

**DESIGNING PRICING MECHANISMS IN THE PRESENCE OF
RATIONAL CUSTOMERS WITH MULTI-UNIT DEMANDS**

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**DESIGNING PRICING MECHANISMS IN THE PRESENCE OF
RATIONAL CUSTOMERS WITH MULTI-UNIT DEMANDS**

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to my parents; Ayten and Hüseyin Gülcü

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SUMMARY

We study the design of optimal pricing mechanisms in the presence of rational customers with multi-unit demands.

First, we analyze the optimal design of a markdown pricing mechanism with pre-announced prices. In the presence of limited supply, buyers who choose to purchase at a lower price may face a scarcity in supply. Our focus is on the structure of the optimal markdown mechanisms in the presence of rational or “strategic” buyers who demand multiple units. We first examine a complete information setting where the set of customer valuations is known but the seller does not know the valuation of each individual customer (i.e., cannot exercise perfect price discrimination). We then generalize our analysis to an incomplete valuation information setting where customer valuations are drawn from known distributions. For both settings, we compare the seller’s profit resulting from the optimal markdown mechanism and the optimal single price. We provide a number of managerial insights into designing profitable markdown mechanisms.

Next chapter focuses on the purchasing behavior of the customers and the optimal pricing decisions of the seller assuming that the seller has incomplete information about the customer demand. Each buyer demands multiple units of the homogeneous product that the seller is offering via a priority pricing mechanism with multiple prices, where the only difference is the availability/scarcity of the supply at each price. We provide managerial insights based on the results from a stylized model.

Final chapter builds on the incomplete demand information setting and focuses on the value of improved information about the customer demand to the seller. We investigate whether improved information benefits the seller and if the seller would prefer to share the improved demand information with the customers.

CHAPTER I

INTRODUCTION

Until late 1800's, the standard method for determining how much one should pay for a product or service was bargaining. Final prices were determined based on what the seller and the buyer considered a reasonable payment at the time of transaction. Then came the age of supermarkets and mass marketing with fixed/static prices, as the advances in production and distribution technologies made it impractical to bargain before every purchase [53].

Abundance of information and the advance of data processing technologies has led to a trend where sellers started to explore pricing mechanisms other than charging the same fixed/static price at any time. Widespread use of the Internet and reduced transaction costs associated with changing prices enable the sellers to experiment with pricing practices where they change fixed/static prices rapidly or offer the same product/service at multiple prices as they take advantage of increased data availability for the markets they operate in.

How sellers price their products or services is not the only impact of the recent advances in information technology. Contemporary consumers have more to choose from and they can gather more information about these alternatives before they make their decision. Thanks to the tools provided by accessible computing technology, today's customers are shopping experts who are making "strategic" purchasing decisions instead of just satisfying their demand. They are trying very hard to get the best "bang for their buck" [41].

Motivated by these trends, we analyze the pricing decision of a seller (monopolist), who has K identical units of a product or service for sale. Initially, K is assumed to be exogenously given; this would be the case, for example, if the K units were comprised of excess inventory for an end-of-season item or service capacity that cannot be easily adjusted in the short run. Later we explore relaxing this assumption under some information settings.

In its most general form, the seller would like to implement an m -step pricing mechanism with price p_k at step k , where $p_1 > p_2 > \dots > p_m$. The seller faces N rational buyers with

valuations (per unit) $v_1 > v_2 > \dots > v_N$ where buyer j wishes to purchase at most $D_j > 0$ units, $\forall j$. Given all the prices, the buyers decide how much to buy at a given price.

The goal of the seller is to design a pricing mechanism such that high(er) valuation customers purchase at high(er) prices. The seller allocates the existing capacity to customers starting from the requests at the highest price and following with lower prices. If the total bids/requests at any price step exceeds availability, the seller uses a random allocation scheme, where she picks a buyer randomly and satisfies his request and repeats this process until all units are allocated.

We assume that customers facing a mechanism with multiple price/availability options make their purchases/bids with the objective of maximizing their individual surpluses, i.e., they act strategically. We consider a business-to-business market which consists of a small number of customers hence, each customer's demand, valuation and bidding strategy has an impact on the decisions of the other customers.

In the following chapters, we analyze versions of this base model under different information settings with the objective of providing answers to the following questions.

- How will rational buyers behave (bid) when facing multiple prices?
- How many price steps should there be and what should those prices be?
- Under what conditions would the seller be better off using a single price?

Complete information (CI) (Chapter 2): Both the buyers and the seller know the set of customer valuations but the seller cannot associate them with individual customers.

Incomplete valuation information (IV) (Chapter 2): The seller does not know the customers' valuations, but knows their cumulative distribution (CDF) and probability density (pdf) functions, $F_i(v_i)$ and $f_i(v_i)$, respectively, with support over $[\underline{v}_i, \bar{v}_i]$, $\forall i$, where the support intervals for any two customers i, j do not overlap, i.e., $\underline{v}_i < \bar{v}_i < \underline{v}_j < \bar{v}_j$. Each customer knows his own valuation with certainty and also knows the pdf and CDF of the other customer valuations.

Incomplete demand information (ID) (Chapter 3): Exact demands of the customers are *not* known by the seller. She believes that each customer's private demand is drawn from a common distribution, with density function $f_i(D) \forall i$, with support over the

interval $[L_i, H_i]$. Each customer shares this belief with the seller regarding the demand of the other customers but knows his own exact demand realization.

In Chapter 4, we investigate the value of improved information on customer demands to the seller based on the insights gained from analyzing pricing mechanism under incomplete demand information.

CHAPTER II

DESIGNING OPTIMAL PRE-ANNOUNCED MARKDOWNS IN THE PRESENCE OF RATIONAL CUSTOMERS WITH MULTI-UNIT DEMANDS

2.1 Introduction

‘Pricing has traditionally been a high-stakes game based on guesses about costs and competitive activities, resulting in either money left on the table or lost sales’ [53]. In recent years, advances in technology have opened the door for companies to turn their “pricing game” into an intelligent and more profitable strategy with the use of price optimization software [13, 15]. Price optimization software assists sellers to intelligently execute various pricing strategies by studying a wide variety of data, ranging from historical sales to demographics. The increased information available about customers, as well as the reduced transaction costs associated with changing prices over time, is enabling sellers to explore a variety of pricing schemes aside from the commonly used traditional single (static) pricing policy.

When the seller does not know the identity of a buyer – i.e., cannot exercise first or third degree price discrimination – but knows that she is facing a market where customers differ by their willingness to pay, a *markdown pricing* strategy offers a way to potentially improve profits above single-price levels. The idea behind a markdown mechanism is to segment or differentiate customers with diverse valuations by offering different prices over time so as to create scarcity at lower prices, with the goal of inducing high-valuation customers to purchase at higher prices. It is clear that the structure of a markdown mechanism influences buyer behavior and, in turn, the seller’s profits.

The rationale behind and, the advantages of, markdown pricing are well known in the fashion apparel industry, as well as many other business-to-consumer (B2C) markets that sell highly seasonal or short-lived products. As companies begin to explore alternative

pricing mechanisms, it is imperative that we understand how they will perform under various market settings. In this chapter, we consider a markdown mechanism where the price of a good decreases over time according to a *pre-announced* schedule. For example, a seller posts a group of items for an opening price, say, \$800 per item; and buyers bid the quantities they want at that price. After a short duration – e.g., two days, the price drops according to a pre-announced schedule, e.g., to \$600, per item, and then to \$300, and then \$100. That is, markdowns continue until all the items are sold or until the price drops to the minimum level set by the seller. While there may be only a few buyers interested in purchasing at \$800, the threat of many more customers who are willing to pay \$300 or \$100 may induce the *high-valuation* buyers to purchase at a higher price. Retailers who have been using preannounced markdowns for decades (e.g., Filene’s Basement) are now being joined by others (e.g., Sam’s Club) as a result of the ease of use provided by the Internet and increased sophistication of buyers. (An example from Sam’s Club web site and an excerpt from Filene’s Basement’s web site are presented in Appendix A.)

Companies operating in business-to-business (B2B) markets are also seeing the potential advantages of price optimization and markdown pricing, as seen by the pricing tools offered by pricing software vendor Zilliant [51] among others [15]. The majority of the previous research on retail price markdowns and clearance sales assumes that customers are *myopic*, that is, a customer will make a purchase immediately without considering future prices if the price is below his valuation. While a plausible assumption for some B2C markets, an analysis of B2B markets requires a richer characterization of customer behavior; under B2B settings, customers often act *rationally* (or *strategically*), taking into account the entire (expected) price path while deciding when to buy. Furthermore, most papers that previously addressed markdown pricing assume single-unit demand; again, a characterization that is less likely to hold in B2B markets.

In this chapter, we analyze the optimal design of a pre-announced markdown mechanism by exploring a setting where strategic/rational customers have multi-unit demands [18, 28, 46, 47] and the seller may have a limited supply of goods [7, 20, 22]. Our goal is to shed light on three fundamental questions by combining research streams from economics and

operations management.

- How will rational buyers behave (bid) when facing a markdown?
- What is the optimal markdown, i.e., how many price steps should it have and what should those prices be? What is the impact of strategic customers with multi-unit (versus single-unit) demands on the design of the optimal markdown?
- Under what conditions would the seller be better off using a single price versus markdown pricing?

We initially study a markdown mechanism under a setting where the seller has complete information (CI) about the buyers' valuations and demands (Section 2.3). Under CI, we find that for $N \geq 2$ buyers, the optimal markdown has *two* steps and buyers submit *all-or-nothing* bids at each price step (Section 2.3). That is, a buyer never finds it optimal to bid only a portion of his demand at any one price step. We find this to be the case under constant, decreasing, or discounted valuations over time. In Section 2.3.1, we derive the optimal markdown prices when there are two buyers and identify the circumstances under which a markdown pricing strategy is more profitable for the seller than the optimal single price. In Section 2.3.2, we extend our analysis to $N > 2$ buyers and characterize the buyers' equilibrium bidding strategies, and use numerical examples to illustrate the structure of the optimal markdown.

While a CI assumption may appear restrictive, we demonstrate that many of the properties of the optimal markdown and equilibrium bidding behavior of the buyers carry over to a more general incomplete information (IV) setting (Section 2.4). Under the IV setting, each customer's valuation is private information (drawn randomly from non-overlapping intervals), and the cumulative distribution (CDF) and probability density (pdf) functions from which it is drawn are common knowledge. As before, we examine the design of the optimal markdown mechanism, but this time we consider markdowns where there is at most one price step in any one valuation interval (to be referred to as INT markdowns). We then characterize the buyers' equilibrium bidding behavior, and compare the performance of the optimal markdown with the optimal single-price policy. As was the case under CI, we find that the optimal markdown has very few steps. We find that the seller should never

use more than *three* price steps in an INT markdown; furthermore, customers continue to submit all-or-nothing bids at each price step. We conclude with a discussion of managerial insights and future research directions in Section 2.5.

2.1.1 Placement in Literature

Several branches of the economics literature analyze declining price mechanisms, considering (i) myopic customers [30, 37, 38, 49, 50]; (ii) rational/strategic customers [4, 6, 24, 43]; and (iii) quantity bid auctions [11, 25, 27, 33].

The assumption of myopic (nonstrategic) customer behavior allows the seller to ignore the effect of decreasing prices on customer purchases early on, which is detrimental to the seller's revenue. In many settings, customers act strategically (rationally), taking into account the future path of prices when making purchasing decisions. In such cases, we need to incorporate customer rationality or strategic behavior into the seller's pricing decisions. Recently, [3] investigated the impact of using standard revenue management techniques, which ignore strategic behavior of customers, on airline revenues. They consider the case where the airline sets protection levels based on expected marginal seat revenue assuming customers will behave myopically and proceed to show how strategic customers can calculate sellout probabilities at different fare classes and make strategic purchasing decisions. They find that the loss of the airline is higher when the expected demand over capacity ratio is low and when the number of times the allocation decisions are reset during the sales horizon is low. They conclude that airlines should explicitly consider strategic customer behavior when making pricing decisions. The bulk of the operations management literature has focused on the optimal pricing policy of a seller who has a limited supply (with a possibility of replenishment); [7], [19] [22] assuming myopic customer behavior.

