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A CONDITION CONCERNING THE EXISTENCE OF CONTINUOUS SOLUTIONS TO POISSON'S EQUATION

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TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS ii

Chapter

I. INTRODUCTION 1

II. A CONDITION CONCERNING THE EXISTENCE OF CONTINUOUS SOLUTIONS TO POISSON'S EQUATION 6

A. Theorem 1 6

B. Definitions and Theorem 2 9

C. Theorem 3 18

D. Theorem 4 32

BIBLIOGRAPHY 34

CHAPTER I

INTRODUCTION

In the solution of technological problems the second order linear nonhomogeneous partial differential equation

$$\Delta u(x_1, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial^2 u(x_1, \ldots, x_n)}{\partial x_i^2} = - w_n \, g(x_1, \ldots, x_n)$$

frequently arises, either as the result of the formulation of a mathematical model approximating a given physical phenomenon or from some abstract approach in function theory. Here, w_n is a positive constant, dependent upon n, representing the surface area of the unit sphere in n dimensions, and g is a real-valued function of n real variables defined on some bounded n-dimensional domain D which is not necessarily "large." This particular partial differential equation is commonly known as Poisson's equation, or, alternatively, the nonhomogeneous Laplace equation.

In the case in which this partial differential equation has been physically motivated by some particular occurrence, the function g, which describes some physical quantity, will more often than not be bounded and continuous on D and possess other regularity conditions such as belonging to C^1, C^2, or perhaps even C^∞ on D; however, such conditions normally stem from the model-building steps pertinent to the given problem. Under suitably stringent regularity conditions such as the above, it is well known that continuous solutions to Poisson's equation will exist; in particular, one such continuous solution on D can be represented by

$$u(x_1, \ldots, x_n) = \int \underset{D}{\cdots} \int g(x_1', \ldots, x_n') \gamma(r) \, dx_1' \cdots dx_n' ,$$

where $r = [\sum_{i=1}^{n} (x_i' - x_i)^2]^{1/2}$, $\gamma(r) = -\log r$ for $n = 2$, and $\gamma(r) = r^{2-n}$ for $n > 2$. More specifically, it can be shown [3] that the n-fold integral defined above represents a continuous solution to $\Delta u(x_1, \ldots, x_n) = -w_n \, g(x_1, \ldots, x_n)$ on D even in the case in which g satisfies only a Hölder condition with exponent α, $0 < \alpha < 1$, on D; that is,

$$|g(x_1', \ldots, x_n') - g(x_1'', \ldots, x_n'')| \leq K\Big[\sum_{i=1}^{n} (x_i' - x_i'')^2\Big]^{\alpha/2}$$

for every pair of points $(x_1', \ldots, x_n')$ and $(x_1'', \ldots, x_n'')$ in D. Here, as in all that follows, a continuous solution to Poisson's equation on D will be defined as any real-valued function continuous on D which satisfies the differential equation at <u>every</u> point in D.

On the other hand, in the abstract sense where physical intuition is of no benefit, one may not be able to deduce such generous smoothness conditions through his formal analysis, or he may not even have such restrictions at his disposal. In this case the questions arise: Do continuous solutions to Poisson's equation on D still always exist, and if so, does the n-fold integral defined above provide one such solution? For the case in which g is merely continuous and bounded on D the answer is "no" in both cases. Hadamard [4] has exhibited a continuous bounded function g of two real variables defined on a two-dimensional bounded domain D for which the pertinent double integral fails to provide a solution and has then stated, without proof, that no continuous solution to $\Delta u(x_1, x_2) = -w_2\ g(x_1, x_2)$ on D can exist; the verification of this final statement follows from the result of Theorem 4 in the text.

Upon learning of this result, another question quite natural to ask would be: For a given continuous and bounded function g defined on a bounded domain D, under what conditions, if any, can one expect to find a continuous solution to Poisson's equation on D? The answer to this inquiry is stated and proved in detail in the text for the case in which $n = 2$. Wintner [6] and Zaremba [7] have provided a source of ideas for this proof, and enough information is contained within these works to deduce the analogous result for the case in which $n = 3$ by using only minor modifications of the techniques to be demonstrated. The problem when $n > 3$ will not be considered here; however, it should be pointed out that the difficulty experienced in dimensions two and three in the evaluation and estimation of certain multiple integrals leads this author to suspect that the same method of proof might not prove advantageous for the higher dimensional cases. For the case in which $n = 2$ or $n = 3$ the main result is as follows: Let g be a real-valued function of two (three) real variables defined, continuous, and bounded on a bounded domain $D \subset E_2\ (E_3)$; unless

$$u_D(x,y) = \frac{1}{2\pi}\iint_D g(x',y') \log r \, dx' \, dy'$$

$$\left(u_D(x,y,z) = -\frac{1}{4\pi}\iiint_D \frac{g(x',y',z')}{r} \, dx' \, dy' \, dz'\right)$$

represents a solution of $\Delta u(x,y) = g(x,y)$ $\left(\Delta u(x,y,z) = g(x,y,z)\right)$ on D, then there exists no continuous solution to $\Delta u(x,y) = g(x,y)$ $\left(\Delta u(x,y,z) = g(x,y,z)\right)$ on D.

The results needed for the proof of the assertion above (for the case in which $n = 2$) are contained in the preliminary three theorems and two lemmas in the text. These theorems are stated, without proof, by Wintner [6] in a bare outline of a proof for the foregoing assertion; however, in doing so he has made reference to the work of Zaremba [7], from which stem the key ideas and methods used in the verification of Theorems 2 and 3 in the text. Briefly, Theorem 1 is a result necessary for judging the relationship between the two-dimensional Laplacian of a function and the limiting case of the five point Laplace difference operator. Theorem 2 states a more general criterion than is normally used for deducing the harmonic character of a given function, and Theorem 3, together with its accompanying lemmas, shows that $u_D(x,y)$ satisfies the limiting case of the nonhomogeneous five point Laplace difference equation for all (x,y) in D. The verification of the main theorem then readily follows upon utilizing these preliminary results in a certain order. As a by-product, the general structure of all continuous solutions to $\Delta u(x,y) = g(x,y)$, under the given hypotheses on g, is also deduced.

CHAPTER II

A CONDITION CONCERNING THE EXISTENCE OF CONTINUOUS SOLUTIONS TO POISSON'S EQUATION

In this chapter all of the results cited in the Introduction are demonstrated. The first discussion to take place will be that concerning Theorem 1; key use of an extension of a mean value theorem will be made in the verification of this theorem.

Theorem 1: Let f be a real-valued function of one real variable defined and continuous on a bounded interval $[a,b]$. Further, suppose that $f''(x)$ exists for each point x in (a,b). Then $\lim_{h\to o}\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$ for each point x in (a,b).

