ON THE BUCKLING AND VIBRATION BEHAVIOR OF A THIN PLATE WITH NONUNIFORM IN-PLANE TRACTIONS

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OF A THIN PLATE WITH NONUNIFORM IN-PLANE TRACTIONS

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I would like to dedicate this thesis to Carla, my wife. She was very close to me during this work and has been my encouragement and inspiration in its completion.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

The objective of this investigation is to formulate the stability and vibration problems for plates with nonuniform stress.

Solution procedures are discussed, by example, for a rectangular plate which has nonuniform in-plane edge tractions applied along opposite boundaries which are clamped. The two remaining edges are free.

The variation in the natural frequency of transversal vibration with the intensity of the applied load is determined by use of energy methods. Power series are used to develop an approximate solution to the problem. The use of the eigenfunctions for a clamped-clamped beam and those for a free-free beam is discussed and convergence is comparatively investigated.

CHAPTER I

INTRODUCTION

Problems involving the buckling and vibration behavior in structural elements have attracted the attention of investigators for many years. The problem of vibration behavior is particularly important for structural engineers for its great influence on the resistance of materials against fatigue and dynamic magnification of stresses.

The purpose of the present investigation is to study the static stability and the dynamic behavior of thin plates with non-uniform in-plane traction. These problems can arise, for example, in an aircraft fuselage or wing. In such an environment a number of force excitation sources may be encountered. If the frequency of a given source is the same as the natural frequency of vibration of the structural element, a severe cyclic loading condition is superimposed on the initial stress. A condition of this type can be expected to accelerate the element collapse and as a consequence the properties of the dynamic behavior involved are of practical concern.

The problem of vibration behavior of plates has been the subject of a large number of investigations, and many aspects of the problem have been considered. One feature of the problem of interest is the dynamic behavior of a compressed free edge. When a rectangular plate which has opposite boundaries that are respectively restrained and free has an eccentric tensile load along the restrained edges (clamped in our case), a region along one free edge can be in compression. The study of the buckling and vibration behavior of this type of plate is the subject of the present investigation.

In considering the governing differential equation of the motion of the plate, we can have either bending accompanied by the stretching of the middle surface, or bending alone. The first case occurs when the deflected surface is a nondevelopable surface. Consideration of stretching of the middle surface leads to nonlinear differential equations because stretching is represented by nonlinear terms in the strain-displacement relations. The problem is governed by two simultaneous nonlinear coupled equations with the unknowns being the Airy stress function and the lateral deflection. Procedures for approximate solutions by iteration have been developed for nonlinear problems. Motion in these problems is not simple harmonic. For small amplitude vibrations, however, the modes of vibration can be assumed as periodic sinusoidal forms when the equations are linearized and uncoupled.

The present analysis will be restricted to a linear type of problem.

CHAPTER II

THE BUCKLING PROBLEM

The importance of the problem of vibration in plates was mentioned in the Introduction. This problem has been the subject of a large number of investigations and many aspects of it have been considered.

In this section the object of the discussion will be the stability problem of a rectangular plate which has nonuniform in-plane tractions applied along opposite boundaries which are clamped. The two remaining edges are free. The lowest buckling load will be determined by use of a Ritz procedure. The Galerkin procedure for the plate will be also discussed.

Consider a plate which is loaded by tractions as indicated in Figure 1. The resulting distribution can be represented as a system with a resultant force P acting at a distance e from the x axis as shown in Figure 2. In this situation a region along one free edge can be in compression. For the fixed plate width $2b$ the compression region is a function of the load eccentricity e only. The stress distribution is given by

$$\sigma_x = N_{x/h} = \frac{P}{2bh}\left(1 + \frac{3e}{b^2}\,y\right)$$

$$N_y = 0 \quad \text{and} \quad N_{xy} = 0$$

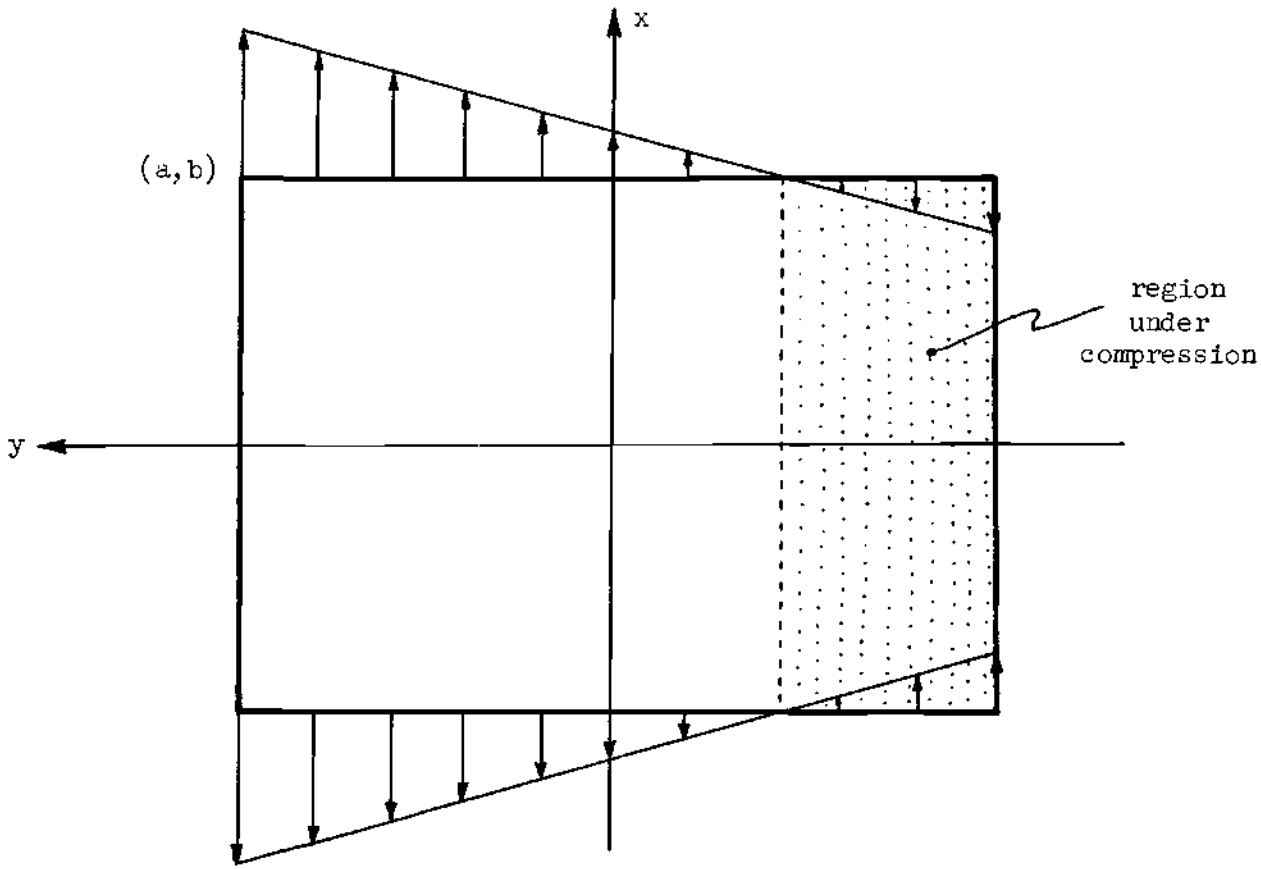

Figure 1. Plate Problem.