One of the first papers that considers *strategic* (rational) customer behavior in a posted price mechanism is [43]. She studies the case of a seller facing a fixed number of potential customers and analyzes the optimal markdown structure. Customers have single-unit demands and the seller (monopolist) has unlimited capacity. [6] extend Stokey's model by assuming that the seller can only make a finite number of price adjustments, i.e, the price

Table 1: Summary of research on dynamic posted pricing mechanisms with myopic customers

	Capacity	Demand Structure	Information Setting		Properties of the Pricing Mechanism
			Demand	Valuations	
Lazear (1986)	Single unit	Single unit	Complete	Incomplete (known common distribution)	Two step markdown mechanism
Pashigian (1988)	Identical to Lazear (1986), the same setting considered in a perfectly competitive market				
Pashigian, Bowen (1991)	Empirical study, no theoretical model for markdowns				
Wang (1993)	Single unit	Single unit	Incomplete (Poisson process with known rate)	Incomplete (known common distribution)	Posted single price and Vickrey auction
Gallego, van Ryzin (1994)	Multi-unit	Aggregate level demand model	Incomplete (Poisson process with known intensity, which is a function of price)	Individual customer valuations are not modeled explicitly	Continuous time price function, not necessarily decreasing
Warner, Barsky (1995)	Empirical study, no theoretical model for markdowns				
Feng, Gallego (1995)	Multi-unit	Aggregate level demand model	Incomplete (Poisson process with known intensity)	Individual customer valuations are not modeled explicitly	A sequence of two prices, not necessarily in decreasing order
Bitran, Mondschein (1997)	Multi-unit	Single unit	Incomplete (Poisson process with known intensity)	Incomplete (known common distribution)	Continuous time price function, not necessarily decreasing
Federgruen, Heching (1999)	Multi-unit	Aggregate level demand model	Incomplete (general stochastic function of price)	Individual customer valuations are not modeled explicitly	A fixed price for each period which is determined at the beginning of the period

path is no longer continuous. As in [43], customers want to purchase at most one unit and the seller has unlimited capacity. These papers conclude that a markdown pricing scheme is optimal. Furthermore, they find that the seller strictly prefers that the number of price adjustments be as few as possible. [24] also analyze a setting where the seller faces a fixed number of strategic buyers with single-unit demands and endogenously derives the form of an optimal pricing mechanism, given that the seller may have capacity constraints. They assume that customers' valuations are private information and find that the determining factor in the structure of the optimal markdown is the seller's capacity constraint. If capacity is exogenously determined and exceeds market demand, then a single price is optimal. If capacity is less than the total demand, then a markdown or Vickrey auction is optimal. If the seller can determine capacity endogenously, then the optimal action is to set capacity equal to the market demand and use a single-price mechanism.

[4] study a model where the seller has a fixed initial inventory. Customers have single-unit demands and they arrive randomly over time according to a Poisson process. The

valuation of a customer who arrives at time t is modeled by a known, deterministic decreasing function. Upon arrival, customers decide whether to buy the product at the time of arrival, return later for a lower price, or not to buy at all. They first consider a model where the seller announces the price path and commits to it. They then consider an alternative model where the seller can choose the timing of when to announce the discount (assuming only one price change is allowed) and the discounted price considering the inventory level (optimal dynamic pricing policy with a single price change). They find that the benefits to the seller from employing this sophisticated strategy are minimal, and hence it is unlikely that the expected benefits merit its implementation. More recently, [44] has investigated pricing in a deterministic setting where a monopolist faces a continuous arrival of customers with single-unit demands who have low or high valuations and are myopic (who make a purchase or leave immediately) or strategic (stay in the system with the goal of maximizing their surplus through purchasing decisions). When supply is exogenous, whether the optimal policy is a markup or a markdown depends on whether the high- or low-type customers are relatively more strategic. When supply is endogenously determined, the optimal policy is either to use a single price or to set the price high until the very end of the sales horizon, and then drop it to capture the strategic low types.

[52] considers a setting where a monopolist faces a known downward-sloping demand curve, comprised of a fixed and large number of customers who arrive in random order. Before any of the customers arrive, the seller puts a separate price tag on each unit for sale; these are fixed prices that remain valid until the item is sold. Customers may desire more than one unit and, upon arrival, will purchase the units at the lowest available price, provided that the marginal benefit of doing so is positive. Although he does not consider a dynamic pricing markdown setting, [52] finds that the seller never needs to charge more than two prices to maximize his revenues. The result we establish for a setting where prices are time-dependent, i.e., the price of each unit changes over time and customers are strategic, resonates with this result. In [52], rational or myopic customers' behavior would be identical, due to the static nature of the prices. The same is not true when prices of individual items change over time, as is the case in our model.

[32] consider a monopolist who uses a two-step markdown with exogenous prices in a market where N risk-averse customers each have a single-unit demand. The seller’s goal is to maximize her profit by deciding on the initial stocking quantity, and implicitly determining the rationing at the second step. The main differences between [32] and our work is that they ignore the strategic interactions among individual customers (and hence do not solve for an equilibrium); assuming that price is given they solve for the seller’s optimal quantity decisions.

Table 2: Summary of research on dynamic posted pricing mechanisms with strategic customers

	Demand Structure	Valuations
Stokey (1979)	Single unit	Complete (known valuations over time)
Besanko, Winston (1990)	Single unit	Incomplete (known common distribution)
Harris, Raviv (1981)	Single unit	Incomplete (known common distribution)
Aviv, Pazgal (2004)	Single unit	Complete (known valuations over time)
Su (2005)	Single unit	Complete (known fixed valuations)
Liu, van Ryzin (2005)	Single unit	Incomplete (known common distribution)
Our work	Multi-unit	Complete (known valuations) and Incomplete (known common distribution) Cannot be associated with individuals

Our work also departs from the papers above by considering a multi-unit demand setting. Tables 1 and 2 summarize relevant research with myopic and strategic customers, respectively. With the goal of complementing the papers from economics and operations management, we seek to characterize the optimal markdown mechanism when the seller has a fixed capacity and faces a (fixed) number of rational customers who demand multiple units.

2.2 Model

We analyze the pricing decision of a seller (monopolist) who has K identical units of an item for sale. The seller’s starting inventory is assumed to be exogenously given; this would be the case, for example, if the K units were comprised of excess inventory for an end-of-season item. We consider mechanisms where the seller announces the prices and the inventory before the sales begin. (The assumption of pre-announced price is an innocuous one under CI, since buyers will rationally solve for equilibrium prices. The same is not true under IV.) We assume that the valuations of the buyers are constant over time and there

is no discounting. (In Section 2.3, we show that our results easily extend to the case of discounting under CI.) In addition, we assume that all the buyers are present at the start of the markdown, and that each buyer remains until the markdown is over or until his entire demand is satisfied. This implies that it is never optimal for the seller to increase prices over time, in contrast to pricing policies that could emerge if customers arrived stochastically over time, e.g. [44]. Hence, in our model the seller's main decisions are the number of price steps and the price at each step.

The seller wishes to implement an m -step markdown mechanism with price p_k at step k , where $p_1 > p_2 > \dots > p_m$. The seller faces N rational buyers with valuations (per unit) $v_1 > v_2 > \dots > v_N$. Buyer j wishes to purchase at most $D_j > 0$ units, $\forall j$, where D_j , $j = 1 \dots N$, is common knowledge. (The assumption of known (or deterministic) demand is commonly used in the pricing literature in order to highlight the strategic interplay between pricing decisions and sales [5, 26, 42].) We define $D[p] = \sum_{\{j|v_j \geq p\}} D_j$ and refer to it as the 'market demand' at price p , i.e., $D[p]$ is the maximum possible demand at price p . At any (price) step k , buyer j submits a quantity bid, q_{jk} , indicating the number of units he wishes to purchase at the current price, p_k . Let \bar{q}_{jk} denote the quantity awarded to buyer j ; note that \bar{q}_{jk} may be smaller than q_{jk} if the total bid quantity ($\sum_{j=1}^N q_{jk}$) is greater than the remaining inventory at step k . In that case, the seller uses the following random rationing rule: Randomly choose a bidder j and assign him the minimum of q_{jk} and the remaining inventory. If there are remaining units, again randomly choose another bidder, and repeat this procedure until the inventory is exhausted. As noted by [35] and [17], this random allocation rule is consistent with the situation when all markdown items are sold on a first-come-first-serve basis at the end of the season when all customers with unsatisfied demand return to the store at the markdown price. It also has the advantage of circumventing the strategic bid inflation that may occur in quantity-proportional allocation rules [9].

Both the seller and the buyers are risk neutral and want to maximize their expected profits. The expected profit (or surplus) of buyer j is the difference between his valuation for the item and the purchase price; that is, $\Pi_j = \sum_{k=1}^m (v_j - p_k) \bar{q}_{jk}$, $j = 1, \dots, N$, where $\sum_{k=1}^m \bar{q}_{jk} \leq D_j$. The seller's expected revenue (or profit) is given by $\Pi_S = \sum_{k=1}^m \sum_{j=1}^N p_k \bar{q}_{jk}$.

We make the following assumptions, commonly adopted in game-theoretic analysis, concerning buyer behavior:

A1. In the last price step, if a buyer is indifferent between purchasing and not purchasing (i.e., $v_j = p_m$ and the buyer's expected profit is the same under both alternatives), the buyer prefers to purchase.

A2. If a buyer is indifferent between bidding his entire demand at step k or $k' > k$, he prefers to bid at step k .

Observation 1. *In an m -step markdown mechanism, all customers with $v_j \geq p_m$ bid all their remaining demand at step m , i.e., customers do not benefit from withholding any of their demand at the last price step.*

2.3 *Markdown Mechanisms under Complete Information (CI)*

In designing the optimal markdown mechanism, a seller who has K units for sale must answer the following two questions: (1) How many price steps should there be? (2) What should the price be at each step? In this section, we address these questions under the CI setting.

We call a markdown mechanism *effective* if it induces positive bids at each price step. Note that if the seller knows ex ante that a price step in a markdown mechanism will be ineffective, then she can remove that price step at which no bids will occur and still obtain the same revenue. We say that there is a *scarcity* of supply at price p_k if the market demand at that price is higher than the number of units available, i.e., $D[p_k] > K$, or equivalently, if $p_k \leq p_c$, where p_c is the *clearing price*, which is the highest price at which market demand exceeds supply.

Observation 2. *If $p_k > p_c$ and the customers know p_c , then customers have no incentive to buy at any step $i < k$, i.e., at any price higher than p_k .*

This result is quite intuitive, since any customer j can postpone his purchases until price step p_k , without the risk of facing a scarcity in supply and receiving less than D_j . In particular, if $D[p_m] \leq K$, then customers can postpone their purchases until the lowest

price step p_m . In such a case, the markdown mechanism is no different than a single-price mechanism with price p_m .

Based on Observation 2, a markdown mechanism cannot be effective if $D[p_m] \leq K$. Similarly, the seller would have no need for a markdown if $D_1 > K$ since she could sell all of her units to Customer 1 at a price of $p = v_1$. Therefore, as we search for the optimal markdown and markets in which it may be an appropriate pricing strategy, it is sufficient to focus on market settings and markdowns satisfying the following conditions.

A3-CI. $D_1 < K$, $D[v_N] > K$, i.e., the total market demand exceeds supply, and $D[p_m] > K$, i.e., there is scarcity at the lowest price step.

Two markdown pricing mechanisms are said to be *equivalent* if the mechanisms yield the same profits (for both the buyers and the seller). A markdown mechanism M is said to *dominate* a markdown mechanism L if M generates more (expected) profits for the seller than L.

Theorem 1. *For any m -step markdown mechanism, $m > 2$, there exists a two-step markdown mechanism that dominates or is equivalent to it, if the customers know the clearing price p_c .*

Proof: Consider an m -step mechanism with prices p_1, \dots, p_m .

Case 1: $p_c \geq p_1$. In this case, the maximum profit the seller can obtain is $p_1 K$. Consider an alternative markdown with two steps, where $\bar{p}_1 = p_c + \epsilon$ ($\epsilon \rightarrow 0$ can be thought of as the minimum possible price increment), and $\bar{p}_2 = p_c$. Under this two-step markdown, the minimum profit of the seller is $p_c K \geq p_1 K$. Hence, the seller is no worse off, and may be better off under the alternative two-step markdown.

Case 2: $p_c \leq p_m$. Under the m -step markdown, the seller's profit is given by $p_m K$ (by Observation 2). The same level of profit can be obtained by a two-step markdown mechanism with prices $\bar{p}_1 = p_{m-1}$ and $\bar{p}_2 = p_m$.

