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AN APPLICATION OF HILBERT'S PROJECTIVE METRIC TO

POSITIVE OPERATORS

A THESIS

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POSITIVE OPERATORS

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INTRODUCTION

The purpose of this thesis is to investigate a recent result by Garrett D. Birkhoff in [1] concerning D. Hilbert's [4] projective metric and its application to positive linear operators. Birkhoff's paper is of particular interest because it ties together ideas from projective geometry with classical results in matrix theory and integro-functional equations. Also, the paper provides a method of attack for standard eigenvalue problems involving positive linear operators. Examples of such operators are matrices with positive entries and the Fredholm integral operator with a positive kernel. The results in Birkhoff's paper are somewhat inaccessible due to two factors. First, they require the reader to have a working knowledge of projective geometry; and second, Birkhoff has an insight into basic relationships that enables him to proceed from result to result with the connection between them vague. The primary goal of this thesis is to give an elucidation of Birkhoff's work. Background material is developed, omitted proofs supplied, and proofs given by Birkhoff are in some cases supplemented and in some cases replaced.

In Chapter I we present a development of the projective line, projective transformations, and cross ratios. The development is taken only as far as suits the needs of this paper. Most of the ideas discussed in this chapter are contained in [2] and [3].

A metric for points in the projective line is given in Chapter II. This metric was first introduced by D. Hilbert and a discussion of it may be found in [4]. Basic facts are noted about this metric, and it is used to define a norm for projective transformations. This approach is along the lines suggested by Birkhoff. In order to investigate the positive operators mentioned earlier, the metric is extended to closed, convex subsets of a Banach space. Under relatively simple hypotheses, such transformations may be viewed in a certain sense as contractive maps. Cauchy sequences necessarily appear in dealing with the contractive property, and the final result in Chapter II is a proof of completeness of the appropriate sets under the extension of Hilbert's metric.

The major theorem of this thesis is the principal concern of Chapter III. This result is called the projective contraction theorem, and consideration is given to its geometric interpretation.

The thesis closes (Chapter IV) with applications of the projective contraction to both finite and infinite dimensional spaces. Part of the classical theorem of Perron-Frobenius regarding the positive eigenvalue associated with positive matrices is obtained, as well as bounds for this eigenvalue. Other results in Chapter IV include explicit formulas to facilitate the evaluation of Hilbert's metric in both finite and infinite dimensional spaces.

CHAPTER I

THE PROJECTIVE LINE

Consider the set of all ordered pairs (x_1,x_2) of real numbers. We will define $(x_1,x_2) \equiv (y_1,y_2)$ if and only if $\lambda x_1 = y_1$ and $\lambda x_2 = y_2$, where λ is some non-zero scalar. This then defines an equivalence relation on this set, each equivalence class consisting of some ordered pair (x_1,x_2) and all its non-zero scalar multiples. Each class will have an infinite number of members, except for the class which contains only $(0,0)$. The equivalence relation implies that two classes are identical if they have one ordered pair in common.

Definition 1.1. The real projective line L is the set of all equivalence classes $\tilde{x}$ as indicated above, with the exception of the class which contains $(0,0)$. An equivalence class $\tilde{x}$ is called a point on the projective line.

Note that the above definition is independent of a geometric model. That is, nothing in the definition of L indicates from what geometric source the classes $\tilde{x}$ are obtained or what geometric meaning is attached to the line. We will show that the Euclidean line can be considered as a subset of the projective line by considering the following. For a Cartesian line X, assign to each point x on the line the equivalence class containing $(x,1)$. Coordinates designated in this manner for a Cartesian line are called homogenous coordinates, and in homogenous coordinates we may view the Cartesian line as a subset of L. Given any point $\tilde{x}$ on L, we may choose as a representative of $\tilde{x}$ on X the point

$(\frac{x_1}{x_2},1)$ for all $\tilde{x}$ except the class whose second coordinate is zero. Thus we may associate each point on X with a class in L; and each class in L, except the one noted above, with a point in X. Intuitively, we would like to adjoin an ideal point, ∞, to X and let it represent the equivalence class containing (1,0).

We will use the above notions to generate a geometric model for L. We will denote the equivalence classes containing (0,1), (1,1), and (1,0) as $\tilde{0}$, $\tilde{1}$, and $\tilde{\infty}$ respectively. Let us represent a Cartesian line in homogenous coordinates and place a unit circle tangent to the origin of the line. We associate points on the line with points on the circle

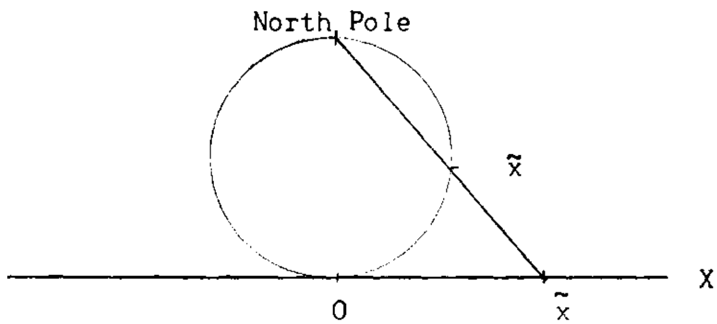

Figure 1.1

in the following manner. For a given point $\tilde{x}$ on X, draw a line segment from $\tilde{x}$ to the North Pole and denote the intersection of the circle and the line segment as $\tilde{\tilde{x}}$. Now we see that attaching the point at ∞ corresponds to closing the line. We simply associate $\tilde{\infty}$ with the point at the North Pole. This defines a one-to-one correspondence from L to the circle and so gives us a geometric model.

For two distinct points $\tilde{x}$, $\tilde{y}$ on the projective line L, take a representative of each x,y. We shall refer to these two representatives as reference points. Now choose a third point $\tilde{z}$ and a representative z. Since both x,y are ordered pairs of real numbers and neither is a scalar multiple of the other, there exist two scalars λ,μ such that $z = \lambda x + \mu y$. If x,y are replaced by other representatives $\lambda' x$, $\mu' y$ of $\tilde{x},\tilde{y}$, then $z' = \lambda\lambda' x + \mu\mu' y$ will, in general, not be a member of $\tilde{z}$, although it will be the representative of some point on the line.

With x,y as reference points, consider a general member of $\tilde{z}$, γz, where γ is arbitrary but non-zero. Since $z = \lambda x + \mu y$, then $\gamma z = \gamma\lambda x + \gamma\mu y$. Note that whatever value of γ we choose, the ratio of the coefficients of x to y remains λ/μ. Hence given any two reference points x,y, we may express all representatives of any point $\tilde{z}$ as a unique linear combination of x,y and each with the same ratio of coordinates.

<u>Definition 1.2.</u> The <u>projective parameter</u> for points in L relative to a fixed set of reference points is defined to be

$$\Theta_z = \Theta_{\tilde{z}} = \mu/\lambda \ .$$

We will allow Θ_z to take on the improper value ∞. That is, $\frac{a}{0} = \infty$ where $a \neq 0$. Note that the ratio $0/0$ will never occur. As we have seen, once reference points have been specified, the projective parameter for a point $\tilde{z}$ is uniquely determined.

For the remainder of the paper, we will assume the following conventions:

$$\frac{a}{0} = \infty, \qquad a \neq 0; \qquad \frac{\infty}{\infty} = 1; \qquad \frac{a}{\infty} = 0, \qquad a \neq \infty;$$

and $$\frac{0}{0} = 1.$$

<u>Definition 1.3.</u> A <u>projective transformation</u> of L onto itself is a transformation taking any point $\tilde{u}$ into some point $\tilde{v}$ by

$$v_1 = a_{11}u_1 + a_{12}u_2$$

$$v_2 = a_{21}u_1 + a_{22}u_2 \quad ,$$

where

$$a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

and (u_1,u_2) is a representative of $\tilde{u}$ and (v_1,v_2) is a representative of $\tilde{v}$.

Note that if we choose another representative of $\tilde{u}$, say $(\gamma u_1,\gamma u_2)$, that we have

$$\gamma v_1 = a_{12}\gamma u_1 + a_{12}\gamma u_2$$

$$\gamma v_2 = a_{21}\gamma u_1 + a_{22}\gamma u_2 \quad ,$$

and $(\gamma v_1, \gamma v_2)$ is another representative of $\tilde{v}$. Thus a projective transformation as defined takes points of L into points of L and is an onto transformation.

Given four distinct points $\tilde{x}$, $\tilde{y}$, $\tilde{u}$, $\tilde{v}$ on L with x, y, u, v as representatives, there exist numbers λ_1, λ_2, μ_1, μ_2, each different than zero, such that

$$u = \lambda_1 x + \lambda_2 y$$

and

$$v = \mu_1 x + \mu_2 y \; .$$

<u>Definition 1.4.</u> The <u>cross ratio</u> of x, y, u, v in the given order is defined as

$$R(u,v,x,y) = \frac{\lambda_2}{\lambda_1} \cdot \frac{\mu_1}{\mu_2} \quad .$$

If x,y remain fixed, but we choose different representatives $\gamma_1 u$, $\gamma_2 v$ of $\tilde{u}$ and $\tilde{v}$, then we have that

$$\gamma_1 u = \gamma_1 \lambda_1 x + \gamma_1 \lambda_2 y$$

and

$$\gamma_2 v = \gamma_2 \mu_1 x + \gamma_2 \mu_2 y \; .$$

Note that

$$R(\gamma_1 u, \gamma_2 v, x, y) = \frac{\lambda_2}{\lambda_1} \cdot \frac{\mu_2}{\mu_1} \quad .$$

Thus it follows that

$$R(\tilde{u},\tilde{v},x,y) = \frac{\lambda_2}{\lambda_1} \cdot \frac{\mu_2}{\mu_1} \quad .$$

That is, given reference points x,y, it does not matter which representative of $\tilde{u},\tilde{v}$ we take to evaluate the cross ratio.

