THE COLORED JONES POLYNOMIAL AND ITS STABILITY

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THE COLORED JONES POLYNOMIAL AND ITS STABILITY

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To Phuong, Nguyen and Khoi.

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SUMMARY

This dissertation studies the colored Jones polynomial of knots and links, colored by representations of simple Lie algebras, and the stability of its coefficients.

In Chapter 1 we provide an explicit formula for the second plethysm of an arbitrary representation of $\mathfrak{sl}_3$, which allows us to give an explicit formula for the colored Jones polynomial of the trefoil, and more generally, for $T(2,n)$ torus knots. We give two independent proofs of our plethysm formula, one of which uses the work of Carini-Remmel. Our formula for the $\mathfrak{sl}_3$ colored Jones polynomial of $T(2,n)$ torus knots allows us to verify the Degree Conjecture for those knots, to efficiently compute the $\mathfrak{sl}_3$ Witten-Reshetikhin-Turaev invariants of the Poincare sphere, and to guess a Groebner basis for recursion ideal of the $\mathfrak{sl}_3$ colored Jones polynomial of the trefoil.

In Chapter 2 we formulate a stability conjecture for the coefficients of the colored Jones polynomial of a knot, colored by irreducible representations in a fixed ray of a simple Lie algebra, and verify it for all torus knots and all simple Lie algebras of rank 2. Our conjecture is motivated by a structure theorem for the degree and the coefficients of a q-holonomic sequence of polynomials given in [20] and by a stability theorem of the colored Jones polynomial of an alternating knot given in [25]. We illustrate our results with sample computations.

In Chapter 3 we give an efficient method to compute those q-series that come from planar graphs (i.e., reduced Tait graphs of alternating links) and compute several terms of those series for all graphs with at most 8 edges. In addition, we give a graph-theory proof of a theorem of Dasbach-Lin which identifies the coefficient of q^k in those series for $k = 0, 1, 2$ in terms of polynomials on the number of vertices, edges and triangles of the graph.

In Chapter 4 we study the structure of the stable coefficients of the Jones polynomial of an alternating link. We start by identifying the first four stable coefficients with polynomial invariants of a (reduced) Tait graph of the link projection. This leads us to introduce a free polynomial algebra of invariants of graphs whose elements give invariants of alternating links which strictly refine the first four stable coefficients. We conjecture that all stable coefficients are elements of this algebra, and give experimental evidence for the fifth and sixth stable coefficient. We illustrate our results in tables of all alternating links with at most 10 crossings and all irreducible planar graphs with at most 6 vertices.

CHAPTER I

THE SL_3 COLORED JONES POLYNOMIAL OF THE TREFOIL

1.1 Introduction

The goal of this chapter is to provide a supply of explicit quantum invariants so as to help in formulating and testing a number of conjectures. The most readily approachable knots in this context are the (m, n) torus knots, particularly when $m = 2$. The aim is to give explicit details for the $\mathfrak{sl}_3$ invariants, as these are potentially the simplest case after the more readily available colored Jones ($\mathfrak{sl}_2$) invariants.

There is a general method of Rosso and Jones to determine any quantum invariant of a torus knot. For the invariant of the (m, n) torus knot with quantum group module V their calculations require knowledge of the decomposition of the module $\psi_m(V)$ into irreducible representations. This is a combinatorial problem depending on the quantum group and the choice of V, which does not always have a readily available explicit formula.

We give here an explicit formula where $m = 2$ and V is a general irreducible $\mathfrak{sl}_3$ module; from this we are able to give a detailed estimate for the extreme degrees of the resulting Laurent polynomial invariant.

We then extend the formula to $\mathfrak{sl}_N$ by reformulating some combinatorial work of Carini and Remmel [10] describing $\psi_2(V)$ for the irreducible $\mathfrak{sl}_N$ modules which correspond to partitions with 2 parts.

1.2 The colored $\mathfrak{sl}_3$ Jones polynomial of the trefoil

In his seminal paper [34], Jones introduced the Jones polynomial of a knot K in 3-space. The Jones polynomial is a Laurent polynomial in a variable q with integer coefficients, which can be generalized to an invariant $J_{K,V}(q) \in \mathbb{Z}[q^{\pm 1}]$ of a (0-framed) knot K colored by a representation V of a simple Lie algebra $\mathfrak{g}$, and normalized to be 1 at the unknot. The definition of $J_{K,V}(q)$ uses the machinery of *quantum groups* and may be found in [51, 53] and also in [33].

Concrete formulas for the colored Jones polynomial $J_{K,V}(q)$ are hard to find in the case of higher rank Lie algebras, and for good reasons. For torus knots T, Jones and Rosso gave a formula for $J_{T,V}(q)$ which involves a plethysm map of V, unknown in general. Our goal is to give an explicit formula for the second plethysm of representations of $\mathfrak{sl}_3$ and consequently to give a formula for the $\mathfrak{sl}_3$ colored Jones polynomial of the trefoil. To state our results, let V_{n_1,n_2} denote the irreducible representation of $\mathfrak{sl}_3$ with highest weight

$$\lambda = n_1\omega_1 + n_2\omega_2 \tag{1}$$

where n_1, n_2 are non-negative integers and ω_1, ω_2 are the fundamental weights of $\mathfrak{sl}_3$ dual to the simple roots α_1, α_2. In coordinates, we have

$$\alpha_1 = (1,-1,0), \qquad \alpha_2 = (0,1,-1), \qquad \omega_1 = \frac{1}{3}(2\alpha_1+\alpha_2), \qquad \omega_2 = \frac{1}{3}(\alpha_1+2\alpha_2)$$

The *quantum integer* $[n]$, the *quantum dimension* d_{n_1,n_2} and the *twist parameter* θ_{n_1,n_2} of V_{n_1,n_2} are defined by

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{2}$$

$$d_{n_1,n_2} = \frac{[n_1+1][n_2+1][n_1+n_2+2]}{[2]} \tag{3}$$

$$\theta_{n_1,n_2} = q^{\frac{1}{3}(n_1^2+n_1n_2+n_2^2)+n_1+n_2} \tag{4}$$

Let $T(m,n)$ denote the *torus knot* associated to a pair of coprime natural numbers m, n, and let $J_{T(m,n),n_1,n_2}(q)$ denote the $\mathfrak{sl}_3$ colored Jones polynomial of the torus knot

$T(m,n)$ colored by V_{n_1,n_2}.

Theorem 1.2.1. *For all odd natural numbers n we have*

$$\begin{aligned} J_{T(2,n),n_1,n_2}(q) &= \frac{\theta_{n_1,n_2}^{-2n}}{d_{n_1,n_2}} \Bigg(\sum_{l=0}^{\min\{n_1,n_2\}} \sum_{k=0}^{n_1-l} (-1)^k d_{2n_1-2k-2l,2n_2+k-2l} \theta_{2n_1-2k-2l,2n_2+k-2l}^{\frac{n}{2}} \\ &\quad + \sum_{l=0}^{\min\{n_1,n_2\}} \sum_{k=0}^{n_2-l} (-1)^k d_{2n_1+k-2l,2n_2-2k-2l} \theta_{2n_1+k-2l,2n_2-2k-2l}^{\frac{n}{2}} \\ &\quad - \sum_{l=0}^{\min\{n_1,n_2\}} d_{2n_1-2l,2n_2-2l} \theta_{2n_1-2l,2n_2-2l}^{\frac{n}{2}} \Bigg). \end{aligned}$$

Theorem 1.2.1 can be used to answer for several problems.

- We can verify the $\mathfrak{sl}_3$-Degree Conjecture of the colored Jones polynomial for the trefoil. Explicitly, we can compute the lowest degree $\delta^*_{T(2,n),n_1,n_2}$ and the highest degree $\delta_{T(2,n),n_1,n_2}$ of the Laurent polynomial $J_{T(2,n),n_1,n_2}(q)$ as follows

$$\delta^*_{T(2,n),n_1,n_2} = \begin{cases} -\frac{n}{2}n_1^2 - \frac{n}{2}n_2^2 - nn_1n_2 - \frac{3n}{2}n_1 - (\frac{5n}{2}-2)n_2 & \text{if } n_1 \geq n_2 \\ -\frac{n}{2}n_1^2 - \frac{n}{2}n_2^2 - nn_1n_2 - \frac{3n}{2}n_2 - (\frac{5n}{2}-2)n_1 & \text{if } n_1 < n_2 \end{cases} \quad (5)$$

$$\delta_{T(2,n),n_1,n_2} = -(n-1)(n_1+n_2) \quad (6)$$

  The above formula verifies that the degree, restricted to each Kostant chamber, is a quadratic quasi-polynomial.

- We can efficiently compute the Witten-Reshetikhin-Turaev invariant of the Poincare sphere, complementing calculations of Lawrence [37].

- We can guess an explicit Groebner basis for the ideal of recursion relations of the 2-variable q-holonomic sequence $J_{T(2,3),n_1,n_2}(q)$; see [22].

Remark 1.2.2. *An alternative formula for the $\mathfrak{sl}_3$ colored Jones polynomial of $T(2,3)$ is given by Lawrence in [37]. Lawrence's formula is derived from the theory of Quantum Groups, and cannot generalize to the case of $T(2,n)$ torus knots. In contrast, the* *plethysm formula of Theorem 1.2.4 below can be generalized to a formula for $\psi_m(V_\lambda)$ which allows for an efficient formula of the $\mathfrak{sl}_3$ colored Jones polynomial of all torus knots..*

Remark 1.2.3. *Theorem 1.2.1 gives an efficient computation of the $\mathfrak{sl}_3$ colored Jones polynomial of the $3_1, 5_1, 7_1$ and 9_1 knots in the Rolfsen notation. In low weights, our answer agrees with the independent computation given by the entirely different methods of the* `KnotAtlas`*; see [6]. This is a consistency check which simultaneously validates the formulas of Theorem 1.2.1 and the data of the* `KnotAtlas`.

1.2.1 An $\mathfrak{sl}_3$ plethysm formula

As mentioned above, Theorem 1.2.1 follows from the Rosso-Jones formula for the colored Jones polynomial of torus knots and the following plethysm computation. Let ψ_m denote the m-plethysm operation.

Theorem 1.2.4. *For λ as in Equation (1) we have*

$$\begin{aligned}\psi_2(V_\lambda) &= \sum_{l=0}^{\min\{n_1,n_2\}} \sum_{k=0}^{n_1-l} (-1)^k V_{2\lambda-k\alpha_1-2l(\alpha_1+\alpha_2)} \\ &+ \sum_{l=0}^{\min\{n_1,n_2\}} \sum_{k=0}^{n_2-l} (-1)^k V_{2\lambda-k\alpha_2-2l(\alpha_1+\alpha_2)} \\ &- \sum_{l=0}^{\min\{n_1,n_2\}} V_{2\lambda-2l(\alpha_1+\alpha_2)}\end{aligned}$$

1.3 The Rosso-Jones formula

The polynomial invariant $J_{K,V}(q)$ of a knot K colored by the representation V of a simple Lie algebra is difficult to compute from its Quantum Group definition even when $K = 4_1$ and $\mathfrak{g} = \mathfrak{sl}_3$. Although it is a finite multi-dimensional sum, a practical computation seems out of reach. Fortunately, there is a class of knots whose quantum group invariant has a simple enough formula that allows us to extract its q-degree. This is the class of *torus knots* $T(m, n)$ where m, n are coprime natural numbers.

The simple formula is due to Rosso and Jones, and also studied by the second named author, [47, 43]. Let d_λ denote the *quantum dimension* of the representation V_λ and θ_λ is the eigenvalue of the *twist* operator on the representation V_λ. d_λ and θ_λ are given by

$$d_\lambda = \prod_{\alpha>0} \frac{[(\lambda+\rho,\alpha)]}{[(\rho,\alpha)]} \tag{7}$$

$$\theta_\lambda = q^{\frac{1}{2}(\lambda,\lambda+2\rho)} \tag{8}$$

where α belongs to the set of positive roots, $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is half the sum of positive roots and $(\cdot,\cdot)$ denotes the $\mathfrak{g}$ invariant inner product on the dual of the Cartan algebra (normalized so that the longest root has length $\sqrt{2}$). When $\mathfrak{g} = \mathfrak{sl}_3$ and λ is given by (1), then the quantum dimension and the twist parameter coincide with (3) and (4). For a natural number m, consider the *m-Adams operation* ψ_m on representations. It is given by (see [17, 40])

$$\psi_m(V_\lambda) = \sum_{\mu\in S_{\lambda,m}} c^\mu_{\lambda,m} V_\mu \tag{9}$$

where $c^\mu_{\lambda,m}$ are non-zero integers. The Rosso-Jones formula is the following (see [47]):

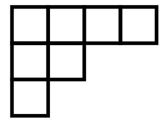
$$J_{T(m,n),\lambda}(q) = \frac{\theta_\lambda^{-mn}}{d_\lambda} \sum_{\mu\in S_{\lambda,m}} c^\mu_{\lambda,m} d_\mu \theta_\mu^{\frac{n}{m}} \tag{10}$$

For related discussion, see also [44].

1.4 Schur functions in $\mathfrak{sl}_3$

1.4.1 A review of Schur functions

Let us recall some well-known properties of Schur functions and their relation to the character of irreducible representations of $\mathfrak{sl}_N$, that can be found in [40, 17]. For a partition λ with parts $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$, let $s_{\lambda_1,\ldots,\lambda_k}(x_1,\ldots,x_N)$ denote the corresponding Schur function. A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ will be depicted as an arrangement of boxes as follows (for $\lambda = (4,2,1)$):



If ω_i denote the fundamental weights of $\mathfrak{sl}_N$ and n_i are nonnegative integers for $i = 1, \ldots, N-1$, and $\lambda = (\sum_{i=1}^{N-1} n_i, \sum_{i=2}^{N-1} n_i, \ldots, \sum_{i=N-1}^{N-1} n_i)$ then

$$\text{character}(V_{\sum_{i=1}^{N-1} n_i \omega_i}) = s_\lambda(x_1, \ldots, x_N) \tag{11}$$

For $\lambda = (4,2,1)$ we then have $(n_1, n_2, n_3) = (2,1,1)$.

The plethysm operation ψ_m is defined by

$$\psi_m(s_\lambda(x_1, \ldots, x_N)) = s_\lambda(x_1^m, \ldots, x_N^m)$$

Note that $s_1 = x_1 + \cdots + x_N$ and $\psi_2(s_1) = s_2 - s_{1,1}$.

In $\mathfrak{sl}_N$ the irreducible modules correspond to partitions λ with at most N parts. The decomposition of $\psi_m(V_\lambda)$ into irreducibles needed for the invariant of the (m,n) torus knot is given by the corresponding expansion of the symmetric function $\psi_m(s_\lambda)$ as a linear combination of Schur functions.

When $N = 3$ the Schur function s_λ vanishes where λ has more than 3 parts, and satisfies $s_{a,b,c} = s_{a+1,b+1,c+1}$. Then $s_{a,b,c} = s_{a-c,b-c}$, so we need only consider partitions with at most 2 parts. All the same, it will be convenient to use 3 parts in what follows.

1.4.2 A reformulation of Theorem 1.2.4

The goal of this section is to give a formula for $\psi_2(s_{m_1,m_2})$ as a linear combination of Schur functions, assuming that $N = 3$.

Definition 1.4.1. *For $m_1 \geq m_2 \geq 0$, let $D(m_1, m_2) \subset \mathbb{N}^3$ denote the set of tuples (a,b,c) that satisfy*

- $a + b + c = 2m_1 + 2m_2,\ 2m_1 \geq a \geq b \geq c \geq 0,\ a \geq 2m_2 \geq c$

- *if* $b \geq 2m_2$ *then* $c \equiv 0 \bmod 2$
- *if* $b \leq 2m_2$ *then* $a \equiv 0 \bmod 2$

Theorem 1.4.2. *In* $\mathfrak{sl}_3$ *for all* $m_1 > m_2$ *we have:*

$$\psi_2(s_{m_1,m_2}) = \sum_{(a,b,c)\in D(m_1,m_2)} (-1)^b s_{a,b,c}$$

It is interesting to note that the coefficient of every Schur function in the expansion of $\psi_2(s_{m_1,m_2})$ is $0, \pm 1$. The same feature proves to be the case for $\psi_2(s_{m_1,m_2})$ in the general case of $\mathfrak{sl}_N$, noted in Subsection 1.5.1.

1.4.3 Theorem 1.4.2 implies Theorem 1.2.4

Since $V^*_{n_1\omega_1+n_2\omega_2} = V_{n_2\omega_1+n_1\omega_2}$, and $J_{K,V^*}(q) = J_{K,V}(1/q)$, it suffices to prove Theorem 1.2.4 when $n_1 > n_2$. Equation (11) for $N = 3$ implies that

$$\text{character}(V_{n_1\omega_1+n_2\omega_2}) = s_{n_1+n_2,n_2}(x_1, x_2, x_3)$$

Fix nonnegative integers n_1 and n_2 and set $(m_1, m_2) = (n_1+n_2, n_2)$ in Theorem 1.4.2.

We can parametrise a tuple $(a, b, c) \in D(m_1, m_2)$ that satisfies $b \geq 2m_2$ by setting $b = 2m_2 + k,\ c = 2l$, to get $a = 2m_1 - k - 2l$, satisfying the inequalities $k, l \geq 0$, $k \leq m_1 - m_2 - l$, $l \leq m_2, m_1 - m_2$. Likewise, we can parametrize a tuple $(a, b, c) \in D(m_1, m_2)$ that satisfies $b \leq 2m_2$ by setting $b = 2m_2-k, a = 2m_1-2l$ to get $c = 2l+k$, satisfying $k, l \geq 0,\ k \leq m_2 - l,\ l \leq m_2, m_1 - m_2$. Thus Theorem 1.4.2 implies the formula of Theorem 1.2.4.

1.4.4 A reformulation of Theorem 1.4.2

To establish Theorem 1.4.2 we first prove Theorem 1.4.3.

Theorem 1.4.3. *For $m_1 > m_2$ we have*

$$\left(\sum_{(a,b,c)\in D(m_1,m_2)} (-1)^b s_{a,b,c}\right) \psi_2(s_1) = \sum_{(a',b',c')\in D(m_1+1,m_2)} (-1)^{b'} s_{a',b',c'} + \sum_{(a',b',c')\in D(m_1,m_2+1)} (-1)^{b'} s_{a',b',c'} + \sum_{\substack{(a',b',c')\in D(m_1-1,m_2-1)\\ m_2>0}} (-1)^{b'} s_{a',b',c'}.$$

In the proof of Theorem 1.4.2 we will need the following special cases of the *Littlewood-Richardson rule* adapted to $\mathfrak{sl}_3$, bearing in mind that Schur functions for partitions with more than 3 parts are 0 in this case; see [40]. In the next lemma and below, we will use the convention that $s_{a_1,a_2,a_3} = 0$ unless $a_1 \geq a_2 \geq a_3$. Furthermore, the notation $s_{a,b,c}|_{a>b}$ (resp. $s_{a,b,c}|_{a=b}$) means $s_{a,b,c}$ when $a > b$ (resp. $a = b$) and zero otherwise.

Lemma 1.4.4. *In $\mathfrak{sl}_3$ we have*

$$\begin{aligned}
s_{a,b,c}s_2 &= s_{a+2,b,c} + s_{a,b+2,c} + s_{a,b,c+2} + s_{a+1,b+1,c}|_{a>b} + s_{a+1,b,c+1} + s_{a,b+1,c+1}|_{b>c} \\
s_{a,b,c}s_{1,1} &= s_{a+1,b+1,c} + s_{a+1,b,c+1} + s_{a,b+1,c+1} \\
s_{m_1,m_2}s_1 &= s_{m_1+1,m_2} + s_{m_1,m_2+1} + s_{m_1,m_2,1}
\end{aligned}$$

Corollary 1.4.5. *For $a \geq b \geq c \geq 0$ we have*

$$s_{a,b,c}(s_2 - s_{1,1}) = s_{a+2,b,c} + s_{a,b+2,c} + s_{a,b,c+2} - s_{a+1,b+1,c}|_{a=b} - s_{a,b+1,c+1}|_{b=c}$$

Corollary 1.4.6. *Since ψ_2 is a ring homomorphism, and $\psi_2(s_1) = s_2 - s_{1,1}$, we have*

$$\begin{aligned}
\psi_2(s_{m_1,m_2})(s_2 - s_{1,1}) &= \psi_2(s_{m_1,m_2})\psi_2(s_1) = \psi_2(s_{m_1,m_2}s_1) \\
&= \begin{cases} \psi_2(s_{m_1+1,m_2}) + \psi_2(s_{m_1,m_2+1}) + \psi_2(s_{m_1,m_2,1}) & \text{if } m_1 > m_2 > 0, \\ \psi_2(s_{m_1+1,m_2}) + \psi_2(s_{m_1,m_2+1}) & \text{if } m_1 > m_2 = 0. \end{cases}
\end{aligned}$$

1.4.5 Theorem 1.4.3 implies Theorem 1.4.2

We deduce Theorem 1.4.2 from Theorem 1.4.3 by induction on m_2.

When $m_2 = 0$ we have $(a, b, c) \in D(m_1, 0)$ iff $c = 0,\ a + b = 2m_1,\ a \geq b \geq 0$. It is known (for example, [11, Eqn.2.30]) that

$$\psi_2(s_m) = \sum_{k=0}^{m} (-1)^k s_{2m-k,k}.$$

This establishes Theorem 1.4.2 for $m_2 = 0$.

Theorem 1.4.3 gives

$$\psi_2(s_{m_1,m_2})\psi_2(s_1) = \psi_2(s_{m_1+1,m_2}) + \sum_{(a',b',c')\in D(m_1,m_2+1)} (-1)^{b'} s_{a',b',c'} + \psi_2(s_{m_1-1,m_2-1})$$

by induction on m_2

Corollary 1.4.6 then shows that

$$\psi_2(s_{m_1,m_2+1}) = \sum_{(a',b',c')\in D(m_1,m_2+1)} (-1)^{b'} s_{a',b',c'},$$

which completes the induction step.

1.4.6 Proof of Theorem 1.4.3

To prove theorem 1.4.3 we sum both sides of the equation in Corollary 1.4.5 over $(a, b, c) \in D(m_1, m_2)$, using the following lemma.

Lemma 1.4.7. *Suppose that* $m_1 > m_2 \geq 0$. *Then*

$$\sum_{(a,b,c)\in D(m_1,m_2)} (-1)^b s_{a+2,b,c} = \sum_{\substack{(a',b',c')\in D(m_1+1,m_2)\\ a'\neq b',a'\neq 2m_2}} (-1)^{b'} s_{a',b',c'} \tag{12}$$

$$\sum_{(a,b,c)\in D(m_1,m_2)} (-1)^b s_{a,b+2,c} = \sum_{\substack{(a',b',c')\in D(m_1,m_2+1)\\ b'\neq c',c'\neq 2m_2+2}} (-1)^{b'} s_{a',b',c'} + \sum_{\substack{(a',b',c')\in D(m_1+1,m_2)\\ a'=2m_2,b'\neq c'}} (-1)^{b'} s_{a',b',c'} \tag{13}$$

$$\sum_{(a,b,c)\in D(m_1,m_2)} (-1)^b s_{a,b,c+2} = \sum_{\substack{(a',b',c')\in D(m_1-1,m_2-1)\\ m_2>0}} (-1)^{b'} s_{a',b',c'} + \sum_{\substack{(a',b',c')\in D(m_1,m_2+1)\\ c'=2m_2+2}} (-1)^{b'} s_{a',b',c'} \tag{14}$$

$$\sum_{\substack{(a,b,c)\in D(m_1,m_2)\\ a=b}} (-1)^{b+1} s_{a+1,b+1,c} = \sum_{\substack{(a',b',c')\in D(m_1+1,m_2)\\ a'=b',a'\neq 2m_2,b'\neq c'}} (-1)^{b'} s_{a',b',c'} \tag{15}$$

$$\sum_{\substack{(a,b,c)\in D(m_1,m_2)\\ b=c}} (-1)^{b+1} s_{a,b+1,c+1} = \sum_{\substack{(a',b',c')\in D(m_1,m_2+1)\\ b'=c',c'\neq 2m_2+2}} (-1)^{b'} s_{a',b',c'} + \sum_{\substack{(a',b',c')\in D(m_1+1,m_2)\\ a'=2m_2,b'=c'}} (-1)^{b'} s_{a',b',c'} \tag{16}$$

The total sum of the left hand sides of the equations in Lemma 1.4.7 is then the left hand side of the equation in theorem 1.4.3, while the terms on the right hand sides make up the right hand side of Theorem 1.4.3.

1.4.7 Proof of Lemma 1.4.7

For each of the five equations we provide a bijective transformation carrying $(a, b, c) \in D(m_1, m_2)$ with the restrictions shown to (a', b', c') satisfying the conditions on the right hand sides.

We make repeated use of the parity rules to ensure that inequalities force a difference of at least 2. With the exception of a couple of less obvious cases we omit proofs that the individual parity rules for (a', b', c') are satisfied, as they generally follow readily from those for (a, b, c) and vice versa. Equally the sum $a' + b' + c'$ is always obviously correct.

Proof. For Equation (12), put $a' = a + 2, b' = b, c' = c$. Let $(a, b, c) \in D(m_1, m_2)$. Then $2m_2 + 2 \geq a' > b' \geq c' \geq 0$, and $a' > 2m_2 \geq c'$. Then $(a', b', c') \in D(m_1 + 1, m_2)$, with $a' \neq b'$ and $a' \neq 2m_2$.

Conversely suppose that $(a', b', c') \in D(m_1+1, m_2)$, with $a' > b'$ and $a' > 2m_2$. By the parity rules, if $b' \leq 2m_2$ then $a' \equiv 0 \bmod 2$, so $a' \geq 2m_2 + 2 \geq b' + 2$. If $b' > 2m_2$ then $a' \equiv b' \bmod 2$, so $a' \geq b' + 2 > 2m_2 + 2$. In any case $2m_1 \geq a' - 2 \geq b' \geq c' \geq 0$, and $a' - 2 \geq 2m_2 \geq c'$. Then $(a, b, c) \in D(m_1, m_2)$. This proves Equation (12).

For Equation (13), put $a' = a, b' = b + 2, c' = c$. Let $(a, b, c) \in D(m_1, m_2)$ with $a \geq b + 2$. If $a = 2m_2$ then $2m_1 + 2 \geq a' \geq b' > c' \geq 0$ and $a' \geq 2m_2 \geq c'$, Then $(a', b', c') \in D(m_1 + 1, m_2)$, with $a' = 2m_2, b' > c'$. Otherwise $a > 2m_2$. If $b \geq 2m_2$ then $a \geq b + 2 \geq 2m_2 + 2$, while if $b < 2m_2$ then $a \equiv 0 \bmod 2$ by the parity rules, so that $a \geq 2m_2 + 2$. Hence $2m_1 \geq a' \geq b' > c' \geq 0$ and $a' \geq 2m_2 + 2 \geq c'$. In this case we check the parity rules explicitly. Here $b' \geq 2m_2 + 2 \implies b \geq 2m_2 \implies c' \equiv c \equiv 0 \bmod 2$ and $b' \leq 2m_2 + 2 \implies b \leq 2m_2 \implies a' \equiv a \equiv 0 \bmod 2$. So $(a', b', c') \in D(m_1, m_2 + 1)$ with $b' > c'$ and $c' < 2m_2 + 2$.

Conversely suppose that $(a', b', c') \in D(m_1, m_2+1)$ with $b' > c'$ and $c' < 2m_2+2$. If $b' \geq 2m_2 + 2$ then $c' \equiv 0 \bmod 2$ so $c' \leq 2m_2 \leq b' - 2$ and if $b' < 2m_2 + 2$ then $b' \equiv c' \bmod 2$ and $c' \leq b' - 2 < 2m_2$. Hence $2m_1 \geq a' \geq b' - 2 \geq c' \geq 0$ and $a' > 2m_2$. A parity check as above shows that then $(a, b, c) \in D(m_1, m_2)$ with $a = a' \geq b' = b+2$ and $a > 2m_2$.

Finally suppose that $(a', b', c') \in D(m_1 + 1, m_2)$, with $a' = 2m_2, b' > c'$. Then $b' \equiv c' \bmod 2$ so $b' - 2 \geq c'$, and $a' = 2m_2 \geq c'$ again giving $(a, b, c) \in D(m_1, m_2)$ with $a = 2m_2 \geq b + 2$. This proves Equation (13).

For Equation (14), put $a' = a, b' = b, c' = c+2$ when $c = 2m_2$, and $a' = a-2, b' = b-2, c' = c$ otherwise. In either case $s_{a,b,c+2} = s_{a',b',c'}$ since we are working in $\mathfrak{sl}_3$. Let $(a, b, c) \in D(m_1, m_2)$ with $b \geq c+2$. If $c = 2m_2$ then $2m_1 \geq a' \geq b' \geq 2m_2+2 = c' \geq 0$, and $(a', b', c') \in D(m_1, m_2+1)$ with $c' = 2m_2+2$. Otherwise $c < 2m_2 \neq 0$. If $b \leq 2m_2$ then $c \leq 2m_2 - 2$. If $b > 2m_2$ then $c \equiv 0 \bmod 2$ by the parity rules, giving again

$c \leq 2m_2 - 2$. Then $2m_1 - 2 \geq a - 2 \geq b - 2 \geq c \geq 0$ and $a - 2 \geq 2m_2 - 2 \geq c$. So $(a', b', c') \in D(m_1 - 1, m_2 - 1)$.

Conversely let $(a', b', c') \in D(m_1 - 1, m_2 - 1)$, with $m_2 \neq 0$. Then $2m_1 \geq a' + 2 \geq b' + 2 \geq c' \geq 0$ and $a' \geq 2m_2 - 2 \geq c'$, so $a' + 2 \geq 2m_2 > c'$. Hence $(a, b, c) \in D(m_1, m_2)$ with $c \neq 2m_2$.

Finally, suppose that $(a', b', c') \in D(m_1, m_2 + 1)$ with $c' = 2m_2 + 2$. Then $2m_1 \geq a' \geq b' \geq 2m_2 = c' - 2 \geq 0$ so that $(a', b', c' - 2) = (a, b, c) \in D(m_1, m_2)$ with $c = 2m_2$. This proves Equation (14).

For Equation (15), put $a' = a + 1, b' = b + 1, c' = c$. Let $(a, b, c) \in D(m_1, m_2)$ with $a = b$. Then $2m_1 + 2 \geq a' \geq b' \geq c' \geq 0$ and $a' > a \geq 2m_2 \geq c' \geq 0$. Since $b' = a' > 2m_2$ and $c' \equiv 0 \bmod 2$ the parity rules are satisfied, and $(a', b', c') \in D(m_1 + 1, m_2)$ with $a' = b', a' > 2m_2, b' \neq c'$.

Conversely let $(a', b', c') \in D(m_1 + 1, m_2)$ with $a' = b', a' > 2m_2, b' \neq c'$. Now $2a' \leq a' + b' + c' = 2m_1 + 2m_2 + 2 \leq 4m_1$, since $m_2 < m_1$. Then $2m_1 > a' - 1 \geq b' - 1 \geq c' \geq 0$ and $a' - 1 \geq 2m_2 \geq c'$. Hence $(a, b, c) \in D(m_1, m_2)$ with $a = b$. This proves Equation (14).

For Equation (16), put $a' = a, b' = b + 1, c' = c + 1$. Let $(a, b, c) \in D(m_1, m_2)$ with $a > b = c$. If $a = 2m_2$ then $2m_1 + 2 > a' \geq b' \geq c' \geq 0$ and $a' = 2m_2 \geq c'$. Hence $(a', b', c') \in D(m_1 + 1, m_2)$ with $a' = 2m_2, b' = c'$. Otherwise $a > 2m_2$, and $a' = a \geq 2m_2 + 2$, since $b = c$, while $2m_2 + 2 \geq c + 2 > c'$. We have also $2m_1 \geq a' \geq b' \geq c' \geq 0$. Hence $(a', b', c') \in D(m_1, m_2 + 1)$ with $b' = c', c' \neq 2m_2 + 2$.

Conversely suppose that $(a', b', c') \in D(m_1, m_2 + 1)$ with $b' = c', c' < 2m_2 + 2$. Now $a' + 2c' = 2m_1 + 2m_2 + 2$ and $a' \leq 2m_1$, so $c' > 0$. Hence $2m_1 \geq a' > b' - 1 \geq c' - 1 \geq 0$ and $a' > 2m_2 \geq c' - 1$. Then $(a, b, c) \in D(m_1, m_2)$ with $a > b = c$ and $a > 2m_2$.

Finally if $(a', b', c') \in D(m_1 + 1, m_2)$ with $a' = 2m_2, b' = c'$ then $b' = c' = m_1 + 1 > 0$

and $(a, b, c) \in D(m_1, m_2)$ with $a = 2m_2 > b = c$. □

1.5 A proof of Theorem 1.4.2 using Carini-Remmel's work

1.5.1 A review of Theorem 5 of [10]

In this section we give an alternative proof of Theorem 1.4.2 using the work [10] of Carini and Remmel. In Theorem 5 of loc.cit., Carini and Remmel give the expansion of the plethysm $\psi_2(s_{a,b})$ for the Schur function of a 2-row partition of $n = a + b$ in terms of Schur functions s_λ, where λ runs through partitions of $2n$ with at most 4 parts. In this expansion each s_λ has coefficient $0, \pm 1$, depending on the parities of the parts of λ and some linear inequalities.

In their paper they use the opposite convention to Macdonald, so that they take $0 \le a \le b$ for the given partition of $n = a+b$ and $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$ for the parts of the partition λ of $2n$. They also use the more common combinatorial notation p_2 rather than ψ_2.

Theorem 5 of [10] can be readily restated as follows, by grouping separately the partitions λ of $2a+2b$ with $\lambda_1+\lambda_3 \ge 2a$ and those with $\lambda_1+\lambda_3 < 2a$ in the expansion of $\psi_2(s_{a,b})$:

- When $\lambda_1 + \lambda_3 \ge 2a$, $\lambda_1 + \lambda_2$ is even and $\lambda_1 + \lambda_2 \le 2a$, the Schur function s_λ has coefficient $(-1)^{\lambda_2+\lambda_3}$.
- When $\lambda_1 + \lambda_3 < 2a$, $\lambda_2 + \lambda_3$ is even, $2a \le \lambda_2 + \lambda_3$ and $2a \le \lambda_1 + \lambda_4$, the Schur function s_λ has coefficient $(-1)^{\lambda_1+\lambda_2}$.
- All other s_λ have coefficient 0.

The first of these cases corresponds to the partitions in (ii) and some of (i) in [10, Thm.5], while the second corresponds to the partitions in (iii) and the remaining partitions in (i).

1.5.2 Reformulation of Carini and Remmel's expansion of $\psi_2(s_{m_1,m_2})$

Theorem 5 of [10] gives rise to an expansion of $\psi_2(s_{m_1,m_2})$, $m_1 \geq m_2$, in Schur functions of $x_1, \dots, x_N$ which is valid for all N.

We can reformulate this further by specifying the support set for the partitions which appear in the expansion in terms of linear inequalities and some parity rules, so that Theorem 1.4.2, the case where $N = 3$, is an immediate corollary.

Using Macdonald's ordering, we take m_1 in place of b and m_2 in place of a from [10], and write $(\lambda_4, \lambda_3, \lambda_2, \lambda_1) = (a, b, c, d) = \lambda$.

Definition 1.5.1. *For* $m_1,\ m_2 \in \mathbb{N}$*, let* $A(m_1, m_2) \subset \mathbb{N}^4$ *denote the set of tuples* (a, b, c, d) *that satisfy*

- $a + b + c + d = 2m_1 + 2m_2,\ a \geq b \geq c \geq d \geq 0,\ 2m_1 \geq a + d \geq 2m_2 \geq c + d$
- *if* $b + d \geq 2m_2$ *then* $c \equiv d \bmod 2$
- *if* $b + d \leq 2m_2$ *then* $a \equiv d \bmod 2$

Theorem 1.5.2. *Let* $m_1 \geq m_2 \geq 0$*. Then*

$$\psi_2(s_{m_1,m_2}) = \sum_{(a,b,c,d) \in A(m_1,m_2)} (-1)^{b+d} s_{a,b,c,d}.$$

Theorem 1.4.2 is an immediate corollary, since Schur functions for partitions with more than 3 rows are 0 in $\mathfrak{sl}_3$, and the support set $A(m_1, m_2)$ becomes $D(m_1, m_2)$ when $d = 0$.

We can see readily that theorem 1.5.2 follows from Theorem 5 of [10] as rearranged above.

Firstly, for $\lambda \in A(m_1, m_2)$ with $b + d \geq 2m_2$ we have $c + d$ even, by the parity rule, and $c + d \leq 2m_2$, while the coefficient of s_λ is $(-1)^{b+d} = (-1)^{b+c}$. This agrees with the first group of partitions above. The condition $2m_1 \geq a + d$ does not impose any extra restriction on this group, since it is equivalent to $b + c \geq 2m_2$.

For $\lambda \in A(m_1, m_2)$ with $b+d \leq 2m_2$ we have $a+d$ even, and hence $b+c$ even, by the parity rule. In addition we have $2m_2 \leq b+c$ since $2m_1 \geq a+d$, and $2m_2 \leq a+d$. Again this agrees with the second group of partitions above, and the coefficient of s_λ is $(-1)^{b+d} = (-1)^{c+d}$ as required there.

1.5.3 Parametrisation

Theorem 1.5.2 can be used to give a parametrisation of these two sets of Schur functions with non-zero coefficient, each in terms of 3 integer parameters satisfying some linear inequalities. These in turn give a parametric formula for $\psi_2(s_{m_1,m_2})$, with a reduction in the case of $\mathfrak{sl}_3$ to the formulae of Theorem 1.2.4.

1.5.3.1 The first group of Schur functions

Parametrise $\{A(m_1, m_2) : b+d \geq 2m_2\}$ by setting $b+d = 2m_2+k, k \geq 0$. Write $c = d+2l, l \geq 0$ to get $c \equiv d \bmod 2$. The condition $c+d \leq 2m_2$ is equivalent to $d+l \leq m_2$. This ensures that $c \leq b$. Then $a = 2m_1 - k - 2l - d$, which satisfies $2m_1 \geq a+d$. To ensure that $a \geq b$ we impose the condition $a+d = 2m_1 - k - 2l \geq b+d = 2m_2 + k$ to finish with parameters $k, l, d \geq 0, d+l \leq m_2, k+l \leq m_1 - m_2$.

The contribution of the partitions λ with $b+d \geq 2m_2$ is then

$$\sum (-1)^k s_\lambda, \text{ where } \lambda = (2m_1 - k - 2l - d, 2m_2 + k - d, 2l + d, d)$$

and k, l, d are integer parameters with $k, l, d \geq 0, d+l \leq m_2, k+l \leq m_1 - m_2$.

1.5.3.2 The second group of Schur functions

Parametrise $\{A(m_1, m_2) : b+d \leq 2m_2\}$ by setting $b+d = 2m_2 - k, k \geq 0$. Write $a+d = 2m_1 - 2l, l \geq 0$ to get $a \equiv d \bmod 2$ and $2m_1 \geq a+d$. Then $b+c = 2m_2+2l$, so $c \geq d$. The condition $2m_2 \leq a+d$ is equivalent to $l \leq m_1 - m_2$. This ensures that $b \leq a$.

Now $b = 2m_2 - k - d$ so $c = 2l + k + d$ so $c \leq b$ is equivalent to $l + k + d \leq m_2$.

The contribution of the partitions λ with $b+d \leq 2m_2$ is

$$\sum(-1)^k s_\lambda, \text{ where } \lambda = (2m_1 - 2l - d, 2m_2 - k - d, 2l + k + d, d)$$

and k, l, d are integer parameters with $k, l, d \geq 0, l + k + d \leq m_2, l \leq m_1 - m_2$.