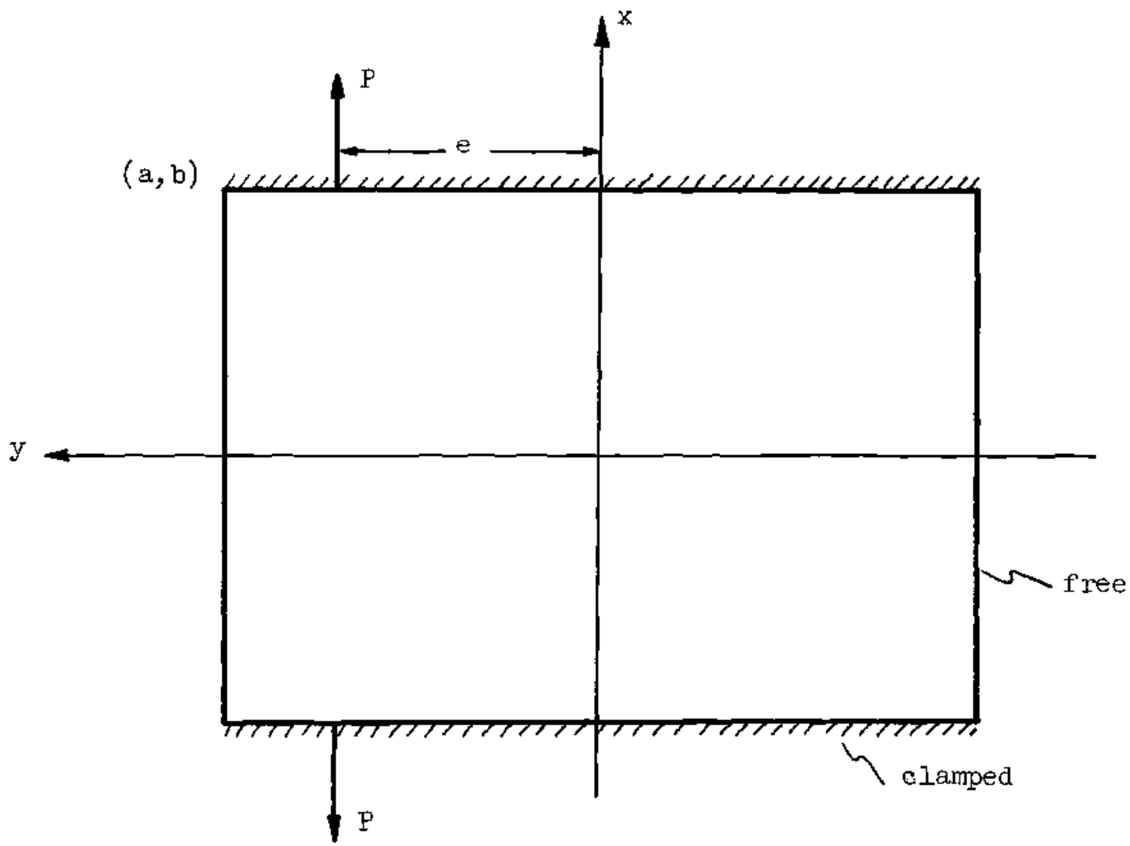

Figure 2. Equivalent Plate Problem.

and the compression region is between

$$y = -\frac{b^2}{3e} \quad \text{and} \quad y = -b$$

Note that it is necessary that

$$e > \frac{1}{3} b$$

Hence, we get a problem with a free edge under compression (Figure 1). Such a problem is conceptually similar to practical problems which do occur; for example, crack problems in which the crack free edges are under compression. Thus, our study has features which are of interest in more complex investigations.

The plate behavior is governed by the von Kármán plate equations, namely:

$$D\nabla^4 w = \left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{1}$$

$$\nabla^4 F = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{2}$$

where F is the Airy stress function for the in-plane, middle surface force per unit length components, and w is the transverse deflection. Equations (1) and (2) are coupled in w and F and need to be solved simultaneously. In our case, however, we assume displacements to be small and that the Gaussian curvature (right side of equation (2)) is small and may be neglected when compared to the term $\nabla^4 F$. Then the equations are uncoupled.

If the Airy stress function is defined such that

$$N_x = \frac{\partial^2 F}{\partial y^2}, \qquad N_y = \frac{\partial^2 F}{\partial x^2}, \qquad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

equation (1) becomes

$$D\nabla^4 w = N_x \frac{\partial^2 w}{\partial x^2} + 2\, N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \tag{1a}$$

In our problem

$$N_x = \frac{P}{2b}\left(1 + \frac{3e}{b^2}\, y\right) \tag{3}$$

$$N_y = 0$$

$$N_{xy} = 0$$

so that equation (1a) simplifies to

$$D\nabla^4 w - N_x \frac{\partial^2 w}{\partial x^2} = 0 \tag{4}$$

This may be considered to be the Euler equation of the functional

$$U[w] = \int_{-a}^{a}\int_{-b}^{b} \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dxdy \tag{5}$$

$$+ \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b} N_x\left(\frac{\partial w}{\partial x}\right)^2 dxdy$$

Note that this functional is equivalent to the total potential energy of the system.

Boundary conditions:

Along the clamped edges ($x = \pm a$),

$$w = 0 \tag{6}$$

$$\frac{\partial w}{\partial x} = 0$$

Along the free edges ($y = \pm b$),

$$M_y = 0 \rightarrow \frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = 0 \tag{7}$$

$$Q_x = 0 \rightarrow \frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y} = 0$$

<u>Energy Approach for Approximate Solution</u>

Let us consider the functional (5) and use the Ritz method to construct an approximate solution to our problem. Let $w_n(x,y)$ be a set of admissible functions (i.e. linearly independent functions which satisfy the geometric boundary conditions (6)) and define

$$w_N = \sum_{n=1}^{N} c_n w_n \tag{8}$$

The convergence of the function w_N to the actual solution for $N \to \infty$ by use of the Ritz procedure has been shown in problems in which the w_n form a complete set. The number N of terms needed for a good approximation of course will depend upon considerations such as the geometric configuration assumed by the system, the constraint behavior and the types of admissible functions which have been chosen.

The trial functions we chose can be written as follows

$$w_n(x,y) = f_n(y)\left(1 + \cos\frac{\pi x}{a}\right)$$

where $f_n(y)$ will be taken to be a term of a power series. Thus,

$$w_N = f_N(y)\left(1 + \cos\frac{\pi x}{a}\right) \tag{9}$$

where

$$f_N(y) = \sum_{n=1}^{N} c_n f_n(y) = c_0 + c_1\left(\frac{y}{b}\right) + c_2\left(\frac{y}{b}\right)^2 + c_3\left(\frac{y}{b}\right)^3 + \quad \dots \tag{10}$$

It seems reasonable to assume that the left edge of the plate, which (as shown in Figure 1) is under tensile stress, will remain nearly flat, i.e. with zero or very small curvature. Now, if we select the first three terms in (10), we can see from Figure 3 that the linear term will tend to cancel out the displacement due to the quadratic, but not its curvature, so that the curvature of the edge under tension is constant and equal to the curvature of the compressed edge. This seems unlikely. Similarly, by droping the linear term and by considering the cubic one, its curvature will tend, by minimizing the functional (5), to compensate the quadratic term at the edge under tension, otherwise adding to it at the compressed edge.