Suppose we have the following four points on L: $x, y, u = \lambda x + \mu y$, and $v = \lambda' x + \mu' y$. We now want to see what happens to the cross ratio of these points under a projective transformation. Letting (x_1, x_2), (y_1, y_2), (u_1, u_2), and (v_1, v_2) represent x, y, u, v, respectively, we have that under a projective transformation

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

$$y_1' = a_{11}y_1 + a_{12}y_2$$

$$y_2' = a_{21}y_1 + a_{22}y_2$$

$$u_1' = a_{11}(\lambda x_1 + \mu y_1) + a_{12}(\lambda x_2 + \mu y_2)$$

$$u_2' = a_{21}(\lambda x_1 + \mu y_1) + a_{22}(\lambda x_2 + \mu y_2) ,$$

or

$$u_1' = \lambda(a_{11}x_1 + a_{12}x_2) + \mu(a_{11}y_1 + a_{12}y_2)$$

$$u_2' = \lambda(a_{21}x_1 + a_{22}x_2) + \mu(a_{21}y_1 + a_{22}y_2)$$

which implies

$$u_1' = \lambda x_1' + \mu y_1' ,$$

$$u_2' = \lambda x_2' + \mu y_2' .$$

Similarly, we have

$$v_1' = \lambda_1' x_1' + \mu' y_1'$$

$$v_2' = \lambda' x_2' + \mu' y_2' \ .$$

Thus the images of $x, y, u = \lambda x + \mu y$, and $v = \lambda' x + \mu' y$ under a projective transformation are $x', y', u' = \lambda x' + \mu y'$, and $v' = \lambda' x' + \mu' y'$. Further, we can calculate immediately that

$$R(u,v,x,y) = R(u',v',x',y') = \frac{\mu}{\lambda} \cdot \frac{\lambda'}{\mu'} \ .$$

Thus we have

<u>Theorem 1.1.</u> The cross ratio of four points on L is preserved under a projective transformation.

For convenience, let us choose for our reference points $x = (1,0)$ and $y = (0,1)$. With $v = (1,1)$ and $u = (u_1,u_2)$ we have

$$v = x + y$$

$$u = u_1 x + u_2 y$$

and so

$$R(u,v,x,y) = \frac{u_2}{u_1} = \theta_u \ .$$

Let r and s be any representatives of the points $\tilde{r}$, $\tilde{s}$ on L with coordinates (r_1,r_2), (s_1,s_2) and consider their images (r_1',r_2'), (s_1',s_2') under a projective transformation

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with

$$a_{11}a_{22} - a_{12}a_{21} = \gamma \quad .$$

Denoting

$$\det \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}$$

by

$$|r,s| = \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix} ,$$

we have

$$|r',s'| = \begin{vmatrix} r'_1 & s'_1 \\ r'_2 & s'_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}r_1 + a_{12}r_2 & a_{11}s_1 + a_{12}s_2 \\ a_{21}r_1 + a_{22}r_2 & a_{21}s_1 + a_{22}s_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix}$$

$$= \gamma |r,s| \ .$$

Now

$$R(u,v,x,y) = \frac{u_2}{u_1}$$

$$= \frac{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & u_1 \\ 0 & u_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & u_1 \\ 1 & u_2 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} y_1 & v_1 \\ y_2 & v_2 \end{vmatrix} \cdot \begin{vmatrix} x_1 & u_1 \\ x_2 & u_2 \end{vmatrix}}{\begin{vmatrix} x_1 & v_1 \\ x_2 & v_2 \end{vmatrix} \cdot \begin{vmatrix} y_1 & u_1 \\ y_2 & u_2 \end{vmatrix}}$$

$$= \frac{|y,v| \cdot |x,u|}{|x,v| \cdot |y,u|} \ .$$

Hence in terms of the images,

$$R(u,v,x,y) = \frac{|y',v'| \cdot |x',u'|}{|x',v'| \cdot |y',u'|} .$$

Evaluating explicitly we have

$$R(u,v,x,y) = \frac{y_1v_2 - y_2v_1}{x_1v_2 - x_2v_1} \cdot \frac{x_1u_2 - x_2u_1}{y_1u_2 - y_2u_1}$$

$$= \frac{\frac{y_1}{y_2} - \frac{v_1}{v_2}}{\frac{x_1}{x_2} - \frac{v_1}{v_2}} \cdot \frac{\frac{x_1}{x_2} - \frac{u_1}{u_2}}{\frac{y_1}{y_2} - \frac{u_1}{u_2}} ,$$

which in terms of the respective projective parameters

$$= \frac{\theta_y - \theta_v}{\theta_x - \theta_v} \cdot \frac{\theta_x - \theta_u}{\theta_y - \theta_u} .$$

Thus we have shown the following.

Theorem 1.2. If $\tilde{x}$, $\tilde{y}$, $\tilde{u}$, $\tilde{v}$ are four distinct points on L, then

$$R(u,v,x,y) = \frac{\theta_x - \theta_u}{\theta_x - \theta_v} \cdot \frac{\theta_y - \theta_v}{\theta_y - \theta_u} .$$

Geometrically, if we take four distinct points a, b, c, d on L and transform under a projective transformation a into $\tilde{\infty}$, b into $\tilde{0}$, and c into $\tilde{1}$, then the fourth, d, will go into some point (d_1,d_2); and since cross ratio is preserved, we have that the cross ratio R(c,d,a,b) is equal to $\frac{d_1}{d_2}$. We can find d_1,d_2 by solving

$$1 = a_{11}a_1 + a_{12}a_2$$

$$0 = a_{21}a_1 + a_{22}a_2$$

$$0 = a_{11}b_1 + a_{12}b_2$$

$$1 = a_{21}b_1 + a_{22}b_2$$

for A and then computing Ad.

<u>Definition 1.5.</u> A <u>linear fractional transformation</u> is a mapping of the form

$$\Theta' = \frac{a\Theta + b}{c\Theta + d}$$

where Θ, a, b, c, d are real and

$$ad - bc \neq 0 .$$

If we map L under a projective transformation, then the projective parameter of each point is mapped by a linear fractional transformation onto the projective parameter of the image. This immediately follows since

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

implies

$$\frac{x_1'}{x_2'} = \frac{a_{11}\dfrac{x_1}{x_2} + a_{12}}{a_{21}\dfrac{x_1}{x_2} + a_{22}} \quad .$$

Further, we immediately have

Theorem 1.3. The cross ratio of the projective parameters of four points on L is invariant under a linear fractional transformation.

CHAPTER II

HILBERT'S METRIC ON THE PROJECTIVE LINE

Basic Properties

Definition 2.1. If a,b,c are three distinct points on L, then a is between b and c if any only if we may write a as a convex linear combination of b and c.

We now want to consider only a segment of the projective line, and for convenience will look at the segment between $\tilde{0}$ and $\tilde{\infty}$ which contains $\tilde{1}$. We will refer to this particular segment as the positive axis. Each point will have representation (u_1,u_2), and again $(\lambda u_1,\lambda u_2)$ for all non-zero λ will represent the same point.

Definition 2.2. Denoting (1,0) as x and (0,1) as y, then if u,v are two distinct points on the positive axis, neither equal to x,y and so that v lies between x and u, define the function

$$d(u,v) = \ln R(u,v,x,y)$$

and

$$d(v,u) = \ln R(v,u,y,x) \ .$$

Theorem 2.1. The function d(u,v) is a metric for the interior of the positive axis of the projective line.

Proof: Since our reference points are unit points, we have that the coordinates of all points on the positive axis may be specified as

$(u_1, 1-u_1)$, $0 \leq u_1 \leq 1$.

(i) Let u,v be two distinct points on the positive axis. Then

$$R(u,v,x,y) = \frac{\theta_x - \theta_u}{\theta_x - \theta_v} \cdot \frac{\theta_y - \theta_v}{\theta_y - \theta_u}$$

$$= \frac{\infty - \frac{u_1}{u_2}}{\infty - \frac{v_1}{v_2}} \cdot \frac{0 - \frac{v_1}{v_2}}{0 - \frac{u_1}{u_2}}$$

$$= \frac{u_2 v_1}{u_1 v_2} = \frac{\theta v}{\theta u} .$$

Suppose v lies between x and u. Then it must be the case that $0 < u_1 < v_1 < 1$ and $0 < v_2 < u_2 < 1$. Further,

$$\frac{u_2 v_1}{u_1 v_2} = \frac{(1 - u_1)\, v_1}{(1 - v_1)\, u_1} > 1,$$

and so d(u,v) is well defined and positive. If we interchange u,v, then to evaluate d(u,v) we first compute

$$R(u,v,y,x) = \frac{\theta_y - \theta_u}{\theta_y - \theta_v} \cdot \frac{\theta_x - \theta_v}{\theta_x - \theta_u}$$

$$= \frac{0 - \theta_u}{0 - \theta_v} \cdot \frac{\infty - \theta_v}{\infty - \theta_u}$$

$$= \frac{\theta_u}{\theta_v} = \frac{u_1}{u_2} \cdot \frac{v_2}{v_1} .$$

But now $u_1 > v_1$ and so $\frac{u_1 v_2}{u_2 v_1} > 1$ and $d(u,v)$ is well defined and positive.