1.5.4 Reduction to the case of $\mathfrak{sl}_3$.

In the special case of $\mathfrak{sl}_3$ we have $d = 0$, and we get two double sums of 3-row Schur functions, one for partitions with $b \geq 2m_2$, and one for those with $b < 2m_2$, to avoid double counting those with $b = 2m_2$. Since we are working in $\mathfrak{sl}_3$ this can be reduced further to sums over 2-row partitions, since $s_{a,b,c} = s_{a-c,b-c}$

Explicitly we have from the first group of partitions the sum

$$\sum(-1)^k s_{2m_1-4l-k, 2m_2-2l+k}$$

taken over $k, l \geq 0, l \leq m_2, k + l \leq m_1 - m_2$. The second group yields

$$\sum(-1)^k s_{2m_1-4l-k, 2m_2-2l-2k}$$

taken over $l \geq 0, k > 0, k + l \leq m_2, l \leq m_1 - m_2$. This gives a second proof of Theorem 1.2.4. It may be preferable all the same to retain the 3-row format when estimating the effects of twists in $\mathfrak{sl}_3$ as then all the partitions have $2m_1 + 2m_2$ cells and thus their twist factors depend only on the total content of the partition.

1.6 *Sample computations*

In this section we give some sample computations of Theorems 1.2.1 and 1.2.4. Theorem 1.2.1 implies that:

$J_{T(2,3),5,7}(1/q) = q^{24} + q^{30} + q^{32} - q^{35} + q^{36} + 2q^{38} - q^{39} - q^{41} + q^{42} - q^{43} + 2q^{44} - q^{45} - 2q^{47} + q^{48} - q^{49} + 2q^{50} - 2q^{51} + q^{52} - 2q^{53} - 2q^{55} + 3q^{56} - 2q^{57} + 2q^{58} - 2q^{59} - q^{60} - q^{61} + 2q^{62} - 4q^{63} + 3q^{64} + q^{66} - q^{67} + q^{68} - 3q^{69} + 3q^{70} - 2q^{71} + 3q^{72} + q^{73} - q^{74} - q^{75} - 2q^{77} + 2q^{78} + q^{79} + 2q^{80} - 2q^{82} - q^{83} + q^{85} + 2q^{86} - 3q^{88} + q^{89} - 2q^{90} - q^{92} + 2q^{93} + q^{94} + 2q^{95} - 3q^{96} + q^{97} - 2q^{98} + q^{99} + q^{100} +$

$2q^{101}-2q^{102}+3q^{103}-5q^{104}-q^{106}+3q^{107}+2q^{108}+4q^{109}-4q^{110}+3q^{111}-3q^{112}-2q^{113}+q^{114}+q^{115}-q^{116}+5q^{117}-5q^{118}-2q^{119}-2q^{121}+2q^{122}+5q^{123}-2q^{124}+q^{125}-q^{126}-4q^{127}-q^{129}-q^{130}+4q^{131}-q^{132}-2q^{133}+2q^{134}-q^{135}+q^{136}+q^{137}-2q^{138}+2q^{139}+3q^{140}-3q^{141}+2q^{142}-2q^{143}-4q^{144}+2q^{145}+6q^{148}-2q^{149}-q^{151}-6q^{152}+3q^{153}+4q^{154}-q^{155}+3q^{156}-4q^{157}-4q^{158}+3q^{159}-3q^{160}+2q^{161}+4q^{162}-3q^{163}+4q^{164}-2q^{165}-4q^{166}+5q^{167}+2q^{170}-6q^{171}+2q^{172}+3q^{173}-4q^{174}+q^{175}+q^{176}-3q^{177}+5q^{178}-2q^{179}-2q^{180}+4q^{181}-2q^{183}-q^{184}-6q^{185}+3q^{186}+2q^{187}+2q^{189}+q^{190}-5q^{191}+2q^{192}-q^{193}-q^{194}+5q^{195}+2q^{196}-q^{197}-q^{198}-5q^{199}+3q^{201}-2q^{202}+q^{203}+3q^{204}-2q^{205}+q^{206}-5q^{208}+4q^{209}+2q^{210}-3q^{213}-3q^{214}+4q^{215}-2q^{216}+2q^{217}+3q^{218}-2q^{219}-4q^{222}+5q^{223}+2q^{224}-2q^{225}-3q^{227}-3q^{228}+3q^{229}-q^{230}+3q^{232}-2q^{233}+q^{234}+2q^{235}-3q^{236}+q^{237}+q^{238}-2q^{239}+3q^{240}-q^{241}-q^{242}+2q^{243}-4q^{244}-2q^{245}+2q^{246}+4q^{248}+2q^{249}-3q^{250}-2q^{252}-2q^{253}+3q^{254}+2q^{256}+2q^{257}-3q^{258}-3q^{259}-2q^{260}+q^{261}+4q^{262}+q^{263}+q^{264}-q^{265}-3q^{266}-2q^{267}+q^{268}+q^{269}+2q^{270}+q^{271}-q^{272}-q^{273}-q^{274}+q^{275}$

Theorem 1.2.4 implies that:

$\psi_2(V_{5,7}) = V_{0,4} - V_{0,7} + V_{0,10} - V_{0,13} + V_{0,16} - V_{0,19} - V_{1,2} + V_{2,0} + V_{2,6} - V_{2,9} + V_{2,12} - V_{2,15} + V_{2,18} - V_{3,4} + V_{4,2} + V_{4,8} - V_{4,11} + V_{4,14} - V_{4,17} - V_{5,0} - V_{5,6} + V_{6,4} + V_{6,10} - V_{6,13} + V_{6,16} - V_{7,2} - V_{7,8} + V_{8,0} + V_{8,6} + V_{8,12} - V_{8,15} - V_{9,4} - V_{9,10} + V_{10,2} + V_{10,8} + V_{10,14} - V_{11,0} - V_{11,6} - V_{11,12} + V_{12,4} + V_{12,10} - V_{13,2} - V_{13,8} + V_{14,0} + V_{14,6} - V_{15,4} + V_{16,2} - V_{17,0}$ where $V_{n_1,n_2} = V_{n_1\omega_1+n_2\omega_2}$.

For future checks with other formulas, Theorem 1.2.1 implies that $J_{2,3,70,70}(1/q)$ is a polynomial of q with exponents with respect to q in the interval $[280, 30100]$ (where the end points are attained), leading and trailing coefficients 1 and coefficients in the interval $[-55196, 65594]$, where the coefficient -55196 is attained at precisely at q^{18854} and q^{18925} and the coefficient 65594 is attained precisely at q^{18165}. In other words, we have

$$J_{2,3,70,70}(1/q) = q^{280}+\cdots+65594q^{18165}+\cdots-55196q^{18854}+\cdots-55196q^{18925}+\cdots+q^{30100}$$

Using Theorem 1.2.1 it is possible to compute the colored Jones polynomials $J_{T(2,3),n_1,n_2}(q)$ for $n_1, n_2 = 0, \ldots, 100$.

CHAPTER II

A STABILITY CONJECTURE FOR THE COLORED JONES POLYNOMIAL

2.1 Introduction

2.1.1 The degree and coefficients of a q-holonomic sequence

Our goal in this chapter is to formulate a stability conjecture for the coefficients of q-holonomic sequences that appear naturally in Quantum Knot Theory [24]. Our conjecture is motivated by

(a) a structure theorem for the degree and coefficients of a q-holonomic sequence of polynomials given in [20],

(b) a stability theorem of the colored Jones polynomial of an alternating knot [25].

To discuss our first motivation, recall [56] that a sequence $(f_n(q))$ is *q-holonomic* if it satisfies a linear recursion

$$\sum_{j=0}^{d} c_j(q^n, q) f_{n+j}(q) = 0$$

for all n where $c_j(u,v) \in \mathbb{Z}[u,v]$ and $c_d \neq 0$. Here, $f_n(q)$ is either in $\mathbb{Z}[q^{\pm 1}]$, the ring of Laurent polynomials with integer coefficients, or more generally in $\mathbb{Q}(q)$, the field of rational functions with rational coefficients or even $\mathbb{Z}((q))$, the ring of Laurent power series in q $\sum_{j \in \mathbb{Z}} a_j q^j$ (with a_j integers, vanishing when j is small enough). $\mathbb{Z}((q))$ has a subring $\mathbb{Z}[[q]]$ of formal power series in q, where $a_j = 0$ for $j < 0$. The *degree* $\delta^*(f(q))$ of $f(q) \in \mathbb{Z}((q))$ is the smallest integer m such that $q^m f(q) \in \mathbb{Z}[[q]]$.

Thus, we can expand every non-zero sequence $(f_n(q))$ in the form

$$f_n(q) = a_0(n) q^{\delta^*(n)} + a_1(n) q^{\delta^*(n)+1} + a_2(n) q^{\delta^*(n)+2} + \dots \tag{17}$$

where $\delta^*(n)$ is the degree of $f_n(q)$ and $a_k(n)$ is the k-th coefficient of $q^{-\delta^*(n)} f_n(q)$, reading from the left. We will often call $a_k(n)$ the k-th *stable coefficient* of the sequence $(f_n(q))$.

In [20] it was proven that if $(f_n(q))$ is q-holonomic, then

- $\delta^*(n)$ is a *quadratic quasi-polynomial* for all but finitely many values of n,
- for every $k \in \mathbb{N}$, $a_k(n)$ is *recurrent* for all but finitely many values of n.

Recall that a quasi-polynomial (of degree at most d) is a function of the form

$$p : \mathbb{N} \longrightarrow \mathbb{Z}, \qquad n \mapsto p(n) = \sum_{j=0}^{d} c_j(n) n^j$$

where $c_j : \mathbb{N} \longrightarrow \mathbb{Q}$ are periodic functions. Let $\mathcal{P}$ denote the ring of integer-valued quasi-polynomials. A recurrent sequence is one that satisfies a linear recursion with constant coefficients. Recurrent sequences are well-known in number theory under the name of *Generalized Exponential Sums* [54, 16]. The latter are expressions of the form

$$a(n) = \sum_{i=1}^{m} A_i(n) \alpha_i^n$$

with *roots* α_i, $1 \leq i \leq m$ distinct algebraic numbers and polynomials A_i. Integer-valued generalized exponential sums form a ring $\mathcal{E}$, which contains a subring $\mathcal{P}$ that consists of integer-valued exponential sums whose roots are complex roots of unity.

2.1.2 Stability of the colored Jones polynomial of an alternating link

The second motivation of our Conjecture 2.1.5 below comes from the stability theorem of [25] that concerns the colored Jones polynomial of an alternating link. Recall the notion of convergence in the completed ring $\mathbb{Z}((q)) = \varprojlim_n \mathbb{Z}[q^{\pm 1}]/(q^n)$. Given $f_n(q), f(q) \in \mathbb{Z}((q))$, we write that

$$\lim_{n\to\infty} f_n(q) = f(q)$$

if there exists C such that $\delta^*(f_n(q)) > C$ for all n, and for every $m \in \mathbb{N}$ there exists $N_m \in \mathbb{N}$ such that

$$f_n(q) - f(q) \in q^m \mathbb{Z}[[q]]\,.$$

The next definition of stability appears in [24] and the notion of its tail is inspired by Dasbach-Lin [12].

Definition 2.1.1. *We say that a sequence $f_n(q) \in \mathbb{Z}[[q]]$ is* stable *if there exists a series $F(x,q) = \sum_{k=0}^{\infty} \Phi_k(q) x^k \in \mathbb{Z}((q))[[x]]$ such that for every $k \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} q^{-k(n+1)} \left(f_n(q) - \sum_{j=0}^{k} \Phi_j(q) q^{j(n+1)} \right) = 0\,. \tag{18}$$

*We will call $F(x,q)$ the (x,q)-*tail *(in short, the tail) of the sequence $(f_n(q))$.*

Examples of stable sequences are the shifted colored Jones polynomials of an alternating link. Let $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1/2}]$ denote the colored Jones polynomial of a link K colored by the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ (see [52, 53]). Let $\delta^*_K(n)$ and $a_{K,0}(n)$ denote the degree and the 0-th stable coefficient of $J_{K,n}(q)$. It is well-known known that $a_{K,0}(n) = (-1)^{c_- n}$ where c_- is the number of negative crossings of K [39].

Theorem 2.1.2. *[25] If K is an alternating link, then the sequence $a_{K,0}(-n) q^{-\delta^*_K(n)} J_{K,n}(q) \in \mathbb{Z}[q]$ is stable.*

2.1.3 c-stability

We are now ready to introduce the notion of c-stability.

Definition 2.1.3. *We say that a sequence $f_n(q) \in \mathbb{Z}((q))$ with q-degree $\delta^*(n)$ is c-*stable *(i.e.,* cyclotomically stable*) if there exists a series $F(n,x,q) = \sum_{k=0}^{\infty} \Phi_k(n,q) x^k \in \mathcal{P}((q))[[x]]$ such that for every $k \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} q^{-k(n+1)} \left(q^{-\delta^*(n)} f_n(q) - \sum_{j=0}^{k} \Phi_j(n,q) q^{j(n+1)} \right) = 0\,. \tag{19}$$

We will call $F(n,x,q)$ the (n,x,q)-tail (in short, tail) of the sequence $(f_n(q))$.

Remark 2.1.4. *The stable coefficients of a c-stable sequence* $(f_n(q))$ *are quasi-polynomials. I.e., with the notation of Equation* (17)*, we have that* $a_k \in \mathcal{P}$ *for all* k*. In fact, if* $(f_n(q))$ *is c-stable and* $l \in \mathbb{N}$*, the stable coefficients of the sequence*

$$q^{-l(n+1)} \left(f_n(q) - \sum_{j=0}^{l-1} \Phi_j(q) q^{j(n+1)} \right)$$

are quasi-polynomials.

2.1.4 Our results

For a knot K in S^3, colored by an irreducible representation V_λ of a simple Lie algebra $\mathfrak{g}$ with highest weight λ, one can define the colored Jones polynomial $J^{\mathfrak{g}}_{K,V_\lambda}(q) \in \mathbb{Z}[q^{\pm 1}]$ [52, 53]. This requires a rescalled definition of q, which depends only on the Lie algebra and not on the knot, and is discussed carefully in [38]. In [24] it was shown that for every knot K and every simple Lie algebra $\mathfrak{g}$, the function $\lambda \mapsto J^{\mathfrak{g}}_{K,V_\lambda}(q)$ (and consequently the sequence $(J^{\mathfrak{g}}_{K,n\lambda}(q))$) is q-holonomic.

Conjecture 2.1.5. *Fix a knot* K*, a simple Lie algebra* $\mathfrak{g}$ *and a dominant weight* λ *of* $\mathfrak{g}$*. Then the sequence* $(J^{\mathfrak{g}}_{K,n\lambda}(q))$ *of colored Jones polynomials is c-stable.*

Theorem 2.1.6. *Conjecture 2.1.5 holds for all torus knots and all rank* 2 *simple Lie algebras.*

For a precise statement and for a computation of the tail, see Theorem 2.6.2.

Remark 2.1.7. *Theorem 2.1.2 implies that if* K *is an alternating knot with* c_- *crossings and* $k \in \mathbb{N}$*, the* k*-th stable coefficient* $a_{K,k}(n)$ *of the sequence* $(J_{K,n}(q))$ *is given by*

$$a_{K,k}(n) = (-1)^{c_- n} \text{coeff}(\Phi_{K,0}(q), q^k)$$

and satisfies the first order linear recurrence relation

$$a_{K,k}(n+1) - (-1)^{c_-} a_{K,k}(n) = 0.$$

Here $\mathrm{coeff}(f(q), q^k)$ *denotes the coefficient of* q^k *in* $f(q) \in \mathbb{Z}((q))$*. The stable coefficients* $c_{K,k}$ *of an alternating knot* K *are studied in [27, 26]. In all examples of the colored Jones polynomial of a knot that have been analyzed (this includes alternating knots, torus knots and the 2-fusion knots), the* k*-stable coefficient is a quasi-polynomial of degree* 0*, i.e., it is constant on suitable arithmetic progressions. One might think that this holds for all simple Lie algebras. Example 2.1.10 below shows that this is not the case, hence the notion of c-stability is necessary.*

2.1.5 A sample of q-series

In this section we give a concrete sample of tails and q-series that appear in our study.

Example 2.1.8. *Consider the theta series given by [9]*

$$\theta_{b,c}(q) = \sum_{s \in \mathbb{Z}} (-1)^s q^{\frac{b}{2}s^2 + cs} \tag{20}$$

In Section 2.9 we will prove the following.

Theorem 2.1.9. *The tail of the c-stable sequence* $(J^{\mathfrak{g}}_{T(2,b),n\lambda_1}(q))$ *for* $b > 2$ *odd is given by*

$$\frac{\theta_{b,\frac{b}{2}-1}(q)(1+q^3x^2) + q^3\theta_{b,\frac{b}{2}+2}(q)x}{(1-q)(1-qx)(1-q^2x)}$$

In particular, for the trefoil, i.e., $b = 3$ *the tail equals to*

$$(q)_\infty \frac{1 - qx + q^3x^2}{(1-q)(1-qx)(1-q^2x)}$$

Example 2.1.10. *The tail of the c-stable sequence* $(J^{A_2}_{T(4,5),n\rho}(q))$ *is given by*

$$\frac{1}{(1-xq)^2(1-x^2q^2)}\left(A_0(q) + nA_1(q)\right)$$

where $A_0(q), A_1(q) \in \mathbb{Z}[[q]]$ *are given explicitly in Proposition 2.9.3. The first few* *terms of those q-series are given by*

$$A_0(q) = 1 - 2q + 2q^3 - q^4 + q^{48} - 2q^{55} - 2q^{57} + 2q^{63} + 2q^{66} + 2q^{69} + 2q^{75} - q^{76} - 2q^{78} - 2q^{81} - 2q^{82} - q^{84} - 2q^{85} + 2q^{87} + \dots$$

$$A_1(q) = 1 - 2q + 2q^3 - q^4 - q^6 + 2q^9 + 2q^{10} - 2q^{12} - 4q^{15} - q^{18} + 2q^{19} + 2q^{21} + 3q^{22} - 2q^{27} - 2q^{30} - 2q^{33} + 4q^{36} - q^{42} - 2q^{46} + q^{48} - 2q^{49} + 2q^{51} + 2q^{55} + 4q^{57} - 2q^{58} + 2q^{60} - 4q^{64} - 2q^{66} - 2q^{69} - 2q^{73} - q^{76} + 4q^{78} + 2q^{81} + 2q^{82} + q^{84} + 2q^{85} - 2q^{87} + \dots$$

It follows that for every fixed k, the k-th stable coefficient $a_k(n)$ of $(J^{A_2}_{T(4,5),n\rho}(q))$ satisfies the linear recursion relation

$$a_k(n+2) - 2a_k(n+1) + a_k(n) = 0$$

for all n.

We leave as an exercise to the reader to show that

$$\begin{aligned} A_1(q) &= \sum_{m_1,m_2\in\mathbb{Z}} q^{20(m_1^2+3m_1m_2+3m_2^2)+2m_1+3m_2}(1-q^{4m_1+1})(1-q^{4m_1+12m_2+1})(1-q^{8m_1+12m_2+2}) \\ &= (q)_\infty \left(\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{15n^2+n}{2}} - \sum_{n\in\frac{3}{5}+\mathbb{Z}} (-1)^n q^{\frac{15n^2+n}{2}} \right) \\ &= (q)_\infty \left((q^7,q^{15})_\infty (q^8,q^{15})_\infty (q^{15},q^{15})_\infty - q(1-q^2)(q^{13},q^{15})_\infty (q^{15},q^{15})_\infty (q^{17},q^{15})_\infty \right) \end{aligned}$$

where as usual, $(x,q)_\infty = \prod_{k=0}^\infty (1-q^k x)$ and $(q)_\infty = (q,q)_\infty$. The book [9] is an excellent source for proving such identities.

2.2 The colored Jones polynomial of a torus knot

2.2.1 The Jones-Rosso formula

To verify Conjecture 2.1.5 for all torus knots $T(a,b)$ (where $0 < a < b$ and a and b are coprime integers), we will use the formula of Jones-Rosso [47]. It states that

$$J^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \frac{\theta_\lambda^{-ab}}{d_\lambda} \sum_{\mu\in S_{\lambda,a}} m^\mu_{\lambda,a} d_\mu \theta_\mu^{\frac{b}{a}} \tag{21}$$

where

- d_λ is the *quantum dimension* of V_λ and θ_λ is the eigenvalue of the *twist* operator on the representation V_λ given by:

$$d_\lambda = \prod_{\alpha>0} \frac{[(\lambda+\rho,\alpha)]}{[(\rho,\alpha)]}, \qquad \theta_\lambda = q^{\frac{1}{2}(\lambda,\lambda+2\rho)}, \qquad [n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \tag{22}$$

- $m^\mu_{\lambda,a} \in \mathbb{Z}$ is the multiplicity of V_μ in the *a-plethysm* of V_λ where where ψ_a denote the a-Adams operation. I.e., we have:

$$\psi_a(ch_\lambda) = \sum_{\mu\in S_{\lambda,a}} m^\mu_{\lambda,a} ch_\mu \tag{23}$$

  where ch_λ is the formal character of V_λ.

To describe the plethysm multiplicity $m^\mu_{\lambda,a}$ and the summation set $S_{\lambda,a}$, recall the Kostant multiplicity formula [36] which expresses the multiplicities m^μ_λ of the μ-weight space of V_λ in terms of the Kostant partition function p:

$$m^\mu_\lambda = \sum_{\sigma\in W} (-1)^\sigma p(\sigma(\lambda+\rho) - \mu - \rho) \tag{24}$$

As usual, W is the Weyl group of the simple Lie algebra $\mathfrak{g}$ and ρ is half the sum of its positive roots.

Lemma 2.2.1. (a) We have:

$$m^\mu_{\lambda,a} = \sum_{\sigma\in W} (-1)^\sigma m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{a}} \tag{25}$$

where the summation is over the elements $\sigma \in W$ such that $\frac{\mu+\rho-\sigma(\rho)}{a}$ is in the weight lattice (but not necessarily a dominant weight).

(b) It follows that

$$S_{\lambda,a} = \left[\bigcup_{\sigma\in W} (\sigma(\rho) - \rho + a\Pi_\lambda) \right] \cap \Lambda^+ \tag{26}$$

where Π_λ is the set of all weights of V_λ.

Remark 2.2.2. *The Jones-Rosso formula* (21) *combined with Equations* (24) *and* (25) *imply that that we can write*

$$J^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \sum_{\sigma,\sigma' \in W} J^{\mathfrak{g}}_{T(a,b),\lambda,\sigma,\sigma'}(q) \tag{27}$$

for some rational functions $J^{\mathfrak{g}}_{T(a,b),\lambda,\sigma,\sigma'}(q)$. It is easy to see that the sequences $(J^{\mathfrak{g}}_{T(a,b),n\lambda,\sigma,\sigma'}(q))$ are q-holonomic (with respect to n) and c-stable. If cancellation of the leading and trailing terms did not occur in Equation (27)*, it would imply a short proof of Theorem 2.1.6 for all torus knots and all simple Lie algebras. Unfortunately, after we perform the sum in Equation* (27) *cancellation occurs and the degree of the summand is much lower than the degree of the sum. This already happens for A_2 and the trefoil, an alternating knot. This cancellation is responsible for the minimizer $\mu_{\lambda,a}$ to be of order $O(\lambda)$ rather than $O(1)$ in case A_2, part (b) of Theorem 2.2.4.*

2.2.2 The degree of the colored Jones polynomial

The Jones-Rosso formula can be written in the form

$$J^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \frac{q^{-\frac{ab}{2}(\lambda,\lambda)-(-1+ab)(\lambda,\rho)}}{\prod_{\alpha\succ 0}(1-q^{(\lambda+\rho,\alpha)})} \sum_{\mu \in S_{\lambda,a}} q^{\frac{b}{2a}(\mu,\mu)+(-1+\frac{b}{a})(\mu,\rho)} \prod_{\alpha\succ 0}(1-q^{(\mu+\rho,\alpha)}) \,. \tag{28}$$

When the dominant weight λ and the torus knot $T(a,b)$ is fixed, the minimum the and maximum degree of the summand are positive-definite quadratic forms $f^*(\mu)$ and $f(\mu)$ given by

$$f^*(\mu) = \frac{b}{2a}(\mu,\mu) + \left(-1+\frac{b}{a}\right)(\mu,\rho) - \frac{ab}{2}(\lambda,\lambda) - (-1+ab)(\lambda,\rho) \tag{29a}$$

$$f(\mu) = \frac{b}{2a}(\mu,\mu) + \left(1+\frac{b}{a}\right)(\mu,\rho) - \frac{ab}{2}(\lambda,\lambda) - (1+ab)(\lambda,\rho) \tag{29b}$$

In Section 2.7 we will prove the following.

Theorem 2.2.3. *Fix a simple Lie algebra $\mathfrak{g}$ and a torus knot $T(a,b)$. The quadratic form $f(\mu)$ achieves maximum uniquely at $M_{\lambda,a} = a\lambda \in S_{a,\lambda}$. Moreover, $m^{M_{\lambda,a}}_{\lambda,a} = 1$.*

The next theorem states that $f^*(\mu)$ has a unique minimizer which we denote by $\mu_{\lambda,a}$ and describes $\mu_{\lambda,a}$ explicitly for all simple Lie algebras of rank 2. Below, $\{\lambda_1, \lambda_2\}$ are the dominant weights of a simple Lie elgebra of rank 2. Its proof is given in Section 2.8 using a case-by-case analysis.

Theorem 2.2.4. *When $\mathfrak{g}$ is a simple Lie algebra of rank 2, then*

(a)The quadratic form $f^*(\mu)$ achieves minimum uniquely at $\mu_{\lambda,a} \in S_{a,\lambda}$ and $m^{\mu}_{\lambda,a} \neq 0$.

(b) For a dominant weight $\lambda = m_1\lambda_1 + m_2\lambda_2$, we have

For A_2:

$$\mu_{\lambda,2} = \begin{cases} (m_1 - m_2)\lambda_2 & \text{if } m_1 \geq m_2 \\ (m_2 - m_1)\lambda_1 & \text{if } m_1 \leq m_2 \end{cases} \qquad \mu_{\lambda,3} = 0$$

and

$$\mu_{\lambda,a} = \begin{cases} 0 & \text{if } m_1 \equiv m_2 \bmod 3 \\ (a-3)\lambda_1 & \text{if } m_1 \equiv m_2 + 1 \bmod 3 \\ (a-3)\lambda_2 & \text{if } m_1 \equiv m_2 + 2 \bmod 3 \end{cases} \quad \text{for } a \geq 4$$

For B_2:

$$\mu_{\lambda,2} = \begin{cases} \lambda_1 & \text{if } m_1 = 0\,, m_2 \equiv 1 \bmod 2 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_{\lambda,3} = \begin{cases} 0 & \text{if } m_1, m_2 \equiv 0 \bmod 2 \\ 2\lambda_2 & \text{if } m_1 \equiv 1 \bmod 2\,, m_2 \equiv 0 \bmod 2 \\ \lambda_1 + \lambda_2 & \text{if } m_2 \equiv 1 \bmod 2 \end{cases}$$

$$\mu_{\lambda,4} = 0 \qquad \mu_{\lambda,a} = \begin{cases} 0 & \text{if } m_2 \equiv 0 \bmod 2 \\ (a-4)\lambda_2 & \text{if } m_2 \equiv 1 \bmod 2 \end{cases} \quad \text{for } a \geq 5$$

For G_2:

$$\mu_{\lambda,a} = 0 \text{ for all } a \geq 2$$

Theorem 2.2.4 part (b) implies the following.

Corollary 2.2.5. *$\mu_{n\lambda,a}$ is a piecewise quasi-linear function of n for $n \gg 0$.*

Let $\delta^*_K(\lambda)$ and $\delta_K(\lambda)$ denote the *minimum* and the *maximum* degree of the colored Jones polynomial $J^{\mathfrak{g}}_{K,V_\lambda}(q)$ with respect to q.

Corollary 2.2.6. *We have:*

$$\delta^*_{T(a,b)}(\lambda) = f^*(\mu_{\lambda,a}) \tag{30a}$$

$$\delta_{T(a,b)}(\lambda) = f(a\lambda)\,. \tag{30b}$$

2.3 Some lemmas about stability

In this section we collect some lemmas about stable sequences.

Lemma 2.3.1. *Fix natural numbers c and d and consider $g_n(q) = \frac{f_n(q)}{1-q^{cn+d}}$. Then $(f_n(q))$ is stable if and only if $(g_n(q))$ is stable. In that case, their corresponding tails $F(x,q)$ and $G(x,q)$ satisfy*

$$G(x,q) = \frac{F(x,q)}{1-q^d x^c}\,. \tag{31}$$

Proof. Let

$$F(x,q) = \sum_{k=0}^{\infty} \phi_k(q)x^k, \qquad G(x,q) = \sum_{k=0}^{\infty} \psi_k(q)x^k\,.$$

If F and G satisfy Equation (31), collecting powers of x^k on both sides implies that

$$\psi_k(q) = \sum_{i+jc=k} \phi_i(q)q^{jd}\,. \tag{32}$$

Assume that $f_n(q)$ is stable, and define $\psi_k(q)$ by Equation (32). We will prove by induction on k that $g_n(q)$ is k-stable with corresponding limit $\psi_k(q)$.

Let

$$\begin{aligned}
\alpha_{0,n}(q) =& f_n(q) \\
\alpha_{k,n}(q) =& q^{-n}(\alpha_{k-1,n} - \phi_{k-1}(q)) \\
=& q^{-kn}\left(f_n(q) - \sum_{l=0}^{k-1}\phi_l(q)q^{ln}\right), \quad k \geq 1
\end{aligned}$$

and

$$\begin{aligned}\beta_{0,n}(q) =& g_n(q)\\ \beta_{k,n}(q) =& q^{-n}(\beta_{k-1,n} - \psi_{k-1}(q))\\ =& q^{-kn}\left(g_n(q) - \sum_{l=0}^{k-1}\psi_l(q)q^{ln}\right), \quad k \geq 1\end{aligned}$$

For $k = 0$, the $0-$limit of $g_n(q)$ is $\lim_{n\to\infty} g_n(q) = \lim_{n\to\infty} \frac{f_n(q)}{1-q^{cn+d}} = \phi_0(q) = \psi_0(q)$.

Assuming by induction that $g_n(q)$ is $(k-1)$-stable, we have

$$\begin{aligned}\beta_{k,n}(q) =& q^{-kn}\left(\frac{f_n(q)}{1-q^{cn+d}} - \sum_{l=0}^{k-1}\sum_{i+jc=l}\phi_i(q)q^{jd}q^{(i+jc)n}\right)\\ =& q^{-kn}\left(f_n(q)\sum_{j=0}^{\infty}q^{j(cn+d)} - \sum_{0\leq i+jc\leq k-1}\phi_i(q)q^{in}q^{j(cn+d)}\right)\\ =& q^{-kn}\sum_{j=0}^{\lfloor\frac{k-1}{c}\rfloor}q^{j(cn+d)}\left(f_n(q) - \sum_{i=0}^{k-1-jc}\phi_i(q)q^{in}\right) + q^{-kn}\sum_{j>\lfloor\frac{k-1}{c}\rfloor}q^{j(cn+d)}f_n(q)\\ =& \sum_{j=0}^{\lfloor\frac{k-1}{c}\rfloor}q^{jd}q^{-(k-jc)n}\left(f_n(q) - \sum_{i=0}^{k-1-jc}\phi_i(q)q^{in}\right) + q^{-kn}\sum_{j>\lfloor\frac{k-1}{c}\rfloor}q^{j(cn+d)}f_n(q)\\ =& \sum_{j=0}^{\lfloor\frac{k-1}{c}\rfloor}q^{jd}\alpha_{k-jc,n}(q) + q^{-kn}\sum_{j>\lfloor\frac{k-1}{c}\rfloor}q^{j(cn+d)}f_n(q)\\ =& \sum_{j=0}^{\lfloor\frac{k-1}{c}\rfloor}q^{jd}\alpha_{k-jc,n}(q) + q^{-kn}\sum_{\lfloor\frac{k-1}{c}\rfloor<j\leq\frac{k}{c}}q^{j(cn+d)}f_n(q) + q^{-kn}\sum_{j>\frac{k}{c}}q^{j(cn+d)}f_n(q)\\ =& \sum_{i+jc=k}q^{jd}\alpha_{i,n}(q) + \sum_{j>\frac{k}{c}}q^{n(jc-k)+jd}f_n(q)\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\beta_{k,n}(q) = \sum_{i+jc=k}q^{jd}\phi_i(q) = \psi_k(q)\,.$$

Conversely, if $(g_n(q))$ is stable, so is $(f_n(q))$. □

Lemma 2.3.2. *Fix a rational polytope $P \subset [0,\infty)^r$ that intersects the interior of every positive coordinate ray and a positive definite quadratic function $Q : \mathbb{Z}^r \longrightarrow \mathbb{Z}$.*

Let $c : \mathbb{N} \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$ be such that for each fixed $v \in \mathbb{Z}^r$ and for $n \gg 0$, $c(n,v) = t(n,v)$ where $n \mapsto t(n,v)$ is a quasi-polynomial. For each natural number n define

$$T_n(q) = \sum_{v \in nP \cap \mathcal{L}} c(n,v) q^{Q(v)} .$$

Then $(T_n(q))$ is c-stable and its (n,x,q)-tail is given by

$$F(n,x,q) = \sum_{v \in \mathcal{L} \cap \mathbb{R}^r_+} t(n,v) q^{Q(v)} .$$

Proof. Let $\phi_0(n,q) = \sum_{v \in \mathcal{L} \cap \mathbb{R}^r_+} t(n,v) q^{Q(v)}$. We need to prove that for all $k \geq 0$, we have

$$\lim_{n \to \infty} q^{-kn}(T_n(q) - \phi_0(n,q)) = 0 .$$

We have

$$q^{-kn}(T_n(q) - \phi_0(n,q)) = q^{-kn} \sum_{v \in P_n} (c(n,v) - t(n,v)) q^{Q(v)} - \sum_{v \in (\mathcal{L} \cap \mathbb{R}^r_+) \setminus P_n} t(n,v) q^{Q(v) - kn} \tag{33}$$

$$= - \sum_{v \in (\mathcal{L} \cap \mathbb{R}^r_+) \setminus P_n} t(n,v) q^{Q(v) - kn} \tag{34}$$

for n large enough. Let us first assume that Q is a quadratic form and let d be the minimum of Q on $\mathbb{R}^r \setminus P^\circ$. We will prove that $d > 0$. Indeed, since Q is a positive definite form we only need to minimize Q over the union F of the faces of P that are not in the coordinate planes. Since F is compact, Q attains its minimum at some $v_0 \in F$ and $d = Q(v_0) > 0$ since $v_0 \neq 0$. If $v \in \mathbb{R}^r \setminus nP^\circ$ (where P° is the interior of P) then $v = nv', v' \in \mathbb{R}^r \setminus P^\circ$, so $Q(v) = Q(nv') = n^2 Q(v') \geq dn^2$. Therefore the limit of the right hand side of Equation (33) as n approaches infinity is zero.

If Q is not a quadratic form we can write $Q = Q_2 + Q_1$ where Q_2 is the quadratic part of Q. Then if $v \in \mathbb{R}^r \setminus nP^\circ$ we have $Q(v) = Q_2(v) + Q_1(v) \geq dn^2 + Q_1(v) > (d+1)n^2$ for n large enough. □

Remark 2.3.3. *Let $p \in P$. The tangent cone $Tan(P, p)$ is defined to be the set of all directions v that one can go and stay in P:*

$$Tan(P, p) = \{v \in \mathbb{R}^r | p + \epsilon v \in P \, for \; small \, \epsilon > 0\}$$

Lemma 2.3.2 still holds if we replace nP with $n(P - p)$ or $nP - p$ and $\mathcal{L}$ with a union of a finite number of translates of $\mathcal{L}$. In this setting, the stable series is

$$F(n, x, q) = \sum_{v \in Tan(P,p) \cap \mathbb{Z}^r} t(n, v) q^{Q(v)}.$$

Remark 2.3.4. *Supose that $f_n(q)$ satisfies $\delta^*(f_n(q)) \geq cn^2$ for some $c > 0, n \geq 0$ then $g_n(q)$ is c-stable if $g_n(q) + f_n(q)$ is c-stable and they have the same tails.*

2.4 Stability of the multiplicity

2.4.1 Lie algebra notation

Let us recall some standard notation from [8, 32]. Let $\mathfrak{g}$ denote a simple Lie algebra of rank r with weight lattice Λ, root lattice Λ_r and normalized inner product $(\cdot, \cdot)$ on Λ. Let W be its Weyl group and Λ^+ the set of all the dominant weights with respect to a fixed Weyl chamber. Let α_i (resp., λ_i), $1 \leq i \leq r$, be the set of simple roots (resp., fundamental weights) of $\mathfrak{g}$.The root lattice Λ_r has the partial order given by $\beta \prec \alpha$ if and only if $\alpha - \beta = \sum_{i=1}^{r} n_i \alpha_i$ where $n_i \in \mathbb{N}$, $i = 1, \ldots, r$.

For a dominant weight $\lambda \in \Lambda^+$, let V_λ denote the corresponding irreducible representation V_λ and let Π_λ denote the set of all of the weights of V_λ.

The *Kostant partition function* $p(\alpha)$ of an element of the root lattice α is the sum of all ways of writing α as a nonnegative integer linear combination of positive roots [36].

2.4.2 A formula for the plethysm multiplicity

In this section we prove Lemma 2.2.1.

Proof. (of Lemma 2.2.1) (a) We have

$$\psi_a(ch_\lambda) = \psi_a(\sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_\mu) = \sum_{\mu\in\Pi_\lambda} m_\lambda^\mu \psi_a(e_\mu) = \sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu}\,. \tag{35}$$

From Equations (23) and (35) we have

$$\sum_\mu m_{\lambda,a}^\mu ch_\mu = \sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu}\,. \tag{36}$$

Let us define $\omega(\mu) := \sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\mu)}$ by for $\mu \in \Lambda^+$. The Weyl character formula states that [32]:

$$\omega(\rho)ch_\lambda = \omega(\lambda+\rho)\,.$$

Multiplying both sides of Equation (36) with $\omega(\rho)$ and applying Weyl's formula we have

$$\sum_\mu m_{\lambda,a}^\mu \omega(\mu+\rho) = (\sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu})\omega(\rho) \tag{37}$$

Replacing $\omega(\mu+\rho)$ with $\sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\mu+\rho)}$ and $\omega(\rho)$ with $\sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\rho)}$ in Equation (37) we have

$$\sum_\mu \sum_{\sigma\in W}(-1)^\sigma m_{\lambda,a}^\mu e_{\sigma(\mu+\rho)} =(\sum_{\mu\in\Pi_\lambda} m_\lambda^\mu e_{a\mu})(\sum_{\sigma\in W}(-1)^\sigma e_{\sigma(\rho)}) \tag{38}$$

$$= \sum_{\mu\in\Pi_\lambda}\sum_{\sigma\in W}(-1)^\sigma m_\lambda^\mu e_{a\mu+\sigma(\rho)} \tag{39}$$

Setting $\sigma(\mu+\rho) = \nu+\rho$ on the left hand side of Equation (39) and $a\mu+\sigma(\rho) = \nu+\rho$ on right hand side we have

$$\sum_\nu \sum_{\sigma\in W}(-1)^\sigma m_{\lambda,a}^{\sigma^{-1}(\nu+\rho)-\rho} e_{\nu+\rho} = \sum_\nu \sum_{\sigma\in W}(-1)^\sigma m_\lambda^{\frac{\nu+\rho-\sigma(\rho)}{a}} e_{\nu+\rho} \tag{40}$$

But we want $\sigma^{-1}(\nu+\rho)-\rho$ to be a dominant weight, which can happen only when $\sigma = 1$. Therefore Equation (40) becomes

$$\sum_\nu m_{\lambda,a}^\nu e_{\nu+\rho} = \sum_\nu \sum_{\sigma\in W}(-1)^\sigma m_\lambda^{\frac{\nu+\rho-\sigma(\rho)}{a}} e_{\nu+\rho} \tag{41}$$

Identifying the coefficients of $e_{\nu+\rho}$ on both sides of Equation (41) gives us the desired equality.

(b) This follows from the fact that $m_\lambda^\nu \neq 0$ if and only if $\nu \in \Pi_\lambda$. If $\nu = \frac{\mu+\rho-\sigma(\rho)}{a}$ this means that $\mu \in \sigma(\rho) - \rho + a\Pi_\lambda$. □

2.4.3 Stability of the plethysm multiplicity

In this section we will prove that the coefficients $m_{n\lambda,a}^{\mu+n\nu}$ is a piecewise quasi-polynomial for $n \gg 0$ where $\lambda \in \Lambda^+$, $\mu, \nu \in \Lambda$. A piecewise quasi-polynomial function on a rational vector space is a rational polyhedral fan together with a quasi-polynomial function on each chamber of the fan. Piecewise quasi-polynomials appear naturally as *vector partition functions* [49]. The *Kostant partition function* of a simple Lie algebra $\mathfrak{g}$ is a vector partition function (see [36]), hence a piecewise quasi-polynomial.