This kind of consideration leads to a choice

$$w(x,y) = \left[c_0 + c_2\left(\frac{y}{b}\right)^2 + c_3\left(\frac{y}{b}\right)^3\right]\left(1 + \cos\frac{\pi x}{a}\right) \tag{11}$$

Substitute now (11) into (5). We begin by setting the first variation of (5) equal to zero;

$$\delta U = 0$$

But, $U = U\ (c_0,\ c_2,\ c_3)$, thus

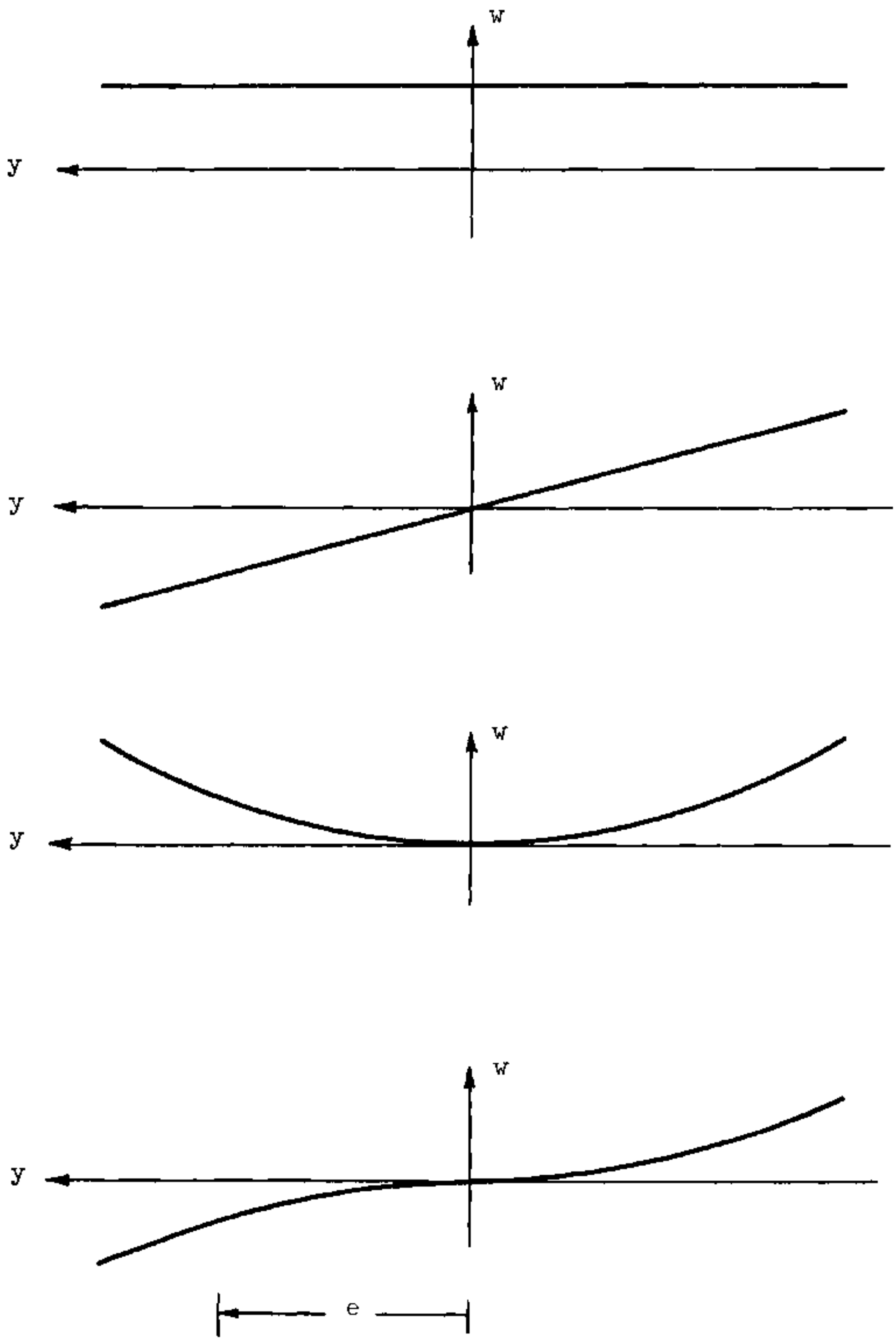

Figure 3. Mode Shapes in the y-Direction.

$$\delta U = \frac{\partial U}{\partial c_0}\,\delta c_0 + \frac{\partial U}{\partial c_2}\,\delta c_2 + \frac{\partial U}{\partial c_3}\,\delta c_3 = 0$$

which, because the δc_k are independent, implies that

$$\frac{\partial U}{\partial c_0} = 0$$

$$\frac{\partial U}{\partial c_2} = 0$$

$$\frac{\partial U}{\partial c_3} = 0$$

So, we get a set of three coupled equations in the three unknown c_0, c_2, c_3

$$(\pi^2\alpha^2 + \frac{M}{2}\,\beta)\;c_0 + \frac{1}{3}(\pi^2\alpha^2 - 6\nu + \frac{M}{2}\,\beta)\;c_2 + \frac{3M}{10}\,c_3 = 0 \qquad (12)$$

$$\frac{1}{3}(\pi^2\alpha^2 - 6\nu + \frac{M}{2}\,\beta)\;c_0 + (\frac{\pi^2\alpha^2}{5} + \frac{4(2-3\nu)}{3} + \frac{6}{\pi^2\alpha^2} + \frac{M}{10}\,\beta)\;c_2 + \frac{3M}{14}\,c_3 = 0$$

$$\frac{3M}{10}\,c_0 + \frac{3M}{14}\,c_2 + (\frac{\pi^2\alpha^2}{7} + \frac{6(3-5\nu)}{5} + \frac{36}{\pi^2\alpha^2} + \frac{M}{14}\,\beta)\;c_3 = 0$$

where

$$\alpha = \frac{b}{a}$$

$$\beta = \frac{b}{e}$$

$$M = \frac{Pe}{D}$$

Consider now a plate in which

$$\alpha = \frac{10}{7}$$

$$\beta = \frac{10}{7}$$

If $\nu = 1/3$ equations (12) give

$$(1398 + 49.57M)\ c_0 + (419.7 + 16.52M)\ c_2 + (20.82M)\ c_3 = 0 \qquad (13)$$

$$(10492 + 413.09M)\ c_0 + (9819 + 247.84M)\ c_2 + (371.8M)\ c_3 = 0$$

$$(2602M)\ c_2\ (1859M)\ c_2 + (54343 + 885M)\ c_3 = 0$$

By setting the determinant of coefficients of (13) equal to zero, we get

$$M^3 - 3.8104\ M^2 - 3189\ M - 46564 = 0 \qquad (14)$$

which solved gives the following result

$$M = 63$$

Since we defined $M = Pe/D$, we get

$$P = 63 \frac{D}{e}$$

But $e = 7/10\ b$, so

$$P = 90 \frac{D}{b} \tag{15}$$

which is valid for all plates having the given values for the ratios α and β previously defined. By example, if we take a plate which has width $2b = 10''$ and thickness $h = 0.05''$, we can use the definition of the flexural rigidity D and compute

$$D = \frac{Eh^3}{12(1-\nu^2)} = 123.047 \text{ lbs.in.}$$

then we get the following buckling load

$$P_c = 2{,}220 \text{ lbs.} \tag{15a}$$

The hypothesis previously made about the choice of the approximate solution can be checked for the above particular case by considering different trial functions

(a) For

$$w = \left[c_0 + c_3\left(\frac{y}{b}\right)^3\right]\left(1 + \cos\frac{\pi x}{a}\right)$$

with

$$\frac{\partial U}{\partial c_0} = 0$$

$$\frac{\partial U}{\partial c_3} = 0$$

gives a value for the buckling load

$$P_c = 14{,}070 \text{ lbs.}$$

(b) For

$$w = \left[c_0 + c_1\left(\frac{y}{b}\right)\right]\left(1 + \cos\frac{\pi x}{a}\right)$$

with

$$\frac{\partial U}{\partial c_0} = 0$$

$$\frac{\partial U}{\partial c_1} = 0$$

gives

$$P_c = 5{,}130 \text{ lbs.}$$

(c) Taking

$$w = \left[c_0 + c_1\left(\frac{y}{b}\right) + c_2\left(\frac{y}{b}\right)^2\right]\left(1 + \cos\frac{\pi x}{a}\right)$$

with

$$\frac{\partial U}{\partial c_0} = 0$$

$$\frac{\partial U}{\partial c_1} = 0$$

$$\frac{\partial U}{\partial c_2} = 0$$

gives

$$P_c = 4{,}230 \text{ lbs.}$$

A comparison of the above values shows that the trial function (11) is by far the best, and suggests that a power series type of a solution converges rapidly. Moreover cases (b) and (c), which both contain a linear term lead to a value of the buckling load much higher than (15a). This indicates that the presence of the linear term introduces constraints which effectively slow down the convergence in this problem.

