A LOCAL VARIATIONAL PRINCIPLE OF PRESSURE AND ITS APPLICATIONS TO EQUILIBRIUM STATES

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ABSTRACT. We prove a local variational principle of pressure for any given open cover. More precisely, for a given dynamical system (X,T), an open cover $\mathcal{U}$ of X, and a continuous, real-valued function f on X, we show that the corresponding local pressure $P(T,f;\mathcal{U})$ satisfies

$$P(T,f;\mathcal{U}) = \sup\{h_\mu(T,\mathcal{U}) + \int_X f(x)d\mu(x) : \mu \text{ is a } T\text{-invariant measure}\},$$

and moreover, the supremum can be attained by a T-invariant ergodic measure. By establishing the upper semi-continuity and affinity of the entropy map relative to an open cover, we further show that

$$h_\mu(T,\mathcal{U}) = \inf_{f\in C(X,\mathbb{R})}\{P(T,f;\mathcal{U}) - \int_X f d\mu\}$$

for any T-invariant measure μ of (X,T), i.e., local pressures determine local measure-theoretic entropies. As applications, properties of both local and global equilibrium states for a continuous, real-valued function are studied.

1. Introduction and main result

Topological pressure is a generalization to topological entropy for a dynamical system. The notion was first introduced by Ruelle [24] for expansive dynamical system and later by Walters [26] for general case. Let (X,T) be a *topological dynamical system* (TDS for short) in the sense that X is a compact metric space and $T : X \to X$ is a surjective and continuous map. It is known that certain results concerning topological entropy can be generalized to topological pressure. In particular, Waters [26] generalized the classical variational principle of entropy [15, 16, 21] to obtain the following variational principle of pressure:

$$P(T,f) = \sup_\mu\{h_\mu(T) + \int_X f(x)d\mu(x) : \mu \text{ is a } T\text{-invariant measure}\}, \tag{1.1}$$

where f is a continuous, real-valued function on X, $P(T,f)$ is the *topological pressure* of f, and, for each T-invariant measure μ, $h_\mu(T)$ is the *measure-theoretic entropy* of μ.

With the notion of entropy pairs [2, 4] in both topological and measure-theoretic situations, a notable amount of attention has recently been paid to the study of local version of the variational principle of entropy. Given a TDS (X,T) and an open cover $\mathcal{U}$ of X. It was first shown in [3] that there is a T-invariant measure μ such that

$$\inf_{\alpha\succeq\mathcal{U}} h_\mu(T,\alpha) \ge h_{\text{top}}(T,\mathcal{U}), \tag{1.2}$$

where $h_{\text{top}}(T,\mathcal{U})$ is the *topological entropy relative to* $\mathcal{U}$, $h_\mu(T,\alpha)$ is the *measure-theoretic entropy of* μ *relative to a finite Borel partition* α of X, and $\alpha \succeq \mathcal{U}$ means that α is finer than $\mathcal{U}$. A somewhat converse statement to (1.2) is given in [18] as the following: if μ is a T-invariant measure and $h_\mu(T,\alpha) > 0$ for each partition α which is finer than $\mathcal{U}$, then $\inf_{\alpha\succeq\mathcal{U}} h_\mu(T,\alpha) > 0$ and $h_{\text{top}}(T,\mathcal{U}) > 0$. To make a general investigation on the converse to (1.2), Romagnoli [23] introduced two types of

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measure-theoretic entropies relative to $\mathcal{U}$: $h_\mu(T,\mathcal{U})$ and $h_\mu^+(T,\mathcal{U})$, satisfying $h_\mu(T,\mathcal{U}) \le h_\mu^+(T,\mathcal{U})$ and $h_\mu(T,\mathcal{U}) \le h_{\rm top}(T,\mathcal{U})$, and proved that

$$h_{\rm top}(T,\mathcal{U}) = \max\{h_\mu(T,\mathcal{U}) : \mu \text{ is a } T\text{-invariant measure}\}. \tag{1.3}$$

Later, by proving that both $h_\mu(T,\mathcal{U})$ and $h_\mu^+(T,\mathcal{U})$ have the properties of ergodic decomposition, it was shown in [17] that the maximum in (1.3) can be in fact attained by a T-invariant ergodic measure. Recently, Glasner and Weiss [12] proved that if the system (X,T) is *invertible*, i.e, T is a homeomorphism, then the local variational principle is also true for $h_\mu^+(T,\mathcal{U})$, i.e.,

$$h_{\rm top}(T,\mathcal{U}) = \sup\{h_\mu^+(T,\mathcal{U}) : \mu \text{ is a } T\text{-invariant measure}\} \tag{1.4}$$

(see also [19] for a relative version). It also follows from [17] that the supremum in (1.4) can be attained by a T-invariant ergodic measure.

The main purpose of this paper is to generalize the above local variational principles of entropy to the case of pressure. Our main results state as follows.

Theorem 1 (Upper semi-continuity and affinity). *The local entropy map $h_{\{\cdot\}}(T,\mathcal{U})$ is upper semi-continuous and affine on the space of T-invariant measures.*

Theorem 2 (Local variational principle). *For any $f \in C(X,\mathbb{R})$, the local pressure $P(T,f;\mathcal{U})$ of f relative to $\mathcal{U}$ satisfies*

$$P(T,f;\mathcal{U}) = \sup\{h_\mu(T,\mathcal{U}) + \int_X f d\mu(x) : \mu \textit{ is a } T\textit{-invariant measure}\}$$

and the supremum can be attained by a T-invariant ergodic measure.

We will also show that local pressures determine local measure-theoretic entropies, i.e., the following holds.

Theorem 3 (Determining measure-theoretic entropy). *Let μ be a T-invariant measure of (X,T). The following holds.*

a)

$$h_\mu(T,\mathcal{U}) = \inf_{f\in C(X,\mathbb{R})}\{P(T,f;\mathcal{U}) - \int_X f d\mu\};$$

b) *If, in addition, (X,T) is invertible, then*

$$h_\mu^+(T,\mathcal{U}) \le \inf_{f\in C(X;\mathbb{R})}\{P(T,f;\mathcal{U}) - \int_X f d\mu\}.$$

Theorem 3 immediately leads to the following.

Corollary ([17, 19]). *If (X,T) is invertible, then*

$$h_\mu^+(T,\mathcal{U}) = h_\mu(T,\mathcal{U}).$$

The local variational principle of pressure stated in Theorem 2 guarantees the existence of equilibrium states with respect to a local pressure. Such an existence is not true in general for a topological pressure, unless some additional properties (such as expansivity) of the TDS are assumed. Applying our main results, another purpose of the paper is to study properties of equilibrium states for local pressures and to establish their connections with equilibrium states for topological pressures. We refer the readers to [25] for physical relevance of topological pressures and equilibrium states.

The paper is organized as follows. Section 2 is a preliminary section in which basic properties of both local and global entropies and pressures are studied in both topological and measure-theoretic situations. We prove Theorem 1 in Section 3, Theorem 2 in Section 4, and Theorem 3 in Section 5. Some applications of our main results to equilibrium states for both local and global topological pressures are given in Section 6.

2. Entropies and pressures

Throughout the section, we let (X,T) be a TDS and $\mathcal{B}_X$ be the collection of all Borel subsets of X.

2.1. **Topological entropies and pressures.** Recall that a *cover* of X is a finite family of Borel subsets of X whose union is X, and, a *partition* of X is a cover of X whose elements are pairwise disjoint. We denote the set of covers, partitions, and open covers, of X, respectively, by $\mathcal{C}_X$, $\mathcal{P}_X$, and $\mathcal{C}_X^0$, respectively. For given two covers $\mathcal{U},\mathcal{V}\in\mathcal{C}_X$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$ (denote by $\mathcal{U}\succeq\mathcal{V}$) if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Let $\mathcal{U}\vee\mathcal{V}=\{U\cap V: U\in\mathcal{U}, V\in\mathcal{V}\}$. Given integers M,N with $0\le M\le N$ and $\mathcal{U}\in\mathcal{C}_X$ or $\mathcal{P}_X$, we use $\mathcal{U}_M^N$ to denote $\bigvee_{n=M}^N T^{-n}\mathcal{U}$.

For $\mathcal{U}\in\mathcal{C}_X$, we define $N(\mathcal{U})$ as the minimum among the cardinals of the sub-covers of $\mathcal{U}$ and define $H(\mathcal{U})=\log N(\mathcal{U})$. Clearly, for each $\mathcal{U},\mathcal{V}\in\mathcal{C}_X$, $H(\mathcal{U}\vee\mathcal{V})\le H(\mathcal{U})+H(\mathcal{V})$, and if $\mathcal{V}\succeq\mathcal{U}$, then $H(\mathcal{V})\ge H(\mathcal{U})$.

Since $a_n=H(\mathcal{U}_0^{n-1})$ is a bounded, sub-additive sequence, i.e., $a_{n+m}\le a_n+a_m$ for any $n,m\in\mathbb{N}$, the quantity

$$h_{\text{top}}(T,\mathcal{U})=\lim_{n\to\infty}\frac{1}{n}H(\mathcal{U}_0^{n-1})=\inf_{n\ge1}\frac{1}{n}H(\mathcal{U}_0^{n-1}),$$

called the *topological entropy of* $\mathcal{U}$, is well defined (see [1]). The *topological entropy of* (X,T) is defined by

$$h_{\text{top}}(T)=\sup_{\mathcal{U}\in\mathcal{C}_X^0} h_{\text{top}}(T,\mathcal{U}).$$

Let $C(X,\mathbb{R})$ be the Banach space of all continuous, real-valued functions on X endowed with the supremum norm. For $f\in C(X,\mathbb{R})$ and $\mathcal{U}\in\mathcal{C}_X^o$, we define

$$P_n(T,f;\mathcal{U})=\inf\{\sum_{V\in\mathcal{V}}\sup_{x\in V}e^{f_n(x)}:\ \mathcal{V}\in\mathcal{C}_X \text{ and } \mathcal{V}\succeq\mathcal{U}_0^{n-1}\},$$

where $f_n(x)=\sum_{j=0}^{n-1}f(T^jx)$. It is clear that if f is the null function, then $P_n(T,0;\mathcal{U})=\log N(\mathcal{U}_0^{n-1})=H(\mathcal{U}_0^{n-1})$.

For $\mathcal{V}\in\mathcal{C}_X$, we let α be the Borel partition generated by $\mathcal{V}$ and define

$$\mathcal{P}^*(\mathcal{V})=\{\beta\in\mathcal{P}_X:\beta\succeq\mathcal{V} \text{ and each atom of } \beta \text{ is the union of some atoms of } \alpha\}. \tag{2.1}$$

Lemma 2.1. *$\mathcal{P}^*(\mathcal{V})$ is a finite set, and, for each $n\in\mathbb{N}$,*

$$\inf_{\beta\in\mathcal{C}_X,\beta\succeq\mathcal{V}}\sum_{B\in\beta}\sup_{x\in B}e^{f_n(x)}=\min\{\sum_{B\in\beta}\sup_{x\in B}e^{f_n(x)}:\ \beta\in\mathcal{P}^*(\mathcal{V})\}.$$

Proof. Let $\mathcal{V}=\{V_1,V_2,\cdots,V_k\}$. For any $\beta=\{B_1,B_2,\cdots,B_l\}\in\mathcal{C}_X$ with $\beta\succeq\mathcal{V}$, we let $i_1\in\{1,2,\cdots,l\}$ be such that $\sup_{x\in B_{i_1}}f_n(x)=\sup_{x\in X}f_n(x)$. Since $\beta\succeq\mathcal{V}$, there exists a $j_1\in\{1,2,\cdots,k\}$ such that $B_{i_1}\subset V_{j_1}$. Let

$$\beta(j_1)=\{V_{j_1}\}\cup\{B_i\setminus V_{j_1}:\ i\in\{1,2,\cdots,l\}\text{ and } B_i\setminus V_{j_1}\ne\emptyset\}.$$

Then $\beta(j_1)\in\mathcal{C}_X$, $\beta(j_1)\succeq\mathcal{V}$, and,

$$\sum_{B\in\beta}\sup_{x\in B}e^{f_n(x)}\ge\sum_{B\in\beta(j_1)}\sup_{x\in B}e^{f_n(x)}.$$

Let $\{i\in\{1,2,\cdots,l\}:\ B_i\setminus V_{j_1}\ne\emptyset\}=\{1\le r_1<r_2<\cdots<r_{l_1}\le l\}$. Denote $B_i^1=B_{r_i}\setminus V_{j_1}$, $i=1,2,\cdots,l_1$. Then $B_i^1\cap V_{j_1}=\emptyset$, $i=1,2,\cdots,l_1$, and, $\beta(j_1)=\{V_{j_1},B_1^1,B_2^1,\cdots,B_{l_1}^1\}$.

If $X\setminus V_{j_1}=\emptyset$, then $\beta(j_1)=\{V_{j_1}\}\in\mathcal{P}^*(\mathcal{V})$. If $X\setminus V_{j_1}\ne\emptyset$, then we let $i_2\in\{1,2,\cdots,l_1\}$ be such that $\sup_{x\in B_{i_2}^1}e^{f_n(x)}=\sup_{x\in X\setminus V_{j_1}}e^{f_n(x)}$. Since $\beta(j_1)\succeq\mathcal{V}$ and $B_i^1\cap V_{j_1}=\emptyset, i=1,2,\cdots,l_1$, there exists a $j_2\in\{1,2,\cdots,k\}\setminus\{j_1\}$ such that $B_{i_2}^1\subset V_{j_2}\setminus V_{j_1}$. Let

$$\beta(j_1,j_2)=\{V_{j_1},V_{j_2}\setminus V_{j_1}\}\cup\{B_i^1\setminus V_{j_2}:\ i\in\{1,2,\cdots,l_1\}\text{ and } B_i\setminus V_{j_2}\ne\emptyset\}.$$

Then $\beta(j_1,j_2)\in\mathcal{C}_X$, $\beta(j_1,j_2)\succeq\mathcal{V}$, and,

$$\sum_{B\in\beta(j_1)}\sup_{x\in B}e^{f_n(x)}\geq\sum_{B\in\beta(j_1,j_2)}\sup_{x\in B}e^{f_n(x)}.$$

If $X\setminus(V_{j_1}\bigcup V_{j_2})=\emptyset$, then $\beta(j_1,j_2)=\{V_{j_1},V_{j_2}\setminus V_{j_1}\}\in\mathcal{P}^*(\mathcal{V})$. If $X\setminus(V_{j_1}\bigcup V_{j_2})\neq\emptyset$, then we continue the above procedure. By induction, we obtain a sequence $\{j_1,j_2,\cdots,j_r\}\subseteq\{1,2,\cdots,k\}$, where $r\leq k$, such that $\bigcup_{i=1}^{r-1}V_{j_i}\neq X$, $\bigcup_{i=1}^{r}V_{j_i}=X$, and

$$\beta(j_1,j_2,\cdots,j_r)=\{V_{j_1},V_{j_2}\setminus V_{j_1},\cdots,V_{j_r}\setminus(\bigcup_{i=1}^{r-1}V_{j_i})\}$$

satisfies

$$\sum_{B\in\beta}\sup_{x\in B}e^{f_n(x)}\geq\sum_{B\in\beta(j_1,\cdots,j_r)}\sup_{x\in B}e^{f_n(x)}.$$

Clearly, $\beta(j_1,j_2,\cdots,j_r)\in\mathcal{P}^*(\mathcal{V})$.

Hence

$$\inf_{\beta\in\mathcal{C}_X,\beta\succeq\mathcal{V}}\sum_{B\in\beta}\sup_{x\in B}e^{f_n(x)}=\min_{\{j_1,j_2,\cdots,j_r\}\subseteq\{1,2,\cdots,k\}}\sum_{B\in\beta(j_1,j_2,\cdots,j_r)}\sup_{x\in B}e^{f_n(x)},$$

where $\{j_1,j_2,\cdots,j_r\}$ are such that $|\{j_1,j_2,\cdots,j_r\}|=r\leq k$, $\bigcup_{i=1}^{r-1}V_{j_i}\neq X$ and $\bigcup_{i=1}^{r}V_{j_i}=X$. □

In particular, by taking $\mathcal{V}=\mathcal{U}_0^{n-1}$ in Lemma 2.1, we have that $\mathcal{P}^*(\mathcal{U}_0^{n-1})$ is a finite set, and

$$P_n(T,f;\mathcal{U})=\min\{\sum_{B\in\beta}\sup_{x\in B}\ e^{f_n(x)}:\ \beta\in\mathcal{P}^*(\mathcal{U}_0^{n-1})\}. \tag{2.2}$$

If, in addition, $\mathcal{U}$ is a partition, then

$$P_n(T,f;\mathcal{U})=\sum_{U\in\mathcal{U}_0^{n-1}}\sup_{x\in U}\ e^{f_n(x)}.$$

Lemma 2.2. *For any $f\in C(X,\mathbb{R})$ and $\mathcal{U}\in\mathcal{C}_X^o$,*

$$P(T,f;\mathcal{U})\equiv\lim_{n\to\infty}\frac{1}{n}\log P_n(T,f;\mathcal{U})$$

exists and equals $\inf_{n\geq1}\frac{1}{n}\log P_n(T,f;\mathcal{U})$.

Proof. For any $n,m\in\mathbb{N}$, $\mathcal{V}_1\succeq\mathcal{U}_0^{n-1},\mathcal{V}_2\succeq\mathcal{U}_0^{m-1}$, we have $\mathcal{V}_1\vee T^{-n}\mathcal{V}_2\succeq\mathcal{U}_0^{n+m-1}$. It follows that

$$\begin{aligned}P_{n+m}(T,f;\mathcal{U})&\leq\sum_{V_1\in\mathcal{V}_1}\sum_{V_2\in\mathcal{V}_2}\sup_{x\in V_1\cap T^{-n}V_2}e^{f_{n+m}(x)}\\&=\sum_{V_1\in\mathcal{V}_1}\sum_{V_2\in\mathcal{V}_2}\sup_{x\in V_1\cap T^{-n}V_2}e^{f_n(x)+f_m(T^nx)}\\&\leq\sum_{V_1\in\mathcal{V}_1}\sum_{V_2\in\mathcal{V}_2}\sup_{x\in V_1}e^{f_n(x)}\cdot\sup_{z\in V_2}e^{f_m(z)}\\&=(\sum_{V_1\in\mathcal{V}_1}\sup_{x\in V_1}e^{f_n(x)})(\sum_{V_2\in\mathcal{V}_2}\sup_{z\in V_2}e^{f_m(z)}).\end{aligned}$$

Since $\mathcal{V}_1,\mathcal{V}_2$ are arbitrary, $P_{n+m}(T,f;\mathcal{U})\leq P_n(T,f;\mathcal{U})P_m(T,f;\mathcal{U})$, i.e., $\log P_n(T,f;\mathcal{U})$ is sub-additive. This proves the lemma. □

Using Lemma 2.2, we immediately have the following.