Proof: It should first be noted that $\varphi(h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$ is symmetric in h for each fixed x in (a,b); that is $\varphi(h) = \varphi(-h)$ for each fixed x in (a,b) and every number h such that $x+h$ and $x-h$ lie in (a,b). Consequently, there is no loss in generality in assuming that $h > 0$.

Let x be any arbitrary fixed point in (a,b) and define two functions F and G on (a,b) by the equations

$$F(z) = f(x + z) - 2f(x) + f(x - z)$$

and

$$G(z) = z^2 .$$

Here, z is any positive number such that $x + z$ and $x - z$ lie in (a,b). Then

$$F'(z) = f'(x + z) - f'(x - z),$$

due to the hypotheses governing f, and

$$G'(z) = 2z .$$

Using an extension of a mean value theorem, one can infer the existence of a number η, $0 < \eta < h$, such that

$$\frac{F(h) - F(0)}{G(h) - G(0)} = \frac{F'(\eta)}{G'(\eta)} ;$$

hence,

$$\frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = \frac{f'(x + \theta h) - f'(x - \theta h)}{2\theta h} \qquad (1)$$

for θ such that $0 < \theta < 1$. Addition and subtraction of $f'(x)$ to the right-hand member of the preceding equation, together with a slight rearrangement, then lead to the result

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \frac{1}{2}\left[\frac{f'(x+\theta h) - f'(x)}{\theta h} + \frac{f'(x) - f'(x-\theta h)}{\theta h}\right] .$$

The last equation is valid for all $h > 0$ such that $x + h$ and $x - h$ lie in (a,b). This implies the validity of the equation upon passage to the limit, provided the component limits exist. Thus,

$$\lim_{h \to 0^+} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} =$$

$$= \frac{1}{2}\left[\lim_{h \to o^+} \frac{f'(x + \theta h) - f'(x)}{\theta h} + \lim_{h \to o^+} \frac{f'(x) - f'(x - \theta h)}{\theta h}\right]$$

$$= \frac{1}{2}\left[f''(x) + f''(x)\right]$$

$$= f''(x) \ ,$$

where the hypothesis concerning the existence of $f''(x)$ for each point x in (a,b) has been utilized. The conclusion of the theorem then follows since x was any arbitrary point in (a,b).

The terms used in the proof of the ensuing theorems will necessitate the following definition.

Definition: Let u be a real-valued function of two real variables defined and continuous on a two-dimensional domain D. If u satisfies $\Delta u(x,y) = 0$ for all (x,y) in D, then u is *harmonic in D*. The function u is *harmonic at a point* (x_o, y_o) in D if u is harmonic in some neighborhood of (x_o, y_o).

It should be noted here that the definition above, while superficially not the same as that advanced in many potential theory and complex variable texts, is consistent with the usual definition; this fact is well known from complex variable theory.

The technique of proof used in the verification of Theorem 2 is a rather indirect one. First, an intermediary function ψ is introduced which is a linear combination of a harmonic function v satisfying certain boundary conditions, a solution w to Poisson's equation satisfying other boundary conditions, and the function u which is to be shown harmonic. Various properties of ψ are then derived, the most important of which is that $\psi(x,y) \leq 0$ on a disk in D. Lastly, the function u is compared with v, and the two are shown to be identical on a disk in D through the use of the maximum property of ψ; this leads to the desired result.

<u>Theorem 2</u>: Let u be a real-valued function of two real variables defined and continuous on a bounded domain $D \subset E_2$. Suppose

$$\Delta^* u(x,y) = \lim_{h \to o} \frac{u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)}{h^2}$$
$$= 0$$

for every point (x,y) in D. Then u is a harmonic function on D.

<u>Proof</u>: Let (x_o, y_o) be any arbitrary fixed point in D and denote by $\rho_{x_o y_o}$ the minimum distance from (x_o, y_o) to ∂D. Let

$$\bar{C}_{x_o y_o} = \{(x,y) \mid (x - x_o)^2 + (y - y_o)^2 \leq \frac{\rho_{x_o y_o}}{2}\}$$

and $C_{x_o y_o} = \bar{C}_{x_o y_o} - \partial\bar{C}_{x_o y_o}$. Then define on $\bar{C}_{x_o y_o}$ the real-valued continuous function v satisfying the following two properties: $\Delta v(x,y) = 0$ in $C_{x_o y_o}$ and $v(x,y) = u(x,y)$ on $\partial\bar{C}_{x_o y_o}$. That such a function v exists and is unique follows immediately from the results of Dirichlet's problem in complex variable theory. The function v is then harmonic in $C_{x_oy_o}$ according to the preceding definition. Further, let w be a real-valued continuous function defined on $\bar{C}_{x_oy_o}$ having the properties that $\Delta w(x,y) = 1$ in $C_{x_oy_o}$ and $w(x,y) = 0$ on $\partial\bar{C}_{x_oy_o}$. Again such a function exists as is easily seen by considering the function w defined on $\bar{C}_{x_oy_o}$ by

$$w(x,y) = \frac{1}{4}[(x - x_o)^2 + (y - y_o)^2] + h(x,y) ,$$

where h is a harmonic function in $C_{x_oy_o}$ having value $\frac{-\rho_{x_oy_o}}{8}$ on $\partial\bar{C}_{x_oy_o}$.

Now consider the function ψ defined on $\bar{C}_{x_oy_o}$ by

$$\psi(x,y) = \eta[u(x,y) - v(x,y)] + t^2w(x,y) ,$$

where $\eta = \pm 1$ and t is any real nonzero number. It will first be shown that $\Delta^*\psi(x,y) = t^2$ for any point (x,y) in $C_{x_oy_o}$ and every real nonzero number t.

Let (x,y) be any arbitrary point in $C_{x_oy_o}$. By definition

$$\begin{aligned}\Delta^*\psi(x,y) = \lim_{h\to o} \{ &\eta[u(x + h,y) - v(x + h,y)] + t^2w(x + h,y) \\ &+ \eta[u(x,y+h) - v(x,y+h)] + t^2w(x,y+h) + \eta[u(x-h, y) - v(x-h,y)] \\ &+ t^2w(x-h,y) + \eta[u(x,y-h) - v(x,y-h)] + t^2w(x,y-h) \\ &-4\eta[u(x,y) - v(x,y)] - 4t^2w(x,y)\} .\end{aligned}$$

Rearrangement of the terms, together with applications of standard limit theorems, leads to

$$\begin{aligned}\Delta^*\psi(x,y) = \eta\Big\{\Delta^*u(x,y) &- \lim_{h\to o}\left[\frac{v(x+h,y) - 2v(x,y) + v(x-h,y)}{h^2}\right] \\ &- \lim_{h\to o}\left[\frac{v(x,y+h) - 2v(x,y) + v(x,y-h)}{h^2}\right]\Big\} \\ &+ t^2\Big\{\lim_{h\to o}\left[\frac{w(x+h,y) - 2w(x,y) + w(x-h,y)}{h^2}\right] \\ &+ \lim_{h\to o}\left[\frac{w(x,y+h) - 2w(x,y) + w(x,y-h)}{h^2}\right]\Big\}.\end{aligned}$$