Theorem 2.4.1. *Let $\lambda \in \Lambda^+$, $\mu, \nu \in \Lambda$, then $m_{n\lambda,a}^{\mu+n\nu}$ is a piecewise quasi-polynomial in n for $n \gg 0$.*

Proof. We have

$$m_{n\lambda,a}^{\mu+n\nu} = \sum_{\sigma \in W} (-1)^\sigma m_{n\lambda}^{\frac{\mu+n\nu+\rho-\sigma(\rho)}{a}} \tag{42}$$

and by Kostant's multiplicity formula in [36], we have

$$\begin{aligned} m_{n\lambda}^{\frac{\mu+n\nu+\rho-\sigma(\rho)}{a}} &= \sum_{\sigma' \in W} (-1)^{\sigma'} p\left(\sigma'(n\lambda+\rho) - (\frac{\mu+n\nu+\rho-\sigma(\rho)}{a} + \rho)\right) \\ &= \sum_{\sigma' \in W} (-1)^{\sigma'} p\left(n\sigma'(\lambda) - (\frac{\mu+n\nu+\rho-\sigma(\rho)}{a} + \rho - \sigma'(\rho))\right) \\ &= \sum_{\sigma' \in W} (-1)^{\sigma'} p\left(n(\sigma'(\lambda) - \frac{\nu}{a}) - (\frac{\mu+\rho-\sigma(\rho)}{a} + \rho - \sigma'(\rho))\right) \\ &= \sum_{\sigma' \in W} (-1)^{\sigma'} p\left(n\lambda' - \alpha'\right) \end{aligned}$$

Assume that $n\lambda' - \alpha'$ can be written as a linear combination of positive roots of $\mathfrak{g}$ so that $p(n\lambda' - \alpha') \neq 0$. For $n \gg 0$, $n\lambda' - \alpha'$ stays in some fixed Kostant chamber and it follows from Theorem 1 in [49] that $p(n\lambda' - \alpha')$ is a quasi-polynomial in n. Since $m_{n\lambda,a}^{\mu+n\nu}$ is a finite sum of quasi-polynomials in n, it is also a quasi-polynomial in n. □

2.5 The summation set

2.5.1 A lattice point description of the summation set

In this section give a lattice point description of the summation set $S_{\lambda,a}$. Let P_λ denote the convex polytope defined by the convex hull of $\Pi_\lambda \cap \Lambda^+$.

Lemma 2.5.1. *For all λ, a we have:*

$$S_{\lambda,a} \subseteq \mathcal{L}_{\lambda,a} \cap P_{a\lambda} \tag{43}$$

where

$$\mathcal{L}_{\lambda,a} = \bigcup_{\sigma \in W} (a\lambda + \sigma(\rho) - \rho + a\Lambda_r) \tag{44}$$

is a finite union of translates of the lattice $a\Lambda_r$. Let

$$R_{\lambda,a} = (\mathcal{L}_{\lambda,a} \cap P_{a\lambda}) \setminus S_{\lambda,a} \tag{45}$$

denote the set of missing points.

Proof. Recall that P_λ consists of all α that satisfy (see [32]),

$$(\alpha, \alpha_i) \geq 0 \qquad (\lambda - \alpha, \lambda_i) \qquad \geq 0 \tag{46}$$

for all $i = 1, \dots, r$. We first prove that $S_{\lambda,a} \subseteq P_{a\lambda}$. By Lemma 2.2.1(b), we can write $\mu = a\nu + \sigma(\rho) - \rho \in \Lambda^+$ where $\nu \in \Pi_\lambda$. Since $\mu \in \Lambda^+$, Inequality (46) holds trivially. To prove the second part of Inequality (46), it suffices to show that $(\mu, \lambda_i) \leq (a\lambda, \lambda_i)$ for every $1 \leq i \leq r$. We have

$$\begin{aligned}(a\lambda, \lambda_i) - (\mu, \lambda_i) &= (a\lambda - \mu, \lambda_i) \\ &= (a(\lambda - \mu) + \rho - \sigma(\rho), \lambda_i) \\ &\geq 0\end{aligned}$$

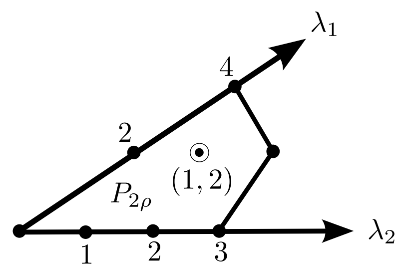
since $a(\lambda - \mu) + \rho - \sigma(\rho)$ is a $\mathbb{N}$-linear combination of positive roots.

Let $\mu = a\nu + \sigma(\rho) - \rho \in S_{\lambda,a}$ where $\nu \in \Pi_\lambda$ and $\sigma \in W$. Then $\mu = a(\lambda - \alpha) + \sigma(\rho) - \rho$ where α is some positive root. It follows that $\mu \in a\Lambda_r + a\lambda + \sigma(\rho) - \rho \subset \mathcal{L}_{\lambda,a}$. This proves that $S_{\lambda,a} \subseteq \mathcal{L}_{\lambda,a}$ and completes the proof of the lemma. ∎

Remark 2.5.2. *The inclusion in Equation (43) is not an equality in general. For example, consider $\mathfrak{g} = B_2, \lambda = \rho, a = 2$. In weight coordinates we have*

$$S_{\rho,2} = \cup_{\sigma \in W} (\sigma(\rho) - \rho + 2\Pi_\rho) \cap \Lambda^+ \tag{47}$$

$$= \{(2,2),(0,4),(3,0),(2,0),(0,2),(1,0),(0,0)\} \quad (\textit{see figure below}) \tag{48}$$



It is clear that $(1,2) \in P_{2\rho}$. We show that $(1,2) = \lambda_1 + 2\lambda_2 \in \mathcal{L}_{\rho,2}$ and hence this a missing. Indeed, by the definition of $\mathcal{L}_{\rho,2}$, we only need to find $\sigma \in W$ and a root α such that

$$\lambda_1 + 2\lambda_2 = 2\alpha_1 + 3\alpha_2 = 2\rho + \sigma(\rho) - \rho + 2\alpha \tag{49}$$

In root coordinates we have

$$2\rho = 3\alpha_1 + 4\alpha_2,\ \rho - \sigma(\rho) \in \{0, \alpha_1, \alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 4\alpha_2\}$$

So by choosing $\alpha = \alpha_2$ and σ such that $\rho - \sigma(\rho) = \alpha_1 + 3\alpha_2$ we have equality (49).

Nevertheless, equality holds when $\mathfrak{g} = A_2, a = 2, \lambda = \lambda_1$. This is the content of the next section.

2.5.2 A special case: no missing points

Proposition 2.5.3. *For A_2, we have: $S_{n\lambda_1,2} = \mathcal{L}_{n\lambda_1,2} \cap P_{2n\lambda_1}$.*

Proof. Let $\mathcal{L}_{n\lambda_1,2} \cap P_{2n\lambda_1} \ni \mu = 2n\lambda_1 + \sigma(\rho) - \rho + 2\alpha = 2\nu - (\rho - \sigma(\rho))$ where $\nu = n\lambda_1 + \alpha$ and some $\sigma \in W$. As σ runs over the Weil group W, $\rho - \sigma(\rho)$ is expressed in weight and root coordinates as follows

weight	(0, 0)	(2, −1)	(−1, 2)	(0, 3)	(3, 0)	(2, 2)
root	(0, 0)	(0, 1)	(1, 0)	(1, 2)	(2, 1)	(2, 2)

(50)

Since $\mu \in P_{2n\lambda_1}$, from inequality (46) we have $(2\nu - (\rho - \sigma(\rho)), \alpha_i) \geq 0$, i.e., $2(\nu, \alpha_i) \geq (\rho - \sigma(\rho), \alpha_i)$, $i = 1, 2$. Looking at the first row of the above table we see that this forces $(\nu, \alpha_i) \geq 0$, $i = 1, 2$. Therefore we have $\nu \in \Lambda^+$.

From inequality (46) we have $(2n\lambda_1 - \mu, \lambda_i) \geq 0$, $i = 1, 2$. This implies that $(-2\alpha + \rho - \sigma(\rho), \lambda_i) \geq 0$ or equivalently

$$(\rho - \sigma(\rho), \lambda_i) \geq 2(\alpha, \lambda_i), \tag{51}$$

$i = 1, 2$. We consider the following cases.

Case 1: If $\rho - \sigma(\rho) = 0, \alpha_1$ or α_2 then inequalities (51) imply that $(\alpha, \lambda_i) \leq 0$ for all i so $\alpha \prec 0$. Since $\nu = n\lambda_1 + \alpha \in \Lambda^+$, it follows from [32, §13.4] that $\nu \in \Pi_{n\lambda_1}$ and hence $\mu \in S_{n\lambda_1,2}$.

Case 2: If $\rho - \sigma(\rho) = \alpha_1 + 2\alpha_2$ then from (51) we have $(\alpha, \lambda_1) \leq 0$ and $(\alpha, \lambda_2) \leq 1$. If we also have $(\alpha, \lambda_2) \leq 0$ then by a similar the argument to Case 1 we conclude that $\mu \in S_{n\lambda_1,2}$. If $(\alpha, \lambda_2) = 1$ we can write $\alpha = -x\alpha_1 + \alpha_2$, where $x \in \mathbb{N}$. It follows that $\mu = 2n\lambda_1 + 2\alpha - (\rho - \sigma(\rho)) = 2n\lambda_1 - 2x\alpha_1 + 2\alpha_2 - \alpha_1 - 2\alpha_2 = 2(n\lambda_1 - x\alpha_1) - \alpha_1$. Since $\nu = n\lambda_1 + \alpha \in \Lambda^+$, from inequality (46) we have $(n\lambda_1 - x\alpha_1 + \alpha_2, \alpha_1) \geq 0$, i.e., $n - 2x - 1 \geq 0$. We have $\langle n\lambda_1, \alpha_1 \rangle = \frac{2(n\lambda_1, \alpha_1)}{(\alpha_1, \alpha_1)} = n \geq 2x + 1 > x$, therefore $n\lambda_1 - x\alpha_1 \in \Pi_{n\lambda_1}$ (see [32, § 13.4]). Since we can choose σ' such that $\rho - \sigma'(\rho) = \alpha_1$, we have $\mu = 2(n\lambda_1 - x\alpha_1) - (\rho - \sigma'(\rho)) \in S_{n\lambda_1,2}$.

Case 3: If $\rho - \sigma(\rho) = 2\alpha_1 + \alpha_2$ then by a similar argument to the above we can write $\alpha = \alpha_1 - x\alpha_2$, $x \in \mathbb{N}$. We show that α cannot have this form. Indeed, since $\nu = n\lambda_1 + \alpha \in \Lambda^+$, we have $(n\lambda_1 + \alpha_1 - x\alpha_2, \alpha_2) \geq 0$, i.e., $-1 - 2x \geq 0$. This is in contradiction to the fact that $x \in \mathbb{N}$.

Case 4: If $\rho - \sigma(\rho) = 2\alpha_1 + 2\alpha_2 = 2\rho$ then $(\alpha, \lambda_1) \leq 1$ and $(\alpha, \lambda_2) \leq 1$. If either $(\alpha, \lambda_1) \leq 0$ or $(\alpha, \lambda_2) \leq 0$ then the same argument as in Cases 2 and 3 above apply. If $(\alpha, \lambda_1) = (\alpha, \lambda_2) = 1$ then $\alpha = \alpha_1 + \alpha_2 = \rho$ and $\mu = 2n\lambda_1 + 2\alpha - (\rho - \sigma(\rho)) = 2n\lambda_1 + 2\rho - 2\rho = 2n\lambda_1 \in \Pi_{2n\lambda_1} \subseteq S_{n\lambda_1,2}$. □

2.5.3 An estimate for the missing points

The next proposition shows that the norm of the missing points in $R_{n\lambda,a}$ is bounded below by a quadratic function of n.

Proposition 2.5.4. *For every $\lambda \in \Lambda^+$ there exists a simple root β such that if $\mu \in R_{n\lambda,a}$ and $n \gg 0$ then*

$$(\mu, \mu) \geq a^2 n^2 ((\lambda, \lambda) - \frac{(\lambda, \beta)^2}{(\beta, \beta)} - 1)\,.$$

Proof. Let $\mu = a\alpha + an\lambda + \sigma(\rho) - \rho = a(n\lambda + \alpha) + \sigma(\rho) - \rho$ for some $\alpha \in \Lambda_r$ and $\sigma \in W$. Since $\mu \notin S_{n\lambda,a}$, we have that $n\lambda + \alpha \notin \Pi_{n\lambda}$. The ray $n\lambda + \alpha$ meets one of the facets of the convex hull of $\Pi_{n\lambda}$ at some point, say λ_n. There exist $\sigma_1, \sigma_2 \in W$ such that $\sigma_1(n\lambda), \sigma_2(n\lambda)$ are the vertices of this facet, and we have

$$\begin{aligned}
(n\lambda + \alpha, n\lambda + \alpha) &\geq (\lambda_n, \lambda_n) \\
&\geq (\frac{\sigma_1(n\lambda) + \sigma_2(n\lambda)}{2}, \frac{\sigma_1(n\lambda) + \sigma_2(n\lambda)}{2}) \\
&= \frac{n^2}{4}((\sigma_1(\lambda), \sigma_1(\lambda)) + (\sigma_2(\lambda), \sigma_2(\lambda)) + 2(\sigma_1(\lambda), \sigma_2(\lambda))) \\
&= \frac{n^2}{4}(2(\lambda, \lambda) + 2(\sigma_1(\lambda), \sigma_2(\lambda))) \\
&= \frac{n^2}{2}((\lambda, \lambda) + (\sigma_1(\lambda), \sigma_2(\lambda)))
\end{aligned}$$

Since $\sigma_1(\lambda), \sigma_2(\lambda)$ are in two nearby chambers, there exists a simple root β such that

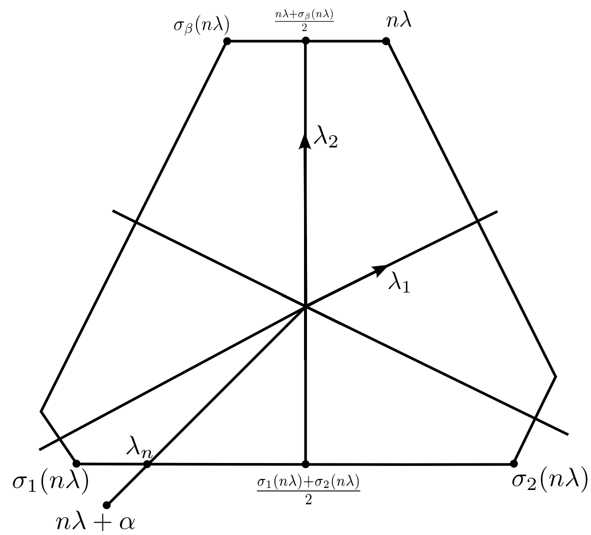
$$(\sigma_1(\lambda), \sigma_2(\lambda)) = (\lambda, \sigma_\beta(\lambda))$$

We have

$$(\lambda, \sigma_\beta(\lambda)) = (\lambda, \lambda - 2\frac{(\lambda, \beta)}{(\beta, \beta)}\beta) = (\lambda, \lambda) - 2\frac{(\lambda, \beta)^2}{(\beta, \beta)}$$

So

$$(n\lambda + \alpha, n\lambda + \alpha) \geq n^2((\lambda, \lambda) - \frac{(\lambda, \beta)^2}{(\beta, \beta)})$$



Therefore

$$\begin{aligned}(\mu,\mu) &= (a(n\lambda+\alpha)-(\rho-\sigma(\rho)), a(n\lambda+\alpha)-(\rho-\sigma(\rho)))\\ &= a^2(n\lambda+\alpha, n\lambda+\alpha) - 2a(n\lambda+\alpha, \rho-\sigma(\rho)) + (\rho-\sigma(\rho), \rho-\sigma(\rho))\\ &\geq a^2n^2((\lambda,\lambda) - \frac{(\lambda,\beta)^2}{(\beta,\beta)} - 1)\end{aligned}$$

for large enough n. □

Recall that $\hat{S}_{\lambda,a} = S_{\lambda,a} - \mu_{\lambda,a}$. Let

$$\hat{\mathcal{L}}_{\lambda,a} = \mathcal{L}_{\lambda,a} - \mu_{\lambda,a}, \qquad \hat{P}_{a\lambda} = P_{a\lambda} - \mu_{\lambda,a}, \qquad \hat{R}_{\lambda,a} = R_{\lambda,a} - \mu_{\lambda,a}. \tag{52}$$

Remark 2.5.5. *From now we fix a natural number n_0 and we work with $n \equiv n_0$* mod *da where d is the order of the fundamental group Λ/Λ_r. Theorem 2.2.4 implies that for such n, we have:*

- $\mu_{n\lambda,a} = n\nu^1_{\lambda,a} + \nu^0_{\lambda,a}$ *for some fixed weights $\nu^1_{\lambda,a}, \nu^0_{\lambda,a} \in \Lambda^+$.*

- $\hat{\mathcal{L}}_{n\lambda,a} = \hat{\mathcal{L}}_{n_0\lambda,a}$. *Indeed, we have*

$$\begin{aligned}
\hat{\mathcal{L}}_{n\lambda,a} &= \mathcal{L}_{n\lambda,a} - \mu_{n\lambda,a} \\
&= n\lambda + \sigma(\rho) - \rho + a\Lambda_r - n\nu^1_{\lambda,a} - \nu^0_{\lambda,a} \\
&= n_0\lambda + \sigma(\rho) - \rho + a\Lambda_r - n_0\nu^1_{\lambda,a} - \nu^0_{\lambda,a} + (n-n_0)(\lambda - \nu^1_{\lambda,a}) \\
&= n_0\lambda + \sigma(\rho) - \rho + a\Lambda_r - \mu_{n_0\lambda,a} + k(a.d)(\lambda - \nu^1_{\lambda,a}),\ k \in \mathbb{N} \\
&= n_0\lambda + \sigma(\rho) - \rho + a\Lambda_r - \mu_{n_0\lambda,a} \quad (\textit{since } d(\lambda - \nu^1_{\lambda,a}) \in \Lambda_r) \\
&= \mathcal{L}_{n_0\lambda,a} - \mu_{n_0\lambda,a} = \hat{\mathcal{L}}_{n_0\lambda,a}
\end{aligned}$$

Corollary 2.5.6. (1) $\hat{S}_{n\lambda,a} \subset \hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}$.

(2) Let $\hat{R}_{n\lambda,a} = (\hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}) \setminus \hat{S}_{n\lambda,a}$. If $\hat{\mu} \in \hat{R}_{n\lambda,a}$ then

$$(\hat{\mu}, \hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) \geq a^2n^2 \left((\lambda, \lambda) - \frac{(\lambda, \beta)^2}{(\beta, \beta)} - 1 \right) - (\mu_{n\lambda,a}, \mu_{n\lambda,a})$$

for some simple root β.

Proof. Part (1) follows from Lemma 2.5.1(b) and Remark 2.5.5:

$$\hat{S}_{n\lambda,a} \subset \hat{\mathcal{L}}_{n\lambda,a} \cap \hat{P}_{an\lambda} = \hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}$$

For part (2), recall that

$$(\mu, \mu) = (\hat{\mu} + \mu_{n\lambda,a}, \hat{\mu} + \mu_{n\lambda,a})$$

and therefore if $\hat{\mu} \in \hat{R}_{n\lambda,a}$ then

$$\begin{aligned}
(\hat{\mu}, \hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) &= (\mu, \mu) - (\mu_{n\lambda,a}, \mu_{n\lambda,a}) \\
&\geq a^2n^2 \left((\lambda, \lambda) - \frac{(\lambda, \beta)^2}{(\beta, \beta)} - 1 \right) - (\mu_{n\lambda,a}, \mu_{n\lambda,a})
\end{aligned}$$

by Proposition 2.5.4. □

Proposition 2.5.7. *If $\mathfrak{g}$ has rank 2 and $\hat{\mu} \in \hat{R}_{n\lambda,a}$ then*

$$(\hat{\mu}, \hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) \geq n^2$$

Proof. We can prove this by a direct computation for the rank 2 simple Lie algebras A_2, B_2 and G_2 using Theorem 2.2.4 that gives an explicit formula for $\mu_{\lambda,a}$.

For A_2 and $m_1 \geq m_2$, from Theorem 2.2.4 we have

$$(\mu_{n\lambda,a}, \mu_{n\lambda,a}) \leq (n(m_1 - m_2)\lambda_2, n(m_1 - m_2)\lambda_2) = \frac{2}{3}n^2(m_1 - m_2)^2$$

By Corollary 2.5.6 we have

$$\begin{aligned}(\hat{\mu}, \hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) &\geq a^2n^2((\lambda, \lambda) - \frac{(\lambda, \alpha_1)^2}{(\alpha_1, \alpha_1)} - 1) - (\mu_{n\lambda,a}, \mu_{n\lambda,a}) \\ &\geq a^2n^2(\frac{2}{3}(m_1^2 + m_1m_2 + m_2^2) - \frac{m_1^2}{2} - 1) - \frac{2}{3}n^2(m_1 - m_2)^2 \\ &= n^2(\frac{a^2 - 4}{6}m_1^2 + \frac{2}{3}(a^2 + 2)m_1m_2 + \frac{2}{3}(a^2 - 1)m_2^2 - 1) \\ &\geq n^2(4m_1m_2 + 2m_2^2 - 1) \\ &\geq n^2\end{aligned}$$

except when $a = 2$ and $m_2 = 0$. In the later case, Proposition 2.5.3 says that $R_{n\lambda,a} = \emptyset$ and the inequality holds trivially. The argument is similar for the case $m_1 \leq m_2$.

For B_2, $\frac{(\lambda,\beta)^2}{(\beta,\beta)}$ is either $\frac{m_1^2}{2}$ or m_2^2. We have

$$\begin{aligned}(\hat{\mu}, \hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) &\geq a^2n^2((\lambda, \lambda) - \frac{(\lambda, \alpha_1)^2}{(\alpha_1, \alpha_1)} - 1) - (\mu_{n\lambda,a}, \mu_{n\lambda,a}) \\ &= a^2n^2(m_1^2 + m_1m_2 + \frac{m_2^2}{2} - \max\{\frac{m_1^2}{2}, \frac{m_2^2}{4}\} - 1) - (\mu_{n\lambda,a}, \mu_{n\lambda,a}) \\ &\geq n^2\end{aligned}$$

where in the last inequality we have used the fact that $(\mu_{\lambda,a}, \mu_{\lambda,a})$ is bounded for B_2, see Theorem 2.2.4.

For G_2, $\frac{(\lambda,\beta)^2}{(\beta,\beta)}$ is either $\frac{m_1^2}{2}$ or $\frac{m_2^2}{6}$. Therefore we have

$$\begin{aligned}(\hat{\mu}, \hat{\mu}) + 2(\hat{\mu}, \mu_{n\lambda,a}) &\geq a^2n^2((\lambda, \lambda) - \frac{(\lambda, \alpha_1)^2}{(\alpha_1, \alpha_1)} - 1) - (\mu_{n\lambda,a}, \mu_{n\lambda,a}) \\ &= a^2n^2(2m_1^2 + 6m_1m_2 + 6m_2^2 - \max\{\frac{m_1^2}{2}, \frac{3m_2^2}{2}\} - 1) - (\mu_{n\lambda,a}, \mu_{n\lambda,a}) \\ &\geq n^2\end{aligned}$$

since $\mu_{\lambda,a} = 0$ for G_2, see Theorem 2.2.4. □

2.6 Proof of Theorem 2.1.6

In this section we will prove Theorem 2.1.6 assuming Theorem 2.2.4. Corollary 2.2.6 implies that the shifted colored Jones polynomial defined by

$$\hat{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q) = q^{-\delta^*_{T(a,b)}(\lambda)} J^{\mathfrak{g}}_{T(a,b),\lambda}(q) \in \mathbb{Z}[q] \tag{53}$$

satisfies

$$\hat{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \frac{1}{\prod\limits_{\alpha \succ 0} (1 - q^{(\lambda+\rho,\alpha)})} \check{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q)$$

where

$$\check{J}^{\mathfrak{g}}_{T(a,b),\lambda}(q) = \sum_{\hat{\mu} \in \hat{S}_{\lambda,a}} m^{\hat{\mu}+\mu_{\lambda,\alpha}}_{\lambda,a} q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{\lambda,\alpha})} \prod_{\alpha \succ 0} (1 - q^{(\hat{\mu}+\mu_{\lambda,a}+\rho,\alpha)}) \tag{54}$$

with $\hat{S}_{\lambda,a} = S_{\lambda,a} - \mu_{\lambda,a}$ and $\hat{\mu} = \mu - \mu_{\lambda,a}$.

Fix a natural number n, observe that $(f_n(q))$ is c-stable if and only if $(f_{Mn+n_0}(q))$ is c-stable for all $n_0 = 0, 1, ..., M$. In what follows, we will use $M = ad$ and fix $n \equiv n_0$ mod ad.

Proposition 2.6.1. *$(\hat{J}^{\mathfrak{g}}_{T(a,b),n\lambda}(q))$ is c-stable if and only if*

$$\frac{1}{\prod\limits_{\alpha \succ 0} (1 - q^{(n\lambda+\rho,\alpha)})} \sum_{\hat{\mu} \in \hat{\mathcal{L}}_{n_0\lambda,a} \cap \hat{P}_{an\lambda}} m^{\hat{\mu}+\mu_{\lambda,\alpha}}_{\lambda,a} q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{\lambda,\alpha})} \prod_{\alpha \succ 0} (1 - q^{(\hat{\mu}+\mu_{\lambda,a}+\rho,\alpha)}) \tag{55}$$

is c-stable. In that case, they have the same tails.

Proof. Fix a, b, λ and let $g_n(q)$ denote the difference between $\hat{J}_{T(a,b),n\lambda}(q)$ and Equation (55). Then

$$g_n(q) = \frac{1}{\prod\limits_{\alpha \succ 0} (1 - q^{(n\lambda+\rho,\alpha)})} \sum_{\hat{\mu} \in \hat{R}_{n\lambda,a}} m^{\hat{\mu}+\mu_{n\lambda,\alpha}}_{\lambda,a} q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\mu_{n\lambda,\alpha})} \prod_{\alpha \succ 0} (1 - q^{(\hat{\mu}+\mu_{n\lambda,a}+\rho,\alpha)}) \tag{56}$$

Proposition 2.5.7 implies that the minimum degree of the summands of Equation (56) is greater or equal to $\frac{b}{2a}n^2$ for $n \gg 0$. The proof then follows from Remark 2.3.4.

Proposition 2.5.7 implies that we can replace the summation set $\hat{S}_{n\lambda,a}$ by $\hat{\mathcal{L}}_{n\lambda,a} \cap \hat{P}_{an\lambda}$ without affecting the stability of $\hat{J}^{\mathfrak{g}}_{T(a,b),n\lambda}(q)$: if $\hat{\mu} \in (\hat{\mathcal{L}}_{\lambda,a} \cap \hat{P}_{an\lambda}) \setminus \hat{S}_{n\lambda,a}$ then the minimum degree of the summand of Equation (55) is

$$\begin{aligned}\frac{b}{2a}(\hat{\mu},\hat{\mu}) + (-1+\frac{b}{a})(\hat{\mu},\rho) + \frac{b}{a}(\hat{\mu},\mu_{n\lambda,\alpha}) &= \frac{b}{2a}((\hat{\mu},\hat{\mu}) + 2(\hat{\mu},\mu_{\lambda,a})) - (\hat{\mu},\rho) \\ &\geq \frac{b}{2a}n^2 - (\hat{\mu},\rho) = \frac{b}{2a}n^2 + O(n)\end{aligned}$$

where the last inequality follows from Proposition 2.5.7. By Remark 2.5.5 we have $\hat{\mathcal{L}}_{n\lambda,a} = \hat{\mathcal{L}}_{\lambda,a}$ and the Proposition follows. □

Let $t_{\lambda,\hat{\mu},a}(n) = m^{\hat{\mu}+\mu_{n\lambda,a}}_{n\lambda,a}$. Theorem implies that $t_{\lambda,\hat{\mu},a}$ is a quasi-polynomial. From Lemma 2.3.2, Proposition 2.5.7, Proposition 2.6.1 and together with the special case given in Section 2.9.1 we conclude that

Theorem 2.6.2. *Fix a rank* 2 *simple Lie algebras* $\mathfrak{g}$*, a dominant weight* λ*, and a torus knot* $T(a,b)$*. The colored Jones polynomial* $\hat{J}^{\mathfrak{g}}_{T(a,b),n\lambda}(q)$ *is c-stable and its* (n,x,q)*-tail is given by*

$$F_{T(a,b),\lambda}(n,x,q) = \frac{1}{\prod_{\alpha\succ 0}(1-x^{(\lambda,\alpha)}q^{(\rho,\alpha)})} \sum_{\hat{\mu}\in\hat{\mathcal{L}}_{\lambda,a}\cap\Lambda^+} t_{\lambda,\hat{\mu},a}(n) q^{\frac{b}{2a}(\hat{\mu},\hat{\mu})+(-1+\frac{b}{a})(\hat{\mu},\rho)+\frac{b}{a}(\hat{\mu},\nu^0_{\lambda,a})} x^{\nu^1_{\lambda,a}} \tag{57}$$

$$\prod_{\alpha\succ 0}(1-q^{(\hat{\mu}+\nu^0_{\lambda,a}+\rho,\alpha)}x^{\nu^1_{\lambda,a}}),$$

where $\mu_{n\lambda,a} = n\nu^1_{\lambda,a} + \nu^0_{\lambda,a}$.

2.7 Proof of Theorem 2.2.3

In this section we prove Theorem 2.2.3. Since λ is fixed, it suffices to maximize

$$g(\mu) = \frac{b}{4}(\mu,\mu) + \left(-1+\frac{b}{2}\right)(\mu,\rho)$$

on the set $S_{\lambda,a}$.

Lemma 2.7.1. *Let $\mu \in \Lambda^+$ and $\alpha \succ 0$ be a positive root such that $\mu + \alpha \in \Lambda^+$. Then we have*

$$(\mu, \mu) < (\mu + \alpha, \mu + \alpha)$$

Proof. We have:

$$(\mu + \alpha, \mu + \alpha) - (\mu, \mu) = 2(\mu, \alpha) + (\alpha, \alpha)$$

Now $(\mu, \alpha) > 0$ since μ is dominant and α is a positive root and $(\alpha, \alpha) > 0$ since $(\cdot, \cdot)$ is positive definite. □

If $\nu \in \Pi_\lambda$ then $\nu = \lambda - \alpha'$ where $\alpha' \succ 0$. Since $\rho - \sigma(\rho) \succ 0$, we have $\mu = a\lambda - \alpha$ where $\mu \in S_{a,\lambda}$ and $\alpha \succ 0$. It follows from the above lemma that $M_{\lambda,a} = a\lambda$ is the unique maximizer of $f(\mu)$.

Next, we compute the plethysm multiplicity $m_{\lambda,a}$. From Lemma 2.2.1 we have

$$\begin{aligned} m_{\lambda,a}^{a\lambda} &= \sum_{\sigma \in W} (-1)^\sigma m_\lambda^{\frac{a\lambda + \rho - \sigma(\rho)}{a}} \\ &= \sum_{\sigma \in W} (-1)^\sigma m_\lambda^{\lambda + \frac{\rho - \sigma(\rho)}{a}} \\ &= 1 \end{aligned}$$

since $\lambda + \frac{\rho - \sigma(\rho)}{a} \succ \lambda$ if $\frac{\rho - \sigma(\rho)}{a} \in \Lambda_r$, with equality only when $\sigma = 1$. This concludes the proof of Theorem 2.2.3. □

2.8 Proof of Theorem 2.2.4

This section is devoted to the proof of Theorem 2.2.4, done by a case-by-case analysis for a fixed simple Lie algebra $\mathfrak{g}$ of rank 2. Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ and $\mu = u_1\lambda_1 + u_2\lambda_2$ be dominant weights. Since λ is fixed, it suffices to minimize

$$g^*(\mu) = \frac{b}{4}(\mu, \mu) + \left(-1 + \frac{b}{2}\right)(\mu, \rho)$$

on the set $S_{\lambda,a}$. We use the following lemma and its consequence, Corollary 2.8.2, in the proof of Theorem 2.2.4.

Lemma 2.8.1. *$g^*(\mu) \geq 0$ with equality if and only if $\mu = 0$.*

Proof. $g^*(\mu)$ is non-negative since $(\cdot, \cdot)$ is a positive-definite form and $(\mu, \rho) \geq 0$ since μ is a dominant weight and ρ is a linear combination of simple roots with positive coefficients. If $g^*(\mu) = 0$ then $(\mu, \mu) = 0$ which implies that $\mu = 0$. □

Corollary 2.8.2. *If $m^0_{\lambda,a} \neq 0$ then $\mu_{\lambda,a} = 0$ is the unique minimizer of $g^*(\mu)$.*

We give the proof of Theorem 2.2.4 in Section 2.8.1 below.

2.8.1 Theorem 2.2.4 for A_2

2.8.1.1 Plethysm multiplicities for A_2

There are two simple roots $\{\alpha_1, \alpha_2\}$ of A_2 and three positive roots $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$ shown in Figure 1. The Kostant function $p(u, v) = p(u\alpha_1 + v\alpha_2)$ is given by

$$p(u, v) = 1 + \min(u, v) \,.$$

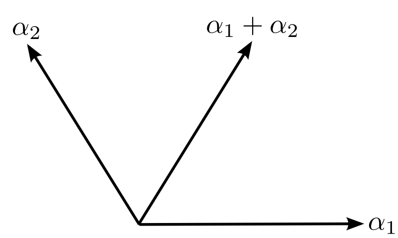


Figure 1: The two chambers of the Kostant partition function of A_2. Kostant chambers from left to right: $u \leq v$, $u \geq v$.

Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ denote a dominant weight and $m_1 \geq m_2$. Assuming $\mu =$

$u_1\lambda_1 + u_2\lambda_2 \in \Pi_\lambda$, by Kostant's formula we have

$$\begin{aligned}
m_\lambda^\mu &= \sum_{\sigma\in W}(-1)^\sigma p(\sigma(\lambda+\rho)-\mu-\rho) \\
&= p(\frac{2m_1+m_2}{3}-\frac{2u_1+u_2}{3}, \frac{m_1+2m_2}{3}-\frac{u_1+2u_2}{3}) \\
&- p(\frac{2m_1+m_2}{3}-\frac{2u_1+u_2}{3}, \frac{m_1-m_2}{3}-\frac{u_1+2u_2}{3}-1) \\
&= \begin{cases} 1+\frac{2m_1+m_2}{3}-\frac{2u_1+u_2}{3} & \text{if } m_1-m_2 < u_1-u_2 \\ 1+\frac{m_1+2m_2}{3}-\frac{u_1+2u_2}{3} & \text{if } u_1-u_2 \le m_1-m_2 \le u_1+2u_2+3 \\ 1+m_2 & \text{if } m_1-m_2 > u_1+2u_2+3 \end{cases}
\end{aligned}$$

Lemma 2.2.1 gives

$$\begin{aligned}
m_{\lambda,2}^\mu &= \sum_{\sigma\in S_3}(-1)^\sigma m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{2}} \\
&= m_\lambda^{\frac{1}{2}(u_1,u_2)} - m_\lambda^{\frac{1}{2}(u_1+2,u_2-1)} - m_\lambda^{\frac{1}{2}(u_1-1,u_2+2)} + m_\lambda^{\frac{1}{2}(u_1,u_2+3)} + m_\lambda^{\frac{1}{2}(u_1+3,u_2)} - m_\lambda^{\frac{1}{2}(u_1+2,u_2+2)}
\end{aligned}$$

Let us consider $\mu \in S_{\lambda,2}$. There are four cases.

Case 1: u_1, u_2 are even.

$$m_{\lambda,2}^\mu = m_\lambda^{(\frac{u_1}{2},\frac{u_2}{2})} - m_\lambda^{(\frac{u_1+2}{2},\frac{u_2+2}{2})} = \begin{cases} 1 & \text{if } u_1+2u_2 \ge 2(m_1-m_2) \\ 0 & \text{if } u_1+2u_2 < 2(m_1-m_2) \end{cases} \tag{58}$$

Case 2: u_1 even and u_2 odd.

$$\begin{aligned}
m_{\lambda,2}^\mu &= m_\lambda^{(\frac{u_1}{2},\frac{u_2+3}{2})} - m_\lambda^{(\frac{u_1+2}{2},\frac{u_2-1}{2})} \\
&= \begin{cases} -1 & \text{if } u_1-u_2 \le 2(m_1-m_2) \le u_1+2u_2 \\ 0 & \text{if } 2(m_1-m_2) < u_1-u_2 \text{ or } 2(m_1-m_2) > u_1+2u_2 \end{cases}
\end{aligned} \tag{59}$$

Case 3: u_1 odd and u_2 even.

$$m_{\lambda,2}^\mu = m_\lambda^{(\frac{u_1+3}{2},\frac{u_2}{2})} - m_\lambda^{(\frac{u_1-1}{2},\frac{u_2+2}{2})} = \begin{cases} -1 & \text{if } 2(m_1-m_2) < u_1-u_2 \\ 0 & \text{if } 2(m_1-m_2) \ge u_1-u_2 \end{cases}$$

Case 4: u_1 and u_2 are odd.

$$m^{\mu}_{\lambda,2} = 0$$

Corollary 2.8.3. *For A_2, if $m^{\mu}_{\lambda,2} \neq 0$ then $u_1 + 2u_2 \geq 2(m_1 - m_2)$.*

If $m_1 \leq m_2$ we have a similar corollary:

Corollary 2.8.4. *For A_2, if $m^{\mu}_{\lambda,2} \neq 0$ then $2u_1 + u_2 \geq 2(m_2 - m_1)$.*

2.8.1.2 The minimizer for A_2

Case 1: $a = 2$. By Corollary 2.8.3 it suffices to minimize $g^*(\mu)$ over subset $\{\mu \in S_{\lambda,2} : u_1, u_2 \in \mathbb{N}, u_1 + 2u_2 \geq 2(m_1 - m_2)\}$ of $S_{\lambda,2}$. We have

$$\begin{aligned}
g^*(\mu) &= \frac{b}{4}(\mu,\mu) + \left(-1 + \frac{b}{2}\right)(\mu,\rho) \\
&= \frac{b}{6}(u_1^2 + u_1u_2 + u_2^2) + (-1 + \frac{b}{2})(u_1 + u_2) \\
&= \frac{b}{6}((u_2 + \frac{u_1}{2})^2 + \frac{3u_1^2}{4}) + \frac{b-2}{4}(u_1 + u_1 + 2u_2) \\
&\geq \frac{b}{8}u_1^2 + \frac{b-2}{4}u_1 + \frac{b}{6}(m_1 - m_2)^2 + \frac{b-2}{2}(m_1 - m_2) \\
&\geq \frac{b}{6}(m_1 - m_2)^2 + \frac{b-2}{2}(m_1 - m_2)
\end{aligned}$$

with equality if and only if $u_1 = 0, u_2 = m_1 - m_2$.

Next we show that $\mu_{\lambda,2} = (m_1 - m_2)\lambda_2 \in S_{\lambda,2}$. Indeed,

1. If $m_1 - m_2 \equiv 0 \pmod 2$ then $\mu_{\lambda,2} = 2\nu - (\rho - \sigma(\rho)) \in S_{\lambda,2}$ where $\nu = \frac{m_1 - m_2}{2}\lambda_2 \in \Pi_\lambda$ and $\sigma = 1$.

2. If $m_1 - m_2 \equiv 1 \pmod 2$ then $\mu_{\lambda,2} = 2\nu - (\rho - \sigma(\rho)) \in S_{\lambda,2}$ where $\nu = \frac{m_1 - m_2 + 3}{2}\lambda_2 \in \Pi_\lambda$ and ρ such that $\rho - \sigma(\rho) = 3\lambda_2$.