Lemma 2.3. *$P(T^k,f_k;\mathcal{U}_0^{k-1})=kP(T,f;\mathcal{U})$ for any $f\in C(X,\mathbb{R})$, $\mathcal{U}\in\mathcal{C}_X^o$, and $k\in\mathbb{N}$.*

We refer to $P(T,f;\mathcal{U})$ as the *topological pressure of f relative to $\mathcal{U}$* and to

$$P(T,f)=\sup_{\mathcal{U}\in\mathcal{C}_X^o}P(T,f;\mathcal{U})$$

as the *topological pressure of f*. We note that the definition of $P_n(T,f;\mathcal{U})$ (hence $P(T,f;\mathcal{U})$) above is slightly different than the one given in [27]. However, it is easy to see that the topological pressures $P(T,f)$ defined above are the same as the ones defined in [27], and moreover, if f is the null function, then $P(T,0;\mathcal{U}) = h_{\mathrm{top}}(T,\mathcal{U})$. An advantage of our definition of $P_n(T,f;\mathcal{U})$ is its monotonicity, i.e., if $\mathcal{U} \succeq \mathcal{V}$, then $P_n(T,f;\mathcal{U}) \geq P_n(T,f;\mathcal{V})$, which is essential for the validity of the local variational principle stated in Theorem 2.

2.2. **Measure-theoretic entropies.** Let $\mathcal{M}(X)$, $\mathcal{M}(X,T)$, and $\mathcal{M}^e(X,T)$, respectively, be the set of all Borel probability measures, T-invariant Borel probability measures, and T-invariant ergodic measures, on X, respectively. Then $\mathcal{M}(X)$ and $\mathcal{M}(X,T)$ are all convex, compact metric spaces when endowed with the weak*-topology; $\mathcal{M}^e(X,T)$ is a G_δ subset of $\mathcal{M}(X,T)$.

For given partitions $\alpha,\beta \in \mathcal{P}_X$ and $\mu \in \mathcal{M}(X)$, let

$$H_\mu(\alpha) = \sum_{A\in\alpha} -\mu(A)\log\mu(A) \text{ and } H_\mu(\alpha|\beta) = H_\mu(\alpha\vee\beta) - H_\mu(\beta).$$

One standard fact is that $H_\mu(\alpha|\beta)$ increases with respect to α and decreases with respect to β. When $\mu \in \mathcal{M}(X,T)$, it is not hard to see that $H_\mu(\alpha_0^{n-1})$ is a non-negative and sub-additive sequence for a given $\alpha \in \mathcal{P}_X$. The *measure-theoretic entropy of* μ *relative to* α is defined by

$$h_\mu(T,\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\alpha_0^{n-1}),$$

and the *measure-theoretic entropy of* μ is defined by

$$h_\mu(T) = \sup_{\alpha\in\mathcal{P}_X} h_\mu(T,\alpha). \tag{2.3}$$

For a given $\mathcal{U} \in \mathcal{C}_X$, Romagnoli [23] introduced the following two types of *measure-theoretic entropies relative to* $\mathcal{U}$,

$$h_\mu(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}) \text{ and } h_\mu^+(T,\mathcal{U}) = \inf_{\alpha\succeq\mathcal{U},\alpha\in\mathcal{P}_X} h_\mu(T,\alpha),$$

where

$$H_\mu(\mathcal{U}) = \inf_{\alpha\in\mathcal{P}_X,\alpha\succeq\mathcal{U}} H_\mu(\alpha).$$

As to be seen below, many properties of $H_\mu(\alpha)$ for a partition α can be extended to $H_\mu(\mathcal{U})$ for a cover $\mathcal{U}$.

Lemma 2.4. *Let* $\mu \in \mathcal{M}(X)$. *The following holds for any* $\mathcal{U},\mathcal{V} \in \mathcal{C}_X$.

1. $0 \leq H_\mu(\mathcal{U}) \leq \log N(\mathcal{U})$.
2. *If* $\mathcal{U} \succeq \mathcal{V}$, *then* $H_\mu(\mathcal{U}) \geq H_\mu(\mathcal{V})$.
3. $H_\mu(\mathcal{U}\vee\mathcal{V}) \leq H_\mu(\mathcal{U}) + H_\mu(\mathcal{V})$.
4. $H_\mu(T^{-1}\mathcal{U}) \leq H_{T\mu}(\mathcal{U})$, *and, the equality holds when* (X,T) *is invertible.*

Proof. See [23], Lemma 8. □

For a given $\mathcal{U} \in \mathcal{C}_X$, $\mu \in \mathcal{M}(X,T)$, it follows easily from Lemma 2.4 that $H_\mu(\mathcal{U}_0^{n-1})$ is a sub-additive function of $n \in \mathbb{N}$. Hence the local measure-theoretic entropy $h_\mu(T,\mathcal{U})$ is well defined. This extension of local measure-theoretic entropy from partitions to covers allows the generalization of the local variational principle of entropy to the local variational principle of pressure stated in Theorem 2.

Lemma 2.5. *For any* $\mathcal{V} \in \mathcal{C}_X$ *and* $\mu \in \mathcal{M}(X,T)$,

$$H_\mu(\mathcal{V}) = \min_{\beta\in\mathcal{P}^*(\mathcal{V})} H_\mu(\beta).$$

Proof. The proof is very similar to that of Lemma 2.1. Let $\phi(x) = -x\log x$, $x \geq 0$. We first observe that if $0 < x \leq y$ and $0 < \delta \leq x$, then

$$\phi(x-\delta) + \phi(y+\delta) < \phi(x) + \phi(y). \tag{2.4}$$

Let $\mathcal{V} = \{V_1, V_2, \cdots, V_k\}$. For any $\beta = \{B_1, B_2, \cdots, B_l\} \in \mathcal{P}_X$ with $\beta \succeq \mathcal{V}$, we let $i_1 \in \{1,2,\cdots,l\}$ be such that $\mu(B_{i_1}) = \max_{1\leq i\leq l} \mu(B_i)$. Choose j_1 and $\beta(j_1) = \{V_{j_1}, B_1^1, B_2^1, \cdots, B_{l_1}^1\}$

the same way as in the proof of Lemma 2.1. Then $\beta(j_1) \in \mathcal{P}_X$ and $\beta(j_1) \succeq \mathcal{V}$. Moreover, it follows from (2.4) that $H_\mu(\beta) \geq H_\mu(\beta(j_1))$.

If $X \setminus V_{j_1} = \emptyset$, then $\beta(j_1) = \{V_{j_1}\} \in \mathcal{P}^*(\mathcal{V})$. Otherwise, we let $i_2 \in \{1, 2, \cdots, l_1\}$ be such that $H_\mu(B^1_{i_2}) = \max_{1\leq i\leq l_1} H_\mu(B^1_i)$, and choose j_2, $\beta(j_1, j_2)$ the same way as in the proof of Lemma 2.1. Then $\beta(j_1, j_2) \in \mathcal{P}_X$ and $\beta(j_1, j_2) \succeq \mathcal{V}$. Also by (2.4), $H_\mu(\beta(j_1)) \geq H_\mu(\beta(j_1, j_2))$.

Continuing the above procedure inductively, we find a sequence $\{j_1, j_2, \cdots, j_r\} \subseteq \{1, 2, \cdots, k\}$, where $r \leq k$, such that $\bigcup_{i=1}^{r-1} V_{j_i} \neq X$, $\bigcup_{i=1}^{r} V_{j_i} = X$, and

$$\beta(j_1, j_2, \cdots, j_r) = \{V_{j_1}, V_{j_2} \setminus V_{j_1}, \cdots, V_{j_r} \setminus (\bigcup_{i=1}^{r-1} V_{j_i})\}$$

satisfies that $H_\mu(\beta) \geq H_\mu(\beta(j_1, \cdots, j_r))$ and $\beta(j_1, j_2, \cdots, j_r) \in \mathcal{P}^*(\mathcal{V})$.

Hence $H_\mu(\mathcal{V}) = \min_{\{j_1,j_2,\cdots,j_r\}\subseteq\{1,2,\cdots,k\}} H_\mu(\beta(j_1, j_2, \cdots, j_r))$, where $\{j_1, j_2, \cdots, j_r\}$ are such that $|\{j_1, j_2, \cdots, j_r\}| = r \leq k$, $\bigcup_{i=1}^{r-1} V_{j_i} \neq X$ and $\bigcup_{i=1}^{r} V_{j_i} = X$. □

Some properties of the local measure-theoretic entropies relative to covers are summarized as follows.

Lemma 2.6. *The following holds for all $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$.*

1) $h_\mu(T, \mathcal{U}) = \frac{1}{M} h_\mu(T^M, \mathcal{U}_0^{M-1})$ *for all* $M \geq 1$;
2) $h^+_\mu(T, \mathcal{U}) \geq \frac{1}{M} h^+_\mu(T^M, \mathcal{U}_0^{M-1})$ *for all* $M \geq 1$;
3) $h_\mu(T, \mathcal{U}) = \lim_{M\to\infty} \frac{1}{M} h^+_\mu(T^M, \mathcal{U}_0^{M-1})$;
4) $h_\mu(T, \mathcal{U}) \leq h^+_\mu(T, \mathcal{U})$;
5) $h^+_\mu(T, \mathcal{U} \vee \mathcal{V}) \leq h^+_\mu(T, \mathcal{U}) + h^+_\mu(T, \mathcal{V})$ *and* $h_\mu(T, \mathcal{U} \vee \mathcal{V}) \leq h_\mu(T, \mathcal{U}) + h_\mu(T, \mathcal{V})$;
6) $h_\mu(T, \mathcal{U}) \geq h_\mu(T, \mathcal{V})$ *whenever* $\mathcal{U} \succeq \mathcal{V}$.

Proof. See [23]. □

Since a partition of X is also a cover, we have that $h_\mu(T) = \sup_{\mathcal{U}\in\mathcal{C}_X} h_\mu(T, \mathcal{U})$. In fact, the following holds.

Lemma 2.7. *For $\mu \in \mathcal{M}(X, T)$, $h_\mu(T) = \sup_{\mathcal{U}\in\mathcal{C}_X^0} h_\mu(T, \mathcal{U})$.*

Proof. Let $\mathcal{U} \in \mathcal{C}_X^0$ and α be the Borel partition generated by $\mathcal{U}$. Then $\alpha \succeq \mathcal{U}$ and hence $h_\mu(T) \geq h_\mu(T, \alpha) \geq h_\mu(T, \mathcal{U})$. It follows that $h_\mu(T) \geq \sup_{\mathcal{U}\in\mathcal{C}_X^0} h_\mu(T, \mathcal{U})$.

Conversely, let $\alpha = \{A_1, A_2, \cdots, A_k\} \in \mathcal{P}_X$ and $\epsilon > 0$. Then there exists a $\delta_1 = \delta_1(k, \epsilon) > 0$ such that whenever $\beta_1 = \{B^1_1, B^1_2, \cdots, B^1_k\}$ and $\beta_2 = \{B^2_1, B^2_2, \cdots, B^2_k\}$ are k-measurable partitions with $\sum_{i=1}^k \mu(B^1_i \Delta B^2_i) < \delta_1$, then $H_\mu(\beta_1|\beta_2) \leq \epsilon$ (see e.g., [27], Lemma 4.15).

Since μ is regular, we can take closed subsets $B_i \subset A_i$ with $\mu(A_i \setminus B_i) < \frac{\delta_1}{2k^2}$, $i = 1, 2, \cdots, k$. Let $B_0 = X \setminus \bigcup_{i=1}^k B_i$, $U_i = B_0 \cup B_i, i = 1, 2, \cdots, k$. Then $\mu(B_0) < \frac{\delta_1}{2k}$ and $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$ is an open cover of X.

For any integer $j \geq 0$ and any finite measurable partition β which is finer than $T^{-j}\mathcal{U}$ as a cover, we can find a measurable partition $\beta' = \{C_1, C_2, \cdots, C_k\}$ satisfying $C_i \subset T^{-j}U_i,\ i = 1, 2, \cdots, k$ and $\beta \succeq \beta'$. Hence $H_\mu(T^{-j}\alpha|\beta) \leq H_\mu(T^{-j}\alpha|\beta')$. Since $T^{-j}U_i \supset C_i \supset X \setminus \bigcup_{l\neq i} T^{-j}U_l = T^{-j}B_i$,

$$\mu(C_i \Delta T^{-j}A_i) \leq \mu(T^{-j}A_i \setminus T^{-j}B_i) + \mu(T^{-j}B_0) = \mu(A_i \setminus B_i) + \mu(B_0) < \frac{\delta_1}{2k} + \frac{\delta_1}{2k^2} \leq \frac{\delta_1}{k},$$

i.e., $\sum_{i=1}^k \mu(C_i \Delta T^{-j}A_i) < \delta_1$. It follows that $H_\mu(T^{-j}\alpha|\beta') \leq \epsilon$ and hence $H_\mu(T^{-j}\alpha|\beta) \leq \epsilon$. Above all, we have shown that there exists an open cover $\mathcal{U}$ of X consisting of k elements such that for any $j \geq 0$ and any k-measurable partition β which is finer than $T^{-j}\mathcal{U}$ as a cover, we have $H_\mu(T^{-j}\alpha|\beta) \leq \epsilon$.

Now for each $n \in \mathbb{N}$ and a finite measurable partition $\beta_n \succeq \mathcal{U}_0^{n-1}$, we have $\beta_n \geq T^{-j}\mathcal{U}$ for $j = 0, 1, \cdots, n-1$, and

$$\begin{aligned} H_\mu(\alpha_0^{n-1}) &\leq H_\mu(\beta_n) + H_\mu(\alpha_0^{n-1}|\beta_n) \\ &\leq H_\mu(\beta_n) + \sum_{j=0}^{n-1} H_\mu(T^{-j}\alpha|\beta_n) \\ &\leq H_\mu(\beta_n) + n\epsilon. \end{aligned}$$

Since β_n is arbitrary, $H_\mu(\alpha_0^{n-1}) \leq H_\mu(\mathcal{U}_0^{n-1}) + n\epsilon$. Hence

$$\begin{aligned} h_\mu(T,\alpha) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}) \leq \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}) + \epsilon \\ &= h_\mu(T,\mathcal{U}) + \epsilon \leq \sup_{\mathcal{U}\in\mathcal{C}_X^0} h_\mu(T,\mathcal{U}) + \epsilon. \end{aligned}$$

Since α and ϵ are arbitrary, $h_\mu(T) \leq \sup_{\mathcal{U}\in\mathcal{C}_X^0} h_\mu(T,\mathcal{U})$. □

2.3. **Conditional entropies.** For a non-empty set $Y \subset X$ and covers $\mathcal{U}$, $\mathcal{V} \in \mathcal{C}_X$, define

$$\begin{aligned} N(\mathcal{U}|Y) &= \min\{\mathrm{card}\,\mathcal{U}' : \mathcal{U}' \subset \mathcal{U},\ Y \subset \cup_{U'\in\mathcal{U}'} U'\}, \\ N(\mathcal{U}|\mathcal{V}) &= \max_{V\in\mathcal{V}} N(\mathcal{U}|V). \end{aligned}$$

Clearly, $N(\mathcal{U}|X) = N(\mathcal{U})$ and $N(\mathcal{U}|\{X\}) = N(\mathcal{U})$.

Lemma 2.8. *Let* $\mathcal{U}, \mathcal{V}, \mathcal{U}_1, \mathcal{V}_1 \in \mathcal{C}_X$. *Then*

$$\begin{aligned} &N(\mathcal{U}|\mathcal{V}) \leq N(\mathcal{U}_1|\mathcal{V}_1) \text{ for } \mathcal{U}_1 \succeq \mathcal{U} \text{ and } \mathcal{V} \succeq \mathcal{V}_1, \\ &N(T^{-1}\mathcal{U}|T^{-1}\mathcal{V}) = N(\mathcal{U}|\mathcal{V}), \\ &N(\mathcal{U}\vee\mathcal{U}_1|\mathcal{V}\vee\mathcal{V}_1) \leq N(\mathcal{U}|\mathcal{V})N(\mathcal{U}_1|\mathcal{V}_1). \end{aligned}$$

Proof. See [22]. □

Lemma 2.9. *For any* $\mu \in \mathcal{M}(X,T)$ *and* $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$,

$$H_\mu(\mathcal{V}) \leq H_\mu(\mathcal{U}) + \log N(\mathcal{V}|\mathcal{U}).$$

Proof. Let $\beta = \{B_1, B_2, \cdots, B_n\} \in \mathcal{P}_X$ with $\beta \succeq \mathcal{U}$ and denote $\mathcal{V} = \{V_1, V_2, \cdots, V_m\}$. For each $i = 1, 2, \cdots, n$, we choose $I_i = \{j_1^i, j_2^i, \cdots, j_{l_i}^i\} \subset \{1, 2, \cdots, m\}$ such that

$$l_i \leq N(\mathcal{V}|\beta) \text{ and } \bigcup_{r=1}^{l_i} V_{j_r^i} \supseteq B_i.$$

Let $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$, where $\gamma_i = \{V_{j_1^i} \cap B_i, (V_{j_2^i} \setminus V_{j_1^i}) \cap B_i, \cdots, (V_{j_{l_i}^i} \setminus \bigcup_{r=1}^{l_i-1} V_{j_r^i}) \cap B_i\}$, $i = 1, 2, \cdots, n$. It is easy to see that $\gamma \in \mathcal{P}_X$, $\gamma \succeq \mathcal{V}$ and $N(\gamma|\beta) = N(\mathcal{V}|\beta)$. Since $\beta \succeq \mathcal{U}$, we have by Lemma 2.8 that

$$N(\gamma|\beta) = N(\mathcal{V}|\beta) \leq N(\mathcal{V}|\mathcal{U}). \tag{2.5}$$

For simplicity, we denote $\gamma = \{A_1, A_2, \cdots, A_k\}$. It follows from (2.5) that

$$\begin{aligned} H_\mu(\gamma) &\leq H_\mu(\beta) + H_\mu(\gamma|\beta) \\ &\leq H_\mu(\beta) + \sum_{j=1}^{n} \mu(B_j)\Big(\sum_{i=1}^{k} -\frac{\mu(A_i\cap B_j)}{\mu(B_j)} \log \frac{\mu(A_i\cap B_j)}{\mu(B_j)}\Big) \\ &= H_\mu(\beta) + \sum_{j=1}^{n} \mu(B_j)\Big(\sum_{i, A_i\cap B_j\neq\emptyset} -\frac{\mu(A_i\cap B_j)}{\mu(B_j)} \log \frac{\mu(A_i\cap B_j)}{\mu(B_j)}\Big) \\ &\leq H_\mu(\beta) + \sum_{j=1}^{n} \mu(B_j) \log |\{i \in \{1, 2, \cdots, k\} : A_i \cap B_j \neq \emptyset\}| \end{aligned}$$

$$= H_\mu(\beta) + \sum_{j=1}^{n} \mu(B_j) \log N(\gamma|B_j)$$
$$\leq H_\mu(\beta) + \log N(\gamma|\beta) \leq H_\mu(\beta) + \log N(\mathcal{V}|\mathcal{U}).$$

Hence $H_\mu(\mathcal{V}) \leq H_\mu(\gamma) \leq H_\mu(\beta) + \log N(\mathcal{V}|\mathcal{U})$. The lemma follows since β is arbitrary. □

For any covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, it follows from Lemma 2.8 that the sequence $\{\log N(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1})\}_{n=1}^\infty$ is sub-additive. Hence the quantity

$$h(T, \mathcal{U}|\mathcal{V}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U}_0^{n-1}|\mathcal{V}_0^{n-1}),$$

called the *conditional entropy of* $\mathcal{U}$ *with respect to* $\mathcal{V}$, is well defined, and moreover,

$$h(T, \mathcal{U}|\mathcal{V}) \leq h(T, \mathcal{U}_1|\mathcal{V}_1) \text{ whenever } \mathcal{U}_1 \succeq \mathcal{U} \text{ and } \mathcal{V} \succeq \mathcal{V}_1. \tag{2.6}$$

It is clear that if $\mathcal{U} \in \mathcal{C}_X^o$ then $h(T, \mathcal{U}|\{X\}) = h_{\mathrm{top}}(T, \mathcal{U})$.

