.362	3.283	3.216	3.160	3.110	3.087	3.028	2.992	2.960	2.929	2.901	2.874	2.848	2.824	2.800
1.50	0.070	7.039	6.881	6.735	6.601	6.478	6.366	6.264	6.171	6.086	6.009	5.938	5.874	5.816	5.762	5.713	5.668	5.626	5.588	5.553	5.520

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

From the previous investigation certain definite conclusions can be made. These conclusions concern both the evaluation of $|M|$ and M_2 as measures of system memory length and some characteristics of memory length itself. The conclusions, however, are general in nature. No attempt has been made to enumerate results as was done previously.

From an over-all standpoint it must be concluded that M_2 is the better measure of memory length. This conclusion is largely a result of the fact that a value for M_2 can be obtained for a large number of nonmonotone systems, while in general derivations of $|M|$ for these systems are difficult to impossible. It is true that M_2 is swayed in many cases by large values in the weighting function sufficiently to cause near violations of criterion (ii). However, systems where this occurs are not common and are not likely to be encountered in practice. Also the investigation in Chapter IV, related to the second-order system, indicated that M_2 was capable of choosing the best performing system (and hence also the one of minimum memory length). The percentage points for M_2 offered no objectional behavior. These points for $|M|$ and M_2 differed only in their range of values.

For monotone systems on the other hand the measure $|M|$ is very attractive. It is free from the troublesome near violation of

criterion (ii). In general for any system its formula is easy to derive and simple in form.

The previous investigation also rendered certain information in regards to relationships (v) and (vi). For monotone systems cascaded, $|M|$ indicated that memory length is the sum of the memory lengths for each individual component systems. This was not the indication of M_2 , though M_2 and $|M|$ differ by only a small amount when a large number of systems are cascaded. For nonmonotone systems there seemed to be no connection between component system memories and over-all system memory. It also seems that memory length for parallel system combinations does indeed behave according to relationship (vi). Only under certain strained conditions did this relationship not hold.

Recommendations

If in the event that systems are classified or compared on the basis of memory length and these systems are all monotone, then use of the measure $|M|$ would be recommended. The ease of derivation and simple form of the formula make this an obvious recommendation. Another recommendation is the use of $|M|$ as an approximation to M_2 for the case of monotone systems in cascade.

It should be kept in mind that many of the results presented in this investigation can be generalized to other systems with weighting functions of a similar form. This includes especially those results pertaining to the relative values and percentage points.

APPENDIX

APPENDIX

Derivation of $|M|$: Second-Order System (Chapter IV)Case I ($\zeta > 1.0$)

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2-1}} (e^{-s_1 t} - e^{-s_2 t})$$

$$-s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \quad \text{and} \quad -s_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

$$\begin{aligned} |M| &= \frac{\int_0^{\infty} (te^{-s_1 t} - te^{-s_2 t}) dt}{\int_0^{\infty} (e^{-s_1 t} - e^{-s_2 t}) dt} = \frac{1/s_1^2 - 1/s_2^2}{1/s_1 - 1/s_2} \\ &= 1/s_1 + 1/s_2 = \frac{s_1 + s_2}{s_1 s_2} = \frac{2\omega_n \zeta}{\omega_n} = 2\zeta. \end{aligned}$$

Case II ($\zeta < 1.0$)

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t} \sin \omega_r t$$

$$\omega_r = \omega_n \sqrt{1-\zeta^2}$$

$$|M| = \frac{\int_0^{\infty} te^{-\omega_n \zeta t} |\sin \omega_r t| dt}{\int_0^{\infty} e^{-\omega_n \zeta t} |\sin \omega_r t| dt}$$

$$\int_0^{\infty} t e^{-\omega_n \zeta t} |\sin \omega_r t| dt = \sum_{n=0}^{\infty} (-1)^n \int_{n\pi/\omega_r}^{(n+1)\pi/\omega_r} t e^{-\omega_n \zeta t} \sin \omega_r t dt$$

$$\int_0^{\infty} e^{-\omega_n \zeta t} |\sin \omega_r t| dt = \sum_{n=0}^{\infty} (-1)^n \int_{n\pi/\omega_r}^{(n+1)\pi/\omega_r} e^{-\omega_n \zeta t} \sin \omega_r t dt$$

But

$$\int_{n\pi/\omega_r}^{(n+1)\pi/\omega_r} t e^{-\omega_n \zeta t} \sin \omega_r t dt = (2\zeta\sqrt{1-\zeta^2} (-1)^n + n\pi(-1)^n)$$

$$\left[\frac{r^{n+1} + r^n}{\omega_n^2} \right] + \left[\pi(-1)^n \right] \left[\frac{r^{n+1}}{\omega_n^2} \right].$$