The first term in the right-hand member of the preceding equation is zero due to the given hypotheses regarding u, and application of Theorem 1 shows that the remaining terms in brackets are $\frac{\partial^2 v(x,y)}{\partial x^2}$, $\frac{\partial^2 v(x,y)}{\partial y^2}$, $\frac{\partial^2 w(x,y)}{\partial x^2}$, and $\frac{\partial^2 w(x,y)}{\partial y^2}$ respectively. Hence,

$$\Delta^*\psi(x,y) = -\eta\left[\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2}\right] + t^2\left[\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}\right],$$

or

$$\begin{aligned}\Delta^*\psi(x,y) &= -\eta(0) + t^2(1) \\ &= t^2 .\end{aligned}$$

This last result is true for all points (x,y) in $C_{x_o y_o}$ and every real nonzero number t, as was to be demonstrated.

Being continuous on the compact set $\bar{C}_{x_o y_o}$, ψ will attain a maximum on $\bar{C}_{x_o y_o}$. It is now asserted that ψ cannot attain this maximum in $C_{x_oy_o}$; this result will be proved through the use of contradiction.

Suppose that ψ does attain a maximum at some point (x_1, y_1) in $C_{x_oy_o}$. Then there exist numbers δ_1, $0 < \delta_1 < \frac{\rho_{x_oy_o}}{2}$, and δ_2, $0 < \delta_2 < \frac{\rho_{x_oy_o}}{2}$, such that $\psi(x_1, y_1) \geq \psi(x,y)$ for all (x,y) in $C_{x_oy_o}$ such that $|x - x_1| < \delta_1$ and $|y - y_1| < \delta_2$. Let h be any number such that $0 < |h| < \delta^*$, where $\delta^* = \min(\delta_1, \delta_2)$. Then

$$\frac{\psi(x_1+h, y) + \psi(x_1,y_1+h) + \psi(x_1-h,y_1) + \psi(x_1,y_1-h) - 4\psi(x_1,y_1)}{h^2}$$

$$= \frac{\psi(x_1+h,y_1) - \psi(x_1,y_1)}{h^2} + \frac{\psi(x_1,y_1+h) - \psi(x_1,y_1)}{h^2}$$

$$+ \frac{\psi(x_1-h,y_1) - \psi(x_1,y_1)}{h^2} + \frac{\psi(x_1,y_1-h) - \psi(x_1,y_1)}{h^2}$$

$$< 0$$

for every number h such that $0 < |h| < \delta^*$. This implies that $\Delta^*\psi(x_1,y_1) \leq 0$, a contradiction to the fact that $\Delta^*\psi(x,y) = t^2 > 0$ for all (x,y) in $C_{x_oy_o}$. By the very principle of contradiction it follows that ψ does not attain a maximum in $C_{x_oy_o}$; hence, the maximum of ψ occurs on $\partial\overline{C}_{x_oy_o}$. But $\psi(x,y) = 0$ on $\partial\overline{C}_{x_oy_o}$ by its very definition; therefore, $\psi(x,y) \leq 0$ for all points (x,y) in $\overline{C}_{x_oy_o}$.

Finally, it will be demonstrated that $u(x,y) - v(x,y) = 0$ for all (x,y) in $C_{x_0y_0}$. As before, the method of contradiction will be utilized.

Suppose that $u(x,y) - v(x,y)$ is not identically zero on $C_{x_0y_0}$. There then exists a point (x_1, y_1) in $C_{x_0y_0}$ such that either $u(x_1, y_1) - v(x_1, y_1) < 0$ or $u(x_1, y_1) - v(x_1, y_1) > 0$. If $u(x_1, y_1) - v(x_1, y_1) < 0$, choose $\eta = -1$ and t so small that $t^2|w(x_1, y_1)| < (-1)[u(x_1, y_1) - v(x_1, y_1)]$. This choice of η and t gives

$$\psi(x_1, y_1) = (-1)[u(x_1, y_1) - v(x_1, y_1)] + t^2w(x_1, y_1) > 0 ,$$

a contradiction to the result that $\psi(x,y) \leq 0$ for all (x,y) in $\overline{C}_{x_0y_0}$. Similarly, if $u(x_1, y_1) - v(x_1, y_1) > 0$, choose $\eta = 1$ and t such that $t^2|w(x_1, y_1)| < u(x_1, y_1) - v(x_1, y_1)$. Then

$$\psi(x_1, y_1) = [u(x_1, y_1) - v(x_1, y_1)] + t^2 w(x_1, y_1) > 0 ,$$

which is again a contradiction to the fact that $\psi(x,y) \leq 0$ for all (x,y) in $\overline{C}_{x_0y_0}$.

From the analysis of the preceding paragraph it is evident that the original assumption, $u(x,y) - v(x,y)$ being not identically zero on $C_{x_0y_0}$, is invalid. Thus, $u(x,y) = v(x,y)$ for all (x,y) in $C_{x_0y_0}$. Consequently, u is harmonic in $C_{x_0y_0}$. But (x_0, y_0) was arbitrarily chosen in D, so the argument above is valid for any point in D. Therefore, u is harmonic in some neighborhood of any point in D which implies that u is harmonic at each point in D. This result shows that u is harmonic in D, as was to be demonstrated.

The verification of Theorem 3 is, by far, the most tedious of the entire text. Two lemmas are first demonstrated to facilitate matters in the main body of this proof.

Lemma 1: For any number s such that $0 \leq s < 1$ and any real number t,

$$\log(1 - 2s \cos t + s^2)^{1/2} = - \sum_{n=1}^{\infty} \frac{s^n \cos nt}{n} .$$

Proof: It should first be noted that

$$1 - 2s \cos t + s^2 \geq (1 - s)^2 > 0$$

for $0 \leq s < 1$ so that the logarithm of this argument is, in fact, well defined. Now consider the series $\frac{1}{2} + \sum_{n=1}^{\infty} z^n$, where $z = se^{i\theta}$ is any complex number having magnitude less than one. Since $|z| = s < 1$, the complex geometric series converges, and

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1}{2} + \frac{z}{1 - z} = \frac{1 + z}{2(1 - z)} .$$

Expansion of z into its polar form and simplification lead to

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1 - s^2 + 2 \text{ is } \cos \theta}{2(1 - 2s \cos \theta + s^2)} \tag{2}$$

for all s such that $0 \leq s < 1$ and all θ. On the other hand,

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1}{2} + \sum_{n=1}^{\infty} s^n(\cos n\theta + i \sin n\theta) \tag{3}$$

through use of DeMoivre's theorem. The equating of imaginary parts in Equations (2) and (3) now allows one to infer that

$$\sum_{n=1}^{\infty} s^n \sin n\theta = \frac{s \sin \theta}{1 - 2s \cos \theta + s^2} \ . \tag{4}$$

