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7/25/68

AN ANALYSIS OF THE SYSTEM EFFECTIVENESS OF A SEQUENTIAL MANPOWER TRAINING MODEL

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

by

David Strother Grieshop

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in the School of Industrial Engineering

Georgia Institute of Technology

January, 1972

AN ANALYSIS OF THE SYSTEM EFFECTIVENESS OF A SEQUENTIAL MANPOWER TRAINING MODEL

Approved:

Chairman

Date approved by Chairman: 1/26/72

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Dr. Norman R. Baker, who served as my thesis committee chairman, for his valuable guidance, criticism, encouragement, and patience throughout the thesis development.

Appreciation is extended to Dr. Lynwood A. Johnson, Dr. Robert B. Cooper, and Dr. Douglas C. Montgomery, members of my thesis reading committee, for their critical appraisals and constructive criticisms throughout the course of my research.

To my wife, Midge, I owe a great debt for taking the time that was rightfully hers, but who always provided the encouragement, patience, and understanding during my graduate studies.

To my typist, Mrs. Lydia Geeslin, my thanks for an excellent typing job.

This thesis was partially supported by Department of the Army, contract number DAHC 04-70-C-0018.

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SUMMARY

Rules for efficiently improving the measure of system effectiveness for a sequential manpower training system are developed, proven, and demonstrated. The system permits trainees of varying aptitudes to train in a series of events at a pace commensurate with their abilities. The cumulative probability of success for an event is a monotonically increasing function of the number of times the event is attempted and is independent of the order of the event within the sequence of all events. An incremental cost is incurred each time an event is attempted and accrues until a trainee either successfully completes the system or is eliminated as a failure. The measure of system effectiveness, the efficiency index, is the expected cost per successful trainee.

An algorithm is presented where the decision variables are the number of times an event may be attempted and the sequence of the events. The cost of training and probabilities of success are considered fixed. A sufficient condition, which when satisfied insures the algorithm yields an optimal solution, is developed and proven. An alternative method for solution improvement is outlined when the sufficiency condition is not satisfied.

CHAPTER I

INTRODUCTION

One requisite of an ideal training system is that the conditions of instruction match well the characteristics of the trainee--his abilities, interests, prior knowledge, and learning style. It is unlikely that any training system will ever fully embody this ideal (2). Perhaps the most compelling reason is the economic cost associated with individualized training.

Most training instruction takes a standardized form (10). That is, the training methods used are common for all trainees, regardless of their individual characteristics. There are advantages and disadvantages associated with such standardization (1,11). The former have to do with administrative and economic considerations, while the latter are concerned with training effectiveness.

Military training is particularly likely to suffer from standardized training programs (1,10). First, manpower requirements often dictate large inputs to certain training programs, which means that selection criteria may have to be disregarded and thus aptitude differences among trainees may be large (11). Second, instructors in military training systems are not likely to be trained or experienced teachers. Third, administrative or traditional considerations frequently take precedence over training objectives, usually to the detriment of training (10).

Miller (7) proposed a manpower training model which represented a compromise between standardized and completely individualized training. He hypothesized a training model for basic combat training (BCT) in the U. S. Army in which trainees of varying aptitude levels (tracks) were permitted to train in a series of subjects (events) at a pace commensurate with their abilities. Within each event, a trainee was tested at the completion of the training (trial). If he successfully passed the trial, he proceeded to the next event in the BCT program. If not, he was permitted to complete the training again and was then retested.

As a measure of the system effectiveness, Miller defined an efficiency index which was the expected cost per successful trainee. The efficiency index of this stochastic system may be controlled in four ways: changing the costs of training, modifying the probabilities of success, regulating the number of trials permitted in any event, and varying the order of sequence in which the events are completed.

This thesis presents an algorithm for efficiently improving the measure of system effectiveness for the above training system. The algorithm permits the manager of the system to control the number of trials permitted in each event and the sequence in which the events are performed. The training costs and probabilities of success are considered fixed.

Chapter II presents the formulation of certain theorems and lemmas required to develop the algorithm. A sufficient condition for optimality, which when satisfied guarantees an optimal solution, is developed and proven. A method is outlined that permits solution improvement when the condition for optimality is not satisfied.

Literature Search

In 1959, Price (9) formally stated the concept of a least-cost testing sequence. He proposed that, for a series of tests where the tests were independent, there existed a least-cost sequence for testing items in acceptance inspection. Independence meant that the results of one test did not affect the probability of the outcome of any subsequent test. Fundamental to the concept was the cost of conducting and the probability of failing a specific test. His procedure called for testing an item according to a specific sequence until the item failed to pass a test, when it was rejected and no further tests conducted. A schematic representation of the procedure is given in Figure 1.

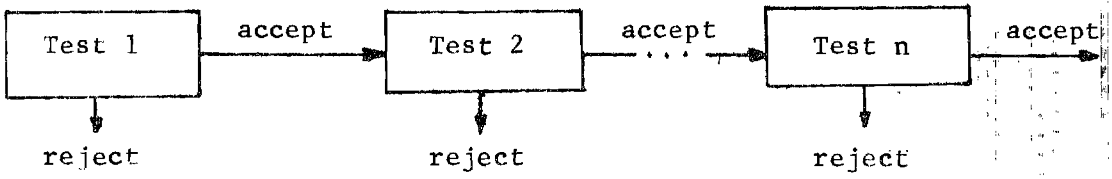

Figure 1. Testing Sequence

An expected total cost for operating a general testing sequence was defined as:

$$\text{COST} = NC_1 + NP_1C_2 + NP_1P_2C_3 + \ldots + N(P_1P_2 \ldots P_{n-1})C_n$$

where

COST = expected total cost of operating the sequence

N = number of items entering the sequence

C_n = per-item cost of test n

P_n = $(1-R_n)$; acceptance probability of test n

R_n = rejection probability of test n.

C was defined as the expected cost per item for the entire sequence. Then,

$$C = COST/N$$

and

$$C = C_1 + P_1C_2 + P_1P_2C_3 + \ldots + (P_1P_2\ldots P_{n-1})C_n \ .$$

The least-cost sequence then was the sequence which minimized C. To determine the least-cost sequence, Price stated that all sequence combinations must be evaluated and the minimum cost sequence selected. This sequence, by definition, was the least-cost sequence.

Mitten (8) proved there existed a simple analytic solution to the problem formulated by Price. Mitten stated:

> The following procedure yields the least-cost sequence:
> 1. For each test j, compute the ratio C_j/R_j;
> 2. Run the test with the smallest value for the above ratio first, the one with the next smallest ratio second, . . ., and the test with the largest ratio last.

Hence, Mitten was able to demonstrate a rapid solution to the problem; whereas, the solution method of Price involved complete enumeration.

Miller (7) extended the procedure developed by Mitten, while investigating the feasibility of a proposed manpower training model. The proposal is that discussed in the previous section. Schematically, the procedure is as depicted in Figure 2.

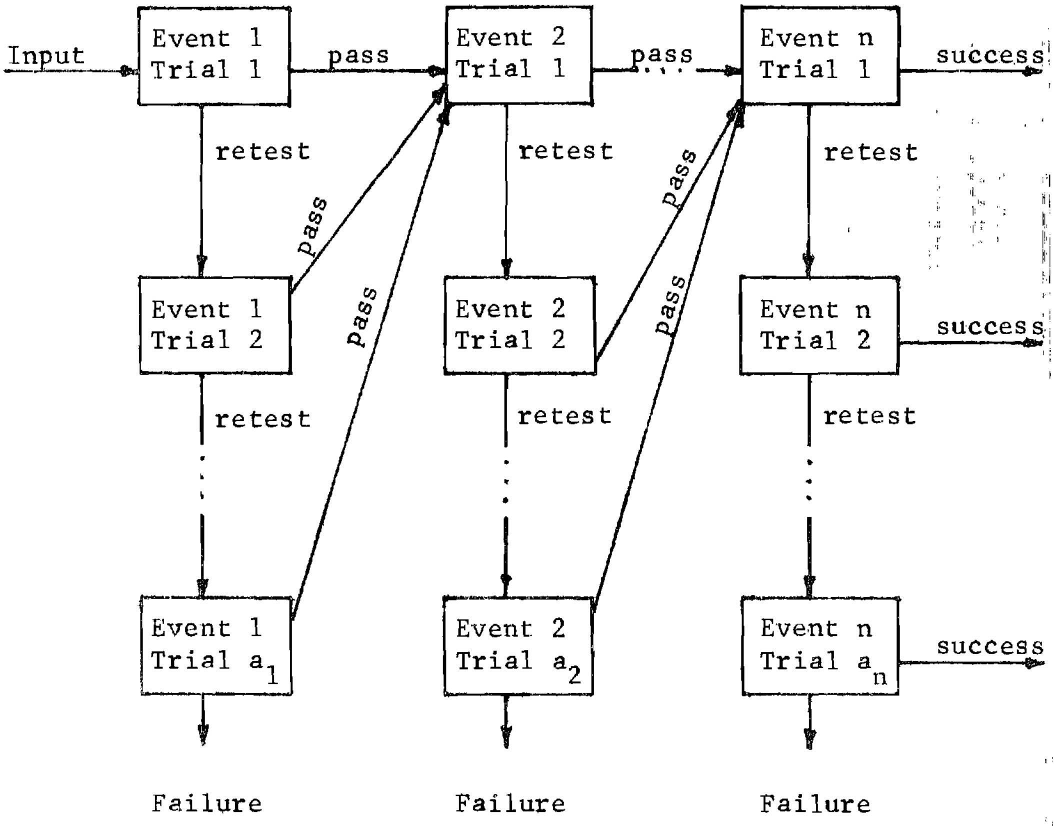

Figure 2. Sequential Training Model for Any Aptitude Level

From Figure 2 it can be seen that success is possible only when completing all n events and failure may occur at any event k after having failed a_k consecutive trials.