(ii) $$d(u,u) = \ln R(u,u,x,y)$$
$$= \ln 1 = 0 .$$

(iii) Suppose $d(u,v) = 0$. Then $R(u,v,x,y) = 1$ which implies $\frac{\Theta_u}{\Theta_v} = 1$, or $\Theta_u = \Theta_v$. Since the projective parameter is uniquely determined, we have $u=v$.

(iv) Suppose u,v are distinct and v lies between x and u. Then

$$d(u,v) = \ln R(u,v,x,y) = \ln \frac{\Theta_v}{\Theta_u} .$$

By definition,

$$d(v,u) = \ln R(v,u,y,x) = \ln \frac{\Theta_v}{\Theta_u} .$$

Thus

$$d(u,v) = d(v,u).$$

The case with u,v permuted follows similarly.

(v) Suppose we have three distinct points u,v,z on L ordered y,u,v,z,x. Then

$$\Theta_u = \frac{\lambda_u}{1 - \lambda_u} , \qquad \Theta_v = \frac{\lambda_v}{1 - \lambda_v} , \qquad \text{and } \Theta_z = \frac{\lambda_z}{1 - \lambda_z}$$

where $0 < \lambda_u, \lambda_v, \lambda_z < 1$.

Now it follows that $\lambda_u < \lambda_v < \lambda_z$ and $1 - \lambda_u > 1 - \lambda_v > 1 - \lambda_z$, and so

$$\frac{\lambda_u}{1 - \lambda_u} < \frac{\lambda_v}{1 - \lambda_v} < \frac{\lambda_z}{1 - \lambda_z} \quad .$$

This implies $\Theta_u < \Theta_v < \Theta_z$. Now we can see that

$$d(u,z) \leq d(u,v) + d(v,z)$$

since

$$\ln \frac{\Theta_z}{\Theta_u} \leq \ln \frac{\Theta_v}{\Theta_u} + \ln \frac{\Theta_z}{\Theta_v} = \ln \frac{\Theta_v \Theta_z}{\Theta_u \Theta_v}$$

$$= \ln \frac{\Theta_z}{\Theta_u} \ .$$

Similarly,

$$d(u,v) \leq d(u,z) + d(z,v)$$

since

$$\ln \frac{\Theta_v}{\Theta_u} \leq \ln \frac{\Theta_z}{\Theta_u} + \ln \frac{\Theta_z}{\Theta_v}$$

and

$$\Theta_v < \Theta_z$$

and

$$\frac{\Theta_z}{\Theta_v} > 1 \ .$$

The other case follows in a similar manner. Thus $d(u,v)$ is a metric.

The above metric was first discussed by D. Hilbert [4]. For points y,u,v,x ordered on L as given,

$$R(u,v,x,y) = \frac{\Theta_v}{\Theta_u} \quad .$$

As $v \to x$, $\Theta_v \to \infty$ and $d(u,v) \to \infty$. As $u \to x$, $\Theta_u \to 0$ and again $d(u,v) \to \infty$. Therefore, the metric $d(u,v)$ is not defined for the end points of the positive axis.

We would like to use $d(u,v)$ to investigate the effect of mapping the positive axis of the projective line into itself under a projective transformation. As noted in Chapter I, such a transformation corresponds to mapping the projective parameter of each point into a new value under a linear fractional transformation, and it is this view that we will investigate in detail.

Consider the following linear fractional transformation

$$\Theta' = P\Theta = \frac{a\Theta + b}{c\Theta + d} \qquad ad - cb \neq 0$$

which maps $0 \leq \Theta \leq \infty$ into a proper subinterval of $0 < \Theta' < \infty$. Since P is either increasing or decreasing and, the projection $\Theta \to \frac{1}{\Theta}$ preserves cross ratio, we can assume $\Theta > \Theta'$ implies $P\Theta \geq P\Theta'$. Since $P(0) = \frac{b}{d}$, then b and d have the same sign. Since $P(\Theta) = \infty$ has no solution, c and d have the same sign. Hence, we can assume a,b,c,d are all positive. Since we have assumed P to be increasing, we have that since

$$P'(\theta) = \frac{ad - bc}{(c\theta + d)^2} \quad ,$$

then

$$ad > bc \ .$$

<u>Definition 2.3.</u> The norm of a projective transformation P which maps the positive axis into itself is defined as

$$N(P) = \sup_{u,v} \left\{ \frac{d(Pu,Pv)}{d(u,v)} : d(u,v) < \infty \right\} ,$$

where u,v are in the positive axis of the projective line.

Thus we are interested in finding the least upper bound of

$$\frac{d(P\theta_1,P\theta_2)}{d(\theta_1,\theta_2)} = \frac{\ln \dfrac{c\theta_1 + d}{a\theta_1 + b} \cdot \dfrac{a\theta_2 + b}{c\theta_2 + d}}{\ln \dfrac{\theta_2}{\theta_1}}$$

$$= H(\theta_1,\theta_2) \qquad \text{where} \quad 0 < \theta_1,\ \theta_2 < \infty \ .$$

We now take partial derivatives of $H(\theta_1,\theta_2)$ with respect to each variable and set them equal to zero. Thus

$$\frac{\partial H}{\partial \theta_1} = \left(\frac{c}{c\theta_1+d} - \frac{a}{a\theta_1+b}\right)(\ln\theta_2 - \ln \theta_1)$$

$$+ \frac{1}{\theta_1}\left(\ln(a\theta_2+b) - \ln(c\theta_2+d) + \ln(c\theta_1+d)-\ln(a\theta_1+b)\right) = 0$$

$$\frac{\partial H}{\partial \theta_2} = \left(\frac{a}{a\theta_2 + b} - \frac{c}{c\theta_2 + d}\right)(\ln \theta_2 - \ln \theta_1)$$

$$- \frac{1}{\theta_2}\Big(\ln(a\theta_2 + b) - \ln(c\theta_2 + d)$$

$$+ \ln(c\theta_1 + d) - \ln(a\theta_1 + b)\Big)$$

$$= 0 \ .$$

Thus,

$$\theta_1\left(\frac{a}{a\theta_1 + b} - \frac{c}{c\theta_1 + b}\right) = \theta_2\left(\frac{a}{a\theta_2 + b} - \frac{c}{c\theta_2 + d}\right)$$

$$\theta_1(ac\theta_2^2 + ad\theta_2 + bc\theta_2 + bd) = \theta_2(ac\theta_1^2 + ad\theta_1 + bc\theta_1 + bd)$$

$$ac(\theta_1\theta_2^2 - \theta_1^2\theta_2) = bd(\theta_2 - \theta_1)$$

Hence,

$$\theta_1\theta_2 = \frac{bd}{ac} \ .$$

Thus the maximum value occurs when $\theta_1\theta_2 = \frac{bd}{ac}$. Now let us consider the graph of $H(\theta_1,\theta_2)$ for fixed θ_2. Both numerator and denominator are non-negative and are zero at $\theta_1 = \theta_2$. The derivative of the denominator with respect to θ_2 is $\frac{1}{\theta_2}$, and the derivative of the numerator with respect to θ_2 is

$$\frac{ad - cb}{ac\theta_2^2 + (ad + bc)\theta_2 + bd} < \frac{1}{\theta_2}$$

since

$$ad\theta_2 - bc\theta_2 < ac\theta_2^2 + ad\theta_2 + bc\theta_2 + bd$$

and

$$0 < ac\theta_2^2 + 2bc\theta_2 + bd \quad .$$

Thus it follows that as $\theta_2 \to \theta_1$, the numerator decreases at a slower rate than the denominator, and as $\theta_1 < \theta_2 \to \infty$, the numerator increases at a faster rate than the denominator. Both numerator and denominator have a cusp at $\theta_1 = \theta_2$. Taking note of their concavity, we have the graphs illustrated in Figure 2.3.

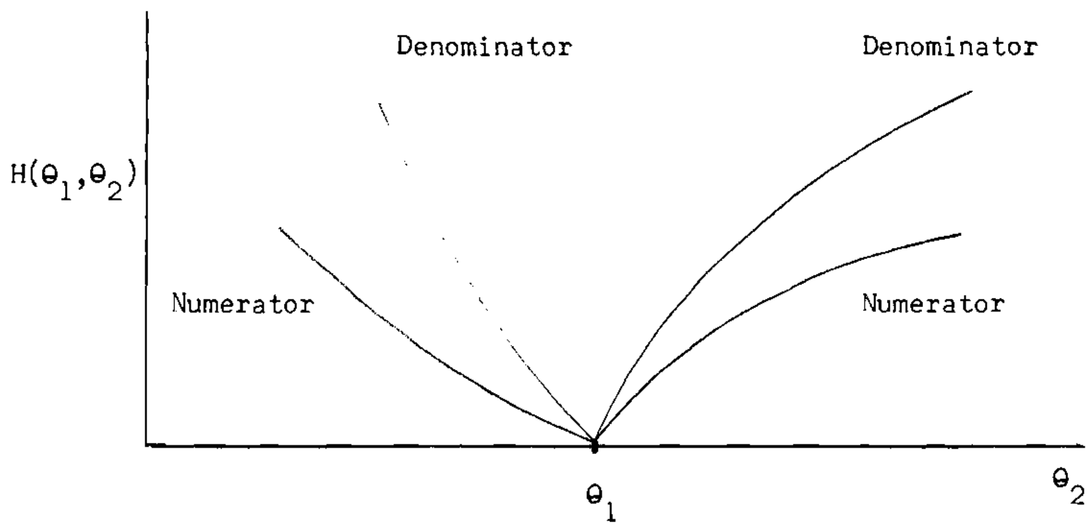

Figure 2.3.