Equation (4) is an identity in s and θ for $0 \leq s < 1$ and all θ; hence, definite integration with respect to θ over both members retains this equality. Thus,

$$\int_0^t \sum_{n=1}^{\infty} \frac{s^n \sin n\theta}{n} \, d\theta = \ln(1 - 2s \cos t + s^2)^{1/2} - \ln(1 - s) \ .$$

Since the infinite series above converges uniformly in θ for all θ, the order of limit and integration operations may be inverted; this inversion yields

$$-\sum_{n=1}^{\infty} \frac{s^n \cos nt}{n} + \sum_{n=1}^{\infty} \frac{s^n}{n} = \ln(1 - 2s \cos t + s^2)^{1/2} - \ln(1 - s) \tag{5}$$

However, $\sum_{n=1}^{\infty} \frac{s^n}{n}$ is nothing more than the unique Taylor Series expansion about zero of $-\ln(1 - s)$ for all s such that $0 \leq s < 1$. Equation (5) must then reduce to

$$-\sum_{n=1}^{\infty} \frac{s^n \cos nt}{n} = \ln(1 - 2s \cos t + s^2)^{1/2} ,$$

which is the desired result.

Lemma 2. Let h be any arbitrary fixed positive number and define $\Delta(\log r,h)$ by

$$\Delta(\log r,h) = \frac{f(x'+h,y') + f(x',y'+h) + f(x'-h,y') + f(x',y'-h) - 4f(x',y')}{h^2},$$

where $f(x',y') = \log\sqrt{(x'-x_0)^2 + (y'-y_0)^2} = \log r$. Let θ be the angle having initial side the ray from (x_0,y_0) to (x_0+h,y_0) and terminal side the ray from (x_0,y_0) to (x',y'). Then, for $r > h$

$$\Delta(\log r,h) = -\sum_{k=1}^{\infty} \frac{h^{4k-2}}{r^{4k}} \frac{\cos 4k\theta}{4k} ,$$

and for $r < h$

$$\Delta(\log r,h) = \frac{4(\log h - \log r)}{h^2} - \sum_{k=1}^{\infty} \frac{r^{4k}}{h^{4k+2}} \frac{\cos 4k\theta}{4k} .$$

Proof. Suppose first that $r > h$. Consideration of the accompanying diagram, use of the law of cosines, and utilization of standard properties of logarithms give

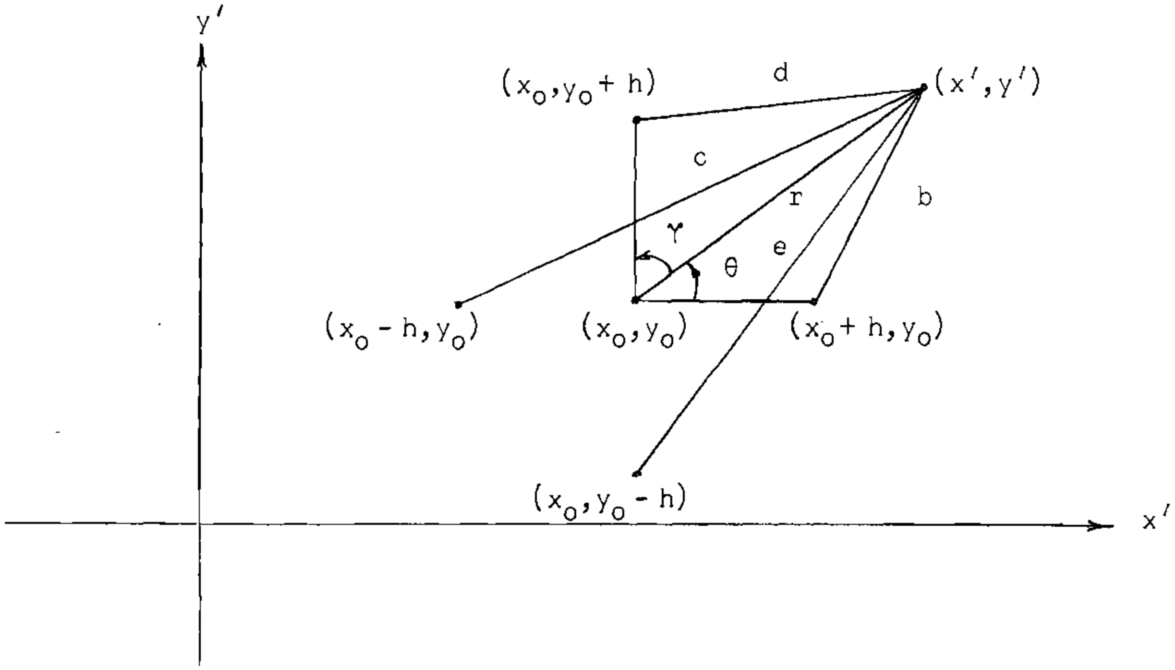
$$\log b = \log r + \log\sqrt{1 - \frac{2h}{r} \cos \theta + \frac{h^2}{r^2}} ,$$

$$\log c = \log r + \log\sqrt{1 - \frac{2h}{r} \cos (\pi - \theta) + \frac{h^2}{r^2}} ,$$

$$\log d = \log r + \log\sqrt{1 - \frac{2h}{r} \cos \gamma + \frac{h^2}{r^2}} ,$$

and

$$\log e = \log r + \log \sqrt{1 - \frac{2h}{r}\cos(\pi - \gamma) + \frac{h^2}{r^2}} .$$



Thus,

$$\Delta(\log r,h) = \frac{1}{h^2}\left[\log\sqrt{1 - \frac{2h}{r}\cos\theta + \frac{h^2}{r^2}} + \log\sqrt{1 - \frac{2h}{r}\cos(\pi-\theta) + \frac{h^2}{r^2}}\right.$$

$$\left. + \log\sqrt{1 - \frac{2h}{r}\cos\gamma + \frac{h^2}{r^2}} + \log\sqrt{1 - \frac{2h}{r}\cos(\pi-\gamma) + \frac{h^2}{r^2}}\right] .$$

Use of Lemma 1 then simplifies the expression above to

$$\Delta(\log r,h) = -\frac{1}{h^2}\left[\sum_{n=1}^{\infty}\left(\frac{h}{r}\right)^n \frac{\cos n\theta}{n} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{h}{r}\right)^n \frac{\cos n\theta}{n}\right.$$

$$\left. + \sum_{n=1}^{\infty}\left(\frac{h}{r}\right)^n \frac{\cos n\gamma}{n} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{h}{r}\right)^n \frac{\cos n\gamma}{n}\right] .$$

If the convergent series in $\cos n\theta$ and $\cos n\gamma$ are combined and the identity $\gamma = \pi/2 - \theta$ is employed, it follows that

$$\begin{aligned}\Delta(\log r,h) &= -\frac{1}{h^2}\left[\sum_{n=1}^{\infty}\left(\frac{h}{r}\right)^{2n}\frac{\cos 2n\theta}{2n} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{h}{r}\right)^{2n}\frac{\cos 2n\theta}{2n}\right] \\ &= -\sum_{n=1}^{\infty}\frac{h^{4n-2}}{r^{4n}}\,\frac{\cos 4n\theta}{4n}\end{aligned}$$

for $r > h$. For the case $r < h$ the procedure is identical with that which was demonstrated above except the expansion now is in powers of r/h. This completes the proof of Lemma 2.