Case 3: $p_1 > p_c > p_m$. Let p_k be the largest price where $p_k \leq p_c$. If $k > 2$, i.e., $D[p_i] \leq K \forall i < k$, buyers have no incentive to buy until p_{k-1} since they do not expect a scarcity of supply before the price reaches p_k . Hence, a markdown mechanism with prices p_{k-1}, \dots, p_m

would result in the same profit for the seller (Observation 2). Now suppose that $k < m$, i.e., $D[p_i] > K, i = k, \dots, m$. (Note that the cases $k > 2$ and $k < m$ are not necessarily mutually exclusive.) We claim that the seller would be better off by eliminating the last price step p_m . Since $D[p_i] > K \forall p_i \leq p_k$, eliminating p_m does not decrease the number of units sold. We also need to show that the seller is guaranteed to sell all K units at the same or higher prices than with the m -step markdown. Customers originally bidding at step m may bid at higher price steps or they may not bid at all in the new markdown with fewer steps. If they do not bid in the new mechanism, then the competition at the higher price levels is unaffected. On the other hand, if they bid at higher price steps, then the competition can only increase since there would be more customers bidding at a price step. As a result, at any price step $j < m$, the competition either increases or remains the same, preventing the seller's revenue from decreasing. Hence, the revenue of the seller is not made any worse by eliminating the lowest price step from the markdown. Using these arguments repeatedly, any m -step markdown mechanism can be reduced to a two-step markdown mechanism with prices p_{k-1}, p_k , yielding the seller equal or higher revenue. \square

Although based on a different market model, Theorem 1 parallels some of the earlier results in the literature. [52] analyzes a static pricing mechanism that assigns prices for each unit in the inventory. He shows that the seller never needs to charge more than two prices to maximize revenues. [6] find that when customers demand at most one unit, a seller's expected profit increases as the number of price adjustments decreases. This is because as the number of possible price adjustments increases, so does the customers' price elasticity, which dampens the seller's ability to exercise her market power. We find a similar result for the multi-unit demand case; fewer markdowns are preferred to multiple markdowns. [20] study the problem where a set of allowable prices, as well as the initial price, is given and the goal is to find the optimal timing of the price change. [22] study a similar but more general problem than in [20] where the optimal initial price and the price path have to be chosen from a discrete set of allowable prices. They find that under the assumption of a deterministic demand function, the optimal solution has two price points: (1) some p_{k^*} for a specified period of time and (2) a neighboring price p_{k^*+1} for the rest of the selling season.

We find a similar result in the case of strategic customer behavior and multi-unit demands.

Remark: The results in Observation 2 and Theorem 1 require only that the customers know the clearing price p_c , i.e., the informational requirements are less restrictive than CI.

Theorem 1 allows us to narrow our search for the optimal markdown from a general m -step mechanism to a specific two-step markdown mechanism under CI. Before we can characterize the optimal two-step markdown, we need to analyze the customers' bidding behavior. For expositional ease, we state our main results in the main text and relegate most of the proofs to the Appendix A.

Theorem 2. *Under a two-step markdown mechanism, it is a dominant strategy for buyer j , $j = 1, \dots, N$, to submit an all-or-nothing bid, i.e., to submit either all or none of his demand at a price step.*

The proof of Theorem 2 is comprised of solving for the buyers' best response bidding strategies. We show that the expected profit of a buyer as a function of his bid quantity at any step is convex, and is maximized at one of the boundary points regardless of its opponents' bids. Therefore, submitting all-or-nothing bids is a dominant strategy. The result also holds if the customer valuations decrease over time (i.e., due to a discount factor $0 \leq \delta \leq 1$ or customer j 's valuation drops to δv_j in step 2, $\forall j$). Furthermore, we will show in Theorem 7 that this result carries over to the setting when there is incomplete information about customer valuations (Section 2.4).

Summarizing our results so far: (1) When p_c is known, as is the case under CI, it is sufficient for the seller to focus on two-step markdowns, since any additional price steps in the markdown will not improve profits. (2) Under a two-step markdown (p_1, p_2) , a buyer will either submit all of his demand at p_1 or at p_2 . Hence, we have already answered two out of our initial three questions. The question that remains is: What are the optimal prices in an effective markdown? (Recall that an effective markdown induces positive bids at each price step. If a two-step markdown is *ineffective* at one of its prices, then it is equivalent to and can be replaced by a single-price policy.)

Observation 3. *In a two-step markdown mechanism, the optimal price for the second step,*

p_2^* , is equal to the valuation of some customer, i.e., $p_2^* = v_j$ for some j .

Next, we characterize the optimal p_1 .

Observation 4. *In an optimal effective two-step markdown mechanism, the prices satisfy $D[p_1] < K$ and $D[p_2] > K$.*

Proof Consider a two-step markdown mechanism (p_1, p_2) , where $D[p_1] \geq K$. Consider an alternative mechanism (\bar{p}_1, \bar{p}_2) where $\bar{p}_2 = p_1$ and $\bar{p}_1 > p_1$ such that $0 < D[\bar{p}_1] < K$. The maximum profit the seller can obtain with prices p_1 and p_2 is Kp_1 . Conversely, with prices \bar{p}_1 and \bar{p}_2 , the minimum profit the seller can obtain is Kp_1 . The fact that $D[p_2] > K$ follows directly from A3-CI. \square

Observation 4 implies that in an effective optimal markdown, there is no scarcity at the high price but there is scarcity at the low price, motivating the high types to buy at p_1 .

To find the optimal value of p_1 , we must first identify the buyers' subgame-perfect Nash equilibrium (SPNE) bidding strategies. In Section 2.3.1, we focus on the case of two customers, characterize the equilibria in a two-step optimal markdown, and compare the markdown mechanism with the optimal single price in terms of seller's profits. In Section 2.3.2, we extend some of these results to multiple customers.

2.3.1 Two Customers

In this subsection, we consider a seller who faces two customers ($N = 2$) with valuations $v_1 > v_2$ and demands of D_1 and D_2 units. Following A3-CI and observations 3 and 4 we assume that $D_1 < K$ and $D_1 + D_2 > K$ and that the seller employs a markdown satisfying $D[p_1] < K$ and $D[p_2] > K$. Without loss of generality, we also assume $D_2 \leq K$ because additional demand above K from Customer 2 does not further increase the competition at step 2. Hence, we focus our attention on markdown mechanisms that satisfy the following condition: $v_1 > p_1 > v_2 = p_2$.

Obviously, p_1 should be within the range (v_1, v_2) if the markdown mechanism is to be effective. If $p_1 > v_1$, neither buyer will submit a positive bid at p_1 ; if $p_1 = v_1$, the high valuation buyer is made strictly better off waiting and submitting all of his demand at p_2 ,

rendering the two-step markdown mechanism ineffective. Borrowing from the language in [18], we label such an equilibrium to be a *pooling* outcome, since both customers bid their entire demand in the second step and the markdown fails to separate the bids of the two types.

We next characterize the set of prices that induce Customer 1 to bid at p_1 .

Proposition 3. *Given a markdown mechanism with prices p_1 and $p_2 = v_2$, Customer 1 bids his entire demand at step 1 if and only if $p_1 \leq \hat{p}_{CI}(v_1, v_2)$, where*

$$\hat{p}_{CI}(v_1, v_2) = p_2 + (v_1 - v_2) \frac{D_1 + D_2 - K}{2D_1}. \quad (1)$$

Proof. Since we have $p_2 = v_2$, buyer 2 will bid all of his demand at step 2 ($q_{21} = 0$ and $q_{22} = D_2$). From Theorem 2, buyers submit all-or-nothing bids; hence, buyer 1 has two options: (a) Bid D_1 at step 1 and get a guaranteed surplus of $\Pi_{11} = D_1(v_1 - p_1)$, or (b) bid zero at step 1 and D_1 in step 2. If the buyer chooses option (b), with probability 0.5, he has the priority in the random allocation and gets D_1 ; similarly, with probability 0.5, buyer 2 has the priority and buyer 1 gets $K - D_2$. This leads to an expected surplus of $\Pi_{12} = \frac{D_1 + K - D_2}{2}(v_1 - v_2)$ for Customer 1 under option (2). Customer 1 bids in step 1 if and only if $\Pi_{11} \geq \Pi_{12}$ i.e., $p_1 \leq \hat{p}_{CI}(v_1, v_2)$. \square

Customer 1's expected unmet demand is $\frac{D_1 + D_2 - K}{2}$ if he bids at step 2. The expression $\frac{D_1 + D_2 - K}{2D_1}$ in Equation (1) denotes *the ratio of Customer 1's expected unmet demand to D_1* , if he chooses to bid at p_2 . Hence, $(v_1 - v_2) \frac{D_1 + D_2 - K}{2D_1}$ is premium charged by the seller, which is the maximum additional amount (above $p_2 = v_2$) per unit Customer 1 is willing to pay to secure his demand by bidding at p_1 . Table 3 summarizes all possible outcomes of a markdown mechanism with prices p_1 and $p_2 = v_2$.

Table 3: Optimal quantity bids of the customers under a two-step markdown mechanism with $p_2 = v_2$.

	Non-Pooling $p_1 \leq \hat{p}_{CI}(v_1, v_2)$	Pooling $p_1 > \hat{p}_{CI}(v_1, v_2)$
Step 1 Bids (q_{11}, q_{21})	$(D_1, 0)$	$(0, 0)$
Step 2 Bids (q_{12}, q_{22})	$(0, D_2)$	(D_1, D_2)

Proposition 4. *Under an effective markdown, the optimal price at step 1 is $p_1^* = \hat{p}_{CI}(v_1, v_2)$.*

The proof of Proposition 4 follows by showing that the profit function of the seller is strictly monotonically increasing (in p_1) up to $\hat{p}_{CI}(v_1, v_2)$, after which it sharply drops to v_2K and remains constant. Hence, this break point $\hat{p}_{CI}(v_1, v_2)$ is the optimal price p_1^* at which Customer 1 is indifferent between buying at step 1 or step 2. By our earlier assumption (A2), Customer 1 bids his entire demand at step 1. The optimal first step price, p_1^* , increases with v_1 , v_2 , D_1 , and D_2 and decreases with K . These dynamics indicate that price increases as the customers' demand or willingness to pay increases, and decreases if the supply increases.

Note: Recall that multi-unit demand is an important generalization in our model compared to the previous literature. One may question whether our results trivially follow if we were to replace a single customer who demands D_1 units with D_1 customers who each demand one unit. In the case of two customers, we showed that the optimal first step price is $p_1^* = v_1 - (v_1 - v_2) \frac{D_1 + K - D_2}{2D_1}$ (by rearranging the terms in Equation (1)). Consider instead a setting where there are D_1 customers with single-unit demand and a valuation of v_1 , and D_2 customers with single-unit demand and a valuation of v_2 . If the buyer uses a random allocation rule at price p_2 , then the optimal first step price that induces all v_1 customers to purchase at that price is given by $p_1^{single} = v_1 - (v_1 - v_2) \frac{K - D_1 + 1}{D_2 + 1}$. As is evident from the optimal prices, setting prices for single unit demand is not identical to pricing for multi-unit demand. Furthermore, we can establish the relationship between p_1^{single} and p_1^* . We find that $p_1^{single} > p_1^*$ and hence the seller's revenue is higher from single-unit demand customers when $\frac{K - D_1 + 1}{D_2 + 1} < \frac{D_1 + K - D_2}{2D_1}$: From A3-CI, this condition reduces to $D_1 > \frac{D_2 + 1}{2}$. Hence, our results for the multi-unit demand case require a separate analysis and do not follow by a transformation of the model to the case of multiple customers with single-unit demands.

Extensions: Our results easily extend to the case of decreasing valuations over time (or discounting). Suppose customer j 's valuation drops to δv_j in step 2, $0 < \delta \leq 1$. In this case, the threshold step 1 price becomes:

$$\hat{p}_{CI}(v_1, v_2, \delta) = (1 - \delta)v_1 + \delta v_2 + \delta(v_1 - v_2) \frac{D_1 + D_2 - K}{2D_1} \quad (2)$$

If $\delta = 1$, i.e., no discounting, then Equation (2) is equivalent to Equation (1). If $\delta = 0$, i.e., the customer receives no value from the good if he waits for the second step, then $\hat{p}_{CI}(v_1, v_2, \delta) = v_1$. If $\delta \in (0, 1)$, we have $\frac{\partial \hat{p}_{CI}(v_1, v_2, \delta)}{\partial \delta} = -(v_1 - v_2) \frac{D_1 + K - D_2}{2D_1}$, which is always negative (since $D_2 \leq K$ without loss of generality). Hence, the threshold step 1 price is decreasing in δ , implying that Customer 1 is willing to pay a higher premium in the first step, if his value for the good is less in the second step. If the customers are risk-averse with constant valuations over time, similar observations as in the case of discounting hold, since the portion of the revenue generated in the second step of the markdown will be discounted in a similar way due to risk aversion.