Miller proposed the following BCT procedure:

> The proposed training system is a multi-ability-track, variable-completion-time one. The procedural rules are:
> (1) Divide the input population in three ability or aptitude groups based on their performance. . . . These groups are identified as N_1: high aptitude, N_2: average aptitude, and N_3: low aptitude.

(2) Begin training each ability group in a training program designed to emphasize those learning methods most appropriate for each group. Generally, this implies a higher instructor to student density and slower rates of learning for the less apt trainees.

(3) As quickly as a man reaches the criterion skill level in an event (or subject) move him to the next event in the training program.

(4) When a man has successfully completed all events in the training program, he is output from the training system.

(5) If a man is incapable of satisfactorily mastering an event after a reasonable number of attempts, he is dropped from the training system as a failure. He does not move from one track to the next. The three tracks are operated independently (7).

The following assumptions were made by Miller:

1. Men can be categorized into aptitude groups. . . . Once categorized they remain in a group.
2. Slower, less-apt trainees require more instructors and more time to learn a subject than do higher aptitude men.
3. There is no probabilistic dependency between events. The probability of success for an event is independent of the event order in the event sequence.
4. Men are not recycled between events.
5. As quickly as a man successfully completes a trial test in any event, he proceeds to the next event in the sequence.
6. A trial is a uniform (fixed) time period of instruction-testing (for a specified aptitude and event). Instructors are of equal capability and use identical teaching methods within an event. (Probability of success is independent of instructors.)
7. Men do not repeat trials in an event indefinitely. A maximum permissible number of trials is established for each event. (Beyond this maximum either all trainees have successfully completed the event or the probability of successful completion with additional trials is assumed to be so small that additional trials are impractical.)
8. Instructor costs accrue only when men are training. Idle instructors have zero costs.

Each aptitude track was operated independently of all others; hence any aptitude track may be considered. Given the proposed model, trainees were trained and tested one trial at a time within an event. For a specified event and trial, there was a known probability of success. As the number of trials increased for an event, there was an associated cumulative probability of success. This cumulative probability was the probability of successfully completing an event in a specified number of trials or less. Miller defined P_{jk} to be the cumulative probability of success where,

j = trial number; that is, the number of trials completed;

$j = 1,2, \ldots, a_k$.

k = event number or index; $k = 1,2, \ldots, n$.

The value of P_{jk} was a function of trial and event. For any event, P_{jk} monotonically increased as the number of trials j increased. The maximum cumulative probability of success for any event was a function of the number of trials permitted.

The expected number of trainees input to an event, say event m, was equal to the number that successfully completed the preceding m-1 events. That is,

$$N_{(m)} = N \prod_{k=1}^{m-1} P_{a_k k}, \qquad (m = 2,3, \ldots, n)$$

where

$N_{(m)}$ = expected number of trainees entering event m

N = initial input of trainees

$P_{a_k k}$ = maximum cumulative probability of success for event k.

The expected number of trainees being trained in trial j of event m was equal to the expected number of trainees starting event m minus the expected number of trainees who had successfully completed event m during any of the preceding j-1 trials. That is,

$$N_{jm} = N_{(m)}(1 - P_{j-1,m})$$

where

N_{jm} = expected number of trainees being trained in trial j of event m.

The expected successful output, N_o, was the product of the maximum cumulative probabilities of success for all events times the initial input. That is,

$$N_o = N \prod_{k=1}^{n} P_{a_k k} .$$

An incremental training cost was incurred each time a trainee completed a trial. The cost, C_k, varied for each event k, but was constant for any trial j within an event. The cost continued to accumulate until a trainee reached the established performance criterion or until he was dropped from the system as a failure at the end of an event. The cost, C_k, was the cost per trainee per trial for event k.

The expected cost of a single event, say event m, was the sum of all trials costs for the event. That is,

$$C_{(m)} = C_m N_{(m)} \sum_{j=0}^{a_m - 1} (1 - P_{jm}) \; ; \qquad P_{0m} = 0$$

or

$$C_{(m)} = C_m N \prod_{k=1}^{m-1} P_{a_k k} \sum_{j=0}^{a_m - 1} (1 - P_{jm}) .$$

The expected total cost for all events, C, was the sum of the expected costs for all events. That is,

$$C = \sum_{k=1}^{n} C_{(k)}$$

or

$$C = NC_1 \sum_{j=0}^{a_1-1} (1 - P_{j1}) + NC_2 P_{a_1 1} \sum_{j=0}^{a_2-1} (1 - P_{j2})$$

$$+ NC_3 P_{a_1 1} P_{a_2 2} \sum_{j=0}^{a_3-1} (1 - P_{j3}) + \dots + NC_n P_{a_1 1} P_{a_2 2} \dots P_{a_{n-1} n-1}$$

$$\sum_{j=0}^{a_n-1} (1 - P_{jn})$$

or

$$C = N \sum_{k=1}^{n} C_k \prod_{\ell=0}^{k-1} P_{a_\ell \ell} \sum_{j=0}^{a_k-1} (1 - P_{jk}) \ , \qquad (1)$$

subject to a_ℓ, $N \geqq 0$, and $0 \leqq P_{jk} \leqq 1$; $P_{a_0 0} = 1$.

Miller proposed that an efficiency index (EI) be the measure of system effectiveness. The EI was defined as the expected cost per successful trainee.

$$EI = \frac{C}{N_o} = \frac{\text{Expected Total Cost}}{\text{Expected Output}}$$

or

$$EI = \frac{\sum_{k=1}^{n} C_k \prod_{\ell=0}^{k-1} P_{a_\ell \ell} \sum_{j=0}^{a_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{n} P_{a_\ell \ell}} \qquad (2)$$

Having developed the training system and defined the above cost and EI equations, Miller presented and proved his optimal event ordering theorem. This theorem extended the analytic solution of Mitten, in that repetitive trials were now considered, instead of a single accept-reject test. Miller's optimal event ordering theorem was as follows:

> If the events . . . are ordered by the following procedure, the cost of training any given input is minimized.
>
> (i) For each event k, compute the ratio
>
> $$\frac{1 - P_{a_k k}}{C_k \sum_{j=0}^{a_k - 1} (1 - P_{jk})}$$
>
> (ii) Place the event with the largest value for the above ratio first, the one with the second largest ratio second, . . ., and the event with the smallest ratio last.

Miller addressed the problem of constraining output by two methods. The first method was simply to reduce the initial input of trainees such that the desired output constraint was satisfied. Second, he showed that the constraint could be satisfied by reducing the probability of success by reducing the number of trials permitted in one or more events. Miller stated:

> The probability of success in one or all events may be regulated by varying the number of trials completed. As the number of trials decreases in any event, the probability of success decreases. The probabilities of success should be regulated such that equation (1) is satisfied as nearly as possible. . . . The following method of reducing trials to constrain output produces the least-cost outcome:
>
> (1) Determine the maximum system output (N_o) when unconstrained.
> (2) If $N_o > N_o^*$, the desired output, then go to step 4.
> (3) If $N_o < N_o^*$, the quantity . . . of the input must be (increased).
> (4) Select all the $P_{\cdot \ell}$ combinations which satisfy (or as nearly as possible)

$$N_o^* = N \prod_{\ell=0}^{n} P_{\cdot \ell}$$

(5) Reorder the event sequence for each $P_{\cdot \ell}$ combination by the optimal sequence rule. The optimally ordered $P_{\cdot \ell}$ combinations constitute the constrained output strategies.

(6) Calculate the expected cost, expected output, and EI for each strategy.

(7) Select the most efficient strategy (7).

The above procedure involved complete enumeration of all possible combinations of $P_{\cdot \ell}$ that satisfied the constraint.

Miller implicitly assumed that the optimal EI was attained when all trials were permitted for all events. Determining the optimal EI, disregarding any constraint, was not addressed.

General Problem Statement

Develop a method for determining the optimal efficiency index for the above training system and associated assumptions, where the decision variables are the number of trials and event sequence. The method or algorithm should permit, when possible, continual improvement in the EI. The algorithm should be relatively efficient and easily executed.

The EI was chosen because it was conjectured that one purpose of the training system was to successfully train personnel at a minimum expected cost. Since successful trainees were the desired and only valuable output from the system, it was considered appropriate to charge all costs to the successful trainees. The EI reflected such a costing procedure.

CHAPTER II

SYSTEM EFFICIENCY ALGORITHM

This chapter states pertinent definitions and notations followed by the development and proofs of theorems and lemmas required for the development of the algorithm. The algorithm is presented along with a condition which identifies when an optimal solution is obtained. An alternative method is outlined for solution improvement when the condition for optimality is not satisfied. The algorithm is applicable to all aptitude tracks in the system.

Definitions and Notation

C_k: The cost incurred to train one trainee for any trial in event k; $k = 1, 2, \ldots, n$. $C_k > 0$.