The actual proof of Theorem 3 will be divided into two main parts, the second of which is further subdivided into two smaller sections. In the first segment it is shown, with the aid of Theorem 1, that $\Delta^* u_D(x_0,y_0) = 0$ on D_2, where (x_0,y_0) is any arbitrary point located in D and D_2 is domain D excluding a disk D_1 centered at (x_0,y_0) with judiciously chosen radius ρ. For the second portion the pertinent integral over D_1 is rewritten as the sum of two other integrals; one integral is actually evaluated while the other is shown to have limit zero as $h \to 0^+$. The totality of these results then leads to the desired conclusion.

Theorem 3. Let g be a real-valued, continuous, and bounded function defined on a bounded domain $D \subset E_2$. Define $u_D(x,y)$ by

$$u_D(x,y) = \frac{1}{2\pi}\iint_D g(x',y')\log r \;dx'\,dy' ,$$

where $r = \sqrt{(x'-x)^2 + (y'-y)^2}$, for all points (x,y) in D. Then u_D satisfies $\Delta^* u_D(x,y) = g(x,y)$ for all points (x,y) in D.

Proof. It is first noted that the double integral does, in fact, define a continuous function on D. Actually, the function so defined belongs to C' on D; however, this fact will not be needed here. Secondly,

$$\Delta(u_D(x,y),h) = \frac{u_D(x+h,y) + u_D(x,y+h) + u_D(x-h,y) + u_D(x,y-h) - 4u_D(x,y)}{h^2}$$

is symmetric in h for each fixed point (x,y) in D and every number h such that $x+h$, $x-h$, $y+h$, and $y-h$ lie in D. As a result, there is no loss in generality in assuming that $h > 0$. This simplification will be adhered to throughout the proof of this theorem. Lastly, it is clear that

$$\Delta(u_D(x,y),h) = \frac{1}{2\pi} \iint_D g(x',y')\, \Delta(\log r,h)\, dx'\, dy'$$

for all (x,y) in D and h such that $x+h$, $x-h$, $y+h$, and $y-h$ lie in D.

Let (x_0,y_0) be any arbitrary fixed point in D and let $\varepsilon > 0$ be given. Designate by $\rho_{x_0y_0}$ the minimum distance from (x_0,y_0) to ∂D and fix ρ, $0 < \rho < \frac{\rho_{x_0y_0}}{2}$, such that $(x'-x_0)^2 + (y'-y_0)^2 \leq \rho^2$ implies that $|g(x',y') - g(x_0,y_0)| < \varepsilon$. This choice of ρ is possible due to the continuity of g on D. Now construct a circle having center at (x_0,y_0) and radius ρ and denote this circle and its interior by D_1. Then, $D = D_1 \cup D_2$, where $D_2 = D - D_1$. Ultimately, a limit passage will be considered; therefore, only those values of h for which $0 < h < \rho$ will be considered.

If the linearity of the integral is utilized, $\Delta(u_D(x_0,y_0),h) = I_1(h) + I_2(h)$, where $I_1(h)$ and $I_2(h)$ correspond to the integration of $g(x',y')\,\Delta(\log r,h)$ over domains D_1 and D_2 respectively. Since the computation of $\Delta^*(u_D(x_0,y_0),h)$ is desired, the existence and evaluation of $\lim_{h\to 0^+} I_1(h)$ and $\lim_{h\to 0^+} I_2(h)$ must be demonstrated. The task of verifying that $\lim_{h\to 0^+} I_2(h)$ exists and equals zero will first be undertaken.

By Equation (1) of Theorem 1 there exist numbers θ_1, $0 < \theta_1 < 1$, and θ_2, $0 < \theta_2 < 1$, such that

$$\begin{aligned}\Delta(\log r,h) &= \frac{1}{h^2}\Big[\log\sqrt{(x'-x_0+h)^2+(y'-y_0)^2}+\log\sqrt{(x'-x_0-h)^2+(y'-y_0)^2}\\ &\quad+\log\sqrt{(x'-x_0)^2+(y'-y_0+h)^2}+\log\sqrt{(x'-x_0)^2+(y'-y_0-h)^2}\\ &\quad-4\log\sqrt{(x'-x_0)^2+(y'-y_0)^2}\Big]\\ &= \frac{1}{2\theta_1 h}\left[\frac{x'-x_0+\theta_1 h}{(x'-x_0+\theta_1 h)^2+(y'-y_0)^2}-\frac{x'-x_0-\theta_1 h}{(x'-x_0-\theta_1 h)^2+(y'-y_0)^2}\right]\\ &\quad+\frac{1}{2\theta_2 h}\left[\frac{y'-y_0+\theta_2 h}{(x'-x_0)^2+(y'-y_0+\theta_2 h)^2}-\frac{y'-y_0-\theta_2 h}{(x'-x_0)^2+(y'-y_0-\theta_2 h)^2}\right].\end{aligned}$$

Combination of the terms in each bracket over a common denominator then leads to the equation

$$\Delta(\log r,h) \tag{6}$$

$$= \left[\frac{(y'-y_o)^2 - (x'-x_o)^2 + \theta_1^2h^2}{L_1}\right] + \left[\frac{(x'-x_o)^2 - (y'-y_o)^2 + \theta_2^2h^2}{L_2}\right],$$

where

$$L_1 = [(x'-x_o+\theta_1h)^2 + (y'-y_o)^2]\,[(x'-x_o-\theta_1h)^2 + (y'-y_o)^2]$$

and

$$L_2 = [(x'-x_o)^2 + (y'-y_o+\theta_2h)^2]\,[(x'-x_o)^2 + (y'-y_o-\theta_2h)^2]\ .$$

Note here that L_1 and L_2 are bounded away from zero on domain D_2; in fact, $\frac{1}{L_1} < \frac{1}{\rho^4}$ and $\frac{1}{L_2} < \frac{1}{\rho^4}$. Further simplification of the terms in (6), together with a slight rearrangement, shows that

$$\Delta(\log r,h) = [(x'-x_o)^2 - (y'-y_o)^2]\left(\frac{L_1 - L_2}{L_1L_2}\right) + h^2\left(\frac{\theta_2^2L_1 + \theta_1^2L_2}{L_1L_2}\right), \tag{7}$$

and expansion of terms L_1 and L_2 yields

$$L_1 - L_2 = -2h^2(\theta_1^2 + \theta_2^2)\left[(x'-x_o)^2 - (y'-y_o)^2 - \frac{h^2}{2}(\theta_1^2 - \theta_2^2)\right]. \tag{8}$$

From combination of (7) and (8) it is then evident that

$$\Delta(\log r,h) = \frac{-2h^2}{L_1L_2}[(x'-x_o)^2 - (y'-y_o)^2]\,(\theta_1^2+\theta_2^2)\,[(x'-x_o)^2 \tag{9}$$

$$- (y'-y_o)^2 - \frac{h^2}{2}(\theta_1^2-\theta_2^2)] + \frac{h^2\theta_2^2}{L_2} + \frac{h^2\theta_1^2}{L_1}\ .$$

Equation (9) will serve as a point of departure for calculating $\lim_{h \to 0^+} I_2(h)$.