2.3.1.1 Comparing Markdown and Single Price Mechanisms

We compare the performance of our markdown mechanism to that of the commonly used optimal single price mechanism. The structure of the optimal single (monopoly) price is well-studied in the economics literature [29, 48]. For the sake of completeness, we restate the following observations about the optimal single price p^* when the seller faces discrete demand.

Observation 5. *In the CI setting, $p^* \in \{v_1, \dots, v_N\}$.*

Based on this observation, we can easily compute the optimal single price p^* as follows: Recall that $D[v_j]$ denotes the market demand if the price is set equal to v_j , and let $\Pi_S^0[j] = \min\{D[v_j], K\}v_j$ denote the corresponding total revenue (profit) of the seller. Then, $p^* = \arg \max_{v_j} \Pi_S^0[j]$. Note that p^* can be higher than the *clearing price* p_c .

Since the optimal single price is equal to one of the customer valuations, the seller has two choices for the single-price p ; v_1 or v_2 .

If $p = v_1$, only the high-valuation customer can afford to buy at v_1 , and the seller effectively excludes the low-valuation customer from the market. We denote this alternative with (SP1) and it corresponds to a revenue of $v_1 D_1$. On the other hand, if $p = v_2$, then both customers can afford the product. We call this (SP2) and it yields a revenue of $v_2 K$ for the seller.

The seller chooses (SP1), if $v_1 D_1 > v_2 K$, and chooses (SP2), otherwise.

Observation 6. *If (SP2) is the optimal single price mechanism, then the markdown mechanism with prices (p_1^*, v_2) dominates the optimal single price.*

Observation 7. *If (SP1) is the optimal single price mechanism, then the markdown mechanism with prices (p_1^*, v_2) dominates the optimal single price if and only if $(v_1 + v_2)D_1 < (v_1 - v_2)D_2 + K(3v_2 - v_1)$, i.e., $\frac{v_2}{v_1} > \frac{K+D_1-D_2}{3K-(D_1+D_2)}$.*

Observation 6 is quite intuitive: The seller's profits under the markdown mechanism will be at least v_2K if both customers were to buy in step 2. But since the optimal markdown is effective, Customer 1 will buy in step 1 at price $p_1^* > v_2$, increasing the seller's profits above v_2K .

When (SP1) is the optimal single price (Observation 7), we find that a two-step markdown mechanism is more likely to dominate the optimal single price when the following are true: (1) D_2 is very large relative to D_1 , (2) the valuations of the two customers are close, and/or (3) K is close to the total demand of both customers, $D_1 + D_2$. Conversely, SP1 will outperform the optimal markdown if these three conditions are not met. As Conditions (1)-(3) become stronger, the optimal single price will switch to (SP2), and a markdown will always be optimal. Figure 12 in Appendix A plots the seller's revenues under the optimal markdown, and illustrates when a markdown or a single price is optimal.

2.3.2 Multiple Customers

In this section, we generalize some of our earlier results and insights to multiple customers. It is important to recall that Theorems 1 and 2 are general results for any $N \geq 2$ customers, i.e., the seller can restrict her search for the optimal markdown to a two-step markdown, and customers submit all-or-nothing bids in equilibrium. A consequence of this is that, as opposed to the case of two customers, we can no longer hope to design a markdown where each customer bids at a different price step. The best that we can achieve under an effective markdown is a partitioning of customers into three groups, with the first group purchasing all of their demand at p_1 , the second group purchasing at p_2 , and the third group (possibly an empty set) not making any purchases.

Following Observation 4, in this section, we consider markdown mechanisms (p_1, p_2) with

$D[p_1] < K$ and $D[p_2] > K$, i.e., customers are guaranteed to receive their bid quantities in the first step and there is scarcity in the second step. Hence, let v_n , $n \leq N$, be the smallest valuation greater than or equal to p_2 . Let $S_3 = \{n+1, \dots, N\}$ be the set of customers with valuations less than p_2 . In equilibrium, the customers in S_3 bid nothing at either price step. Therefore, in the remainder of this section, we focus on customers $1, \dots, n$.

Theorem 5. *Consider a markdown mechanism (p_1, p_2) with $D[p_1] < K$ and $D[p_2] > K$. Given the conditions (C1) and (C2) below, if there exists a partition $\{S_1, S_2\}$ of customers $\{1, \dots, n\}$ such that (C1) is satisfied for all $t \in S_1$ and (C2) is satisfied for all $s \in S_2$, then the following bidding strategy is the (subgame-perfect) Nash equilibrium for customers $1, \dots, n$.*

Equilibrium Strategy (ES):

- (1) If $i \in S_1$, bid D_i at step 1 and nothing at step 2.
- (2) If $i \in S_2$, bid nothing at step 1 and D_i at step 2.

where

$$D_t(v_t - p_1) \geq (v_t - p_2)E[A_t] \quad \forall t \in S_1 \quad (C1)$$

$$D_s(v_s - p_1) < (v_s - p_2)E[A_s] \quad \forall s \in S_2 \quad (C2)$$

and $E[A_j]$ is the expected allocation of customer j in step 2 if all customers (except j) bid according to ES and j bids his entire demand at step 2. We have

$$E[A_j] = \frac{1}{n!} \sum_{\forall \pi \in U} \min \left\{ \left(K - \sum_{i \in B_\pi^j} D_i - \sum_{i \in S_1 \setminus \{j\}} D_i \right)^+, D_j \right\},$$

where U denotes the set of all permutations of $\{j\} \cup S_2$ and $B_\pi^j \subseteq S_2$ denotes the set of customers whose bid quantities are satisfied before customer j in step 2 in some permutation $\pi \in U$.

Theorem 5 formally states how customers would determine at which step to bid by comparing their expected profits from bidding at step 1 versus step 2. Note that it is possible to have a partition where S_1 is empty, i.e., all customers bid at step 2.

In general, we would be interested in situations where “high” types bid at step 1 and “low” types bid at step 2. Hence, we would like to determine conditions under which

$S_1 = \{0, 1, \dots, j\}$ and $S_2 = \{j + 1, \dots, n\}$ for some $j \geq 0$ (where 0 denotes a dummy customer who always bids at step 1 and $S_1 = \{0\}$ indicates that all customers bid at step 2). Proposition 6 provides a sufficient condition for such a high-low partitioning of customers to exist.

Proposition 6. *Consider a markdown mechanism (p_1, p_2) with $D[p_1] < K$ and $D[p_2] > K$. If $D_1 \geq D_2 \geq \dots \geq D_n$, then there exists a partitioning of customers into $\{S_1, S_2\}$ such that $S_1 = \{0, \dots, j\}$ and $S_2 = \{j + 1, \dots, n\}$ for some $0 \leq j < n$ and customers in S_i bid at step i , $i = 1, 2$, in equilibrium.*

The search for the optimal prices is equivalent to searching for the optimal sets S_1, S_2 , and S_3 . If the seller wants to induce a particular partition $\{S_1, S_2, S_3\}$ in equilibrium, using conditions (C1) and (C2), she first needs to characterize the range of prices where the optimal p_1 falls, for a given p_2 . Given its feasible range, the seller can then determine p_1 as a function of p_2 . From Observation 3, we know that $p_2 = v_k$ for some $k \leq N$. The seller can find the optimal price pair $\{p_1, p_2\}$ by searching over all valuations for p_2 (in at most N steps) and the corresponding optimal p_1 .

The proposed search method for finding the optimal price pair would work efficiently if the seller only needs to consider a reasonable number of partitions and if p_1 can be found easily for a given p_2 . For example, when $D_1 = \dots = D_N = D$ and $K = rD$ (where r is a positive integer), the seller only needs to consider partitions $S_1 = \{0, \dots, j\}$ and $S_2 = \{j + 1, \dots, k\}$ for $j \leq r < k \leq N$ (from Proposition 6); hence, the optimal p_1 can be found in at most r steps for each possible $p_2 = v_k$, $k > r$. As a result, the seller can identify the optimal markdown efficiently in $O(N^2)$ time, which is polynomial in the number of customer types. The price range for p_1 that induces the partition $\{S_1, S_2\}$ is $[v_{j+1} - (v_{j+1} - p_2)\frac{r-j}{n-j}, v_j - (v_j - p_2)\frac{r-j+1}{n-j+1}]$. It is important to note that this range may be empty, implying that partition $\{S_1, S_2\}$ cannot be supported in equilibrium under any price p_1 for $p_2 = v_k$. For any partition supportable in equilibrium under equal demands, the seller's profits are maximized by setting p_1 as follows:

$$p_1 = p_2 + (v_j - p_2)\frac{n-r}{n-j+1} \quad (3)$$

This has an interpretation similar to the optimal p_1 for the two-customer setting given in Equation (1). The expression $\frac{n-r}{n-j+1}$ in Equation (3) denotes the ratio of customer j 's expected unmet demand to his total demand, if he chooses to bid at p_2 . Hence, $(v_j - p_2)\frac{n-r}{n-j+1}$ is the maximum additional amount (above p_2) per unit that customer j is willing to pay to secure his entire demand by bidding at p_1 .

The choice of p_2 determines which customers are excluded (S_3) and included (S_1 and S_2) in the market. There are two countervailing forces working in the selection of p_2 . A lower p_2 increases the seller's market (total demand), and hence increases the scarcity at p_2 , providing the incentive for a high-type customer to bid at p_1 . However a lower p_2 also implies reduced revenues for the seller from the units sold at that step and provides an incentive for a high type to bid at p_2 .

2.3.3 Numerical Experiments

To gain insights on the structure and performance of the optimal markdown mechanism, we conducted numerical experiments. We examined how the optimal markdown prices vary according to the distribution of valuations and the supply level (Figure 1(a)-(c)) and the conditions under which markdown revenues exceed single price revenues (Figure 1(d)). In our experiments, we assumed that customers are drawn from two groups, *high*- and *low*-valuation customers. Hence, we randomly selected customer valuations from a bimodal distribution, where half of the customers have valuations drawn from a uniform distribution over $[100, 100 + \delta]$ and the remaining half have valuations drawn from a uniform distribution over $[200 - \delta, 200]$. Such clustering of valuations allows us to model multiple customers while keeping a link to the two-customer case for comparison purposes. Note that as δ increases, the difference between the valuations of the two customer groups decreases. We ran the experiments with 10 different values of $\delta = 5, 10, 15, \dots, 50$. For each δ value, we tested fifty instances with 20 customers, each with a demand of 10 units; hence, the total demand in the market is $20 \times 10 = 200$. We examined six different supply scenarios where the seller has enough supply to meet $Q\%$ of the total demand in the market (i.e., $K = \frac{QD}{100}$), where $Q = \{30, 40, 50, 60, 70 \text{ and } 80\}$. Figure 1 presents the properties of the optimal markdown

averaged over 50 randomly drawn instances.

From Figure 1(a), we observe that *as δ increases*, i.e., the expected customer valuations get closer to each other, p_1 *decreases* for all supply levels. The effect of δ on p_2 and the depth of the markdown critically depend on the supply level (Figures 1(b) and (c)). When supply levels are low ($Q < 50$), we find that p_2 is set very high for $\delta = 5$ and then decreases in δ ; in addition, the *depth of the markdown* $\frac{p_1 - p_2}{p_1}$ increases in δ . The opposite is true when supply levels are moderate to high ($Q \geq 50$) where p_2 is set low for $\delta = 5$, p_2 increases and the depth of markdown decreases in δ . It is interesting to note that the depth of the markdown is the greatest for $Q = 50$. However, when $\delta = 50$ the depth of the average markdown is between 7-10% for all supply levels.