P_{jk}: The cumulative probability of success after the completion of j trials in event k; $j = 1, 2, \ldots, a_k$.

a_k: The maximum number of trials permitted in event k. The corresponding cumulative probability of success after a_k trials is expressed $P_{a_k k}$ or P_{a_k}.

n_k: Any predetermined number of trials considered in event k. $1 \leqq n_k \leqq a_k$. Correspondingly, $P_{n_k k}$ or P_{n_k}.

$\overline{C}_{jk}$: The expected cost per successful trainee after j trials for event k. For example, if $j = n_k$, then:

$$\overline{C}_{n_k k} = \overline{C}_{n_k} = \frac{C_k \sum_{j=0}^{n_k-1} (1 - P_{jk})}{P_{n_k}}$$

m_k: The number of trials in event k corresponding to the minimum $\overline{C}_{jk}$; $1 \leqq m_k \leqq n_k \leqq a_k$. Correspondingly, $P_{m_k k}$ or P_{m_k}.

J_k: The number of trials in event k when the optimal efficiency index has been determined; $1 \leqq m_k \leqq J_k \leqq a_k$. Correspondingly, $P_{J_k k}$ or P_{J_k}.

$\overline{C}_{m_k}$: The minimum expected cost per successful trainee for event k. $\overline{C}_{m_k k} = \overline{C}_{m_k} = \min_j \{\overline{C}_{jk}\}$, $1 \leqq j \leqq a_k$.

SS: The system state or number of trials associated with indexed event k, k = 1,2, ..., r, ..., n, before the possible addition of trials to event k. SS = $\{n_1, n_2, \ldots, n_r, \ldots, n_n\}$, where n_1 is the number of trials in indexed event 1, etc.

SS^+: The system state or number of trials associated with indexed event k, k = 1,2, ..., r, ..., n, after adding trials to an event. $SS^+ = \{n_1, n_2, \ldots, n_r+x, n_s, \ldots, n_n\}$, where n_1 is the number of trials in indexed event 1, etc.

SS^*: The system state when the optimal efficiency index, EI^*, has been determined. $SS^* = \{J_1, J_2, \ldots, J_n\}$.

SS_k: The partial system state for SS for indexed events 1 through k.

SS_k^+: The partial system state for SS^+ for indexed events 1 through k.

S: The optimal indexed sequence, as determined by the event ordering theorem, for SS. S = 1, ..., p,q,r,s, ..., n.

S^+: The optimal indexed sequence, as determined by the event ordering theorem, for SS^+. $S^+ = 1, \ldots, p, q, s, \ldots, r, t, \ldots, n$.

S^*; The optimal sequence, as determined by the event ordering theorem, for SS^*.

S_k: The partial optimal indexed sequence of S for indexed events 1 through k.

S_k^+: The partial optimal indexed sequence of S^+ for indexed events 1 through k.

EI: The expected cost per successful trainee for system state, SS, and the associated optimal sequence, S. The EI associated with SS and S will be denoted as EI ~ (SS,S).

$$EI = \frac{\sum_{k=1}^{n} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{n} P_{n_\ell}}$$

EI^+: The system efficiency index for SS^+ and S^+. $EI^+ \sim (SS^+, S^+)$.

EI^*: The optimal system efficiency index for SS^* and S^*. $EI^* \sim (SS^*, S^*)$.

EI_{n_k-x}: The partial efficiency index for SS_k and S_k computed up to and including event k with n_k-x trials. $EI_{n_k-x} \sim (SS_k, S_k)$.

EI_{n_k} : The partial efficiency index for SS_k^+ and S_k computed up to and including event k with n_k trials. $EI_{n_k} \sim (SS_k^+, S_k)$.

x: An integer; $1 \leqq x \leqq m_k$ or $1 \leqq x \leqq a_k - m_k$, as appropriate.

R_{jk}: The value of the ordering ratio for trial j in event k. For example, for $j = n_k$:

$$R_{n_k} = \frac{1 - P_{n_k}}{C_k \sum_{j=0}^{n_k - 1} (1 - P_{jk})}$$

Development and Proofs of Theorems and Lemmas Required for Construction of the Algorithm

Optimal Event Ordering Theorem

The optimal event ordering theorem and proof developed by Miller are presented for the reader's convenience.

Theorem 1: If the events in any given aptitude track are ordered by the following procedure, the cost of training any given input is minimized.

(i) For each event k, compute the ratio

$$\frac{1 - P_{a_k}}{C_k \sum_{j=0}^{a_k - 1} (1 - P_{jk})}$$

(ii) Place the event with the largest value for the above ratio first, the one with the second largest ratio value second, . . ., and the event with the smallest ratio last.

Proof: Suppose one sequence called s has a minimum cost of operation, C_s^*, and another sequence, s', exists which is feasible and has a cost, C_s, not a minimum. Further suppose that the only difference between these two sequences is that events h and $h+1$ are reversed in the feasible nonoptimal sequence. Then

$$C_s^* \leqq C_s, \quad \text{or} \quad C_s^* - C_s \leqq 0 \tag{3}$$

Substituting equation (1) into equation (3) and expanding yields:

$$N\Big[C_1 P_o \sum_{j=0}^{a_1-1} (1 - P_{j1}) + \ldots + C_h \prod_{\ell=0}^{h-1} P_{a_\ell} \sum_{j=0}^{a_h-1} (1 - P_{jn})$$

$$+ C_{h+1} \prod_{\ell=0}^{h-1} P_{a_\ell} P_{a_h} \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1}) + \ldots$$

$$+ C_n \prod_{\ell=0}^{n-1} P_{a_\ell} \sum_{j=0}^{a_n-1} (1 - P_{jn})\Big] - N\Big[C_1 P_o \sum_{j=0}^{a_1-1} (1 - P_{j1}) + \ldots$$

$$+ C_{h+1} \prod_{\ell=0}^{h-1} P_{a_\ell} \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1})$$

$$+ C_h \prod_{\ell=0}^{h-1} P_{a_\ell} P_{a_{h+1}} \sum_{j=0}^{a_h-1} (1 - P_{jh}) + \ldots + C_n \prod_{\ell=0}^{n-1} P_{a_\ell} \sum_{j=0}^{a_n-1} (1 - P_{jn})\Big] \leqq 0 ,$$

where $P_o = 1$.

Performing the indicated operations and dividing both sides by the common terms N and $\prod_{\ell=0}^{h-1} P_a$ yields:

$$\Big[C_h \sum_{j=0}^{a_h-1} (1 - P_{jh}) + C_{h+1} (P_{a_h}) \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1})\Big]$$

(continued)

$$- \left[C_{h+1} \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1}) + C_h (P_{a_{h+1}}) \sum_{j=0}^{a_h-1} (1 - P_{jh}) \right] \leqq 0 ,$$

which may be further reduced to

$$C_h (1 - P_{a_{h+1}}) \sum_{j=0}^{a_h-1} (1 - P_{jh}) \leqq C_{h+1} (1 - P_{a_h}) \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1}) .$$

Dividing both sides of the equation above by $(1 - P_{a_{h+1}})(1 - P_{a_h})$ gives

$$\frac{C_h \sum_{j=0}^{a_h-1} (1 - P_{jh})}{(1 - P_{a_h})} \leqq \frac{C_{h+1} \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1})}{(1 - P_{a_{h+1}})}$$

or

$$\frac{(1 - P_{a_h})}{C_h \sum_{j=0}^{a_h-1} (1 - P_{jh})} \geqq \frac{(1 - P_{a_{h+1}})}{C_{h+1} \sum_{j=0}^{a_{h+1}-1} (1 - P_{j,h+1})} .$$

QED

Lemma 1

Lemma 1: If event k precedes event k+1 in sequence S, and the number of trials for event k is increased from n_k to n_k+x and a shift is indicated by Theorem 1, then in the new sequence S^+, event k+1 must precede event k.

Proof: Let $k = r$, $k+1 = s$, $n_r+x = J_r$, $P_{n_r} < 1$.

Assume SS^+ and SS.

Theorem 1 requires the calculation of the ratio:

$$\frac{1 - P_{n_k}}{C_k \sum_{j=0}^{n_k-1} (1 - P_{jk})}, \qquad k = 1, 2, \ldots, n.$$

As the number of trials is increased from n_r to J_r, then $(1 - P_{n_r}) \geqq (1 - P_{J_r})$ since P_{jr} is monotonically increasing. Also,

$$C_r \sum_{j=0}^{J_r-1} (1 - P_{jr}) > C_r \sum_{j=0}^{n_r-1} (1 - P_{jr})$$

since

$$C_r \sum_{j=0}^{J_r-1} (1 - P_{jr}) = C_r \sum_{j=0}^{n_r-1} (1 - P_{jr}) + C_r \sum_{j=n_r}^{J_r-1} (1 - P_{jr})$$

$$> C_r \sum_{j=0}^{n_r-1} (1 - P_{jr}) .$$

Hence,

$$\frac{1 - P_{n_r}}{C_r \sum_{j=0}^{n_r-1} (1 - P_{jr})} > \frac{1 - P_{J_r}}{C_r \sum_{j=0}^{J_r-1} (1 - P_{jr})} .$$

Therefore, the ordering ratio strictly decreases as trials are added to an event. Theorem 1 requires that the events be completed in a sequence according to their respective ratios.