Since g is bounded on D, it follows that

$$\frac{1}{2\pi}\left|\iint_{D_2} g(x',y')\,\Delta(\log r,h)\,dx'\,dy'\right| \tag{10}$$

$$\leq \frac{1}{2\pi}\iint_{D_2} |g(x',y')|\,|\Delta(\log r,h)|\,dx'\,dy'$$

$$\leq M\iint_{D_2} |\Delta(\log r,h)|\,dx'\,dy' ,$$

where M is a finite uniform upper bound for $\frac{|g(x',y')|}{2\pi}$ on D. Application of Equation (9), Inequality (10), and the remark concerning the bounding of $1/L_1$ and $1/L_2$ by $1/\rho^4$ then leads to

$$M\iint_{D_2} |\Delta(\log r,h)|\,dx'\,dy' \tag{11}$$

$$< M\iint_{D_2} \frac{2h^2}{\rho^8}\,|[(x'-x_o)^2-(y'-y_o)^2]\,(\theta_1^2+\theta_2^2)$$

$$[(x'-x_o)^2-(y'-y_o)^2-\frac{h^2}{2}(\theta_1^2-\theta_2^2)]+\frac{h^2}{\rho^4}\,(\theta_1^2+\theta_2^2)|dx'dy' .$$

Upon utilizing the linearity of the integral and the triangle inequality, it is now easily seen that

$$M\iint_{D_2} |\Delta(\log r,h)|\,dx'\,dy' \tag{12}$$

$$< \frac{2Mh^2}{\rho^8}\,(\theta_1^2+\theta_2^2)\iint_{D_2}[(x'-x_o)^2-(y'-y_o)^2]^2 +$$

$$+ \frac{h^2}{2} |\theta_1^2 - \theta_2^2| \, |(x' - x_0)^2 - (y' - y_0)^2| \, dx' \, dy'$$

$$+ \frac{Mh^2}{\rho^4} (\theta_1^2 + \theta_2^2) \iint_{D_2} dx' \, dy' \, .$$

Since D is bounded, so also is D_2; moreover, recall that $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. Hence, Inequality (12) becomes

$$M \iint_{D_2} |\Delta(\log r,h)| \, dx' \, dy' < \frac{2Mh^2}{\rho^4} \left(\frac{2N^2}{\rho^4} + \frac{h^2N}{\rho^4} + 1\right) \mu(D_2), \tag{13}$$

where N is a finite uniform bound of $|(x' - x_0)^2 - (y' - y_0)^2|$ on D_2 and $\mu(D_2)$ is the finite two-dimensional Lebesgue measure of D_2. Inequality (13) together with Inequality (10) then yields

$$\begin{aligned} \frac{1}{2\pi} \left| \iint_{D_2} g(x',y') \, \Delta(\log r,h) \, dx'dy' \right| &< \frac{2Mh^2}{\rho^4} \left(\frac{2N^2}{\rho^4} + \frac{h^2N}{\rho^4} + 1\right) \mu(D_2) \qquad (14) \\ &< \frac{2Mh^2}{\rho^4} \left(\frac{2N^2}{\rho^4} + \frac{N}{\rho^2} + 1\right) \mu(D_2) \, , \end{aligned}$$

since only those h for which $0 < h < \rho$ are being considered.

Inequality (14) is true independent of (x',y') in D_2; therefore, if $\varepsilon_1 > 0$ is given there exists a positive number

$$\delta < \min \left(\rho, \rho^4 \sqrt{\frac{\varepsilon_1}{2\mu(D_2) \, M(2N^2 + N\rho^2 + \rho^4)}}\right)$$

which is dependent only upon ε_1 and ε such that when $0 < h < \delta$ it is true that

$$\frac{1}{2\pi}\left|\iint\limits_{D_2} g(x',y')\ \Delta(\log r,h)\ dx'\ dy'\right| < \varepsilon_1 .$$

This is true for every positive number ε_1; hence,

$$\lim_{h\to o^+}\frac{1}{2\pi}\iint\limits_{D_2} g(x',y')\ \Delta(\log r,h)\ dx'\ dy' = \lim_{h\to o^+} I_2(h) = 0 ,$$

as desired.

Since $\lim_{h\to o^+} I_2(h) = 0$, it remains to verify that $\lim_{h\to o^+} I_1(h) = g(x_o,y_o)$ to deduce the result of the theorem. Let

$$A(h) = \frac{g(x_o,y_o)}{2\pi}\iint\limits_{D_1} \Delta(\log r,h)\ dx'\ dy'$$

and

$$B(h) = \frac{1}{2\pi}\iint\limits_{D_1} \Delta(\log r,h)\ [g(x',y') - g(x_o,y_o)]\ dx'\ dy'.$$

Then $I_1(h) = A(h) + B(h)$ by the linearity property of the integral. The problem of proving that $\lim_{h\to o^+} I_1(h) = g(x_o,y_o)$ has now been modified to a similar one of showing that $\lim_{h\to o^+} A(h) = g(x_o,y_o)$ and $\lim_{h\to o^+} (B(h) = 0$. This revised problem is that which will be solved; the order will be that of proving that $\lim_{h\to o^+} A(h) = g(x_o,y_o)$ and then verifying that $\lim_{h\to o^+} B(h) = 0$.