Buckled Shape

By substituting the solution (15) into (13) and solving for c_0, c_2, and c_3 we get

$$c_0 = 1.000$$

$$c_2 = -67.12$$

$$c_3 = 71.25$$

Thus, the function (11) becomes

$$w(x,y) = \left[1 - 67.12\left(\frac{y}{b}\right)^2 + 71.25\left(\frac{y}{b}\right)^3\right]\left(1 + \cos\frac{\pi x}{a}\right)$$

The values of w/w_{max} and of the curvature

$$\frac{1}{r_y} = \frac{\partial^2 w}{\partial y^2}$$

versus y/b for y/b going from -1 to 1 are tabulated in Table 1. The buckled configuration is shown in Figure 4.

Table 1. Deflections and Curvatures in y-Direction

$\frac{y}{b}$	-1	-0.5	0	0.5	1
$\frac{w}{w_{max}}$	-1.00	-0.180	0.00730	-0.0500	0.0373
$\frac{1}{r_y}$	-44.9	-27.8	-10.7	6.4	23.5

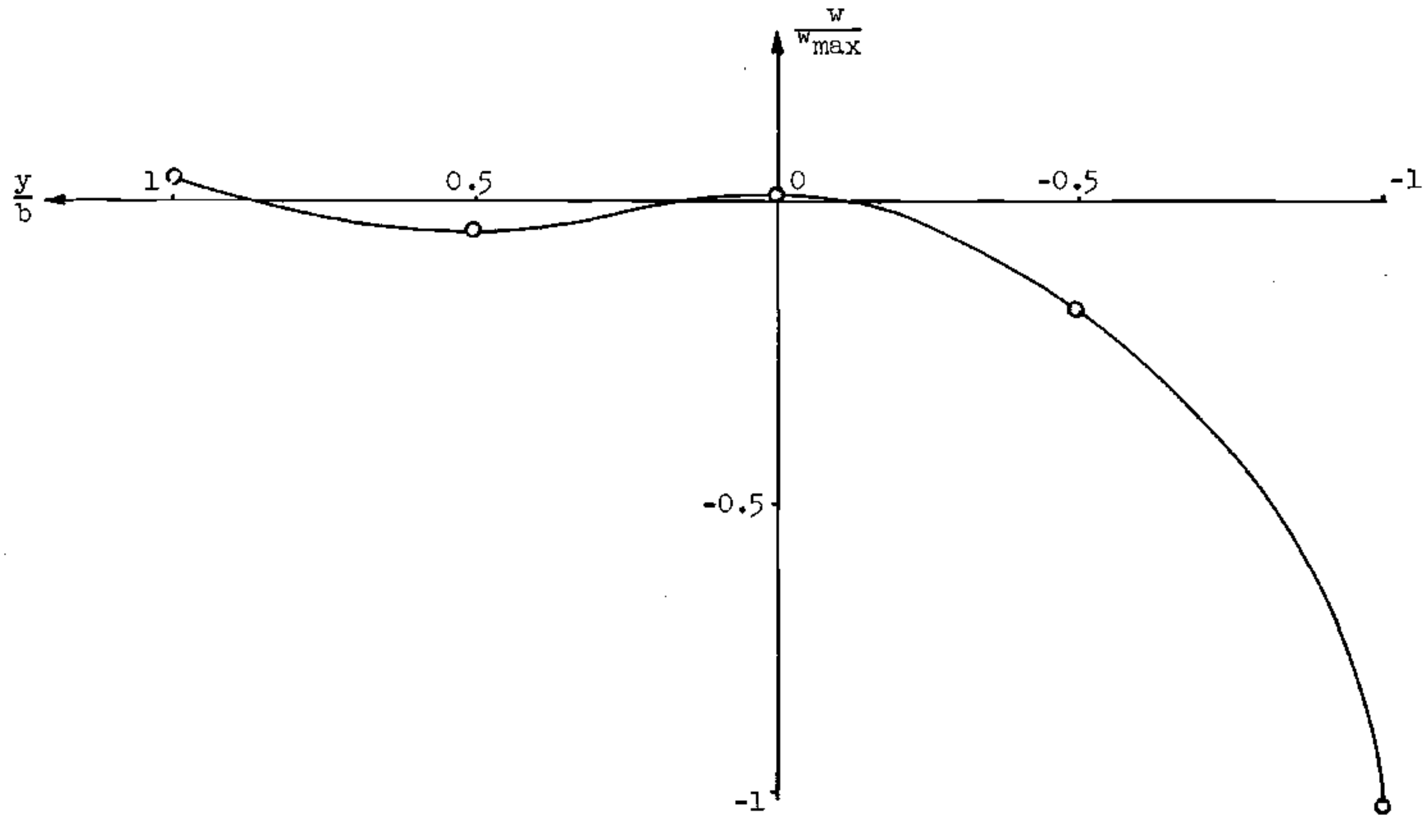

Figure 4. Buckled Configuration for Constant x.

An Alternate Approach to the Stability Problem

In this section the use of the eigenfunctions for a clamped-clamped beam and those for a free-free beam will be discussed. The use of a Galerkin procedure will be also illustrated.

Let us consider the total potential energy U = U[w] given by the functional (5), and let $\bar{w}(x,y)$ be any function chosen within the class of admissible functions. If $w(x,y)$ is the displacement field for which U[w] attains a stationary value, then in some neighborhood of w we can express $\bar{w}$ as

$$\bar{w}(x,y) = w(x,y) + \epsilon\eta(x,y)$$

Then

$$U[\bar{w}] = U[w + \epsilon\eta] = U(\epsilon)$$

For equilibrium we must require $U[\bar{w}]$ to be stationary at $\epsilon = 0$. This leads to the condition

$$\left.\frac{dU}{d\epsilon}\right|_{\epsilon=0} = 0$$

By this condition we can develop a Galerkin procedure for the stability problem of the given plate. The formulation of this type of procedure implies the selection of a series of admissible functions. The approximate solution is

$$w_{MN} = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn}\varphi_{mn} \tag{16}$$

where φ_{mn} can be chosen such that

$$\varphi_{mn}\ (x,y) = X_m(x) \cdot Y_n(y) \tag{17}$$

Let us consider for the $X_m(x)$ the characteristic functions for the free vibration of a clamped-clamped beam

$$X_m = \cosh \beta_m(x+a) - \cos \beta_m(x+a) - \alpha_m\left[\sinh \beta_m(x+a) - \sin \beta_m(x+a)\right]$$

where β_m and α_m are the corresponding characteristic numbers. These satisfy the boundary conditions at $x = \pm a$. Then select for the $Y_n(y)$ the characteristic functions for a free-free beam

$$Y_n = \cosh \beta_n(y+b) + \cos \beta_n(y+b) - \alpha_m\left[\sinh \beta_n(y+b) + \sin \beta_n(y+b)\right]$$