We see that the maximum value of $H(\theta_1,\theta_2)$ for fixed θ_1 is $\lim_{\theta_2 \to \theta_1} H(\theta_1,\theta_2)$. $H(\theta_1,\theta_1)$ is 0/0 so we can not evaluate directly. From Figure 2.3, we expect $\lim_{\theta_2 \to \theta_1} H(\theta_1,\theta_2) \leq 1$ since the numerator is always smaller than the denominator. Since we know the maximum value occurs when $\Theta_1\Theta_2 = \frac{bd}{ac}$, we will set

$$\Theta_1 = \left(\frac{bd}{ac}\right)^{\frac{1}{2}}$$

and take limit of $H(\Theta_1,\Theta_2)$ as $\Theta_2 \to \left(\frac{bd}{ac}\right)^{\frac{1}{2}}$.

$$\lim_{\Theta_2 \to \left(\frac{bd}{ac}\right)^{\frac{1}{2}}} \frac{\ln \dfrac{c\left(\frac{bd}{ac}\right)^{\frac{1}{2}} + d}{a\left(\frac{bd}{ac}\right)^{\frac{1}{2}} + b} \cdot \dfrac{a\Theta_2 + b}{c\Theta_2 + d}}{\ln \dfrac{\Theta_2}{\left(\frac{bd}{ac}\right)^{\frac{1}{2}}}} =$$

$$= \lim_{\Theta_2 \to \left(\frac{bd}{ac}\right)^{\frac{1}{2}}} \frac{\dfrac{c\Theta_2+d}{a\Theta_2+b} \cdot \dfrac{a\left(\frac{bd}{ac}\right)^{\frac{1}{2}}+b}{c\left(\frac{bd}{ac}\right)^{\frac{1}{2}}+d} \cdot \dfrac{c\left(\frac{bd}{ac}\right)^{\frac{1}{2}}+d}{a\left(\frac{bd}{ac}\right)^{\frac{1}{2}}+b} \cdot \dfrac{ad - bc}{(c\Theta_2+d)^2}}{\dfrac{1}{\Theta_2}}$$

$$= \frac{ad - bc}{\left(\frac{ac}{bd}\right)^{\frac{1}{2}}\left(ac \cdot \frac{bd}{ac} + ad\left(\frac{bd}{ac}\right)^{\frac{1}{2}} + bc\left(\frac{bd}{ac}\right)^{\frac{1}{2}} + bd\right)}$$

$$= \frac{\frac{ad}{bc} - 1}{\frac{ad}{bc} + 2\left(\frac{ad}{bc}\right)^{\frac{1}{2}} + 1} .$$

Now letting $v = \frac{ad}{bc} > 1$, we have

$$N(P) = \frac{v - 1}{v + 2v^{\frac{1}{2}} + 1} \quad .$$

Note that

$$\frac{v - 1}{v + 2v^{\frac{1}{2}} + 1} = \frac{v - 1}{(v^{\frac{1}{2}} + 1)^2} = \frac{(v^{\frac{1}{2}} + 1)(v^{\frac{1}{2}} - 1)}{(v^{\frac{1}{2}} + 1)^2}$$

$$= \frac{v^{\frac{1}{2}} - 1}{v^{\frac{1}{2}} + 1}$$

Now letting $e^{\lambda} = v$, we have

$$N(P) = \frac{e^{\frac{\lambda}{2}} - 1}{e^{\frac{\lambda}{2}} + 1} = \frac{1 - e^{-\frac{\lambda}{2}}}{1 + e^{-\frac{\lambda}{2}}} = \tanh\left(\frac{\lambda}{4}\right) < 1 \ .$$

The value $\lambda = \ln(\frac{ad}{bc})$ is the length under the projective metric of the image of $0 \leq \Theta_1 \leq \infty$ under P.

Extensions to Banach Spaces

Let C be a closed, bounded convex subset of a real Banach space X and assume C has a non-empty interior. We are interested in extending the notion of the projective metric to the interior of such a set.

Let Γ be the boundary of C. Let u,v be two distinct points interior to C. Draw a line from u through v until it intersects Γ. Call this intersection x. Extend the same line in the opposite direction until it intersects Γ and call this point y. See Figure 2.2. Now we define the function $D(u,v) = \ln R(u,v,x,y)$. We will later show that this is a metric for points interior to C.

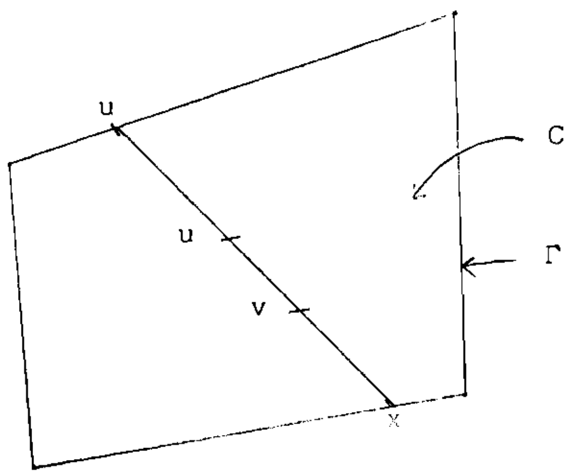

Figure 2.2.

Theorem 2.2. For the above conditions,

$$R(u,v,x,y) = \frac{||x - u||}{||x - v||} \cdot \frac{||y - v||}{||y - u||} ,$$

where $||\cdot||$ is the norm associated with the Banach space X.

Proof: Since we have four points on a line, we can view the line and the points as a subset of the projective line. We will have

$$u = \lambda_1 x + (1 - \lambda_1)y$$

$$v = \lambda_2 x + (1 - \lambda_2)y \qquad 0 < \lambda_1, \lambda_2 < 1 ,$$

and so by definition,

$$R(u,v,x,y) = \frac{1 - \lambda_1}{\lambda_1} \ \frac{\lambda_2}{1 - \lambda_2} \ .$$

Now consider

$$\frac{||x - u||}{||x - v||} \cdot \frac{||y - v||}{||y - u||}$$

$$= \frac{||x - \lambda_1 x - (1 - \lambda_1)y||}{||x - \lambda_2 x - (1 - \lambda_2)y||} \cdot \frac{||y - \lambda_2 x - (1 - \lambda_2)y||}{||y - \lambda_1 x - (1 - \lambda_1)y||}$$

$$= \frac{||(1 - \lambda_1)(x - y)||}{||(1 - \lambda_2)(x - y)||} \cdot \frac{||\lambda_2(x - y)||}{||\lambda_1(x - y)||}$$

$$= \frac{1 - \lambda_1}{\lambda_1} \cdot \frac{\lambda_2}{1 - \lambda_2} \ .$$

This completes the proof.

There is no essential distinction between $d(u,v)$ and $D(u,v)$. However, in order to verify that $D(u,v)$ is a metric in C, we must establish that the triangle inequality holds since three arbitrary points are no longer necessarily collinear.

Theorem 2.3. The function $D(u,v)$ is a metric in C.

Proof: Let a,b,c be three non-collinear points in C. These three points will determine a plane. Call x the intersection of the line from a through b with Γ. Let y be the intersection of the same line extended in the opposite direction with Γ. We have assumed c does not lie on the line from x to y. See Figure 2.3. Let us denote the line from x through y as l(x,y). Let u,v,z, and w be the respective intersections of Γ with

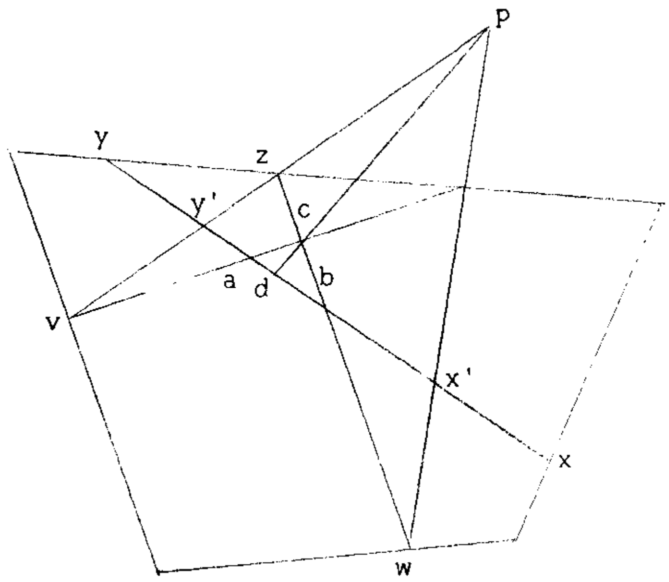

Figure 2.3.

l(a,c), l(c,a), l(b,c), and l(c,b). Set p equal to the intersection of the line l(v,z) and l(w,u). If l(v,z) and l(w,u) did not intersect, then a,b,c would be collinear. Now we need the following lemma.

<u>Lemma 2.1.</u> A geometric projection of a line onto a line, as indicated in Figure 2.4, where $x \to x'$, $y \to y'$, $u \to u'$, and $v \to v'$, preserves cross ratio.