Let

$$h(T|\mathcal{V}) = \sup_{\mathcal{U}\in\mathcal{C}_X^o} h(T, \mathcal{U}|\mathcal{V})$$

be the *conditional entropy of* T *with respect to a cover* $\mathcal{V} \in \mathcal{C}_X$, and

$$h^*(T) = \inf_{\mathcal{V}\in\mathcal{C}_X^o} h(T|\mathcal{V})$$

be the *conditional topological entropy of* T. Then by (2.6),

$$h(T|\mathcal{V}) \leq h(T|\mathcal{V}_1) \text{ whenever } \mathcal{V} \succeq \mathcal{V}_1. \tag{2.7}$$

Clearly, $h(T|\{X\}) = h_{\mathrm{top}}(T)$.

Lemma 2.10. *For any* $\mu \in \mathcal{M}(X, T)$ *and* $\mathcal{U} \in \mathcal{C}_X$,

$$h_\mu(T) \leq h_\mu(T, \mathcal{U}) + h(T|\mathcal{U}).$$

Proof. Let $\mathcal{V} \in \mathcal{C}_X^o$. By Lemma 2.9,

$$H_\mu(\mathcal{V}_0^{n-1}) \leq H_\mu(\mathcal{U}_0^{n-1}) + \log N(\mathcal{V}_0^{n-1}|\mathcal{U}_0^{n-1}) \tag{2.8}$$

for all $n \in \mathbb{N}$. By dividing (2.8) by n then passing the limit $n \to \infty$, we obtain that

$$h_\mu(T, \mathcal{V}) \leq h_\mu(T, \mathcal{U}) + h(T, \mathcal{V}|\mathcal{U}) \leq h_\mu(T, \mathcal{U}) + h(T|\mathcal{U}).$$

It follows from Lemma 2.7 that $h_\mu(T) = \sup_{\mathcal{V}\in\mathcal{C}_X^o} h_\mu(T, \mathcal{V}) \leq h_\mu(T, \mathcal{U}) + h(T|\mathcal{U})$. □

3. Upper semi-continuity and affinity of a local entropy map

The section is devoted to the proof of Theorem 1. Throughout the section, we let (X, T) be a TDS.

For a fixed $\mathcal{U} = \{U_1, U_2, \cdots, U_M\} \in \mathcal{C}_X^o$, we let $\mathcal{U}^* = \{\{A_1, A_2, \cdots, A_M\} \in \mathcal{P}_X : A_m \subseteq U_m, m \in \{1, \cdots, M\}\}$, where A_m can be empty for some values of $m \in \{1, 2, \cdots, M\}$.

The following lemma will be used in the computation of $H_\mu(\mathcal{U})$ and $h_\mu(T, \mathcal{U})$.

Lemma 3.1. *Let* $G : \mathcal{P}_X \to \mathbb{R}$ *be monotone in the sense that* $G(\alpha) \geq G(\beta)$ *whenever* $\alpha \succeq \beta$. *Then*

$$\inf_{\alpha\in\mathcal{P}_X, \alpha\succeq\mathcal{U}} G(\alpha) = \inf_{\alpha\in\mathcal{U}^*} G(\alpha).$$

Proof. See [17], Lemma 2. □

Lemma 3.2. *Let* $K \subseteq X$ *be a closed subset and* $\{U_i^0\}_{i=1}^n$ *be a set of non-empty open subsets of* X *which covers* K. *Then for any* $\delta > 0$ *there exist open subsets* $\{V_i^0\}_{i=1}^n$ *of* X *such that* $\{V_i^0\}_{i=1}^n$ *covers* K, $V_i^0 \subset U_i^0$, $\mu(\partial V_i^0) = 0$, *and* $\mu(U_i^0 \Delta V_i^0) < \delta$ *for all* $i = 1, 2, \cdots, n$.

Proof. We prove the lemma by induction on n. When $n = 1$, we take a closed set K' such that $K \subseteq K' \subseteq U_1^0$ and $\mu(U_1^0 \setminus K') < \delta$. For each $x \in K'$ there exists an $\epsilon_x > 0$ such that $\mu(\partial B(x, \epsilon_x)) = 0$ and $B(x, \epsilon_x) \subset U_1^0$, where $B(x, \epsilon_x) = \{y \in X : d(x, y) < \epsilon_x\}$. Hence $\{B(x, \epsilon_x)\}_{x \in K'}$ is an open cover of K. Using compactness of K, we can find a finite sub-cover $\{B(x_i, \epsilon_{x_i})\}_{i=1}^k$, for some k, such that $x_i \in K'$, $i = 1, 2, \cdots, k$, and $\bigcup_{i=1}^k B(x_i, \epsilon_{x_i}) \supseteq K'$. Let $V_1^0 = \bigcup_{i=1}^k B(x_i, \epsilon_{x_i})$. Then V_1^0 satisfies the properties stated in the lemma.

Now, assume that for some positive integer m the lemma holds for $n = m$. We consider the case $n = m + 1$. Let $K_m = K \cap (X \setminus U_{m+1}^0)$. For any $\delta > 0$, since $\{U_i^0\}_{i=1}^m$ covers K_m, by induction hypothesis there exist open subsets $V_1^0, V_2^0, \cdots, V_m^0$ of X such that $\{V_i^0\}_{i=1}^m$ covers K_m, $V_i^0 \subset U_i^0$, $\mu(\partial V_i^0) = 0$, and $\mu(U_i^0 \Delta V_i^0) < \delta$ for all $i = 1, 2, \cdots, m$.

Let $K'_m = K \cap (X \setminus \bigcup_{i=1}^m V_i^0)$. Then K'_m is a closed subset of X and $K'_m \subseteq U_{m+1}^0$. By induction hypothesis there exists an open subset V_{m+1}^0 such that $K'_m \subseteq V_{m+1}^0 \subseteq U_{m+1}^0$, $\mu(\partial V_{m+1}^0) = 0$, and $\mu(U_{m+1}^0 \Delta V_{m+1}^0) < \delta$. As $\{V_i^0\}_{i=1}^{m+1}$ covers K, the lemma holds for $n = m + 1$. □

Lemma 3.3. *If $\mathcal{U} = \{U_1, U_2, \cdots, U_M\} \in \mathcal{C}_X^o$, then for any $\epsilon > 0$ there exists a $\delta = \delta(M, \epsilon) > 0$ such that if $\mathcal{V} = \{V_1, V_2, \cdots, V_M\} \in \mathcal{C}_X^o$ with $\sum_{i=1}^M \mu(U_i \Delta V_i) < \delta$, then*

$$|H_\mu(\mathcal{U}) - H_\mu(\mathcal{V})| \leq \epsilon.$$

Proof. By Lemma 4.15 in [27], there exists a $\delta' = \delta'(M, \varepsilon) > 0$ such that whenever $\alpha, \beta \in \mathcal{P}_X$ are two M-measurable partitions with $\mu(\alpha \Delta \beta) < \delta'$ then $|H_\mu(\beta) - H_\mu(\alpha)| < \varepsilon$.

Let $\delta = \frac{\delta'}{M}$ and $\mathcal{V} = \{V_1, V_2, \cdots, V_M\}$ be a measurable cover of X such that $\mu(\mathcal{U} \Delta \mathcal{V}) < \delta$.

We first *claim* that for every M-measurable partition $\alpha \succeq \mathcal{U}$ there exists a finite measurable partition $\beta \succeq \mathcal{V}$ such that $H_\mu(\alpha) \geq H_\mu(\beta) - \epsilon$.

By Lemma 3.1, it is sufficient to prove the claim for $\alpha = \{A_1, A_2, \cdots, A_M\} \in \mathcal{U}^*$. Define the partition $\beta = \{B_1, B_2, \cdots, B_M\} \in \mathcal{V}^*$:

$$\begin{aligned} B_1 &= V_1 \setminus (\bigcup_{k>1} (A_k \cap V_k)), \\ B_i &= V_i \setminus (\bigcup_{k>i} (A_k \cap V_k) \cup \bigcup_{j<i} B_j), \ i = 2, 3, \cdots, M. \end{aligned}$$

For each $m = 1, 2, \cdots, M$, we clearly have $A_m \cap V_m \subseteq B_m$, and hence $A_m \setminus B_m \subseteq A_m \setminus (A_m \cap V_m) \subseteq U_m \setminus V_m$ and $B_m \setminus A_m \subseteq \bigcup_{k \neq m} (A_k \setminus B_k) \subseteq \bigcup_{k \neq m} (U_k \setminus V_k)$. It follows that $A_m \Delta B_m \subseteq \bigcup_{k=1}^M (U_k \Delta V_k)$, $m = 1, 2, \cdots, M$, i.e., $\mu(\alpha \Delta \beta) \leq M \mu(\mathcal{U} \Delta \mathcal{V}) < \delta'$, from which the claim follows.

Now, for each $\alpha \succeq \mathcal{U}$, we let β be as in the above claim. Then

$$H_\mu(\alpha) \geq H_\mu(\beta) - \varepsilon \geq H_\mu(\mathcal{V}) - \varepsilon.$$

Since such an $\alpha \succeq \mathcal{U}$ is arbitrary, we have that $H_\mu(\mathcal{U}) \geq H_\mu(\mathcal{V}) - \varepsilon$. Exchanging the roles of $\mathcal{U}$ and $\mathcal{V}$ implies that $H_\mu(\mathcal{V}) \geq H_\mu(\mathcal{U}) - \varepsilon$. Hence $|H_\mu(\mathcal{U}) - H_\mu(\mathcal{V})| \leq \varepsilon$. □

We note that under the weak*-topology $\mathcal{M}(X, T)$ is a compact metric space and $\mathcal{M}^e(X, T)$ is a G_δ-subset of $\mathcal{M}(X, T)$. For each $\mu \in \mathcal{M}(X, T)$, there exists a unique Borel probability measure m on $\mathcal{M}^e(X, T)$ such that $\mu = \int_{\mathcal{M}^e(X,T)} \theta dm(\theta)$, i.e., μ admits an *ergodic decomposition*. The ergodic decomposition of μ also gives rise to an ergodic decomposition of the μ-entropy relative to $\alpha \in \mathcal{P}_X$:

$$h_\mu(T, \alpha) = \int_{\mathcal{M}^e(X,T)} h_\theta(T, \alpha) dm(\theta). \tag{3.1}$$

In fact, ergodic decompositions of the μ-entropies relative to $\mathcal{U} \in \mathcal{C}_X$ also hold.

Lemma 3.4. *Let $\mu \in \mathcal{M}(X, T)$ and $\mathcal{U} \in \mathcal{C}_X$. If $\mu = \int_{\mathcal{M}^e(X,T)} \theta dm(\theta)$ is the ergodic decomposition of μ, then*

$$h_\mu^+(T, \mathcal{U}) = \int_{\mathcal{M}^e(X,T)} h_\theta^+(T, \mathcal{U}) dm(\theta) \ \text{and} \ h_\mu(T, \mathcal{U}) = \int_{\mathcal{M}^e(X,T)} h_\theta(T, \mathcal{U}) dm(\theta).$$

Proof. See [17], Proposition 5. □

We are now ready to prove Theorem 1, i.e., *for any* $\mathcal{U} \in \mathcal{C}_X^o$*, the entropy map* $h_{\{\cdot\}}(T,\mathcal{U}) : \mathcal{M}(X,T) \to \mathbb{R}_+$ *is upper semi-continuous and affine.*

Proof of Theorem 1. We first prove the upper semi-continuity. Fix a $\mu_0 \in \mathcal{M}(X,T)$ and let $\epsilon > 0$. We let $N \in \mathbb{N}$ be sufficiently large such that

$$\frac{1}{N} H_{\mu_0}(\mathcal{U}_0^{N-1}) < h_{\mu_0}(T,\mathcal{U}) + \epsilon.$$

By Lemmas 3.2 and 3.3, there exists a finite open cover $\mathcal{V}_N \succeq \mathcal{U}_0^{N-1}$ such that

$$H_{\mu_0}(\mathcal{V}_N) < H_{\mu_0}(\mathcal{U}_0^{N-1}) + \epsilon \text{ and } \mu_0(\partial V) = 0 \text{ for all } V \in \mathcal{V}_N.$$

Then by Lemma 2.5,

$$H_\mu(\mathcal{V}_N) = \min_{\beta \in \mathcal{P}^*(\mathcal{V}_N)} H_\mu(\beta) \text{ for all } \mu \in \mathcal{M}(X,T). \tag{3.2}$$

Since $\mu_0(\partial V) = 0$ for each $V \in \mathcal{V}_N$, we have that $\mu_0(\partial B) = 0$ for any $B \in \beta$ when $\beta \in \mathcal{P}^*(\mathcal{V}_N)$. It follows from (3.2) and the finiteness of $\mathcal{P}^*(\mathcal{V}_N)$ that the function $H_{\{\cdot\}}(\mathcal{V}_N) : \mathcal{M}(X,T) \to \mathbb{R}_+$ is continuous at μ_0. Hence

$$\begin{aligned} \limsup_{\mu\to\mu_0} h_\mu(T,\mathcal{U}) &\leq \limsup_{\mu\to\mu_0} \tfrac{1}{N} H_\mu(\mathcal{U}_0^{N-1}) \leq \limsup_{\mu\to\mu_0} \tfrac{1}{N} H_\mu(\mathcal{V}_N) \\ &= \tfrac{1}{N} H_{\mu_0}(\mathcal{V}_N) \leq \tfrac{1}{N} H_{\mu_0}(\mathcal{U}_0^{N-1}) + \epsilon \\ &\leq h_{\mu_0}(T,\mathcal{U}) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the entropy map $h_{\{\cdot\}}(T,\mathcal{U})$ is upper semi-continuous at $\mu_0 \in \mathcal{M}(X,T)$.

We now prove the affinity. Given $\mu_1, \mu_2 \in \mathcal{M}(X,T)$ and $\lambda \in (0,1)$. Let $\mu_i = \int_{\mathcal{M}^e(X,T)} \theta dm_i(\theta)$ be the ergodic decomposition of μ_i, $i = 1,2$. Consider $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ and $m = \lambda m_1 + (1-\lambda)m_2$. It is clear that m is a Borel probability measure on $\mathcal{M}^e(X,T)$ and $\mu = \int_{\mathcal{M}^e(X,T)} \theta dm(\theta)$. By Lemma 3.4,

$$\begin{aligned} h_\mu(T,\mathcal{U}) &= \textstyle\int_{\mathcal{M}^e(X,T)} h_\theta(T,\mathcal{U}) dm(\theta) \\ &= \lambda \textstyle\int_{\mathcal{M}^e(X,T)} h_\theta(T,\mathcal{U}) dm_1(\theta) + (1-\lambda) \textstyle\int_{\mathcal{M}^e(X,T)} h_\theta(T,\mathcal{U}) dm_2(\theta) \\ &= \lambda h_{\mu_1}(T,\mathcal{U}) + (1-\lambda) h_{\mu_2}(T,\mathcal{U}). \end{aligned}$$

This shows that the entropy map $h_{\{\cdot\}}(T,\mathcal{U})$ is affine on $\mathcal{M}(X,T)$. □

4. A local variational principle of pressure

Our aim in this section is to prove Theorem 2. Let (X,T) and (Y,S) be two TDS. A continuous map $\pi : X \to Y$ is called *a homomorphism* or *a factor map* from (X,T) to (Y,S) if it is onto and $\pi T = S\pi$. (X,T) is called an *extension* of (Y,S) and (Y,S) is called a *factor* of (X,T). If π is also injective then it is called an *isomorphism*.

Lemma 4.1. *Let* $\pi : (X,T) \to (Y,S)$ *be a factor map between two TDS and* $\mathcal{U} \in \mathcal{C}_Y^o$*. Then for any* $\mu \in \mathcal{M}(X,T)$*,* $h_\mu(T,\pi^{-1}\mathcal{U}) = h_{\pi\mu}(S,\mathcal{U})$*.*

Proof. See [23], Proposition 6. □

Lemma 4.2. *Let* $\pi : (X,T) \to (Y,S)$ *be a factor map between two TDS,* $f \in C(Y,\mathbb{R})$ *and* $\mathcal{U} \in \mathcal{C}_Y^o$*. Then* $P(T, f\circ\pi; \pi^{-1}\mathcal{U}) = P(S,f;\mathcal{U})$*.*

Proof. Fix an $n \in \mathbb{N}$. If $\mathcal{V} \in \mathcal{C}_Y$, $\mathcal{V} \succeq \mathcal{U}_0^{n-1}$, then $\pi^{-1}\mathcal{V} \in \mathcal{C}_X$ and $\pi^{-1}\mathcal{V} \succeq (\pi^{-1}\mathcal{U})_0^{n-1}$. Hence

$$\sum_{V\in\mathcal{V}} \sup_{y\in V} e^{f_n(y)} = \sum_{V\in\mathcal{V}} \sup_{z\in\pi^{-1}V} e^{(f\circ\pi)_n(z)} \geq P_n(T, f\circ\pi; \pi^{-1}\mathcal{U}).$$

Since $\mathcal{V}$ is arbitrary, we have that $P_n(S,f;\mathcal{U}) \geq P_n(T, f\circ\pi; \pi^{-1}\mathcal{U})$.

Conversely, we note that $P_n(T, f \circ \pi; \pi^{-1}\mathcal{U}) = \inf_{\beta \in \mathcal{P}^*((\pi^{-1}\mathcal{U})_0^{n-1})} \{\sum_{B\in\beta} \sup_{x\in B} e^{f_n(x)}\}$. Let $\beta = \{B_1, B_2, \cdots, B_m\} \in \mathcal{P}^*((\pi^{-1}\mathcal{U})_0^{n-1})$. Since $\pi\beta = \{\pi(B_1), \pi(B_2), \cdots, \pi(B_m)\} \in \mathcal{C}_Y$ and $\pi\beta \succeq \mathcal{U}_0^{n-1}$,

$$\sum_{i=1}^m \sup_{x\in B_i} e^{(f\circ\pi)_n(x)} = \sum_{i=1}^m \sup_{y\in\pi(B_i)} e^{f_n(y)} \geq P_n(S, f; \mathcal{U}).$$

Since β is arbitrary, $P_n(T, f \circ \pi; \pi^{-1}\mathcal{U}) \geq P_n(S, f; \mathcal{U})$.