These do <u>not</u> satisfy the boundary conditions at $y = \pm b$. For each of the $M \times N$ selected functions φ_{ij} given by (17) the Galerkin procedure applied to our problem leads to the following equation:

$$\int_{-a}^{a}\int_{-b}^{b} \left(D\nabla^4 w_{mn} - N_x \frac{\partial^2 w_{MN}}{\partial x^2}\right) \varphi_{ij}\ dxdy + \tag{18}$$

$$+ D \int_{a}^{-a} \left(\frac{\partial^2 w_{MN}}{\partial y^2} + \nu \frac{\partial^2 w_{MN}}{\partial x^2}\right) \frac{\partial \varphi_{ij}}{\partial y} \Bigg|_{y=b} dx +$$

$$+ D \int_{-a}^{a} \left(\frac{\partial^2 w_{MN}}{\partial y^2} + \nu \frac{\partial^2 w_{MN}}{\partial x^2}\right) \frac{\partial \varphi_{ij}}{\partial y} \Bigg|_{y=-b} dx +$$

$$- D \int_{a}^{-a} \left[\frac{\partial^3 w_{MN}}{\partial y^3} + (2 - \nu) \frac{\partial^3 w_{MN}}{\partial x^2 \partial y}\right] \varphi_{ij} \Bigg|_{y=b} dx +$$

$$- D \int_{-a}^{a} \left[\frac{\partial^3 w_{MN}}{\partial y^3} + (2 - \nu) \frac{\partial^3 w_{MN}}{\partial x^2 \partial y}\right] \varphi_{ij} \Bigg|_{y=-b} dx = 0$$

$$i = 1,2, \ldots M$$

$$j = 1,2, \ldots N$$

where w_{mn} is given by (16). Note that in Equation (18) the terms

$$\frac{\partial^2 w_{MN}}{\partial y^2} \text{ and } \frac{\partial^3 w_{MN}}{\partial y^3}$$

are zero. By substituting (16) and (17) into (18) and by taking account of the properties of the characteristic functions we obtain the following $M \times N$ equations

$$D \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \left(\beta_m{}^4 + \beta_n{}^4\right) \int_{-a}^{a} X_m X_i \, dx \int_{-b}^{b} Y_n Y_j \, dy + \tag{19}$$

$$+ 2D \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \int_{-a}^{a} X_i \frac{d^2X_m}{dx^2} \, dx \int_{-b}^{b} Y_j \frac{d^2Y_n}{dy^2} \, dy +$$

$$- \frac{P}{2b} \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \int_{-a}^{a} X_i \frac{d^2X_m}{dx^2} \, dx \int_{-b}^{b} \left(1 + \frac{3e}{b^2} y\right) Y_n Y_j \, dy +$$

$$+ D \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \left[\nu \left(Y_n \frac{dY_j}{dy} \Bigg|_{y=-b} - Y_n \frac{dY_j}{dy} \Bigg|_{y=b} \right) + \right.$$

$$\left. - (2-\nu) \left(Y_j \frac{dY_n}{dy} \Bigg|_{y=-b} - Y_j \frac{dY_n}{dy} \Bigg|_{y=b} \right) \right] \int_{-a}^{a} X_i \frac{d^2X_m}{dx^2} \, dx = 0$$

Let us now consider as a first approximation two terms of (16), i.e. one mode for the $X_m(x)$ and two for the $Y_n(y)$, such that

$$w_{mn} = c_{11} X_1 Y_1 + c_{12} X_1 Y_2$$

and substitute this function into (19). The values of the above integrals have been computed by Felgar (see Bibliography). Substitution for

$$a = 3.5''$$

$$b = 5.0''$$

$$h = 0.05''$$

$$\nu = \frac{1}{3}$$

leads to the following two equations for our plate respectively for $\varphi_{ij} = \varphi_{11}$ and $\varphi_{ij} = \varphi_{12}$

$$c_{11}\,[1687 + (1.757 + 1.054\,e)P] - c_{12}\,(0.486\,Pe) = 0$$

$$c_{11}\,(0.486\,Pe) - c_{12}[5251 + (1.757 + 1.054\,e)P] = 0$$

For $e = 3.5''$ the characteristic determinant equal to zero gives

$$P^2 + 1411\,P + 330650 = 0$$

This quadratic does not have positive roots, i.e. the use of this approximation in the energy method does not provide the occurrence of buckling instability for any load applied at the eccentricity, $e = 3.5''$.

This indicates that to get a reasonably good result, we need a large number of terms of the series (16). For the given problem, then, this representation is not as useful as the power series type representation used earlier. The difficulty encountered here can be traced to the mode shapes of the free-free beam (see Figure 5). An examination of these indicates that many terms would be required to provide a reasonably good representation of the buckled configuration.

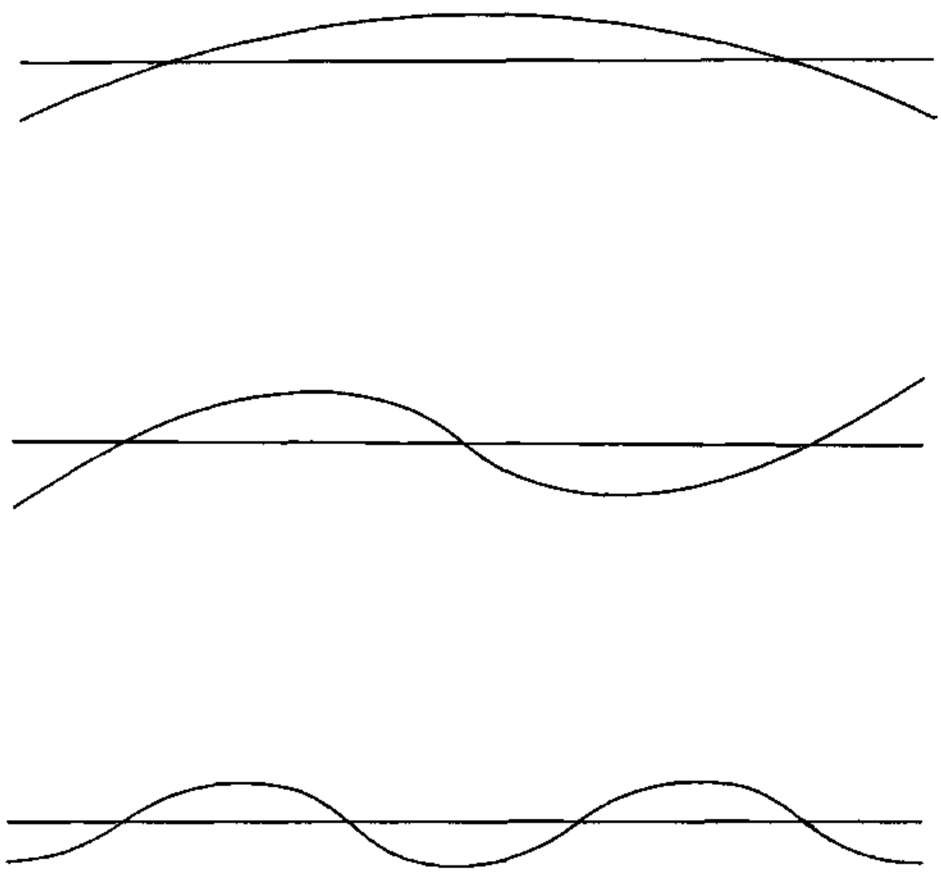

Figure 5. Mode Shapes of a Free-Free Beam.