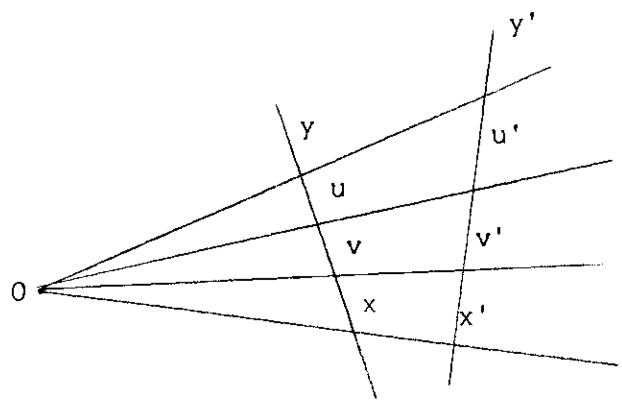

Figure 2.4.

<u>Proof:</u>

$$u = \lambda_1 x + (1 - \lambda_1) y$$

$$v = \lambda_2 x + (1 - \lambda_2) y$$

$$u' = \mu x' + (1 - \mu_1) y'$$

$$v' = \mu_2 x' + (1 - \mu_2) y'$$

We have $x' = \alpha x$, $y' = \beta y$, $u' = \gamma u$, $v' = \delta v$. Substituting in the above we have

$$\gamma u = \mu_1 \alpha x + (1 - \mu_1) \beta y$$

$$\delta v = \mu_2 \alpha x + (1 - \mu_2) \beta y \quad ,$$

and so

$$u = \frac{\mu_1 \alpha}{\gamma} x + \frac{(1 - \mu_1)\beta}{\gamma} y$$

$$v = \frac{\mu_2 \alpha}{\delta} x + \frac{(1 - \mu_2)\beta}{\delta} y \quad .$$

Therefore,

$$\lambda_1 = \frac{\mu_1 \alpha}{\gamma} , \qquad \lambda_2 = \frac{\mu_2 \alpha}{\delta} , \qquad (1 - \lambda_1) = \frac{(1-\mu_1)\beta}{\gamma} ,$$

and

$$(1 - \lambda_2) = \frac{(1 - \mu_2)\beta}{\delta} \quad .$$

Thus

$$\frac{(1 - \mu_1)\frac{\beta}{\gamma}}{\mu_1 \frac{\alpha}{\gamma}} \cdot \frac{\mu_2 \frac{\alpha}{\delta}}{(1 - \mu_2)\frac{\beta}{\delta}} = \frac{1 - \lambda_1}{\lambda_1} \cdot \frac{\lambda_2}{1 - \lambda_2} \quad ,$$

and cross ratio is preserved.

Returning to the proof of Theorem 2.3, we may view a,c,u,v on the line l(u,v) as a geometric projection into a,d,x',y' on the line l(x,y) from p. Thus,

$$R(a,c,u,v) = R(a,d,x',y')$$

$$R(c,b,w,z) = R(d,b,x',y') \quad .$$

Now we need

<u>Lemma 2.2.</u> For points y,a,b,x',x ordered as written, R(a,b,x',y) > R(a,b,x,y).

<u>Proof:</u>

$$R(a,b,x,y) = \frac{\theta_a - \theta_x}{\theta_b - \theta_x} \cdot \frac{\theta_b - \theta_y}{\theta_a - \theta_y} \quad .$$

Note that $\theta_{x'} < \theta_x$ and $\theta_{x'} > \theta_a, \theta_b$. Thus

$$R(a,b,x',y) = \frac{\theta_a - \theta_{x'}}{\theta_b - \theta_{x'}} \cdot \frac{\theta_b - \theta_y}{\theta_a - \theta_y} > R(a,b,x,y) \quad .$$

Again returning to the proof of the theorem we have

$$R(a,c,u,v) \geq R(a,d,x,y)$$

$$R(c,b,w,z) \geq R(d,b,x,y) \quad .$$

Multiplying, we have

$$R(a,c,u,v)R(c,b,w,z) \geq R(a,d,x,y)R(d,b,x,y) \quad .$$

But

$$R(a,d,x,y)\cdot R(d,b,x,y) = \frac{\theta_a-\theta_x}{\theta_d-\theta_x} \cdot \frac{\theta_d-\theta_y}{\theta_a-\theta_y} \cdot \frac{\theta_d-\theta_x}{\theta_b-\theta_x} \cdot \frac{\theta_b-\theta_y}{\theta_d-\theta_y}$$

$$= \frac{\theta_a-\theta_x}{\theta_b-\theta_x} \cdot \frac{\theta_b-\theta_y}{\theta_a-\theta_y}$$

$$= R(a,b,x,y) \quad .$$

Thus

$$R(a,c,u,v)\cdot R(c,b,w,z) \geq R(a,b,x,y) \quad ,$$

and

$$\ln R(a,c,n,v) + \ln R(c,b,w,z) \geq \ln R(a,b,x,y) \quad .$$

Thus

$$D(a,c) + D(c,b) \geq D(a,b) \quad ,$$

and $D(u,v)$ is a metric in C.

Let us note that if a,b,c are three collinear points in the given order in a Banach space, then

$$||a - c|| = ||a - b|| + ||b - c|| \quad .$$

We are going to use this fact to prove the following.

Theorem 2.4. C is complete under $D(u,v)$.

Proof: Suppose $\{a_n\}_{n\geq 1}$ is a sequence in C so that $D(a_n,a_m) \to 0$ as $n,m \to \infty$. The theorem implies that there is an a in C so that $D(a_n,a) \to 0$. Suppose $D(a_n,a_m) \to 0$ as $n,m \to \infty$, but there is some $\varepsilon > 0$ and subsequence $\{a_{n_j}\}_{n_j\geq 1}$ so that $||a_{n_j} - a_{n_k}|| \geq \varepsilon$ for all elements in the subsequence.

To keep the notation simple, let us assume

$$\{a_{n_j}\}_{n_j\geq 1} = \{a_n\}_{n\geq 1} \quad .$$

Let us denote the intersection of the boundary of C with the line from a_n through a_m as x_n^m. We will define y_n^m as the intersection of the line from a_m through a_n with the boundary of C. Then

$$\frac{||a_n - x_n^m||}{||a_m - x_n^m||} \cdot \frac{||a_m - y_n^m||}{||a_n - y_n^m||} \geq \frac{||a_m - y_n^m||}{||a_n - y_n^m||}$$

Let us denote

$$\lambda_{n,m} = ||a_n - y_n^m|| \quad .$$

Then

$$R(a_n,a_m,x_n^m,y_m^n) \geq \frac{\lambda_{n,m} + \varepsilon}{\lambda_{n,m}} = 1 + \frac{\varepsilon}{\lambda_{n,m}} > 1 \quad .$$

Since C is bounded $\lambda_{n,m}$ is bounded for all values n,m. This implies

$D(a_n,a_m)$ does not go to zero which is a contradiction. Hence, $||a_n - a_m||$ must go to zero as $n,m \to \infty$. Since C is a closed subset of a Banach space, there is some a in C so that $||a_n - a|| \to 0$ as $n \to \infty$. Consider

$$D(a_n,a) = \ln \frac{||x_u - a_n||}{||x_n - a||} \cdot \frac{||y_n - a||}{||y_n - a_n||} \quad .$$

As $n \to \infty$, we have

$$\frac{||x_n - a_n||}{||x_n - a||} \; , \; \frac{||y_n - a||}{||y_n - a_n||} \to 1 \quad .$$

Hence $D(a_n,a) \to 0$, and C is complete with respect to the metric $D(u,v)$.

CHAPTER III

PROJECTIVE CONTRACTION THEOREM

Let X be a real Banach space.

Definition 3.1. A closed subset C of X is said to be a closed, convex cone if and only if

$$\text{(i)} \quad C + C \subset C \quad ,$$

$$\text{(ii)} \quad \lambda C \subset C$$

for each positive real number λ, and

$$\text{(iii)} \quad C \cap (-C) = 0 \quad , \qquad \text{where 0 is}$$

the origin of X.

Note that in E_n the positive quadrant, that is, those vectors

$$x = (x_1, \ldots, x_n)$$

so that $x_i \geq 0$ for $i = 1, \ldots, n$, forms a closed, convex cone.

Definition 3.2. Let f be a bounded linear functional on X. The set of points $\{x : f(x)=b, \text{ b a real scalar}\}$ is said to be a hyperplane in X.

We want to consider the intersection of a closed, convex cone C with a hyperplane H. We will choose H so that it intersects at one point each ray which starts at the origin and is in C. It follows that $C \cap H$ is a closed, bounded convex set, and the metric $D(u,v)$ that we developed in Chapter II is applicable to it.