When $\delta = 50$, the customers' valuations are generally evenly distributed across the entire range [100,200] - the distinction between a 'high' and 'low' types becomes weaker. As a result, inducing a separation between these two groups requires a smaller price difference, which implies that the optimal markdown is almost revenue equivalent to the optimal single price (Figure 1(d)). It is worth pointing out that while the depth of the markdown is fairly consistent across supply levels (when $\delta = 50$), the actual price values *do* depend on K , as is clearly illustrated in Figures 1(a) and (b).

The interplay between K and p_2 is a very interesting one. Recall that as the seller decreases p_2 , she increases the number of customers who are able to purchase at p_2 and hence increases the 'scarcity' at p_2 . All else being equal, an increase in scarcity at p_2 creates greater incentives for a high valuation customer to purchase at p_1 , whereas a reduction in p_2 increases his incentive to purchase at p_2 . When supply levels are low ($Q < 50$), the seller's optimal balancing act between these two forces weighs in favor of a high p_2 ; while the seller includes only a few customers in the market to purchase at a high p_2 , this allows her to keep p_1 high as well and maintain an effective markdown. As K increases, the increase in supply induces the seller to include more customers in the market by decreasing p_2 . If the seller were to decrease p_2 by a small amount, the scarcity level at p_2 would not be enough to support any purchases at p_1 , i.e., the markdown would not be effective. Hence, the seller finds it optimal to decrease p_2 substantially. While she receives a lower revenue (per unit)

from sales at p_2 , the increased scarcity at p_2 allows her to support an effective markdown where p_1 is still significantly greater than p_2 .

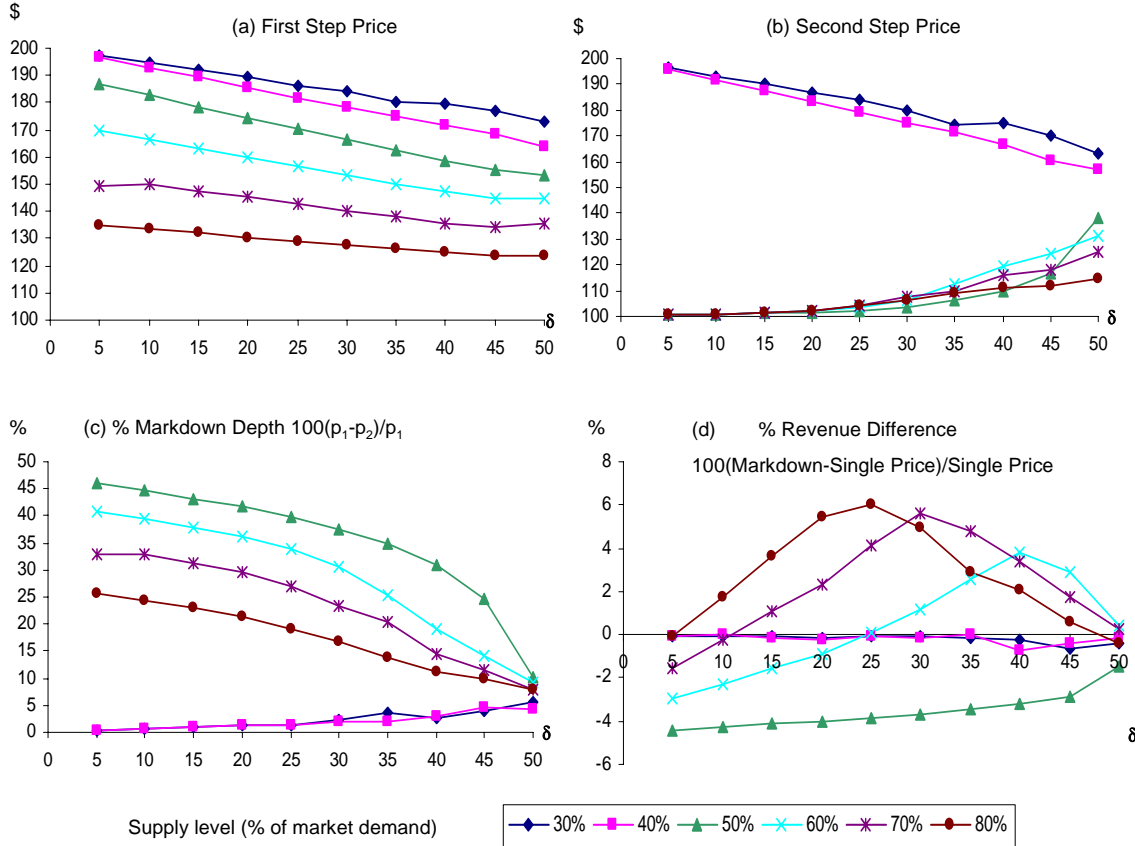


Figure 1: Average optimal prices and revenues for each valuation distribution and supply scenario. Each data point represents the average of fifty instances.

When we look at the difference between the revenues of markdown and single-price mechanisms (Figure 1(d)), we see that whether or not a markdown dominates the optimal single-price mechanism depends on supply levels as well as the dispersion of valuations in the market. When supply levels are low ($Q < 50$), the markdown and single price are almost equivalent for all δ . This is fairly intuitive, since the depth of the markdown is very small, rendering it equivalent to a single price. When $Q = 50$ the single price outperforms the markdown for all δ . When $Q = 50$, the optimal single price is most likely to be the valuation of the 10^{th} customer, v_{10} (lowest-valuation ‘high’ type). However, p_2 must be less than v_{10} under an effective markdown in order to guarantee scarcity at p_2 and to induce

some ‘high’-valuation customers to purchase at p_1 . As a result, the performance of the markdown suffers. When supply levels are higher ($Q > 50$) a markdown yields higher revenues over a wide range of settings. The difference in revenues is a unimodal function of δ , and the markdown performs best when the depth of the markdown is between 20-25% for all K . The maximum percentage difference occurs at $\delta = 25, 30,$ and 40 for $Q = 80, 70,$ and $60,$ respectively. In summary, we observe that a markdown is almost equivalent to a single price when supply levels are low, and that a markdown will dominate the optimal single price when either (i) the supply is moderate and δ is high, or (ii) the supply is high and δ is moderate. (Note that condition (i) is similar to the condition in Observation 7 for the case of two customers.)

2.4 Equilibrium Bidding Behavior and Mechanism Design under Incomplete Information (IV)

In the previous section, we answered the following questions for a complete information setting: (1) How will rational buyers bid when facing a markdown? (2) How many price steps are there in an optimal markdown? (3) Under what conditions would the seller be better off implementing a single price vs. markdown pricing? In this section we extend our analysis to an incomplete valuation information (IV) setting. Under IV, we assume that the valuation of customer j is drawn from $[\underline{v}_j, \bar{v}_j]$ with probability distribution function $F_j(\cdot)$, $\forall j$, where $\underline{v}_j > \bar{v}_{j+1}$, i.e., valuations are drawn from non-overlapping intervals (Figure 2). Each customer knows his own valuation with certainty and both the seller and the customers know the pdf and CDF of other customers’ valuations.

First, we show that the all-or-nothing bidding result we had for two-step markdowns under CI (Theorem 2) carries over to the IV setting and m -step markdowns.

Theorem 7. *Under IV, in an m -step markdown with prices $p_1 > \dots > p_m$ customers submit all-or-nothing bids in equilibrium.*

In Section 2.3 we showed that under CI a two-step markdown was sufficient to maximize the seller’s revenue (Theorem 1). This result relies on the fact that buyers know the clearing price, p_c , under CI. When the customer valuations are unknown, p_c may not be known with

certainty; hence, we cannot claim that a two-step markdown is always optimal. However, we can easily determine in which customer's valuation interval p_c falls and use this information to gain insights into the properties of the optimal markdown. Let k be such that $\sum_{j=1}^{k-1} D_j < K$ and $\sum_{j=1}^k D_j \geq K$. That is, $p_c \in [\underline{v}_k, \bar{v}_k]$.

Observation 8. *In an effective optimal m -step markdown*

- (i) *there is at most one price exceeding \bar{v}_k*
- (ii) *there is at most one price less than \underline{v}_k and it is in the interval $[\underline{v}_j, \bar{v}_j]$ for some $j \in \{k, \dots, N\}$*
- (iii) *all other intermediate prices occur in the range $[\underline{v}_k, \bar{v}_k]$.*

Observation 8 states that in an effective optimal markdown, there exists at most one price (p_1) at which the buyers are guaranteed to receive their bid quantities. In addition, the lowest price p_m is in the valuation interval of *some* customer $j \geq k$. Note that we cannot simply replace the lowest price p_m with \underline{v}_k (or any other higher price) without potentially having an adverse effect on seller's revenue (i.e., increasing the lowest price may reduce the competition at the last step, which could prevent customers who were originally bidding at higher prices from doing so). All other (intermediate) prices must occur within the interval $[\underline{v}_k, \bar{v}_k]$, which is the interval containing the clearing price (Figure 2). Note that p_1 and p_m may themselves fall within $[\underline{v}_k, \bar{v}_k]$.

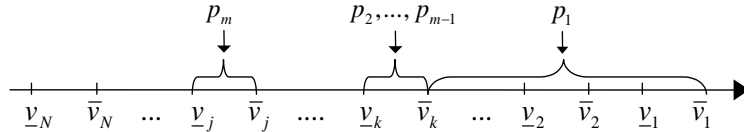


Figure 2: Candidate structure for price steps in an effective optimal m -step markdown under IV.

Since all intermediate prices must occur in the interval $[\underline{v}_k, \bar{v}_k]$, we do not know if a two, three, or m -step markdown is optimal. In B2C markets, we see that price reductions are typically rather large, e.g., a retailer does not drop the price of a \$44 item to \$43, but rather to \$29. This suggests that the retailer is trying to access a new customer group with each price reduction. Motivated by these practices, we next consider a narrower yet

intuitive setting of markdown mechanisms, where there is at most one price step in each customer valuation interval. More formally, we consider a family of interval markdowns, INT markdowns, defined as follows:

INT Markdown There is at most one price drawn from each interval $[\bar{v}_j, \bar{v}_{j-1}]$, $\forall j = 2, \dots, N + 1$, where $\bar{v}_{N+1} = \underline{v}_N$.

Observation 9. *In an effective INT markdown, there are at most three price steps. Furthermore, if the markdown has three steps, then $p_1 > \bar{v}_k$, $p_2 \in [\underline{v}_k, \bar{v}_k]$, and $p_3 \in [\underline{v}_j, \bar{v}_j]$ for some $j \geq k$, where $[\underline{v}_k, \bar{v}_k]$ is the interval containing the clearing price.*

The proof follows directly from Observation 8.

Next, we would like to understand when an optimal INT markdown will have two steps vs. three steps. Although we are not able to answer this question in general, we state a sufficient condition for a two-step INT markdown to be optimal.

Observation 10. *If $p_c \in [\underline{v}_N, \bar{v}_N]$, then in an effective INT markdown, there are at most two price steps, where $p_1 > \bar{v}_N$ and $p_2 \in [\underline{v}_N, \bar{v}_N]$.*

From Observation 10, when $D_1 + D_2 + \dots + D_{N-1} \leq K$, in an m -step markdown customers are partitioned into two or fewer groups, depending on whether or not they may bid at a given price step. For example, if $\bar{v}_j > p_1 > \bar{v}_{j+1}$, customers are partitioned into $\{1, \dots, j\}$ and $\{j, \dots, N\}$ where the first group *may* bid at p_1 , whereas the second group can only bid at p_2 .

Corollary 8. *If $D_1 + D_2 + \dots + D_{N-1} \leq K$, then a two-step INT markdown is equivalent to or dominates any 3-step INT markdown.*

It remains to characterize the optimal markdown. For the remainder of this section, we characterize the optimal markdown and compare its performance against the optimal single price when $N = 2$. As will quickly become clear, designing the optimal markdown under IV is quite complicated even when $N = 2$.

2.4.1 Two Customers under IV

Given that a seller is facing two customers, an INT markdown has 2 steps. While we cannot guarantee that a two-step markdown will be optimal, we believe this to be a reasonable

framework for analysis given their common use in practice [20], and the theoretical [6, 22] and empirical results [20, 42] supporting their near-optimal performance under various market settings.

Following the conditions and the intuition laid out in A3-CI, we focus only on market settings where $D_1 < K$ and $D_1 + D_2 > K$, and markdown mechanisms, where $\bar{D}[p_1] < K$ and $\bar{D}[p_2] > K$, i.e., Customer 1 is guaranteed to receive his entire quantity bid if he should bid at the first price step, but may face scarcity if he bids at the second price step. These conditions imply that we are searching for the optimal INT markdown that satisfies: $\bar{v}_1 \geq p_1 > \bar{v}_2 > p_2 \geq \underline{v}_2$.