CHAPTER III

THE VIBRATION BEHAVIOR PROBLEM

The subject matter of this section is the investigation of the dynamic behavior of eccentrically loaded plates. Since the vibration characteristics are influenced by the presence of an initial stress, the nature of the stress state is important. Furthermore, the presence of a compressive zone can be expected to have a dominant effect on the character of the modes of vibration. It has been shown in the previous section that the presence of the compressive zone may create the necessity for considering the possibility of buckling at some critical value of load. In a thin plate with initial nonuniform stress, as shown in Figure 1, the compressive stress reaches its maximum value at one free edge and remains constant along the complete length of the edge. Thus, when the overall tensile load on the plate is increased, the compressive stress along the free edge increases and can reach a critical value for buckling. In the linear theory of vibrations the buckled state corresponds to a zero frequency. It should be noted, however, that the solution of the coupled nonlinear static problem shows the existence of stable equilibrium branches for loads above the critical values. In the analysis which follows we will restrict ourselves to linear vibrations about the flat configuration.

Consider the same plate previously investigated now subjected to linear free vibration. The equation of motion in the transverse direction becomes

$$D\nabla^4 w = N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - \rho \ddot{w} \tag{20}$$

where ρ represents the mass per unit area of the plate

$$\rho = \frac{\mu h}{g} \tag{21}$$

and

μ = weight per unit volume of the plate

h = thickness of the plate

g = acceleration of gravity

Let us assume that the traction boundary conditions are still given by (3); moreover, if the plate is assumed to undergo harmonic motion, we can write

$$w(x,y,t) = W(x,y) \cos \omega t \tag{22}$$

By substituting (22) into (20) and defining the quantity

$$K = \rho\omega^2 \tag{23}$$

we obtain the following governing equation for the motion of the plate

$$D\nabla^4 W - N_x \frac{\partial^2 W}{\partial x^2} - KW = 0 \tag{24}$$

Similarly to the static case, this equation may be considered to be the Euler equation of the functional

$$U[W] = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x \partial y}\right)^2\right]\right\} dxdy \tag{25}$$

$$- \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b} KW^2 dxdy + \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b} N_x \left(\frac{\partial W}{\partial x}\right)^2 dxdy$$

where N_x is taken to be less than the critical load. If N_x is taken as greater than the critical load, motion will no longer be simple harmonic. Note that equation (24) represents a problem which is equivalent to a static problem. The constant K plays the role of a sort of elastic transversal foundation with a difference in the sign due to the different direction of an inertia term compared to the restoring force. Our problem leads, however, to analytical developments which are similar to the mentioned static case. The solution procedure adopted will be substantially the same adopted earlier. Thus, let us consider (again) for an approximate solution to the given problem, a trial function

$$W(x,y) = (c_0 + c_2y^2 + c_3y^3)(1 + \cos\frac{\pi x}{a}) \tag{26}$$

and substitute this function into (25). Taking a first variation of U[w] leads to the conditions

$$\frac{\partial U}{\partial c_0} = 0$$

$$\frac{\partial U}{\partial c_2} = 0$$

$$\frac{\partial U}{\partial c_3} = 0$$

By using these conditions we get the following set of equations

$$\left(\frac{D\pi^4 b}{a^3} - 3Kab + \frac{P\pi^2}{2a}\right)c_0 + \left(\frac{D\pi^4 b^3}{3a^3} - \frac{2\pi^2 D\nu b}{a} - Kab^3 + \frac{P\pi^2 b^2}{6a}\right)c_2 + \tag{27}$$

$$+ \left(\frac{3Pe\pi^2 b^2}{10a}\right)c_3 = 0$$

$$\left(\frac{D\pi^4 b^3}{3a^3} - \frac{2D\nu\pi^2 b}{a} - Kab^3 + \frac{P\pi^2 b^2}{6a}\right)c_0 + \left(\frac{D\pi^4 b^5}{5a^3} + \frac{4D(2-3\nu)\pi^2 b^3}{3a} +\right.$$

$$\left. + 6Dab - \frac{3}{5}Kab^5 + \frac{P\pi^2 b^4}{10a}\right)c_2 + \left(\frac{3Pe\pi^2 b^4}{14a}\right)c_3 = 0$$

$$\left(\frac{3\pi^2 Peb^2}{10a}\right)c_0 + \left(\frac{3\pi^2 Peb^4}{14a}\right)c_2 + \left(\frac{D\pi^4 b^7}{7a^3} + \frac{6D(3-5\nu)\pi^2 b^5}{5a} +\right.$$

$$\left. + 36\ Dab^3 - \frac{3}{7}\ Kab^7 + \frac{P\pi^2 b^6}{14a}\right)c_3 = 0$$

By substituting

$D = 123.047$ lbs. in.

$a = 3.5$ in.

$b = 5.0$ in.

$\nu = \frac{1}{3}$

$K = \rho\omega^2$

$$\rho = \frac{\mu h}{g} = \frac{0.101 \cdot 0.05}{386.4} = 13.07 \quad 10^{-6} \text{ lbs. sec}^2 \text{ in.}^{-3}$$

$\pi = 3.1416$

we get:

$$(1398 - 52.5\ K + 1.41\ P)\ c_0 + (10492 - 437.5\ K + 11.75\ P)\ c_2 + \qquad (28)$$

$$+ (21.15\ Pe)\ c_3 = 0$$

$$(10492 - 437.5\ K + 11.75\ P)\ c_0 + (245470 - 6562.5\ K + 176.24\ P)\ c_2 +$$

$$+ (377.7\ Pe)\ c_3 = 0$$

$$(21.15\ Pe)\ c_0 + (377.7\ Pe)\ c_2 + (6792940 - 117187\ K + 3147\ P)\ c_3 = 0$$

Assume now $e = 3.5$ in. and set the characteristic determinant of (28) equal to zero. Hence, we obtain the following cubic in K, P

$$K^3 - (8.01395\ 10^{-2}\ P + 1.420975 \cdot 10^2)\ K^2 +$$

$$+ (1.429896\ 10^{-2}\ P^2 + 7.591074\ P + 6.398952\ 10^3)\ K +$$

$$- (-4.3601\ 10^{-5} P^3 + 5.840632 \cdot 10^{-3}\ P^2 + 1.71821\ 10^2\ P +$$

$$+ 8.821808\ 10^4) = 0$$

The values of K and ω versus P for P going from 0 to P_c are tabulated in Table 2 and shown graphically in Figures 6 and 7.

Table 2. K and ω Versus P Values.

P	0	250	500	750	1000	1250	1500	1750	2000	2220
K	26.32	21.79	16.01	11.79	8.37	5.35	3.036	1.3636	0.377	0
ω	1420	1270	1120	950	800	640	482	323	170	0