Note that from the formula for $m^{\mu}_{\lambda,2}$ in Equations (58) and (59) we have $m^{(m_1 - m_2)\lambda_2}_{\lambda,2} = 1$ which proves part (a). Part (b) is obvious. The case $m_1 \leq m_2$ is similar.

Case 2: $a = 3$. From Equation (26), we have

$$m^0_{\lambda,3} = m^0_\lambda + m^{\lambda_1}_\lambda + m^{\lambda_2}_\lambda$$

Since the fundamental group for A_2 consists of only three elements $0,\ \lambda_1,\ \lambda_2$, at least one of the terms on the right hand side is greater than zero. Therefore $m^0_{\lambda,3} > 0$ and it follows from Lemma 2.8.1 that $\mu_{\lambda,3} = 0$ for all λ. Therefore part (b) follows. Part (a) follows from Corollary 2.8.2 and the fact that $m^0_{\lambda,3} > 0$.

Case 3: $a \geq 4$.

Claim. At most one term on the right hand side of Equation (25) is nonzero.

Proof. Indeed, if there are σ_1, σ_2 in the Weyl group for A_2 such that $m_\lambda^{\frac{\mu+\rho-\sigma_1(\rho)}{a}} \neq 0$ and $m_\lambda^{\frac{\mu+\rho-\sigma_2(\rho)}{a}} \neq 0$ then $\frac{\mu+\rho-\sigma_1(\rho)}{a} - \frac{\mu+\rho-\sigma_2(\rho)}{a} \in \Lambda_r$. Equivalently, $(\rho-\sigma_1(\rho))-(\rho-\sigma_2(\rho)) \in a\Lambda_r$. This is a contradiction since $a \geq 4$ and by [36],

$$\rho - \sigma(\rho) = \sum_{\alpha\in\Delta^+:\sigma^{-1}(\alpha)\in\Delta^-} \alpha$$

which do not belong to $a\Lambda_r$ if $a \geq 4$. Here Δ^+ is the set of positive roots and $\Delta^- = -\Delta^+$. □

Case 3.1: $\lambda \in \Lambda_r$, i.e., $m_1 - m_2 \equiv 0 \bmod 3$. By the above claim we have $m^0_{\lambda,a} = m_\lambda^{\frac{\rho-\sigma(\rho)}{a}}$ for some σ. It's easy to see that the only σ for which $\frac{\rho-\sigma(\rho)}{a}$ is a weight is when $\sigma = 1$ and therefore $m^0_{\lambda,a} = m^0_\lambda > 0$. It follows from Lemma 2.8.1 that $\mu_{\lambda,a} = 0$. Therefore part (b) follows for this case. Part (a) follows from Corollary 2.8.2 and the fact that $m^0_{\lambda,3} > 0$.

Case 3.2: If $\lambda \notin \Lambda_r$, or equivalently $m_1 - m_2 \not\equiv 0 \bmod 3$ then $m^0_{\lambda,a} = m^0_\lambda = 0$ so $\mu_{\lambda,a} \neq 0$. By the above claim, we have

$$m^\mu_{\lambda,a} = (-1)^\sigma m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{a}}$$

for someσ. Furthermore, $m_\lambda^{\frac{\mu+\rho-\sigma(\rho)}{a}} \neq 0$ if and only if $\frac{\mu+\rho-\sigma(\rho)}{a} = \nu \in \Pi_\lambda$ or equivalently, $\mu = a\nu - (\rho - \sigma(\rho))$. Let $\rho - \sigma(\rho) = s\lambda_1 + t\lambda_2$, where

(s,t)	$(0,0)$	$(-1,2)$	$(1,-2)$	$(0,3)$	$(3,0)$	$(2,2)$
$(-1)^\sigma$	1	-1	-1	1	1	-1

(60)

So if $\nu = v_1\lambda_1 + v_2\lambda_2$ then $\mu = (av_1 - s)\lambda_1 + (av_2 - t)\lambda_2$. Since μ is a positive weight, we have we have

$$av_1 - s \geq 0$$

$$av_2 - t \geq 0$$

Since $a \geq 4$ and $|s|, |t| \leq 3$, these inequalities imply that $v_1, v_2 \geq 0$, i.e., ν is also a positive weight. There are two possibilities for λ.

Case 3.2.1: $\lambda_1 \in \Pi_\lambda$, i.e., $m_1 \equiv m_2 + 1 \bmod 3$. Then we can choose $\nu_0 = \lambda_1$ and σ_0 to be the unique element in W such that $\rho - \sigma_0(\rho) = 3\lambda_1$. We will prove that $\mu_{\lambda,a} = a\nu_0 - (\rho - \sigma_0(\rho)) = (a-3)\lambda_1$ is the minimizer. Indeed, let $\mu = a\nu - (\rho - \sigma(\rho)) \in S_{\lambda,a}$ where $\nu \in \Pi_\lambda$ as above.

Case 3.2.1.1: If $\nu = \lambda_1$ then for μ to be a dominant weight we should have, according to Table (60),

$$\rho - \sigma(\rho) = \begin{cases} 0 & \text{which gives } \mu = a\lambda_1 \\ \lambda_1 - 2\lambda_2 & \text{which gives } \mu = (a-1)\lambda_1 + 2\lambda_2 \\ 3\lambda_1 & \text{which gives } \mu = (a-3)\lambda_1 = \mu_{\lambda,a} \end{cases}$$

It is easy to check that $g^*(\mu) > g^*(\mu_{\lambda,a})$ for the first two values of μ.

Case 3.2.1.2: If $\nu \neq \lambda_1$, let $\nu = v_1\lambda_1 + v_2\lambda_2$ then we have $v_1, v_2 \geq 0$ and $v_1 + v_2 \geq 3$, since the only cases where $v_1 + v_2 < 3$ are $\nu = \lambda_2$ and $\lambda_1 + \lambda_2$ but these weights donot belong in Π_λ. Let $\nu = a\nu - (\rho - \sigma(\rho)) = (av_1 - s)\lambda_1 + (av_2 - t)\lambda_2$ as before. We have

$$\begin{aligned} g^*(\mu) &= \frac{b}{2a}(\mu,\mu) + \left(-1 + \frac{b}{a}\right)(\mu,\rho) \\ &= \frac{b}{3a}(u_1^2 + u_1u_2 + u_2^2) + (-1 + \frac{b}{a})(u_1 + u_2) \\ &= \frac{b}{3a}(a^2(v_1^2 + v_1v_2 + v_2^2) - 2a(v_1 + v_2)(s+t) + s^2 + st + t^2) \\ &+ (-1 + \frac{b}{a})(a(v_1 + v_2) - s - t) \end{aligned}$$

It is easy to check that for all $(s,t) \in \{(0,0), (-1,2), (1,-2), (0,3), (3,0), (2,2)\}$ and $(v_1, v_2) : v_1, v_2 \geq 0, v_1 + v_2 \geq 3$, we have

$$a^2(v_1^2 + v_1v_2 + v_2^2) - 2a(v_1 + v_2)(s+t) + s^2 + st + t^2 > (a-3)^2$$

$$a(v_1 + v_2) - s - t > a - 3$$

and therefore $g^*(\mu) > \frac{b}{3a}(a-3)^2 + (-1 + \frac{b}{a})(a-3) = g^*(\mu_{\lambda,a})$ for all $\mu \neq \lambda_1$.

The above argument showed that $\mu_{\lambda,a} = (a-3)\lambda_1$ is the unique minimizer, and note that $m_{\lambda,a}^{(a-3)\lambda_1} = m_\lambda^{\lambda_1} \neq 0$ since $\lambda_1 \in \Pi_\lambda$. This proves parts (a) and (b) for Case 3.2.1.

Case 3.2.2: $\lambda_2 \in \Pi_\lambda$ or equivalently, $m_1 \equiv m_2 + 2 \bmod 3$. The proof for this is identical to the one above.

This completes the proof of Theorem 2.2.4 for A_2. □

2.8.2 Theorem 2.2.4 for B_2

There are two simple roots $\{\alpha_1, \alpha_2\}$ and four positive roots $\{a_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$ of B_2 shown in Figure 2. The Kostant partition function $p(u,v) = p(u\alpha_1 + v\alpha_2)$ is given by [50]

$$p(u,v) = \begin{cases} b(v) & \text{if } u \geq v \\ b(v) - \frac{(v-u)(v-u+1)}{2} & \text{if } u \leq v \leq 2u \\ \frac{(u+1)(v+2)}{2} & \text{if } 2u \leq v \end{cases}, \qquad b(n) = \frac{n^2}{4} + n + \begin{cases} 1 & \text{if } 2|n \\ \frac{3}{4} & \text{if } 2 \not| n \end{cases}. \tag{61}$$

There are three Kostant chambers shown in Figure 2. Let $\lambda = m_1\lambda_1 + m_2\lambda_2$ denote

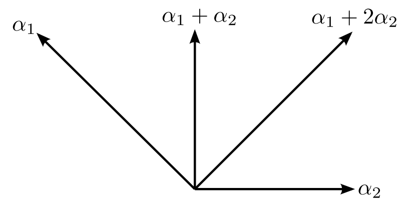


Figure 2: The three chambers of the Kostant partition function of B_2. Kostant chambers from left to right: $u \geq v$, $u \leq v \leq 2u$, $u \geq 2v$.

a dominant weight. In weight coordinates we have

$$\rho - \sigma(\rho) = s\lambda_1 + t\lambda_2$$

where

(s,t)	$(0,0)$	$(2,-2)$	$(-1,2)$	$(-1,4)$	$(3,-2)$	$(3,0)$	$(0,4)$	$(2,2)$
$(-1)^\sigma$	1	−1	−1	1	1	−1	−1	1

(62)

Lemma 2.2.1 implies that

$$m^0_{\lambda,a} = m^0_\lambda + \begin{cases} -m^{\lambda_2}_\lambda - m^{2\lambda_2}_\lambda + m^{\lambda_1+\lambda_2}_\lambda & \text{if} \quad a = 2 \\ -m^{\lambda_1}_\lambda & \text{if} \quad a = 3 \\ -m^{\lambda_2}_\lambda & \text{if} \quad a = 4 \\ 0 & \text{if} \quad a \geq 5 \end{cases} \tag{63}$$

Case 1: $a = 2$. Equation (63) implies that

$$m^0_{\lambda,2} = m^0_\lambda - m^{2\lambda_2}_\lambda - m^{\lambda_2}_\lambda + m^{\lambda_1+\lambda_2}_\lambda\,. \tag{64}$$

Case 1.1: $\lambda \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$. In this case, we have $\lambda_1 + \lambda_2$, $\lambda_2 \notin \Lambda_r$, and therefore $m_\lambda^{\lambda_2} = m_\lambda^{\lambda_1+\lambda_2} = 0$. Equation (64) becomes

$$m_{\lambda,2}^0 = m_\lambda^0 - m_\lambda^{2\lambda_2} = 1\,.$$

where the later equality comes from formula (61) and the Kostant multiplicity formula (24). It follows from Lemma 2.8.1 that $\mu_{\lambda,2} = 0$ which proves part (b). Part (a) follows from Corollary 2.8.2 and the fact that $m_{\lambda,2}^0 = 1 \neq 0$.

Case 1.2: $\lambda \notin \Lambda_r$, i.e., $m_2 \equiv 1 \bmod 2$. Since $m_\lambda^0 = m_\lambda^{2\lambda_2} = 0$ we have

$$m_{\lambda,2}^0 = m_\lambda^{\lambda_1+\lambda_2} - m_\lambda^{\lambda_2} = -1\,.$$

If $m_1 > 0$ then choose $\nu = \lambda_1 + \lambda_2 \in \Pi_\lambda$ and σ such that $\rho - \sigma(\rho) = 2\lambda_1 + 2\lambda_2$ we obtain $\mu_{\lambda,2} = 2\nu - (\rho - \sigma(\rho)) = 2(\lambda_1 + \lambda_2) - (2\lambda_1 + 2\lambda_2) = 0$. If otherwise $m_1 = 0$ then we choose $\nu = \lambda_2 \in \Pi_\lambda$, σ such that $\rho - \sigma(\rho) = -\lambda_1 + 2\lambda_2$ and get $\mu_{\lambda,2} = 2\lambda_2 - (-\lambda_1 + 2\lambda_2) = \lambda_1$. This proves part (b). Part (a) follows from Corollary 2.8.2 and the fact that $m_{\lambda,2}^0 = -1 \neq 0$.

Case 2: $a = 3$. Consider two small cases.

Case 2.1: If $\lambda = m_1\lambda_1 + m_2\lambda_2 \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$ then we have

$$m_{\lambda,3}^0 = m_\lambda^0 - m_\lambda^{\lambda_1} = \frac{1}{2} + \frac{(-1)^{m_1+m_2} + (-1)^{m_1+m_2+2}}{4}$$

If $m_1 \equiv 0 \bmod 2$ then $m_{\lambda,3}^0 = 1$. It follows from Lemma 2.8.1 that $\mu_{\lambda,3} = 0$ and this completes part (b). Part (a) follows from Corollary 2.8.2 and the fact that $m_{\lambda,3}^0 = 1 \neq 0$.

If $m_1 \equiv 1 \bmod 2$ then $m_{\lambda,3}^0 = 0$. By a similar argument to the one in Case 3 for A_2 it can be shown that $\mu_{\lambda,3} = 2\lambda_2$ is the unique minimizer and parts (a) and (b) follow.

Case 2.2: If $\lambda = m_1\lambda_1 + m_2\lambda_2 \notin \Lambda_r$, i.e., $m_2 \not\equiv 0 \bmod 2$ then by a similar argument to the one in Case 3 for A_2 we have $\mu_{\lambda,3} = \lambda_1 + \lambda_2$ is the unique minimizer and $m_{\lambda,3}^{\lambda_1+\lambda_2} \neq 0$ which completes the proof.

Case 3: $a = 4$. From Equation (26) we have

$$m_{\lambda,4}^0 = m_\lambda^0 - m_\lambda^{\lambda_2}$$

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$ then we have $m_{\lambda,4}^0 = m_\lambda^0 - m_\lambda^{\lambda_2} = m_\lambda^0 > 0$, since $0 \in \Pi_\lambda$.

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \notin \Lambda_r$, i.e., $m_2 \not\equiv 0 \bmod 2$ then $m_{\lambda,4}^0 = m_\lambda^0 - m_\lambda^{\lambda_2} = -m_\lambda^{\lambda_2} < 0$, since $\lambda_2 \in \Pi_\lambda$.

It follows from Lemma 2.8.1 that $\mu_{\lambda,4} = 0$, which completes part (b). Part (a) follows from Corollary 2.8.2.

Case 4: $a \geq 5$. The only σ for which $\mu = \frac{\rho-\sigma(\rho)}{a}$ is a weight is $\sigma = 1$ and hence $\mu = 0$. So from Equation (26) we have $m_{\lambda,a}^0 = m_\lambda^0$.

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \in \Lambda_r$, i.e., $m_2 \equiv 0 \bmod 2$ then $m_{\lambda,a}^0 = m_\lambda^0 > 0$. It follows from Lemma 2.8.1 that $\mu_{\lambda,a} = 0$, which completes part (b). Part (a) follows from Corollary 2.8.2.

If $\lambda = m_1\lambda_1 + m_2\lambda_2 \notin \Lambda_r$, i.e., $m_2 \not\equiv 0 \bmod 2$ then by a similar argument to the one in Case 3 for A_2 we have that $\mu_{\lambda,a} = (a-4)\lambda_2$ is the unique minimizer and $m_{\lambda,a}^{(a-4)\lambda_2} = m_\lambda^{\lambda_2} \neq 0$. This completes both parts (a) and (b).

This completes the proof of Theorem 2.2.4 for B_2. □

2.8.3 Theorem 2.2.4 for G_2

There are two simple roots $\{\alpha_1, \alpha_2\}$ and six positive roots $\{a_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ of G_2 shown in Figure 3. The Kostant partition function

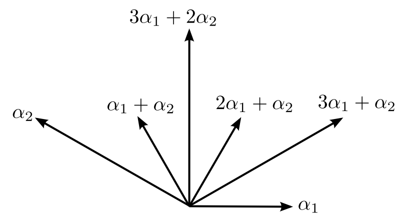


Figure 3: The five chambers of the Kostant partition function of G_2. Kostant chambers from left to right: $u \le v$, $v \le u \le \frac{3}{2}v$, $\frac{3}{2}v \le u \le 2v$, $2v \le u \le 3v$, $3v \le u$.

$p(u, v) = p(u\alpha_1 + v\alpha_2)$ is given by [50]

$$p(u,v) = \begin{cases} g(u) & \text{if} \quad u \le v \\ g(u) - h(u-v-1) & \text{if} \quad v \le u \le \frac{3}{2}v \\ h(v) - g(3v-u-1) + h(2v-u-2) & \text{if} \quad \frac{3}{2}v \le u \le 2v \\ h(v) - g(3v-u-1) & \text{if} \quad 2v \le u \le 3v \\ h(v) & \text{if} \quad 3v \le u \end{cases} \tag{65}$$

where

$$g(n) = \begin{cases} \frac{1}{432}(n+6)(n^3+14n^2+54n+72) & \text{if} \quad n \equiv 0 \bmod 6 \\ \frac{1}{432}(n+5)^2(n^2+10n+13) & \text{if} \quad n \equiv 1 \bmod 6 \\ \frac{1}{432}(n+4)(n^3+16n^2+74n+68) & \text{if} \quad n \equiv 2 \bmod 6 \\ \frac{1}{432}(n+3)^2(n+5)(n+9) & \text{if} \quad n \equiv 3 \bmod 6 \\ \frac{1}{432}(n+2)(n+8)(n^2+10n+22) & \text{if} \quad n \equiv 4 \bmod 6 \\ \frac{1}{432}(n+1)(n+5)(n+7)^2 & \text{if} \quad n \equiv 5 \bmod 6 \end{cases} \tag{66}$$

and

$$h(n) = \begin{cases} \frac{1}{48}(n+2)(n+4)(n^2+6n+6) & \text{if} \quad n \equiv 0 \bmod 2 \\ \frac{1}{48}(n+1)(n+3)^2(n+5) & \text{if} \quad n \equiv 1 \bmod 2 \end{cases} \tag{67}$$

From Lemma 2.2.1 we have

$$m^0_{\lambda,a} = m^0_\lambda + \begin{cases} -m_\lambda^{3\alpha_1+\alpha_2} - m_\lambda^{2\alpha_1+2\alpha_2} + m_\lambda^{5\alpha_1+3\alpha_2} & \text{if} \quad a=2 \\ -m_\lambda^{3\alpha_1+2\alpha_2} & \text{if} \quad a=3 \\ -m_\lambda^{\alpha_1+\alpha_2} & \text{if} \quad a=4 \\ -m_\lambda^{2\alpha_1+\alpha_2} & \text{if} \quad a=5 \\ 0 & \text{if} \quad a \geq 6 \end{cases} \tag{68}$$

From now on, let us consider $\lambda = u\alpha_1 + v\alpha_2 \in \Lambda^+$, so $\frac{3}{2}v \leq u \leq 2v$.

Case 1: $a = 2$. We have

$$m^0_{\lambda,2} = m_\lambda - m_\lambda^{3\alpha_1+\alpha_2} - m_\lambda^{2\alpha_1+2\alpha_2} + m_\lambda^{5\alpha_1+3\alpha_2} \tag{69}$$

Using the Kostant multiplicity formula we can calculate the weight multiplicities on the right hand side of Equation (69), we have, for example

$$\begin{aligned} m^0_\lambda &= \sum_{\sigma\in W} (-1)^\sigma p(\sigma(\lambda+\rho)-\rho) \\ &= p(u,v) - p(-u+3v-1,v) - p(u,u-v-1) + p(3v-u-1,2v-u-2) \\ &+ p(2u-3v-4,u-v-1) \\ &= \frac{u^4}{9} - \frac{29u^3v}{36} - \frac{7u^3}{36} + \frac{17u^2v^2}{8} + \frac{2u^2v}{3} - \frac{19u^2}{24} - \frac{29uv^3}{12} - \frac{uv^2}{2} + 3uv \\ &+ v^4 - \frac{v^3}{12} - \frac{21v^2}{8} + c_{1,0}(u)u + c_{0,1}(v)v + c_{0,0}(u,v) \end{aligned}$$

where

$$c_{1,0}(u) = \begin{cases} \frac{1}{4} & \text{if } u \equiv 0 \bmod 3 \\ \frac{17}{36} & \text{if } u \equiv 1 \bmod 3 \\ \frac{25}{36} & \text{if } u \equiv 2 \bmod 3 \end{cases}, \quad c_{0,1}(u) = \begin{cases} \frac{1}{12} & \text{if } u \equiv 0 \bmod 3 \\ -\frac{13}{36} & \text{if } u \equiv 1 \bmod 3 \\ -\frac{29}{36} & \text{if } u \equiv 2 \bmod 3 \end{cases}$$

$$c_{0,0}(u,v) = \begin{cases} 1 & \text{if } u \equiv 0 \bmod 6,\ v \equiv 0 \bmod 2 \\ \frac{29}{72} & \text{if } u \equiv 1 \bmod 6,\ v \equiv 0 \bmod 2 \\ \frac{5}{9} & \text{if } u \equiv 2 \bmod 6,\ v \equiv 0 \bmod 2 \\ \frac{5}{8} & \text{if } u \equiv 3 \bmod 6,\ v \equiv 0 \bmod 2 \\ \frac{7}{9} & \text{if } u \equiv 4 \bmod 6,\ v \equiv 0 \bmod 2 \\ \frac{13}{72} & \text{if } u \equiv 5 \bmod 6,\ v \equiv 0 \bmod 2 \end{cases}, \quad c_{0,0}(u,v) = \begin{cases} \frac{5}{8} & \text{if } u \equiv 0 \bmod 6,\ v \equiv 1 \bmod 2 \\ \frac{5}{18} & \text{if } u \equiv 1 \bmod 6,\ v \equiv 1 \bmod 2 \\ \frac{13}{72} & \text{if } u \equiv 2 \bmod 6,\ v \equiv 1 \bmod 2 \\ \frac{5}{8} & \text{if } u \equiv 3 \bmod 6,\ v \equiv 1 \bmod 2 \\ \frac{29}{72} & \text{if } u \equiv 4 \bmod 6,\ v \equiv 1 \bmod 2 \\ \frac{13}{72} & \text{if } u \equiv 5 \bmod 6,\ v \equiv 1 \bmod 2 \end{cases}$$

$m_\lambda^{3\alpha_1+\alpha_2}, m_\lambda^{2\alpha_1+2\alpha_2}, m_\lambda^{5\alpha_1+3\alpha_2}$ can be computed similarly to show that $m^0_{\lambda,2} = 1$. This confirms part (b). Part (a) follows Corollary 2.8.2.

Case 2: $a = 3$. We have

$$\begin{aligned} m^0_{\lambda,3} =& m^0_\lambda - m_\lambda^{[3,2]} \\ =& -u^2 + \frac{7uv}{2} + \frac{u}{2} - 3v^2 - \frac{v}{2} + c_{0,0}(u,v) \end{aligned}$$

where

$$c_{0,0}(u,v) = \begin{cases} 1 & \text{if } u \equiv 0, v \equiv 0 \bmod 2 \\ \frac{1}{2} & \text{if } u \equiv 1, v \equiv 0 \bmod 2 \\ \frac{1}{2} & \text{if } v \equiv 1 \bmod 2 \end{cases}$$

Note that since $\frac{3v}{2} \le u \le 2v$, $-u^2 + \frac{7uv}{2} + \frac{u}{2} - 3v^2 - \frac{v}{2} = (-u^2 + \frac{7uv}{2} - 3v^2) + \frac{u-v}{2} \ge 0$ and therefore $m^0_{3,\lambda} > 0$ for all λ. Part (a) follows from Lemma 2.8.1 and part (b) follows from Corollary 2.8.2.

Case 3: $a = 4, 5$. The arguments are similar to that of Case 2.

Case 4: $a \ge 6$, can be done without computations. Indeed, we have $m^0_{\lambda,a} = m^0_\lambda > 0$ since $\lambda \in \Lambda_r$; see [32, §13.4,Lem.B]. Parts (a) and (b) follow from Lemma 2.8.1 and Corollary 2.8.2.

This completes the proof of Theorem 2.2.4 for G_2. □

2.9 *Some tails of the* $T(2,3)$ *and* $T(4,5)$ *torus knots*

2.9.1 The tail for A_2 and the trefoil

In this section we compute the tail of the c-stable sequence $J^{A_2}_{T(2,b),n\lambda_1}(q)$ for $b > 2$ odd. From Proposition 2.2.4 we have $\mu_{n\lambda_1,2} = n\lambda_2$ so Equation (53) gives

$$\hat{J}^{A_2}_{T(2,b),n\lambda_1}(q) = \frac{1}{(1-q)(1-q^{n+1})(1-q^{n+2})}\check{J}^{A_2}_{T(2,b),n\lambda_1}(q)$$

where

$$\check{J}^{A_2}_{T(2,b),n\lambda_1}(q) = \sum_{u_1\lambda_1+u_2\lambda_2\in S_{n\lambda_1,2}} c(u_1,u_2)q^{\frac{b}{6}(u_1^2+u_1u_2+u_2^2-n^2)+(\frac{b}{2}-1)(u_1+u_2-n)} \cdot (1-q^{u_1+1})(1-q^{u_2+1})(1-q^{u_1+u_2+2})$$

and from Cases 1-4 of Section 2.8.1.1,

$$c(u_1,u_2) = m^{u_1\lambda_1+u_2\lambda_2}_{n\lambda_1,2} = \begin{cases} 1 & \text{if } u_1+2u_2 \geq 2n,\ u_1,u_2 \text{ are even} \\ 0 & \text{if } u_1+2u_2 < 2n,\ u_1,u_2 \text{ are even} \\ -1 & \text{if } u_1+2u_2 \geq 2n,\ u_1 \text{ even, } u_2 \text{ odd} \\ 0 & \text{if } u_1+2u_2 < 2n,\ u_1 \text{ even, } u_2 \text{ odd} \\ 0 & \text{if } u_1 \text{ is odd} \end{cases}$$

Lemma 2.9.1. *If* $\mu = u_1\lambda_1 + u_2\lambda_2 \in S_{n\lambda_1,2}$ *then* $u_1 + 2u_2 \leq 2n$.

Proof. By Lemma 2.5.1, we have $\mu \in S_{n\lambda_1,2} \subset P_{2n\lambda_1}$. So by Inequality (46) we have

$$(2n\lambda_1 - u_1\lambda_1 - u_2\lambda_2, \lambda_2) \geq 0 \quad \text{i.e.,} \quad u_1 + 2u_2 \leq 2n$$

□

From Corollary 2.8.3 and Lemma 2.9.1 we have

Corollary 2.9.2. $c(u_1,u_2) \neq 0$ *if and only if* $u_1 + 2u_2 = 2n$.

Proof of Theorem 2.1.9. Set $s = \frac{u_1}{2} = n - u_2$, then $u_1^2 + u_1u_2 + u_2^2 - n^2 = 3s^2$ and we have

$$\check{J}^{\mathfrak{g}}_{T(2,b),n\lambda_1}(q) = \frac{\sum_{s=0}^{n}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}(1-q^{2s+1})(1-q^{n-s+1})(1-q^{n+s+2})}{(1-q)(1-q^{n+1})(1-q^{n+2})}$$

Replacing q^n by x and using Lemma 2.3.1 it follows that $(\hat{J}^{A_2}_{T(2,b),n\lambda_1}(q))$ is c-stable and its tail $G_b(x,q)$ is given by

$$\begin{aligned} G_b(x,q) &= \frac{\sum_{s=0}^{\infty}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}(1-q^{2s+1})(1-xq^{1-s})(1-xq^{s+2})}{(1-q)(1-qx)(1-q^2x)} \\ &= \frac{\sum_{s=0}^{\infty}(-1)^s((q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s} - q^{\frac{b}{2}s^2+(\frac{b}{2}+1)s+1})(1+q^3x^2) + (q^{\frac{b}{2}s^2+(\frac{b}{2}+2)s+3} - q^{\frac{b}{2}s^2+(\frac{b}{2}-2)s+1})x)}{(1-q)(1-qx)(1-q^2x)} \end{aligned}$$

Using $s = t+1$, we have

$$\sum_{s=0}^{\infty}(-1)^{s+1}q^{\frac{b}{2}s^2+(\frac{b}{2}+1)s+1} = \sum_{t=-1}^{\infty}(-1)^{-t}q^{\frac{b}{2}(t+1)^2-(\frac{b}{2}+1)(t+1)+1} = \sum_{s\leq -1}(-1)^t q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s}$$

Therefore,

$$\sum_{s=0}^{\infty}(-1)^s(q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s} - q^{\frac{b}{2}s^2+(\frac{b}{2}+1)s+1}) = \sum_{s=-\infty}^{\infty}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}-1)s} = \theta_{b,\frac{b}{2}-1}(q)$$

Similarly,

$$\sum_{s=0}^{\infty} q^{\frac{b}{2}s^2+(\frac{b}{2}+2)s+3} - q^{\frac{b}{2}s^2+(\frac{b}{2}-2)s+1} = \sum_{s=-\infty}^{\infty}(-1)^s q^{\frac{b}{2}s^2+(\frac{b}{2}+2)s+3} = q^3\theta_{b,\frac{b}{2}+2}(q)$$

Thus,

$$G_b(x,q) = \frac{\theta_{b,\frac{b}{2}-1}(q)(1+q^3x^2) + q^3\theta_{b,\frac{b}{2}+2}(q)x}{(1-q)(1-qx)(1-q^2x)}$$

Note that by replacing s with $s+1$ or s by $-s$ in Equation (20) it follows that

$$\theta_{b,c}(q) = -q^{\frac{b}{2}+c}\theta_{b,b+c}(q), \qquad \theta_{b,-c}(q) = \theta_{b,c}(q)$$

To compute $G_3(x,q)$, use $b = 3, c = \frac{1}{2}$ in the above equation and *Euler's Pentagonal Theorem* (discussed in detail in [7]) to obtain that

$$q^2\theta_{3,\frac{7}{2}}(q) = -\theta_{3,\frac{1}{2}}(q) = -(q)_\infty$$

This completes the proof of Theorem 2.1.9. $\square$

2.9.2 The tail for A_2 and the $T(4,5)$ torus knot

In this section we compute the tail for the c-stable sequence $(J^{A_2}_{T(4,b),n\rho}(q))$ for $b > 4$ odd. This example shows that c-stability is a necessary notion for Conjecture 2.1.5.

Let

$$A_{b,0}(q) = \sum_{(s,t)} \sum_{(u_1,u_2)} \epsilon_{s,t} c_{s,t}(u_1,u_2) q^{\frac{b}{12}(u_1^2+u_1u_2+u_2^2)+(\frac{b}{4}-1)(u_1+u_2)}(1-q^{u_1+1})(1-q^{u_2+1})(1-q^{u_1+u_2+2})$$

$$A_{b,1}(q) = \sum_{(s,t)} \sum_{(u_1,u_2)} \epsilon_{s,t} q^{\frac{b}{12}(u_1^2+u_1u_2+u_2^2)+(\frac{b}{4}-1)(u_1+u_2)}(1-q^{u_1+1})(1-q^{u_2+1})(1-q^{u_1+u_2+2})$$

where the (s,t) summation is over the set

(s,t)	$(0,0)$	$(2,-1)$	$(-1,2)$	$(0,3)$	$(3,0)$	$(2,2)$
$\epsilon_{s,t}$	1	-1	-1	1	1	-1

(70)

and $(u_1,u_2) \in \mathbb{N}^2$ satisfies $u_1 \equiv -s \bmod 4$, $u_1 - u_2 \equiv t - s \bmod 12$ and

$$c_{s,t}(u_1,u_2) = \begin{cases} 1 - \frac{2u_1+u_2+2s+t}{12} & \text{if } u_1+s \geq u_2+t \\ 1 - \frac{u_1+2u_2+s+2t}{12} & \text{if } u_1+s \leq u_2+t \end{cases}$$

Proposition 2.9.3. *The tail of the c-stable sequence $(\hat{J}^{A_2}_{T(4,b),n\rho}(q))$ is given by*

$$\frac{1}{(1-xq)^2(1-x^2q^2)}\left(A_{b,0}(q) + nA_{b,1}(q)\right)$$

Proof. We will use Theorem 2.6.2 and unravel its notation. To begin with, for $a = 4$, we have

$$\mathcal{L}_{n\rho,4} = \bigcup_{\sigma \in W} 4n\rho + \sigma(\rho) - \rho + 4\Lambda_r = \bigcup_{\sigma\in W} \sigma(\rho) - \rho + 4\Lambda_r = \bigcup_{\sigma \in W} \{\mu \in \Lambda | \mu + \rho - \sigma(\rho) \in 4\Lambda_r\}$$

Since $\rho = \alpha_1 + \alpha_2 \in \Lambda_r$, we have $\mathcal{L}_{n\rho,4} = \mathcal{L}_{\rho,4}$ for all natural numbers n. Let $\mu = u_1\lambda_1 + u_2\lambda_2$ and $\rho - \sigma(\rho) = s\lambda_1 + t\lambda_2$ where (s,t) are given in (70) and $(-1)^\sigma = \epsilon_{s,t}$ as in (70). In weight coordinates we have

$$\mathcal{L}_{\rho,4} = \bigcup_{(s,t)} \{(u_1,u_2) \in \mathbb{Z}^2 : u_1 \equiv -s \bmod 4, u_1 - u_2 \equiv t - s \bmod 12\} \qquad (71)$$

Next we compute the plethysm multiplicities. Equation (25) implies that

$$m^{\mu}_{n\rho,4} = \sum_{\sigma\in W} (-1)^{\sigma} m_{n\rho}^{\frac{\mu+\rho-\sigma(\rho)}{4}}$$
$$= m_{n\rho}^{\frac{\mu}{4}} - m_{n\rho}^{\frac{\mu+2\lambda_1-\lambda_2}{4}} - m_{n\rho}^{\frac{\mu-\lambda_1+2\lambda_2}{4}} + m_{n\rho}^{\frac{\mu+3\lambda_1}{4}} + m_{n\rho}^{\frac{\mu+3\lambda_2}{4}} - m_{n\rho}^{\frac{\mu+2\lambda_1+2\lambda_2}{4}}$$

Since $n\rho \in \Lambda_r$, $m^{\nu}_{n\rho} \neq 0$ only if $\nu \in \Lambda_r$. Therefore at most one of the terms in the above equation is non-zero. Equation (24) gives

$$m^{\mu}_{n\rho} = \begin{cases} 1 + \frac{2m_1+m_2}{3} - \frac{2u_1+u_2}{3} & \text{if} \quad u_1 \geq u_2 \\ 1 + \frac{m_1+2m_2}{3} - \frac{u_1+2u_2}{3} & \text{if} \quad u_1 \leq u_2 \end{cases}$$

Therefore

$$m^{\mu}_{n\rho,4} = \epsilon_{s,t} \begin{cases} 1 + n - \frac{2u_1+u_2+2s+t}{12} & \text{if } u_1 \equiv -s \bmod 4, u_1 - u_2 \equiv t - s \bmod 12\,, u_1 + s \geq u_2 + t \\ 1 + n - \frac{u_1+2u_2+s+2t}{12} & \text{if } u_1 \equiv -s \bmod 4, u_1 - u_2 \equiv t - s \bmod 12\,, u_1 + s \leq u_2 + t \end{cases}$$

where $\epsilon_{s,t}$ is given from (70). Since $\mu_{n\rho,4} = 0$, we have $\hat{\mathcal{L}}_{n\rho,4} = \mathcal{L}_{n\rho,4}$, $\hat{P}_{n\rho} = P_{n\rho}$, $\hat{S}_{n\rho,4} = S_{n\rho,4}$. Theorem 2.6.2 concludes the proof of Proposition 2.9.3. □

Exercise 2.9.4. *Show that*

$$A_{b,1}(q) = \sum_{m_1,m_2\in\mathbb{Z}} q^{4b(m_1^2+3m_1m_2+3m_2^2)+(b-4)(2m_1+3m_2)}(1-q^{4m_1+1})(1-q^{4m_1+12m_2+1})(1-q^{8m_1+12m_2+2}) \tag{72}$$

The above equation shows that $A_{b,0}(q)$ is a sum of theta series of rank 2, hence a modular form of weight 1; see [9].

CHAPTER III

ALTERNATING KNOTS, PLANAR GRAPHS AND Q-SERIES

3.1 Introduction

3.1.1 q-series in Quantum Knot Theory

Recent developments in Quantum Topology associate q-series to a knot K in at least three different ways:

- via stability of the coefficients of the colored Jones polynomial of K,
- via the 3D index of K,
- via the conversion of state-integrals of the quantum dilogarithm to q-series.

The first method is developed of alternating knots in detail, see [3, 2, 4] and also [23]. The second method uses the 3D index of an ideal triangulation introduced in [14, 13], with necessary and sufficient conditions for its convergence established in [18] and its topological invariance (i.e., independence of the ideal triangulation) for hyperbolic 3-manifolds with torus boundary proven in [21]. The third method was developed in [22].

In all three methods, the q-series are multi-dimensional q-hypergeometric series of generalized Nahm type; see [23, Sec.1.1]. Their modular and the asymptotic properties remains unknown. Some empirical results and relations among these q-series are given in [29, 30].

This chapter focuses on the q-series obtained by the first method. For some alternating knots, the q-series obtained by the first method can be identified with a finite product of unary theta or false theta series; see [4, 1]. This was observed independently by the first author and Zagier in 2011 for all alternating knots in the Rolfsen table [46] up to the knot 8_4. Ideally, one might expect this to be the case for all alternating knots. For the knot 8_5 however, the first 100 terms of its q-series failed to identify it with a reasonable finite product of unary theta or false theta series. This computation was performed by the first author at the request of Zagier and the result was announced in [19, Sec.6.4].

The purpose of the chapter is to give the details of the above computation and to extend it systematically to all alternating knots and links with at most 8 crossings. Our computational approach is similar to the computation of the index of a knot given in [21, Sec.7].

3.1.2 Rooted plane graphs and their q-series

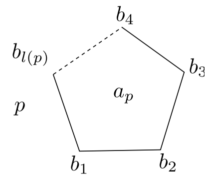
By *planar graph* we mean an abstract graph, possibly with loops and multiple edges, which can be embedded on the plane. A *plane graph* (also known as a planar map) is an embedding of a planar graph to the plane. A *rooted plane map* is a plane map together with the choice of a vertex of the unbounded region.

In [23] Le and the first author introduced a function

$$\Phi : \{\text{Rooted plane graphs}\} \longrightarrow \mathbb{Z}[[q]], \qquad G \mapsto \Phi_G(q)$$

For the precise relation between $\Phi_G(q)$ and the colored Jones function of the corresponding alternating link L_G, see Section 3.2. To define $\Phi_G(q)$, we need to introduce some notation. An *admissible state* (a, b) of G is an integer assignment a_p for each face p of G and b_v for each vertex v of G such that $a_p + b_v \geq 0$ for all pairs (v, p) such that v is a vertex of p. For the unbounded face p_∞ we set $a_\infty = 0$ and thus $b_v = a_\infty + b_v \geq 0$ for all $v \in p_\infty$. We also set $b_v = 0$ for a fixed vertex v of p_∞. In the formulas below, v, w will denote vertices of G, p a face of G and p_∞ is the unbounded face. We also write $v \in p$, $vw \in p$ if v is a vertex and vw is an edge of p.

For a polygon p with $l(p)$ edges and vertices $b_1, \dots, b_{l(p)}$ in counterclockwise order



we define

$$\gamma(p) = l(p)a_p^2 + 2a_p(b_1 + b_2 + \cdots + b_{l(p)}) \,.$$

Let

$$A(a,b) = \sum_p \gamma(p) + 2 \sum_{e=(v_i v_j)} b_{v_i} b_{v_j} \tag{73}$$

where the p-summation (here and throughout the chapter) is over the set of *bounded* faces of G and the e-summation is over the set of edges $e = (v_i v_j)$ of p, and

$$B(a,b) = 2\sum_v b_v + \sum_p (l(p)-2)a_p \tag{74}$$

where the v-summation is over the set of vertices of G and the p-summation is over the set of bounded faces of G.