For S,

$$\ldots > \frac{1 - P_{n_q}}{C_q \sum_{j=0}^{n_q-1} (1 - P_{jq})} > \frac{1 - P_{n_r}}{C_r \sum_{j=0}^{n_r-1} (1 - P_{jr})} > \frac{1 - P_{n_s}}{C_s \sum_{j=0}^{n_s-1} (1 - P_{js})} > \ldots$$

by Theorem 1. When the number of trials is increased to J_r in event r, then if,

$$\ldots > \frac{1 - P_{n_q}}{C_q \sum_{j=0}^{n_q-1} (1 - P_{jq})} > \frac{1 - P_{J_r}}{C_r \sum_{j=0}^{J_r-1} (1 - P_{jr})} > \frac{1 - P_{n_s}}{C_s \sum_{j=0}^{n_s-1} (1 - P_{js})} > \ldots$$

in which case the new sequence $S^+ = S$. If

$$\ldots > \frac{1 - P_{n_q}}{C_q \sum_{j=0}^{n_q-1} (1 - P_{jq})} > \frac{1 - P_{n_s}}{C_s \sum_{j=0}^{n_s-1} (1 - P_{js})} > \ldots$$

$$\ldots > \frac{1 - P_{J_r}}{C_r \sum_{j=0}^{J_r-1} (1 - P_{jr})} > \ldots$$

in which case the new sequence $S^+ \neq S$ and event s precedes event r. QED

Minimization Theorem

Theorem 2: If $\bar{C}_{m_k} < \bar{C}_{m_k-x}$ for $k = 1, 2, \ldots, n$ and all x, $1 \leqq x \leqq m_k-1$, and $S^+ = S$, then $EI^+ < EI$.

Proof: Let: $k = r$.

$S^+ = S = 1, 2, \ldots, q, r, s, \ldots, n$

$SS^+ = \{m_1, m_2, \ldots, m_q, m_r, m_s, \ldots, m_n\}$

$SS = \{m_1, m_2, \ldots, m_q, m_r - x, m_s, \ldots, m_n\}$

$EI^+ \sim (SS^+, S^+)$; $EI \sim (SS, S)$.

By definition, $\overline{C}_{m_r} = \min_j \{\overline{C}_{jr}\}$.

$$\frac{C_r \sum_{j=0}^{m_r-1} (1 - P_{jr})}{P_{m_r}} < \frac{C_r \sum_{j=0}^{m_r-x-1} (1 - P_{jr})}{P_{m_r-x}}$$

Multiply by:

$$\frac{\prod_{\ell=0}^{q} P_{m_\ell}}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell}} ; \qquad P_{m_o} = 1 .$$

$$\frac{C_r \prod_{\ell=0}^{q} P_{m_\ell} \sum_{j=0}^{m_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} < \frac{C_r \prod_{\ell=0}^{q} P_{m_\ell} \sum_{j=0}^{m_r-x-1} (1 - P_{jr})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}}$$

Add to both sides:

$$\frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{m_\ell} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} + \frac{\sum_{k=s}^{n} C_k \prod_{\substack{\ell=0\\ \ell\neq r}}^{k-1} P_{m_\ell} P_{m_r-x} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} .$$

$$\frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{m_\ell} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} + \frac{C_r \prod_{\ell=0}^{q} P_{m_\ell} \sum_{j=0}^{m_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} \tag{4}$$

$$+ \frac{\sum_{k=s}^{n} C_k \prod_{\substack{\ell=0\\ \ell\neq r}}^{k-1} P_{m_\ell} P_{m_r-x} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} < \frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{m_\ell} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}}$$

$$+ \frac{C_r \prod_{\ell=0}^{q} P_{m_\ell} \sum_{j=0}^{m_r-x-1} (1 - P_{jr})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} + \frac{\sum_{k=s}^{n} C_k \prod_{\substack{\ell=0\\ \ell\neq r}}^{k-1} P_{m_\ell} P_{m_r-x} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} .$$

Inspection of the right inequality of equation (4) reveals it is equivalent to EI. In the last term of the left inequality, replace P_{m_r-x} with P_{m_r}.

$$\frac{\sum\limits_{k=1}^{q} C_k \prod\limits_{\ell=0}^{k-1} P_{m_\ell} \sum\limits_{j=0}^{m_k-1} (1 - P_{jk})}{\prod\limits_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} + \frac{C_r \prod\limits_{\ell=0}^{q} P_{m_\ell} \sum\limits_{j=0}^{m_r-1} (1 - P_{jr})}{\prod\limits_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} \tag{5}$$

$$+ \frac{\sum\limits_{k=s}^{n} C_k \prod\limits_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{m_\ell} P_{m_r} \sum\limits_{j=0}^{m_k-1} (1 - P_{jk})}{\prod\limits_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} < EI \ .$$

Examine the first term of the left inequality of equation (5). Since P_{m_r-x} and P_{m_r} are cumulative probabilities, $P_{m_r} \geqq P_{m_r-x}$. Hence

$$\frac{\sum\limits_{k=1}^{q} C_k \prod\limits_{\ell=0}^{k-1} P_{m_\ell} \sum\limits_{j=0}^{m_k-1} (1 - P_{jk})}{\prod\limits_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} \leqq \frac{\sum\limits_{k=1}^{q} C_k \prod\limits_{\ell=0}^{k-1} P_{m_\ell} \sum\limits_{j=0}^{m_k-1} (1 - P_{jk})}{\prod\limits_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r-x}} \ . \tag{6}$$

Substituting P_{m_r} for $P_{m_r - x}$ in the first term of equation (5) and using the property of equation (6) yields:

$$\frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{m_\ell} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} + \frac{C_r \prod_{\ell=0}^{q} P_{m_\ell} \sum_{j=0}^{m_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} \tag{7}$$

$$+ \frac{\sum_{k=s}^{n} C_k \prod_{\substack{\ell=0\\ \ell\neq r}}^{k-1} P_{m_\ell} P_{m_r} \sum_{j=0}^{m_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{m_\ell} P_{m_r}} < EI .$$

Inspection of the left inequality of equation (7) reveals that it is equivalent to EI^+.

Therefore, $EI^+ < EI$. QED

Corollary 2a: If $\bar{C}_{m_k} < \bar{C}_{m_k - x}$ for $k = 1,2, \ldots, n$, and all x, $1 \leqq x \leqq m_k - 1$, and $S^+ \neq S$, then $EI^+ < EI$.

<u>Proof</u>: Let k = r.

Define: $EI' \sim (SS^+, S)$.

$S^+ = 1,2, \ldots, q,s, \ldots, r, \ldots, n$ (by Lemma 1)

By Theorem 2, $EI' < EI$. By Theorem 1, sequence S is not the optimal sequence for SS^+. Let S^+ be the optimal sequence for SS^+. Then by Theorem

1, $EI^+ < EI'$. Hence, $EI^+ < EI' < EI$.

Therefore, $EI^+ < EI$. QED

Note that $S^+ = S$ in Theorem 2, whereas $S^+ \neq S$ in Corollary 2a.

Addition Theorem

Theorem 3: If (a) $EI_{n_k+x} < EI_{n_k}$ for sequences S_k^+ and S_k, respectively, (b) each partial sequence contains all like events, and (c) $S_k^+ \neq S_k$, the $EI^+ < EI$.

Proof: Let $k = r$.

Define: $S = 1,\ldots,p,r,q,s,\ldots,t,v,\ldots,n$

$S^+ = 1,\ldots,p,q,s,\ldots,t,r,v,\ldots,n$ (by Lemma 1)

$SS = \{n_1,n_2,\ldots,n_q,n_r,n_s,\ldots,n_n\}$

$SS^+ = \{n_1,n_2,\ldots,n_q,n_r+x,n_s,\ldots,n_n\}$

$$EI_{n_r} = \frac{\sum_{k=1}^{p} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} P_{n_r}} + \frac{C_r \prod_{\ell=0}^{p} P_{n_\ell} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} P_{n_r}}$$

$$+ \frac{\sum_{k=q}^{t} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} P_{n_r}} .$$

$$EI_{n_r+x} = \frac{\sum\limits_{k=1}^{p} C_k \prod\limits_{\ell=0}^{k-1} P_{n_\ell} \sum\limits_{j=0}^{n_k-1} (1 - P_{jk})}{\prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} P_{n_r+x}} + \frac{\sum\limits_{k=q}^{t} C_k \prod\limits_{\ell=0}^{k-1} P_{n_\ell} \sum\limits_{j=0}^{n_k-1} (1 - P_{jk})}{\prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} P_{n_r+x}}$$

$$+ \frac{C_r \prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} \sum\limits_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} P_{n_r+x}} .$$

Given $EI_{n_r+x} < EI_{n_r}$.