Above all, $P_n(T, f \circ \pi; \pi^{-1}\mathcal{U}) = P_n(S, f; \mathcal{U})$ for each $n \in \mathbb{N}$, from which the lemma follows. □

Lemma 4.3. *Let $a_1, a_2, \cdots, a_k$ be given real numbers. If $p_i \geq 0$, $i = 1, 2, \cdots, k$, and $\sum_{i=0}^k p_i = 1$, then*

$$\sum_{i=1}^k p_i(a_i - \log p_i) \leq \log(\sum_{i=1}^k e^{a_i}),$$

and equality holds iff $p_i = \dfrac{e^{a_i}}{\sum_{i=1}^k e^{a_i}}$ for all $i = 1, 2, \cdots, k$.

Proof. See [27], Lemma 9.9. □

Proposition 4.1. *Let (X, T) be a TDS, $f \in C(X, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. Then for any $\mu \in \mathcal{M}(X, T)$,*

$$P(T, f; \mathcal{U}) \geq h_\mu(T, \mathcal{U}) + \int_X f(x)d\mu(x).$$

Proof. Let $\mu \in \mathcal{M}(X, T)$. For any $n \in \mathbb{N}$ and $\epsilon > 0$, we have by (2.2) that there exists a finite partition $\beta \in \mathcal{P}^*(\mathcal{U}_0^{n-1})$ such that $\sum_{B\in\beta} \sup_{x\in B} e^{f_n(x)} = P_n(T, f; \mathcal{U})$. It follows from Lemma 4.3 that

$$\begin{aligned}
\log(P_n(T, f; \mathcal{U})) &= \log(\sum_{B\in\beta} \sup_{x\in B} e^{f_n(x)}) \\
&\geq \sum_{B\in\beta} \mu(B)(\sup_{x\in B} f_n(x) - \log\mu(B)) \\
&= H_\mu(\beta) + \sum_{B\in\beta} \sup_{x\in B} f_n(x) \cdot \mu(B) \\
&\geq H_\mu(\beta) + \textstyle\int_X f_n(x)d\mu(x) \\
&\geq H_\mu(\mathcal{U}_0^{n-1}) + n\textstyle\int_X f(x)d\mu(x).
\end{aligned}$$

The proof is complete by dividing the above by n then passing the limit $n \to \infty$. □

A subset A of X is called *clopen* if it is both closed and open in X. A partition is called *clopen* if it consists of clopen sets.

Lemma 4.4. *Let (X, T) be a zero-dimensional TDS, $\mu \in \mathcal{M}(X, T)$, $f \in C(X, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^0$. Assume that for some $K \in \mathbb{N}$, $\{\alpha_l\}_{l=1}^K$ is a sequence of finite clopen partitions of X which are finer than $\mathcal{U}$. Then for each $N \in \mathbb{N}$, there exists a finite subset B_N of X such that each atom of $(\alpha_l)_0^{N-1}$, $l = 1, 2, \cdots, K$, contains at most one point of B_N, and, $\sum_{x\in B_N} e^{f_N(x)} \geq \frac{P_N(T,f;\mathcal{U})}{K}$.*

Proof. For each $x \in X$ and $l = 1, 2, \cdots, K$, we let $A_l(x)$ be the atom of $(\alpha_l)_0^{N-1}$ containing x. Then for any $x_1, x_2 \in X$ and $l = 1, 2, \cdots, K$, x_1 and x_2 are contained in the same atom of $(\alpha_l)_0^{N-1}$ iff $A_l(x_1) = A_l(x_2)$.

To construct the set B_N we first let $x_1 \in X$ be such that $e^{f_N(x_1)} = \max_{x\in X} e^{f_N(x)}$. If $\bigcup_{l=1}^K A_l(x_1) = X$, then we take $B_N = \{x_1\}$. Otherwise, we take $X_1 = X \setminus \bigcup_{l=1}^K A_l(x_1)$. In either cases, X_1 is closed subset of X. Next, let $x_2 \in X_1$ be such that $e^{f_N(x_2)} = \max_{x\in X_1} e^{f_N(x)}$. If $\bigcup_{l=1}^K A_l(x_2) \supseteq X_1$, then we take $B_N = \{x_1, x_2\}$. Otherwise, we take $X_2 = X_1 \setminus \bigcup_{l=1}^K A_l(x_2)$. In either cases, X_2 is a closed subset of X. Since $\{A_l(x) : 1 \leq l \leq K, x \in X\}$ is a finite set, we can continue the above procedure inductively to obtain a set $B_N = \{x_1, x_2, \cdots x_m\}$ and non-empty closed sets X_j, $j = 1, 2, \cdots, m-1$, such that

(1) $e^{f_N(x_1)} = \max_{x\in X} e^{f_N(x)}$ and $X_1 = X \setminus \bigcup_{l=1}^K A_l(x_1)$,

(2) $e^{f_N(x_{j+1})} = \max_{x\in X_j} e^{f_N(x)}$ and $X_{j+1} = X_j \setminus \bigcup_{l=1}^K A_l(x_{j+1})$ for $j = 1, 2, \cdots, m-1$, and,

(3) $\bigcup_{j=1}^m \bigcup_{l=1}^K A_l(x_j) = X$.

From the construction of B_N, it is easy to see that each atom of $(\alpha_l)_0^{N-1}$, $l = 1, 2, \cdots, K$, contains at most one point of B_N. Now let $\beta = \{X_{j-1} \cap A_l(x_j)\}_{1\le j\le m, 1\le l\le K}$, where $X_0 = X$. Then β is a cover of X which is finer than $\mathcal{U}_0^{n-1}$. Therefore,

$$\sum_{x\in B_N} e^{f_N(x)} = \sum_{j=1}^m e^{f_N(x_j)} \ge \sum_{j=1}^m \frac{1}{K} \sum_{l=1}^K \sup_{x\in A_l(x_j)\cap X_{j-1}} e^{f_N(x)} = \frac{1}{K} \sum_{B\in\beta} \sup_{x\in B} e^{f_N(x)} \ge \frac{P_N(T, f; \mathcal{U})}{K}.$$

□

Proposition 4.2. *Let (X, T) be an invertible, zero-dimensional TDS, $f \in C(X, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^0$. Then there exists a $\mu \in \mathcal{M}(X, T)$ such that*

$$h_\mu^+(T, \mathcal{U}) + \int_X f(x)d\mu(x) \ge P(T, f; \mathcal{U}). \tag{4.1}$$

Proof. Let $\mathcal{U} = \{U_1, U_2, \cdots, U_d\}$ and define

$$\mathcal{U}^* = \{\alpha \in \mathcal{P}_X : \alpha = \{A_1, A_2, \cdots, A_d\},\ A_m \subset U_m,\ m = 1, 2, \cdots, d\}.$$

Since X is zero-dimensional, the family of partitions in $\mathcal{U}^*$, which are finer than $\mathcal{U}$ and consist of clopen sets, is countable. We let $\{\alpha_l : l \ge 1\}$ denote an enumeration of this family.

Let $n \in \mathbb{N}$. By Lemma 4.4, there exists a finite subset B_n of X such that

$$\sum_{x\in B_n} e^{f_n(x)} \ge \frac{P_n(T, f; \mathcal{U})}{n}, \tag{4.2}$$

and, each atom of $(\alpha_l)_0^{n-1}$ contains at most one point of B_n, for all $l = 1, 2, \cdots, n$. Let

$$\nu_n = \sum_{x\in B_n} \lambda_n(x)\delta_x,$$

where $\lambda_n(x) = \frac{e^{f_n(x)}}{\sum_{y\in B_n} e^{f_n(y)}}$ for $x \in B_n$, and let $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i\nu_n$. Since $\mathcal{M}(X, T)$ is compact we can choose a subsequence $\{n_j\}$ of natural numbers such that $\mu_{n_j} \to \mu$ in the weak*-topology of $\mathcal{M}(X, T)$.

We wish to show that μ satisfies (4.1). By Lemma 3.1 and the fact that

$$h_\mu^+(T, \mathcal{U}) = \inf_{\beta\succeq\mathcal{U}^*} h_\mu(T, \beta) = \inf_{l\in\mathbb{N}} h_\mu(T, \alpha_l),$$

it is sufficient to show that

$$P(T, f; \mathcal{U}) \le h_\mu(T, \alpha_l) + \int_X f(x)d\mu(x)$$

for each $l \in \mathbb{N}$.

Fix a $l \in \mathbb{N}$. For each $n > l$, we know from the construction of B_n that each atom of $(\alpha_l)_0^{n-1}$ contains at most one point in B_n, and,

$$\begin{aligned}\sum_{x\in B_n} -\lambda_n(x)\log\lambda_n(x) &= \textstyle\sum_{x\in B_n} -\nu_n(\{x\})\log\nu_n(\{x\}) \\ &= H_{\nu_n}((\alpha_l)_0^{n-1}).\end{aligned} \tag{4.3}$$

Moreover, it follow from (4.2), (4.3) that

$$\begin{aligned}\log P_n(T, f; \mathcal{U}) - \log n &\le \log(\textstyle\sum_{x\in B_n} e^{f_n(x)}) \\ &= \textstyle\sum_{x\in B_n} \lambda_n(x)(f_n(x) - \log\lambda_n(x)) \\ &= H_{\nu_n}((\alpha_l)_0^{n-1}) + \textstyle\sum_{x\in B_n} \lambda_n(x)f_n(x) \\ &= H_{\nu_n}((\alpha_l)_0^{n-1}) + \textstyle\int_X f_n(x)d\nu_n(x). \\ &= H_{\nu_n}((\alpha_l)_0^{n-1}) + n\textstyle\int_X f(x)d\mu_n(x).\end{aligned}$$

Hence

$$\log P_n(T,f;\mathcal{U}) - \log n \le H_{\nu_n}((\alpha_l)_0^{n-1}) + n\int_X f(x)d\mu_n(x). \tag{4.4}$$

Fix natural numbers m, n with $n > l$ and $1 \le m \le n-1$. Let $a(j) = [\frac{n-j}{m}]$, $j = 0, 1, \cdots, m-1$, where $[a]$ denotes the integral part of a real number a. Then

$$\bigvee_{i=0}^{n-1} T^{-i}\alpha_l = \bigvee_{r=0}^{a(j)-1} T^{-(mr+j)}(\alpha_l)_0^{m-1} \vee \bigvee_{t\in S_j} T^{-t}\alpha_l, \tag{4.5}$$

where $S_j = \{0, 1, \cdots, j-1\} \cup \{j + ma(j), \cdots, n-1)\}$. Since $|S_j| \le 2m$, it follows from (4.4) and (4.5) that

$$\begin{aligned} &\log P_n(T,f;\mathcal{U}) - \log n \le H_{\nu_n}((\alpha_l)_0^{n-1}) + n\textstyle\int_X f(x)d\mu_n(x) \\ \le &\sum_{r=0}^{a(j)-1} H_{\nu_n}(T^{-(mr+j)}(\alpha_l)_0^{m-1}) + H_{\nu_n}(\bigvee_{l\in S_j} T^{-l}\alpha_l) + n\textstyle\int_X f(x)d\mu_n(x) \\ \le &\sum_{r=0}^{a(j)-1} H_{T^{(mr+j)}\nu_n}((\alpha_l)_0^{m-1}) + n\textstyle\int_X f(x)d\mu_n(x) + 2m\log d, \end{aligned} \tag{4.6}$$

where d is the cardinality of $\mathcal{U}$. Summing up (4.6) over j from 0 to $m-1$ then dividing the sum by m yields that

$$\begin{aligned} \log P_n(T,f;\mathcal{U}) - \log n &\le \frac{1}{m}\sum_{j=0}^{m-1}\sum_{r=0}^{a(j)-1} H_{T^{(mr+j)}\nu_n}((\alpha_l)_0^{m-1}) + n\textstyle\int_X f(x)d\mu_n(x) + 2m\log d \\ &\le \frac{1}{m}\sum_{j=0}^{n-1} H_{T^j\nu_n}((\alpha_l)_0^{m-1}) + n\textstyle\int_X f(x)d\mu_n(x) + 2m\log d, \end{aligned}$$

i.e.,

$$\log P_n(T,f;\mathcal{U}) \le \frac{1}{m}\sum_{j=0}^{n-1} H_{T^j\nu_n}((\alpha_l)_0^{m-1}) + n\textstyle\int_X f(x)d\,\mu_n(x) + (2m\log d + \log n). \tag{4.7}$$

Since $H_{\{\cdot\}}((\alpha_l)_0^{m-1})$ is concave on $\mathcal{M}(X,T)$,

$$\frac{1}{n}\sum_{j=0}^{n-1} H_{T^j\nu_n}((\alpha_l)_0^{m-1}) \le H_{\mu_n}((\alpha_l)_0^{m-1}). \tag{4.8}$$

Now by dividing (4.7) by n then combining it with (4.8), we obtain

$$\frac{1}{n}\log P_n(T,f;\mathcal{U}) \le \frac{1}{m}H_{\mu_n}((\alpha_l)_0^{m-1}) + \int_X f(x)d\mu_n(x) + \frac{(2m\log d + \log n)}{n}. \tag{4.9}$$

Since α_l is clopen, it follows that

$$\lim_{j\to\infty} H_{\mu_{n_j}}((\alpha_l)_0^{m-1}) = H_\mu((\alpha_l)_0^{m-1}).$$

By substituting n with n_j in (4.9) and passing the limit $j \to \infty$, we have that

$$\begin{aligned} P(T,f;\mathcal{U}) &= \lim_{j\to\infty} \frac{1}{n_j}\log P_{n_j}(T,f;\mathcal{U}) \\ &\le \lim_{j\to\infty}\left(\frac{1}{m}H_{\mu_{n_j}}((\alpha_l)_0^{m-1}) + \textstyle\int_X f(x)d\mu_{n_j}(x) + \frac{(2m\log d + \log n_j)}{n_j}\right) \\ &= \frac{1}{m}H_\mu((\alpha_l)_0^{m-1}) + \textstyle\int_X f(x)d\mu(x). \end{aligned} \tag{4.10}$$

The proof is now complete by passing the limit $m \to \infty$ in (4.10). □

Proposition 4.3. *Let (X,T) be an invertible TDS, $f \in C(X,\mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. Then there exists a $\mu \in \mathcal{M}(X,T)$ such that*

$$h_\mu(T,\mathcal{U}) + \int_X f(x)d\mu(x) = P(T,f;\mathcal{U}).$$

Proof. We follow the arguments in the proof of Theorem 4, [17]. Let $\mathcal{U} = \{U_1, U_2, \cdots, U_M\} \in \mathcal{C}_X^o$.

We first consider the case that X is zero-dimensional, i.e., there exists a fundamental base of the topology made of clopen sets. Since the set of clopen subsets of X is countable, the family of partitions in $\mathcal{U}^*$ consisting of clopen sets is countable. Let $\{\alpha_l : l = 1, 2, \cdots\}$ be an enumeration of this family. Then, for any $k \in \mathbb{N}$ and $\mu \in \mathcal{M}(X, T)$,

$$h^+_\mu(T^k, \bigvee_{i=0}^{k-1} T^{-i}\mathcal{U}) = \inf_{s_k \in \mathbb{N}^k} h_\mu(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}). \tag{4.11}$$

For any $k \in \mathbb{N}$ and $s_k \in \mathbb{N}^k$, let

$$\begin{aligned} M(k, s_k) = \quad & \{\mu \in \mathcal{M}(X,T) : \tfrac{1}{k}(h_\mu(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}) + \textstyle\int_X f_k(x)d\mu(x)) \\ & \geq \tfrac{1}{k}P(T^k, f_k; \mathcal{U}_0^{k-1})\}. \end{aligned}$$

We note by Lemma 2.3 that $\frac{1}{k}P(T^k, f_k; \mathcal{U}_0^{k-1}) = P(T, f; \mathcal{U})$.

By Proposition 4.2, there exists a $\mu_k \in \mathcal{M}(X, T^k)$ such that

$$h^+_{\mu_k}(T^k, \mathcal{U}_0^{k-1}) + \int_X f_k(x)d\mu_k(x) \geq P(T^k, f_k; \mathcal{U}_0^{k-1}).$$

Since $\bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}$ is finer than $\mathcal{U}_0^{k-1}$ for each $s_k \in \mathbb{N}^k$, we have

$$h_{\mu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}) + \int_X f_k(x)d\mu_k(x) \geq P(T^k, f_k; \mathcal{U}_0^{k-1}). \tag{4.12}$$

Let $\nu_k = \frac{\mu_k + T\mu_k + \cdots + T^{k-1}\mu_k}{k}$. Since $T^i\mu_k \in \mathcal{M}(X, T^k)$, $i = 0, 1, \cdots, k-1$, we have $\nu_k \in \mathcal{M}(X, T)$. For $s_k \in \mathbb{N}^k$ and $j = 1, 2, \cdots, k-1$, let

$$\begin{aligned} P^0 s_k &= s_k \\ P^j s_k &= \underbrace{s_k(k-j)s_k(k-j+1)\cdots s_k(k-1)}_{j}\underbrace{s_k(0)s_k(1)\cdots s_k(k-1-j)}_{k-j} \in \mathbb{N}^k. \end{aligned}$$

It is easy to see that

$$h_{T^j\mu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}) = h_{\mu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{P^j s_k(i)}) \text{ and } \int_X f_k(x)dT^j\mu_k(x) = \int_X f_k(x)d\mu_k(x)$$

for all $j = 0, 1, \cdots, k-1$. It follows from (4.12) that

$$\begin{aligned} & h_{T^j\mu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}) + \textstyle\int_X f_k(x)dT^j\mu_k(x) \\ &= h_{\mu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{P^j s_k(i)}) + \textstyle\int_X f_k(x)d\mu_k(x) \\ &\geq P(T^k, f_k; \mathcal{U}_0^{k-1}). \end{aligned}$$

Moreover, for each $s_k \in \mathbb{N}^k$,

$$\begin{aligned} & h_{\nu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}) + \textstyle\int_X f_k(x)d\nu_k(x) \\ &= \tfrac{1}{k}\textstyle\sum_{j=0}^{k-1}(h_{T^j\mu_k}(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}) + \int_X f_k(x)dT^j\mu_k(x)) \\ &\geq P(T^k, f_k; \mathcal{U}_0^{k-1}). \end{aligned}$$

Hence $\nu_k \in \bigcap_{s_k \in \mathbb{N}^k} M(k, s_k)$.

Let $M(k) = \bigcap_{s_k \in \mathbb{N}^k} M(k, s_k)$. For each $s_k \in \mathbb{N}^k$, since $\bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)}$ is a clopen cover, by Theorem 1, the map $\mathcal{M}(X,T) \to \mathbb{R}$: $\mu \mapsto h_\mu(T^k, \bigvee_{i=0}^{k-1} T^{-i}\alpha_{s_k(i)})$ is upper semi-continuous. It follows that $M(k,s_k)$ is a closed subset of $\mathcal{M}(X,T)$ for $s_k\in\mathbb{N}^k$, which implies that $M(k)$ is a non-empty, closed subset of $\mathcal{M}(X,T)$.