Definition 3.1.1. *[23] With the above notation, we define*

$$\Phi_G(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod_{(p,v):v\in p} (q)_{a_p+b_v}} \tag{75}$$

where the sum is over the set of all admissible states (a,b) of G, and in the product $(p,v) : v \in p$ means a pair of face p and vertex v such that p contains v. Here, c_2 is the number of edges of G and

$$(q)_\infty = \prod_{n=1}^{\infty} (1-q)^n = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} \dots$$

Convergence of the q-series of Equation (75) in the formal power series ring $\mathbb{Z}[[q]]$ is not obvious, but was shown in [23]. Below, we give effective (and actually optimal) bounds for convergence of $\Phi_G(q)$. To phrase them, let $b_p = \min\{b_v : v \in p\}$ where p denotes a face of G.

Theorem 3.1.2. (a) We have

$$A(a,b) = \sum_p \left(l(p)(a_p+b_p)^2 + 2(a_p+b_p)\sum_{v\in p}(b_v-b_p) \right. \tag{76}$$
$$\left. + \sum_{vv'\in p}(b_v-b_p)(b_{v'}-b_p) \right) + \sum_{vv'\in p_\infty} b_v b_{v'} \,.$$

Each term in the above sum is manifestly nonnegative.

(b) $B(a,b)$ can also be written as a finite sum of manifestly nonnegative linear forms on (a,b).

(c) If $\frac{1}{2}(A(a,b)+B(a,b)) \leq N$ for some natural number N, then for every i and every j there exist c_i, c_i' and c_j, c_j' (computed effectively from G) such that

$$c_i N \leq b_i \leq c_i' N, \qquad c_j'\sqrt{N} \leq a_j \leq c_j N + c_j'\sqrt{N} \,.$$

For a detailed illustration of the above Theorem, see Section 3.5.1.

3.1.3 Properties of the q-series of a planar graph

The next lemma summarizes some properties of the series $\Phi_G(q)$. Part (a) of the next lemma is taken from [23, Thm.1.7] [23, Lem.13.2]. Parts (b) and (c) were observed in [4] and [23] and follow easily from the behavior of the colored Jones polynomial under disjoint union and under a connected sum. Note that we use the normalization that the colored Jones polynomial of the unknot is 1. Part (d) was proven in [4] and [23, Lem.13.3].

Lemma 3.1.3. *[4, 23]* (a) The series $\Phi_G(q)$ depends only on the abstract planar graph G and not on the rooted plane map.

(b) If $G = G_1 \sqcup G_2$ is disconnected, then

$$(1-q)\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q) \,.$$

(c) If G has a separating edge (also known as a bridge) e and $G \setminus \{e\} = G_1 \sqcup G_2$, then

$$\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q) \,.$$

(d) If G is a planar graph (possibly with multiple edges and loops) and G' denotes the corresponding simple graph obtained by removing all loops and replacing all edges of multiplicity more than with edges of multiplicity one, then

$$\Phi_G(q) = \Phi_{G'}(q)\,.$$

So, we can focus our attention to simple, connected planar graphs. In the remaining of the chapter, unless otherwise stated, G will denote a *simple* planar graph. Let $\langle f(g) \rangle_k$ denote the coefficient of q^k of $f(q) \in \mathbb{Z}[[q]]$. The next theorem was proven in [12] using properties of the Kauffman bracket skein module. We give an independent proof using combinatorics of planar graphs in Section 3.4. Our proof allows us to compute the coefficient of q^3 in $\Phi_G(q)$, observing a new phenomenon related to induced embeddings, and guess the coefficients of q^4 and q^5 in $\Phi_G(q)$. This is discussed in a subsequent publication [28].

Theorem 3.1.4. *[12] If G is a planar graph, we have*

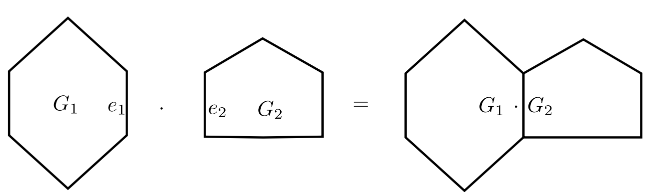
$$\langle \Phi_G(q) \rangle_0 = 1 \tag{77a}$$

$$\langle \Phi_G(q) \rangle_1 = c_1 - c_2 - 1 \tag{77b}$$

$$\langle \Phi_G(q) \rangle_2 = \frac{1}{2}\left((c_1 - c_2)^2 - 2c_3 - c_1 + c_2\right) \tag{77c}$$

where c_1, c_2 and c_3 denotes the number of vertices, edges and 3-cycles of G.

If G_1 and G_2 are two planar graphs with distinguished boundary edges e_1 and e_2, let $G_1 \cdot G_2$ denote their edge connected sum along $e_1 = e_2$ depicted as follows:



Let P_r denote a planar polygon with r edges when $r \geq 3$ and let P_2 denote the connected graph with two vertices and one edge, a reduced form of a bigon. For a positive natural number b, consider the unary theta (when b is odd) and false theta series (when b is even) $h_b(q)$ given by

$$h_b(q) = \sum_{n \in \mathbb{Z}} \varepsilon_b(n)\, q^{\frac{b}{2}n(n+1)-n}$$

where

$$\varepsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd} \\ 1 & \text{if } b \text{ is even and } n \geq 0 \\ -1 & \text{if } b \text{ is even and } n < 0 \end{cases}$$

Observe that

$$h_1(q) = 0, \qquad h_2(q) = 1, \qquad h_3(q) = (q)_\infty\,.$$

The following lemma (observed independently by Armond-Dasbach) follows from the Nahm sum for $\Phi_G(q)$ combined with a q-series identity (see Equation (88) below). This identity was proven by Armond-Dasbach [4, Thm.3.7] and Andrews [1].

Lemma 3.1.5. *For all planar graphs G and natural numbers $r \geq 3$ we have:*

$$\Phi_{G \cdot P_r}(q) = \Phi_G(q)\Phi_{P_r}(q) = \Phi_G(q)h_r(q)\,.$$

Question 3.1.6. *Is it true that for all planar graphs G_1 and G_2 we have:*

$$\Phi_{G_1 \cdot G_2}(q) = \Phi_{G_1}(q)\Phi_{G_2}(q)?$$

As an illustration of Lemma 3.1.5, for the three graphs of Figure 4, we have:

$$\Phi_{L8a8}(q) = \Phi_{8_{13}}(q) = h_4(q)h_3(q)^2\,.$$

Remark 3.1.7. *Observe that the alternating planar projections of the graphs G_1 and G_2 of Figure 4 are related by a* flype move *[42, Fig.1].*

Flyping a planar alternating link projection corresponds to the operation on graphs shown in Figure 5.

If the planar graphs G and G' are related by flyping, then $\Phi_G(q) = \Phi_{G'}(q)$, since the corresponding alternating links are isotopic.

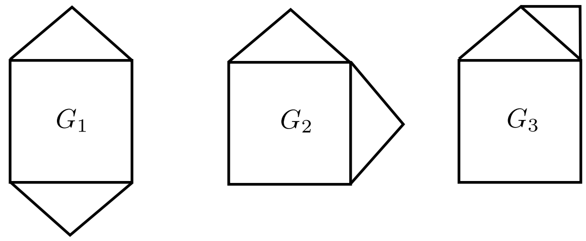


Figure 4: Three graphs G_1, G_2, G_3 and the corresponding alternating links $L8a8$, $L8a8$ and 8_{13}.

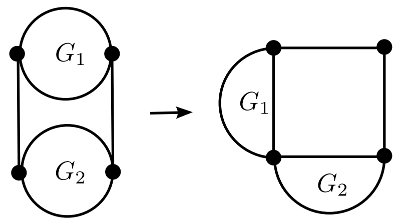
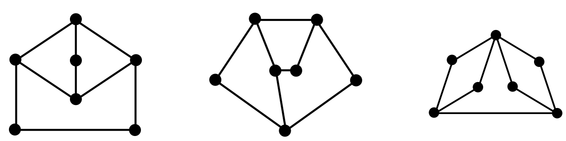


Figure 5: A flyping move on a planar graph.

Remark 3.1.8. *Theorem 3.1.4 might temp one to conjecture that $\Phi_G(q)$ depends on the number of vertices and edges of G and on the number of k-faces of G for $k \geq 3$. This is not true. For example, consider the three graphs G^9_9, G^9_{11} and G^9_{13} of Figure 11 shown here:*



All three graphs have 7 vertices, 9 edges, 2 square faces and 2 pentagonal faces. The DT codes of the corresponding links are given by:

G^9_{10}	*DTCode[{16, 10, 14, 12, 2, 18, 6}, {4, 8}]*
G^9_{12}	*DTCode[{6, 10, 14, 18, 4, 16, 8, 2, 12}]*
G^9_{16}	*DTCode[{6, 10}, {4, 12, 18, 2, 16}, {8, 14}]*

On the other hand, the colored Jones function of the corresponding alternating links [6] gives that

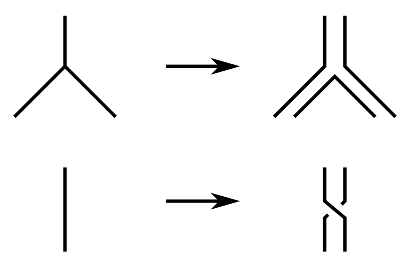
G	$\Phi_G(q)$
G^9_{10}	$1 - 3q + 3q^2 + 2q^3 - 7q^4 + 3q^5 + \dots$
G^9_{12}	$1 - 3q + 3q^2 + q^3 - 7q^4 + 6q^5 + \dots$
G^9_{16}	$1 - 3q + 3q^2 + q^3 - 8q^4 + 6q^5 + \dots$

3.2 The connection between $\Phi_G(q)$ and alternating links

In this section explain connection between $\Phi_G(q)$ and the colored Jones function of the alternating link L_G following [23].

3.2.1 From planar graphs to alternating links

Given a planar graph G (possibly with loops or multiple edges), there is an alternating planar projection of a link L_G given by:



3.2.2 From alternating links to planar (Tait) graphs

Given a diagram D of a *reduced alternating non-split* link L, its Tait graph can be constructed as follows: the diagram D gives rise to a polygonal complex of $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. Since D is alternating, it is possible to label each polygon by a color b (black) or w (white) such that at every crossing the coloring looks as follows:

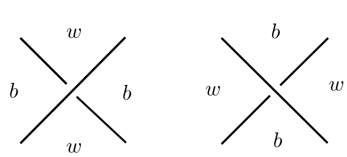


Figure 6: The checkerboard coloring of a link diagram

There are exactly two ways to color the regions of D with black and white colors. In this note we will work with the one whose unbounded region has color w. In each b-colored polygon (in short, b-polygon) we put a vertex and connect two of them with an edge if there is a crossing between the corresponding polygons. The resulting graph is a planar graph called the Tait graph associated with the link diagram D. Note that the Tait graph is always planar but not necessarily reduced. Although the reduction of the Tait graph may change the alternating link and its colored Jones polynomial, it does not change the limit of the shifted colored Jones function in Theorem 3.2.1 because of Lemma 3.1.3.

3.2.3 The limit of the shifted colored Jones function

When L is an alternating link, the colored Jones polynomial $J_{L,n}(q) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$ (normalized to be 1 at the unknot, and colored by the n-dimensional irreducible representation of $\mathfrak{sl}_2$ [23]) has lowest q-monomial with coefficient ± 1, and after dividing by this monomial, we obtain the *shifted* colored Jones polynomial $\hat{J}_{L_G,n}(q) \in 1 + q\mathbb{Z}[q]$. Let $(f(q))_N$ denotes the coefficient of q^N in $f(q)$. The limit $f(q) = \lim_n f_n(q) \in \mathbb{Z}[[q]]$ of a sequence of polynomials $f_n(q) \in \mathbb{Z}[q]$ is defined as follows [23]. For every natural number N, there exists a natural number $n_0(N)$ such that $(f_n(q))_N = (f(q))_N$ for all $n \geq n_0(N)$.

Theorem 3.2.1. *[23, Thm.1.10] Let L be an alternating link projection and G be its Tait graph. Then the following limit exists*

$$\lim_{n\to\infty} \hat{J}_{L,n}(q) = \Phi_G(q) \in \mathbb{Z}[[q]] \tag{78}$$

Remark 3.2.2. (a) The convergence statement in the above theorem holds in the following strong form [23]: for every natural number N, and for $n > N$ we have:

$$(\hat{J}_{L,n}(q))_N = (\Phi_G(q))_N\,. \tag{79}$$

(b) $\Phi_G(q)$ is the *reduced* version of the one in [23, Thm.1.10] and differs from the unreduced version $\Phi_G^{\mathrm{TQFT}}(q)$ by

$$\Phi_G(q) = (1-q)\Phi_G^{\mathrm{TQFT}}(q)\,,$$

where

$$\Phi_G^{\mathrm{TQFT}}(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod\limits_{(p,v):v\in p} (q)_{a_p+b_v}} \tag{80}$$

and the summation (a,b) is over all admissible states where we do not assume that $b_v = 0$ for a fixed vertex v in the unbounded face of G.

3.3 Proof of Theorem 3.1.2

In this section we prove Theorem 3.1.2. Part (a) follows from completing the square in Equation (73):

$$\begin{aligned}
A(a,b) &= \sum_p (l(p)a_p^2 + 2a_p(\sum_{v\in p} b_v)) + 2\sum_{e=(v_iv_j)} b_{v_i}b_{v_j} \\
&= \sum_p (l(p)(a_p+b_p)^2 + 2a_p(\sum_{v\in p} b_v - l(p)b_p) - l(p)b_p^2 + 2\sum_{e=(v_iv_j)} b_{v_i}b_{v_j}) \\
&= \sum_p (l(p)(a_p+b_p)^2 + 2(a_p+b_p)(\sum_{v\in p} b_v - l(p)b_p) + \sum_{e=(v_iv_j)\in p} (b_{v_i}-b_p)(b_{v_j}-b_p)) \\
&\quad + \sum_{e=(v_iv_j)\in p_\infty} b_{v_i}b_{v_j}
\end{aligned}$$

For the remaining parts of Theorem 3.1.2, fix a 2-connected planar graph G, a vertex v_0 of G and a bounded face p_0 of G that contains v_0.

Lemma 3.3.1. *There exists a graph Γ which depends on G, v_0, p_0 such that:*

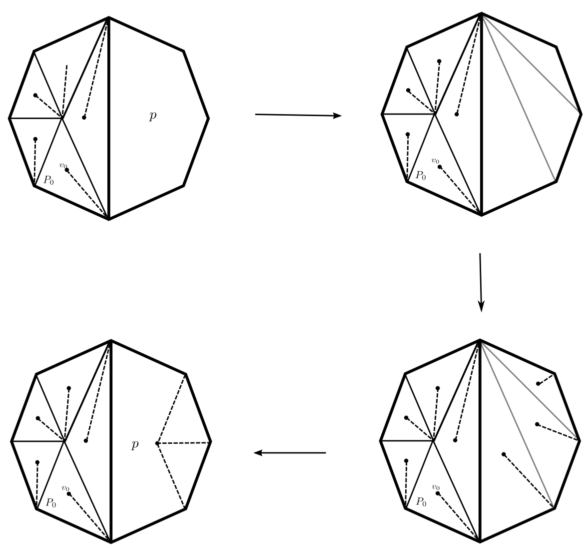
- *The vertices of Γ are vertices of G as well as one vertex v_p for each bounded face p of G.*
- *The edges of Γ are of the form vv_p where v is a vertex of G and p is a bounded face that contains v.*

- *$v_0 v_{p_0}$ is an edge of Γ.*

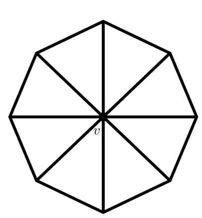
- *Every vertex v in G has degree n_v in Γ where*

$$n_v = \begin{cases} 2 & \text{if } v \text{ is not a boundary vertex} \\ \leq 2 & \text{if } v \text{ is a boundary vertex} \end{cases}$$

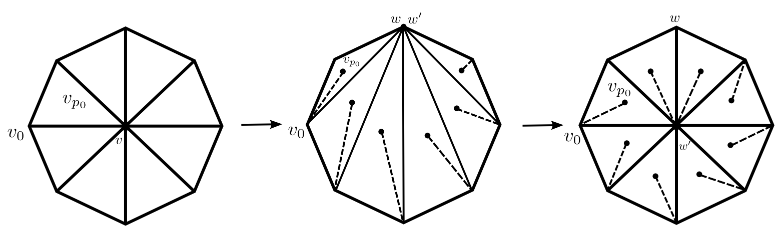
Proof. First note we can assume that each face p of G is a triangle. Indeed, if a face p is not a triangle, we can divide it into a union of triangles by creating new edges inside p. Once we have succeeded in constructing a Γ for the resulted graph, we can remove the added edges in p and collapse all the interior vertices of the newly created triangles in p into one single vertex v_p. The figures below illustrate the above process.



Now assuming that all faces of G are triangles, let us proceed by induction on the number of vertices of G. If there is no interior vertex in G then since the unbounded face p_∞ is also a triangle, G itself is a triangle and we are done. Therefore let us assume that there is an interior vertex v of G. Locally the graph at v looks like the following:



Next we remove v and all of the edges incident to it from G and denote the resulted face by p. Let w be a vertex of p and connect w to each of the vertices of p by an edge. Denote the resulted graph by G_w. By induction hypothesis, there exists a graph Γ_w for G_w. At w make another copy of the vertex called w'. Now drag w' into the interior of p while keeping it connected to vertices of p and at the same time delete the edges that are incident to w and that lie in the interior of p. This has to be done in such a way that all the vertices of Γ_w still lie in the interior of the new triangles that have w' as a vertex. Create two new vertices in the interior of the two triangles in p that contain w as a vertex and connect them to w'. The resulted graph satisfies the requirements of the lemma. The figures below explain the process.



□

Proof. (of part (b) of Theorem 3.1.2) We can decompose $B(a, b)$ into a finite sum of nonnegative terms as follows

$$B(a, b) = \sum_{\hat{e}=(vv_p)} (a_p + b_v) + \sum_{v} (2 - n_v) b_v \tag{81}$$

where the summation is over all edges of Γ. □

Corollary 3.3.2. *For a pair (p, v) where p is a face of G and v is a vertex of p then $B(a, b) \geq a_p + b_v$.*

Proof. This is a direct consequence of Equation (81) since by Lemma 3.3.1 there exists a graph Γ that contains vv_p as an edge. □

Proof. (of part (c) of Theorem 3.1.2) Let us prove the linear bound on the b_v first. Let us set $b_{v_0} = 0$ where v_0 is a boundary vertex of G. Let p_0 be a bounded face that contains v_0, so we have $a_{p_0}+b_{v_0} \geq 0$. Since $0 \leq B(a,b) \leq 2N$ by part (b) of Theorem 3.1.2 and Corollary 3.3.2 we have that $0 \leq a_{p_0}+b_{v_0} \leq 2N$. Since $b_{v_0}=0$ this means that $0 \leq a_{p_0} \leq 2N$. Similarly if v is another vertex of p_0 then by Corollary 3.3.2 we have $0 \leq a_{p_0}+b_v \leq 2N$ which implies that $-2N \leq b_v \leq 2N$. Let G' be the graph obtained from G by removing the boundary edges of p_0. Choose a face p' of G' and a vertex $v' \in p'$ that also belongs to the removed face p_0. Repeat the above process with (p', v') we have that $-4N \leq b_{v''} \leq 4N$ for any $v'' \in p'$. Continuing this process until all faces of g are covered have that $|b_v| \leq dN$ for all vertices v of G.

To prove the bound for the a_p's, note that from part (a) of Theorem 3.1.2 we have that $\frac{e(p)}{2}(a_p+b_v)^2 \leq N$ for all bounded faces p and all vertices v of G. This implies that $|a_p+b_v| \leq \sqrt{\frac{2}{e_p}}\sqrt{N}$. Since $|b_v| \leq dN$ this implies that $|a_p| \leq \sqrt{\frac{2}{e_p}}\sqrt{N}+dN$. For the lower bound of a_p, note that since $a_p+b_v \geq 0$ we have $a_p \geq -b_v \geq -dN$. □

3.4 The coefficients of 1, q and q^2 in $\Phi_G(q)$

3.4.1 Some lemmas

In this section we prove Theorem 3.1.4, using the unreduced series $\Phi_G^{\mathrm{TQFT}}(q)$ of Equation (80). Our admissible states (a,b) in this section do not satisfy the property that $b_v=0$ for some vertex v of the unbounded face of G.

Since $A(a,b)+B(a,b) \geq 0$ for an admissible state (a,b) with equality if and only if $(a,b)=(0,0)$ (as shown in Theorem 3.1.2), it follows that the coefficient of q^0 in $\Phi_G(q)$ is 1. For the remaining of the proof of Theorem 3.1.4 we will use several lemmas.

Lemma 3.4.1. *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a,b) is an admissible state such that*

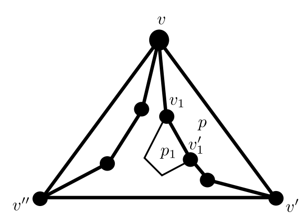
1. *$b_v=b_{v'}=1$ where vv' is an edge of p_∞,*
2. *$a_p+b_p=0$ for any face p of G,*

3. $(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ *for any face* p *of* G *and edge* v_1v_2 *of* p*,*

then

- $b_v \geq 1$ *for all vertices* v*,*
- $a_p = -1$ *for all faces* $p \neq p_\infty$*, and*
- $B(a,b) \geq 2 + V_\infty$*.*

Proof. Let p be the bounded face that contains v, v'. We have $(b_v - b_p)(b_{v'} - b_p) = 0$ so $b_p = 1$ since $b_v = b_{v'} = 1$. (2) then implies that $a_p = -b_p = -1$ and thus $b_w \geq b_p = 1$ for all $w \in p$. Let v_1v_1' be another edge of p and let $p_1 \neq p$ bea bounded face that



contains v_1v_1'. Since $(b_{v_1} - b_p)(b_{v_1'} - b_p) = 0$ we have $\min\{b_{v_1}, b_{v_1'}\} = b_p = 1$. So from $(b_{v_1} - b_{p_1})(b_{v_1'} - b_{p_1}) = 0$ we have that $b_{p_1} = 1$. Therefore $a_{p_1} = -1$ and $b_w \geq b_{p_1} = 1$ for any vertex $w \in p_1$. By a similar argument we can show that $b_v \geq 1$ for every vertex v and $a_p = -1$ for every face p of G. Let $p_1, p_2, \ldots, p_f$ be the bounded faces of G, where $f = F_G - 1$. Then from Equation (74) we have

$$\begin{aligned} B(a,b) &= -\sum_{j=1}^{f}(l(p_j) - 2) + 2\sum_{v} b_v \\ &\geq -\sum_{j=1}^{f} l(p_j) + 2f + 2c_1 \\ &= -(2c_2 - V_\infty) + 2F_G - 2 + 2c_1 \\ &= 2(c_1 - c_2 + F_G) - 2 + V_\infty \\ &= 2 + V_\infty \end{aligned}$$

□

The proof of the next lemma is similar to the one of Lemma 3.4.1 and is therefore omitted.

Lemma 3.4.2. *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a, b) is an admissible state such that*

1. *$b_v = b_{v'} = 0$ and $(b_v - b_p)(b_{v'} - b_p) = 1$ where p is a boundary face and vv' is a boundary edge that belongs to p,*
2. *$a_p + b_p = 0$ for any face p of G,*
3. *$(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ for any face p of G and edge v_1v_2 not on the boundary of p.*

Then $b_w \geq -1$ for all vertices w, $a_p = 1$ for all faces $p \neq p_\infty$ and $B(a, b) \geq V_\infty - 2$. Furthermore $B(a, b) = V_\infty - 2$ if and only if

- *$b_v = 0$ for all boundary vertices v and $b_w = -1$ for all other vertices w.*
- *$a_p = 1$ for all faces p.*

Lemma 3.4.3. *Let G be a 2-connected planar graph, p_0 be a boundary face and (a, b) be an admissible state such that*

1. *$a_{p_0} + b_{p_0} = 0$,*
2. *There exists a boundary edge vv' of p_0 such that $b_v b_{v'} = 0$ and $(b_v - b_{p_0})(b_{v'} - b_{p_0}) = 0$,*
3. *Let G_0 be the graph obtained from G by deleting the boundary edges of p_0 and let (a_0, b_0) be the restriction of the admissible state (a, b) on G_0.*

Then,

(a) (a_0, b_0) is an admissible state for G_0,

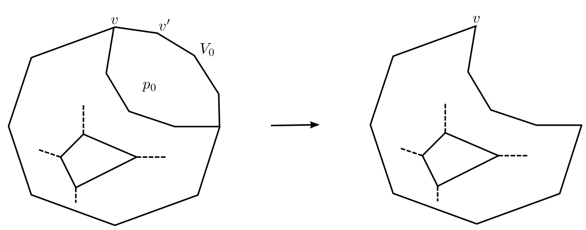
(b) $A(a_0, b_0) = A(a, b) - \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'}$,

(c) $B(a_0, b_0) = B(a, b) - 2 \sum_{v \in V_0} b_v$, *where* V_0 *is the set of boundary vertices of* p_0 *that do not belong to any other bounded face,*

(d) $B(a, b) \geq 2 \sum_{v \in V_0} b_v$,

(e) *If furthermore* $B(a, b) \leq 1$ *then* $A(a, b) = A(a_0, b_0), B(a, b) = B(a_0, b_0)$.

Proof. From (2) we have either $b_v = 0$ or $b_{v'} = 0$ and it follows from $(b_v - b_{p_0})(b_{v'} - b_{p_0}) = 0$ that $b_{p_0} = 0$. This means that we have $b_v \geq 0$ for all $v \in p_0$. This implies (a). Furthermore (1) implies that $a_{p_0} = 0$ and thus $A(a, b) - A(a_0, b_0) = l(p_0)a_{p_0}^2 + 2a_{p_0}(\sum_{v \in p_0} b_v) + \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'} = \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'}$ and $B(a, b) - B(a_0, b_0) = a_{p_0} + 2 \sum_{v \in V_0} b_v = 2 \sum_{v \in V_0} b_v$. This proves (b) and (c). (d) follows from (c) since we have $0 \leq B(a_0, b_0) = B(a, b) - 2 \sum_{v \in V_0} b_v$, (e) is a consequence of (b), (c) and (d) since $1 \geq B(a, b) \geq 2 \sum_{v \in V_0} b_v$ implies that $\sum_{v \in V_0} b_v = 0$.



□

3.4.2 The coefficient of q in $\Phi_G(q)$

We need to find the admissible states (a, b) such that $\frac{1}{2}(A(a, b) + B(a, b)) = 1$. Parts (a) and (b) of Theorem 3.1.2 imply that $A(a, b), B(a, b) \in \mathbb{N}$. Thus, if $\frac{1}{2}(A(a, b) + B(a, b)) = 1$ then we have the following cases:

$A(a,b)$	2	1	0
$B(a,b)$	0	1	2

Case 1: $(A(a, b), B(a, b)) = (2, 0)$. Since $l(p) \geq 3$, we should have $a_p + b_p = 0$ for all faces p. This implies that $a_p + b_v = a_p + b_p + b_v - b_p = b_v - b_p$ and it follows from

Corollary 3.3.2 that $0 = B(a,b) \geq a_p + b_v = b_v - b_p$. This means $b_v - b_p = a_p + b_v = 0$ for all faces p and vertices v of p, so Equation (76) is equivalent to

$$\sum_{vv' \in p_\infty} b_v b_{v'} = 2\,. \tag{82}$$

If vv' is an edge of G and p is a face that contains vv' then we have $a_p + b_v = 0 = a_p + b_{v'}$ and therefore $b_v = b_{v'}$. So by Equation (82) there exists a boundary edge vv' such that $b_v = b_{v'} = 1$. Lemma 3.4.1 implies that $B(a,b) \geq 2 + V_\infty > 0$ which is impossible. Therefore there are no admissible states (a,b) that satisfy $(A(a,b), B(a,b)) = (2,0)$.

Case 2: $(A(a,b), B(a,b)) = (1,1)$. As above we have that $a_p + b_p = 0$ for all faces p. Since $A(a,b) = 1$, there is either a bounded face p_1 with an edge $v_1 v_1'$ such that $(b_{v_1} - b_{p_1})(b_{v_1'} - b_{p_1}) = 1$ or a boundary edge $v_2 v_2'$ such that $b_{v_2} b_{v_2'} = 1$ and all other terms in Equation (76) are equal to zero. Let p_2 be the bounded face that contains $v_2 v_2'$ and let $p \neq p_1, p_2$ be a bounded face. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a,b) on G'. By part (e) of Lemma 3.4.3, we have $A(a',b') = A(a,b)$ and $B(a',b') = B(a,b)$. Continue this process until either $G = p_1$ or $G = p_2$. If $G = p_2$ then $b_{v_2} b_{v_2'} = 1$ and therefore $B(a,b) \geq 2(b_{v_2} + b'_{v_2}) = 4$ which is impossible. If $G = p_1$ then v_1, v_2 are now boundary vertices and so $b_{v_1} b_{v_1'} = 0$ and we can assume that $b_{v_1} = 0$. But this implies that $-b_{p_1}(b_{v_1'} - b_{p_1}) = 1$ hence $b_{p_1} = -1$. This is impossible since b_{p_1} is a boundary vertex. Thus there are no admissible states (a,b) that satisfy $(A(a,b), B(a,b)) = (1,1)$.

Case 3: $(A(a,b), B(a,b)) = (0,2)$. Since $A(a,b) = 0$ we should have

- $a_p + b_p = 0$ for all faces p,
- $b_v b_{v'} = 0$ for all boundary edges vv',
- $(b_v - b_p)(b_{v'} - b_p) = 0$ for all bounded faces p and and edges $vv' \in p$.

Let p be a bounded face of G. Let G' be the graph obtained from G by deleting the boundary edges of G and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 3.4.3, we have $A(a', b') = A(a, b)$ and $B(a', b') = B(a, b) - 2n_p$ where $n_p \in \mathbb{N}$. Since $B(a, b) = 2$, $n_p \leq 1$ and $n_p = 1$ if and only if there exists exactly one boundary vertex $v \in p$ such that $b_v = 1$ and $b'_v = 0$ for any other boundary vertex v' of p. Continuing this process it is easy to show that an admissible state (a, b) such that $(A(a, b), B(a, b)) = (0, 2)$ must satisfy the following:

- $a_p = 0$ for all p,
- $b_v = 1$ for a vertex v and $b_{v'} = 0$ for any other vertex v' of G.

The contribution of this state to $\Phi_G(q)$ is $\frac{q}{(1-q)^{\deg(v)}} = q + O(q^2)$.

Thus from Theorem 3.2.1 and cases 1-3 we have

$$\begin{aligned}\langle \Phi_G^{\mathrm{TQFT}}(q) \rangle_1 &= \left\langle (q)_\infty^{c_2} \left(1 + \sum_v q + O(q^2) \right) \right\rangle_1 \\ &= c_1 - c_2 \,.\end{aligned}$$

Therefore,

$$\langle \Phi_G(q) \rangle_1 = \langle (1-q) \Phi_G^{\mathrm{TQFT}}(q) \rangle_1 = c_1 - c_2 - 1 \,.$$

3.4.3 The coefficient of q^2 in $\Phi_G(q)$

We need to find the admissible states (a, b) such that $\frac{1}{2}(A(a, b) + B(a, b)) = 2$. Since $A(a, b), B(a, b) \in \mathbb{N}$ we have the following cases:

$A(a,b)$	4	3	2	1	0
$B(a,b)$	0	1	2	3	4

Case 1: $(A(a, b), B(a, b)) = (4, 0)$. If there is a face p such that $a_p + b_p > 0$ then by Corollary 3.3.2 we have $B(a, b) \geq a_p + b_v \geq a_p + b_p > 0$ where v is a vertex of p. Therefore $a_p + b_p = 0$ for all faces p. Similarly, if there exists a face p and a vertex

$v \in p$ such that $b_v - b_p > 0$ then $0 = B(a,b) \geq a_p + b_v = a_p + b_p + b_v - b_p \geq b_v - b_p > 0$. Therefore $a_p + b_v = b_v - b_p = 0$ for all $v \in p$. Thus $A(a,b) = 4$ is equivalent to

$$\sum_{vv' \in p_\infty} b_v b_{v'} = 4\,. \tag{83}$$

If vv' is an edge of G and p is a bounded face that contains vv' then we have $a_p + b_v = 0 = a_p + b_{v'}$ and therefore $b_v = b_{v'}$. So by Equation (82) there exists a boundary edge vv' such that $b_v = b_{v'} = 1$. Lemma 3.4.1 implies that $B(a,b) \geq 2 + V_\infty > 0$ which is impossible. Therefore there are no admissible states (a,b) that satisfy $(A(a,b), B(a,b)) = (4,0)$.

Case 2: $(A(a,b), B(a,b)) = (3,1)$. If there exists a face p_0 such that $a_{p_0} + b_{p_0} > 0$ then we must have $l(p_0) = 3$ and

- $a_{p_0} + b_{p_0} = 1$, $a_p + b_p = 0$ for any $p \neq p_0$,
- $b_v b_{v'} = 0$ for all boundary edges vv',
- $(b_v - b_p)(b_{v'} - b_p) = 0$ for all bounded faces p and and edges $vv' \in p$.

Let $p \neq p_0$ be a bounded face of G. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a,b) on G'. By part (e) of Lemma 3.4.3, we have $A(a',b') = A(a,b)$ and $B(a',b') = B(a,b)$. We can continue this process until $G = p_0$. Let v_0, v_0', v_0'' be the vertices of p_0 then $b_{v_0} b_{v_0'} = 0$ so we can assume that $b_{v_0} = 0$. Since $(b_{v_0} - b_{p_0})(b_{v_0'} - b_{p_0}) = 0$ we have $b_{p_0} = 0$ and hence $a_{p_0} = a_{p_0} + b_{p_0} = 1$. Since $1 = B(a,b) = a_{p_0} + 2(b_{v_0} + b_{v_0'} + b_{v_0''})$ it implies that $b_{v_0'} = b_{v_0''} = 0$. This gives us the following set of admissible states (a,b):

- $a_p = 1$ for a triangular face p, $a_{p'} = 0$ for $p' \neq p$,
- $b_v = 0$ for all vertices v,

The contribution of this state to $\Phi_G(q)$ is $(-1)^1 \frac{q^2}{(1-q)^{l(p)}} = -\frac{q^2}{(1-q)^3} = -q^2 + O(q^3)$.

On the other hand if $a_p + b_p = 0$ for all p then we have

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 3\,. \tag{84}$$

There are at most three positive terms in the above equation. If a boundary face p has a boundary edge vv' that does not correspond to any positive term then we have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$ so $b_p = 0$ which implies that $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 3.4.3, we have $A(a', b') = A(a, b)$ and $B(a', b') = B(a, b)$. We can continue to do this until all boundary edges of G are $v_i v_i'$, $i = 1, 2, 3$. This only happens if these three edges together form a triangle. Let us denote the triangle's vertices by v, v', v'' and let p, p', p'' be the bounded faces that contain $vv', v'v'', v''v$ respectively. Note that since the positive terms in Equation (84) correspond to different edges, we must have

$$b_v b_{v'} + (b_v - b_p)(b_{v'} - b_p) = 1$$

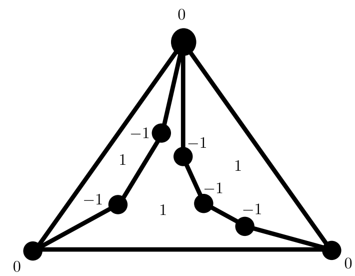
$$b_{v'} b_{v''} + (b_{v'} - b_{p'})(b_{v''} - b_{p'}) = 1$$

$$b_{v''} b_v + (b_{v''} - b_{p''})(b_v - b_{p''}) = 1$$

Case 2.1: If the positive terms are $b_v b_{v'}, b_{v'} b_{v''}, b_{v''} b_v$ then we must have simultaneously $b_v b_{v'} = b_{v'} b_{v''} = b_{v''} b_v = 1$ and $(b_w - b_{\tilde{p}})(b_{w'} - b_{\tilde{p}}) = 0$ for all faces $\tilde{p}$ and edge ww'. The former implies that $b_v = b_{v'} = b_{v''} = 1$. Therefore from Lemma 3.4.1 we have $B(a, b) \geq 2 + 3 = 5$ which is impossible.

Case 2.2: If, for instance, $b_v b_{v'} = 0$ then we must also have $(b_v - b_p)(b_{v'} - b_p) = 1$. Thus we can assume that $b_v = 0$ and so $-b_p(b_{v'} - b_p) = 1$. This implies that $b_p = -1$ and $b_{v'} = 0$. In particular, we have $b_{v'} b_{v''} = 0$ and hence $(b_{v'} - b_{p'})(b_{v''} - b_{p'}) = 1$. Since $b_v b_{v''} = 0$ we also have $(b_{v''} - b_{p''})(b_v - b_{p''}) = 1$. In particular, this implies that $(b_w - b_{\tilde{p}})(b_{w'} - b_{\tilde{p}}) = 0$ for all faces $\tilde{p}$ and edges $ww' \in \tilde{p}$ not on the boundary. Since

$B(a,b) = 1$ Lemma 3.4.2 implies that we must have $b_w = -1$ for all $w \neq v, v', v''$ and $a_p = 1$ for all $p \neq p_\infty$.



This corresponds to the following admissible state of G:

- $a_p = 1$ for all bounded faces p,
- $b_v = b_{v'} = b_{v''} = 0$ where v, v', v'' are the vertices of a 3-cycle in G,
- $b_w = -1$ for all vertices w inside the 3-circle mentioned above,
- $b_{\tilde{w}} = 0$ for any other vertex w.

The contribution of this state to $\Phi_G(q)$ is

$$(-1)^1 \frac{q^2}{(1-q)^{\deg_\Delta(v)+\deg_\Delta(v')+\deg_\Delta(v'')-3}} = -q^2 + O(q^3)$$

where $\deg_\Delta(v)$ is the degree of v in the triangle $\Delta = vv'v''$.

Case 3: We consider the two cases $(A(a,b), B(a,b)) = (2,2)$ and $(A(a,b), B(a,b)) = (1,3)$ together. Since $A(a,b) \leq 2$ we should have $a_p + b_p = 0$ for all faces p and $A(a,b) = 2$ is equivalent to

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 2$$

There are at most two positive terms in the above equation. If a boundary face p has a boundary edge vv' that does not correspond to any positive term then we have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$ so $b_p = 0$ which implies that $a_p = 0$. By part (d) of Lemma 3.4.3, it follows that if w is a boundary vertex of p then $B(a,b) \geq 2b_w$ and since $B(a,b) \leq 3$ we have $b_w = 0$ or 1. Therefore by parts (b,c) of Lemma 3.4.3 we can remove the boundary edges of p to obtain a new graph G' that satisfies $A(a,b) = A'(a,b)$ and $B(a,b) = B'(a,b)$ or $B(a,b) = B'(a,b)+1$ where $A'(a,b), B'(a,b)$ are the restrictions of $A(a,b)$ and $B(a,b)$ on G'. By continuing this process until $G = \emptyset$, it is easy to see that we must have $A(a,b) = 0$, $B(a,b) \leq 1$ and $B(a,b) = 1$ if and only if there exists a unique boundary vertex w of p such that $b_w = 1$. Thus there are no admissible states that satisfy $(A(a,b), B(a,b)) = (2,2)$ or $(A(a,b), B(a,b)) = (1,3)$.

Case 4: $(A(a,b), B(a,b)) = (0,4)$. Since $A(a,b) = 0$, we should have

$$a_p + b_p = 0 \text{ for all faces } p \tag{85}$$

$$(b_v - b_p)(b_{v'} - b_p) = 0 \text{ for all faces } p \text{ and edges } vv' \in p \tag{86}$$

$$b_v b_{v'} = 0 \text{ for all edges } vv' \in p \tag{87}$$

Let p be a boundary face of G and $vv' \in p$ be a boundary edge. Equations (86) and (87) imply that $b_p = 0$ and so $a_p = 0$ by Equation (85). Let G' be the graph obtained from G by deleting the boundary edges of G and (a', b') be the restriction of (a,b) on G'. By part (e) of Lemma 3.4.3 we have $A(a',b') = A(a,b)$, $B(a',b') = B(a,b) - 2n_p$ where $n_p \in \mathbb{N}$. Since $B(a,b) = 4$ we have $n_p \leq 2$ and

- $n_p = 2$ if and only if there exist either exactly two boundary vertices $v, w \in p$ that are not connected by an edge such that $b_v = b_{v'} = 1$ or exactly one boundary vertex $v \in p$ such that $b_v = 2$ and $b_{v'} = 0$ for all other boundary vertices $v' \in p$

- $n_p = 1$ if and only if there exists exactly one boundary vertex $v \in p$ such that $b_v = 1$ and $b'_v = 0$ for any other boundary vertex v' of p.