Divide both sides by $\prod\limits_{\ell=v}^{n} P_{n_\ell}$ and add

$$\frac{\sum\limits_{k=v}^{n} C_k \prod\limits_{\substack{\ell=0 \\ \ell \neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum\limits_{j=0}^{n_k-1} (1 - P_{jk})}{\prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} \prod\limits_{\ell=v}^{n} P_{n_\ell} P_{n_r}} .$$

$$\frac{\sum\limits_{k=1}^{p} C_k \prod\limits_{\ell=0}^{k-1} P_{n_\ell} \sum\limits_{j=0}^{n_k-1} (1 - P_{jk})}{\prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} \prod\limits_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} + \frac{\sum\limits_{k=q}^{t} C_k \prod\limits_{\ell=0}^{k-1} P_{n_\ell} \sum\limits_{j=0}^{n_k-1} (1 - P_{jk})}{\prod\limits_{\ell=0}^{p} P_{n_\ell} \prod\limits_{\ell=q}^{t} P_{n_\ell} \prod\limits_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} \qquad (8)$$

(continued)

$$+ \frac{C_r \prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} + \frac{\sum_{k=v}^{n} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r}}$$

$$< \frac{\sum_{k=1}^{p} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r}} + \frac{C_r \prod_{\ell=0}^{p} P_{n_\ell} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r}}$$

$$+ \frac{\sum_{k=q}^{t} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r}} + \frac{\sum_{k=v}^{n} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{p} P_{n_\ell} \prod_{\ell=q}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r}}$$

$$= \frac{\sum_{k=1}^{n} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{n} P_{n_\ell}} .$$

The right inequality of equation (8) is equivalent to EI ~ (SS,S). In the last term of the left inequality of equation (8), replace P_{n_r} by P_{n_r+x}.

$$\frac{\sum_{k=1}^{p} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} + \frac{\sum_{k=q}^{t} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} \qquad (9)$$

$$+ \frac{C_r \prod_{\ell=0}^{t} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\ell=0}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} + \frac{\sum_{k=v}^{n} C_k \prod_{\substack{\ell=0 \\ \ell \neq r}}^{k-1} P_{n_\ell} P_{n_r+x} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{t} P_{n_\ell} \prod_{\ell=v}^{n} P_{n_\ell} P_{n_r+x}} < EI .$$

The left inequality of equation (9) is now equivalent to $EI^+ \sim (SS^+, S^+)$.

Therefore, $EI^+ < EI$. QED

Corollary 3a: If (a) $EI_{n_k+x} < EI_{n_k}$, (b) $S_k^+ = S_k$ for computing EI_{n_k+x} and EI_{n_k}, respectively, and (c) $S^+ = S$, then $EI^+ < EI$.

Proof: Let $k = r$.

Define: $S^+ = 1, 2, \ldots, p, q, r, s, \ldots, n$

$SS^+ = \{n_1, n_2, \ldots, n_q, n_r+x, n_s, \ldots, n_n\}$

$SS = \{n_1, n_2, \ldots, n_q, n_r, n_s, \ldots, n_n\}$

$$EI_{n_r+x} = \frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{q} P_{n_\ell} P_{n_r+x}} + \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\ell=0}^{q} P_{n_\ell} P_{n_r+x}} .$$

$$EI_{n_r} = \frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{q} P_{n_\ell} P_{n_r}} + \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\ell=0}^{q} P_{n_\ell} P_{n_r}} .$$

Given $EI_{n_r+x} < EI_{n_r}$.

Divide both sides by $\prod_{\ell=s}^{n} P_{n_\ell}$ and add

$$\frac{\sum_{k=s}^{n} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}} .$$

$$\frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} + \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} \tag{10}$$

$$+ \frac{\sum_{k=s}^{n} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}} < \frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}}$$

(continued)

$$+ \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}} + \frac{\sum_{k=s}^{n} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}}$$

$$= \frac{\sum_{k=1}^{n} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{n} P_{n_\ell}} .$$

The right inequality of equation (10) is equivalent to EI ~ (SS,S). In the last term of the left inequality, replace P_{n_r} by P_{n_r+x}.

$$\frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} + \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} \tag{11}$$

$$+ \frac{\sum_{k=s}^{n} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} P_{n_r+x} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} < \text{EI} .$$

The left inequality of equation (11) is now equivalent to $EI^+ \sim (SS^+,S) = (SS^+,S^+)$.

Therefore, $EI^+ < EI$. QED

The converse of the corollary, that is, if $EI_{n_k+x} > EI_{n_k}$ then $EI^+ > EI$, can be shown in a similar manner.

Corollary 3b: If (a) $EI_{n_k+x} < EI_{n_k}$, (b) $S_k^+ = S_k$ for computing EI_{n_k+x} and EI_{n_k}, respectively, and (c) $S^+ \neq S$, then $EI^+ < EI$.

Proof: Let $k = r$.

Define: $EI' \sim (SS^+,S)$

$$S = 1,2,\ldots,q,r,s,\ldots,n$$

$$S^+ = 1,2,\ldots,q,s,\ldots,r,\ldots,n \quad \text{(by Lemma 1)}$$

Given $EI_{n_r+x} < EI_{n_r}$. By Corollary 3a, $EI' < EI$.

By Theorem 1, sequence S is not the optimal sequence for SS^+. Let S^+ be the optimal sequence for SS^+. Then by Theorem 1,

$$EI^+ < EI' .$$

Hence, $EI^+ < EI' < EI$.

Therefore, $EI^+ < EI$. QED

Lemma 2

Lemma 2: If $EI_{n_k+x} < EI_{n_k}$ for partial sequence S_k, then $EI'_{n_k+x} < EI'_{n_k}$, where EI'_{n_k+x} is computed for any other partial sequence, S'_k, and event k shifts by Lemma 1.

Proof: Let $k = r$.

Define: $S = 1,\ldots,q,r,s,\ldots,t,v,\ldots,n$

$S' = 1,\ldots,q,s,\ldots,t,r,v,\ldots,n$ (by Lemma 1)

$EI_{n_r+x} \sim (SS^+,S)$

$EI_{n_r} \sim (SS,S)$

$EI'_{n_r+x} \sim (SS^+,S')$

$EI'_{n_r} \sim (SS,S)$, with events s through t added.

Given $EI_{n_r+x} < EI_{n_r}$ for partial sequence S, and that S' is the sequence for SS^+ by Theorem 1.

By Corollary 3b, $EI^+ < EI$. That is:

$$\frac{\sum_{k=1}^{t} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} + \frac{C_r \prod_{\ell=0}^{t} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} \tag{12}$$

$$+ \frac{\sum_{k=v}^{n} C_k \prod_{\substack{\ell=0\\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r+x} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r+x}} < \frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0\\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}}$$

(continued)

$$+ \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}} + \frac{\sum_{k=s}^{t} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}}$$

$$+ \frac{\sum_{k=v}^{n} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{n} P_{n_\ell} P_{n_r}} .$$

The last terms of each inequality in equation (12) are equal. Deleting the last term of each inequality and multiplying the remaining terms by $\prod_{\ell=v}^{n} P_{n_\ell}$ yields:

$$\frac{\sum_{k=1}^{t} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\ell=0}^{t} P_{n_\ell} P_{n_r+x}} + \frac{C_r \prod_{\ell=0}^{t} P_{n_\ell} \sum_{j=0}^{n_r+x-1} (1 - P_{jr})}{\prod_{\ell=0}^{t} P_{n_\ell} P_{n_r+x}} \tag{13}$$

$$< \frac{\sum_{k=1}^{q} C_k \prod_{\ell=0}^{k-1} P_{n_\ell} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{t} P_{n_\ell} P_{n_r}} + \frac{C_r \prod_{\ell=0}^{q} P_{n_\ell} \sum_{j=0}^{n_r-1} (1 - P_{jr})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{t} P_{n_\ell} P_{n_r}}$$

(continued)

$$+ \frac{\sum_{k=s}^{t} C_k \prod_{\substack{\ell=0 \\ \ell\neq r}}^{k-1} P_{n_\ell} P_{n_r} \sum_{j=0}^{n_k-1} (1 - P_{jk})}{\prod_{\substack{\ell=0 \\ \ell\neq r}}^{t} P_{n_\ell} P_{n_r}} .$$

By definition, the left inequality of equation (13) is EI'_{n_r+x} and the right inequality is EI'_{n_r}.

Therefore, $EI'_{n_r+x} < EI'_{n_r}$.

Construction of the Algorithm

This section presents a step by step development of the algorithm. A discussion is included with each step to show the rationale for that step. A hypothetical problem, Table 1, is given as an aid to the presentation of the algorithm.

Table 1. Expected Event Costs and Ordering Ratios

Trial	Event			
	1	2	. . .	n
1	$\bar{C}_{11}$	$\bar{C}_{12}$	. . .	$\bar{C}_{1n}$
2	$\bar{C}_{21}$	$\bar{C}_{22}$	. . .	$\bar{C}_{2n}$
⋮	⋮	⋮		⋮
m_k	$\bar{C}_{m_1}/R_{m_1}$	$\bar{C}_{m_2}/R_{m_2}$	. . .	$\bar{C}_{m_n}/R_{m_n}$
⋮	⋮	⋮		⋮
n_k	$\bar{C}_{n_1}/R_{n_1}$	$\bar{C}_{n_2}/R_{n_2}$	. . .	$\bar{C}_{n_n}/R_{n_n}$
⋮	⋮	⋮		⋮
a_k	$\bar{C}_{a_1}/R_{a_1}$	$\bar{C}_{a_2}/R_{a_2}$	. . .	$\bar{C}_{a_n}/R_{a_n}$

Step by Step Development of the Algorithm

Step 1: Compute $\bar{C}_{jk}$; $j = 1,2,\ldots,a_k$; $k = 1,2,\ldots,n$. Determine $\bar{C}_{m_k} = \min_j \{\bar{C}_{jk}\}$; $k = 1,2,\ldots,n$. For the set of all $\bar{C}_{m_k}$, SS^m, compute the ordering ratios and determine the optimal sequence, S, by Theorem 1. For any $m_k = a_k$, set $J_k = a_k$. Index sequence, S, as $S = 1,2,\ldots,n$. If $m_k = a_k$ for all k, go to step 5.