From Observation 1, we know that Customer 2 will bid his entire demand at step 2 with probability $1 - F_2(p_2)$, i.e., only if $v_2 \geq p_2$; otherwise, he does not bid at all. Furthermore, we know from Theorem 7 that Customer 1 with valuation v_1 bids his entire demand at either p_1 or p_2 . Proposition 9 summarizes customer 1's bidding behavior under IV.

Proposition 9. *Given a markdown mechanism with prices p_1 and p_2 , Customer 1 bids D_1 at step 1 if and only if $p_1 \leq \hat{p}_{IV}(v_1, p_2)$, or equivalently, $v_1 \geq \hat{v}_1(p_1, p_2)$; otherwise, he bids D_1 at step 2, where*

$$\hat{p}_{IV}(v_1, p_2) = p_2 + (v_1 - p_2)(1 - F_2(p_2)) \frac{D_1 + D_2 - K}{2D_1} \quad (4)$$

$$\hat{v}_1(p_1, p_2) = p_2 + \frac{(p_1 - p_2)}{1 - F_2(p_2)} \frac{2D_1}{D_1 + D_2 - K} \quad (5)$$

A markdown mechanism $(\hat{p}_{IV}(v_1, p_2), p_2)$ makes customer 1 indifferent between bidding at steps 1 and 2. That is, $\hat{p}_{IV}(\cdot)$ is the *threshold step 1 price* above which Customer 1 will not bid at step 1. Note the similarity between the threshold step 1 price under IV (equation (4)) and CI (equations (1) and (3)) settings. In Equation (4), the term $(1 - F_2(p_2)) \frac{D_1 + D_2 - K}{2D_1}$ indicates the “scarcity” at step 2, i.e., it is the ratio of the expected unmet demand of Customer 1 to his entire demand if he bids his entire demand at step 2. Since the scarcity increases in D_1 and D_2 and decreases in K , the threshold step 1 price increases in D_1 and D_2 , and decreases in K . As in the CI setting, the second term of the threshold step 1 price is the premium that the high-type customer is willing to pay to secure his entire demand

at step 1. Similarly, $\hat{v}_1(p_1, p_2)$ is the *threshold valuation* below which Customer 1 would not bid at step 1.

For a markdown not to be pooling, the threshold valuation should not exceed \bar{v}_1 , or equivalently, the step 1 price should be below the threshold step 1 price for the highest-possible type.

Corollary 10. *If $\hat{v}_1(p_1, p_2) > \bar{v}_1$, equivalently, if $p_1 > \hat{p}_{IV}(\bar{v}_1, p_2)$, then $F_1(\hat{v}_1) = 1$ and Customer 1 never bids at p_1 ; that is, the markdown is always pooling.*

From Corollary 10, a markdown mechanism where $p_1 > \hat{p}_{IV}(\bar{v}_1, p_2)$ is always pooling (for all possible realizations of customer valuations), and hence, is equivalent to or dominated by the optimal single price. Therefore, in our search for the optimal non-pooling INT markdown, we focus our search on markdowns satisfying the following:

$$\mathbf{A4-IV} \quad \hat{p}_{IV}(\bar{v}_1, p_2) \geq p_1 > \bar{v}_2 > p_2 \geq \underline{v}_2$$

It is also interesting to know when *all* Customer 1 types will bid at p_1 . This happens when the threshold valuation is below the lowest Customer 1 type \underline{v}_1 (or equivalently, when the step 1 price is below the threshold step 1 price for the lowest Customer 1 type).

Corollary 11. *If $\hat{v}_1(p_1, p_2) \leq \underline{v}_1$, equivalently, if $p_1 \leq \hat{p}_{IV}(\underline{v}_1, p_2)$, then $F_1(\hat{v}_1) = 0$ and every Customer 1 type will bid at p_1 .*

When the threshold valuation for a given (p_1, p_2) falls in $(\underline{v}_1, \bar{v}_1)$, then only a strict subset of customer 1 types will bid at p_1 (Figure 3).

Corollary 12. *If $\underline{v}_1 < \hat{v}_1(p_1, p_2) \leq \bar{v}_1$, equivalently, if $\hat{p}_{IV}(\underline{v}_1, p_2) < p_1 \leq \hat{p}_{IV}(\bar{v}_1, p_2)$ then $0 < F_1(\hat{v}_1) < 1$ and customer 1 bids at p_1 if and only if $\hat{v}_1(p_1, p_2) \leq v_1$; that is, the resulting markdown is potentially separating and only some subset of Customer 1 types will submit a quantity bid at p_1 .*

From Corollaries 11 and 12, the seller's choice of p_1 , for a given p_2 , determines whether a markdown mechanism will induce *all* or only *some* Customer 1 types to purchase at p_1 , which we refer to as *totally separating* (TS) and *potentially separating* (PS) markdowns, respectively (see Figure 3). In a TS markdown, all Customer 1 types bid at p_1 whereas in

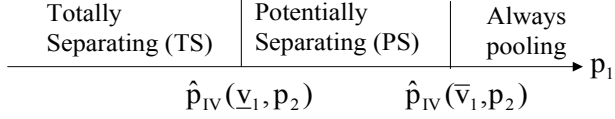


Figure 3: Seller’s alternative choices for p_1 , given p_2 .

a PS markdown only Customer 1 types within $[\hat{v}_1, \bar{v}_1]$ bid at p_1 , with the remaining types bidding at p_2 .

Given the customers’ equilibrium behavior, the seller wishes to design a markdown that maximizes her expected profits. The seller’s problem is significantly more complicated under an IV setting than a CI two-customer setting, for the seller must decide on both the optimal p_2 and the optimal p_1 . Under IV, it is no longer clear that a seller can or *should* try to design a markdown that is TS (as was the case under CI). Interestingly, we find conditions/examples that illustrate that a seller may be better off with a PS markdown, whereby Customer 1 will purchase at p_2 for some realizations of valuation. An additional challenge facing the seller is the choice of p_2 . Under CI, $p_2^* = v_2$ always, and hence Customer 2 always bids at the second price step. Under IV, to induce all Customer 2 types to bid at p_2 the seller must set $p_2 = \underline{v}_2$. However, a higher p_2 *may* allow the seller to charge a higher p_1 and increase his profits. Note that this is somewhat similar to the problem faced by the seller under CI with $N > 2$ customers, i.e., the choice of p_2 determines which customers are excluded from and included in the market. Under IV, the effect of increasing p_2 on the surplus of Customer 1 is two-fold: the expected per unit surplus of Customer 1 from bidding at step 2 decreases, but his expected allocation increases since the probability that Customer 2 can bid at step 2 decreases. Therefore $\hat{p}_{IV}(v_1, p_2)$ might be decreasing or increasing in p_2 . We provide examples below that illustrate that it may be optimal to set $p_2 > \underline{v}_2$.

In solving the seller’s problem, we must

- identify the p_2 ranges for which TS and PS markdowns are feasible (i.e., exist satisfying A4-IV),
- identify the optimal TS and PS markdowns, namely, (p_1^{*TS}, p_2^{*TS}) and (p_1^{*PS}, p_2^{*PS}) ,

respectively, over their respective feasible regions, and

- determine whether a TS or PS markdown or single price leads to higher (expected) revenues for the seller.

Figure 4 shows (for uniformly distributed valuations) how the existence of PS and TS markdowns depends on the “scarcity” of supply measured by $\frac{D_1+D_2-K}{2D_1}$ (the ratio of customer 1’s expected unmet demand to his entire demand, if both customers bid at step 2): If scarcity is low, it is more likely to see a pooling outcome. (Figure 4 is based on Results 1 and 2 presented in Appendix A. Figure 4 assumes $\bar{v}_1 - \underline{v}_1 < \bar{v}_2 - \underline{v}_2$, which does not necessarily need to hold. If $\bar{v}_1 - \underline{v}_1 \geq \bar{v}_2 - \underline{v}_2$, then (c) would be replaced by (c’) where only PS exists for all p_2 over $\frac{\bar{v}_2 - \underline{v}_2}{\bar{v}_1 - \bar{v}_2} < \frac{D_1 + D_2 - K}{2D_1} < \frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \underline{v}_2}$.)

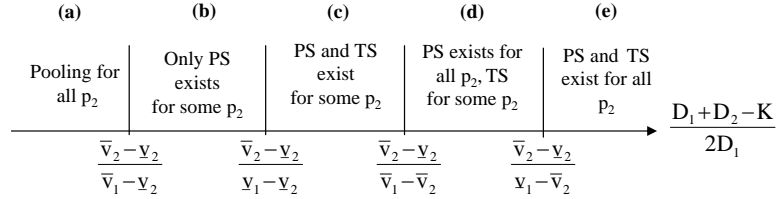


Figure 4: Existence of INT markdowns satisfying A4-IV as a function of market parameters.

The analysis that goes into answering the three questions above is quite involved and lengthy. So as not to encumber the reader with these derivations, we relegate most of this analysis to Appendix A and provide our main results and insights below via numerical instances that are representative of the general results found in Section 2.4.3. Given the intractability of finding closed-form solutions for generic distribution functions in this setting, the analysis that follows assumes that the customer valuations are drawn from uniform distributions. Before we present our numerical examples, we characterize the optimal single price under IV.

2.4.2 Comparison to the Optimal Single Price

As in Section 2.3, we compare the performance of an INT markdown to that of the optimal single price. We first characterize the optimal single price as a function of the market setting when there are $N \geq 2$ customers.

Observation 11. *In the (IV) setting, $p^* \in [\underline{v}_j, \bar{v}_j]$ for some customer j , $j = 1, \dots, N$.*

Note that $p^* \leq \bar{v}_1$, otherwise, the seller would not be able to sell any units. Assume $\underline{v}_i > p^* > \bar{v}_{i+1}$ for some $i < N$. The seller could increase her profits by setting $p' = \underline{v}_i$, and sell the same number of units as she did at price p^* , thereby contradicting the optimality of p^* .

Using this observation, the seller can easily compute the optimal single price p^* as follows. Define the maximum potential market demand at price p as $\bar{D}[p] = \sum_{i:\bar{v}_i \geq p} D_i$. Let p^{*i} be the best single price that falls within the valuation interval of customer i , that is, $p^{*i} = \arg \max_{\underline{v}_i \leq p \leq \bar{v}_i} \Pi_S^0[i]$, where $\Pi_S^0[i] = \{\min\{\bar{D}[p], K\}(1 - F_i(p)) + \min\{(\bar{D}[p] - D_i), K\}F_i(p)\} p$. Then the optimal single price is $p^* = p^{*k}$, where $\Pi_S^0[k] \geq \Pi_S^0[i]$ for all $i \neq k$.

With $N = 2$, the structure of the optimal single price implies that the seller has two choices for the single-price p :

(SP1) : Set $p \in [\underline{v}_1, \bar{v}_1]$; excluding the low-valuation customer from the market to receive a revenue of $\Pi_S^0[1]$.

(SP2) : Set $p \in [\underline{v}_2, \bar{v}_2]$ and *possibly* sell to both customers and receive revenue $\Pi_S^0[2]$. It is important to point out the different interpretations of (SP1) and (SP2) under IV from CI when $N = 2$. Under IV, the seller cannot guarantee that Customer 1 will purchase under SP1, provided that $p > \underline{v}_1$. A similar statement is true for Customer 2 and (SP2).

Proposition 13. *If (SP2) is the optimal single-price mechanism with p^* , then a TS or PS markdown mechanism, should it exist for $p_2 = p^*$, dominates the optimal single price.*

Proof: The seller's profits under a markdown mechanism would be the same as $\Pi_S^0[2]$ if all Customer 1 types purchased at p_2 . However, a PS or TS markdown, by design, leaves some customer 1 types $\hat{v}_1(p_1, p_2)$ indifferent between buying in step 1 or step 2. Consequently, a TS or PS markdown - with $p_2 = p^*$ and Customer 1 types $v_1 \geq \hat{v}_1(p_1, p^*)$ purchasing in step 1 - yields the seller a higher expected revenue than $\Pi_S^0[2]$; *ex post* the seller's expected revenue is bounded below by $\Pi_S^0[2]$. \square

Proposition 13 is quite intuitive and extends Observation 6 to IV. Note that if a PS or a TS markdown does not exist for $p_2 = p^*$, the optimal markdown revenue may not

exceed the (SP2) revenue. When (SP1) is the optimal single-price mechanism, we cannot make a theoretical comparison of the markdown and single price revenues since we do not have a closed-form solution for the optimal markdown revenue in general. For additional insights on the comparison of markdown versus single-price revenues, we turn to numerical examples.