Similarly, by continuing this process it is easy to show that an admissible state (a,b) such that $(A(a,b), B(a,b)) = (0,4)$ must satisfy one the following.

- $b_v = b_{v'} = 1$ for a pair of vertices that are not connected by an edge of G, $b_w = 0$ for any other vertex w,

- $a_p = 0$ for all faces p.

The contribution of this state to $\Phi_G(q)$ is $\frac{q^2}{(1-q)^{\deg(v)+\deg(v')}} = -q^2 + O(q^3)$.

- $b_v = 2$ for a vertex v, $b_w = 0$ for any other vertex w,

- $a_p = 0$ for all faces p.

The contribution of this state to $\Phi_G(q)$ is $\frac{q^2}{(1-q)_2^{\deg(v)}} = -q^2 + O(q^3)$.

It follows from Theorem 3.2.1, Section 3.4.2 and cases 1-4 that

$$\begin{aligned}\langle \Phi_G^{\text{TQFT}}(q)\rangle_2 &= \langle (q)_\infty^{c_2}(1 + \sum_v \frac{q}{(1-q)^{\deg(v)}} + (-c_3 + c_1 + \frac{c_1(c_1-1)}{2} - c_2))q^2\rangle_2 \\ &= \langle (q)_\infty^{c_2}(1 + q(c_1 + 2c_2 q) + (\frac{c_1(c_1+1)}{2} - c_2 - c_3)q^2\rangle_2 \\ &= \langle (1 - c_2 q + \frac{c_2(c_2-3)}{2}q^2)(1 + c_1 q + (\frac{c_1(c_1+1)}{2} + c_2 - c_3)q^2\rangle_2 \\ &= \frac{(c_1-c_2)^2}{2} - c_3 + \frac{c_1 - c_2}{2}\,.\end{aligned}$$

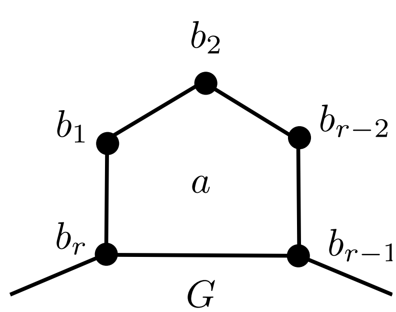
Therefore,

$$\begin{aligned}\langle \Phi_G(q)\rangle_2 &= \langle (1-q)\Phi_G^{\text{TQFT}}(q)\rangle_2 \\ &= \Big\langle (1-q)(1 + (c_1 - c_2)q + (\frac{(c_1-c_2)^2}{2} - c_3 + \frac{c_1-c_2}{2})q^2)\Big\rangle_2 \\ &= \frac{1}{2}((c_1-c_2)^2 - 2c_3 - c_1 + c_2)\,.\end{aligned}$$

This completes the proof of Theorem 3.1.4. $\square$

3.4.4 Proof of Lemma 3.1.5

Fix a planar graph G and consider $G \cdot P_r$ where P_r is a polygon with r sides and vertices $b_1, \ldots, b_r$ as in the following figure



Consider the corresponding portion $S(b_{r-1}, b_r)$ of the formula of $\Phi_{G\cdot P_r}(q)$

$$S(b_{r-1}, b_r) = \sum_{a,b_1,\dots,b_{r-2}} (-1)^{ra} \frac{q^{\frac{r}{2}a^2+a(b_1+\dots b_r)+\sum_{i=1}^{r-2} b_i b_{i+1}+b_1 b_r+\sum_{i=1}^{r-2} b_i+\frac{r-2}{2}a}}{(q)_{b_1}(q)_{b_2}\dots(q)_{b_{r-2}}(q)_{b_1+a}(q)_{b_2+a}\dots(q)_{b_r+a}} \tag{88}$$

for fixed $b_{r-1}, b_r \geq 0$. Armond-Dasbach [4, Thm3.7] and Andrews [1] prove that

$$S(b_{r-1}, 0) = (q)_\infty^{-r+1} h_r(q)$$

for all $b_{r-1} \geq 0$. Summing over the remaining variables in the formula for $\Phi_{G\cdot P_r}(q)$ concludes the proof of the Lemma. □

3.5 The computation of $\Phi_G(q)$

3.5.1 The computation of $\Phi_{L8a7}(q)$ in detail

In this section we explain in detail the computation of $\Phi_{L8a7}(q)$. Consider the planar graph of the alternating link $L8a7$ shown in Figure 7, with the marking of its vertices by b_i for $i = 1, \dots, 6$ and its bounded faces by a_j for $j = 1, 2, 3$.

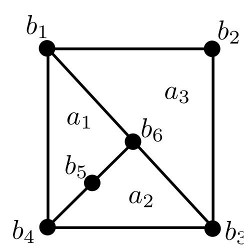


Figure 7: The planar graph of the link $L8a7$.

Consider the minimum values of the b-variables at each bounded face:

$$\bar{b}_1 = \min\{b_1, b_4, b_5, b_6\}$$

$$\bar{b}_2 = \min\{b_3, b_4, b_5, b_6\}$$

$$\bar{b}_3 = \min\{b_1, b_2, b_3, b_6\}\,.$$

We have

$$
\begin{aligned}
\frac{1}{2}A(a,b) &= 2(a_1+\bar{b}_1)^2 + (a_1+\bar{b}_1)(b_1+b_4+b_5+b_6-4\bar{b}_1) \\
&+ 2(a_2+\bar{b}_2)^2 + (a_1+\bar{b}_2)(b_3+b_4+b_5+b_6-4\bar{b}_2) \\
&+ 2(a_3+\bar{b}_3)^2 + (a_3+\bar{b}_3)(b_1+b_2+b_3+b_6-4\bar{b}_3) \\
&+ \frac{1}{2}(b_1-\bar{b}_1)(b_6-\bar{b}_1) + (b_6-\bar{b}_1)(b_5-\bar{b}_1) + (b_5-\bar{b}_1)(b_4-\bar{b}_1) + (b_4-\bar{b}_1)(b_1-\bar{b}_1) \\
&+ \frac{1}{2}(b_3-\bar{b}_2)(b_4-\bar{b}_2) + (b_4-\bar{b}_2)(b_5-\bar{b}_2) + (b_5-\bar{b}_2)(b_6-\bar{b}_2) + (b_6-\bar{b}_2)(b_3-\bar{b}_2) \\
&+ \frac{1}{2}(b_1-\bar{b}_3)(b_2-\bar{b}_3) + (b_2-\bar{b}_3)(b_3-\bar{b}_3) + (b_3-\bar{b}_3)(b_6-\bar{b}_3) + (b_6-\bar{b}_3)(b_1-\bar{b}_3) \\
&+ \frac{1}{2}(b_1b_2+b_2b_3+b_3b_4+b_4b_1) \\
&= C(a_1,a_2,a_3,b_1,b_2,b_3,b_4,b_5,b_6) + D(b_1,b_2,b_3,b_4,b_5,b_6) \qquad (89)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}B(a,b) &= a_1+a_2+a_3+b_1+b_2+b_3+b_4+b_5+b_6 \\
&= \frac{a_1+b_1}{2} + \frac{a_1+b_5}{2} + \frac{a_2+b_5}{2} + \frac{a_2+b_6}{2} + \frac{a_3+b_1}{2} + \frac{a_3+b_6}{2} + b_2+b_3+b_4\,. \qquad (90)
\end{aligned}
$$

If $\frac{1}{2}(A(a,b)+B(a,b)) \leq N$ then $\frac{1}{2}B(a,b) \leq N$, so

$$
\begin{aligned}
0 \leq & b_2 \leq N \qquad (91) \\
0 \leq & b_3 \leq N - b_2 \qquad (92) \\
0 \leq & b_4 \leq N - b_2 - b_3\,. \qquad (93)
\end{aligned}
$$

Let us set

$$b_1 = 0\,. \qquad (94)$$

Equation (90) implies that $0 \leq \frac{a_1+b_1}{2} \leq N - b_2 - b_3 - b_4$ which implies that $0 \leq a_1 \leq 2(N-b_2-b_3-b_4)$. It follows from $0 \leq \frac{a_1+b_5}{2} \leq N$ that

$$-2(N-b_2-b_3-b_4) \leq b_5 \leq 2(N-b_2-b_3-b_4)\,. \qquad (95)$$

Since $0 \le \frac{a_2+b_5}{2} \le N - b_2 - b_3 - b_4$ from (95) we have $-2(N - b_2 - b_3 - b_4) \le a_2 \le 4(N - b_2 - b_3 - b_4)$. Therefore, since $0 \le a_2 \le \frac{a_2+b_6}{2}$ we have

$$-4(N - b_2 - b_3 - b_4) \le b_6 \le 4(N - b_2 - b_3 - b_4)\,. \tag{96}$$

Equations (91)-(96) in particular bound b_2, b_3, b_4, b_5 and b_6 from above and from below by linear forms in N. But even better, Equations (91)-(96) allow for an iterated summation for the b_i variables which improves the computation of the $\Phi_{L8a7}(q)$ series.

To bound a_1, a_2, a_3 we will use the auxiliary function

$$u(c,d) = \left[\frac{-c + \sqrt{c^2 + 2d}}{2}\right]$$

where the integer part $[x]$ of a real number x is the biggest integer less than or equal to x. The argument of $u(c,d)$ inside the integer part is one of the solutions to the equation $2x^2 + cx - d = 0$. Let

$$\begin{aligned}
\tilde{b}_1 &= b_1 + b_4 + b_5 + b_6 - 4\bar{b}_1 \\
\tilde{b}_2 &= b_3 + b_4 + b_5 + b_6 - 4\bar{b}_2 \\
\tilde{b}_3 &= b_1 + b_2 + b_3 + b_6 - 4\bar{b}_3 \\
\tilde{D} &= D(b_1, b_2, b_3, b_4, b_5, b_6) + b_2 + b_3 + b_4
\end{aligned}$$

Since

$$2(a_1 + \bar{b}_1)^2 + (a_1 + \bar{b}_1)\tilde{b}_1 \le N - \tilde{D}$$

we have

$$-\bar{b}_1 \le a_1 \le -\bar{b}_1 + u(\tilde{b}_1, N - \tilde{D}) \tag{97}$$

where the left inequality follows from the fact that $a_1 \ge -b_i, i = 1, 4, 5, 6$. Similarly we have

$$-\bar{b}_2 \le a_2 \le -\bar{b}_2 + u(\tilde{b}_1, N - \tilde{D} - 2(a_1 + \bar{b}_1)^2 - (a_1 + \bar{b}_1)\tilde{b}_1) \tag{98}$$

and

$$-\bar{b}_3 \le a_3 \le -\bar{b}_3 + u(\tilde{b}_1, N - \tilde{D} - 2(a_1 + \bar{b}_1)^2 - (a_1 + \bar{b}_1)\tilde{b}_1 - 2(a_2 + \bar{b}_2)^2 - (a_2 + \bar{b}_2)\tilde{b}_2) \tag{99}$$

Note that Equations (97)-(99) allow for an iterated summation in the a_i variables, and in particular imply that the span of the a_i variables is bounded by a linear form of $\sqrt{N}$.

It follows that

$$\Phi_{L8a7}(q) + O(q)^{N+1} = (q)_\infty^8 \sum_{(a,b)} \frac{q^{\frac{1}{2}(A(a,b)+B(a,b))}}{(q)_{a_1+b_1}(q)_{a_1+b_4}(q)_{a_1+b_5}(q)_{a_1+b_6}(q)_{a_2+b_3}(q)_{a_2+b_4}(q)_{a_2+b_5}(q)_{a_2+b_6}}$$
$$\cdot \frac{1}{(q)_{a_3+b_1}(q)_{a_3+b_2}(q)_{a_3+b_3}(q)_{a_3+b_6}(q)_{b_1}(q)_{b_2}(q)_{b_3}(q)_{b_4}} + O(q)^{N+1}$$

where $(a,b) = (a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6)$ satisfy the inequalities (91)-(96) and (97)-(99). We give the first 21 terms of this series in the Table 15.

3.5.2 The computation of $\Phi_G(q)$ by iterated summation

Our method of computation requires not only the planar graph with its vertices and faces (which is relatively easy to automate), but also the inequalities for the b_i and a_j variables which lead to an iterated summation formula for $\Phi_G(q)$. Although Theorem 3.1.2 implies the existence of an iterated summation formula for every planar graph, we did not implement this algorithm in general.

Instead, for each of the 11 graphs that appear in Figures 9 and 10, we computed the corresponding inequalities for the iterated summation by hand. These inequalities are too long to present them here, but we have them available. A consistency check of our computation is obtained by Equation (79), where the shifted colored Jones polynomial of an alternating link is available from [6] for several values. Our data matches those values.

3.6 Tables

In this section we give various tables of graphs, and their corresponding alternating knots (following Rolfsen's notation [46]) and links (following Thistlethwaite's notation [6]) and several terms of $\Phi_G(q)$. In view of an expected positive answer to Question

3.1.6, we will list *irreducible* graphs, i.e., simple planar 2-connected graphs which are not of the form $G_1 \cdot G_2$ (for the operation $\cdot$ defined in Section 3.1.3).

- The first table gives number of alternating links with at most 10 crossings and the number of irreducible graphs with at most 10 edges

crossings = edges	3	4	5	6	7	8	9	10
alternating links	1	2	3	8	14	39	96	297
irreducible graphs	1	1	1	3	3	8	17	41

(100)

To list planar graphs, observe that they are *sparse*: if G is a planar graph which is not a tree, with V vertices and E edges then

$$V \leq E \leq 3V - 6\,.$$

- The next table gives the number of planar 2-connected irreducible graphs with at most 9 vertices

vertices	3	4	5	6	7	8	9
graphs	1	2	5	19	106	897	10160

(101)

- Tables 8, 9, 10 and 11 give the list of irreducible graphs with at most 9 edges. These tables were constructed by listing all graphs with $n \leq 9$ vertices, selecting those which are planar, and further selecting those that are irreducible. Note that if G is a planar graph with $E \leq 9$ edges, V vertices and F faces then $E - V = F - 2 \geq 0$ hence $V \leq E \leq 9$.

- Tables 12 and 13 give the reduced Tait graphs of all alternating knots and links (and their mirrors) with at most 8 crossings. Here P_r is the planar polygon with r sides and $-K$ denotes the mirror of K. Moreover, the notation $G = G_1 \cdot G_2 \cdot G_3$ indicates that $\Phi_G(q) = \Phi_{G_1}(q)\Phi_{G_2}(q)\Phi_{G_3}(q)$ by Lemma 3.1.5.

- Table 14 gives the alternating knots and links with at most 8 crossings for the irreducible graphs with at most 8 edges.

- Table 15 gives the first 21 terms of of $\Phi_G(q)$ for all irreducible graphs with at most 8 edges. Many more terms are available from

`http://www.math.gatech.edu/~stavros/publications/phi0.graphs.data/`

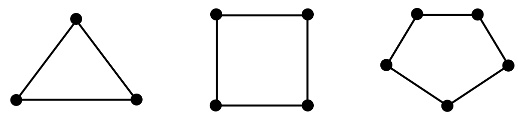


Figure 8: The irreducible planar graphs G_0^3, G_0^4 and G_0^5 with 3, 4 and 5 edges.

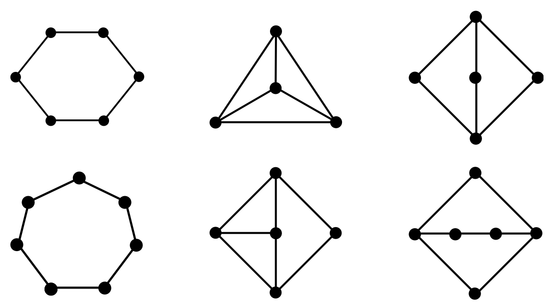


Figure 9: The irreducible planar graphs with 6 and 7 edges: G_0^6, G_1^6, G_2^6 on the top and G_0^7, G_1^7, G_2^7 on the bottom.

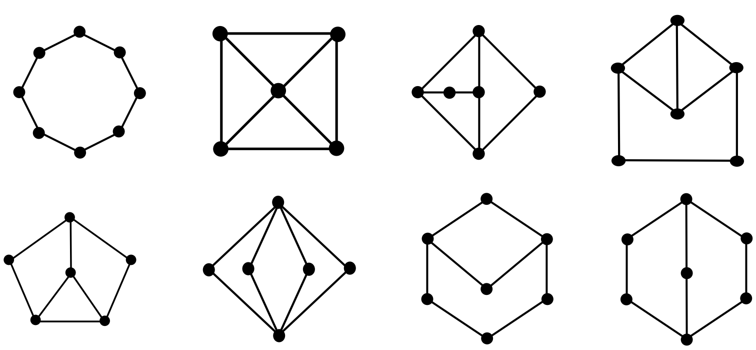


Figure 10: The irreducible planar graphs with 8 edges: $G_0^8, \dots, G_3^8$ on the top (from left to right) and $G_4^8, \dots, G_7^8$ on the bottom.

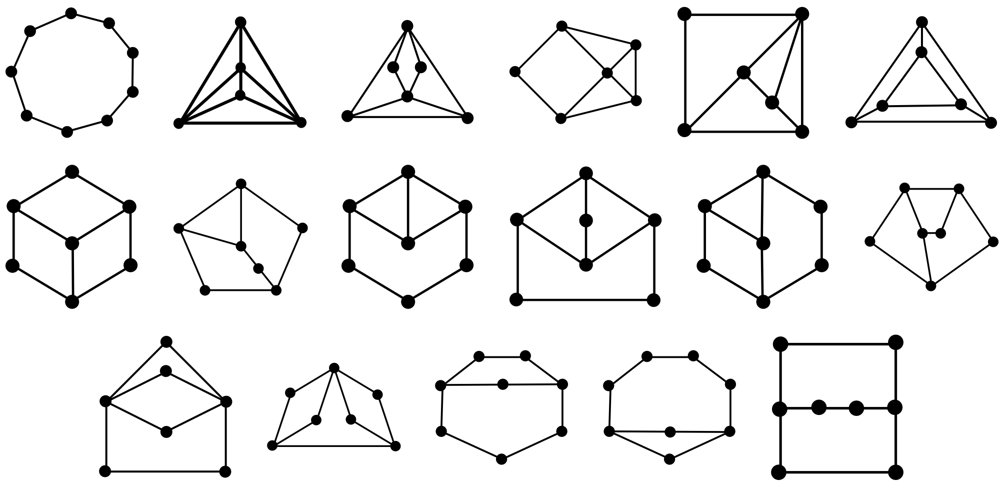


Figure 11: The irreducible planar graphs with 9 edges: $G_0^9, \dots, G_5^9$ on the top, $G_6^9, \dots, G_{11}^9$ on the middle and $G_{12}^9, \dots, G_{16}^9$ on the bottom.

K	G	$-G$	K	G	$-G$	K	G	$-G$	K	G	$-G$
0_1	P_2	P_2	7_2	P_6	P_3	8_4	P_3	$P_4{\cdot}P_5$	8_{13}	$P_3{\cdot}P_3{\cdot}P_4$	$P_3{\cdot}P_3$
3_1	P_3	P_2	7_3	P_5	P_4	8_5	G_7^8	P_3	8_{14}	$P_3{\cdot}P_4$	$P_3{\cdot}P_3{\cdot}P_3$
4_1	P_3	P_3	7_4	$P_4{\cdot}P_4$	P_3	8_6	$P_3{\cdot}P_4$	P_5	8_{15}	$P_3{\cdot}P_3{\cdot}P_3$	G_2^6
5_1	P_5	P_2	7_5	$P_3{\cdot}P_4$	P_4	8_7	$P_3{\cdot}P_5$	P_4	8_{16}	G_4^8	G_1^6
5_2	P_4	P_3	7_6	$P_3{\cdot}P_4$	$P_3{\cdot}P_3$	8_8	$P_3{\cdot}P_5$	$P_3{\cdot}P_3$	8_{17}	G_1^7	G_1^7
6_1	P_5	P_3	7_7	$P_3{\cdot}P_3{\cdot}P_3$	$P_3{\cdot}P_3$	8_9	$P_3{\cdot}P_4$	$P_3{\cdot}P_4$	8_{18}	G_1^8	G_1^8
6_2	$P_3{\cdot}P_4$	P_3	8_1	P_7	P_3	8_{10}	G_2^7	$P_3{\cdot}P_3$			
6_3	$P_3{\cdot}P_3$	$P_3{\cdot}P_3$	8_2	$P_3{\cdot}P_6$	P_3	8_{11}	$P_3{\cdot}P_4$	$P_3{\cdot}P_4$			
7_1	P_7	P_2	8_3	P_5	P_5	8_{12}	$P_3{\cdot}P_4$	$P_3{\cdot}P_4$			

Figure 12: The reduced Tait graphs of the alternating knots with at most 8 crossings

L	G	$-G$	L	G	$-G$	L	G	$-G$	L	G	$-G$
$2a1$	P_2	P_2	$7a2$	$P_3{\cdot}P_3$	G_2^6	$8a4$	$P_3{\cdot}P_4$	$P_3{\cdot}P_3{\cdot}P_3$	$8a13$	$P_4{\cdot}P_4$	P_4
$4a1$	P_4	P_2	$7a3$	G_2^7	P_3	$8a5$	P_4	$P_3{\cdot}P_3{\cdot}P_4$	$8a14$	P_8	P_2
$5a1$	$P_3{\cdot}P_3$	P_3	$7a4$	P_5	$P_3{\cdot}P_3$	$8a6$	P_6	$P_3{\cdot}P_3$	$8a15$	P_5	$P_3{\cdot}P_3{\cdot}P_3$
$6a1$	P_4	$P_3{\cdot}P_3$	$7a5$	$P_3{\cdot}P_4$	$P_3{\cdot}P_3$	$8a7$	G_2^8	G_1^6	$8a16$	G_3^8	G_1^6
$6a2$	P_4	P_4	$7a6$	$P_3{\cdot}P_5$	P_3	$8a8$	$P_3{\cdot}P_4{\cdot}P_3$	$P_3{\cdot}P_3$	$8a17$	$P_3{\cdot}P_4$	G_2^6
$6a3$	P_6	P_2	$7a7$	P_4	G_2^6	$8a9$	$P_3{\cdot}P_3{\cdot}P_3$	$P_3{\cdot}P_3{\cdot}P_3$	$8a18$	G_6^8	P_3
$6a4$	G_1^6	G_1^6	$8a1$	G_1^7	$P_3{\cdot}G_1^6$	$8a10$	$P_3{\cdot}P_4$	$P_3{\cdot}P_3$	$8a19$	G_1^7	G_1^7
$6a5$	P_3	G_2^6	$8a2$	$P_3{\cdot}P_3$	$P_3{\cdot}G_2^6$	$8a11$	$P_3{\cdot}P_5$	P_4	$8a20$	G_2^6	G_2^6
$7a1$	G_1^7	G_1^6	$8a3$	G_2^7	$P_3{\cdot}P_3$	$8a12$	P_6	P_4	$8a21$	P_4	G_5^8

Figure 13: The reduced Tait graphs of the alternating links with at most 8 crossings

G_1^6	$L6a4$ $-L6a4$ $-L7a1$ $-L8a7$ -8_{16} $-L8a16$
G_2^6	$-L6a5$ $-L7a2$ $-L7a7$ $-L8a17$ -8_{15} $L8a20$ $-L8a20$
G_1^7	$L7a1$ $L8a1$ 8_{17} -8_{17} $L8a19$ $-L8a19$
G_2^7	8_{10} $L7a3$ $L8a3$
G_1^8	8_{18} -8_{18}
G_2^8	$L8a7$
G_3^8	$L8a16$
G_4^8	8_{16}
G_5^8	$-L8a21$
G_6^8	$L8a18$
G_7^8	8_5

Figure 14: The irreducible planar graphs with at most 8 edges and the corresponding alternating links

G	$\Phi_G(q) + O(q)^{21}$
G_1^6	$1 - 3q - q^2 + 5q^3 + 3q^4 + 3q^5 - 7q^6 - 5q^7 - 8q^8 - 6q^9 + 6q^{10}$ $+7q^{11} + 12q^{12} + 15q^{13} + 16q^{14} - 3q^{15} - q^{16} - 15q^{17} - 21q^{18} - 31q^{19} - 30q^{20}$
G_2^6	$1 - 2q + q^2 + 3q^3 - 2q^4 - 2q^5 - 3q^6 + 3q^7 + 4q^8 + q^9 + 3q^{10}$ $-6q^{11} - 5q^{12} - 3q^{13} + q^{15} + 7q^{16} + 9q^{17} + 3q^{18} - 6q^{20}$
G_1^7	$1 - 3q + q^2 + 5q^3 - 3q^4 - 3q^5 - 6q^6 + 6q^7 + 8q^8 + 3q^9 + 6q^{10}$ $-13q^{11} - 14q^{12} - 9q^{13} - q^{14} + 3q^{15} + 21q^{16} + 27q^{17} + 14q^{18} + 3q^{19} - 17q^{20}$
G_2^7	$1 - 2q + q^2 + q^3 - 3q^4 + q^5 + q^6 + 3q^7 - 2q^8 - 4q^9 + q^{10}$ $+4q^{12} + 5q^{13} - 2q^{14} - 5q^{15} - 4q^{16} - 2q^{17} - 2q^{18} + 5q^{19} + 8q^{20}$
G_1^8	$1 - 4q + 2q^2 + 9q^3 - 5q^4 - 8q^5 - 14q^6 + 10q^7 + 21q^8 + 14q^9 + 19q^{10}$ $-29q^{11} - 42q^{12} - 42q^{13} - 20q^{14} + 3q^{15} + 64q^{16} + 104q^{17} + 88q^{18} + 55q^{19} - 25q^{20}$
G_2^8	$1 - 3q + 3q^2 + 4q^3 - 8q^4 - 2q^5 + 2q^6 + 12q^7 + 3q^8 - 15q^9 - 4q^{10}$ $-14q^{11} + 10q^{12} + 25q^{13} + 15q^{14} - 18q^{16} - 22q^{17} - 39q^{18} - 12q^{19} + 19q^{20}$
G_3^8	$1 - 3q + q^2 + 3q^3 - 3q^4 + 3q^5 + 4q^7 - 6q^8 - 10q^9 + q^{10}$ $-q^{11} + 9q^{12} + 13q^{13} + 3q^{14} - 9q^{15} - 3q^{16} - 6q^{17} - 4q^{18} + 5q^{19} + 13q^{20}$
G_4^8	$1 - 3q + 2q^2 + 3q^3 - 6q^4 + q^5 + 2q^6 + 8q^7 - 3q^8 - 13q^9$ $-3q^{11} + 13q^{12} + 19q^{13} + q^{14} - 15q^{15} - 20q^{16} - 16q^{17} - 13q^{18} + 15q^{19} + 37q^{20}$
G_5^8	$1 - 3q + 3q^2 + 5q^3 - 8q^4 - 5q^5 - q^6 + 15q^7 + 12q^8 - 8q^9 - 7q^{10}$ $-31q^{11} - 11q^{12} + 14q^{13} + 30q^{14} + 35q^{15} + 27q^{16} + 8q^{17} - 48q^{18} - 66q^{19} - 72q^{20}$
G_6^8	$1 - 2q + q^2 + q^3 - q^4 + 2q^5 - 2q^6 - q^7 - 2q^8 + 2q^9 + 5q^{10}$ $-q^{11} - q^{12} - 3q^{13} - 2q^{14} + 5q^{16} - 2q^{18} - q^{19} - q^{20}$
G_7^8	$1 - 2q + q^2 - 2q^4 + 3q^5 - 3q^8 + q^9 + 4q^{10}$ $-q^{11} - 2q^{12} - 2q^{13} - 3q^{14} + 3q^{15} + 7q^{16} + 2q^{17} - 4q^{18} - 4q^{19} - 4q^{20}$

Figure 15: The first 21 terms of $\Phi_G(q)$ for the irreducible planar graphs with at most 8 edges

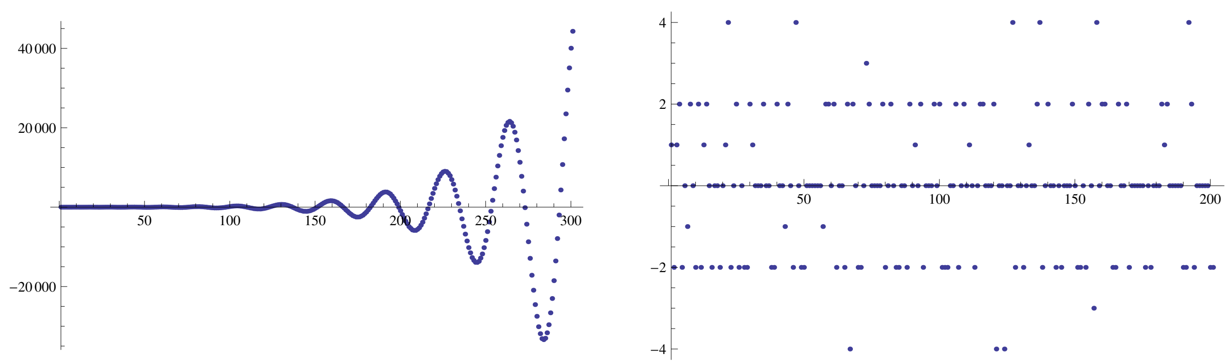


Figure 16: Plot of the coefficients of $\Phi_{G_2^6}(q)$ on the left and $h_4(q)^2$ (keeping in mind that G_2^6 has two bounded square faces) on the right.

CHAPTER IV

FLAG ALGEBRAS AND THE STABLE COEFFICIENTS OF THE JONES POLYNOMIAL

4.1 Introduction

4.1.1 The stable coefficients of the Jones polynomial

The Jones polynomial of a knot [34] is a powerful knot invariant that takes values (when properly normalized) in the ring $\mathbb{Z}[q^{\pm 1}]$. If $\sum_j a_j q^j$ denotes the Jones polynomial of a knot (a finite sum where a_j are integers), then the coefficients (a_j) are mysterious and fascinating integer-valued knot invariants. In the nineties, the Jones polynomial was studied from the point of view of perturbative Chern-Simons theory and finite-type invariants [5]. This amounts to studying sums $\sum_j j^k a_j$ for natural numbers k. Whereas such sums for all k uniquely recover the Jones polynomial and hence its coefficients, this sheds little light into the geometric meaning of the coefficients (a_j). In the early 2000, the categorification ideas of Khovanov [35] introduced a refinement $a_j(t) \in \mathbb{Z}[t]$ of the coefficients of the Jones polynomial such that $a_j = a_j(-1)$. In contrast to the theory of finite type invariants, the coefficients of the Jones polynomial (and its Khovanov Homology) are non-perturbative knot invariants.

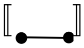
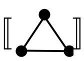
When L is an alternating link, the coefficients of the (shifted) colored Jones polynomial $\hat{J}_{L,n}(q)$ stabilize, in the sense that the limit

$$\Phi_L(q) = \lim_n \hat{J}_{L,n}(q)$$

exists. Even more, for every $k \in \mathbb{N}$, the coefficient of q^k in $\hat{J}_{L,n}(q) \in \mathbb{Z}[q]$ is independent of n for $n > k$. The existence of $\Phi_L(q)$ and its extension to $\Phi_G(q)$ for $G \in \mathcal{G}$ (where G is the reduced Tait graph of a planar projection of L) was shown in [23], see also [2, 4]. We will denote by $\phi_{G,k}$ (resp., $\phi_{L,k}$) the coefficient of q^k in $\Phi_G(q)$ (resp.,

$\Phi_G(q)$), and we will often call it the k-th stable coefficient of G (resp., L). Thus, we have $\Phi(q) = \sum_{k=0}^{\infty} \phi_k q^k$.

In [12] the first three stable coefficients $\phi_k : G \mapsto \phi_{G,k}$ for $k = 0, 1, 2$ were expressed in terms of the number of vertices, edges and 3-cycles of G. The proof used properties of the Kauffman bracket skein module. An independent proof was given in [27]. To express the answer, and to motivate the polynomial algebra $\mathcal{P}$ introduced below, consider the elements $c_1, c_2, c_3 \in \mathcal{P}$ given by

$(c_1, c_2, c_3) = ([\![\bullet]\!],$ $[\![$  $]\!]$, $[\![$  $]\!]$$)\,.$ (102)

c_1, c_2, c_3 count the number of vertices, edges and triangles in a graph G. Then, we have [12]

$$(\phi_0, \phi_1, \phi_2) = \left(1, c_1 - c_2 - 1, \frac{1}{2}\left((c_1 - c_2)^2 - 2c_3 - c_1 + c_2\right)\right). \tag{103}$$

It is natural to ask for a formula for the next coefficient ϕ_3. The answer is given in Theorem 4.1.3 below. What's more, Theorem 4.1.3

(a) motivates us to introduce the algebra $\mathcal{P}$ of polynomial invariants of graphs, in the spirit of flag algebras of [45]. $\mathcal{P}$ turns out to be a free polynomial algebra, see Theorem 4.1.2.

(b) shows that ϕ_3 is determined by ϕ_k for $k \leq 2$ and $-c_{41} + 2c_{42}$. The latter is an integer linear combination of the refined alternating link invariants c_{41}, c_{42}; see Proposition 4.2.2

(c) motivates us to write $\Phi(q)$ as an infinite product and conjecture that its exponents are linear forms on the set of irreducible planar graphs, see Conjecture 4.1.6 and its explicit form, Conjecture 4.1.4. The latter is verified by explicit computation for all alternating links with at most 10 crossings and all irreducible graphs with at most 7 vertices.

(d) raises the question of how Rozansky's categorification $\Phi_L(t,q)$ of $\Phi_L(q) = \Phi_L(-1,q)$ (see [48]) can further refine Conjecture 4.1.6. Since this categorification has not been computed, we cannot make this question more precise.

4.1.2 An algebra $\mathcal{P}$ of polynomial invariants of graphs

Let $\mathcal{G}$ denote the set of simple finite graphs, i.e., abstract non-embedded graphs with no loops and no multiple edges, and unlabeled vertices and edges. For H and G in $\mathcal{G}$, an *embedding* $f : H \to G$ is an injection $f : V(H) \hookrightarrow V(G)$ (where $V(G)$ denotes the set of vertices of G) such that for every $v, v' \in V(H)$ (v, v') is an edge of H if and only if $(f(v), f(v'))$ is an edge of G. Let $i(H,G)$ denote the number of embeddings of H in G, divided by the number of automorphisms of H. Varying G, we get a function $[H] : \mathcal{G} \to \mathbb{N}$ given by $G \in \mathcal{G} \mapsto [H](G) = i(H,G)$. The *degree* of $[H]$ is the number of vertices of H. Let $[\mathcal{G}]$ denote the set $\{[H] \,|H \in \mathcal{G}\}$. Likewise we define $[\mathcal{G}^c]$ where $\mathcal{G}^c$ is the set of connected graphs. $\mathcal{P}$ denote the $\mathbb{Q}$-vector space on the set $[\mathcal{G}]$.

Proposition 4.1.1. (a) $\mathcal{P}$ is a commutative algebra. In fact,

$$[H_1][H_2] = \sum_H c_H [H] \tag{104}$$

where H is a graph on on at most $|V(H_1)| + |V(H_2)|$ vertices and c_H is the number of ordered pairs of induced subgraphs (F_1, F_2) of H (possibly sharing some vertices) such that F_i is isomorphic to H_i for $i = 1, 2$ and moreover $V(F_1) \cup V(F_2) = V(H)$.
(b) It follows that $\mathcal{P}$ is a quotient of the polynomial algebra on $[\mathcal{G}^c]$.

Equation (104) shows that the structure constants of the multiplication in $\mathcal{P}$ are natural numbers. For instance we have:

$$\frac{1}{2}([\bullet]^2 - [\bullet]) = [\bullet\!\!-\!\!\!-\!\!\bullet] + [\bullet\ \bullet]$$

This holds both sides of the above equation evaluated on $G \in \mathcal{G}$ equal to the number of pairs of vertices of G and such a pair is either connected by an edge or not. More generally, if H is a graph on k vertices then

$$[H][\bullet] = k[H] + \sum c_F[F]$$

where the sum is over all graphs F on $k+1$ vertices and c_F is equal to the number of induced subgraphs of F isomorphic to H.

Theorem 4.1.2. *$\mathcal{P}$ is a free polynomial algebra on the set $[\mathcal{G}^c]$.*

Real valued functions on $\mathcal{G}$ are also called *graph parameters* and linear combinations of graphs are also called *quantum graphs* in the context of graph theory. The algebra $\mathcal{P}$ is reminiscent to the *flag algebras* of graph theory [45].

Since alternating links involve planar graphs only, let $\mathcal{G}^{\mathrm{pl}}$ denote the set of simple planar graphs. For $H \in \mathcal{G}^{\mathrm{pl}}$, we denote by $[\![H]\!]$ the restriction of the function $[H] : \mathcal{G} \to \mathbb{N}$ to $\mathcal{G}^{\mathrm{pl}} \subset \mathcal{G}$, and $\mathcal{P}^{\mathrm{pl}}$ the vector space generated by $[\![H]\!]$ for $H \in \mathcal{G}^{\mathrm{pl}}$. $\mathcal{P}^{\mathrm{pl}}$ is also an algebra, however it is not free. Indeed, it was pointed out by Armond-Dasbach that if $(c_1 - c_2 - 1)(G) = 0$ for a planar graph G, then G is a tree hence $c_3(G) = 0$. The structure of the algebra $\mathcal{P}^{\mathrm{pl}}$ is an interesting and challenging problem.

4.1.3 A formula for ϕ_3

Let $c_{4,i} = [\![Gv_i^4]\!]$ for $i = 1, 2$ where Gv_i^4 are shown in Figure 17.

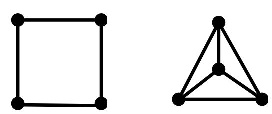


Figure 17: The irreducible planar graphs Gv_1^4 (left) and Gv_2^4 (right) with 4 vertices.

Theorem 4.1.3. *We have:*

$$\phi_3 = c_{41} - 2c_{42} + \frac{c_2}{6} + c_3 c_2 - \frac{c_2^3}{6} - \frac{c_1}{6} - c_3 c_1 + \frac{c_2^2 c_1}{2} - \frac{c_2 c_1^2}{2} + \frac{c_1^3}{6} \tag{105}$$

Equations (103) *and* (105) *are equivalent to*

$$\Phi(q) = (1-q)^{1-c_1+c_2}(1-q^2)^{c_3}(1-q^3)^{c_3-c_{41}+2c_{42}} + O(q)^4 \,. \tag{106}$$

4.1.4 A conjecture for ϕ_4, ϕ_5 and ϕ_k

A comparison of Equations (105) and (106) suggests us to write $\Phi_G(q)$ as an infinite product

$$\Phi(q) = (1-q)^{1-c_1+c_2} \prod_{k=2}^{\infty} (1-q^k)^{C_k} \tag{107}$$

where $C_k(G) \in \mathbb{Z}$ for all k. This is possible since $\Phi_G(q) \in 1 + q\mathbb{Z}[[q]]$. Theorem 4.1.3 gives an expression for C_k for $k = 2, 3$. To phrase our conjecture for C_k for $k = 4, 5$, recall the notion of an irreducible planar graph from [27]. The latter is a planar graph which is not a vertex connected sum or an edge connected sum of planar graphs as in Figure 18. The table of irreducible planar graphs with at most 10 edges is given in Figures 28, 29, 30, 31 and 32, and with at most 6 vertices is given in Figures 17, 26 and 27.

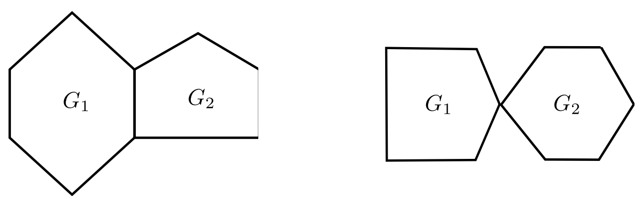


Figure 18: A vertex connected sum (on the left) and an edge-connected sum on the right.