Discussion. Step 1 is an application of Theorem 2. By Theorem 2, there exists no m_k-x trials for any event k which can yield an EI which is better than the EI associated with $SS^m = \{m_1,m_2,\ldots,m_n\}$. Theorem 2 permits the immediate consideration of SS^m without having to consider any intermediate system state, say, $SS = \{m_1,m_2,\ldots,m_r-x,\ldots,m_n\}$. All numbers of trials less than m_k may be disregarded, since they will only yield an EI that is greater (worse) than the EI ~ (SS^m,S).

If $m_k = a_k$, then no additional trials are available for future consideration. m_k is set equal to J_k to conform to the notation associated with the eventual optimal solution.

The last part of step 1 requires indexing the sequence S. If the optimal sequence for SS^m is, say, $S = 4,3,7,\ldots,n,\ldots,5$, then four corresponds to the event number four, etc. Index the sequence, S, means to set event number four as the first event in the sequence, event number three as the second event in the sequence, . . ., and event number five as event n or the last event in the sequence. In subsequent discussions, event k means indexed event k. Such indexing is done to facilitate other steps in the algorithm.

Step 2: For all $m_k < a_k$, determine $\bar{C}_{n_k} = \min_{m_k+1 \leqq j \leqq a_k} \{\bar{C}_{jk}\}$ and sequentially compute, beginning at $k = 2$, EI_{n_k} until there exists $EI_{n_k} < EI_{m_k}$; go to step 3. If there exists no $EI_{n_k} < EI_{m_k}$, $k = 2,\ldots,n$, then for all $m_k < a_k$, set $J_k = m_k$; go to step 4.

Discussion. Step 2 applies Corollaries 3a and 3b. If there exists $EI_{n_k} < EI_{m_k}$, then by the Corollaries, the EI can be improved by adding additional trials to event k.

The step permits computing EI_{n_k} beginning at $k = 2$. This is allowable, since for $k = 1$, there exists no $EI_{n_1} < EI_{m_1}$. For this case, $EI_{n_1} = \overline{C}_{n_1}$ and $EI_{m_1} = \overline{C}_{m_1}$, and by definition, $\overline{C}_{m_1} < \overline{C}_{n_1}$. Hence, by Corollary 3a or 3b, no improvement in the EI is possible for additional trials in event 1. By Lemma 1, the addition of trials to an event cannot shift that event in the sequence so that it precedes event 1. Event 1 remains event 1 in all future sequences.

The step also permits computing EI_{n_k} without having to compute any intermediate $EI_{n_k - x}$, where $n_k - x > m_k$. This is permissible by Theorem 2.

The situations when $EI_{n_k} < EI_{m_k}$ and when there exists no $EI_{n_k} < EI_{m_k}$ will be analyzed in the discussions associated with steps 3 and 4, respectively.

Step 3: For $EI_{n_k} < EI_{m_k}$, add the $n_k - m_k$ additional trials to event k; compute the new ordering ratio and determine S^+. If $n_k = a_k$, set $J_k = n_k$. If $n_k < a_k$, set $m_k = n_k$. Set $S = S^+$; index the new S; return to step 2.

Discussion. Step 3 adds the additional trials to event k when Corollary 3a or 3b indicates an improved EI is possible. The last part of this step is accomplished to set the new initial conditions for repeating step 2.

As long as there exists an $EI_{n_k} < EI_{m_k}$, steps 2 and 3 will be systematically repeated. Consider the process during a complete cycle of these two steps. In step 2, EI_{n_k} is sequentially computed until there exists an $EI_{n_k} < EI_{m_k}$. If there exists no $EI_{n_2} < EI_{m_2}$, then by either Corollary 3a or 3b, no improvement can be made to the EI by adding more trials to event 2. By Corollary 3a or 3b and Lemma 1, event 2 will always remain event 2 in future sequences. Events 1 and 2 are now fixed. If $EI_{n_2} < EI_{m_2}$, then step 3 is completed. If $n_2 = a_2$, then event 2 has no additional trials for future consideration. Event 2 cannot improve the EI regardless of its position in any new sequence S^+ which satisfies Lemma 2. If $n_2 < a_2$, then by Theorem 1, either $S = S^+$ or $S \neq S^+$. If $S = S^+$, step 2 is repeated with no change in the indexing of S. If $S \neq S^+$, then by Lemma 1, event 3 of S must become event 2 of S^+, and subsequently event 2 is the new indexed sequence S.

For step 2 to lead to step 3, trials must be added to some event k. As the cycle continues between steps 2 and 3, either all trials will be required in an event to improve the EI or there will exist no $EI_{n_k} < EI_{m_k}$, $k < n$. Eventually, event k becomes event n. For event n, either $n_n = a_n$ or there exists no $EI_{n_n} < EI_{m_n}$. Step 2 then leads to step 4.

Step 4: For $SS' = \{J_1, J_2, \ldots, J_n\}$ and $S' = 1,2,\ldots,n$, determine if there exists and $J_k < a_k$, $k = 1,2,\ldots,n$. If there exists no $J_k < a_k$, go to step 5. If there exists $J_k < a_k$ and the addition of $n_k - J_k$ trials, $J_k < n_k \leqq a_k$, to event k causes no shift in the sequence S' by Theorem 1, go to step 5. If the addition of $n_k - J_k$ trials causes a shift in the sequence S' by Theorem 1, go to step 6.

Discussion. SS' and S' specify a possible optimal solution. If all $J_k = a_k$, then this solution is the optimal solution, EI^*. This follows from Theorem 2 and the fact that no additional trials are available for consideration. If there exists $J_k < a_k$ and the addition of $n_k - J_k$ trials, $J_k < n_k \leqq a_k$, to event k causes no shift in sequence S', then SS' and S' yield an optimal solution. This sufficient condition follows from Corollary 3a because there exists no $EI_{n_k} < EI_{J_k}$, and hence, no additional improvement in the EI is possible. The sufficient condition is proven in detail in a subsequent section.

If there exists $J_k < a_k$ and the addition of $n_k - J_k$ trials causes a shift in sequence S', the solution may or may not be optimal. The relationship of EI_{n_k} and EI_{J_k} must be investigated by Theorem 3. A method for possible solution improvement of SS' and S' is outlined in a subsequent section.

Step 5: Stop. Set $SS^* = SS'$ and $S^* = S'$. For SS^* and S^*, compute the optimal EI, EI^*.

Step 6: Compute $EI' \sim (SS', S')$. This solution may or may not be optimal. Further investigation is required to see if solution improvement is possible.

Discussion of the Algorithm

Theorem 3 is a stronger condition for EI improvement than are the conditions in Corollaries 3a and 3b. That is, any opportunities for improvement identified by application of the Corollaries would also be identified by application of Theorem 3; but, Theorem 3 may identify opportunities not identified by the Corollaries. This result follows since in

Theorem 3, EI_{n_k+x} is based on the new partial sequence S_k^+; whereas, in the Corollaries EI_{n_k+x} is based on the previous partial sequence S_k. Thus, if $S_k^+ \neq S_k$, Theorem 3 can identify improvement possibilities which might be overlooked by the Corollaries.

The algorithm is based on successive application of Corollaries 3a and 3b, despite the fact that Theorem 3 is a stronger condition. This is done to ensure that the algorithm will terminate in a finite number of iterations. The finiteness property follows from Lemma 2 and from the iterative nature of steps 2 and 3 of the algorithm. Specifically, steps 2 and 3 assure that the partial sequence S_k, which resulted in addition of trials to event k at some iteration, will be maintained for all future iterations. Lemma 2 assures that, if S_k is maintained, then the augmented trials need not be reconsidered in subsequent iterations.

The finiteness property may[*] not result if the algorithm were based on Theorem 3 rather than on Corollaries 3a and 3b. This follows due to calculations based on S_k^+, the new partial sequence, rather than on the current partial sequence, S_k. By construction of the algorithm, S_k is maintained in all future iterations; however, S_k^+ need not be maintained.

As an example, suppose that event r is under consideration and that the current sequence is $S = 1,2,\ldots,q,r,s,\ldots,n$. Let n_r be the current number of trials in event r. Suppose that, if x trials are added, then a new sequence, $S' = 1,2,\ldots,q,s,r,\ldots,n$ results and that $EI_{n_r+x} <$

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[*] See Chapter III, Recommendations.

EI_{n_r}, where EI_{n_r+x} is based on $S_r^+ = 1,2,\ldots,q,s,r$ and EI_{n_r} is based on $S = 1,2,\ldots,q,r,s$. The next iteration of the algorithm could result in the addition of trials to event s and lead to a new sequence in which event r precedes event s. The partial sequence S_r^+ is now destroyed and the basis on which the x trials were added to event r no longer exists. Accordingly, it is possible that the x trials should be removed. Clearly, a cycle could result in which the algorithm would continue adding and deleting trials to events r and s, thus destroying the finiteness property. Since Corollaries 3a and 3b are concerned only with S_k, cycling is impossible.

In summary, even though Theorem 3 is a stronger condition, the algorithm is based on Corollaries 3a and 3b in order to assure that the algorithm will terminate in a finite number of iterations. Further, the algorithm generally does not require investigating the addition of trials, one trial at a time. By Theorem 2, all numbers of trials less than m_k can be disregarded. If $EI_{n_k} < EI_{m_k}$ by Corollaries 3a and 3b, all numbers of trials between m_k and n_k, where $\overline{C}_{n_k} = \min_j \{\overline{C}_{jk}\}$; $m_k+1 \leqq j \leqq a_k$, can be disregarded by Theorem 2.