And

$$\int_{n\pi/\omega_r}^{(n+1)\pi/\omega_r} e^{-\omega_n \zeta t} \sin \omega_r t dt = \left[\frac{\sqrt{1-\zeta^2}}{\omega_n} \right] \left[r^{n+1} + r^n \right] (-1)^n$$

$$r = e^{-\phi}, \quad \phi = \zeta\pi/\sqrt{1-\zeta^2}$$

Thus

$$\int_0^{\infty} t e^{-\omega_n \zeta t} |\sin \omega_r t| dt = \left[\frac{2\zeta\sqrt{1-\zeta^2}}{\omega_n^2} \right] \left[\frac{r+1}{1-r} \right] + \left[\frac{\pi}{\omega_n^2} \right] \left[\frac{2r}{(1-r)^2} \right]$$

$$\int_0^{\infty} e^{-\omega_n \zeta t} |\sin \omega_r t| dt = \left[\frac{\sqrt{1-\zeta^2}}{\omega_n} \right] \left[\frac{r+1}{1-r} \right]$$

$$\therefore |M| = \frac{2\zeta}{\omega_n} + \frac{2\pi}{\omega_r} \left[\frac{r}{1-r^2} \right]$$

Derivation of M_2 : Second-Order System (Chapter IV)

Case I ($\zeta > 1.0$)

$$h^2(t) = \frac{\omega_n^2}{4(\zeta^2-1)} \left[e^{-s_1 t} - e^{-s_2 t} \right]^2$$

$$M_2 = \frac{\int_0^{\infty} t(e^{-s_1 t} - e^{-s_2 t})^2 dt}{\int_0^{\infty} (e^{-s_1 t} - e^{-s_2 t})^2 dt} = \frac{1/4s_1^2 - 2/(s_1+s_2)^2 + 1/4s_2^2}{1/2s_1 - 2/(s_1+s_2) + 1/2s_2}$$

$$= \frac{(s_1^2+s_2^2)(s_1+s_2)^2 - 8s_1^2s_2^2}{2s_1s_2(s_1+s_2)^3 - 8s_1^2s_2^2(s_1+s_2)}$$

after some algebra

$$= \frac{2\zeta^4 - \zeta^2 - 1}{2\zeta\omega_n(\zeta^2-1)}$$

$$= \frac{\zeta}{\omega_n} + \frac{1}{2\zeta\omega_n}$$

Case II ($\zeta < 1.0$)

$$h^2(t) = \frac{\omega_n^2}{(1-\zeta^2)} e^{-2\omega_n \zeta t} \sin^2 \omega_r t$$

$$M_2 = \frac{\int_0^{\infty} t e^{-2\omega_n \zeta t} \sin^2 \omega_r t dt}{\int_0^{\infty} e^{-2\omega_n \zeta t} \sin^2 \omega_r t dt}$$

$$= \frac{1/8 \left[\frac{(2\zeta^2+1)(1-\zeta^2)}{\omega_n^2 \zeta^2} \right]}{1/4 \left[\frac{1-\zeta^2}{\omega_n \zeta} \right]}$$

$$= \zeta/\omega_n + 1/2\omega_n \zeta$$

Partial Derivation of $\omega_n M_2$: First- and Second-Order
Systems Cascaded (Chapter V)

$$h(t) = \left[\frac{\omega_n^2/T}{(1/T - \zeta\omega_n)^2 + \omega_r^2} \right] \left[e^{-t/T} + \frac{\sqrt{(1/T - \zeta\omega_n)^2 + \omega_r^2}}{\omega_r} e^{-\zeta\omega_n t} \sin(\omega_r t - \phi) \right]$$

$$t \geq 0$$

$$\phi = \text{Tan}^{-1} \left[\frac{\omega_r}{1/T - \zeta\omega_n} \right]$$

$$h^2(t) = e^{-2t/T} + \frac{2r}{\omega_r} e^{-at} (\sin \omega_r t \cos \phi - \cos \omega_r t \sin \phi) \\ + \frac{\operatorname{cosec}^2 \phi}{2} e^{-2a_0 t} (1 - \cos 2\omega_r t \cos 2\phi - \sin 2\omega_r t \sin 2\phi)$$