We now show that if $k_1,k_2\in\mathbb{N}$, k_1 divides k_2, then $M(k_2)\subseteq M(k_1)$. Indeed, let $\mu\in M(k_2)$ and $k=\frac{k_2}{k_1}$. For any $s_{k_1}\in\mathbb{N}^{k_1}$, we take $s_{k_2}=\underbrace{s_{k_1}\cdots s_{k_1}}_{k}\in\mathbb{N}^{k_2}$. Then

$$\begin{aligned}&\tfrac{1}{k_1}(h_\mu(T^{k_1},\bigvee_{i=0}^{k_1-1}T^{-i}\alpha_{s_{k_1}(i)})+\textstyle\int_X f_{k_1}(x)d\mu(x))\\ &=\tfrac{1}{k_1}\tfrac{1}{k}h_\mu(T^{kk_1},\bigvee_{j=0}^{k-1}T^{-jk_1}\bigvee_{i=0}^{k_1-1}T^{-i}\alpha_{s_{k_1}(i)})+\textstyle\int_X f(x)d\mu(x)\\ &=\tfrac{1}{k_2}(h_\mu(T^{k_2},\bigvee_{i=0}^{k_2-1}T^{-i}\alpha_{s_{k_2}(i)})+\textstyle\int_X f_{k_2}(x)d\mu(x))\\ &\geq\tfrac{1}{k_2}P(T^{k_2},f_{k_2};\mathcal{U}_0^{k_2-1})=P(T,f;\mathcal{U})\\ &=\tfrac{1}{k_1}P(T^{k_1},f_{k_1};\mathcal{U}_0^{k_1-1}).\end{aligned}$$

Hence $\mu\in M(k_1,s_{k_1})$ for each $s_{k_1}\in\mathbb{N}^{k_1}$ and $\mu\in M(k_1)$. This shows that $M(k_2)\subseteq M(k_1)$.

Since $\emptyset\neq M(k_1k_2)\subseteq M(k_1)\cap M(k_2)$ for any $k_1,k_2\in\mathbb{N}$, we have that $\bigcap_{k\in\mathbb{N}}M(k)\neq\emptyset$.

Let $\nu\in\bigcap_{k\in\mathbb{N}}M(k)$ and $k\in\mathbb{N}$. By (4.11), we have that

$$\begin{aligned}&\tfrac{1}{k}h_\nu^+(T^k,\mathcal{U}_0^{k-1})+\textstyle\int_X f(x)d\nu(x)\\ &=\tfrac{1}{k}(h_\nu^+(T^k,\mathcal{U}_0^{k-1})+\textstyle\int_X f_k(x)d\nu(x))\\ &=\inf_{s_k\in\mathbb{N}^k}\tfrac{1}{k}(h_\nu(T^k,\bigvee_{i=0}^{k-1}T^{-i}\alpha_{s_k(i)})+\textstyle\int_X f_k(x)d\nu(x))\\ &\geq P(T,f;\mathcal{U}).\end{aligned}$$

It follows from Lemma 2.6 that

$$h_\nu(T,\mathcal{U})+\int_X f(x)d\nu(x)=\lim_{k\to\infty}\frac{1}{k}(h_\nu^+(T^k,\mathcal{U}_0^{k-1})+\int_X f_k(x)d\nu(x))\geq P(T,f;\mathcal{U}).$$

Combining this with Proposition 4.1, we have proved the proposition in the case that X is zero-dimensional.

We now treat the general case. It is well known that there exists an invertible TDS (Z,R), with Z being zero-dimensional, and a continuous surjective map $\varphi:Z\to X$ such that $\varphi\circ R=T\circ\varphi$ (see e.g., [3]). For the TDS (Z,R), we have already shown that there exists a $\nu\in\mathcal{M}(Z,R)$ such that

$$h_\nu(R,\varphi^{-1}(\mathcal{U}))+\int_Z f\circ\varphi(z)d\nu(z)=P(R,f\circ\varphi;\varphi^{-1}\mathcal{U}).$$

Let $\mu=\varphi\nu$. Then $\mu\in\mathcal{M}(X,T)$. Since, by Lemma 4.1, $h_\mu(T,\mathcal{U})=h_\nu(R,\varphi^{-1}(\mathcal{U}))$, we have

$$h_\mu(T,\mathcal{U})+\int_X f(x)d\mu(x)=h_\nu(R,\varphi^{-1}(\mathcal{U}))+\int_Z f\circ\varphi(z)d\nu(z)=P(T,f\circ\varphi;\varphi^{-1}\mathcal{U}).\tag{4.13}$$

Now, by Lemma 4.2, we also have

$$P(R,f\circ\varphi;\varphi^{-1}\mathcal{U})=P(T,f;\mathcal{U}).\tag{4.14}$$

The proof is now complete by combining (4.13) and (4.14). □

We are now ready to prove the local variational principle of pressure stated in Theorem 2, i.e., *for any TDS* (X,T), $f\in C(X,\mathbb{R})$ *and* $\mathcal{U}\in\mathcal{C}_X^o$,

$$P(T,f;\mathcal{U})=\sup_{\mu\in\mathcal{M}(X,T)}\{h_\mu(T,\mathcal{U})+\int_X fd\mu(x)\}\tag{4.15}$$

and the supremum can be attained by a T-invariant ergodic measure.

Let d be the metric on X and define $\widetilde{X} = \{(x_1, x_2, \cdots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$. It is clear that $\widetilde{X}$ is a subspace of the product space $\prod_{i=1}^{\infty} X$ with the metric d_T defined by

$$d_T((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}.$$

Let $\sigma_T : \widetilde{X} \to \widetilde{X}$ be the shift homeomorphism, i.e., $\sigma_T(x_1, x_2, \cdots) = (T(x_1), x_1, x_2, \cdots)$. We refer the TDS $(\widetilde{X}, \sigma_T)$ as the *natural extension* of (X, T). For each $i \in \mathbb{N}$, we denote $\pi_i : \widetilde{X} \to X$ as the natural projection which projects each element of $\widetilde{X}$ onto its i-th component. Then $\pi_1 : (\widetilde{X}, \sigma_T) \to (X, T)$ is a factor map.

Proof of Theorem 2. Let $(\widetilde{X}, \sigma_T)$ be the natural extension of (X, T) defined above. By Proposition 4.2, there exists a $\nu \in \mathcal{M}(\widetilde{X}, \sigma_T)$ such that

$$h_\nu(\sigma_T, \pi_1^{-1}(\mathcal{U})) + \int_{\widetilde{X}} f \circ \pi_1(\widetilde{x}) d\nu(\widetilde{x}) = P(\sigma_T, f \circ \pi_1; \pi_1^{-1}\mathcal{U}).$$

Let $\mu = \pi_1 \nu$. Then $\mu \in \mathcal{M}(X, T)$. Since, by Lemma 4.1, $h_\mu(T, \mathcal{U}) = h_\nu(\sigma_T, \pi_1^{-1}(\mathcal{U}))$, we have

$$h_\mu(T, \mathcal{U}) + \int_X f(x) d\mu(x) = h_\nu(\sigma_T, \pi_1^{-1}(\mathcal{U})) + \int_{\widetilde{X}} f \circ \pi_1(\widetilde{x}) d\nu(\widetilde{x}) = P(\sigma_T, f \circ \pi_1; \pi_1^{-1}\mathcal{U}). \tag{4.16}$$

But by Lemma 4.2,

$$P(\sigma_T, f \circ \pi_1; \pi_1^{-1}\mathcal{U}) = P(T, f; \mathcal{U}). \tag{4.17}$$

Combining (4.16) and (4.17), we have

$$h_\mu(T, \mathcal{U}) + \int_X f(x) d\mu(x) = P(T, f; \mathcal{U}). \tag{4.18}$$

Let $\mu = \int_{\mathcal{M}^e(X,T)} \theta dm(\theta)$ be the ergodic decomposition of μ. Then by Lemma 3.4 and (4.18),

$$\begin{aligned} &\int_{\mathcal{M}^e(X,T)} (h_\theta(T, \mathcal{U}) + \int_X f(x) d\theta(x)) dm(\theta) \\ &= \int_{\mathcal{M}^e(X,T)} h_\theta(T, \mathcal{U}) dm(\theta) + \int_{\mathcal{M}^e(X,T)} \int_X f(x) d\theta(x) dm(\theta) \\ &= h_\mu(T, \mathcal{U}) + \int_X f(x) d\mu(x) = P(T, f; \mathcal{U}). \end{aligned}$$

Hence there exists a T-invariant ergodic measure θ such that

$$h_\theta(T, \mathcal{U}) + \int_X f(x) d\theta(x) \geq P(T, f; \mathcal{U}).$$

The proof of the theorem is now complete by applying Proposition 4.1. □

We remark that Theorem 2 generalizes the topological variational principle of pressure given in [27], i.e., the following holds.

Corollary 4.1. (Topological variational principle of pressure, [27]) *Let (X, T) be a TDS and $f \in C(X, \mathbb{R})$. Then*

$$P(T, f) = \sup_{\mu \in \mathcal{M}(X,T)} \{h_\mu(T) + \int_X f(x) d\mu(x)\}.$$

Proof. The proof follows immediately from Theorem 2 and Lemma 2.7 by taking the supremum over all open covers in (4.15). □

Another immediate consequence of Theorem 2 is the following.

Corollary 4.2. *Let (X, T) be a uniquely ergodic TDS and let μ be the unique invariant probability measure on X. Then for each $\mathcal{U} \in \mathcal{C}_X^0$ and $f \in C(X, \mathbb{R})$,*

$$P(T, f; \mathcal{U}) = h_\mu(T, \mathcal{U}) + \int_X f(x) d\mu(x) = h_{top}(T, \mathcal{U}) + \int_X f(x) d\mu(x).$$

Using this corollary, one can also give an alternative proof to the following classical ergodic theorem of Oxtoby (see also [10]).

Corollary 4.3. (Ergodic theorem) *Let* (X,T) *be a uniquely ergodic TDS and* $f \in C(X,\mathbb{R})$*. Then* $\frac{f_n(x)}{n}$ *converges uniformly to* $\int_X f(x)d\mu(x)$ *as* $n \to \infty$*, where* μ *is the unique invariant probability measure on* X*.*

Proof. Take $\mathcal{U} = \{X\}$ in Corollary 4.2. Then $h_\mu(T,\{X\}) = 0$. Recall that

$$P_n(T,f;\{X\}) = e^{\max_{x\in X} f_n(x)},$$

and hence $P(T,f;\{X\}) = \lim\limits_{n\to\infty} \frac{\max_{x\in X} f_n(x)}{n}$. We have by Corollary 4.2 that

$$\lim_{n\to\infty} \frac{\max_{x\in X} f_n(x)}{n} = P(T,f;\{X\}) = \int_X f(x)d\mu(x). \tag{4.19}$$

By replacing f in (4.19) with $-f$, we also have

$$\lim_{n\to\infty} \frac{\min_{x\in X} f_n(x)}{n} = \int_X f(x)d\mu(x).$$

Hence $\frac{f_n(x)}{n}$ converges uniformly to $\int_X f(x)d\mu(x)$. □

5. Local pressures determine local measure-theoretic entropies

We will prove Theorem 3 in this section. Throughout the section, we let (X,T) be a TDS.

Lemma 5.1. *Let* $\mathcal{U} \in \mathcal{C}_X^o$*. The following holds for any* $f,g \in C(X,\mathbb{R})$ *and* $c \in \mathbb{R}$*.*

1) $P(T,0;\mathcal{U}) = h_{top}(T,\mathcal{U})$.
2) *If* $f \le g$*, then* $P(T,f;\mathcal{U}) \le P(T,g;\mathcal{U})$*. In particular,*
$$h_{top}(T,\mathcal{U}) + \min_{x\in X} f(x) \le P(T,f;\mathcal{U}) \le h_{top}(T,\mathcal{U}) + \max_{x\in X} f(x).$$
3) $P(T,f+c;\mathcal{U}) = P(T,f;\mathcal{U}) + c$.
4) $|P(T,f;\mathcal{U}) - P(T,g;\mathcal{U})| \le ||f-g||$.
5) $P(T,\cdot;\mathcal{U})$ *is convex.*
6) $P(T,f+g\circ T-g;\mathcal{U}) = P(T,f;\mathcal{U})$.
7) $P(T,f+g;\mathcal{U}) \le P(T,f;\mathcal{U}) + P(T,g;\mathcal{U})$.
8) $P(T,cf;\mathcal{U}) \le cP(T,f;\mathcal{U})$ *if* $c \ge 1$*, and,* $P(T,cf;\mathcal{U}) \ge cP(T,f;\mathcal{U})$ *if* $c \le 1$.
9) $|P(T,f;\mathcal{U})| \le P(T,|f|;\mathcal{U})$.

Proof. 1), 2) and 3) easily follow from the definition of $P(T,f;\mathcal{U})$.

By Theorem 2, there exists a $\mu \in \mathcal{M}(X,T)$ such that $h_\mu(T,\mathcal{U}) + \int_X f(x)d\mu(x) = P(T,f;\mathcal{U})$. Hence

$$\begin{aligned} P(T,f;\mathcal{U}) - P(T,g;\mathcal{U}) &\le (h_\mu(T,\mathcal{U}) + \textstyle\int_X f(x)d\mu(x)) - (h_\mu(T,\mathcal{U}) + \int_X g(x)d\mu(x)) \\ &= \textstyle\int_X (f-g)(x)d\mu(x) \le ||f-g||. \end{aligned}$$

Similarly, we have $P(T,g;\mathcal{U}) - P(T,f;\mathcal{U}) \le ||f-g||$. This proves 4).

Let $a \in [0,1]$. By Theorem 2 there exists a $\mu \in \mathcal{M}(X,T)$ such that

$$h_\mu(T,\mathcal{U}) + \int_X (af+(1-a)g)(x)d\mu(x) = P(T,af+(1-a)g;\mathcal{U}).$$

It follows that

$$\begin{aligned} P(T,af+(1-a)g;\mathcal{U}) &= h_\mu(T,\mathcal{U}) + \textstyle\int_X (af+(1-a)g)(x)d\mu(x) \\ &= a(h_\mu(T,\mathcal{U}) + \textstyle\int_X f(x)d\mu(x)) + (1-a)(h_\mu(T,\mathcal{U}) + \int_X f(x)d\mu(x)) \\ &\le aP(T,f;\mathcal{U}) + (1-a)P(T,g;\mathcal{U}). \end{aligned}$$

Since a, f and g are arbitrary, 5) follows.

To prove 6), we note that $\int_X (g\circ T - g)(x)d\mu(x) = 0$ for each $\mu \in \mathcal{M}(X,T)$. Then

$$\begin{aligned} P(T, f + g\circ T - g; \mathcal{U}) &= \sup_{\mu\in\mathcal{M}(X,T)}\{h_\mu(T,\mathcal{U}) + \textstyle\int_X (f + g\circ T - g)(x)d\mu(x)\} \\ &= \sup_{\mu\in\mathcal{M}(X,T)}\{h_\mu(T,\mathcal{U}) + \textstyle\int_X f(x)d\mu(x)\} \\ &= P(T, f; \mathcal{U}). \end{aligned}$$

7), ¡img src=”/horde/themes/graphics/emoticons/cool.png” alt=”8)” title=”8)” align=”middle” /¿ and 9) can be proved similarly by applying Theorem 2. We omit the details. □

Recall that a *finite signed measure* on X is a map $\mu : \mathcal{B}_X \to \mathbb{R}$ which is countably additive. The following result says that, for each $\mathcal{U} \in \mathcal{C}_X^o$, $P(T, \cdot; \mathcal{U})$ determines the members of $\mathcal{M}(X,T)$.

Proposition 5.1. *Let $\mathcal{U} \in \mathcal{C}_X^o$ and $\mu : \mathcal{B}_X \to \mathbb{R}$ be a finite signed measure on X. Then $\mu \in \mathcal{M}(X,T)$ iff $\int_X f(x)d\mu(x) \le P(T,f;\mathcal{U})$ for all $f \in C(X,\mathbb{R})$.*

Proof. The proof follows completely from that of Theorem 9.11 in [27]. □

Corollary 5.1. *Let $\mu : \mathcal{B}(X) \to \mathbb{R}$ be a finite signed measure on X. Then $\mu \in \mathcal{M}(X,T)$ iff $\int_X f(x)d\mu(x) \le \frac{\max_{x\in X} f_n(x)}{n}$ for all $n \in \mathbb{N}$ and $f \in C(X,\mathbb{R})$.*

Proof. Take $\mathcal{U} = \{X\}$ in Proposition 5.1. Since $P(T,f;\{X\}) = \lim_{n\to\infty} \frac{\max_{x\in X} f_n(x)}{n}$, the corollary follows. □

We first prove part a) of Theorem 3, i.e., *for given $\mathcal{U} \in \mathcal{C}_X^o$ and $\mu \in \mathcal{M}(X,T)$, $P(T,\cdot;\mathcal{U})$ determines the μ-entropy relative to $\mathcal{U}$ in the sense that*

$$h_\mu(T,\mathcal{U}) = \inf\{P(T,f;\mathcal{U}) - \int_X f(x)d\mu(x) : f \in C(X,\mathbb{R})\}.$$

Proof of part a) of Theorem 3. We follow the arguments in the proof of Theorem 9.12, [27]. By Theorem 2, we first have

$$h_\mu(T,\mathcal{U}) \le \inf\{P(T,f;\mathcal{U}) - \int_X f(x)d\mu(x) : f \in C(X,\mathbb{R})\}.$$

To prove the opposite, we let

$$C = \{(\mu,t) \in \mathcal{M}(X,T)\times\mathbb{R} :\ 0 \le t \le h_\mu(T,\mathcal{U})\}.$$

Since, by Theorem 1, the entropy map $h_{\{\cdot\}}(T,\mathcal{U}) : \mathcal{M}(X,T) \to \mathbb{R}^+$ is an affine map, C is a convex set. Let $C(X,\mathbb{R})^*$ be the dual space of $C(X,\mathbb{R})$ endowed with the weak*-topology and view C as a subset of $C(X,\mathbb{R})^* \times R$. Take $b > h_\mu(T,\mathcal{U})$. Since, by Theorem 1, the entropy map $h_{\{\cdot\}}(T,\mathcal{U})$ is upper semi-continuous at μ, we have that $(\mu,b) \notin \mathrm{cl}(C)$. Let $V = C(X,\mathbb{R})^* \times \mathbb{R}$, $K_1 = \mathrm{cl}(C)$, and $K_2 = \{(\mu,b)\}$. Then V is a locally convex, linear topological space, and K_1, K_2 are disjoint, closed, and convex subsets of V. It follows from [9] (pp. 417) that there exists a continuous, real-valued, linear functional F on V such that $F(x) < F(y)$ for all $x \in K_1, y \in K_2$, i.e., $F : C(X,\mathbb{R})^* \times \mathbb{R} \to \mathbb{R}$ is a continuous linear functional such that $F(\mu_*,t) < F(\mu,b)$ for all $(\mu_*,t) \in \mathrm{cl}(C)$. Note that under the weak*-topology on $C \subset C(X,\mathbb{R})^*$, F must have the form $F(\mu_*,t) = \int_X f(x)d\mu_*(x) + td$ for some $f \in C(X,\mathbb{R})$ and some $d \in \mathbb{R}$. It follows that $\int_X f(x)d\mu_*(x) + dt < \int_X f(x)d\mu(x) + db$ for all $(\mu_*,t) \in \mathrm{cl}(C)$. In particular, $\int_X f(x)d\mu_*(x) + dh_{\mu_*}(T,\mathcal{U}) < \int_X f(x)d\mu(x) + db$ for all $\mu_* \in \mathcal{M}(X,T)$. By taking $\mu_* = \mu$, we have that $dh_\mu(T,\mathcal{U}) < db$. Hence $d > 0$ and

$$h_{\mu_*}(T,\mathcal{U}) + \int_X \frac{f(x)}{d} d\mu_*(x) < b + \int_X \frac{f(x)}{d} d\mu(x), \text{ for all } \mu_* \in \mathcal{M}(X,T).$$

Using Theorem 2, we have

$$P(T, \frac{f}{d}; \mathcal{U}) \le b + \int_X \frac{f(x)}{d} d\mu(x),$$

i.e.,

$$b \geq P(T, \frac{f}{d}; \mathcal{U}) - \int_X \frac{f(x)}{d} d\mu(x) \geq \inf\{P(T, g; \mathcal{U}) - \int_X g(x) d\mu(x) : \ g \in C(X, \mathbb{R})\}.$$

Since the above inequality is true for any $b > h_\mu(T, \mathcal{U})$, the above implies that $h_\mu(T, \mathcal{U}) \geq \inf\{P(T, g; \mathcal{U}) - \int_X g(x) d\mu(x) : \ g \in C(X, \mathbb{R})\}$. □

Next, we show part b) of Theorem 3, i.e., *if* (X, T) *is invertible, then for given* $\mathcal{U} \in \mathcal{C}_X^o$ *and* $\mu \in \mathcal{M}(X, T)$,

$$h_\mu^+(T, \mathcal{U}) \leq \inf_{f \in C(X; \mathbb{R})} \{P(T, f; \mathcal{U}) - \int_X f d\mu\}.$$

We need the following classical result of Rohlin.