2.4.3 Numerical Examples

In this section, we present representative numerical examples (see Table 4 for the parameters) to provide insights on the structure and performance of optimal markdown mechanisms. We first present a base instance, solve for the optimal markdown, identify whether it is totally or partially separating, and compare its performance against the optimal single price.

(Base Instance) Example 1: TS optimal. In this setting, $p_2 = 2.41 > 2 = v_2$, i.e., with positive probability, Customer 2 will not make a purchase. However, the optimal first price is to induce all Customer 1 types to purchase at the first price step, i.e., $p_1 = \hat{p}_{IV}(12, 2.41)$, and hence the optimal markdown in this instance is TS. Given the relatively large demand from Customer 2 compared to Customer 1, the seller finds it optimal to set the optimal single price to $p^* = 2.76$. A TS markdown exists for all $p_2 \in [2, 3)$ and we have $p^* = 2.76 \in [2, 3)$; hence, it follows from Proposition 13 that the optimal markdown yields higher profits than SP2.

Table 4: Optimal markdown and single price for specific instances under IV

$K, (D_1, D_2), (v_2, \bar{v}_2), (v_1, \bar{v}_1)$	Markdown type	(p_1, p_2)	$\Pi_{(p_1, p_2)}$	Single price	p^*	Π_{p^*}
Example 1: 20, (3,19), (2,5), (12,18)	TS	(5.17, 2.41)	50.88	SP2	2.76	43.31
Example 2: 20, (3,19), (2, 4), (12,18)	TS	(5.33, 2.00)	50.00	SP2	2.18	40.27
Example 3: 20, (3,19), (2, 7), (12,18)	PS	(7.01, 3.00)	49.80	SP2	3.94	52.81
Example 4: 20, (3,19), (2,5), (12, 23)	PS	(5.37, 2.41)	50.92	SP2	2.76	43.31
Example 5: 20, (8 ,19), (2,5), (12,18)	TS	(6.38, 2.00)	75.00	SP1	12.00	96.00

In each of the subsequent four examples, we alter one parameter from this base instance (highlighted in bold in Table 4). These examples are constructed to illustrate how the

existence and the optimal structure of the PS and TS markdowns may change, as well as the performance of a markdown when compared to the optimal single prices. These examples were drawn from a larger numerical example set (see Appendix A), generated by increasing/decreasing one parameter at a time from the base instance, until one of the following assumptions of our model was violated.

- Customer valuations overlap (i.e., INT markdowns do not exist).
- D_1 or D_2 exceeds K (i.e., there are no leftover units after the first customer's demand is satisfied).
- K exceeds $D_1 + D_2$ (i.e., there is no scarcity in the market).
- The valuation interval of a customer reduces to a singleton (i.e., there is no uncertainty in valuations).

Example 2: TS optimal. Example 2 complements the base case presented above. Here the optimal markdown is also TS, and it performs better than the optimal single price. However, unlike the base case, the shrinking support for Customer 2's valuations (by a decrease in \bar{v}_2) allows the optimal second price step under TS to decrease to $p_2 = \underline{v}_2 = 2$, i.e., all Customer 2 types are now guaranteed to bid at p_2 . Since the optimal markdown remains a TS one, both customers' bids are independent of their valuation realization and the seller receives a guaranteed revenue. A decrease in \bar{v}_2 causes a decrease in the expected valuation of Customer 2. As a result: (i) The seller finds it optimal to decrease p_2 to \underline{v}_2 to increase the competition at the second step by including all Customer 2 types. In the meantime, given the increased competition at a lower p_2 , the seller finds it optimal to increase p_1 from 5.17 to 5.33. Hence, as the gap between the expected valuations increases, so does the gap between the price steps. (ii) The optimal single price is of type SP2, but the price decreases to 2.18. (iii) Both the expected optimal markdown and the single-price revenues decrease. (iv) TS markdown now exists for all $p_2 \in [2, 4]$, and hence Proposition 13 implies that the optimal markdown should yield a higher profit than SP2.

As $\bar{v}_2 \rightarrow 2$, a TS markdown with $p_2 = 2$ continues to be optimal and dominates the

optimal single price (which is of type SP2). Hence, as the uncertainty about the lower-valuation customer's valuation decreases, the seller is better able to design both an effective as well as a separating markdown, similar to the CI setting.

Example 3: SP2 optimal. In contrast to Example 2, here we expand the support of Customer 2's valuations by increasing \bar{v}_2 . Since Customer 2's valuation range is now wider, the seller is no longer able to design a TS markdown; this is because the first price step necessary to induce all Customer 1 types to bid at p_1 falls into $(\underline{v}_2, \bar{v}_2)$, thereby violating A4-IV. The seller's ability to design a PS markdown is also constricted by this increased uncertainty in Customer 2's valuation since the set of feasible PS markdown decreases. We find that a PS markdown exists only for very low values of p_2 , specifically, $p_2 \in [2, 3]$. However, the increase in customer 2's expected valuation positively affects the seller's optimal single price. A SP2 single-price continues to be optimal, but it is now at the increased price of $p^* = 3.94$, and a single price mechanism outperforms the optimal PS markdown. In summary, an increased range and expectation of Customer 2 valuations implies that it is difficult for the seller to design an effective INT markdown. The only effective INT markdowns require such substantially low p_2 prices (relative to SP2), that the seller is better off using SP2.

As \bar{v}_2 continues to grow, the seller is then incapable of designing an INT markdown, i.e., the optimal markdown will be characterized by a $p_1 < \bar{v}_2$. This implies that there is a positive probability of competition at each price step, and hence the buyer is no longer able to guarantee any high-valuation customer that his entire demand will be satisfied at p_1 .

Example 4: PS optimal. In this case \bar{v}_1 increases by five units compared to Example 1. Let us first remind the reader that the optimal prices in a TS markdown do not depend on \bar{v}_1 , but rather are driven by \underline{v}_1 . Hence, as the range of Customer 1 valuation increases, as well as its expected value, a PS markdown becomes more attractive since it offers the seller the freedom to set a higher p_1 . Intuitively, given the increase in the expected valuation of Customer 1, the seller finds it more profitable to increase p_1 (i.e., increase the gap between the price steps) and induce only a subset of Customer 1 types (12.97, 23) to bid at p_1 , and increase her expected markdown revenue. Despite the increase in \bar{v}_1 , the large value of D_2

implies that the optimal single price is still SP2 with $p^* = 2.76$ and $\Pi_S^o[2] = 43.31$, and hence the PS markdown outperforms the single price.

As \bar{v}_1 continues to increase, a PS markdown continues to be optimal (for the reasons articulated above) and dominates the optimal single price for a wide range of \bar{v}_1 . We find that \bar{v}_1 needs to be greater than 40 for the optimal single price to switch to SP1 and it must be greater than 63 for SP1 to be optimal. In such a setting, the expected value of Customer 1's valuation is so high as to make targeting that customer alone optimal.

Example 5: SP1 optimal. Although the valuation ranges in this example are the same as in the base case, the increase in D_1 from three to eight results in an increase in the optimal first price step under TS. Recall that we observed a similar effect of D_1 under CI (see Equation (1)). Furthermore, we find that the demand from the high-valuation customers is now large enough to have SP1 become the optimal single price. Under this particular market setting, we find that SP1 dominates the optimal (TS) markdown. From our expanded example set, we find that SP1 outperforms the optimal TS markdown when $D_1 \geq 5$ (given the other parameters in the base instance).

Although we do not have any theoretical results that rank the performance of SP1 with the optimal markdown, in general, this example clearly illustrates that an optimal markdown may fail to perform better than SP1 when the demand from the high-valuation customer is relatively large, the valuation ranges of the two customers are far apart, and K is small relative to $D_1 + D_2$. Conversely, a markdown tends to dominate the optimal single price if D_1 is small relative to D_2 and the space between the customers' valuations, $(\underline{v}_1 - \bar{v}_2)$ is 'moderately' far apart. Note that these conditions are the same as those found in Observation 7 for the optimal markdown to dominate the optimal single price under CI.

2.5 Conclusions and Future Research Directions

As businesses operating in both the B2C and B2B markets face increasingly sophisticated buyers, there is a need for sellers to consider buyers' strategic behavior in their pricing strategies. In this chapter, we study the optimal design of markdowns with pre-announced prices and their suitability in the presence of strategic buyers with multi-unit demands. We

find that:

- The optimal markdown has 2 steps if buyers know the clearing price (the price at which the demand exceeds the available supply, p_c).
- If the buyers do not know p_c , but know that at most one price will occur within any one customer's valuation range (INT markdown), then the optimal INT markdown has no more than 3 steps.
- Under either of these settings, it is optimal for a buyer to submit all-or-nothing bids at each price step.

There is a common intuition behind the first step prices under CI (equations (1) and (3)) and IV (Equation (4)). “The ratio of a customer's expected unmet demand to his total demand if he chooses to bid at p_2 ” can be perceived as a measure of *scarcity*, which the seller can use to induce purchases at the first step. Hence, the seller can charge a premium proportional to the scarcity that is equal to the maximum additional amount (above p_2) per unit the customer is willing to pay to secure his demand by bidding at p_1 . We would also like to point out the similarity of trade-offs involved in the case of CI with multiple customers and IV. In both cases, design of the optimal markdown involves deciding which customers/types to exclude from the market and which to encourage to bid at higher price steps.

Future Research Directions Future work in this area could consider the optimal markdown design under IV when $N > 2$. From the results established in this chapter, we know that the optimal INT markdown will have at most three steps (Observation 9) and that customers will submit all-or-nothing bids (Theorem 7). We conjecture that the performance of a markdown will improve (vis-a-vis a single price policy) as the number of customer types increases; that is, the added pricing flexibility offered by markdowns will outweigh the strategic opportunities it creates for the buyers.

We were able to prove that a seller would never need to use more than three-steps in an optimal INT markdown (Observation 9) under IV, which brings up the question: How well does a two-step markdown perform compared to a three-step markdown? Under which

market settings is the difference in seller profits negligible: Conversely, when can they be substantial?

One of the limitations of the model considered in this chapter is the assumption that the seller and buyers have complete information on customer demand. When customer valuations are also known by the seller, this assumption enables the seller to identify the clearing price with certainty. However, if the customer demand information is incomplete, both the seller and the buyers are unable to identify at which price the market would clear. We believe an important extension of this work is to the setting where customer demands are private information. Furthermore, we assume that all customers are present at the start of the markdown and remain until either they meet all of their demands, or the markdown is over. Our two- or three-step markdown result is partly due to this longevity of customers. Future work could consider the random arrival of customers to the system and its effect on the optimal number of markdown steps. (Note that in the Sam's Club example in Appendix A there are five price steps. This is possibly a result of the stochastic arrival of customers with unknown valuations to their website.)

We focused on the design of INT markdowns under the IV setting; this implied that the high-valuation customer was guaranteed to have his entire demand satisfied *if* he bid at p_1 . For some market settings, in particular when the uncertainty surrounding low-valuation customer valuations is high, INT markdowns fail to exist. Under these settings, we would need to understand how customers will behave when they may face competition at *each* price step, and compare the performance of a single-price mechanism to the optimal markdown.

Finally, we study the optimal design and use of markdown with pre-announced prices. While this format has been adopted by some companies, there are many other applications where price drops are not announced. Therefore, a natural extension of our work is to study the design and performance of unannounced markdowns, and to contrast them with preannounced price drops. This form of analysis would allow us to understand if and when a seller is better off sharing the price path information with his strategic buyers.

Given the recent increase in the popularity and use of dynamic pricing, we believe that exploration of these research directions will be crucial in designing and participating in such

mechanisms.

CHAPTER III

ALLOCATING CAPACITY TO STRATEGIC CUSTOMERS USING PRIORITY PRICING UNDER INCOMPLETE DEMAND INFORMATION

3.1 Introduction

In this chapter, we analyze the pricing decision of a seller who owns a certain type of capacity for service or production. The seller announces a set of prices with varying availability guarantees. Customers place quantity bids to purchase at these prices. Bids at the highest price carry an allocation guarantee and the allocation at lower prices is done by rationing the remaining capacity.