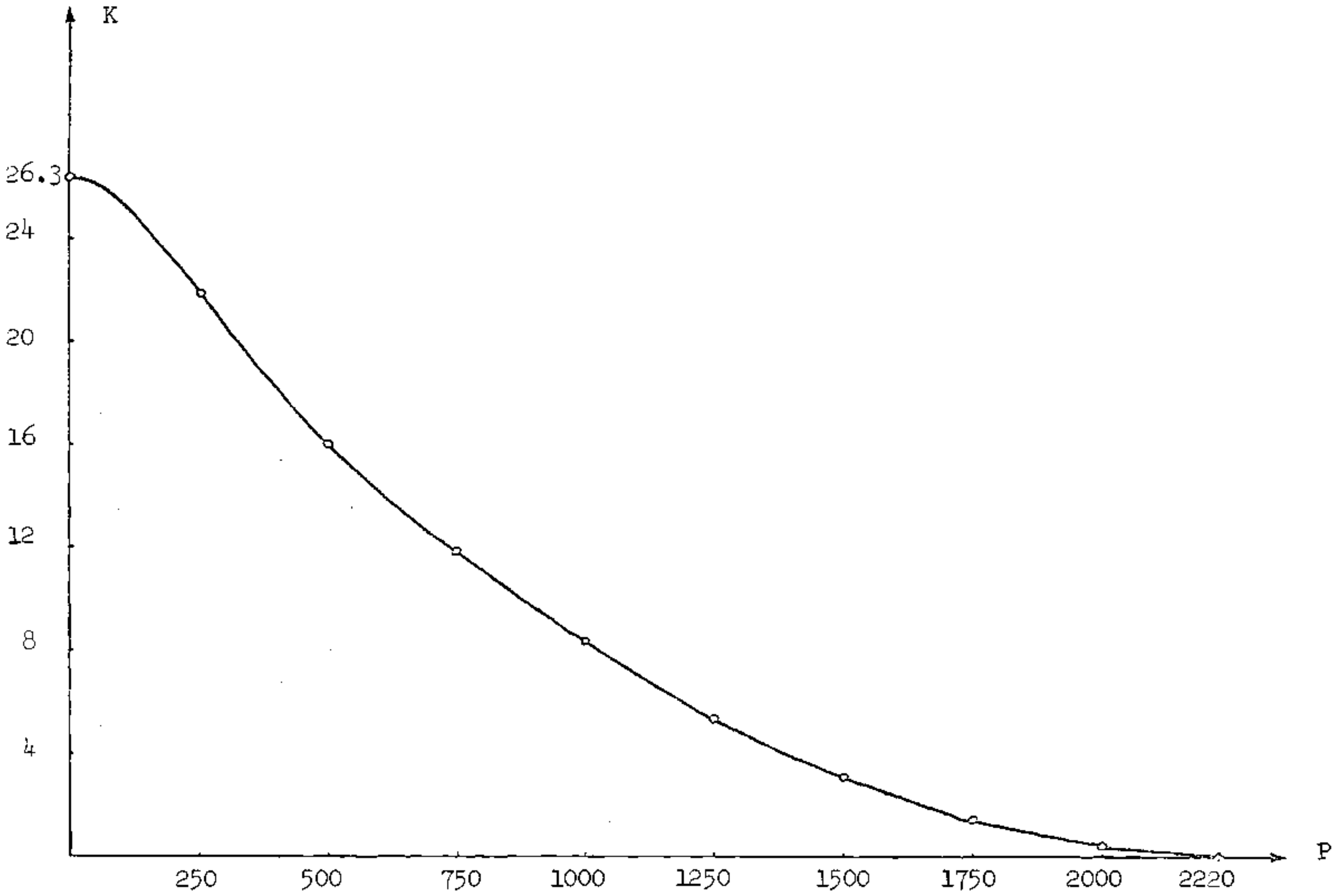

Figure 6. K Versus P Plot.

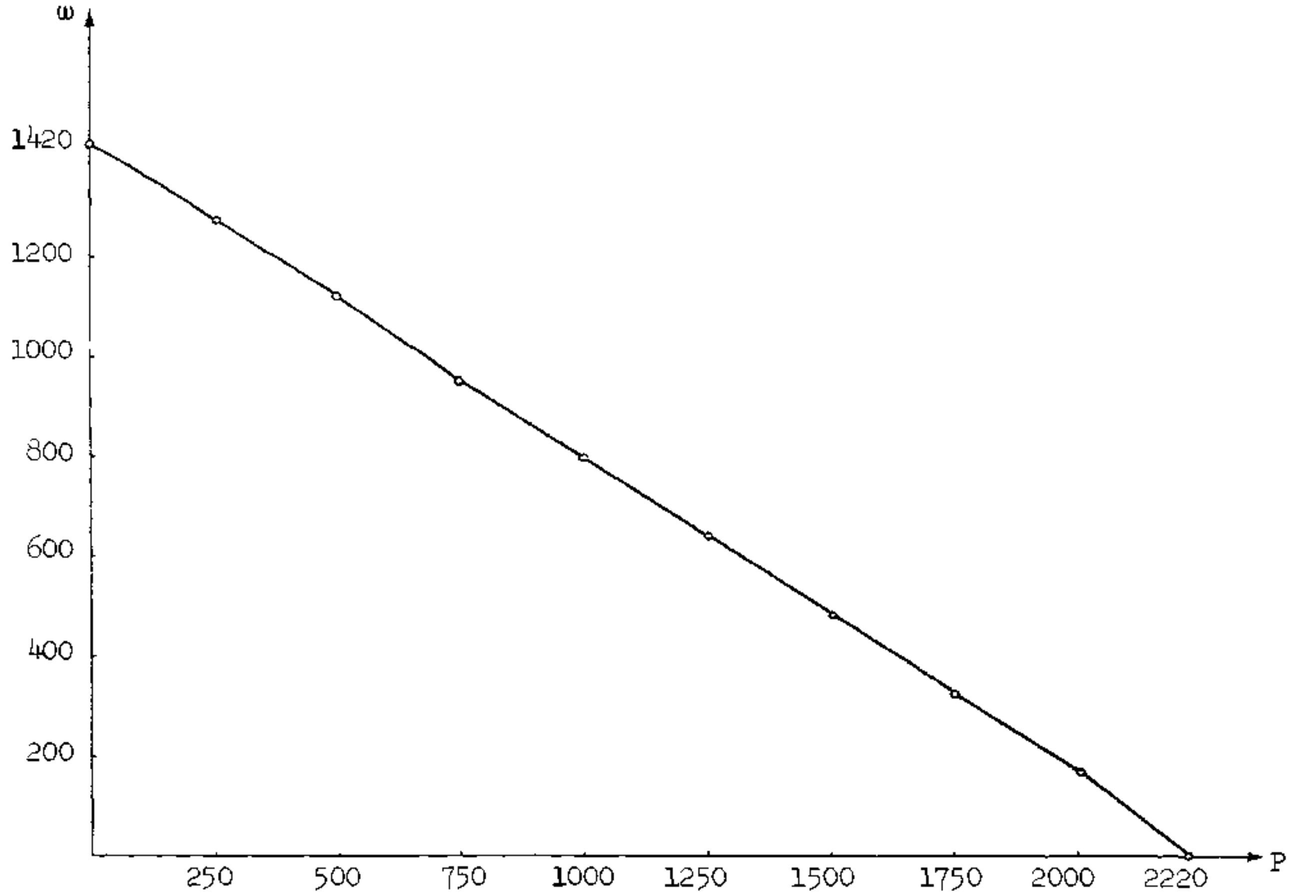

Figure 7. ω Versus P Plot.

CHAPTER IV

DISCUSSION

The approximation used in treating our problem seems to be reasonably good. The earlier discussion of the buckling problem showed that the use of a three term power series in the energy method provides a result which illustrates the use of the solution procedure.

The use of a simple one-term series like

$$W(w,y) = c_0(1 + \cos \frac{\pi x}{a})$$

gives a value for the frequency at P = 0.

$$\omega = 1425 \text{ sec}^{-1}$$

and a two terms series

$$W(x,y) = (c_0 + c_2y^2)(1 - \cos \frac{\pi x}{a})$$

gives

$$\omega = 1421 \text{ sec}^{-1}$$

Both are very close to the value obtained by the three terms power series. This seems to indicate that additional terms do not provide a much better representation of the plate configuration at $P = 0$.

If we investigate comparatively the vibration behavior of a plate subjected, respectively, to (a) a tensile uniform stress, and (b) a lateral moment (zero load with infinite eccentricity), as shown in Figure 8, it is reasonable to expect that the frequency-load behavior will be represented by curves of the type shown respectively in Figure 8a and Figure 8b.

In our case the eccentricity is large enough to have a region under compression, but tensile stresses are still dominant in the plate; i.e. this is a loading case in between the above two extremes. Thus, it seems consistent to the continuity of plate's behavior to expect that, for all eccentricities between zero and ∞, the vibration behavior must be a combination of (a) and (b). The slope of the curve (ω, P) at $P = 0$ will depend on the value of e, and it can be expected to decrease as e increases as shown in Figure 9.