The task of calculating $\lim_{h \to o^+} A(h)$ is essentially that of evaluating $\iint_{D_1} \Delta(\log r,h)\, dx'\, dy'$. To do so, a change of coordinate systems proves advantageous; in particular, the coordinate system related to the X'Y'-system by the transformation equations

$$x' = x_o + r \cos\theta \tag{15}$$

and

$$y' = y_o + r \sin\theta\ , \tag{16}$$

where $0 < r \leq \rho$ and $0 \leq \theta < 2\pi$, will be used. The use of (15), (16), and well-known theorems of analysis then lead to

$$\iint_{D_1} \Delta(\log r,h)\, dx'\, dy' = \int_o^\rho r \int_o^{2\pi} \Delta(\log r,h)\, d\theta\, dr\ . \tag{17}$$

If the standard properties of logarithms and Equations (15) and (16) are utilized, Equation (17) can be rewritten as

$$\int_o^\rho r \int_o^{2\pi} \Delta(\log r,h)\, d\theta\, dr \tag{18}$$

$$= \frac{1}{2h^2} \int_o^\rho r \int_o^{2\pi} \{\log[(r\cos\theta + h)^2 + r^2\sin^2\theta]$$

$$[(r\cos\theta - h)^2 + r^2\sin^2\theta][r^2\cos^2\theta + (r\sin\theta + h)^2]$$

$$[r^2\cos^2\theta + (r\sin\theta - h)^2] - 4\log r^2\}\, d\theta\, dr\ .$$

Expansion and combination of the terms in the integrand of (18), together with the use of the basic trigonometric identities $\sin^2\theta + \cos^2\theta = 1$ and

$\cos^2 \theta - \sin^2 \theta = \cos 2\theta$, then shows that

$$\int_0^\rho r \int_0^{2\pi} \Delta(\log r,h)\, d\theta\, dr \tag{19}$$

$$= \frac{1}{2h^2} \int_0^\rho r \int_0^{2\pi} \log(r^4 - 2h^2r^2 \cos 2\theta + h^4)$$

$$\cdot\, (r^2 + 2r^2h^2 \cos 2\theta + h^4) - 4 \log r^2\, d\theta\, dr$$

$$= \frac{1}{2h^2} \int_0^\rho r \int_0^{2\pi} \log \left[\frac{(r^4 + h^4)^2 - 4r^4h^4 \cos^2 2\theta}{r^8}\right] d\theta\, dr\,,$$

where standard properties of logarithms have again been used in the last computation. If the identity $\frac{1 + \cos 4\theta}{2} = \cos^2 2\theta$ is substituted for $\cos^2 2\theta$, Equation (19) further reduces to

$$\int_0^\rho r \int_0^{2\pi} \Delta(\log r,h)\, d\theta\, dr \tag{20}$$

$$= \frac{1}{2h^2} \int_0^\rho r \int_0^{2\pi} \log\left(1 + \frac{h^8}{r^8} - \frac{2h^4}{r^4} \cos 4\theta\right) d\theta\, dr\,.$$

At this point it should be noted that $0 \leq \left(1 - \frac{h^4}{r^4}\right)^2$ so that $1 + \frac{h^8}{r^8} \geq \frac{2h^4}{r^4}$. A change of variable, $x = 4\theta$, is performed, and use of the periodic character of the cosine function is made. Then, Equation (20) can be modified to

$$\int_0^\rho r \int_0^{2\pi} \Delta(\log r,h)\, d\theta\, dr \tag{21}$$

$$= \frac{1}{2h^2} \int_0^\rho r \int_0^{2\pi} \log\left(1 + \frac{h^8}{r^8} - \frac{2h^4}{r^4} \cos x\right) dx\, dr\,,$$

and, since the cosine function is even and periodic of period 2π, this last equation may be further reduced to

$$\int_0^\rho r \int_0^{2\pi} \Delta(\log r, h)\ d\theta\ dr \tag{22}$$

$$= \frac{1}{h^2} \int_0^\rho r \int_0^\pi \log\left(1 + \frac{h^8}{r^8} - \frac{2h^4}{r^4} \cos x\right) dx\ dr\ .$$

Use of Equation (403) of [2] now allows integration over the variable x to be accomplished. One can then see that

$$\int_0^\rho r \int_0^{2\pi} \Delta(\log r, h)\ d\theta\ dr \tag{23}$$

$$= \frac{\pi}{h^2} \int_0^\rho r \log \left[\frac{1 + \frac{h^8}{r^8} + \sqrt{\left(1 + \frac{h^8}{r^8}\right)^2 - 4\frac{h^8}{r^8}}}{2}\right] dr$$

$$= \frac{\pi}{h^2} \int_0^\rho \log \left(\frac{1 + \frac{h^8}{r^8} + \left|\frac{h^8}{r^8} - 1\right|}{2}\right) dr\ .$$

The integration over r can be accomplished by utilizing the linearity property of the integral; namely, integrate from zero to h and then from h to ρ. If this decomposition is performed, Equation (23) becomes

$$\int_0^\rho r \int_0^{2\pi} \Delta(\log r, h)\ d\theta\ dr = \frac{\pi}{h^2} \int_0^h r \log \frac{r^8}{h^8}\ dr$$

$$= \frac{8\pi}{h^2} \int_0^h (r \log h - r \log r)\ dr\ .$$

Integration of the last expression gives the value 2π for every h such that $0 < h < \rho$; hence,

$$A(h) = \frac{g(x_o,y_o)}{2\pi} \iint_{D_1} \Delta(\log r,h)\ dx'\ dy' = g(x_o,y_o)$$

for all h such that $0 < h < \rho$. This implies that $\lim_{h\to o^+} A(h) = g(x_o,y_o)$, as was to be shown.

Lastly, the demonstration that $\lim_{h \to o^+} B(h) = 0$ will be performed. First, decompose domain D_1 into two concentric annular subdomains D_1' and D_1''' and a circle D_1'', where

$$D_1' = \{(x',y') \mid 0 \le (x' - x_o)^2 + (y' - y_o)^2 < h^2\} ,$$

$$D_1'' = \{(x',y') \mid (x' - x_o)^2 + (y' - y_o)^2 = h^2\} ,$$

and

$$D_1''' = \{(x',y') \mid h^2 < (x' - x_o)^2 + (y' - y_o)^2 \le \rho^2\} .$$

Now designate by $\psi_1(h)$, $\psi_2(h)$, and $\psi_3(h)$ the integral of $[g(x',y') - g(x_o,y_o)]\ \Delta(\log r,h)$ over D_1', D_1'', and D_1''' respectively. Then, by the linearity property of the integral, it follows that $B(h) = \psi_1(h) + \psi_3(h)$. It must now be shown that $|B(h)|$ can be made arbitrarily "small" for all $h > 0$ sufficiently "close" to zero; to do so, investigation of the behavior of ψ_1 and ψ_3 is undertaken.