Conjecture 4.1.4. *We conjecture that*

$$C_4 = c_3 - c_{41} + 5c_{42} + c_{51} - c_{52} - 2c_{53} - 3c_{54} \tag{108}$$

$$\begin{aligned} C_5 = {} & c_3 - c_{41} + 12c_{42} + c_{51} - 4c_{53} - 9c_{54} \\ & - c_{61} + c_{62} - 2c_{63} - c_{64} + 2c_{65} + 3c_{66} + 4c_{68} - 4c_{69} \\ & + 2c_{610} + c_{611} - 3c_{612} + 4c_{613} + c_{614} - 5c_{616} - 16c_{618} + c_{619} \end{aligned} \tag{109}$$

where $c_{j,i} = [\![Gv_i^j]\!]$ *and* Gv_i^5 *and* Gv_i^6 *are* irreducible *planar graphs with* 5 *and* 6 *vertices shown in Figures 26 and 27.*

Independently of the above conjecture, each term that appears in the right hand side of Equations (108)-(109) is an alternating link invariant; see Proposition 4.2.2 below.

The expression for C_4 and C_5 is the unique linear combination of irreducible planar graphs with 5 and 6 vertices (and this is how it was found) which fits the stable coefficients of the Jones polynomial of all alternating links with at most 10 crossings and all alternating links whose reduced Tait graph has at most 6 vertices. For details, see Section 3.5.

The reader may observe that the graph Gv_5^5 is missing from C_4. This motivates the following question.

Question 4.1.5. *Is it true that*

$$(\Phi_{G_1^6})^2 = \Phi_{G_1^9}\Phi_{G_0^3}$$

A direct computation confirms this up to $O(q)^{31}$.

Conjecture 4.1.6. *For all* $k \geq 2$, C_k *are linear forms with integer coefficients on the set of irreducible planar graphs with at most* $k+1$ *vertices.*

The above conjecture has an equivalent formulation.

Conjecture 4.1.7. Φ *is multiplicative under vertex and edge connected sum, and for every connected irreducible planar graph* H *there exist* $\Psi_H(q) \in 1 + q^{\deg(H)-1}\mathbb{Z}[[q]]$ *such that*

$$\Phi(q) = (1-q)^{1-c_1+c_2} \prod_H \Psi_H(q)^{[\![H]\!]} \tag{110}$$

where the product is taken over the set of irreducible planar graphs.

4.2 The algebra $\mathcal{P}$

4.2.1 Proof of Proposition 4.1.1

A *subgraph of a graph* G *induced by* $S \subseteq V(G)$ is a graph $G[S]$ such that $V(G[S]) = S$ and two vertices in S are joined by an edge in $G[S]$ if and only if they are joined by an edge in G. The value $i(H,S)$ can be equivalently defined as the number of sets $S \subseteq V(G)$ such that $G[S]$ is isomorphic to H.

To show that (111) holds we need to show that

$$i(H_1,G)i(H_2,G) = \sum_{H} c_H i(H,G) \tag{111}$$

for every graph G. Note that $i(H_1,G)i(H_2,G)$ equals the number of pairs (S_1,S_2) of subsets of $V(G)$ such that $G[S_i]$ is isomorphic to H_i for $i = 1,2$. We claim that for a fixed graph H the number of pairs as above, such that $G[S_1 \cup S_2]$ is isomorphic to H, is equal to $c_H i(H,G)$. The equation (111) immediately follows from this claim. The claim holds as the number of sets $S \subseteq V(G)$ such that $G[S]$ is isomorphic to H is equal to $i(H,G)$. Further, for given $S \subseteq V(G)$ the number of pairs (S_1,S_2) defined above with $S = S_1 \cup S_2$ equals c_H, by definition. □

4.2.2 Proof of Theorem 4.1.2

The proof of the theorem is derived from the results of [15]. We start by introducing the additional notation, which will allow us to state the necessary results. Let

$$\gamma(H,G) = i(H,G)/\binom{|V(G)|}{|V(H)|}.$$

Let k be a fixed integer and let $H_1, H_2, \ldots, H_m$ be all connected graphs with $|V(H_i)| \leq k$. Given a graph G define a vector

$$\gamma(k,G) = (\gamma(H_1,G), \gamma(H_2,G), \ldots, \gamma(H_m,G)).$$

Let S_k be defined as the set of all vectors $\mathbf{v} \in \mathbb{R}^m$ such that there exists an infinite sequence of graphs $G_1, G_2, \ldots G_n, \ldots$, such that $|V(G_n)| \to \infty$ and $\gamma(k,G) \to \mathbf{v}$. The following lemma follows immediately from [15, Theorems 1 and 3].

Lemma 4.2.1. *Let k be a positive integer, let m be the number of connected graphs on at most k vertices and let $S_k \subseteq \mathbb{R}^m$ be as defined above. Then S_k contains an m-dimensional ball of positive radius.*

We are now ready to prove Theorem 4.1.2.

Proof. (of Theorem 4.1.2) Let k be a positive integer and let $H_1, H_2, \ldots, H_m$ be all connected graphs on at most m vertices, as before. It suffices to show that for every $p \in \mathbb{R}[x_1, x_2, \ldots, x_m]$, $p \not\equiv 0$, we have $p([H_1], [H_2], \ldots, [H_m]) \not\equiv 0$. Suppose for a contradiction that for some polynomial $p_0 \in \mathbb{R}[x_1, x_2, \ldots, x_m]$, $p_0 \not\equiv 0$ we have

$$p_0(i(H_1, G), i(H_2, G), \ldots, i(H_m, G)) = 0$$

for every graph G. As $i(H, G) = \binom{|V(G)|}{|V(H)|}\gamma_H(G)$, there exists an $(m+1)$-variable polynomial $p_1 \in \mathbb{R}[x_1, x_2, \ldots, x_m, y]$, $p_1 \not\equiv 0$ such that

$$p_0(i(H_1, G), i(H_2, G), \ldots, i(H_m, G))) = p_1(\gamma(H_1, G), \gamma(H_2, G), \ldots, \gamma(H_m, G), |V(G)|)$$
$$(= p_1(\gamma(k, G), |V(G)|)).$$

Let

$$p_1(x_1, x_2, \ldots, x_m, y) = \sum_{i=1}^{t} r_i(x_1, \ldots, x_m) y^i,$$

Suppose without loss of generality that r_t is not identically zero. We claim that r_t is identically zero on S_k, in contradiction with Lemma 4.2.1.

To prove the claim, consider $\mathbf{v} \in S_k$ and let $G_1, G_2, \ldots G_n, \ldots$ be a sequence of graphs such that $|V(G_n)| \to \infty$ and $\gamma(k, G) \to \mathbf{v}$, as in the definition of S_k. Let $f(G_n) = p_1(\gamma(k, G_n), |V(G)|)/|V(G)|^t$. Clearly, $\lim_{n\to\infty} f(G_n) = r_t(\mathbf{v})$. On the other hand, $f(G_n) = 0$ for every n by the choice of p_1. It follows that $r_t(\mathbf{v}) = 0$, as desired. This establishes the claim and the theorem. □

4.2.3 A subalgebra $\mathcal{P}^{\text{fl}}$ of $\mathcal{P}$

In this section we introduce a subalgebra $\mathcal{P}^{\text{fl}}$ of $\mathcal{P}$ which is motivated by knot theory. Consider a *flype move* on a graph shown in Figure 19.

The importance of the flype move is Tait's Conjecture proven by Menasco-Thisthlethwaite [42]: every two reduced S^2 projections of an alternating link are connected by a sequence of flype moves. Closely related to a flype move is a *Whitney flip* move [55],

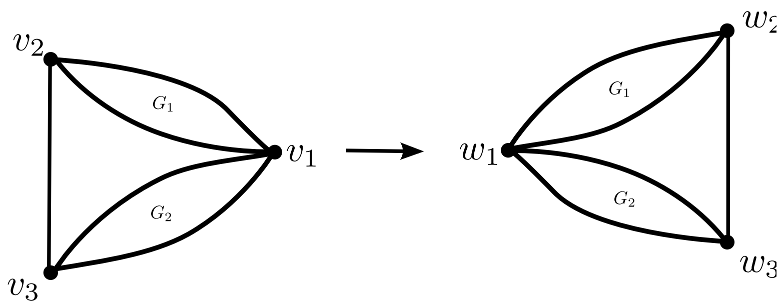


Figure 19: A flype move on a planar graph.

illustrated in Figure 20.

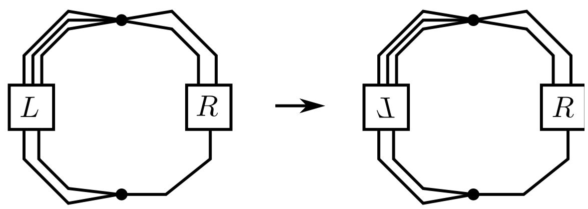


Figure 20: A Whitney flip on a graph.

In [23] it was shown that

- a Whitney flip on a planar graph corresponds to a Conway mutation for the corresponding alternating links.
- A flype move can be obtained by two Whitney flip moves.

Menasco [41] shows that there are two types of Conway mutation, type I (visible in an alternating link projection) and type II (hidden from the link projection). It was pointed out to us by F. Bonahon and J. Greene that a type II mutation can be achieved by two type I mutations. Independent of this fact, in [31, Thm.1.1] Greene proves that the Tait graph gives a 1-1 correspondence between the set of alternating links, modulo Conway mutation and the set of planar graphs modulo flips. A Conway mutation does not change the colored Jones polynomial, hence $\Phi_G(q)$ does not change under Whitney flips on G.

Let $\mathcal{G}^{\mathrm{fl}}$ denote the set of equivalence classes on $\mathcal{G}$ induced by the Whitney flip equivalence relation. Let $\mathcal{P}^{\text{fl}}$ denote the subalgebra of $\mathcal{P}$ that consists of all polynomials $P : \mathcal{G} \to \mathbb{Q}$ (where $P \in \mathcal{P}$) that satisfy $P(G) = P(G')$ whenever G and G' are related by a Whitney flip.

The above discussion gives rise to a map

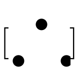
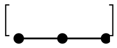
$$\mathcal{P}^{\text{fl}} \times \{\text{Alternating links}\}/\{\text{Conway mutation}\} \longrightarrow \mathbb{Q} \tag{112}$$

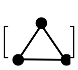
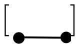
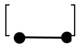
Proposition 4.2.2. *If H is 2-connected and isomorphic to every one of its Whitney flips, then $[H] \in \mathcal{P}^{\text{fl}}$.*

Corollary 4.2.3. *The terms c_{4i}, $c_{5,j}$ and $c_{6,k}$ that appear in Equations* (108) *and* (109) *all belong to $\mathcal{P}^{\text{fl}}$ and therefore they are invariants of alternating links.*

Proof. Since Whitney flips preserve the number of vertices, by Proposition 4.2.2 it suffices to show that no two of the graphs Gv_i^5 (and similarly Gv_j^6) differ by Whitney flips. In [23, Sec.13.2] it was shown that if two planar graphs differ by Whitney flips, the corresponding alternating links are Conway mutant, and hence they have equal colored Jones polynomial, hence equal $\Phi(q)$ invariant. Inspection shows that the 5 irreducible graphs with 5 vertices shown in Figure 26 and the 19 irreducible graphs with 6 vertices shown in Figure 27 all have different Jones polynomial. Therefore, no two graphs are flip equivalent. □

Let

$(\gamma, \delta) = ([$  $], [$  $])$

Lemma 4.2.4. *(a)* $\gamma - \delta = \frac{1}{6}[\bullet]^3 + 2[$  $] - [$  $][\bullet] + 2[$  $] - \frac{1}{2}[\bullet]^2 + \frac{1}{3}[\bullet]$.
(b) $\gamma - \delta \in \mathcal{P}^{fl}$ is an invariant of alternating links, polynomially determined by c_1, c_2, c_3.

Proof. (a) By the multiplication formula (104) we have

$$[\bullet][\bullet] = 2[\bullet\ \bullet] + 2[\bullet\!-\!\bullet] + [\bullet] \tag{113}$$

$$[\bullet\ \bullet][\bullet] = 2[\bullet\ \bullet] + [\cdot\!-\!\cdot\!-\!\cdot] + 2[\bullet\!-\!\bullet\ \bullet] + 3[\bullet\ \bullet\ \bullet] \tag{114}$$

$$[\bullet\!-\!\bullet][\bullet] = 2[\bullet\!-\!\bullet] + 3[\triangle] + 2[\cdot\!-\!\cdot\!-\!\cdot] + 2[\bullet\!-\!\bullet\ \bullet] \tag{115}$$

It follows that

$$\begin{aligned}
[\bullet]^3 &= 2[\bullet\ \bullet][\bullet] + 2[\bullet\!-\!\bullet][\bullet] + [\bullet][\bullet] \\
&= 2(2[\bullet\ \bullet] + [\cdot\!-\!\cdot\!-\!\cdot] + 2[\bullet\!-\!\bullet\ \bullet] + 3[\bullet\ \bullet\ \bullet]) \\
&+ 2(2[\bullet\!-\!\bullet] + 3[\triangle] + 2[\cdot\!-\!\cdot\!-\!\cdot] + 2[\bullet\!-\!\bullet\ \bullet]) \\
&+ 2[\bullet\ \bullet] + 2[\bullet\!-\!\bullet] + [\bullet] \\
&= 6[\triangle] + 6[\bullet\!-\!\bullet\ \bullet] + 6[\bullet\ \bullet\ \bullet] + 6[\cdot\!-\!\cdot\!-\!\cdot] + 6[\bullet\ \bullet] + 6[\bullet\!-\!\bullet] + [\bullet]
\end{aligned}$$

Therefore

$$6[\bullet\ \bullet\ \bullet] = [\bullet]^3 - 6[\triangle] - 6[\bullet\!-\!\bullet\ \bullet] - 6[\cdot\!-\!\cdot\!-\!\cdot] - 6[\bullet\ \bullet] - 6[\bullet\!-\!\bullet] - [\bullet] \tag{116}$$

On the other hand, from Equation (113) we have

$$[\bullet\ \bullet] = \frac{1}{2}[\bullet]^2 - [\bullet\!-\!\bullet] - \frac{1}{2}[\bullet][\bullet] \tag{117}$$

and from Equation (115)

$$[\bullet\!-\!\bullet\ \bullet] = [\bullet\!-\!\bullet][\bullet] - 2[\bullet\!-\!\bullet] - 3[\triangle] - 2[\cdot\!-\!\cdot\!-\!\cdot] \tag{118}$$

Equations (116),(117),(118) give

$$\begin{aligned}
6[\bullet\ \bullet\ \bullet] &= [\bullet]^3 - 6[\triangle] - 6([\bullet\!-\!\bullet][\bullet] - 2[\bullet\!-\!\bullet] - 3[\triangle] - 2[\cdot\!-\!\cdot\!-\!\cdot]) \\
&- 6[\cdot\!-\!\cdot\!-\!\cdot] - 6(\frac{1}{2}[\bullet]^2 - [\bullet\!-\!\bullet] - \frac{1}{2}[\bullet][\bullet]) - 6[\bullet\!-\!\bullet] - [\bullet] \\
&= [\bullet]^3 + 12[\triangle] - 6[\bullet\!-\!\bullet][\bullet] + 12[\bullet\!-\!\bullet] - 3[\bullet]^2 + 2[\bullet] + 6[\cdot\!-\!\cdot\!-\!\cdot]
\end{aligned}$$

So

$$[\bullet\ \bullet\ \bullet] - [\cdot\!-\!\cdot\!-\!\cdot] = \frac{1}{6}[\bullet]^3 + 2[\triangle] - [\bullet\!-\!\bullet][\bullet] + 2[\bullet\!-\!\bullet] - \frac{1}{2}[\bullet]^2 + \frac{1}{3}[\bullet]$$

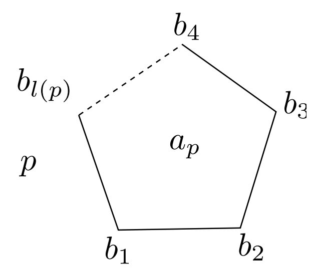
(b) This follows from (a) and Proposition 4.2.2. □

4.3 A review of the q-series $\Phi_G(q)$

4.3.1 The q-series $\Phi_G(q)$

In this section we will review the definition of the q-series $\Phi_G(q)$ of [23] following our earlier work [27]. An *admissible state* (a, b) of G is an integer assignment a_p for each face p and b_v for each vertex v of G such that $a_p + b_v \geq 0$. For the unbounded face p_∞ we set $a_\infty = 0$ and thus $b_v = a_\infty + b_v \geq 0$ for all $v \in p_\infty$. We also set $b_v = 0$ for a fixed vertex v of p_∞. In the formulas below, v, w will denote vertices of G, p a face of G and p_∞ is the unbounded face. We also write $v \in p$, $vw \in p$ if v is a vertex and vw is an edge of p.

For a polygon p with $l(p)$ edges and vertices $b_1, \dots, b_{l(p)}$ in counterclockwise order



we define

$$\gamma(p) = l(p)a_p^2 + 2a_p(b_1 + b_2 + \cdots + b_{l(p)}) \,.$$

Let

$$A(a, b) = \sum_p \gamma(p) + 2 \sum_{e=(v_i v_j)} b_{v_i} b_{v_j}$$

where the summation is over the faces p of G and edges $e = (v_i v_j)$ of p, and

$$B(a, b) = 2 \sum_v b_v + \sum_p (l(p) - 2)a_p \tag{119}$$

where the summation is over the vertices v and faces p of G.

Definition 4.3.1. *[23] With the above notation, we define*

$$\Phi_G(q) = (q)_\infty^{c_2} \sum_{(a,b)} (-1)^{B(a,b)} \frac{q^{\frac{1}{2}A(a,b)+\frac{1}{2}B(a,b)}}{\prod_{(p,v):v \in p} (q)_{a_p+b_v}} \tag{120}$$

where the sum is over the set of all admissible states (a, b) of G, and in the product $(p, v) : v \in p$ means a pair of face p and vertex v such that p contains v.

Let $b_p = \min\{b_v : v \in p\}$.

Theorem 4.3.2. *[27]* (a) We have

$$A(a,b) = \sum_p \left(l(p)(a_p + b_p)^2 + 2(a_p + b_p) \left(\sum_{v \in p} (b_v - b_p) \right) + \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} \right). \tag{121}$$

Each term in the above sum is manifestly nonnegative.

(b) $B(a, b)$ can also be written as a finite sum of manifestly nonnegative linear forms on (a, b).

(c) If $\frac{1}{2}(A(a, b) + B(a, b)) \leq N$ for some natural number N, then for every i and every j there exist c_i, c'_i and c_j, c'_j (computed effectively from G) such that

$$c_i N \leq b_i \leq c'_i N, \qquad c'_j \sqrt{N} \leq a_j \leq c_j N + c'_j \sqrt{N}\,.$$

4.3.2 Some lemmas from [27]

In this section we review the statements of some lemmas from [27] which we use for the proof of Theorem 4.1.3.

Lemma 4.3.3. *[27, Cor.3.2] For a pair (p, v) a 2-connected graph G where p is a face and v is a vertex of p we have $B(a, b) \geq a_p + b_v$.*

The proofs of the three lemmas below can be found in [27, Sec.4].

Lemma 4.3.4. *Let G be a 2-connected planar graph whose unbounded face has V_∞ vertices. If (a, b) is an admissible state such that*

1. *$b_v = b_{v'} = 1$ where vv' is an edge of p_∞,*

2. *$a_p + b_p = 0$ for any face p of G,*

3. $(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ *for any face* p *of* G *and edge* v_1v_2 *of* p,

then $b_v \geq 1$ *for all vertices* v, $a_p = -1$ *for all faces* $p \neq p_\infty$ *and* $B(a, b) \geq 2 + V_\infty$.

Lemma 4.3.5. *Let* G *be a 2-connected planar graph whose unbounded face has* V_∞ *vertices. If* (a, b) *is an admissible state such that*

1. $b_v = b_{v'} = 0$ *and* $(b_v - b_p)(b_{v'} - b_p) = 1$ *where* p *is a boundary face and* vv' *is a boundary edge that belongs to* p,

2. $a_p + b_p = 0$ *for any face* p *of* G,

3. $(b_{v_1} - b_p)(b_{v_2} - b_p) = 0$ *for any face* p *of* G *and edge* v_1v_2 *not on the boundary of* p.

Then $b_w \geq -1$ *for all vertices* w, $a_p = 1$ *for all faces* $p \neq p_\infty$ *and* $B(a, b) \geq V_\infty - 2$. *Furthermore* $B(a, b) = V_\infty - 2$ *if and only if*

- $b_v = 0$ *for all boundary vertices* v *and* $b_w = -1$ *for all other vertices* w.

- $a_p = 1$ *for all faces* p.

Lemma 4.3.6. *Let* G *be a 2-connected planar graph,* p_0 *be a boundary face and* (a, b) *be an admissible state such that*

1. $a_{p_0} + b_{p_0} = 0$,

2. *There exists a boundary edge* vv' *of* p_0 *such that* $b_v b_{v'} = 0$ *and* $(b_v - b_{p_0})(b_{v'} - b_{p_0}) = 0$,

3. *Let* G_0 *be the graph obtained from* G *by deleting the boundary edges of* p_0 *and let* (a_0, b_0) *be the restriction of the admissible state* (a, b) *on* G_0.

Then,

(a) (a_0, b_0) *is an admissible state for* G_0, *(b)* $A(a_0, b_0) = A(a, b) - \sum_{e=(vv'):v,v' \in p_0 \cap p_\infty} b_v b_{v'}$,

(c) $B(a_0, b_0) = B(a, b) - 2 \sum_{v \in V_0} b_v$, *where* V_0 *is the set of boundary vertices of* p_0 *that do not belong to any other bounded face,*

(d) $B(a, b) \geq 2 \sum_{v \in V_0} b_v$,

(e) If furthermore $B(a, b) \leq 1$ *then* $A(a, b) = A(a_0, b_0), B(a, b) = B(a_0, b_0)$.

4.4 The coefficient q^3 in $\Phi_G(q)$

4.4.1 Analysis of admissible states

In this section we find the admissible states (a, b) such that $\frac{1}{2}(A(a, b) + B(a, b)) = 3$. Since $A(a, b), B(a, b) \in \mathbb{N}$ we have the following cases:

$A(a,b)$	6	5	4	3	2	1	0
$B(a,b)$	0	1	2	3	4	5	6

Case 1: $(A(a, b), B(a, b)) = (6, 0)$. By Lemma 4.3.3 we have $B(a, b) \geq a_p + b_p \geq 0$ and so $a_p + b_p = 0$ for all faces p. Similarly since $B(a, b) \geq a_p + b_v = b_v - b_p \geq 0$ we have $a_p + b_v = b_v - b_p = 0$ for all $v \in p$. Thus $A(a, b) = 6$ is equivalent to

$$\sum_{vv' \in p_\infty} b_v b_{v'} = 6 \tag{122}$$

If vv' is an edge of G and p is a face that contains vv' then we have $b_v = b_p = b_{v'}$. So by Equation (122) there exists a boundary edge vv' such that $b_v = b_{v'} = 1$. Lemma 4.3.4 implies that $B(a, b) \geq 2 + V_\infty > 0$ which is impossible. Therefore there are no admissible states (a, b) that satisfy $(A(a, b), B(a, b)) = (6, 0)$.

Case 2: $(A(a, b), B(a, b)) = (5, 1)$. Since $l(p) \geq 3$ we have $a_p + b_p \leq 1$ for all p.

Case 2.1: There exists a face p_0 such that $a_{p_0} + b_{p_0} = 1$, which implies that $a_p + b_p = 0$ for all $p \neq p_0$.

Case 2.1.1: $l(p_0) = 4$ or 5. We have $B(a,b) \geq (a_{p_0} + b_{v_1}) + (a_{p_0} + b_{v_2}) = 2(a_{p_0} + b_{p_0}) = 2$ which is impossible, here v_1, v_2 are two vertices of p_0.

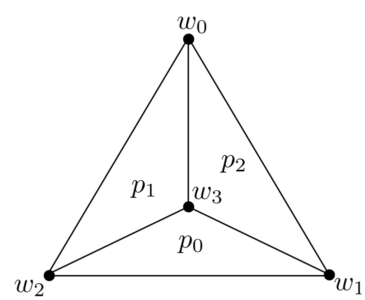
Case 2.1.2: $l(p_0) = 3$. We have

$$5 = A(a,b) = 3 + \sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'}$$

and therefore

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 2 \tag{123}$$

There are at most two positive terms in Equation (123). Let $v_i v_i' \in p_i,\ 1 \leq i \leq 2$ be the edges and bounded faces that appear in these terms. If a bounded face p contains a boundary edge $vv' \neq v_i v_i',\ i = 1, 2$ then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 4.3.6 we have $A(a', b') = A(a, b)$, $B(a', b') = B(a, b)$. Continue this process until G doesn't have any face p with a boundary edge $vv' \neq v_i v_i',\ i = 1, 2$. It's easy to see that the only possibility for this to happen is when $G = p_0 \cup p_1 \cup p_2$, where say $v_i v_i' \in p_i,\ i = 1, 2$. Since p_1, p_2 do not contain any boundary edge other than $v_i v_i',\ i = 1, 2$, G should be isomorphic to the graph in the following figure.



where $w_0 w_1 = v_1 v_1',\ w_0 w_2 = v_2 v_2'$. It follows that $b_{w_1} b_{w_2} = 0$ and let us assume that $b_{w_2} = 0$, and so $b_{w_2} b_{w_0} = 0$. This forces $(b_{w_2} - b_{p_1})(b_{w_0} - b_{p_1}) = 1$ since the edge $w_0 w_2$ corresponds to a positive term in Equation (123), which must equal 1. It follows from the later that $b_{w_0} = b_{w_2} = 0$ and therefore from Equation (123), $2 = \sum_{vv' \in p_\infty} b_v b_{v'} = b_{w_0} b_{w_1} + b_{w_1} b_{w_2} + b_{w_2} b_{w_0} = 0$ which is impossible.

Case 2.2: $a_p + b_p = 0$ for all p. Then we have

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 5 \tag{124}$$

There are at most 5 positive terms in Equation (124). Let $v_i v_i' \in p_i,\ 1 \le i \le 5$ be the edges and bounded faces that appear in these terms. If a bounded face p contains a boundary edge $vv' \neq v_i v_i',\ 1 \le i \le 5$ then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By part (e) of Lemma 4.3.6 we have $A(a', b') = A(a, b)$, $B(a', b') = B(a, b)$. We can continue this process until all the boundary edges of G are among the $v_i v_i'$. This means we can assume that G has m boundary edges where $3 \le m \le 5$. Let us relabel the boundary vertices by $v_1, v_2, .., v_m$.

Case 2.2.1: All the positive terms in Equation (124) correspond to boundary edges. If the positive terms are $b_{v_1} b_{v_2}, ..., b_{v_m} b_{v_1}$ then since $b_{v_1} b_{v_2} + ... + b_{v_m} b_{v_1} = 5$,

- there exists $1 \le i \le m$ such that $b_{v_i} b_{v_{i+1}} = 1$,
- $(b_v - b_p)(b_{v'} - b_p) = 0$ for all faces p and edges vv' of G.

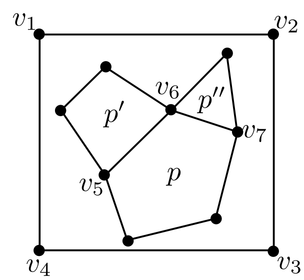
It follows from 4.3.4 that $B(a, b) \ge V_\infty + 2 \ge 5$ which is impossible. On the other hand, if for instance $b_{v_1} b_{v_2} = 0$ then we can assume that $b_{v_1} = 0$. Since the edge $v_1 v_2$ corresponds to a positive term, we have

$$(b_{v_1} - b_{p_1})(b_{v_2} - b_{p_1}) = k \tag{125}$$

where $1 \le k \le 3$ and p_1 is the bounded face that contains $v_1 v_2$. Here $k \neq 4, 5$ since we are assuming that all positive terms correspond to boundary edges and there are at least 3 edges. We claim that $k = 1$. Indeed, let us assume to the contradiction that $k \ge 2$. Equation (125) implies that either $b_{p_1} = -k$ and $b_{v_2} - b_{p_1} = 1$ or $b_{p_1} = -1$ and $b_{v_2} - b_{p_1} = k$. The former is impossible since $b_{v_2} \ge 0$. From the later we have

$b_{v_2} = k-1$ and since $a_{p_1} + b_{p_1} = 0$ we also have $a_{p_1} = 1$. So by Lemma 4.3.3 we have $B(a,b) \geq a_{p_1} + b_{v_2} = k \geq 2$ which is impossible and the claim is proven. Therefore $k = 1$ and hence $b_{v_1} = b_{v_2} = 0$, $b_{p_1} = -1$. It follows that $b_{v_2}b_{v_3} = 0$ which means $(b_{v_2} - b_{p_2})(b_{v_3} - b_{p_2}) = k'$, $1 \leq k' \leq 3$, because the edge v_2v_3 corresponds to a positive term. By a similar argument we can show that $k' = 1$ and $b_{p_2} = -1$, $b_{v_3} = 0$. Similarly we can prove that $b_{v_i} = 0$ and $b_{p_i} = -1$ for all $1 \leq i \leq 5$ for all $1 \leq i \leq m$ where p_i is the boundary face that contains v_iv_{i+1}. In particular, this implies that $m = 5$ and $(b_{v_i} - b_{p_i})(b_{v_{i+1}} - b_{p_i}) = 1$ for $1 \leq i \leq 5$ and therefore $(b_v - b_p)(b_{v'} - b_p) = 0$ for all $(p, vv') \neq (p_i, v_iv_{i+1})$ for all i. So by Lemma 4.3.5 we have $B(a,b) \geq V_\infty - 2 = 3$ which is impossible.

Case 2.2.2: There are 1 or 2 positive terms in Equation (124) that do not correspond to the boundary edges. By a similar argument as the above, we can reduce this to the case where the unbounded face of G has 3 or 4 vertices. Let us consider the case where G has 4 boundary edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1$ that correspond to 4 of the 5 positive terms and the other positive term corresponds to an edge v_5v_6 inside of G as in the figure below. The other cases are completely similar.



If the positive terms that correspond to the boundary edges are $b_{v_1}b_{v_2}, ..., b_{v_4}b_{v_1}$ then since $b_{v_1}b_{v_2} + ... + b_{v_4}b_{v_1} = 4$. This means that each of the terms $b_{v_i}b_{v_{i+1}}$ is equal to 1 and by an argument similar to the one of Case 2.2.1 we can conclude that $B(a,b) \geq V_\infty + 2 = 6$ which is impossible. If, say $b_{v_1}b_{v_2} = 0$ then $(b_{v_1} - b_{p_1})(b_{v_2} - b_{p_1}) = k > 0$ since the edge v_1v_2 corresponds to a positive term, here p_1 is the bounded face that contains v_1v_2. Since we have 4 positive terms and 4 boundary edges, each positive term is equal to 1, hence $k = 1$. Similar to the argument in Case 2.2.1, we can show that $b_p = -1$ for all faces p. Let p be the face that appears in the positive term that contains v_5v_6 and p' be the other face that contains v_5v_6. It follows from $(b_{v_5} - b_p)(b_{v_6} - b_p) = 1$ that $b_{v_5} = b_{v_6} = b_p + 1 = 0$. Since $(b_{v_5} - b_{p'})(b_{v_6} - b_{p'}) = 0$ we have $b_{p'} = 0$ which is impossible.

Case 3: $(A(a, b), B(a, b)) = (4, 2)$.

Case 3.1: There exists a face p_0 such that $a_{p_0} + b_{p_0} = 1$, which implies that $a_p + b_p = 0$ for all $p \neq p_0$. Since $A(a, b) = 4$ we have $l(p_0) \leq 4$.

Case 3.1.1: $l(p_0) = 4$. By a similar argument to the case 2 of Section 4.3 in [27] we can show that this gives us the following set of admissible states (a, b):

- $a_{p_0} = 1$ for a square face p_0, $a_p = 0$ for $p \neq p_0$,

- $b_v = 0$ for all vertices v,

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{l(p_0)}} = \frac{q^3}{(1-q)^4} = q^3 + O(q^4)$$

Case 3.1.2: $l(p_0) = 3$. We have

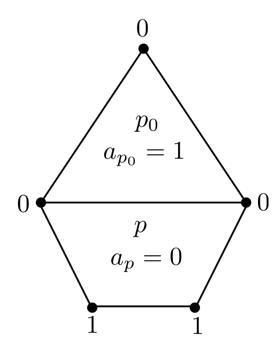
$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 1 \tag{126}$$

There is exactly one positive term in Equation (126). Let $vv' \in p$ be the edge and bounded face that appears in this term. If a bounded face p' contains a boundary edge $ww' \neq vv'$ then we should have $b_w b_{w'} = (b_w - b_{p'})(b_{w'} - b_{p'}) = 0$. This implies that $b_{p'} = 0$ and hence $a_{p'} = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By parts (c) and (d) of Lemma 4.3.6 we have $A(a', b') = A(a, b)$, $B(a', b') = B(a, b) - 2k$, $k \in \{0, 1\}$. Here

- $k = 0$ iff $b_v = 0$ for all removed vertices v,

- $k = 1$ if there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.

We can continue this process until $G = p_0$ if $p = p_0$ or $G = p \cup p_0$ if $p \neq p_0$. Let us consider first the case where $p = p_0$. Let the three vertices of p_0 be v, v', v'' and $b_{p_0} = b_v$. We have $2 \geq B(a,b) = a_{p_0} + 2(b_v + b_{v'} + b_{v''}) = (a_{p_0} + b_{p_0}) + b_{p_0} + 2(b_{v'} + b_{v''}) = 1 + b_{p_0} + 2(b_{v'} + b_{v''})$. It follows that $1 \geq b_{p_0} + 2(b_{v'} + b_{v''})$ and hence $b_{p_0} = b_{v'} = b_{v''} = 0$ since they are all non-negative. This implies that $a_{p_0} = 1$ and so $A(a,b) = 3a_{p_0}^2 + 2a_{p_0}(b_v + b_{v'} + b_{v''}) = 3$ which is impossible. If $p \neq p_0$ then there should exist an edge v_0v_0' of p_0 that does not correspond to a positive term and hence $b_{v_0}b_{v_0'} = 0$. It follows that $b_{p_0} = 0$ and so $a_{p_0} = 1$. This forces $b_v = 0$ for all $v \in p_0$ since otherwise $B(a,b) = a_{p_0} + 2\sum_{v \in p_)} b_v \geq 3$ which is impossible. Similarly there should exist an edge ww' of p such that $b_wb_{w'} = 0$ which implies that $a_p = 0$ and hence $b_p = 0$. If p and p_0 are disjoint then we have $2 = B(a,b) = B^p(a,b) + B^{p_0}(a,b) = B^p(a,b) + 1$ where $B^p(a,b)$ denotes the restriction of $B(a,b)$ on p. It follows that $B^p(a,b) = 1$ and the argument in Lemma 4.3.6 implies that $b_v = 0$ for all $v \in p$. This is impossible since it gives $B(a,b) = a_p + 2\sum_{v \in p} b_v = 0$. So p and p_0 are not disjoint. If v is a vertex of both p and p_0 then $b_v = 0$ and therefore $b_p = 0$ which implies that $a_p = 0$ since $a_p + b_p = 0$.



As before, the argument in Lemma 4.3.6 implies that $b_v = 0$ for all $v \in p$ and so $B(a,b) = B^p(a,b) + B^{p_0}(a,b) = 1$ which is impossible.

Case 3.2: $a_p + b_p = 0$ for all p. Then we have

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 4 \tag{127}$$

There are at most 4 positive terms in Equation (127). If an edge $vv' \in p$ does not correspond to a positive term then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a, b) on G'. By parts (c) and (d) of Lemma 4.3.6 we have $A(a', b') = A(a, b)$, $B(a', b') = B(a, b) - 2k$, $k \in \{0, 1\}$. Here

- $k = 0$ iff $b_v = 0$ for all removed vertices v,
- $k = 1$ if there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.

We can continue to do this until the boundary of G has at most 4 edges all of which correspond to positive terms.

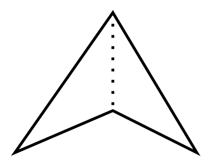
Case 3.2.1: All of the positive terms in Equation (127) correspond to boundary edges.

Case 3.2.1.1 G has 3 vertices on the boundary, say v_1, v_2, v_3. If all the positive terms are equal to 1 then there must exist a boundary edge, for instance, v_1v_2 of G such that $b_{v_1} b_{v_2} = (b_{v_1} - b_{p_1})(b_{v_2} - b_{p_1}) = 1$ where p_1 is the bounded face that contains v_1v_2. This implies that $b_{p_1} = 0$ and hence $a_{p_1} = 0$. Let $vv' \notin \{v_1v_2, v_2v_3, v_3v_1\}$ be another edge of p_1 and let p be the other bounded face that contains vv'. Since vv' does not correspond to a positive term, we have $(b_v - b_{p_1})(b_{v'} - b_{p_1}) = 0$ and so $b_v b_{v'} = 0$. We also have $(b_v - b_p)(b_{v'} - b_p) = 0$ which means $b_p = \min\{b_v, b_{v'}\} = 0$ and hence $a_p = 0$. Similarly we can show that $b_{p'} = a_{p'} = 0$ for all faces p' and in particular $b_w \geq 0$ for all w. It follows that $B(a, b) \geq 2(b_{v_1} + b_{v_2}) = 4$ which is impossible.

If one of the positive terms is equal to 2 then the other two are equal to 1. Without loss of generality we can assume that the edge v_1v_2 corresponds to this term, so either $b_{v_1}b_{v_2} = 2$ or $(b_{v_1} - b_{p_1})(b_{v_2} - b_{p_2}) = 2$. For the former we can assume that $b_{v_1} = 1$ and $b_{v_2} = 2$. This implies that $b_{v_3} = 0$ since otherwise $A(a, b) \geq b_{v_1}b_{v_2} + b_{v_2}b_{v_3} + b_{v_3}b_{v_1} \geq 2+1+2 = 5$ which is impossible. Since $b_{v_2}b_{v_3} = 0$ which means $(b_{v_2} - b_{p_2})(b_{v_3} - b_{p_2}) = 1$ and this leads to $-b_{p_2}(2 - b_{p_2}) = 1$ which is impossible.

Case 3.2.1.2 G has 4 vertices on the boundary, say v_1, v_2, v_3, v_4. By a similar argument to the case 2.2 of Section 4.3 in [27], this corresponds to the following admissible state of G:

- $a_p = 1$ for all bounded faces p,
- $b_{v_1} = b_{v_2} = b_{v_3} = b_{v_4} = 0$ where v_1, v_2, v_3, v_4 are the vertices of a square in G_0 that does not have any diagonal in its interior. We will write $c_{40} = [G_0](G)$.

  $G_0 =$ 

  where the dotted line means G_0 does not contain an internal diagonal,
- $b_w = -1$ for all vertices w inside the 4-circle mentioned above,
- $b_{\tilde{w}} = 0$ for any other vertex w.

The contribution of this state to $\Phi_G(q)$ is

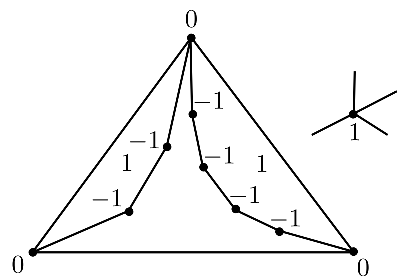
$$\frac{q^3}{(1-q)^{\deg_\square(v_1)+\deg_\square(v_2)+\deg_\square(v_3)+\deg_\square(v_4)-4}} = q^3 + O(q^4)$$

where $\deg_\square(v)$ is the degree of v in the square $\square = v_1v_2v_3v_4$.

Case 3.2.2: One of of the positive terms in Equation (127) does not correspond to any boundary edge. By a similar argument to the caase 2.2.2 we can show that there are no admissible states here.