Our approximation provides a curve which is almost a straight line between $P = 0$ and $P = P_c$. With an improved trial function for $P > 0$, the curve might tend to be curved as shown for e_2 in Figure 9. The type of behavior depicted in Figure 9 could only be studied by including not only more terms in the trial function, but also additional values of eccentricity. This is beyond the scope of the present investigation which has been primarily concerned with methods of solution.

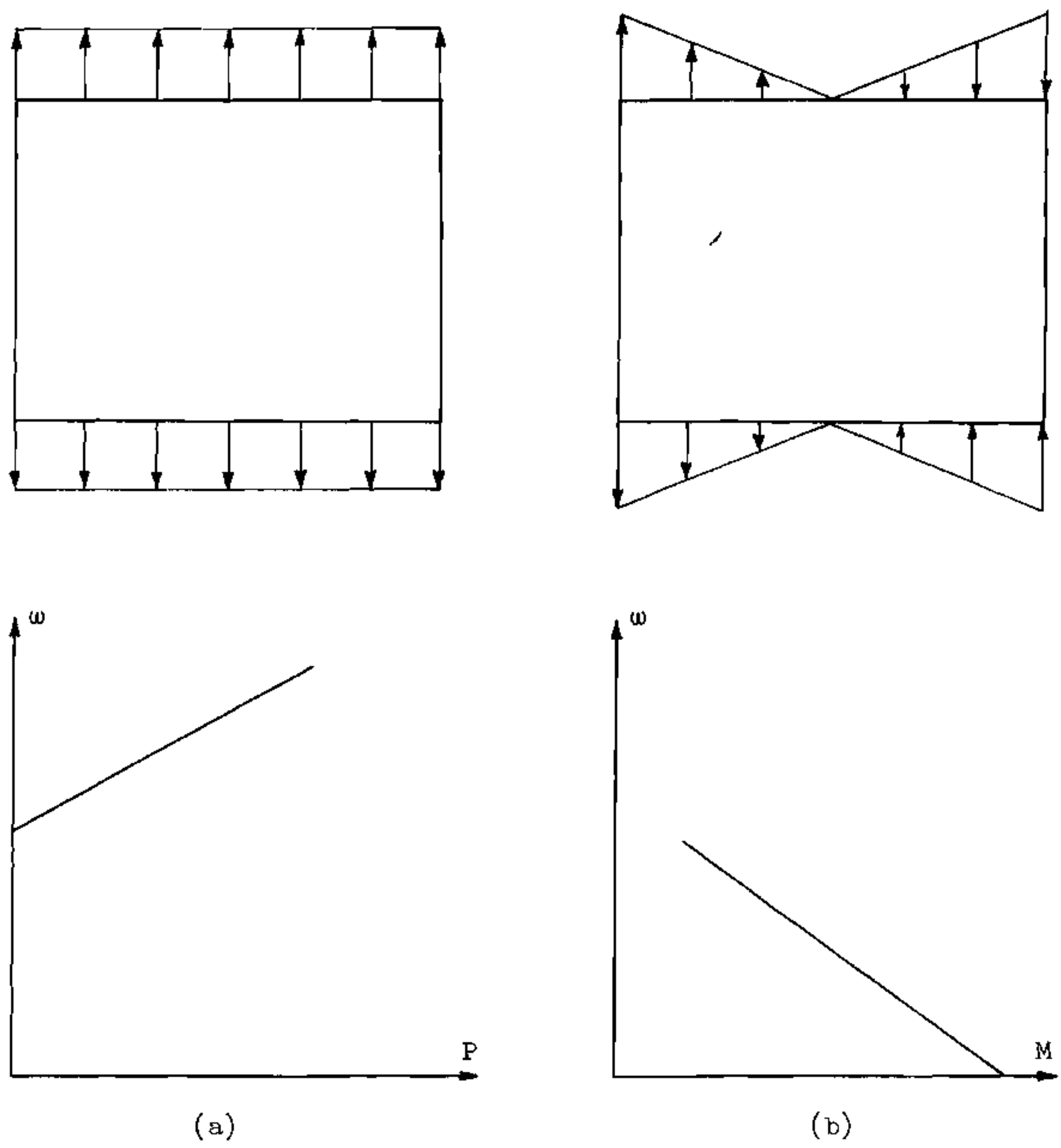

Figure 8. Frequency Versus Load of a Plate Subjected to:
(a) Uniform Tensile Stress,
(b) In-Plane Bending Moment.

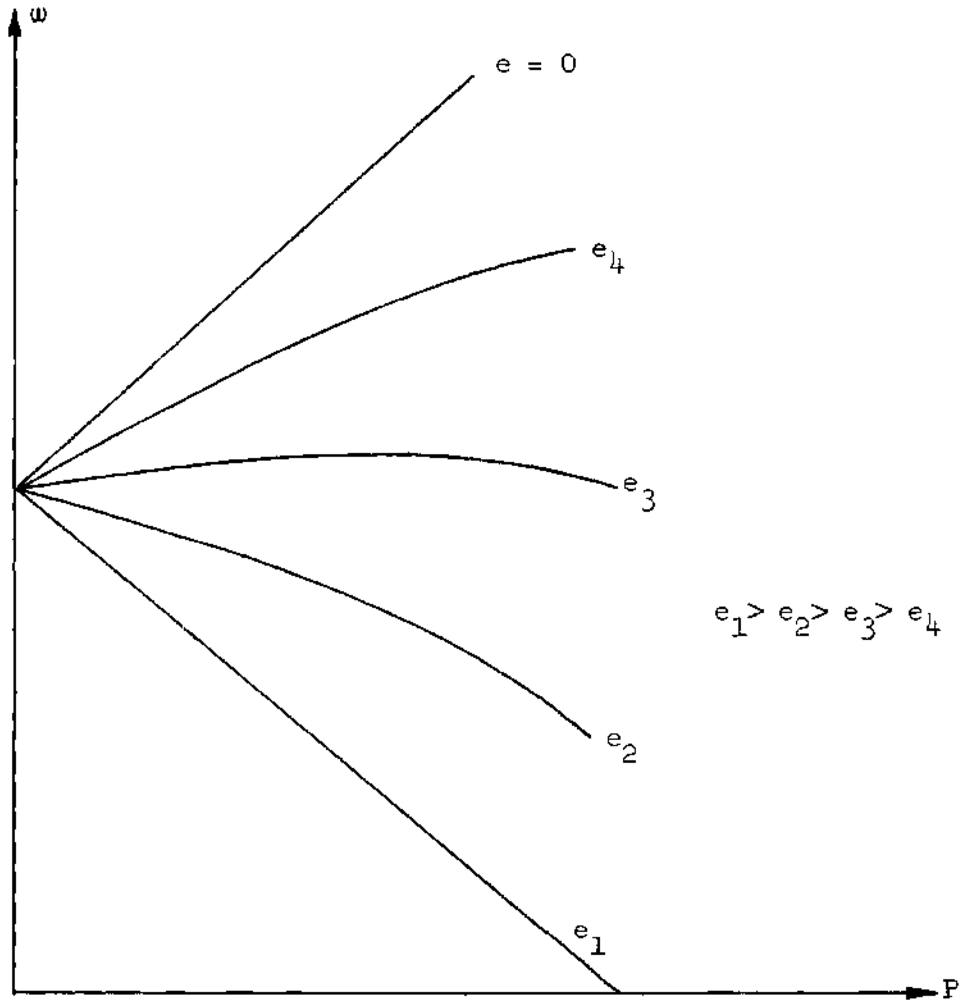

Figure 9. Vibration Behavior of a Plate at Different Eccentricities.

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