Lemma 5.2. *Let* (X, T) *be invertible and* $\mu \in \mathcal{M}^e(X, T)$*. If* μ *is non-atomic (i.e.* $\mu(\{x\}) = 0$ *for each* $x \in X$*), then for any* $N \in \mathbb{N}$ *and* $\epsilon > 0$*, there exists a Borel subset* D *of* X *such that* $D, TD, \cdots, T^{N-1}D$ *are pairwise disjoint and* $\mu(\bigcup_{i=0}^{N-1} T^i D) > 1 - \epsilon$*.*

Proof. See e.g., [11]. □

Proof of part b) of Theorem 3. We follow the arguments in the proof of Proposition 7.10, [12]. Let $\mathcal{U} = \{U_1, U_2, \cdots, U_k\}$. By Lemma 3.4 and Proposition 4.1, we may assume that μ is ergodic and non-atomic. Since $P(T, f + c; \mathcal{U}) - \int_X (f + c) d\mu = P(T, f; \mathcal{U}) - \int_X f d\mu$ for each $c \in \mathbb{R}$, we can assume that $f(x) \geq 0$ for all $x \in X$. Let $f_{\max} = \max_{x \in X} f(x)$. Then $f_{\max} \geq 0$.

For $\epsilon > 0$, we let $N \in \mathbb{N}$ be sufficiently large such that

$$P_N(T, f; \mathcal{U}) \leq 2^{N(P(T, f; \mathcal{U}) + \epsilon)} \text{ and } -(1 - \frac{1}{N}) \log(1 - \frac{1}{N}) - \frac{1}{N} \log \frac{1}{N} < \epsilon, \tag{5.1}$$

and let $1 > \delta > 0$ be sufficiently small such that

$$\sqrt{\delta}(\log k + f_{\max} + \log(k e^{f_{\max}})) < \epsilon. \tag{5.2}$$

For such δ, N chosen, by Lemma 5.2, there is a Borel subset D of X such that $D, TD, \cdots, T^{N-1}D$ are pairwise disjoint, and $\mu(\bigcup_{i=0}^{N-1} T^i D) > 1 - \delta$. Let $\beta \in \mathcal{P}^*(\mathcal{U}_0^{N-1})$ be such that

$$1 \leq \sum_{B \in \beta} \sup_{x \in B} e^{f_N(x)} = P_N(T, f; \mathcal{U}) \tag{5.3}$$

and consider the partition $\beta_D = \{B \cap D : B \in \beta\}$ of D. Hence for each element $P \in \beta_D$ we can find a $s_P \in \{1, 2, \cdots, k\}^N$ such that $P \subseteq (\bigcap_{j=0}^{N-1} T^{-j} U_{i_j}) \cap D$. Using the partition β_D, we define a partition $\alpha = \{A_i : i = 1, 2, \cdots, k\}$ of X as follows. First, for each $i = 1, 2, \cdots, k$, let

$$A_i' = \bigcup_{j=0}^{N-1} \bigcup \{T^j P : \ P \in \beta_D \text{ and } s_P(j) = i\}.$$

We then let $B_1' = U_1$, $B_2' = U_2 \setminus B_1', \cdots, B_k' = U_k \setminus (\bigcup_{j=1}^{k-1} B_j')$. Finally, we let $A_i = A_i' \cup (B_i' \cap (X \setminus \bigcup_{j=0}^{N-1} T^j D))$ for $i = 1, 2, \cdots, k$. Clearly, $\alpha = \{A_i : i = 1, 2, \cdots, k\}$ is a partition of X and $A_i \subset U_i$ for all $i = 1, 2, \cdots, k$. Hence $\alpha \succeq \mathcal{U}$.

For $\beta' \in \mathcal{P}_X$ and $R \subset X$, we let $\beta' \cap R = \{A \cap R : A \in \beta' \text{ and } A \cap R \neq \emptyset\}$. From the construction of α, it is easy to see that $\alpha_0^{N-1} \cap D = \beta_D$, and moreover,

$$\sum_{C \in \alpha_0^{N-1} \cap D} \sup_{x \in C} e^{f_N(x)} = \sum_{P \in \beta_D} \sup_{x \in P} e^{f_N(x)} \leq \sum_{C \in \beta} \sup_{x \in C} e^{f_N(x)} = P_N(T, f; \mathcal{U}). \tag{5.4}$$

Let $E = \bigcup_{i=0}^{N-1} T^i D$. Then $\mu(E) > 1 - \delta$. For any fixed $n \gg N$, we let $G_n = \{x \in X : \frac{1}{n} \sum_{i=0}^{n-1} 1_E(T^i x) > 1 - \sqrt{\delta}\}$. Since $\mu(G_n) + (1 - \sqrt{\delta})(1 - \mu(G_n)) \geq \int_X \frac{1}{n} (\sum_{i=0}^{n-1} 1_E(T^i x)) d\mu(x) > 1 - \delta$, we have

$$\mu(G_n) > 1 - \sqrt{\delta}. \tag{5.5}$$

For each $x \in G_n$, let $S_n(x) = \{i \in \{0,1,\cdots,n-1\} : T^i x \in D\}$ and $U_n(x) = \{i \in \{0,1,\cdots,n-1\} : T^i x \in E\}$. Note that for any $x \in X$ and $i \in \mathbb{Z}$, if $T^i x \in E$ then there exists a $j \in \{0,1,\cdots,N-1\}$ such that $T^{i-j}x \in D$. Using this fact, it is not hard to see that for each $x \in G_n$, $U_n(x) \subseteq \bigcup_{j=0}^{N-1}(S_n(x)+j) \cup \{0,1,\cdots,N-1\}$. Since for each $x \in G_n$, $|U_n(x)| = \sum_{i=0}^{n-1} 1_E(T^i x) > n(1-\sqrt{\delta})$, we have $|\{0,1,\cdots,n-1\} \setminus U_n(x)| \le n\sqrt{\delta}$. Therefore, for each $x \in G_n$,

$$\begin{aligned} &|\{0,1,\cdots,n-1\} \setminus \textstyle\bigcup_{j=0}^{N-1}(S_n(x)+j)| \\ &\le |\{0,1,\cdots,N-1\} \cup (\{0,1,\cdots,n-1\} \setminus U_n(x))| \le n\sqrt{\delta} + N. \end{aligned} \tag{5.6}$$

Let $\mathcal{F}_n = \{S_n(x) : x \in G_n\}$. Since for each $F \in \mathcal{F}_n$, $F \cap (F+i) = \emptyset$, $i = 0,1,\cdots,N-1$, we have $|F| \le \frac{n}{N}+1$. Hence

$$|\mathcal{F}_n| \le \sum_{j=1}^{a_n} \frac{n!}{j!\cdot(n-j)!} \le a_n \frac{n!}{a_n!\cdot(n-a_n)!} \le n\frac{n!}{a_n!\cdot(n-a_n)!},$$

where $a_n = [\frac{n}{N}]+1$. By Stirling's formulation and the second inequality in (5.1), we have

$$\lim_{n\to\infty} \frac{1}{n}\log(n\frac{n!}{a_n!\cdot(n-a_n)!}) = -(1-\frac{1}{N})\log(1-\frac{1}{N}) - \frac{1}{N}\log\frac{1}{N} < \epsilon.$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n}\log|\mathcal{F}_n| \le \lim_{n\to\infty}\frac{1}{n}\log n\frac{n!}{a_n!\cdot(n-a_n)!} \le \epsilon. \tag{5.7}$$

For each $F \in \mathcal{F}_n$, let $B_F = \{x \in G_n : S_n(x) = F\}$. Clearly, $\{B_F\}_{F\in\mathcal{F}_n}$ forms a partition of G_n.

For each $F \in \mathcal{F}_n$, $F = \{s_1 < s_2 < \cdots < s_l\}$, we let $H_F = \{0,1,\cdots,n-1\} \setminus \bigcup_{i=0}^{N-1}(F+i)$. It follows from (5.6) that $l \le \frac{n}{N}+1$, $|H_F| \le n\sqrt{\delta}+N$. Moreover, using (5.4) and the facts that $|\alpha| = k$, $P_N(T,f,\mathcal{U}) \ge 1$, and $B_F \subseteq G_n \cap \bigcap_{j=1}^{l} T^{-s_j}D$, we have

$$\begin{aligned} &\sum_{C\in\alpha_0^{n-1}\cap B_F} \sup_{x\in C} e^{f_n(x)} \le \sum_{C\in \bigvee_{j=1}^{l} T^{-s_j}(\alpha_0^{N-1}\cap D)\vee \bigvee_{r\in H_F} T^{-r}\alpha} \sup_{x\in C} e^{f_n(x)} \\ &\le \prod_{j=1}^{l}(\sum_{C_j\in T^{-s_j}(\alpha_0^{N-1}\cap D)} \sup_{x\in C_j} e^{f_N(T^{s_j}x)}) \cdot \prod_{r\in H_F}(\sum_{C_i\in T^{-r}\alpha}\sup_{x\in C_i} e^{f(T^r x)}) \\ &= \prod_{j=1}^{l}(\sum_{C_j\in\alpha_0^{N-1}\cap D}\sup_{x\in C_j} e^{f_N(x)}) \cdot (\sum_{C_i\in\alpha}\sup_{x\in C_i} e^{f(x)})^{|H_F|} \\ &\le (ke^{f_{\max}})^{|H_F|}\cdot(P_N(T,f,\mathcal{U}))^l \\ &\le (ke^{f_{\max}})^{n\sqrt{\delta}+N}\cdot(P_N(T,f,\mathcal{U}))^{\frac{n}{N}+1}. \end{aligned}$$

Summing the above inequality over $F \in \mathcal{F}_n$ yields that

$$\sum_{F\in\mathcal{F}_n}\sum_{C\in\alpha_0^{n-1}\cap B_F}\sup_{x\in C} e^{f_n(x)} \le |\mathcal{F}_n|\cdot(ke^{f_{\max}})^{n\sqrt{\delta}+N}\cdot(P_N(T,f,\mathcal{U}))^{\frac{n}{N}+1}. \tag{5.8}$$

Since $\mu(X\setminus G_n) < \sqrt{\delta}$ and $|\alpha_0^{n-1}\cap(X\setminus G_n)| \le k^n$,

$$\begin{aligned} &H_\mu(\alpha_0^{n-1}\cap(X\setminus G_n)) + \textstyle\int_{X\setminus G_n} f_n d\mu \\ &\le \sum_{C'\in\alpha_0^{n-1}\cap(X\setminus G_n)} -\mu(C')\log\mu(C') + n\mu(X\setminus G_n)f_{\max} \\ &\le -(\sum_{C'\in\alpha_0^{n-1}\cap(X\setminus G_n)}\mu(C'))\log\frac{(\sum_{C'\in\alpha_0^{n-1}\cap(X\setminus G_n)}\mu(C'))}{|\alpha_0^{n-1}\cap(X\setminus G_n)|} + n\mu(X\setminus G_n)f_{\max} \\ &= \mu(X\setminus G_n)(\log|\alpha_0^{n-1}\cap(X\setminus G_n)| - \log\mu(X\setminus G_n) + nf_{\max}) \\ &\le \sqrt{\delta}(\log k^n + nf_{\max}) - \mu(X\setminus G_n)\log\mu(X\setminus G_n). \end{aligned} \tag{5.9}$$

Let $\gamma = \{B_F\}_{F\in\mathcal{F}_n} \cup \{X \setminus G_n\}$ and $\phi(t) = -t\log t$, $t \geq 0$. Then $\gamma \in \mathcal{P}_X$, and, by (5.8), (5.9) and Lemma 4.3,

$$
\begin{aligned}
& H_\mu(\alpha_0^{n-1}) + \textstyle\int_X f_n d\mu \leq H_\mu(\alpha_0^{n-1} \vee \gamma) + \int_X f_n d\mu \\
(5.10)\qquad & = \sum_{F\in\mathcal{F}_n} (H_\mu(\alpha_0^{n-1} \cap B_F) + \textstyle\int_{B_F} f_n d\mu) + (H_\mu(\alpha_0^{n-1} \cap (X \setminus G_n)) + \int_{X\setminus G_n} f_n d\mu) \\
& \leq \sum_{F\in\mathcal{F}_n} (H_\mu(\alpha_0^{n-1} \cap B_F) + \textstyle\int_{B_F} f_n d\mu) + \sqrt{\delta}(\log k^n + n f_{\max}) + \phi(\mu(X \setminus G_n)) \\
& \leq \sum_{F\in\mathcal{F}_n} \sum_{C\in\alpha_0^{n-1}\cap B_F} \mu(C)(\sup_{x\in C} f_n(x) - \log\mu(C)) \\
& +\mu(X \setminus G_n)(0 - \log\mu(X \setminus G_n)) + n\sqrt{\delta}(\log k + f_{\max}) \\
& \leq \log\Big(\sum_{F\in\mathcal{F}_n} \sum_{C\in\alpha_0^{n-1}\cap B_F} e^{\sup_{x\in C} f_n(x)} + e^{\sup_{x\in X\setminus G_n} 0}\Big) + n\sqrt{\delta}(\log k + f_{\max}) \\
& \leq n(b_n + \sqrt{\delta}(\log k + f_{\max})),
\end{aligned}
$$

where $b_n = \frac{1}{n}\log(|\mathcal{F}_n| \cdot (ke^{f_{\max}})^{n\sqrt{\delta}+N} \cdot (P_N(T, f, \mathcal{U}))^{\frac{n}{N}+1} + 1)$.

Now, by (5.1), (5.2), (5.7) and (5.10),

$$
\begin{aligned}
& h_\mu^+(T,\mathcal{U}) + \textstyle\int_X f(x)d\mu(x) \leq h_\mu(T,\alpha) + \int_X f d\mu \\
& = \lim_{n\to\infty} \tfrac{1}{n}(H_\mu(\alpha_0^{n-1}) + \textstyle\int_X f_n d\mu) \leq \limsup_{n\to\infty} b_n + \sqrt{\delta}(\log k + f_{\max}) \\
& = \limsup_{n\to\infty} \tfrac{1}{n}(\log|\mathcal{F}_n| + (n\sqrt{\delta} + N)\log(ke^{f_{\max}}) + (\tfrac{n}{N} + 1)\log P_N(T, f, \mathcal{U})) + \sqrt{\delta}(\log k + f_{\max}) \\
& = \limsup_{n\to\infty} \tfrac{1}{n}\log|\mathcal{F}_n| + \tfrac{1}{N}\log P_N(T, f, \mathcal{U}) + \sqrt{\delta}(\log k + f_{\max} + \log(ke^{f_{\max}})) \\
& \leq \tfrac{1}{N}\log P_N(T, f, \mathcal{U}) + 2\epsilon \\
& \leq P(T, f; \mathcal{U}) + 3\epsilon.
\end{aligned}
$$

The proof is now complete since $\epsilon > 0$ is arbitrary. □

In [17], the authors showed that for an invertible TDS (X,T), $\mu \in \mathcal{M}(X,T)$ and $\mathcal{U} \in \mathcal{C}_X^0$, i) $h_\mu(T,\mathcal{U}) = h_\mu^+(T,\mathcal{U})$ iff ii) $h_{\mathrm{top}}(T,\mathcal{U}) \geq h_\mu^+(T,\mathcal{U})$. If we let f be the null function in Lemma 4.1, then we have that ii) is true (in fact, this is already shown in Proposition 7.10 of [12]). Hence i) is also true. This gives an alternative proof of the Corollary stated in Section 1, i.e., $h_\mu(T,\mathcal{U}) = h_\mu^+(T,\mathcal{U})$ (see also [19] for a relative version).

A general question is whether the equality $h_\mu(T,\mathcal{U}) = h_\mu^+(T,\mathcal{U})$ still holds for a non-invertible TDS. We believe that the answer to this question is affirmative.

6. Equilibrium states

In this section, we will investigate properties of equilibrium states using our findings in the previous sections. Throughout the section, we let (X,T) be a TDS.

6.1. **Local equilibrium states.** Differing from global equilibrium states, we will show that local equilibrium states always exist and can be characterized by tangential functionals.