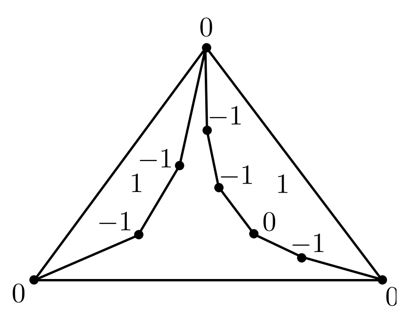
Case 4: $(A(a,b), B(a,b)) = (3,3)$. By a similar argument to the case 2 of Section 4.3 in [27] we can show that the admissible states for this case are

- $a_p = 1$ for all faces p.
- $b_{v_1} = b_{v_2} = b_{v_3} = 0$ where v_1, v_2, v_3 are the vertices of a 3-cycle in G.
- $b_v = -1$ for all v inside the 3-cycle mentioned above.
- $b_{v_0} = 1$ for a fixed vertex w outside of the 3-cycle.
- $b_w = 0$ for all other vertices w.



and

- $a_p = 1$ for all faces p.
- $b_{v_1} = b_{v_2} = b_{v_3} = 0$ where v_1, v_2, v_3 are the vertices of a 3-cycle in G.
- $b_{v_0} = 0$ for a fixed vertex v_0 inside the 3-cycle that is not adjacent to any of the vertices v_1, v_2, v_3 and $b_v = -1$ for all other v also inside the cycle.
- $b_w = 0$ for all other vertices w.



The contribution of both types of states above to $\Phi_G(q)$ is

$$(-1)^3 \frac{q^3}{(1-q)^{\deg_\Delta(v_1)+\deg_\Delta(v_2)+\deg_\Delta(v_3)+\deg_\Delta(v_0)-3}} = -q^3 + O(q^4)$$

where $\deg_\Delta(v)$ is the degree of v in the triangle $\Delta = v_1v_2v_3$.

Case 5: $(A(a,b), B(a,b)) = (2,4)$. Since $A(a,b) = 2$, we have $a_p + b_p = 0$ for all p and

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 2 \tag{128}$$

There are at most 2 positive terms in Equation (128). If a boundary face p contains a boundary edge vv' that does not correspond to a positive term then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a,b) on G'. By parts (c) and (d) of Lemma 4.3.6 we have $A(a',b') = A(a,b) - 2i$, $B(a',b') = B(a,b) - 2k$, $k \in \{0,1,2\}$. Here

- $i = 0$ and $k = 0$ iff $b_v = 0$ for all removed vertices.
- $i = 0$ and $k = 1$ iff there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.
- $i = 0$ and $k = 2$ iff there exist two removed vertices v, v' which are not connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other removed vertices w.
- $i = 1$ and $k = 2$ iff there exist two removed vertices v, v' which are connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other removed vertices w.

It is easy to see that only the last item gives admissible states (a,b) with $(A(a,b), B(a,b)) = (2,4)$. To summarize, the admissible states in this case are those (a,b) that satisfy

- $a_p = 0$ for all faces p.
- There exist two vertices v, v' which are connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other vertices w.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{\deg(v)+\deg(v')}} = q^3 + O(q^4)$$

Case 6: $(A(a,b), B(a,b)) = (1,5)$. Since $A(a,b) = 1$, we have $a_p + b_p = 0$ for all p and

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 1 \tag{129}$$

There is exactly 1 positive term in Equation (129). If the pair (vv', p), $vv' \in p$ does not correspond to this positive term then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Similarly we can show that $a_{p'} = 0$ for all other faces p'. This implies that $5 = B(a,b) = 2\sum_v b_v$ which is impossible. So there are no admissible states in this case.

Case 7: $(A(a,b), B(a,b)) = (0,6)$. Since $A(a,b) = 0$, we have $a_p + b_p = 0$ for all p and

$$\sum_p \sum_{vv' \in p} (b_v - b_p)(b_{v'} - b_p) + \sum_{vv' \in p_\infty} b_v b_{v'} = 0 \tag{130}$$

Let $vv' \in p$ where p is a boundary face then we should have $b_v b_{v'} = (b_v - b_p)(b_{v'} - b_p) = 0$. This implies that $b_p = 0$ and hence $a_p = 0$. Let G' be the graph obtained from G by deleting the boundary edges of p and (a', b') be the restriction of (a,b) on G'. By parts (c) and (d) of Lemma 4.3.6 we have $A(a',b') = A(a,b)$, $B(a',b') = B(a,b) - 2k$, $k \in \{0,1,2,3\}$. Here

- $k = 0$ iff $b_v = 0$ for all removed vertices.
- $k = 1$ iff there exists a removed vertex v such that $b_v = 1$ and $b_w = 0$ for all other removed vertices w.
- $k = 2$ iff there exists a removed vertex v such that $b_v = 2$ or iff there exist two removed vertices v, v' which are not connected by an edge such that $b_v = b_{v'} = 1$ and $b_w = 0$ for all other removed vertices w.

- $k = 3$ iff there exists a removed vertex v such that $b_v = 3$ or iff there exist two removed vertices v, v' which are not connected by an edge such that $b_v = 1,\ b_{v'} = 2$ or iff there exist three removed vertices v, v', v'' none of which are connected by an edge such that $b_v = b_{v'} = b_{v''} = 1$ and $b_w = 0$ for all other removed vertices w.

The above possible values of k lead to the following admissible states (a, b):

- $a_p = 0$ for all faces p.
- There exists a vertex v such that $b_v = 3$ and $b_w = 0$ for all $w \neq v$.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)_3^{\deg(v)}} = q^3 + O(q^4)$$

- $a_p = 0$ for all faces p.
- There exist two vertices v, v' which are not connected by an edge such that $b_v = 1,\ b_{v'} = 2$ and $b_w = 0$ for all other vertices w.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{\deg(v)}(1-q)_2^{\deg(v')}} = q^3 + O(q^4)$$

- $a_p = 0$ for all faces p.
- There exist three vertices v, v', v'' none of which are connected by an edge such that $b_v = b_{v'} = b_{v''} = 1$ and $b_w = 0$ for all other vertices w.

The contribution of this state to $\Phi_G(q)$ is

$$\frac{q^3}{(1-q)^{\deg(v)+\deg(v')+\deg(v'')}} = q^3 + O(q^4)$$

4.4.2 Proof of Theorem 4.1.3

We now give a proof of Theorem 4.1.3 based on cases 1-7 of Section 4.4.1. We write

$$\begin{aligned}\Phi_G(q) &= (1-q)(q)_\infty^{c_2}(1+a_1q+a_2q^2+a_3q^3+\mathrm{O}(q^4))\\ &= (1-q)(1+b_1q+b_2q^2+b_3q^3)(1+a_1q+a_2q^2+a_3q^3)+\mathrm{O}(q^4)\end{aligned}$$

where from [27, Sec.4.2] we have

$$\begin{aligned}a_1 &= c_1\\ a_2 &= \frac{c_1(c_1+1)}{2}+c_2-c_3\end{aligned}$$

and a_3 receives contributions from

- States (a,b) such that $\frac{1}{2}(A+B)=3$. These are discussed in Section 4.4.1.
- States (a,b) such that $\frac{1}{2}(A+B)\leq 2$ which are discussed in [27, Sec.4.2].

By expanding the factor $(q)_\infty^{c_2}$ we have

$$\begin{aligned}b_1 &= -c_2\\ b_2 &= \frac{c_2(c_2-3)}{2}\\ b_3 &= \frac{-c_2^3+9c_2^2-8c_2}{6}\end{aligned}$$

The total contribution of the admissible states found in cases 1-7 to $a_3q^3+O(q^4)$ is

$$(c_{40}+c_1+c_2+2(\frac{c_1(c_1-1)}{2}-c_2)+\gamma-\sum_{C_3=vv'v''}(c_1-\alpha(C_3)))q^3+\mathrm{O}(q^4) \qquad (131)$$

where $\frac{c_1(c_1-1)}{2}-c_2$ is the number of pair of vertices in G that are not connected by an edge. The last term is a summation over 3-cycles $C_3=vv'v''$ of G and $\alpha(C_3)=3+$ the number of vertices contained in C_3 that are adjacent to either v, v' or v''. The admissible states in Sections 4.2 and 4.3 in [27] gives the following contribution to

$a_3 q^3 + O(q^4)$:

$$1 + \sum_v q(1+q+q^2)^{\deg(v)} - q^2 \sum_{C_3 = vv'v''} (1+q)^{\deg_{C3}(v)+\deg_{C3}(v')+\deg_{C3}(v'')-3} \tag{132}$$
$$+ q^2 \sum_{(vv')\neq e} (1+q)^{\deg(v)+\deg(v')} + \sum_v q^2(1+q)^{\deg(v)} + \mathrm{O}(q^4)$$

where by $(vv') \neq e$ we mean a pair of vertices v, v' that are not connected by an edge and $\deg_{C3}(v)$ denotes the degree of v in the subgraph of G that is contained in C_3. Summing up (131) and (132) we get

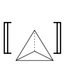
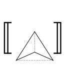
$$a_3 = c_{40} + c_1 + c_2 + 2(\frac{c_1(c_1-1)}{2} - c_2) + \gamma + \delta + c_2 + \sum_{(vv')\neq e} (\deg(v) + \deg(v'))$$
$$+ 2c_2 - \sum_{C_3 = vv'v''} (c_1 + \deg_{C3}(v) + \deg_{C3}(v') + \deg_{C3}(v'') - 3 - \alpha(C_3)) \tag{133}$$

Note that

$$\begin{aligned}\sum_{(vv')\neq e} (\deg(v) + \deg(v')) &= \sum_v \deg(v)(c_1 - 1 - \deg(v)) \\ &= 2c_2(c_1 - 1) - \sum_v (\deg(v))^2 \\ &= 2c_2(c_1-1) - 2\delta\end{aligned}$$

Let us define

$$d_3 = \deg_{C3}(v) + \deg_{C3}(v') + \deg_{C3}(v'') - 3 - \alpha(C_3)$$

and $c'_{40} = [\![$  $]\!]$, $c_{41} = [\![$  $]\!]$.

Lemma 4.4.1. *We have*

(a) $d_3 = c'_{40} + 2c_{42}$.

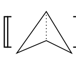
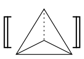
(b) $c_{40} - c'_{40} = c_{41}$.

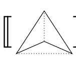
Proof. (a) If w is a vertex in the interior incident to v and v' then it contributes +1 to $\deg(v)$, +1 to $\deg(v')$ and -1 to itself. Hence totally such w's contribute c'_{40}.

If w is a vertex in the interior incident to v, v', v'' then it contributes +1 to each $\deg(v)$, $\deg(v')$, $\deg(v'')$ and -1 to itself. So totally such w's contribute $2c_{42}$. If w is a boundary vertex then its contribution to each of $\deg(v)$, $\deg(v')$, $\deg(v'')$ is +2 and the total contribution of the 3 boundary vertices is +6 which cancels the -6 in d_3. Thus we have

$$d_3 = c'_{40} + 2c_{42}$$

(b) We have

$c_{40} - c'_{40}$ = ⟦  ⟧ − ⟦  ⟧

= ⟦  ⟧

= c_{41}

□

Therefore Equation (133) combined with Lemmas 4.2.4 and 4.4.1 gives that

$$\begin{aligned} a_3 &= c_{41} - 2c_{42} - c_3c_1 + c_1 + c_2 + 2(\frac{c_1(c_1-1)}{2} - c_2) + \gamma + \delta + c_2 + 2c_2(c_1-1) - 2\delta + 2c_2 \\ &= 2c_1c_2 + c_1^2 - c_3c_1 + c_{41} - 2c_{42} + \gamma - \delta \\ &= \frac{{c_1}^3}{6} + \frac{{c_1}^2}{2} + c_1c_2 - c_1c_3 + \frac{c_1}{3} + c_2 - c_3 + c_{41} - 2c_{42} \end{aligned}$$

Therefore the coefficient $\phi_{G,3}$ of q^3 in $\Phi_G(q)$ is given by

$$\begin{aligned} \phi_{G,3} &= a_3 + b_3 + a_1b_2 + a_2b_1 - a_2 - b_2 - a_1b_1 \\ &= c_{41} - 2c_{42} + \frac{c_2}{6} + c_3c_2 - \frac{c_2^3}{6} - \frac{c_1}{6} - c_3c_1 + \frac{c_2^2c_1}{2} - \frac{c_2c_1^2}{2} + \frac{c_1^3}{6} \end{aligned}$$

This completes the proof of Theorem 4.1.3. □

.1 Computations

Tables 22 and 25 illustrate Theorem 4.1.3 and confirm Conjecture 4.1.4 for all alternating links with at most 10 crossings and all irreducible planar graphs with at most 7 vertices. These tables were compiled as follows.

- We use `Sage` to list all irreducible planar graphs with at most 10 edges (using the notation of [27, App.A]).
- We use a `Mathematica` program to compute the corresponding vectors c and C and the series $\Phi_G(q) + O(q)^4$ of Theorem 4.1.3.
- To identify the corresponding alternating links L, we use a `Mathematica` program that converts the adjacency matrix of a planar graph G to the Dowker-Thistlethwaite code of the corresponding alternating link L, and then use `SnapPy` (see [6]) to identify the link with one of the Rolfsen's table [46] (if L has at most 10 crossings) or Thislethwaite's table (if L has more than 10 crossings).
- We compute the stable coefficients $\Phi_L(q) + O(q)^6$ using `KnotAtlas` (see [6]) which computes the colored Jones polynomials of a link.

The equality $\Phi_G(q) = \Phi_L(q)$ of Theorem 4.1.3 is observed up to $O(q^4)$ and Conjecture 4.1.4 is verified for all such graphs.

Remark .1.1. *If G is a connected planar graph with v vertices and e edges, the following inequalities bound e in terms of v and vice-versa*

$$v \leq e \qquad \textit{and} \qquad e \leq 3v - 6$$

.2 Tables of irreducible planar graphs

crossings = edges	3	4	5	6	7	8	9	10
alternating links	1	2	3	8	14	39	96	297
irreducible graphs	1	1	1	3	3	8	17	41

Figure 21: The number of alternating links with at most 10 crossings and the number of irreducible graphs with at most 10 edges.

G	c	C	L	$\Phi_L(q) + O(q^6)$
G_0^3	$3, 3, 1, 0, 0$	$1, 1, 1, 1, 1$	3_1	$1 - q - q^2 + q^5$
G_0^4	$4, 4, 0, 1, 0$	$1, 0, -1, -1, -1$	4_1^2	$1 - q + q^3$
G_0^5	$5, 5, 0, 0, 0$	$1, 0, 0, 1, 1$	5_1	$1 - q - q^4$
G_0^6	$6, 6, 0, 0, 0$	$1, 0, 0, 0, -1$	6_1^2	$1 - q + q^5$
G_1^6	$4, 6, 4, 0, 1$	$3, 4, 6, 9, 16$	6_2^3	$1 - 3q - q^2 + 5q^3 + 3q^4 + 3q^5$
G_2^6	$5, 6, 0, 3, 0$	$2, 0, -3, -4, -3$	6_1^3	$1 - 2q + q^2 + 3q^3 - 2q^4 - 2q^5$
G_0^7	$7, 7, 0, 0, 0$	$1, 0, 0, 0, 0$	7_1	$1 - q$
G_1^7	$5, 7, 2, 2, 0$	$3, 2, 0, -2, -4$	7_6^2	$1 - 3q + q^2 + 5q^3 - 3q^4 - 3q^5$
G_2^7	$6, 7, 0, 1, 0$	$2, 0, -1, 1, 2$	7_4^2	$1 - 2q + q^2 + q^3 - 3q^4 + q^5$

Figure 22: The irreducible graphs G with at most 10 edges, the 6-tuple of polynomial invariants $c = (c_1, c_2, c_3, c_{41}, c_{42})$, $C = (C_1, C_2, C_3, c_4, C_5)$ as defined in Equantion (107), the alternating link L and the 6 stable coefficients of the Jones polynomial of L.

G_0^8	$8, 8, 0, 0, 0$	$1, 0, 0, 0, 0$	8_1^2	$1 - q$
G_1^8	$5, 8, 4, 1, 0$	$4, 4, 3, 0, -6$	8_{18}	$1 - 4q + 2q^2 + 9q^3 - 5q^4 - 8q^5$
G_2^8	$6, 8, 0, 5, 0$	$3, 0, -5, -7, -4$	8_{14}^2	$1 - 3q + 3q^2 + 4q^3 - 8q^4 - 2q^5$
G_3^8	$6, 8, 2, 0, 0$	$3, 2, 2, 4, 6$	8_{15}^3	$1 - 3q + q^2 + 3q^3 - 3q^4 + 3q^5$
G_4^8	$6, 8, 1, 2, 0$	$3, 1, -1, 0, 2$	8_{16}	$1 - 3q + 2q^2 + 3q^3 - 6q^4 + q^5$
G_5^8	$6, 8, 0, 6, 0$	$3, 0, -6, -10, -7$	8_1^4	$1 - 3q + 3q^2 + 5q^3 - 8q^4 - 5q^5$
G_6^8	$7, 8, 0, 1, 0$	$2, 0, -1, -1, -3$	8_1^3	$1 - 2q + q^2 + q^3 - q^4 + 2q^5$
G_7^8	$7, 8, 0, 0, 0$	$2, 0, 0, 2, 1$	8_5	$1 - 2q + q^2 - 2q^4 + 3q^5$

Figure 23: Figure 22 continued.

G^9_1	$5, 9, 7, 0, 2$	$5, 7, 11, 17, 31$	9_{40}	$1 - 5q + 3q^2 + 14q^3 - 6q^4 - 15q^5$
G^9_2	$6, 9, 2, 5, 0$	$4, 2, -3, -9, -13$	9^3_{12}	$1 - 4q + 4q^2 + 7q^3 - 13q^4 - 7q^5$
G^9_3	$6, 9, 3, 1, 0$	$4, 3, 2, 3, 6$	9^2_{42}	$1 - 4q + 3q^2 + 6q^3 - 9q^4$
G^9_4	$6, 9, 2, 4, 0$	$4, 2, -2, -5, -5$	9_{34}	$1 - 4q + 4q^2 + 6q^3 - 13q^4 - 3q^5$
G^9_5	$6, 9, 2, 3, 0$	$4, 2, -1, -1, 3$	9_{40}	$1 - 4q + 4q^2 + 5q^3 - 13q^4 + q^5$
G^9_6	$7, 9, 0, 3, 0$	$3, 0, -3, -3, -4$	9^2_{40}	$1 - 3q + 3q^2 + 2q^3 - 6q^4 + 4q^5$
G^9_7	$7, 9, 1, 0, 0$	$3, 0, -3, -3, -4$	9_{41}	$1 - 3q + 2q^2 + q^3 - 4q^4 + 7q^5$
G^9_8	$7, 9, 2, 0, 0$	$3, 2, 2, 2, 0$	9^2_{31}	$1 - 3q + q^2 + 3q^3 - q^4 + 3q^5$
G^9_9	$7, 9, 0, 3, 0$	$3, 0, -3, -2, 0$	9^2_{36}	$1 - 3q + 3q^2 + 2q^3 - 7q^4 + 3q^5$
G^9_{10}	$7, 9, 1, 1, 0$	$3, 1, 0, 1, 0$	9^2_{35}	$1 - 3q + 2q^2 + 2q^3 - 4q^4 + 4q^5$
G^9_{11}	$7, 9, 0, 2, 0$	$3, 0, -2, 1, 3$	9_{29}	$1 - 3q + 3q^2 + q^3 - 7q^4 + 6q^5$
G^9_{12}	$7, 9, 0, 3, 0$	$3, 0, -3, -1, 3$	9^3_3	$1 - 3q + 3q^2 + 2q^3 - 8q^4 + 3q^5$
G^9_{13}	$7, 9, 0, 2, 0$	$3, 0, -2, 2, 6$	9^3_9	$1 - 3q + 3q^2 + q^3 - 8q^4 + 6q^5$
G^9_{14}	$8, 9, 0, 0, 0$	$2, 0, 0, 1, 0$	9^2_{19}	$1 - 2q + q^2 - q^4 + 2q^5$
G^9_{15}	$8, 9, 0, 1, 0$	$2, 0, 0, 1, 0$	9^2_{13}	$1 - 2q + q^2 + q^3 - q^4$
G^9_{16}	$8, 9, 0, 0, 0$	$2, 0, 0, 0, -3$	9_{35}	$1 - 2q + q^2 + 3q^5$

Figure 24: Figure 22 continued.

G_0^{10}	$6, 10, 5, 2, 1$	$5, 5, 5, 6, 11$	10_{121}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 7q^5$
G_1^{10}	$6, 10, 5, 0, 0$	$5, 5, 5, 6, 10$	10_{123}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 6q^5$
G_2^{10}	$6, 10, 4, 4, 0$	$5, 4, 0, -8, -20$	10^4_{17}	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
G_3^{10}	$6, 10, 4, 4, 0$	$5, 4, 0, -8, -20$	10^2_{155}	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
G_4^{10}	$6, 10, 4, 3, 0$	$5, 4, 1, -3, -6$	10^2_{137}	$1 - 5q + 6q^2 + 9q^3 - 21q^4 - 6q^5$
G_5^{10}	$7, 10, 0, 10, 0$	$4, 0, -10, -20, -15$	10^5_1	$1 - 5q + 6q^2 + 9q^3 - 21q^4 - 6q^5$
G_6^{10}	$7, 10, 0, 8, 0$	$4, 0, -8, -13, -7$	10^3_{25}	$1 - 4q + 6q^2 + 4q^3 - 18q^4 + 3q^5$
G_7^{10}	$7, 10, 0, 7, 0$	$4, 0, -7, -10, -5$	10_{120}	$1 - 4q + 6q^2 + 3q^3 - 17q^4 + 7q^5$
G_8^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 1, 3$	10^2_{33}	$1 - 4q + 4q^2 + 4q^3 - 11q^4 + 5q^5$
G_9^{10}	$7, 10, 3, 0, 0$	$4, 3, 3, 4, 3$	10_{112}	$1 - 4q + 3q^2 + 5q^3 - 6q^4 + 4q^5$
G_{10}^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 0, -1$	10_{116}	$1 - 4q + 4q^2 + 4q^3 - 10q^4 + 5q^5$
G_{11}^{10}	$7, 10, 1, 3, 0$	$4, 1, -2, 1, 7$	10^2_{151}	$1 - 4q + 5q^2 + 2q^3 - 14q^4 + 11q^5$
G_{12}^{10}	$7, 10, 1, 4, 0$	$4, 1, -3, -3, 0$	10_{119}	$1 - 4q + 5q^2 + 3q^3 - 14q^4 + 7q^5$
G_{13}^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 0, -1$	10_{114}	$1 - 4q + 4q^2 + 4q^3 - 10q^4 + 5q^5$
G_{14}^{10}	$7, 10, 1, 3, 0$	$4, 1, -2, 0, 3$	10^2_{156}	$1 - 4q + 5q^2 + 2q^3 - 13q^4 + 11q^5$
G_{15}^{10}	$7, 10, 2, 1, 0$	$4, 2, 1, 4, 7$	10^2_{147}	$1 - 4q + 4q^2 + 3q^3 - 10q^4 + 9q^5$
G_{16}^{10}	$7, 10, 2, 1, 0$	$4, 2, 1, 3, 3$	10_{122}	$1 - 4q + 4q^2 + 3q^3 - 9q^4 + 9q^5$
G_{17}^{10}	$7, 10, 2, 2, 0$	$4, 2, 0, 0, -1$	10^3_{74}	$1 - 4q + 4q^2 + 4q^3 - 10q^4 + 5q^5$
G_{18}^{10}	$7, 10, 1, 4, 0$	$4, 1, -3, -2, 4$	10^2_{28}	$1 - 4q + 5q^2 + 3q^3 - 15q^4 + 7q^5$
G_{19}^{10}	$7, 10, 2, 1, 0$	$4, 2, 1, 5, 11$	10^4_{12}	$1 - 4q + 4q^2 + 3q^3 - 11q^4 + 9q^5$
G_{20}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 3, 4$	10^2_{106}	$1 - 3q + 3q^2 - 6q^4 + 8q^5$
G_{21}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 2, 1$	10^2_{20}	$1 - 3q + 3q^2 - 5q^4 + 8q^5$
G_{22}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 1, -1$	10^2_{141}	$1 - 3q + 3q^2 - 4q^4 + 7q^5$
G_{23}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 2, 2$	10_{93}	$1 - 3q + 3q^2 - 5q^4 + 7q^5$
G_{24}^{10}	$8, 10, 1, 1, 0$	$3, 1, 0, 0, -1$	10_{85}	$1 - 3q + 2q^2 + 2q^3 - 3q^4 + 2q^5$
G_{25}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -1, -1$	10_{100}	$1 - 3q + 3q^2 + q^3 - 5q^4 + 4q^5$
G_{26}^{10}	$8, 10, 1, 0, 0$	$3, 1, 1, 3, 3$	10^3_{33}	$1 - 3q + 2q^2 + q^3 - 3q^4 + 5q^5$
G_{27}^{10}	$8, 10, 2, 0, 0$	$3, 2, 2, 2, 2$	10^3_{40}	$1 - 3q + q^2 + 3q^3 - q^4 + q^5$
G_{28}^{10}	$8, 10, 0, 3, 0$	$3, 0, -3, -4, -5$	10^2_{59}	$1 - 3q + 3q^2 + 2q^3 - 5q^4 + 2q^5$
G_{29}^{10}	$8, 10, 0, 1, 0$	$3, 0, -1, 3, 5$	10^3_{37}	$1 - 3q + 3q^2 - 6q^4 + 7q^5$
G_{30}^{10}	$8, 10, 0, 0, 0$	$3, 0, 0, 4, 2$	10^2_{37}	$1 - 3q + 3q^2 - q^3 - 4q^4 + 10q^5$
G_{31}^{10}	$8, 10, 1, 0, 0$	$3, 1, 1, 2, 0$	10_{108}	$1 - 3q + 2q^2 + q^3 - 2q^4 + 5q^5$
G_{32}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -2, -5$	10^2_{40}	$1 - 3q + 3q^2 + q^3 - 4q^4 + 5q^5$
G_{33}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -2, -6$	10^4_3	$1 - 3q + 3q^2 + q^3 - 4q^4 + 6q^5$
G_{34}^{10}	$8, 10, 0, 3, 0$	$3, 0, -3, -4, -6$	10^4_{22}	$1 - 3q + 3q^2 + 2q^3 - 5q^4 + 3q^5$
G_{35}^{10}	$8, 10, 0, 2, 0$	$3, 0, -2, -2, -6$	10^4_3	$1 - 3q + 3q^2 + q^3 - 4q^4 + 6q^5$
G_{36}^{10}	$9, 10, 0, 1, 0$	$2, 0, -1, -1, -1$	10^3_{69}	$1 - 2q + q^2 + q^3 - q^4$
G_{37}^{10}	$9, 10, 0, 0, 0$	$2, 0, 0, 1, 1$	10_{46}	$1 - 2q + q^2 - q^4 + q^5$
G_{38}^{10}	$9, 10, 0, 0, 0$	$2, 0, 0, 0, -2$	10^3_{65}	$1 - 2q + q^2 + 2q^5$
G_{39}^{10}	$9, 10, 0, 0, 0$	$2, 0, 0, 0, -1$	10_{61}	$1 - 2q + q^2 + q^5$
G_{40}^{10}	$10, 10, 0, 0, 0$	$1, 0, 0, 0, 0$	10^2_1	$1 - q$

G	C	L	$\Phi_L(q) + O(q^6)$
Gv_1^6	$3, 0, -6, -10, -7$	8_1^4	$1 - 3q + 3q^2 + 5q^3 - 8q^4 - 5q^5$
Gv_2^6	$2, 0, -1, 1, 2$	7_4^2	$1 - 2q + q^2 + q^3 - 3q^4 + q^5$
Gv_3^6	$4, 2, -3, -9, -13$	9_{12}^3	$1 - 4q + 4q^2 + 7q^3 - 13q^4 - 7q^5$
Gv_4^6	$1, 0, 0, 0, -1$	6_1^2	$1 - q + q^5$
Gv_5^6	$3, 2, 2, 4, 6$	8_5^3	$1 - 3q + q^2 + 3q^3 - 3q^4 + 3q^5$
Gv_6^6	$4, 3, 2, 3, 6$	9_{42}^2	$1 - 4q + 3q^2 + 6q^3 - 9q^4$
Gv_7^6	$5, 5, 5, 6, 11$	10_{121}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 7q^5$
Gv_8^6	$5, 5, 5, 6, 10$	10_{123}	$1 - 5q + 5q^2 + 10q^3 - 16q^4 - 6q^5$
Gv_9^6	$5, 4, 0, -8, -20$	10_{17}^4	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
Gv_{10}^6	$3, 1, -1, 0, 2$	8_{16}	$1 - 3q + 2q^2 + 3q^3 - 6q^4 + q^5$
Gv_{11}^6	$4, 2, -2, -5, -5$	9_{34}	$1 - 4q + 4q^2 + 6q^3 - 13q^4 - 3q^5$
Gv_{12}^6	$5, 4, 0, -8, -20$	10_{155}^2	$1 - 5q + 6q^2 + 10q^3 - 21q^4 - 11q^5$
Gv_{13}^6	$5, 4, 0, -8, -20$	9_{40}	$1 - 4q + 4q^2 + 5q^3 - 13q^4 + q^5$
Gv_{14}^6	$5, 4, 0, -8, -20$	10_{137}^2	$1 - 5q + 6q^2 + 9q^3 - 21q^4 - 6q^5$
Gv_{15}^6	$6, 7, 8, 8, 9$	11_{314}	$1 - 6q + 8q^2 + 14q^3 - 29q^4 - 17q^5$
Gv_{16}^6	$6, 6, 3, -7, -28$	$L11a520$	$1 - 6q + 9q^2 + 13q^3 - 35q^4 - 17q^5$
Gv_{17}^6	$7, 10, 16, 25, 46$	$L12a1183$	$1 - 7q + 11q^2 + 19q^3 - 43q^4 - 33q^5$
Gv_{18}^6	$7, 8, 5, -13, -65$	$L12a2008$	$1 - 7q + 13q^2 + 16q^3 - 57q^4 - 28q^5$
Gv_{19}^6	$3, 0, -5, -7, -4$	8_{14}^2	$1 - 3q + 3q^2 + 4q^3 - 8q^4 - 2q^5$

Figure 25: The irreducible graphs G with 6 vertices, the vector $C = (C_1, \dots, C_5)$, the alternating link L and the 6 stable coefficients of the Jones polynomial of L.

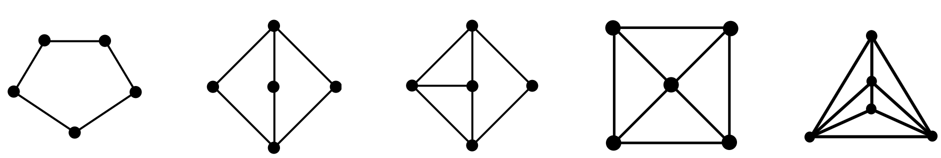


Figure 26: The irreducible planar graphs Gv_i^5 for $i = 1, \dots, 5$ (from the left to the right) with 5 vertices.

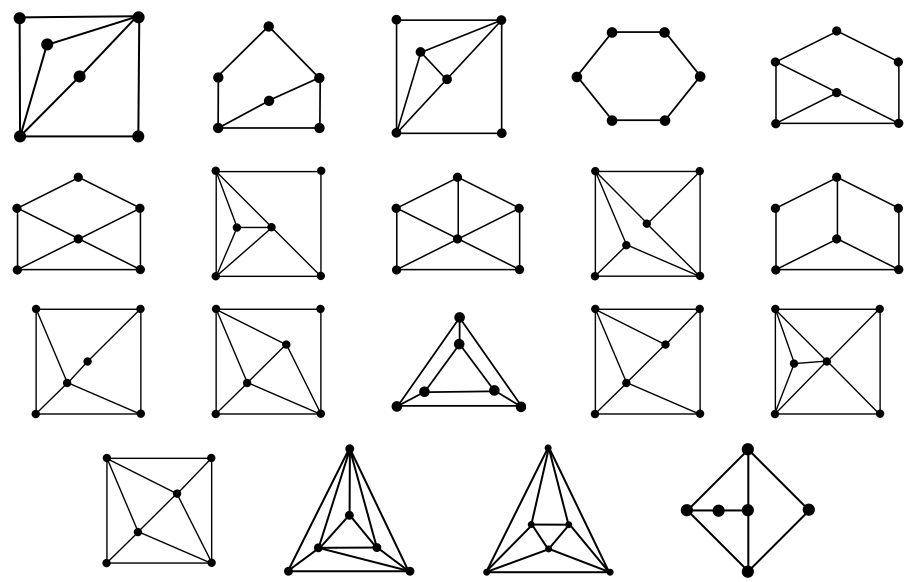


Figure 27: The irreducible planar graphs Gv_i^6 for $i = 1, \ldots, 19$ (from the left to the right) with 6 vertices.

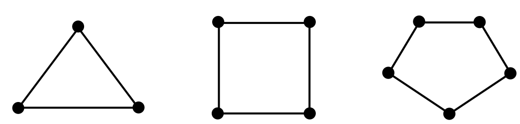


Figure 28: The irreducible planar graphs G_0^3, G_0^4 and G_0^5 with 3, 4 and 5 edges.

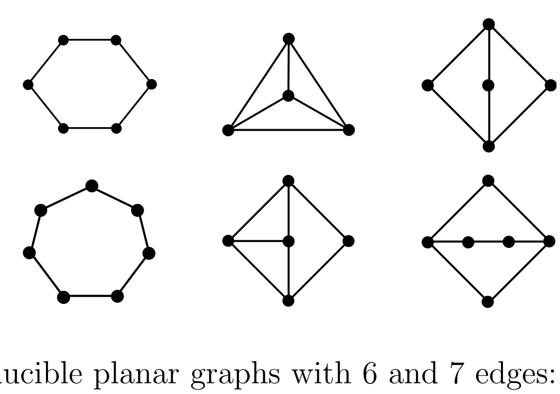


Figure 29: The irreducible planar graphs with 6 and 7 edges: G_0^6, G_1^6, G_2^6 on the top and G_0^7, G_1^7, G_2^7 on the bottom.

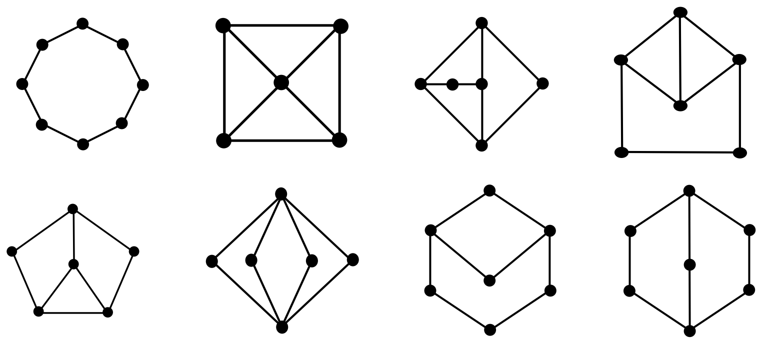


Figure 30: The irreducible planar graphs with 8 edges: $G_0^8, \ldots, G_3^8$ on the top (from left to right) and $G_4^8, \ldots, G_7^8$ on the bottom.

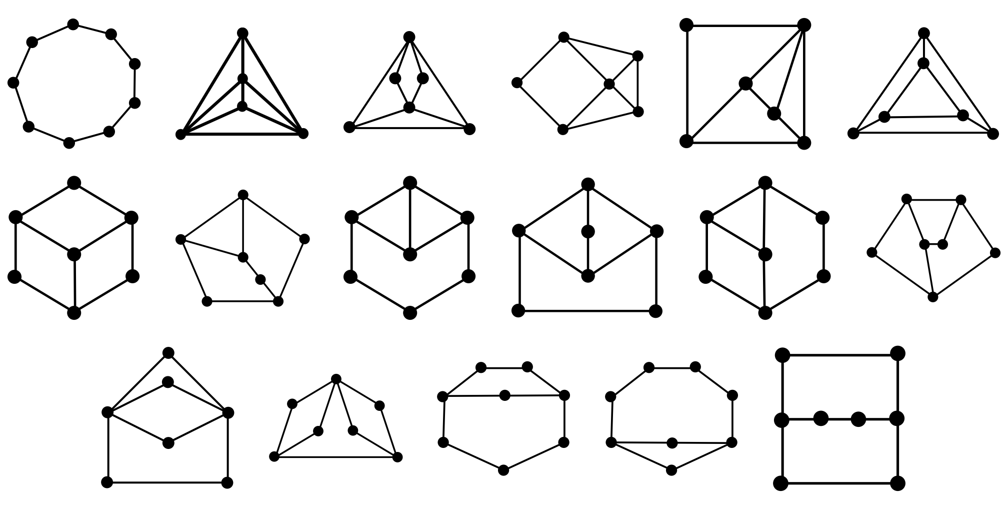


Figure 31: The irreducible planar graphs with 9 edges: $G_0^9, \ldots, G_5^9$ on the top, $G_6^9, \ldots, G_{11}^9$ on the middle and $G_{12}^9, \ldots, G_{16}^9$ on the bottom.

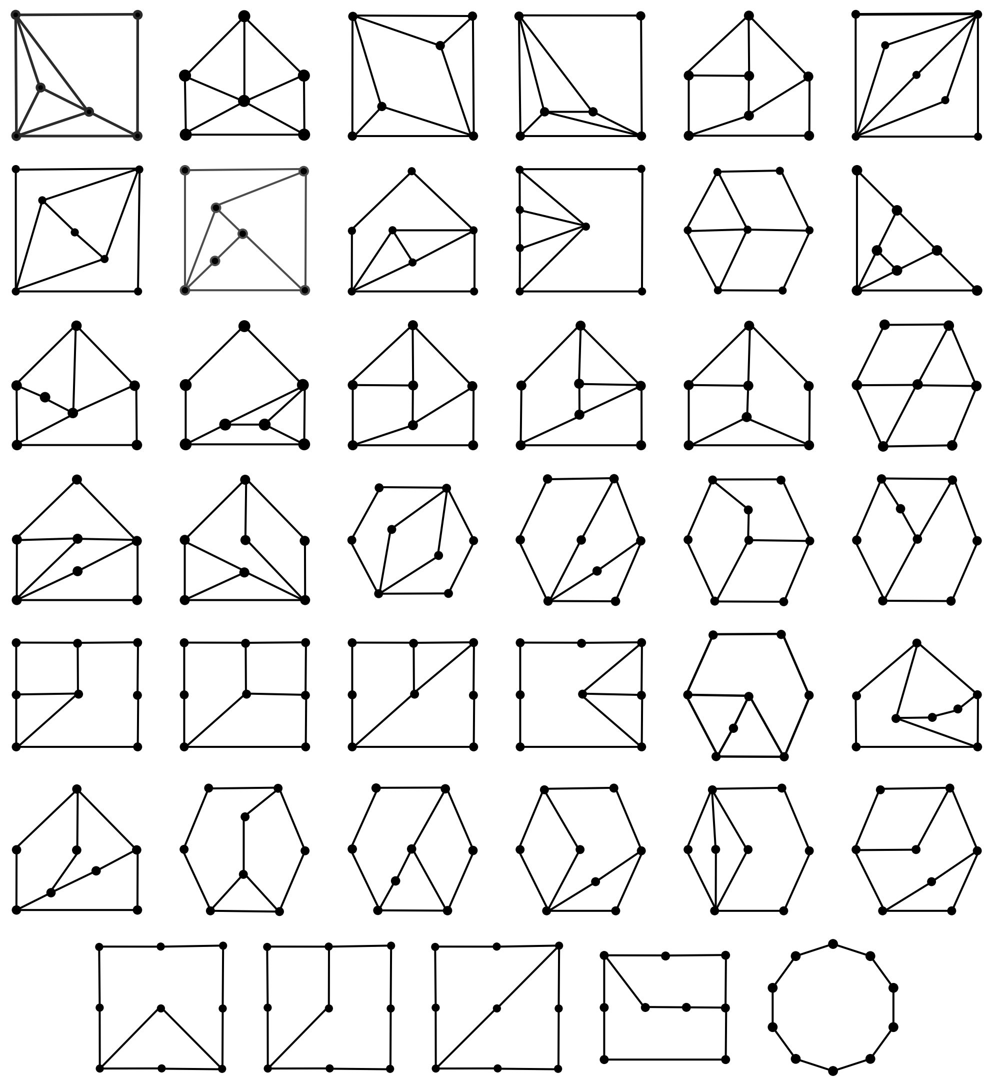


Figure 32: The irreducible planar graphs with 10 edges: $G_0^{10}, \ldots, G_5^{10}$ on the top, $G_6^{10}, \ldots, G_{35}^{10}$ on the middle and $G_{36}^{10}, \ldots, G_{40}^{40}$ on the bottom.

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