Let P be any bounded, linear transformation that maps C into itself. Then $P(C \cap H)$ will be a closed bounded convex subset of C. Let us choose two distinct points u,v interior to $C \cap H$. We would like to apply $D(\cdot,\cdot)$ to the images of u,v under P. As before, draw the line $l(u,v)$ until it intersects the boundary of $C \cap H$ and call the point x. Similarly let y be the intersection of $l(v,u)$ with the boundary of $C \cap H$. Now compute Px, Py, Pu, Pv. We have that x,y,u,v are collinear in $C \cap H$. Since P is linear, Px, Py, Pu, Pv will be collinear in $P(C \cap H)$. Thus we can make a geometric projection through the origin of the line segment joining Px and Py onto $C \cap H$. As we saw in Chapter II, such a projection preserves cross ratio, and hence will preserve the projective distance between Pu and Pv. We will call the composite transformation of P operating on u and the geometric projection of Pu through the origin and back into $C \cap H$ as T_p. Thus we have

$$D(Pu,Pv) = D(T_p u,\ T_p v)$$

and

$$T_p(C \cap H) \subset C \cap H \quad .$$

When we construct $C \cap H$ and evaluate the distance between points there, we are actually computing in a certain sense the distance between rays (through the origin) in C. Choose two rays a,b. Select a hyperplane H_1 with the required cutting property, and denote the intersection of a,b with $H_1 \cap C$ as u,v, respectively. To evaluate $D(u,v)$ we first select x,y in the usual manner. Choose another feasible hyperplane H_2. Call the intersection of a,b with $H_2 \cap C$ u',v', respectively.

To evaluate $D(u',v')$ we first find x',y'. Note we may view u', v', x', y' as a geometric projection u,v,x,y through the origin. Thus it follows that $D(u,v) = D(u',v')$. This number can be considered the distance between the rays a,b since this distance is independent of the hyperplane selected.

<u>Definition 3.3.</u> If P is a bounded, linear transformation that maps C into itself then T_p maps $C \cap H$ into itself, and the <u>diameter</u> Δ of PC is equal to the diameter of $T_p(C \cap H)$ and is given by

$$\Delta = \sup\left\{D(Pu,\ Pv):\ u,v \text{ in } C \cap H\right\}$$

$$= \sup\left\{D(T_p u, T_p v):\ u,v \text{ in } C \cap H\right\} .$$

If x,y lie on the boundary of $C \cap H$, then in order for $D(T_p x, T_p y)$ to be finite, it must be the case that neither $T_p x$ nor $T_p y$ lie on the boundary of $C \cap H$. Suppose for example that $T_p x$ did. Then

$$D(T_p x,\ T_p y) = \ln \quad \frac{||x' - T_p x||}{0} \cdot \frac{||T_p x - T_p y||}{||x' - T_p y||} = \infty \quad .$$

That is, if $T_p(C \cap H)$ is to have finite diameter, $T_p(C \cap H)$ must be a proper subset of $C \cap H$, and in fact, bounded away from the boundary of $C \cap H$.

We would like to establish that T_p is actually a linear fractional transformation of the projective parameter of the points in $C \cap H$ onto the projective parameter of the images.

Choose two distinct points u,v in $C \cap H$. Locate x,y in the usual manner. Now compute $T_p x$, $T_p y$, $T_p u$, and $T_p v$. We will have

$$u = \lambda_1 x + (1 - \lambda_1) y$$

$$(1) \qquad v = \lambda_2 x + (1 - \lambda_2) y, \qquad 0 < \lambda_1, \lambda_2 < 1 \quad ,$$

and so

$$T_p u = \lambda_1 T_p x + (1 - \lambda_1) T_p y$$

$$(2) \qquad T_p v = \lambda_2 T_p x + (1 - \lambda_2) T_p y \quad .$$

Now extend the line from $T_p x$ through $T_p y$ until it intersects the boundary and call this point x'. Similarly generate y'. Thus

$$T_p x = \gamma_1 x' + (1 - \gamma_1) y'$$

$$(3) \qquad T_p y = \gamma_2 x' + (1 - \gamma_2) y', \qquad 0 < \gamma_1, \gamma_2 < 1 \quad .$$

Now substituting (3) into the first equation in (2) we see that

$$T_p u = \lambda_1 \ \gamma_1 x' + (1 - \gamma_1) y' \ + (1 - \lambda_1) \ \gamma_2 x' + (1 - \gamma_2) y'$$

$$= \ \lambda_1 \gamma_1 + (1 - \lambda_1) \gamma_2 \ x' + \lambda_1 (1 - \gamma_1)$$

$$+ (1 - \lambda_1)(1 - \gamma_2) \ \ y' \quad .$$

Note that from (1) $\Theta_u = \dfrac{\lambda_1}{1 - \lambda_1}$. Also note that

$$\theta_{T_p u} = \frac{\lambda_1\gamma_1 + (1 - \lambda_1)\gamma_2}{\lambda_1(1 - \gamma_1)+(1 - \lambda_1)(1-\gamma_2)}$$

$$= \frac{(\frac{\lambda_1}{1 - \lambda_1})\gamma_1 + \gamma_2}{(\frac{\lambda_1}{1 - \lambda_1})(1 - \gamma_1) + (1 - \lambda_1)}$$

$$= \frac{\gamma_1\theta_u + \gamma_2}{(1 - \gamma_1)\theta_n + (1 - \gamma_2)}$$

Hence T_p is a linear fractional transformation of the projective parameter of all points contained on any line in $C \cap H$ onto the projective parameter of the points on the image line.

Definition 3.4. The projective norm of P relative to C is the projective norm of T_p relative to $C \cap H$ and is given by

$$N(P;C) = N(T_p;C \cap H)$$

$$= \sup_{u,v \text{ in } C \cap H} \left\{ \frac{D(T_p u, T_p v)}{D(u,v)} : D(u,v) < \infty \right\}.$$

Lemma 3.1. If the transform of $C \cap H$ under T_p has finite diameter Δ, then

$$N(T_p;C \cap H) = \tanh\ (\tfrac{\Delta}{4}) < 1 \quad .$$

Proof: From our investigation of projective transformations in Chapter II, it immediately follows that $N(T_p;C \cap H) \leq \frac{\Delta}{4}$. To show that equality holds, we take a sequence of inverse images u_n, v_n of suitably nearby pairs of points on line segments from x_n to y_n of lengths $\Delta - 2^{-n}$ or more. We know such segments exist because of the definition of Δ as the sup over all segments. For such segments, we have

$$N(T_p;C \cap H) \geq \tanh\left(\frac{\Delta - 2^{-n}}{4}\right) .$$

Hence, as $N(T_p;C \cap H)$ is the sup over all such segments,

$$N(T_p;C \cap H) = \tanh \frac{\Delta}{4} .$$

Now we are ready to prove the major theorem of this paper.

<u>Theorem 3.1.</u> Let C be a closed, convex cone in a real Banach space X. Let P be any bounded linear transformation so that $PC \subseteq C$. If $N(T_p^r;C \cap H) < 1$ for some r, then for any u in $C \cap H$, the sequence $\left\{T_p^n u\right\}_{n\geq 1}$ converges to a unique fixpoint c in $C \cap H$.

<u>Proof:</u> If $N(T_p^r;C \cap H) < 1$, then $T_p^r(C \cap H)$ has finite diameter by what we have just seen. Further, $D(T_p^r u,\ T_p^{r+1} u) < +\infty$ for arbitrary u in $C \cap H$ since if

$$D(T_p^r u,\ T_p^{r+1} u) = +\infty ,$$

then $T_p^{r+1} u$ is in the boundary of $C \cap H$. But $T_p^r(C \cap H)$ is a proper subset of $C \cap H$ and $T_p^{r+1}(C \cap H) \subseteq T_p^r(C \cap H)$ and so $T_p^{r+1} u$ can not be on the boundary. Note that

$$D(T_p(T_p u),T_p(T_p^2 u)) \leq N(T_p;C \cap H)\cdot D(T_p u,T_p^2 u)$$

since

$$\sup_{u,v} \frac{D(T_p u, T_p v)}{D(u,v)} = N(T_p; C \cap H) \quad .$$

Hence for $n > r$, $n = qr + \gamma$, $0 \leq \gamma < r$,

$$D(T_p^n u, T_p^{n+1} u) = D[T_p^{r(q-1)}(T_p^{r+\gamma} u), T_p^{r(q-1)}(T_p^{r+\gamma+1} u)]$$

$$\leq N(T_p^r; C \cap H)^{q-1} \; D(T_p^{r+\gamma} u, T_p^{r+\gamma+1} u) \quad .$$

Let

$$K = \max_{0 \leq \gamma < r} \; D(T_p^{r+\gamma} u, T_p^{r+\gamma+1} u) \quad .$$

Since

$$N(T_p^r; C \cap H) < 1 \quad ,$$

then K is finite, and it follows that

$$D(T_p^n u, \; T_p^{n+1} n) \leq N(T_p^r; C)^{q-1} K \; .$$

Hence $\left\{T_p^n u\right\}_{n \geq 1}$ is a Cauchy sequence. We established in Chapter II that C H is complete under $D(u,v)$, and so the Cauchy sequence converges to a limit c in C. Thus $c = \lim_{n \to \infty} T_p^n c$. We have that P is bounded. Thus T_p is bounded and continuous. Hence

$$c = T_p \lim_{n \to \infty} T_p^n c = T_p c \quad .$$

Uniqueness is immediate. If $T_p c^* = c^*$, then

$$D(c,c^*) = D(T_p^r c,\ T_p^r c^*) \leq N(T_p^r;C \cap H)D(c,c^*)$$

$$< D(c,c^*) \quad . \quad \text{Hence } D(C,C^*) = 0 \text{ and } C = C^*.$$

Geometrically, the above theorem implies that P has a fixed ray. That is, there is some vector c and some real number $\lambda > 0$ so that $Pc = \lambda c$. In the above theorem, in order that $D(T_p c,c) = 0$, we have that after a geometric projection of Pc through the origin onto $C \cap H$, $T_p c$ and c are the same point. This implies Pc and c lie on the same ray, and $Pc = \lambda c$ for some $\lambda > 0$.