On D_1'''

$$|\psi_3(h)| = \left| \iint_{D_1'''} \Delta(\log r,h)\ [g(x',y') - g(x_o,y_o)]\ dx'\ dy' \right. \tag{24}$$

$$\le \iint_{D_1'''} |\Delta(\log r,h)|\ |g(x',y') - g(x_o,y_o)|\ dx'\ dy' <$$

$$< \varepsilon \iint_{D_1'''} |\Delta(\log r,h)| \, dx' \, dy' ,$$

where the last inequality is true by the definition of D_1. Further, on D_1''' $r > h$; hence, with the aid of Lemma 2 and the boundedness of cos x, Inequality (24) may be written as

$$|\psi_3(h)| < \varepsilon \iint_{D_1'''} \left| \sum_{k=1}^{\infty} \frac{h^{4k-2}}{r^{4k}} \left(\frac{-\cos 4k\theta}{4k} \right) \right| dx' \, dy' \tag{25}$$

$$\leq \varepsilon \iint_{D_1'''} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{r^{4k}} \frac{|\cos 4k\theta|}{4k} dx' \, dy'$$

$$\leq \varepsilon \iint_{D_1'''} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k r^{4k}} dx' \, dy' .$$

Appending a set of two-dimensional Lebesgue measure zero and using Equations (15) and (16), one can easily see that

$$|\psi_3(h)| < 2\pi\varepsilon \int_h^{\rho} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k \, r^{4k-1}} dr .$$

The integrand in the inequality above becomes unbounded at the lower limit of integration; hence, the integral must be treated as an improper integral of the second kind. Now, for each number t such that $h < t \leq \rho$,

$$\int_t^{\rho} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k \, r^{4k-1}} dr = \sum_{k=1}^{\infty} \frac{1}{4k} \int_t^{\rho} \frac{h^{4k-2}}{r^{4k-1}} dr$$

due to the uniform convergence of the infinite series when $\frac{h}{r} < 1$.

Then,

$$\int_t^{\rho} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k\ r^{4k-1}}\, dr = \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k} \left[\frac{1}{(2-4k)\ \rho^{4k-2}} + \frac{1}{(4k-2)\ t^{4k-2}}\right].$$

for all numbers t such that $h < t \le \rho$. As such, the last equation will be true when $t \to h^+$, provided both limits exist. But

$$\begin{aligned}\lim_{t\to h^+} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k} \left[\frac{1}{(2-4k)\ \rho^{4k-2}} + \frac{1}{(4k-2)\ t^{4k-2}}\right] &= \sum_{k=1}^{\infty} \frac{1}{4k} \left[\frac{1}{(2-4k)} \left(\frac{h}{\rho}\right)^{4k-2} + \frac{1}{4k-2}\right] \\ &< \frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)},\end{aligned}$$

since the power series in $1/t$ defines a continuous function of t for all those t for which it converges, namely, $t \ge h$. Thus, $\lim_{t\to h^+} \int_t^{\rho} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k\ r^{4k-1}}\, dr$ exists and is less than $\frac{1}{8} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)}$, a convergent series. By definition of the improper integral of the second kind, it then follows that

$$|\psi_3(h)| < 2\pi\varepsilon \int_h^{\rho} \sum_{k=1}^{\infty} \frac{h^{4k-2}}{4k\ r^{4k-1}}\, dr < \frac{\pi\varepsilon}{4} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)}. \qquad (26)$$

Similarly, on D_1'

$$|\psi_1(h)| = \left| \iint_{D_1'} \Delta(\log r,h)\,[g(x',y') - g(x_0,y_0)]\,dx'\,dy' \right| \tag{27}$$

$$< \varepsilon \left[2\pi \int_0^h \sum_{k=1}^{\infty} \frac{r^{4k+1}}{4k\,h^{4k+2}}\,dr + \frac{8\pi}{h^2} \int_0^h r\,(\log h - \log r)\,dr \right],$$

where the addition of a set of two-dimensional Lebesgue measure zero, application of Equations (15) and (16), and the utilization of the boundedness property of the cosine function have followed an application of Lemma 2 with $r < h$. The first integral in the right-hand member of Inequality (27) must again be thought of as an improper integral of the second kind; however, elementary considerations as demonstrated above on D_1''' show that the improper integral converges and that termwise integration is permissible. Inequality (27) then transforms to

$$|\psi_1(h)| < \frac{\pi\varepsilon}{4} \left[\sum_{k=1}^{\infty} \frac{1}{k(2k+1)} + 8 \right]. \tag{28}$$

Inequalities (26) and (28) show that

$$|\psi_1(h) + \psi_3(h)| \leq |\psi_1(h)| + |\psi_3(h)| < N\varepsilon,$$

where N is a finite number independent of h which is determined by

$$N = \pi \left[\sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} + 2 \right] = \frac{5\pi}{2}.$$

Thus, $|B(h)| < N\varepsilon$ for all numbers h such that $0 < h < \rho$. Since this result is true for each such given $\varepsilon > 0$, $\lim_{h \to 0^+} B(h) = 0$, as was to be shown.

The totality of results obtained in the preceding paragraphs of this proof now leads to the result $\Delta^* u_D(x_0,y_0) = g(x_0,y_0)$. The proof of Theorem 3 is thus complete since (x_0,y_0) was any arbitrary point in D.

At this point the machinery necessary for the verification of the main theorem of this text has been developed. The theorem itself, as stated in the Introduction, will be restated in contrapositive form so as to induce a more concise logical structure for the proof. It should be noted here that if u_D does represent a solution to $\Delta u(x,y) = g(x,y)$ on D, then Poisson's equation will have a continuous solution on D, namely, u_D. In this event other continuous solutions on D might exist; however, the remark made at the end of the proof of this theorem will deal with this possibility in its entirety.

Theorem 4. Let g be a real-valued function of two real variables defined, continuous, and bounded on a bounded domain $D \subset E_2$. If $\Delta u(x,y) = g(x,y)$ has a continuous solution on D, then u_D represents a continuous solution to $\Delta u(x,y) = g(x,y)$ on D.

Proof. Let v be any continuous solution to $\Delta u(x,y) = g(x,y)$ on D; at least one such function will exist due to the given hypotheses. Since v is a continuous solution of $\Delta u(x,y) = g(x,y)$ on D, then $\Delta v(x,y) = g(x,y)$ for all (x,y) in D; hence, by Theorem 1, $\Delta^* v(x,y) = g(x,y)$

for all (x,y) in D. By Theorem 3 $\Delta^* u_D(x,y)$ exists, and $\Delta^* u_D(x,y) = g(x,y)$ for all (x,y) in D. Therefore, $\Delta^* v(x,y) = \Delta^* u_D(x,y)$ on D, or $\Delta^*[v(x,y) - u_D(x,y)] = 0$ on D. Now $v - u_D$ is continuous on D; hence, by Theorem 2, $\Delta[v(x,y) - u_D(x,y)] = 0$ for all (x,y) in D. From the last deduction it readily follows that u_D is a continuous solution to Poisson's equation on D, as was to be shown.

As a point in passing it should be noted here that the proof of Theorem 4 demonstrates the general structure for continuous solutions to $\Delta\, u(x,y) = g(x,y)$ on D. Thus, for a given function g having the same properties as discussed earlier, either $\Delta u(x,y) = g(x,y)$ has no continuous solutions on D or if such a continuous solution does exist, say v, then v must be of the form $v(x,y) = u_D(x,y) + h(x,y)$, where h is any harmonic function on D.

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