Given $f \in C(X,\mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. A member μ of $\mathcal{M}(X,T)$ is called an *equilibrium state for f relative to $\mathcal{U}$* if

$$P(T, f; \mathcal{U}) = h_\mu(T,\mathcal{U}) + \int_X f(x)d\mu(x).$$

We let $\mathcal{M}_f(X,T;\mathcal{U})$ denote the set of all equilibrium states for f relative to $\mathcal{U}$. Since $h_{\{\cdot\}}(T,\mathcal{U})$ is upper semi-continuous on $\mathcal{M}(X,T)$, it is easy to see that

$$\mathcal{M}_f(X,T;\mathcal{U}) = \bigcap_{n=1}^{\infty} \mathrm{cl}(\{\mu \in \mathcal{M}(X,T) :\ h_\mu(T;\mathcal{U}) + \int f d\mu > P(T, f; \mathcal{U}) - \frac{1}{n}\}).$$

A *tangent functional* to the convex function $P(T,\cdot;\mathcal{U})$ at f is a finite signed Borel measure μ on X such that

$$P(T, f + g; \mathcal{U}) - P(T, f; \mathcal{U}) \geq \int_X g(x)d\mu(x), \quad \text{for all } g \in C(X,\mathbb{R}).$$

We let $\mathcal{T}_f(X,T;\mathcal{U})$ denote the set of all tangent functionals to $P(T,\cdot;\mathcal{U})$ at f.

Proposition 6.1. *The following holds.*

1) $\mathcal{M}_f(X,T;\mathcal{U})$ *is a non-empty, compact and convex set.*
2) *The extreme points of* $\mathcal{M}_f(X,T;\mathcal{U})$ *are precisely the ergodic members of* $\mathcal{M}_f(X,T;\mathcal{U})$.
3) *Let* $\mu \in \mathcal{M}_f(X,T;\mathcal{U})$ *and* $\mu = \int_{\mathcal{M}^e(X,T)} \theta dm(\theta)$ *be the ergodic decomposition of* μ. *Then for* m*-a.e.* $\theta \in \mathcal{M}^e(X,T)$, $\theta \in \mathcal{M}_f(X,T;\mathcal{U})$.
4) $\mathcal{M}_f(X,T;\mathcal{U}) = \mathcal{T}_f(X,T;\mathcal{U})$.

Proof. For each $\nu \in \mathcal{M}(X,T)$, we let $L(f,\mathcal{U},\nu) = h_\nu(T,\mathcal{U}) + \int_X f(x)d\nu(x)$.

1) By Theorem 2, $\mathcal{M}_f(X,T;\mathcal{U})$ is non-empty. By Theorem 1, $L(f,\mathcal{U},\cdot) : \mathcal{M}(X,T) \to \mathbb{R}$ is a upper semi-continuous, affine map. Hence $\mathcal{M}_f(X,T;\mathcal{U})$ is a closed, convex subset of the compact metric space $\mathcal{M}(X,T)$.

2) Let μ be an extreme point of $\mathcal{M}_f(X,T;\mathcal{U})$. To show μ is ergodic, it is sufficient to show that μ is an extreme point of $\mathcal{M}(X,T)$. Let $\mu_1,\mu_2 \in \mathcal{M}(X,T)$ and $a \in (0,1)$ such $\mu = a\mu_1 + (1-a)\mu_2$. Then $aL(f,\mathcal{U},\mu_1) + (1-a)L(f,\mathcal{U},\mu_2) = L(f,\mathcal{U},\mu) = P(T,f,\mathcal{U})$. It follows from Theorem 2 that $L(f,\mathcal{U},\mu_1) = L(f,\mathcal{U},\mu_2) = P(T,f,\mathcal{U})$. Hence $\mu_1,\mu_2 \in \mathcal{M}_f(X,T;\mathcal{U})$. Since μ is an extreme point of $\mathcal{M}_f(X,T;\mathcal{U})$, $\mu_1 = \mu_2 = \mu$. It follows that μ is an extreme point of $\mathcal{M}(X,T)$.

3) This follows from the following two facts: a) $L(f,\mathcal{U},\theta) \le P(T,f;\mathcal{U})$ for each $\theta \in \mathcal{M}^e(X,T)$; b) $\int_{\mathcal{M}^e(X,T)} L(f,\mathcal{U},\theta)dm(\theta) = L(f,\mathcal{U},\mu) = P(T,f;\mathcal{U})$.

4) We follow the arguments in the proofs of Theorems 9.14 and 9.15, [27]. Let $\mu \in \mathcal{M}_f(X,T;\mathcal{U})$. By Theorem 2, if $g \in C(X,\mathbb{R})$, then

$$\begin{aligned}P(T,f+g;\mathcal{U}) - P(T,f;\mathcal{U}) &\ge h_\mu(T,\mathcal{U}) + \int_X (f+g)(x)d\mu(x) - (h_\mu(T,\mathcal{U}) + \int_X f(x)d\mu(x))\\ &= \int_X g(x)d\mu(x).\end{aligned}$$

Therefore $\mathcal{M}_f(X,T;\mathcal{U}) \subseteq \mathcal{T}_f(X,T;\mathcal{U})$.

Conversely, let $\mu \in \mathcal{T}_f(X,T;\mathcal{U})$. For any $g \in C(X,\mathbb{R})$ with $g \ge 0$ and any $\epsilon > 0$, we have by Lemma 5.1 2) and 3) that

$$\begin{aligned}\int_X (g+\epsilon)d\mu &= -\int_X -(g+\epsilon)d\mu\\ &\ge -P(T,f-(g+\epsilon);\mathcal{U}) + P(T,f;\mathcal{U})\\ &\ge -[P(T,f;\mathcal{U}) - \min(g+\epsilon)] + P(T,f;\mathcal{U})\\ &= \min g + \epsilon > 0.\end{aligned}$$

Hence $\int_X g d\mu \ge 0$. This implies that μ is a non-negative measure. To show that μ is T-invariant, we note by Lemma 5.1 6) that

$$n\int_X (g\circ T - g)d\mu \le P(T,f+n(g\circ T-g);\mathcal{U}) - P(T,f;\mathcal{U}) = 0$$

for any $n \in \mathbb{Z}$ and $g \in C(X,\mathbb{R})$. Hence, if $n > 0$ then $\int_X g\circ T d\mu \le \int_X g d\mu$ and if $n < 0$ then $\int_X g\circ T d\mu \ge \int_X g d\mu$. This shows that $\int_X g\circ T d\mu = \int_X g d\mu$, i.e., μ is T-invariant.

Next, we show that μ is a probability measure. Note that $\int_X n d\mu \le P(T,f+n;\mathcal{U}) - P(T,f;\mathcal{U}) = n$ for any $n \in \mathbb{Z}$. Hence if $n \ge 1$ then $\mu(X) \le 1$ and if $n \le -1$ then $\mu(X) \ge 1$. Thus $\mu(X) = 1$. Above all, $\mu \in \mathcal{M}(X,T)$.

Now, since $\mu \in \mathcal{T}_f(X,T;\mathcal{U})$, $P(T,f+g;\mathcal{U}) - \int_X (f+g)d\mu \ge P(T,f;\mathcal{U}) - \int_X f d\mu$ for any $g \in C(X,\mathbb{R})$. Hence $P(T,h;\mathcal{U}) - \int_X h d\mu \ge P(T,f;\mathcal{U}) - \int_X f d\mu$ for any $h \in C(X;\mathbb{R})$. It follows from Theorem 2 and part a) of Theorem 3 that $h_\mu(T,\mathcal{U}) = P(T,f;\mathcal{U}) - \int_X f d\mu$, i.e., $P(T,f;\mathcal{U}) = h_\mu(T,\mathcal{U}) + \int_X f d\mu$. Thus $\mu \in \mathcal{M}_f(X,T;\mathcal{U})$. □

Lemma 6.1. *Given* $\mathcal{U} \in \mathcal{C}_X^o$, *there is a dense subset* $\mathcal{C}$ *of* $C(X,\mathbb{R})$ *such that each function in* $\mathcal{C}$ *has a unique equilibrium state relative to* $\mathcal{U}$.

Proof. The lemma follows from Proposition 6.1 4) and the fact that a convex function on a separable Banach space has a unique tangent functional at a dense set of points (see [9], pp. 450). □

Next, we discuss uniqueness of local equilibrium states. Recall that $\mathcal{M}(X,T)$ forms a compact metric space under the weak*-topology. Let d be a compatible metric of $\mathcal{M}(X,T)$ and H_d be the Hausdorff metric of $2^{\mathcal{M}(X,T)}$. Given $\mathcal{U}\in\mathcal{C}_X^o$, define

$$\Phi_{\mathcal{U}}: C(X,\mathbb{R})\to 2^{\mathcal{M}(X,T)}:\ \Phi_{\mathcal{U}}(f)=\mathcal{M}_f(X,T;\mathcal{U}),\ f\in C(X,\mathbb{R}). \tag{6.1}$$

Lemma 6.2. *$\Phi_{\mathcal{U}}$ is upper semi-continuous.*

Proof. Let $f_n\to f$ in $C(X,\mathbb{R})$ and $\mu_n\in\mathcal{M}_{f_n}(X,T;\mathcal{U})$ with $\mu_n\to\mu$ for some $\mu\in\mathcal{M}(X,T)$.

Since $\mu_n\in\mathcal{M}_{f_n}(X,T;\mathcal{U})$,

$$h_{\mu_n}(T,\mathcal{U})+\int_X f_n(x)d\mu_n(x)=P(T,f_n;\mathcal{U}).$$

Let $n\to\infty$ in the above. It follows from Lemma 5.1 4) and Theorem 1 that

$$h_\mu(T,\mathcal{U})+\int_X f(x)d\mu(x)\geq P(T,f;\mathcal{U}).$$

By Proposition 4.1, $\mu\in\mathcal{M}_f(X,T;\mathcal{U})$. □

Proposition 6.2. *For a given $\mathcal{U}\in\mathcal{C}_X^o$, $f\in C(X,\mathbb{R})$ has a unique equilibrium state associated to $\mathcal{U}$ iff f is a point of continuity of $\Phi_{\mathcal{U}}$. Moreover, the subset $\mathcal{C}$ of $C(X,\mathbb{R})$ such that each function in $\mathcal{C}$ has a unique equilibrium state relative to $\mathcal{U}$ is a dense G_δ set.*

Proof. If $\mathcal{M}_f(X,T;\mathcal{U})$ has only one point, then it is clear that $\Phi_{\mathcal{U}}$ is continuous at f, as $\Phi_{\mathcal{U}}$ is upper semi-continuous.

Conversely, let $\Phi_{\mathcal{U}}$ be continuous at $f\in C(X,\mathbb{R})$. By Lemma 6.1 there exists a sequence $f_n\in C(X,\mathbb{R})$ such that $f_n\to f$ and each $\mathcal{M}_{f_n}(X,T;\mathcal{U})$ has only one point. Since $\Phi_{\mathcal{U}}$ is continuous at f, $\mathcal{M}_f(X,T;\mathcal{U})$ has only one point.

Let $\mathcal{C}$ be set of points of continuity of $\Phi_{\mathcal{U}}$. By Lemma 6.1, $\mathcal{C}\subset C(X,\mathbb{R})$ is a dense subset. Since $\Phi_{\mathcal{U}}$ is upper semi-continuous, $\mathcal{C}$ is also a G_δ set. □

We now discuss uniformity of local equilibrium states relative to a fixed $\mathcal{U}\in\mathcal{C}_X^o$. Let

$$\mathcal{M}(X,T;\mathcal{U})=\cup_{f\in C(X,\mathbb{R})}\mathcal{M}_f(X,T;\mathcal{U})$$

denote the set of all equilibrium states relative to $\mathcal{U}$.

Lemma 6.3. *Let $f\in C(X,\mathbb{R})$. Then for any $\mu\in\mathcal{M}(X,T)$ and $\epsilon>0$, there exist $f'\in C(X,\mathbb{R})$ and $\mu'\in\mathcal{M}_{f'}(X,T;\mathcal{U})$ such that*

$$\|\mu-\mu'\|=\sup_{g\in C(X,\mathbb{R}),\|g\|=1}|\int_X gd\mu-\int_X gd\mu'|\leq\epsilon$$

and

$$\|f'-f\|\leq\frac{1}{\epsilon}[P(T,f;\mathcal{U})-(h_\mu(T,\mathcal{U})+\int_X fd\mu)].$$

Proof. We follows the arguments of Theorem 3.16 and Remark 6.15 in [25]. By Lemma 5.1 4) and 5), $P(T,\cdot;\mathcal{U}):C(X,\mathbb{R})\to\mathbb{R}$ is convex and continuous. Since $\int_X gd\mu\leq P(T,g;\mathcal{U})$ for all $g\in C(X,\mathbb{R})$, it follows from a general result of Bishop and Phelps (see [14], pp 112 or [25], A.3.6) on the denseness of tangent functionals of a Banach space that there exists $f'\in C(X,\mathbb{R})$ and $\mu'\in\mathcal{T}_{f'}(X,T;\mathcal{U})=\mathcal{M}_{f'}(X,T;\mathcal{U})$ such that

$$\|\mu-\mu'\|\leq\epsilon$$

and

$$\|f'-f\|\leq\frac{1}{\epsilon}[P(T,f;\mathcal{U})-\int_X fd\mu-\inf_{g\in C(X,\mathbb{R})}\{P(T,g;\mathcal{U})-\int_X gd\mu\}].$$

The lemma follows as $\inf_{g\in C(X,\mathbb{R})}\{P(T,g;\mathcal{U})-\int_X gd\mu\}=h_\mu(T,\mathcal{U})$, by Theorem 3. □

Proposition 6.3. *The following holds.*

1) *The set $\mathcal{M}(X,T;\mathcal{U})$ of all equilibrium states relative to $\mathcal{U}$ is dense in $\mathcal{M}(X,T)$.*
2) *For any finite collection of ergodic measures $\{\mu_1,\mu_2,\cdots,\mu_n\}\subset\mathcal{M}^e(X,T)$, there exists a $f\in C(X,\mathbb{R})$ such that $\{\mu_1,\mu_2,\cdots,\mu_n\}\subset\mathcal{M}_f(X,T;\mathcal{U})$.*

Proof. 1) follows directly from Lemma 6.3.

2) We follow the arguments of Corollary 3.17 and Appendix A.5.5 in [25]. From 1), we know that there exist $f\in C(X,\mathbb{R})$ and $\mu\in\mathcal{M}_f(X,T;\mathcal{U})$ such that

$$\|\mu-\frac{1}{n}(\mu_1+\mu_2+\cdots+\mu_n)\|<\frac{1}{n}.$$

Let $\mu=\int_{\mathcal{M}^e(X,T)}\theta dm(\theta)$ be the ergodic decomposition of μ.

Note that $\omega\equiv m-\frac{1}{n}(\delta_{\mu_1}+\cdots+\delta_{\mu_n})$ is a finite signed Borel measure on $\mathcal{M}^e(X,T)$ and there are finite positive Borel measures ω_+,ω_- on $\mathcal{M}^e(X,T)$ such that $\omega=\omega_+-\omega_-$ and ω_+,ω_- are mutually singular. Let $\nu_+=\int_{\mathcal{M}^e(X,T)}\theta d\omega_+(\theta)$ and $\nu_-=\int_{\mathcal{M}^e(X,T)}\theta d\omega_-(\theta)$. Then ν_+,ν_- are mutually singular, finite, positive Borel measures on X and $\mu-\frac{1}{n}(\mu_1+\mu_2+\cdots+\mu_n)=\nu_+-\nu_-$. Since

$$\begin{aligned}\|\mu-\tfrac{1}{n}(\mu_1+\mu_2+\cdots+\mu_n)\| &=\|\nu_+-\nu_-\|=\|\nu_+\|+\|\nu_-\|=\nu_+(X)+\nu_-(X)\\ &=\omega_+(\mathcal{M}^e(X,T))+\omega_-(\mathcal{M}^e(X,T))\\ &=\|\omega_+\|+\|\omega_-\|=\|\omega_+-\omega_-\|\\ &=\|\omega\|,\end{aligned}$$

we have that

$$\|m-\frac{1}{n}(\delta_{\mu_1}+\cdots+\delta_{\mu_n})\|<\frac{1}{n}.$$

Hence $m(\{\mu_i\})>0$, $i=1,2,\cdots,n$. It follows from Proposition 6.1 3) that $\mu_1,\cdots,\mu_n$ are equilibrium states of f relative to $\mathcal{U}$. □

6.2. **Global equilibrium states.** Let $f\in C(X,\mathbb{R})$. A member μ of $\mathcal{M}(X,T)$ is called an *equilibrium state for f* if

$$P(T,f)=h_\mu(T)+\int_X f(x)d\mu(x).$$

We let $\mathcal{M}_f(X,T)$ denote the set of all equilibrium states for f. We note that $\mathcal{M}_f(X,T)$ can be an empty set (see e.g., [13, 20]). But it is not hard to see that if $h_{\mathrm{top}}(T)=\infty$, then $\mathcal{M}_f(X,T)=\{\mu\in\mathcal{M}(X,T):h_\mu(T)=\infty\}$. So for the rest of the section we assume that $h_{\mathrm{top}}(T)<\infty$.

A finite signed Borel measure μ on X is a *tangent functional* to $P(T,\cdot)$ at $f\in C(X,\mathbb{R})$ if

$$P(T,f+g)-P(T,f)\geq\int_X g(x)d\mu(x)\ \text{ for all } g\in C(X,\mathbb{R}).$$

We let $\mathcal{T}_f(X,T)$ denote the set of all tangent functionals to $P(T,\cdot)$ at f.

Define

$$\begin{aligned}\mathcal{M}^u(X,T)=&\{\mu\in\mathcal{M}(X,T):h_{\{\cdot\}}(T)\text{ is upper semi-continuous at }\mu\},\\ \mathcal{M}^l_f(X,T)=&\{\mu\in\mathcal{M}(X,T):\text{there exist }\mathcal{U}_n\in\mathcal{C}^o_X\text{ with diam}(\mathcal{U}_n)\to0,\ g_n\in C(X,\mathbb{R})\text{ with}\\ &\|g_n-f\|\to0,\text{ and }\mu_n\in\mathcal{M}_{g_n}(X,T;\mathcal{U}_n),\text{ such that }\mu_n\to\mu\},\\ \mathcal{M}^{sl}_f(X,T)=&\{\mu\in\mathcal{M}(X,T):\text{there exist }\mathcal{U}_n\in\mathcal{C}^o_X\text{ with diam}(\mathcal{U}_n)\to0\text{ and}\\ &\mu_n\in\mathcal{M}_f(X,T;\mathcal{U}_n),\text{ such that }\mu_n\to\mu\}.\end{aligned}$$

It follows from the Hahn-Banach theorem that $\mathcal{T}_f(X,T)$ is non-empty. It is also easy to see that both $\mathcal{M}^{sl}_f(X,T)$ and $\mathcal{M}^l_f(X,T)$ are non-empty, closed, and, $\mathcal{M}^{sl}_f(X,T)\subseteq\mathcal{M}^l_f(X,T)$. The set $\mathcal{M}^u(X,T)$ can be empty because the entropy map $h_{\{\cdot\}}(T)$ needs not have any points of upper semi-continuity in general. This is in fact the main obstruction for the existence of an equilibrium state for f.

The general connections of these sets are the following.