Although the algorithm may not yield an optimal solution, the property of adding trials by Corollary 3a or 3b is a most significant advantage. Adding trials by the Corollaries insures finiteness of the algorithm and a solution in a relatively small number of steps.

A Sufficient Condition for Optimality

Since the algorithm is based on the weaker conditions of Corollaries 3a and 3b, it is possible that the algorithm terminates in a nonoptimal solution. Since the essential difference between Theorem 3 and Corollaries 3a and 3b is explicit, a sufficient condition for optimality can be proven. Further, Theorem 3 provides the insight to improve the resulting solution when the sufficient condition is not satisfied.

Theorem 4: Suppose that the algorithm terminates with S' = 1,2, ...,q,r,s,...,n, SS' = $\{J_1,J_2,\ldots,J_n\}$, and EI' ~ (SS',S'). If there does not exist an event k such that the addition of trials to that event will cause a change in S' by Theorem 1, then S', SS', and the resulting EI' are optimal.

Proof:

(i) Theorem 2 guarantees that it cannot be advantageous to reduce J_k below m_k. Any other reduction in J_k must result in a situation considered by the algorithm. That is, event k will be preceded by a partial sequence S_k for which it was advantageous to increase the number of trials to event k by either Corollary 3a or 3b. Thus, for any solution with less than J_k trials, the associated EI, say EI^-, must be such that $EI^- > EI'$. Accordingly, the solution cannot be improved by reducing the number of trials.

(ii) Now consider increasing the number of trials. If an improvement existed due to increasing the number of trials (with no resultant sequence shift), the algorithm could not have terminated (by step 2). By hypothesis, there does not exist an event k such that the addition of trials to that event will cause a sequence shift. Hence, the solution cannot be improved by increasing the number of trials.

It follows from (i) and (ii) that the solution must be optimal under the conditions specified.

System Efficiency Algorithm

Step 1: Compute $\bar{C}_{jk}$ for $j = 1,2,\ldots,a_k$; $k = 1,2,\ldots,n$. Determine $\bar{C}_{m_k} = \min_j \{\bar{C}_{jk}\}$; $k = 1,2,\ldots,n$. Compute the ordering ratios for the set, $\{\bar{C}_{m_k}\}$, and determine S. For $m_k = a_k$, set $J_k = m_k$. Index $S = 1,2,\ldots,n$. If $m_k = a_k$, for all k, go to step 5.

Step 2: For all $m_k < a_k$, determine $\bar{C}_{n_k} = \min_j \{\bar{C}_{jk}\}$, $m_k+1 \leqq j \leqq a_k$. Sequentially compute EI_{n_k}, beginning at $k = 2$, until $EI_{n_k} < EI_{m_k}$; go to step 3. If there exists no $EI_{n_k} < EI_{m_k}$, $k = 2,3,\ldots,n$, then for all $m_k < a_k$, set $J_k = m_k$; go to step 4.

Step 3: For $EI_{n_k} < EI_{m_k}$, add the $n_k - m_k$ additional trials to event k; compute the new ordering ratio and determine S^+. If $n_k = a_k$, set $J_k = n_k$. If $n_k < a_k$, set $m_k = n_k$. Set $S = S^+$, index new S; go to step 2.

Step 4: For $SS' = \{J_1, J_2, \ldots, J_n\}$ and $S' = 1,2,\ldots,n$, determine if there exists any $J_k < a_k$, $k = 1,2,\ldots,n$. If there exists $J_k < a_k$ and the addition of $n_k - J_k$ trials, $J_k < n_k \leqq a_k$, to event k causes no shift in the sequence S' by Theorem 3, go to step 5. If the addition of $n_k - J_k$ additional trials causes a shift in the sequence S' by Theorem 3, go to step 6.

Step 5: Stop. Set $SS^* = SS'$ and $S^* = S'$. Determine EI^*, $EI^* \sim (SS^*, S^*)$, an optimal solution.

Step 6: Compute $EI' \sim (SS', S')$. This solution may or may not be optimal. Further investigation is required to see if solution improvement is possible.

A Method for Improving the EI When the Sufficient Condition for Optimality of the Algorithm Is Not Satisfied

If the condition for optimality is not satisfied in step 4 of the algorithm, the following procedure may improve the algorithm's solution:

1. For SS' and S' in step 4 of the algorithm, determine the first event r where the addition of $K_r - J_r$ additional trials, $J_r < K_r \leqq a_r$, and $\bar{C}_{K_r} = \min_j \{\bar{C}_{jr}\}$. $J_r < j \leqq a_r$, indicates a new sequence S^+; $S^+ \neq S'$. Set all numbers of trials for all events, save event r, equal to m_k as determined by step 1. Assign K_r trials to event r and determine S for $SS = \{m_1,\ldots,m_q,K_r,m_s,\ldots,m_n\}$. Apply the algorithm, adding trials when EI improvement is possible, until the next iteration requires investigating EI_{J_r} and EI_{K_r}. Determine if $EI_{K_r} < EI_{J_r}$ by Theorem 3. If $EI_{K_r} < EI_{J_r}$, add the $K_r - J_r$ additional trials and continue the algorithm until entering step 4. If $EI_{K_r} > EI_{J_r}$, then the addition of $K_r - J_r$ additional trials cannot improve EI'.

2. Determine if the condition for optimality is now satisfied, excluding event r if $EI_{K_r} > EI_{J_r}$ by Theorem 3 above. If the condition is not satisfied, repeat the above procedure. If the condition is satisfied, then the new solution is optimal.

Appendix A contains sample problems demonstrating the algorithm and the alternative method for solution improvement when the condition for optimality is not satisfied.

CHAPTER III

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

This research has developed an algorithm for efficiently improving the system effectiveness for a sequential manpower training system. Rules which indicate when improvement in the system effectiveness is possible have been developed, proven, and demonstrated. A sufficient condition for optimality, which when satisfied guarantees an optimal solution, has been developed and proven. An alternative method for solution improvement, when the sufficient condition for optimality is not satisfied, has been developed and demonstrated.

The essential assumptions of the training system that were required for the development of the algorithm are:

(1) There is no probabilistic dependency between events. The probability of success for an event is independent of the event order in the event sequence.

(2) Trainees are not recycled between events.

(3) As quickly as a trainee successfully completes a trial in any event, he proceeds to the next event in the sequence.

(4) Within an event, a trial is a fixed time period of instruction-testing.

(5) Trainees do not repeat trials in an event indefinitely. A maximum permissible number of trials is established for each event.

The general results of this thesis are:

(1) The optimal system efficiency may be determined, when the sufficiency condition is satisfied, without enumerating all numbers of trials and sequence combinations.

(2) Theorem 2 permits the immediate discarding of all numbers of trials less than the number of trials associated with the minimum expected event cost. The theorem also permits discarding all numbers of trials between the number associated with the minimum expected event cost and the number associated with the next higher minimum expected event cost.

(3) Theorem 3 demonstrates that adding additional trials, while incurring a greater than minimum expected event cost, may improve the measure of system effectiveness.

Recommendations

The assumption that the event costs and probabilities of success are fixed should be relaxed. These costs and probabilities may be changed by modifying instructional techniques, varying the length of trials, adding learning incentives, etc. Determining the optimal system efficiency would then become a four variable problem.

The algorithm should be investigated and modified to permit trial additions by using Theorem 3 in addition to Corollaries 3a and 3b. It is conjectured that such a modified algorithm would still exhibit finiteness and the sufficient condition for optimality could be deleted. This author was not able to prove this conjecture.

The training model should be extended to programmed learning environments. How would the system effectiveness by affected by modifying the "spaced-learning" concept? That is, what would be the effects on costs and probabilities by increasing or decreasing the time allotted to specific subjects? Would the expected total time for success for all such subjects be lengthened or shortened?

The training system and algorithm could possibly be modified to apply to variations of a training or learning environment. For example, if the first trial in an event is not successfully completed, should the trainee attempt the event again immediately after failing or should subsequent events be completed before repeating the failed event? Should subsequent trials in an event be identical (in terms of cost and time) to the first trials, or should the first trial be distinctly different from subsequent trials? How would these changes affect the system efficiency?

The training system and developed theorems may have design implications applicable to production systems. Consider a marketable product that is the valuable output from a production system. How would the efficiency index change if the product was completely rejected during acceptance inspection versus being reworked and re-inspected? If the product is to be reworked, should it be reworked after the first rejection or should all inspections be completed and then reworked? How many rejections would make reworking economically infeasible?

The system should be investigated from the time point of view. The system may be represented as a finite, irreversible, absorbing state Markov chain (3,4,5). Markov theory may be employed to answer specific questions about the system, such as:

1. What is the probability that a trainee will attempt a future trial and event given he is currently training in some earlier trial and event?

2. How many event-trials will a trainee complete before he finally completes or fails the system?

3. What is the probability a trainee will be dropped from the system if he fails a specified event or successfully completes the system given he is currently training in a specific event and trial?

The probability distribution of the time required to successfully complete the system should be investigated. The expected time required to successfully complete the system could be added as a system constraint. That is, if the expected time associated with the optimal system efficiency is greater than some desired expected time, what is the least-cost method of reducing trials to satisfy the time constraint? This problem would be of particular interest when the armed forces must rapidly expand to meet an emergency situation.