CHAPTER IV

APPLICATIONS

Finite Dimensional Cones

In the finite dimensional case, taking as our cone C the positive quadrant and as our linear transformation, P, a square matrix, the requirement that PC have finite diameter is equivalent to requiring that PC be a proper subset of C. This implies P is a positive matrix; that is, that all the elements of P are positive. If e_j is the coordinate vector with all zeros except a one in the j^{th} position, then PC being a proper subset of C implies Pe_j is a positive vector. Thus the j^{th} column of P is positive. The projective contraction theorem then immediately gives us the portion of the classical theorem of Perron-Frobenius that guarantees that a positive matrix has a unique positive eigenvector and eigenvalue. See Varga [6]. The projective contraction theorem is stated so that it is sufficient that P^r take C into a proper subset of itself with finite diameter, provided $P \geq 0$ for some r, which implies P is a primitive matrix; that is, P has only one eigenvalue in modulus equal to the spectral radius of P.

With the positive quadrant as our cone C in E_n and P a positive matrix, we would like to establish methods other than the definition for evaluating D(u,v) and Δ. Let H be the <u>unit hyperplane</u> given by the set of all points $f = (f_1, \ldots, f_n)$ satisfying

$$\sum_{i=1}^{n} f_i = 1, \quad f_i \geq 0 \text{ for } i = 1, \ldots, n .$$

Select two distinct points $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $C \cap H$. Find x,y as in Chapter III. Now find the smallest value μ so that $u \leq \mu v$ and find the largest value λ so that $\lambda v \leq u$. See Figure 4.1.

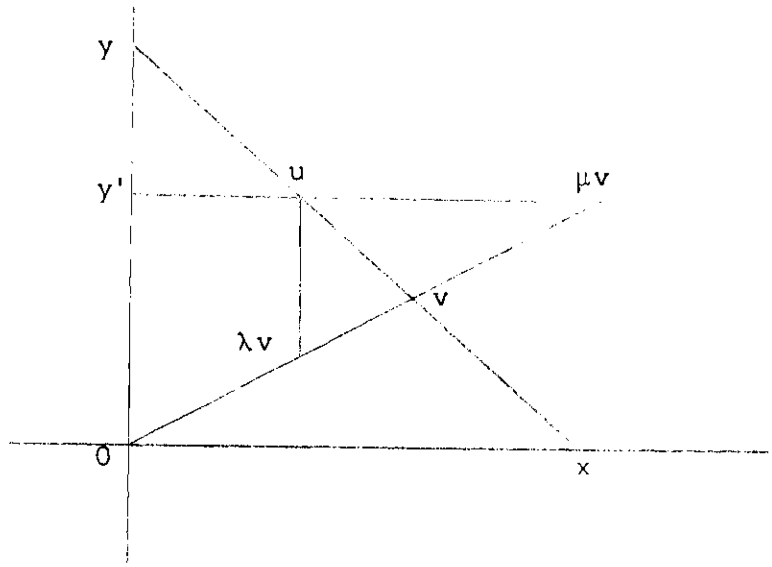

Figure 4.1.

Now construct the line through u and μv. Call the intersection of this line and the boundary of $C \cap H$ y'. This line will be parallel to one of the coordinate faces of C and so $x' = \infty$. Now $y', u, \mu v, \infty$ is a geometric projection of y,u,v,x through the origin, and so

$$R(u,v,x,y) = R(u,\mu v,\infty,y')$$

$$= \frac{||y' - \mu v||}{||y' - u||} .$$

The line connecting u to λv will be parallel to the line connecting y' and the origin. Thus in the triangle with vertices at the origin, y', and μv it follows that

$$\frac{||x' - \mu v||}{||y' - u||} = \frac{||\mu v||}{||\lambda v||} = \frac{\mu}{\lambda} \quad .$$

Hence we have the following computational method.

Theorem 4.1. Let C be the positive quadrant in E_n and H the unit hyperplane. If u,v are two points in $C \cap H$ and $m(\lambda) = \max \lambda$, $M(\mu) = \min \mu$ satisfying $\lambda v \leq u \leq \mu v$, then

$$D(u,v) = \ln \frac{M(\mu)}{m(\lambda)} \quad .$$

It is a simple matter to evaluate $m(\lambda)$ and $M(\mu)$. The inequality $m(\lambda)v \leq u \leq M(\mu)v$ may be written

$$m(\lambda) \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \leq \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \leq M(\mu) \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$$

The largest value that λ may take on is $m(\lambda) = \min\limits_i \frac{u_i}{v_i}$. Similarly, $M(\mu) = \max\limits_j \frac{u_j}{v_j}$. Thus

$$\frac{M(\mu)}{m(\lambda)} = \frac{\max\limits_j \frac{v_j}{v_j}}{\min\limits_i \frac{u_i}{v_i}} = \max_{i,j} \frac{u_i v_j}{u_j v_i} \quad .$$

Therefore another computational scheme for evaluating D(u,v) is

Theorem 4.2. With C,H as in Theorem 4.1, two points $u = [u_1, \ldots, u_n]$ and $v = [v_1, \ldots, v_n]$ in $C \cap H$ will satisfy

$$D(u,v) = \max_{i,j} \frac{u_i v_j}{u_j v_i} .$$

In a similar vein, we will establish a formula for Δ. Again let H be the unit hyperplane, C the positive quadrant, and P a positive square matrix. The image $P(C \cap H)$ will be closed, convex, and bounded by straight lines. An extreme point for such a set is a point in the set that can not be written as a convex linear combination of any two distinct points again in the set; hence in this case an intersection of the boundary lines. Then in order to compute Δ, we need only the distances between all pairs of extreme points in $P(C \cap H)$ because Δ is equal to the maximum of these values. The images of the extreme points in $C \cap H$ will be the extreme points in $P(C \cap H)$. These points will have coordinates given by the column vectors of P. Using Theorem 4.2, we have

Theorem 4.3. Under the above conditions,

$$\Delta = \ln \max_{i,j,k,l} \frac{p_{ik} p_{jl}}{p_{jk} p_{il}} ,$$

where

$$P = [p_{ij}]_{n \times n} , \qquad p_{ij} > 0 \quad \text{all } i,j = 1, \ldots, n .$$

The geometric considerations we have been making throughout this paper yield immediate bounds on the positive eigenvalue λ associated with a positive matrix P, and these bounds are similar to those contained in Varga [6]. There is no loss in generality in assuming the positive eigenvector c corresponding to λ lies in $C \cap H$. The value λ scale c so that it intersects $P(C \cap H)$ and Pc. Since c is interior to $C \cap H$, the value λ is between the smallest and largest of the values that scale the appropriate vectors in $C \cap H$ into the extreme points of $P(C \cap H)$. Since the extreme points of $P(C \cap H)$ are the column vectors of P, we need find for each column of P a vector $u = (u_1, \ldots, u_n)$ in $C \cap H$ and a value λ_i so that

$$\lambda_i(u_1, u_2, \ldots, u_{n-1}, 1 - (u_1 + \ldots + u_{n-1}))$$

$$= (p_{1i}, p_{2i}, \ldots, p_{ni}) \quad ,$$

or we need solve the n equations

$$\lambda_i u_1 = p_{1i}$$

$$\lambda_i u_2 = p_{2i}$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$\lambda_i u_{n-1} = p_{n-1,i}$$

$$\lambda_i(1 - (u_1 + \ldots + u_{n-1})) = p_{ni}$$

Solving the first (n-1) equations for the u_i's and substituting into the n^{th} equation we have

$$\lambda_i(1 - (\frac{p_{1i}}{\lambda_i} + \ldots + \frac{p_{n-1,i}}{\lambda_i})) = p_{ni}$$

or

$$\lambda_i = p_{1i} + \ldots + p_{ni} .$$

It follows that

$$\min_{1 \leq j \leq n} \sum_{i=1}^{n} p_{ij} \leq \lambda \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} p_i .$$

If all column sums are the same, then λ is equal to their common value. In this case $P(C \cap H)$ is parallel to $C \cap H$. Since the eigenvalues of P equal those of P^T, similar remarks are true about the row sums.

Infinite Dimensional Cones

Another cone of interest is the following. Let L be the space of continuous functions or $[0,1]$ with norm $|| \cdot || = \sup_{x \text{ in } [0,1]} |f(x)|$. Consider the set C of non-negative functions. This defines a closed, convex cone complete under $|| \cdot ||$ and so complete under $D(u,v)$. Let P be the operator defined by

$$Pf = \int_0^1 p(x,y)f(y)dy \quad ,$$

where $p(x,y)$ is continuous on $X = [0,1] \times [0,1]$ and

$$0 < \inf_X p(x,y) = \alpha \leq \sup_X p(x,y) = \beta = \gamma\alpha \quad .$$

Now we would like to establish that P takes C into a set with finite diameter Δ so that we can apply the projective contraction theorem and establish that P has a positive eigenvector and eigenvalue. First we need

Lemma 4.1. Let the following six collinear points a,y,u,v,x,b in a Banach space be ordered along the line as listed. Then as y approaches a and/or x approaches b along the line, R(u,v,x,y) decreases.

Proof:

$$R(u,v,x,y) = \frac{||x - u||}{||y - u||} \cdot \frac{||y - v||}{||x - v||} .$$

As $y \to a$, $\frac{||y - v||}{||x - v||}$ remains constant, and $||x - u||$ and $||y - u||$ are increased by the same amount. Thus it follows that $\frac{||x - u||}{||y - u||}$ decreases. Similar remarks are true if $x \to b$.