Proposition 6.4. *The following holds.*

1) $\mathcal{M}_f(X,T) \subseteq \mathcal{T}_f(X,T) \subseteq \mathcal{M}(X,T)$.
2) $\mathcal{T}_f(X,T) = \bigcap_{n=1}^{\infty} cl(\{\mu \in \mathcal{M}(X,T) : \ h_\mu(T) + \int_X f d\mu > P(T,f) - \frac{1}{n}\})$.
3) $\mathcal{M}_f(X,T) = \mathcal{T}_f(X,T) \cap \mathcal{M}^u(X,T)$.
4) $\mathcal{M}_f^l(X,T) = \mathcal{T}_f(X,T)$
5) $\mathcal{M}_f^l(X,T) \cap \mathcal{M}^u(X,T) = \mathcal{M}_f(X,T)$.

Proof. 1) and 2) are precisely the Theorem 9.14 and the Remark of Theorem 9.15 in [27].

3) Using 2) we have that $\mathcal{T}_f(X,T) \cap \mathcal{M}^u(X,T) \subseteq \mathcal{M}_f(X,T)$. Now let $\mu \in \mathcal{M}_f(X,T)$, i.e., $h_\mu(T) + \int_X f(x)d\mu(x) = P(T,f)$. If $\mu_n \in \mathcal{M}(X,T)$, $\mu_n \to \mu$, then by (1.1),

$$h_{\mu_n}(T) + \int_X f(x)d\mu_n(x) \le P(T,f),$$

i.e.,

$$h_{\mu_n}(T) \le h_\mu(T) + (\int_X f(x)d\mu(x) - \int_X f(x)d\mu_n(x)).$$

Hence $\limsup_{n\to\infty} h_{\mu_n}(T) \le h_\mu(T)$, i.e., the entropy map $h_{\{\cdot\}}(T)$ is upper semi-continuous at μ. We thus have $\mu \in \mathcal{T}_f(X,T) \cap \mathcal{M}^u(X,T)$.

4) Let $\mu \in \mathcal{M}_f^l(X,T)$ and let $\mathcal{U}_n \in \mathcal{C}_X^o$, $g_n \in C(X,\mathbb{R})$ and $\mu_n \in \mathcal{M}_{g_n}(X,T;\mathcal{U}_n)$ be such that $\mathrm{diam}(\mathcal{U}_n) \to 0$, $\|g_n - f\| \to 0$ and $\mu_n \to \mu$. Then for any $g \in C(X,\mathbb{R})$, we have

$$P(T, g_n + g;\mathcal{U}_n) - P(T, g_n;\mathcal{U}_n) \ge \int_X g(x)d\mu_n(x),$$

and

$$\begin{aligned} P(T, g_n + g) &\ge P(T, g_n + g;\mathcal{U}_n) \ge P(T, g_n;\mathcal{U}_n) + \int_X g(x)d\mu_n(x) \\ &\ge (P(T, f;\mathcal{U}_n) - \|f - g_n\|) + \int_X g(x)d\mu_n(x). \end{aligned}$$

By taking the limit $n \to \infty$, we have

$$P(T, f + g) \ge P(T,f) + \int_X g(x)d\mu(x), \quad \text{for all } g(x) \in C(X,\mathbb{R}).$$

This shows that $\mu \in \mathcal{T}_f(X,T)$.

Conversely, let $\mu \in \mathcal{T}_f(X,T)$. Then it follows from 2) that

$$\mu \in \mathrm{cl}(\{\nu \in \mathcal{M}(X,T) : \ h_\nu(T) + \int_X f d\nu > P(T,f) - \frac{1}{n^2}\}) \tag{6.2}$$

for all $n \in \mathbb{N}$. Let d be a prescribed compatible metric on $\mathcal{M}(X,T)$. Without loss of generality, we assume that $d(\theta_1,\theta_2) \le \|\theta_1 - \theta_2\|$ for any $\theta_1,\theta_2 \in \mathcal{M}(X,T)$. For each $n \in \mathbb{N}$, we have by (6.2) that there exists a $\mu' \in \mathcal{M}(X,T)$ such that

$$h_{\mu'}(T) + \int_X f d\mu' > P(T,f) - \frac{1}{n^2} \ \text{ and } d(\mu',\mu) < \frac{1}{n}.$$

Then by Lemma 2.7, for each $n \in \mathbb{N}$, there exists a $\mathcal{U}_n \in \mathcal{C}_X^o$ such that $\mathrm{diam}(\mathcal{U}_n) < \frac{1}{n}$ and

$$h_{\mu'}(T,\mathcal{U}_n) \ge h_{\mu'}(T) - (h_{\mu'}(T) + \int_X f d\mu' - P(T,f) + \frac{1}{n^2}),$$

from which we have

$$P(T,f,\mathcal{U}_n) - (h_{\mu'}(T,\mathcal{U}_n) + \int_X f d\mu') \le P(T,f) - (h_{\mu'}(T,\mathcal{U}_n) + \int_X f d\mu') \le \frac{1}{n^2}.$$

For each $n \in \mathbb{N}$, we apply Lemma 6.3 for μ', f and $\mathcal{U}_n$ to obtain $\mu_n \in \mathcal{M}(X,T)$ and $f_n \in C(X,\mathbb{R})$ such that

$$\|\mu_n - \mu'\| < \frac{1}{n},$$

$$\|f_n - f\| \le n(P(T,f,\mathcal{U}_n) - (h_{\mu'}(T,\mathcal{U}_n) + \int_X f d\mu')) \le \frac{1}{n}.$$

Since $d(\mu_n,\mu) \le d(\mu,\mu') + d(\mu',\mu_n) \le d(\mu,\mu') + \|\mu' - \mu_n\| \le \frac{2}{n}$, we have $\mu_n \to \mu$. Hence $\mu \in \mathcal{M}^l_f(X,T)$.

5) follows immediately from 3) and 4). □

From the proof of part 4) of Proposition 6.4, we also have the following.

Proposition 6.5. *Given $\{\mathcal{V}_n\} \subset \mathcal{C}^o_X$ with $\operatorname{diam}(\mathcal{V}_n) \to 0$. Let d be a prescribed compatible metric on $\mathcal{M}(X,T)$. Then*

$$\mathcal{M}^l_f(X,T) = \{\mu \in \mathcal{M}(X,T) : \text{ for any } \epsilon > 0 \text{ and } M \in \mathbb{N} \text{ there exist } g_{\epsilon,M} \in C(X,\mathbb{R}),\ L \ge M \text{ and } \mu_{\epsilon,M} \in \mathcal{M}_{g_{\epsilon,M}}(X,T;\mathcal{V}_L) \text{ such that } \|g_{\epsilon,M} - f\| \le \epsilon \text{ and } d(\mu,\mu_{\epsilon,M}) < \epsilon\}.$$

We remark that points of $\mathcal{M}^l_f(X,T)$ need not be equilibrium states for f in general. But Proposition 6.4 4) asserts that $\mu \in \mathcal{M}^l_f(X,T)$ is an equilibrium state for f iff it is a point of upper semi-continuity of $h_{\{.\}}(T)$. This gives a necessary and sufficient condition for a point of $\mathcal{M}^l_f(X,T)$ to become an equilibrium state for f. Of course, such a condition needs not be satisfied in general. One exception is the case when the entropy map $h_{\{.\}}(T)$ can be realized by local ones in the sense that there exists a $\mathcal{U} \in \mathcal{C}^o_X$ such that

$$h_\mu(T,\mathcal{U}) = h_\mu(T) \quad \text{for all } \mu \in \mathcal{M}(X,T). \tag{6.3}$$

Define

$$h^M(T,\mathcal{U}) = \sup_{\mu\in\mathcal{M}(X,T)} (h_\mu(T) - h_\mu(T,\mathcal{U})),\ \mathcal{U} \in \mathcal{C}^o_X.$$

Then it is clear that $h^M(T,\mathcal{U}) \le h^M(T,\mathcal{V})$ whenever $\mathcal{U} \succeq \mathcal{V}$. Moreover, (6.3) holds for an $\mathcal{U} \in \mathcal{C}^o_X$ iff $h^M(T,\mathcal{U}) = 0$. Also define

$$h^M(T) \equiv \inf_{\mathcal{U}\in\mathcal{C}^o_X} h^M(T,\mathcal{U}).$$

Then $h^M(T) = 0$ gives a weaker notion of realization of the entropy map by local ones. By Lemma 2.10, we have

$$0 \le h^M(T,\mathcal{U}) \le h(T|\mathcal{U}),\ \mathcal{U} \in \mathcal{C}^o_X, \quad \text{and } 0 \le h^M(T) \le h^*(T), \tag{6.4}$$

where $h^*(T) = \inf_{\mathcal{U}\in\mathcal{C}^o_X} h(T|\mathcal{U})$ is the conditional entropy of T.

Proposition 6.6. *If $h^M(T,\mathcal{U}) = 0$ for some $\mathcal{U} \in \mathcal{C}^0_X$, then for each $f \in C(X,\mathbb{R})$*

$$\mathcal{M}^{ls}_f(X,T) = \mathcal{M}^l_f(X,T) = \mathcal{T}_f(X,T) = \mathcal{M}_f(X,T). \tag{6.5}$$

Proof. Note that the condition $h^M(T,\mathcal{U}) = 0$ implies that (6.3) holds. It follows from Theorem 1 that the entropy map $h_{\{.\}}(T)(= h_{\{.\}}(T,\mathcal{U})) : \mathcal{M}(X,T) \to \mathbb{R}$ is upper semi-continuous. Hence $\mathcal{M}^u(X,T) = \mathcal{M}(X,T)$. By (1.1), (6.3), and Theorem 2, we also have $P(T,f;\mathcal{U}) = P(T,f)$ for each $f \in C(X,\mathbb{R})$. Hence $\mathcal{M}_f(X,T) = \mathcal{M}_f(X,T;\mathcal{U})$ and $\mathcal{T}_f(X,T) = \mathcal{T}_f(X,T;\mathcal{U})$ for each $f \in C(X,\mathbb{R})$. It follows from Proposition 6.1 4) and Proposition 6.4 3) that $\mathcal{M}^l_f(X,T) = \mathcal{T}_f(X,T) = \mathcal{M}_f(X,T)$ for each $f \in C(X,\mathbb{R})$.

Let $\{\mathcal{U}_n\} \subset \mathcal{C}^o_X$ be a sequence such that $\mathcal{U}_n \succeq \mathcal{U}$ for each n and $\operatorname{diam}(\mathcal{U}_n) \to 0$. It follows from (6.3) that $h_\mu(T,\mathcal{U}_n) = h_\mu(T)$ for all n and all $\mu \in \mathcal{M}(X,T)$. Again, by (1.1) and Theorem 2, we have $P(T,f;\mathcal{U}_n) = P(T,f)$ and hence $\mathcal{M}_f(X,T) = \mathcal{M}_f(X,T;\mathcal{U}_n)$ for all n and $f \in C(X,\mathbb{R})$. This shows that for $f \in C(X,\mathbb{R})$, $\mathcal{M}^{sl}_f(X,T) \supseteq \mathcal{M}_f(X,T)$, and hence $\mathcal{M}^{sl}_f(X,T) = \mathcal{M}_f(X,T)$. □

Lemma 6.4. *$h^M(T)=0$ iff the entropy map $h_{\{\cdot\}}(T):\mathcal{M}(X,T)\to\mathbb{R}$ is upper semi-continuous.*

Proof. Let $h^M(T)=0$. Fix a $\mu\in\mathcal{M}(X,T)$. For any given $\epsilon>0$, we let $\mathcal{V}\in\mathcal{C}_X^o$ be such that $h^M(T,\mathcal{V})\le\epsilon$. Since $h_{\{\cdot\}}(T,\mathcal{V})$ is upper semi-continuous,

$$\limsup_{\nu\to\mu} h_\nu(T)\le\limsup_{\nu\to\mu} h_\nu(T,\mathcal{V})+\epsilon\le h_\mu(T,\mathcal{V})+\epsilon\le h_\mu(T)+\epsilon.$$

Since ϵ is arbitrary, $\limsup_{\nu\to\mu} h_\nu(T)\le h_\mu(T)$. This shows the upper semi-continuity of the entropy map $h_{\{\cdot\}}(T)$.

Conversely, let $h_{\{\cdot\}}(T):\mathcal{M}(X,T)\to\mathbb{R}$ be upper semi-continuous. If $h^M(T)>0$, then there are sequences $\{\mathcal{U}_n\}_{n=1}^\infty\subset\mathcal{C}_X^o$ with $\operatorname{diam}(\mathcal{U}_n)\to 0$ and $\{\mu_n\}_{n=1}^\infty\subset\mathcal{M}(X,T)$ such that

$$h_{\mu_n}(T)-h_{\mu_n}(T,\mathcal{U}_n)\ge\frac{h^M(T)}{2}\ \text{ for all } n.$$

Without loss of generality, we assume that $\mathcal{U}_{n+1}\succeq\mathcal{U}_n$ for all n.

By taking subsequence if necessary, we let $\mu_n\to\mu$ in the weak*-topology. Fix $m\in\mathbb{N}$. For any $n\ge m$, we have

$$h_{\mu_n}(T)-h_{\mu_n}(T,\mathcal{U}_m)\ge h_{\mu_n}(T)-h_{\mu_n}(T,\mathcal{U}_n)\ge\frac{h^M(T)}{2}.$$

Letting $n\to\infty$, it follows from the upper semi-continuity of $h_{\{\cdot\}}(T)$ and $h_{\{\cdot\}}(T,\mathcal{U}_m)$ that $h_\mu(T)-h_\mu(T,\mathcal{U}_m)\ge\frac{h^M(T)}{2}>0$. But since $h_\mu(T)\le h_{\rm top}(T)<\infty$, $\lim_{m\to\infty}(h_\mu(T)-h_\mu(T,\mathcal{U}_m))=0$, a contradiction. □

Proposition 6.7. *If $h^M(T)=0$, then*

$$\mathcal{M}_f^l(X,T)=\mathcal{T}_f(X,T)=\mathcal{M}_f(X,T).$$

Proof. By Lemma 6.4, the entropy map $h_{\{\cdot\}}(T):\mathcal{M}(X,T)\to\mathbb{R}$ is upper semi-continuous, i.e., $\mathcal{M}^u(X,T)=\mathcal{M}(X,T)$. The proposition now follows from Proposition 6.4 3)-5). □

We note that the condition $h_{\rm top}(T)<\infty$ a prior assumed at the beginning of the subsection is actually implied by the condition $h^M(T)=0$. Indeed, if $h^M(T)=0$, then there exists a $\mathcal{U}\in\mathcal{C}_X^o$ such that $h^M(T,\mathcal{U})\le 1$. Since

$$h^M(T,\mathcal{U})\ge\sup_{\mu\in\mathcal{M}(X,T)}(h_\mu(T)-h_{\rm top}(T,\mathcal{U}))=h_{\rm top}(T)-h_{\rm top}(T,\mathcal{U}),$$

we have

$$h_{\rm top}(T)\le h^M(T,\mathcal{U})+h_{\rm top}(T,\mathcal{U}))\le 1+\log N(\mathcal{U})<\infty.$$

We now discuss two classes of weak expansive systems: the h-expansive and asymptotically h-expansive systems, introduced by Bowen [6] and Misiurewicz [20], respectively. Given $n\in\mathbb{N}$ and $\epsilon>0$. A subset $E\subset X$ is said to (n,ϵ)*-spans* another subset $F\subset X$ (with respect to T), if for each $y\in F$ there is a $x\in E$ so that $d(T^k(x),T^k(y))\le\epsilon$ for all $k=0,1,\cdots,n$. For a compact subset $K\subset X$, we let $r_n(K,\epsilon)=r_n(T,K,\epsilon)=\min\{\operatorname{card}E: E\ (n,\epsilon)\text{-spans K}\}$. Define

$$h(T,K)=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_n(K,\epsilon).$$

It is well known that $h_{\rm top}(T)=h(T,X)$. Let

$$\Phi_\epsilon(x)=\{y\in X: d(T^n(x),T^n(y))\le\epsilon \text{ for } n\ge 0\}$$

and define

$$h_T^*(\epsilon)=\sup_{x\in X} h(T,\Phi_\epsilon(x)).$$

(X,T) is called *h-expansive* if there exists an $\epsilon>0$ such that $h_T^*(\epsilon)=0$, and is called *asymptotically h-expansive* if $\lim_{\epsilon\to 0}h_T^*(\epsilon)=0$.

It is shown by Bowen [6] that expansive systems, expansive homeomorphisms, endomorphisms of a compact Lie group, and Axiom A diffeomorphisms are all h-expansive, by Misiurewicz [22] that every continuous endomorphism of a compact metric group is asymptotically h-expansive if its entropy is finite, and by Buzzi [8] that any C^∞ diffeomorphism on a compact manifold is asymptotically h-expansive.

The following characterization is given by Misiurewicz ([20]).

Lemma 6.5. *The following holds.*

1) *If $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X^o$, $\mathrm{diam}\mathcal{U} < \epsilon < \frac{\delta}{2}$, where δ is the Lebesgue number for $\mathcal{V}$, then $h(T|\mathcal{U}) \leq h_T^*(\epsilon) \leq h(T|\mathcal{V})$;*
2) *(X,T) is h-expansive iff there exists a $\mathcal{U} \in \mathcal{C}_X^o$ such that $h(T|\mathcal{U}) = 0$;*
3) *$h^*(T) = \lim_{\epsilon\to 0} h_T^*(\epsilon)$. Consequently, (X,T) is asymptotically h-expansive iff $h^*(T) = 0$.*

Proof. See [20], Lemma 2.1 and Corollary 2.1. □

Another characterization of asymptotically h-expansivity is recently given by Boyle and Downarowicz [5] as the following: (X,T) is asymptotically h-expansive iff it has a principal extension to a symbolic system.

Proposition 6.8. *The following holds.*

1) *If (X,T) is h-expansive, then there exists a $\mathcal{U} \in \mathcal{C}_X^o$ such that $h^M(T,\mathcal{U}) = 0$. Consequently,* Proposition 6.6 *holds for a h-expansive TDS.*
2) *If (X,T) is asymptotically h-expansive, then $h^M(T) = 0$. Consequently,* Proposition 6.7 *holds for an asymptotically h-expansive TDS.*

Proof. 1) Let (X,T) be h-expansive. Then by Lemma 6.5 2), there exists a $\mathcal{U} \in \mathcal{C}_X^o$ such that $h(T|\mathcal{U}) = 0$. It follows from Lemma 2.10 that $h_\mu(T) = h_\mu(T,\mathcal{U})$ for all $\mu \in \mathcal{M}(X,T)$, i.e., $h^M(T,\mathcal{U}) = 0$.

(2) Let (X,T) be asymptotically h-expansive. Then by Lemma 6.5 3) and (6.4), we have $0 \leq h^M(T) \leq h^*(T) = 0$, i.e., $h^M(T) = 0$. □

We remark that, based on discussions of upper semi-continuity of the entropy map, the existence of global equilibrium states and the equality $\mathcal{T}_f(X,T) = \mathcal{M}_f(X,T)$ are essentially known for an expansive TDS or an expansive homeomorphism (see [7, 24, 25]), and, more generally, for an asymptotically h-expansive TDS (see [22]). Our results above give a general treatment on both h-expansive and asymptotically h-expansive cases with respect to these issues by making use of local entropies and pressures, and meantime provide more information on the characterization of global equilibrium states.

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