Theorems 1, 2, and 3 only consider the case where all events are independent. An extension of the algorithm would be to constrain certain events to a specified region within a sequence. Some work on partially pre-ordered sequences of tests (events) has been done by Mankekar and Mitten (6).

APPENDIX A

SAMPLE PROBLEMS

Three problems are presented. The first problem is based on data from Miller (7, Appendices D and E) for the high aptitude trainees. The second problem is hypothetical and demonstrates when the algorithm yields an optimal solution. The third problem is also hypothetical and demonstrates the alternative method for solution improvement when the algorithm initially does not yield an optimal solution. Throughout all problems, the step number given refers to that specific step of the algorithm to be executed.

Problem 1:

Step 1: Compute $\overline{C}_{jk}$ for all events and trials (see Table 1, p. 33).

Trial	Event					
	1	2	3	4	5	6
1	∞	∞	∞	.052	.063	.0270
2	2.500	2.500	2.500	.048	.051	.0274
3	.925	.740	1.576	.043	.046	.0268
4	.792	.940	1.500			
5	.570	.713	1.333			
6	.403	.354	1.350			
7	.345	.353	1.312			
8	.298	.318				
9	.288	.310				
10		.312				
11		.315				
12		.300				

Choose $\overline{C}_{m_k}$ for all events: SS = {9,12,7,3,3,3}.

Determine R_{m_k} for all $\overline{C}_{m_k}$: All ratios zero; S = any order.

Go to step 5.

Step 5:

$SS^* = \{9,12,7,3,3,3\}$ (by Theorem 2).

S^* = an order (by Theorem 1).

EI^* = 2.054 instructor hours/successful trainee.

Problem 2:

Hypothetical data:

$\underline{P_{jk} \text{ and } C_k}$

Event

	1	2	3	4
C_k ($)	10	15	6	8
Trial				
1	.40	.60	.70	.40
2	.80	.85	.80	.50
3	.90	.86	.90	.80
4	1.00		.95	.85

Step 1: Compute $\bar{C}_{jk}$ for all events and trials:

$\underline{\bar{C}_{jk}/R_{jk}}$

Event

Trial	1	2	3	4
1	25.00	25.00	8.57/.0500	20.00/.0750
2	20.00	24.71/.0071	9.75/.0256	25.60
3	20.00	27.03/.0060	10.00/.0111	21.00/.0119
4	19.00/.0000		10.10/.0052	21.67/.0082

Choose $\bar{C}_{m_k}$ for all events: $SS = \{4,2,1,1\}$ (by Theorem 2).

Determine R_{m_k} for all events: $S = 4,3,2,1$ (by Theorem 1).

For any $m_k = a_k$, set $J_k = a_k$: $m_1 = a_1 = 4$; $J_1 = 4$.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{23} = 34.750 < EI_{13} = 37.143$; go to step 3.

Step 3:

Add second trial to event 3: $SS^+ = \{4,2,2,1\}$ (by Corollary 3a).

Determine S^+: $S^+ = 4,3,2,1$ (by Theorem 1).

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{33} = 32.222 < EI_{23} = 34.750$; go to step 3.

Step 3:

Add third trial to event 3: $SS^+ = \{4,2,3,1\}$ (by Corollary 3a).

Determine S^+: $S^+ = 4,3,2,1$ (by Theorem 1).

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{43} = 31.158 < EI_{33} = 32.222$; go to step 3.

Step 3:

Add last trial to event 3: $SS^+ = \{4,2,4,1\}$ (by Corollary 3b).

Determine S^+: $S^+ = 4,2,3,1$ (by Theorem 1).

$m_3 = a_3$; set $J_3 = a_3$: $J_3 = 4$.

Set $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{32} = 50.291 > EI_{22} = 48.235$

For $k = 3$: $m_3 = a_3$.

For $k = 4$: $m_1 = a_1$.

Set $m_2 = J_2 = 2$; go to step 4.

Step 4:

$R_{44} > R_{22}$ and $R_{32} > R_{43}$: no sequence shift possible. Hence sufficient condition for optimality is satisfied; go to step 5.

Step 5:

$SS^* = \{4,2,4,1\}$; $S^* = 4,2,3,1$.

$EI^* = \$79.88$/successful trainee.

Problem 3:

Hypothetical data:

P_{jk} and C_k

Event

	1	2	3	4
C_k (\$)	10	7	6	3
Trial				
1	.80	.40	.70	.50
2	.85	.50	.75	.80
3	.90	.55	.80	.85
4	.92	.60	.90	.90

Step 1: Compute $\bar{C}_{jk}$ for all events and trials:

$\bar{C}_{jk}/R_{jk}$

Event

Trial				
1	12.50/.0200	17.50/.0857	8.57/.0500	6.00
2	14.12/.0125	22.40/.0446	10.40/.0321	5.63/.0444
3	15.00/.0074	26.73/.0306	11.63/.0215	6.00/.0294
4	15.76/.0055	29.75/.0224	11.67/.0095	6.17/.0180

Choose $\bar{C}_{m_k}$ for all events: SS = {1,1,1,2} (by Theorem 2).

Determine R_{m_k} for all events: S = 2,3,4,1 (by Theorem 1).

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{23} = 33.733 > EI_{13} = 33.571$; $m_3 = 1$, set $J_3 = 1$.

$EI_{34} = 45.496 < EI_{24} = 47.589$; go to step 3.

Step 3:

Add third trial to event 4: $SS^+ = \{1,1,1,3\}$ (by Corollary 3a).

Determine S^+; $S^+ = 2,3,4,1$

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{44} = 44.468 < EI_{34} = 45.496$; go to step 3.

Step 3:

Add last trial to event 4: $SS^+ = \{1,1,1,4\}$ (by Corollary 3b).

Determine S^+: $S^+ = 2,3,1,4$. (by Theorem 1).

$m_4 = a_4$; set $J_4 = 4$.

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{21} = 53.61 < EI_{11} = 54.66$; go to step 3.

Step 3:

Add second trial to event 1: $SS^+ = \{2,1,1,4\}$ (by Corollary 3b).

Determine S^+: $S^+ = 2,3,4,1$ (by Theorem 1).

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{31} = 63.798 < EI_{21} = 65.257$; go to step 3.

Step 3:

Add third trial to event 1: $SS^+ = \{3,1,1,4\}$ (by Corollary 3a).

Determine S^+: $S^+ = 2,3,4,1$ (by Theorem 1).

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{41} = 63.01 < EI_{31} = 63.798$; go to step 3.

Step 3:

Add last trial to event 1: $SS^+ = \{4,1,1,4\}$ (by Corollary 3a).

Determine S^+: $S^+ = 2,3,4,1$ (by Theorem 1).

$m_1 = a_1$; set $J_1 = 4$.

Set: $SS = SS^+$, $S = S^+$; go to step 2.

Step 2:

For sequence S, determine first $EI_{n_k} < EI_{m_k}$.

$EI_{23} > EI_{13}$; $J_4 = a_4$ and $J_1 = a_1$; go to step 4.

Step 4:

$SS' = \{4,1,1,4\}$; $S' = 2,3,4,1$

$R_{22} < R_{13}$; go to step 6 (use alternative method for possible solution improvement).

Set all event trials equal to m_k, save event 2.

$SS = \{1,2,1,1\}$; then $S = 3,2,4,1$ (by Theorem 1).

Compute EI_{22} and EI_{13} by Theorem 3.

$EI_{22} = 39.542 > EI_{13} = 33.571$; do not consider event 2 with 2 trials. $J_2 = 1$.

For SS' and S': $R_{43} < R_{44}$. Set all event trials to m_k, save events 2 and 3. SS = {1,2,4,1}; then S = 2,4,1,3.

Applying steps 2 and 3 of the algorithm:

$EI_{44} = 25.61 < EI_{24} = 27.50$; add all trials to event 4. $J_4 = 4$.

For sequence S, which is now S = 2,1,4,3, check for possible sequence shift for any event subsequent to event 2: $R_{21} < R_{44}$. Compute EI_{21} and EI_{44} by Theorem 3:

$EI_{21} = 44.25 < EI_{44} = 44.36$; add second trial to event 1. For resulting sequence S = 2,4,1,3, check for possible sequence shift for any event subsequent to event 4: $R_{31} < R_{43}$. Compute EI_{31} and EI_{43} by Theorem 3:

$EI_{31} = 59.58 < EI_{43} = 60.83$; add third trial to event 1.

Now for SS = {3,1,4,4} and S = 2,4,3,1, the earlier addition of all trials to event 3 must be rechecked. This follows from the fact that event 3 is now the third event in the sequence, whereas the trials were added to event 3 when it was the fourth event in the earlier sequence. By Theorem 3, $EI_{43} = 37.04 < EI_{44} = 43.47$; all trials remain in event 3. For current SS and S, check for possible solution improvement by adding the last trial to event 1.

$EI_{41} = 59.37 < EI_{31} = 59.58$; add last trial to event 1. $J_1 = 4$.

SS = {4,1,4,4} and S = 2,4,3,1: check for possible sequence shift by adding trials to event 2: $R_{42} > R_{44}$; no shift possible in sequence; go to step 5.

Step 5:

$SS^* = SS = \{4,1,4,4\}$ and $S^* = S = 2,4,3,1.$

$EI^* = \$59.37$/successful trainee.

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