Now we will choose two arbitrary continuous functions $f,g > 0$ on $[0,1]$, and show that D(Pf,Pg) is less than a constant that is independent of f,g. Letting $e(x) \equiv 1$ on $[0,1]$, there will exist real numbers $\lambda(f)$, $\lambda(g) > 0$ so that

$$\lambda(f)e(x) \leq Pf \leq \gamma\, \lambda(f)e(x)$$

$$\lambda(g)e(x) \leq Pg \leq \gamma\, \lambda(g)e(x) \quad ,$$

or

$$(1) \quad e(x) \leq \frac{Pf}{\lambda(f)} \leq \gamma e(x)$$

$$(2) \quad e(x) \leq \frac{Pg}{\lambda(g)} \leq \gamma\, e(x) \quad .$$

It follows that

$$(3) \quad \gamma\, e(x) \leq \frac{\gamma Pf}{\lambda(f)} \leq \gamma^2 e(x)$$

and

$$(4) \quad \gamma\, e(x) \leq \frac{\gamma Pg}{\lambda(g)} \leq \gamma^2 e(x) \quad .$$

Combining (3) with (2) and (4) with (1) we have

$$(5) \quad \frac{\gamma Pf}{\lambda(f)} \geq \frac{Pg}{\lambda(g)}$$

$$(6) \quad \frac{\gamma Pf}{\lambda(g)} \geq \frac{Pf}{\lambda(f)} \quad .$$

Assuming $\gamma > 1$ (otherwise the problem of finding eigenvalue for P is trivial), consider the four points $\frac{\gamma Pf}{\lambda(f)} - \frac{Pg}{\lambda(g)}$, $(\gamma - 1)\frac{Pf}{\lambda(f)}$, $(\gamma - 1)\frac{Pg}{\lambda(g)}$, and $\frac{\gamma Pg}{\lambda(g)} - \frac{Pf}{\lambda(f)}$. All four of these points will be interior to C due to (5) and (6) and assumptions on f, g, γ. Now we need to establist that these four points are collinear. Hence, we must find a real number λ_1 satisfying $0 < \lambda_1 < 1$ so that

$$(\gamma - 1)\frac{Pf}{\lambda(f)} = \lambda_1 \gamma \frac{Pf}{\lambda(f)} - \lambda_1 \frac{Pg}{\lambda(g)} + \frac{\gamma Pg}{\lambda(g)} - \frac{Pf}{\lambda(f)}$$

$$- \lambda_1 \frac{Pg}{\lambda(g)} + \lambda_1 \frac{Pf}{\lambda(f)}$$

or,

$$(\gamma - \lambda_1\gamma - \lambda_1)\frac{Pf}{\lambda(f)} = (\gamma - \lambda_1\gamma - \lambda_1)\frac{Pg}{\lambda(g)} \quad .$$

We can do this by choosing $\lambda_1 = \dfrac{\gamma}{\gamma + 1}$. And second, we need to find λ_2 satisfying $0 < \lambda_2 < 1$ so that

$$(\gamma - 1)\frac{Pg}{\lambda(g)} = \lambda_2 \frac{\gamma Pf}{\lambda(P)} - \lambda_2 \frac{Pg}{\lambda(g)} + \gamma \frac{Pg}{\lambda(g)} - \frac{Pf}{\lambda(f)}$$

$$- \lambda_2 \frac{\gamma Pg}{\lambda(g)} + \lambda_2 \frac{Pf}{\lambda(f)}$$

or

$$(-1 + \lambda_2 + \lambda_2\gamma)\frac{Pg}{\lambda(g)} = (\lambda_2\gamma - 1 + \lambda_2)\frac{Pf}{\lambda(f)} \; .$$

We can do this by choosing

$$\lambda_2 = \frac{1}{\gamma + 1} \quad .$$

Thus,

$$R\left((\gamma - 1)\frac{Pf}{\lambda(f)} , (\gamma - 1)\frac{Pg}{\lambda(g)} , \gamma\frac{P(g)}{\lambda(g)} - \frac{Pf}{\lambda(f)} , \gamma\frac{Pf}{\lambda(P)} - \frac{Pg}{\lambda(g)}\right)$$

$$= \frac{\dfrac{\gamma}{\gamma + 1}}{1 - \dfrac{\gamma}{\gamma + 1}} \cdot \frac{1 - \dfrac{1}{\gamma + 1}}{\dfrac{1}{\gamma + 1}}$$

$$= \gamma^2$$

By Lemma 4.1 and the properties of $D(u,v)$ we have that

$$D((\gamma - 1)\frac{Pf}{\lambda(f)}, (\gamma - 1)\frac{Pg}{\lambda(g)}) = D(Pf,Pg) \leq \ln \gamma^2 \quad .$$

Thus, PC has finite diameter, and the projective contraction theorem applies to guarantee that P, a Fredholm integral operator with a positive, continuous kernel, has a positive eigenvalue and eigenvector in the space of continuous functions on a closed, bounded interval.

This result may be generalized to an arbitrary Banach lattice L. A bounded linear transformation of a vector lattice L into itself will be called uniformly positive if, for some fixed $e > 0$ in L and finite number γ, independent of f, we have

$$(7) \quad \lambda e \leq Pf \leq k\lambda e$$

for any $f > 0$ and some $\lambda = \lambda(f) > 0$. Since our previous development for the specific case was based entirely on an equation of the form (7), we have

Theorem 4.4. Any uniformly bounded linear transformation P of a Banach lattice L into admits a unique positive vector c and scalar $\lambda > 0$ so that

$$Pc = \lambda c \quad .$$

Consider the Volterra operator with positive, continuous kernel, i.e.,

$$Pf = \int_a^x p(x,t)F(t)dt, \qquad a \leq x \leq b \quad .$$

We choose $e(x) = x - a$ and set

$$\alpha = \inf_{x,t \text{ in } [a,b]} p(x,t) \quad ,$$

$$\beta = \sup_{x,t \text{ in } [a,b]} p(s,t) \quad ,$$

and

$$\gamma = \frac{\beta}{\alpha} \quad .$$

Thus P is uniformly positive and Theorem 4.4 applies.

Theorem 4.1 prompts the following investigation. Let L be a Banach lattice. Let L^+ be the set of positive vectors in L. Two elements f,g in L^+ will be called <u>comparable</u> if there exist two positive numbers λ,μ such that

$$\lambda g < f < \mu g \quad .$$

Let Ω be the subset of L^+ containing all comparable functions in L^+. Note that Ω is a convex cone. Let $M(f,g) = \inf \mu$, $m(f,g) = \sup \lambda$. We will establish the following theorem in analogy with Theorem 4.1.

<u>Theorem 4.5.</u>

$$D(f,g) = \ln \frac{M(f,g)}{m(f,g)}$$

is a projective metric in Ω.

<u>Proof:</u> (i) By definition of m(f,g) and M(f,g) we see D(f,g) is well defined and non-negative and equal zero if and only if f = g.

(ii) To establish that we have a projective metric, we need only show that homogenity is satisfied, that is

$$D(\gamma f,g) = D(f,g) \qquad \text{for all } \gamma > 0 \quad .$$

But if

$$m(f,g)g \leq f \leq M(f,g)g,$$

then

$$\gamma m(f,g)g \leq \gamma f \leq \gamma M(f,g)g$$

and

$$m(\gamma f,g) = \gamma m(f,g) \qquad \text{by definition of } \; m(\cdot,\cdot)$$

and similarly

$$M(\gamma f,g) = \gamma M(f,g) \quad .$$

Hence

$$D(\gamma f,g) = \ln \frac{\gamma M(f,g)}{\gamma m(f,g)} = D(f,g) \quad .$$

(iii) Suppose we have $m(f,g)g \leq f \leq M(f,g)g$ and

$$D(f,g) = \ln \frac{M(f,g)}{m(f,g)} \quad .$$

Then

$$\frac{1}{M(f,g)}\, f \leq g \leq \frac{1}{m(f,g)}\, f$$

and

$$D(g,f) = \ln \frac{M(f,g)}{m(f,g)} = D(f,g) \quad .$$

(iv) Let f, g, h be distinct in Ω. Let $\varepsilon > 0$. Then

$$(m(f,h) - \varepsilon)h < f < (M(f,h) + \varepsilon)$$

and

$$(m(h,g) - \varepsilon)g < h < (M(h,g) + \varepsilon) \quad .$$

Here we have assumed $\varepsilon < \min\left\{m(f,h), m(h,g)\right\}$. Thus we have

$$(m(f,h - \varepsilon)(m(h,g) - \varepsilon)g < f < (M(f,h) + \varepsilon)(M(h,g) + \varepsilon)g \ .$$

So by definition of M, m we have

$$(m(f,h) - \varepsilon)(m(h,g) - \varepsilon) \leq m(f,g)$$

and

$$(M(f,h) + \varepsilon)(M(h,g) + \varepsilon) \geq M(f,g) \quad .$$

Letting $\varepsilon \to 0^+$ we have

$$m(f,g) \geq m(f,h) \cdot m(h,g)$$

and

$$M(f,g) \leq M(f,h) \cdot M(h,g) \quad .$$

And so

$$\frac{M(f,g)}{m(f,g)} \leq \frac{M(f,h) \cdot M(h,g)}{m(f,h) \cdot m(h,g)} \quad ,$$

and, therefore, the triangle inequality

$$D(f,g) \leq D(f,h) + D(h,g)$$

is established.

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