$$r = \sqrt{(1/T - \zeta \omega_n)^2 + \omega_r^2}$$

$$\sin \phi = \omega_r / r$$

$$a_0 = \zeta \omega_n$$

$$a = a_0 + 1/T$$

$$\int_0^{\infty} th^2(t) dt = \left(\frac{T}{2}\right)^2 + \frac{K^2}{8a_0^2} + 2 \left[\frac{\cos 2\theta_1}{a^2 + \omega_r^2} \right] \left[(\cot \phi)(\tan 2\theta_1) - 1 \right] \\ - \frac{1}{8} \left[\frac{\cos 2\theta_2}{\omega_n^2} \right] \left[\cot^2 \phi + (2 \tan 2\theta_2)(\cot \phi) - 1 \right] \quad (\neq)$$

$$\int_0^{\infty} h^2(t) dt = T/2 + \frac{K^2}{4a_0} + \left[\frac{2a}{a^2 + \omega_r^2} \right] \left[(\tan \theta_1)(\cot \phi) - 1 \right] \\ - \left[\frac{a_0}{4\omega_n^2} \right] \left[\cot^2 \phi + (2 \tan \theta_2)(\cot \phi) - 1 \right] \quad (\neq)$$

$$K = r/\omega_r = \operatorname{cosec}\phi$$

$$\theta_1 = \operatorname{Sin}^{-1} \left[\frac{\omega_r}{\sqrt{a^2 + \omega_r^2}} \right]$$

$$\theta_2 = \beta = \operatorname{Sin}^{-1} \left[\frac{\omega}{\sqrt{a_o^2 + \omega_r^2}} \right]$$

Let $1/T = k\zeta\omega_n$, then

$$(+)\left\{ \begin{array}{l} \phi = \operatorname{Tan}^{-1} \left[\frac{\tan\beta}{k-1} \right] \\ \cot\phi = (k-1)(\cot\beta) \\ \cos 2\theta_1 = \frac{a^2 - \omega_r^2}{a^2 + \omega_r^2} = \frac{(k^2 + 2k + 2)\cos^2\beta - 1}{(k^2 + 2k)\cos^2\beta + 1} \\ K = r/\omega_r = \frac{\sqrt{(k^2 - 2k)\cos^2\beta + 1}}{\sin\beta} \\ a^2 + \omega_r^2 = \omega_n^2 \left[(k^2 + 2k + 2)\cos^2\beta + 1 \right] \\ \tan\theta_1 = \frac{\tan\beta}{k+1} \end{array} \right.$$

Using equations (+), (≠) and (≠), the desired results can be obtained.

Partial Derivation of $\omega_n M_2$: First- and Second-Order
Systems in Parallel (Chapter V)

$$h(t) = 1/T e^{-t/T} + \frac{\omega_n^2}{\omega_r} e^{-\zeta\omega_n t} \sin\omega_r t, \quad t \geq 0$$

$$h^2(t) = [k\zeta\omega_n]^2 e^{-ka_2 t} + K_1 e^{-a_1 t} \sin\omega_r t + K_2 e^{-a_2 t} \sin^2\omega_r t$$

where

$$k\zeta\omega_n = 1/T$$

$$a_1 = \zeta\omega_n + 1/T = \zeta\omega_n (k+1)$$

$$a_2 = 2\zeta\omega_n$$

$$K_1 = \frac{2\omega_n^2}{T\omega_r} = 2k\omega_n^2 \cot\beta$$

$$K_2 = \frac{\omega_n^4}{\omega_r^2} = \omega_n^2 \operatorname{cosec}^2\beta$$

$$\int_0^\infty h^2(t) dt = 1/4 + K_1 \frac{\sin 2\theta_1}{a_1 + \omega_r^2} + \frac{K_2}{2a_2} - \frac{K_2}{2} \cdot \frac{\cos 2\theta_2}{a_2 + 4\omega_r^2} \quad (\#)$$

$$\int_0^\infty h^2(t) dt = \frac{k\zeta\omega_n}{2} + \frac{K_1\omega_r}{a_1^2 + \omega_r^2} + \frac{2K_2\omega_r^2}{a_2(a_2^2 + 4\omega_r^2)} \quad (\#)$$

where

$$\theta_1 = \sin^{-1} \left[\frac{\omega_r}{\sqrt{a_1^2 + \omega^2}} \right] = \sin^{-1} \left[\frac{\sin \beta}{\sqrt{(k+1)^2 \cos^2 \beta + \sin^2 \beta}} \right]$$

$$\theta_2 = \cos^{-1} \left[\frac{a_2}{\sqrt{a_2^2 + 4\omega^2}} \right] = \cos^{-1}(\cos \beta) = \beta$$

The desired results are easily obtained from (#) and (##).

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