The priority pricing mechanism we study is equivalent to an intertemporal pricing mechanism, specifically a markdown with pre-announced prices. With pre-announced markdowns, the seller announces a declining schedule of prices and when each price in the list becomes effective, customers place quantity bids to purchase at that price. Since the allocation decisions are made sequentially as price decreases, fewer units are available at lower prices. Despite the differences between our setting and common markdown pricing model in the literature, we employ the markdown terminology while discussing the intertemporal dynamics of our model.

Priority pricing is a potentially effective method to improve the profits above what could be achieved with a single price when the seller cannot exercise first or third degree price discrimination. The premise of priority pricing is to segment customers with varying willingness-to-pay to create scarcity at lower prices and ultimately induce customers with high valuations to buy at higher prices.

The idea of offering the same product or service at different prices, where the only difference between these prices is availability/scarcity of supply is not new. Paris Metro, the only subway in the world to have a first class, has had two classes since it opened in

1900 [1]. 1st and 2nd class cars had identical number of seats, and the seats were of the same quality, but 1st class ticket cost was twice the 2nd class. Each ticket holder was entitled to an available seat in the corresponding class car. The end result justified the means, as the 1st class cars were less congested since people who cared about being able get a seat paid the higher ticket price of the 1st class [36].

How sellers determine prices is not the only aspect of the business that has been changing recently. Modern consumers are educated, sophisticated and trying very hard to get the best “bang for their buck” [41]. There is more to choose from as the markets become globally integrated, and the customers can get more information about each possible option. Rather than merely trying to meet their demand, customers are making “strategic” purchasing decisions, wielding various tools made available to them with more accessible computing technology.

We assume that customers facing a mechanism with multiple price/availability options make their purchases/bids with the objective of maximizing their individual surpluses, i.e., they act strategically. We consider a business-to-business market which consists of a small number of customers hence, each customer’s demand, valuation and bidding strategy has an impact on the decisions of the other customers.

In this context, we address the the following research questions:

- How do customers bid when facing priority pricing?
- What are the optimal prices? Under what conditions would the seller be better off using a single price?

One motivating example for this setup originates from the pricing of after-sales services. The service revenue contributes to a very significant portion of the operating income for after-sales service providers. [8] reports: “Manufacturers of – products built to last – find that revenue from after sales (maintenance, repairs etc.) are 30 % or more of their total revenues, and the proportion is increasing ...” It is crucial for the service providers to be able to provide the right mix of service contract options to cater to the needs of different segments [8]. A 25-question survey was sent by an original equipment manufacturer (OEM) to 500 users of varying size which helped identify two of the customer segments/needs

as “basic needs customers” vs. “hand-holders”, where the “hand-holders” were offered guaranteed response times, which outline the basic difference between the potential contract types offered to the customers.

Consider an OEM servicing a range of customers, some with higher valuations for the service than others. For example, these customers could be corporate and small business/home owners of copy machines who demand after-sales service for the equipment they purchased or leased. The service provider may offer different types of service contracts, the most common two of which are:

- Guaranteed service (service capacity available with certainty)
- Best-effort contract (no guarantees, service provided if there is enough capacity)

With both type of contracts, the service provider charges a constant per unit price for maintenance/repair service delivered. Effectively, this is identical to charging different prices at different times for the same unit capacity, where the purchases at the higher price carry an allocation guarantee and the allocation at the lower price is done by rationing the remaining capacity.

In this setting, the service provider faces customers with different valuations for the same basic service and demand from each customer is stochastic due to the nature of their operations. Hence, the general pricing problem described above can be posed as follows: Which contracts should the service provider offer and at what price? Should the seller offer only guaranteed service or best-effort type contract or should she offer both types of contracts and let the customers self-select? How do the valuations and demands of the customers impact these decisions?

The remainder of this chapter is organized as follows. Section 3.2 reviews the literature, and Section 3.3 includes analysis of the bidding behavior of the customers, design of the pricing mechanism that will maximize the expected revenue of the seller and comparison the performance of this mechanism to optimal single price via numerical examples. Section 3.4 provides conclusions and managerial insights.

3.2 Placement in Literature

Not charging the same price for all units of the same product or service capacity is a recurring idea in multiple streams of literature. One stream of literature employing this idea focuses on the problem of allocating a limited resource while enabling customers with varying needs to experience different performance levels is known as congestion pricing or priority pricing. A specific application is the pricing of bandwidth capacity for data transfer, where treating all service requests equally by charging the same price simply results in a wide dissatisfaction with the perceived performance for customers with different needs and expectations. Previous work in this area has devised mechanisms involving complicated pricing structures (a general survey and references available in [21]). Many of these mechanisms are not easy to implement as they consider queueing models, in which pricing for each individual customer depends on the congestion in the queue at the time of arrival or they involve additional control policies governing admission to the system or resource allocation which may also be dynamically altered based on the system state. See [34] and [23] for further discussion and related literature.

Analyses of intertemporal pricing mechanism considering rational/strategic customers, which appear in the economics and operations management literature are also variations on the same basic idea. Essentially, in these settings the seller offers the product or service at a declining schedule of prices which are announced in advance, and the customers decide when and at what price to make their purchase.

Technically, the time dimension does not alter the dynamics of the pricing mechanism as the rational behavior of a customer facing a declining schedule of prices is identical to the case where all prices are offered at once but the allocation of capacity is prioritized starting from the requests at the highest price.

Earlier work which considers customers facing declining prices is in the economics literature. [43] studied a posted price mechanism where a seller with unlimited capacity faces customers with single unit demands. An extension where the the seller is restricted to making limited number of price adjustments was later studied by [6]. Both papers conclude that a declining price schedule is optimal and the seller prefers to have as few number of

price adjustments as possible.

[24] investigates the optimal pricing mechanism of a seller, who may have a capacity constraint, facing a fixed number of buyers with single unit demands. Assuming that customer valuations are private information, they find that if the capacity is exogenously determined and exceeds the market demand, then a single price is optimal. If the capacity is less than the market demand, a declining price schedule or a Vickrey auction is optimal. [52] considers a monopolist facing a downward sloping demand curve which represents the demand from a fixed and large number of randomly arriving customers. Before the customer arrivals, the seller fixes the price of each unit by placing a tag that remains on the item until it is sold. Note that with this mechanism, not only the prices but also how many units will be available at each price is announced in advance. Customers may demand more than one unit and purchase as much as their demand at the lowest price on the tags, upon arrival to the store, as long as doing so has a positive marginal benefit. [52] finds that the seller can maximize the revenue by charging no more than two different prices. Since each item has a tag displaying the price in this setup, the derived result holds without explicitly modeling the customers as myopic or strategic. This is not the case when prices of items change over time.

The last group of relevant previous work is in the operations management literature, even though the bulk of it was content with myopic customer behavior when studying the optimal pricing policy of a seller with limited supply ([7], [19], [22]). In the past few years, several papers focused on different aspects of declining price schedules with strategic consumers and fixed seller inventory. [4] considers continuously declining customer valuations. Upon arrival, customers decide whether to buy at the time of arrival, return later for a lower price or not to buy at all. They investigate whether the seller benefits from not committing to a discount path upfront, and find that alternative mechanisms where the seller decides on the discounted price dynamically considering the inventory level after initial sales. Benefits from this sophisticated strategy is minimal and it would not be a worthwhile alternative to a pre-announced price path.

[44] focuses on a deterministic setting with a monopolist facing a continuous arrival of

customers with single unit demands. Customers are heterogeneous in two dimensions; they may have low or high valuations and either make a purchase and leave (myopic) or stay in the system to maximize surplus with their purchase (strategic). The findings include whether the optimal policy has decreasing or increasing prices depending on which type customers are relatively more strategic when supply is exogenous. In case of endogenous supply, the optimal policy is either a single price or multiple prices with a decrease at the end of the horizon to capture the strategic low types.

[10] considers a two-step mechanism in a market consisting of myopic and strategic customers. Existence of strategic customers in the market induces the seller to stock less, give a smaller discount and make lower profits. The seller is better off by not committing to a price path and deciding on the second step price after the demand in the first step is observed. Furthermore, they find that the seller benefits from a mid-season replenishment at a possibly higher cost than the initial order, when strategic customers are present in the market. Conversely, [14] find that with fixed initial order quantities, posted pricing schemes are nearly optimal and preferred over contingent pricing schemes since they are easier to implement.

Our work departs from all of the mentioned work in operations management literature by considering a multi-unit demand setting. Our model does not consider the individual customers' arrivals, and assumes that all customers are present in the system. Since customers are in a typical B2B sales environment, we do not restrict ourselves to the assumption of single unit demands.

Our approach is closely related to the markdown pricing model in Chapter 2, which focused on the analysis of a declining price mechanism in a complete information (CI) setting as well as under a special incomplete valuation information (IV) setting. However, the fact that demand information is complete in both cases ensured that the seller can always find a price at which the market demand exceeds the available supply. Without complete information on demands, we cannot provide a theoretical limit on the number of price changes in the optimal mechanism when there is an arbitrary number of customers in the market.

The incomplete demand information setting in this chapter allows us to explore issues related to the effect of the relationship between the demand distributions of the customers to the structure of the optimal pricing mechanism and its performance. The underlying problem of the customers is also quite different. Lacking the exact demand of other customers in the market, they have to base their bidding behavior on the demand distributions.

Our work is different from but complementary to the previous priority/congestion pricing research by considering the problem at a higher level, and eliminating the minute details of a queueing model and the flexibility to adjust prices or alter admission or resource allocation policies based on the state of the system. The pricing mechanism we consider and the model assumptions are more along the lines of the economics and operations management literature mentioned above, with the following important distinctions: We explicitly model strategic behavior of the customers, and consider an incomplete demand information setting where the customers are not restricted to single unit demands. By studying an incomplete information setting and incorporating customers who demand multiple units and act strategically, our model yields relevant insights for the real world problems that motivate our research.

3.3 Model

We consider a seller facing two customers in a market. The seller implements a two-step mechanism with price p_k at step k , $k = 1, 2$, where $p_1 > p_2$. The seller's starting inventory is assumed to be exogenously given, which would be the case if the K units were comprised of excess inventory for an end-of-season item or it is the service capacity which cannot be adjusted easily in the short run. Later we discuss the choice of K , show that 2 steps are sufficient for achieving optimal revenue with two customers and how this extends to the case of N customers.

Exact demands of the customers are *not* known by the seller. She believes that each customer's private demand is drawn from a commonly known atomless continuous distribution, with density function $f_i(D)$, with support over the interval $[L_i, H_i]$. Each customer shares this belief with the seller regarding the demand of the other but knows his own exact

demand realization.

Customers have constant marginal valuations, v_1 and v_2 , where $v_1 > v_2$, for up to a certain number of units of the product. The valuations of the customers, the initial supply level, and the unit product procurement or service cost of the seller are common knowledge.

We consider a mechanism where the seller announces the prices and the inventory on hand before the sales begin¹.

We assume that the valuations of the customers are constant over time and there is no discounting. We do this to keep the exposition simple and we later show how our results extend when a discount factor is present or customer valuations decline over time. In addition, we assume that both customers are present at the start of the markdown and that each one remains until the markdown is over or until his demand is satisfied. Hence, in our model, the timing of price changes does not impact the seller's profits, and the seller's main decision (given an initial inventory) is to choose the price at each of the two steps.

We assume that the customers and the seller are risk-neutral, so they maximize expected surplus. At any (price) step, customers submit quantity bids, indicating the number of units they would like to purchase at the current price. q_{jk} denotes the bid of customer j at step k . Π_j stands for the total expected surplus (valuation – cost) over both steps, of the customers ($j = 1, 2$) or the seller ($j = S$).

If the total demand exceeds the available supply at any step, the seller uses a random allocation rule. She chooses one of the customers randomly and gives him the priority in allocation. The possible scarcity in the second step is what induces the buyers to bid sooner in the mechanism. As noted earlier by [35], this rationing rule is consistent with the case where all items are sold at a first-come-first-serve basis when all customers arrive at the store at the beginning of the second step.

To evaluate its suitability, we compare the performance of the optimal two-step mechanism to the (simpler) optimal single price mechanism.

¹Currently eCost.com has a “Bargain Countdown” feature where the available quantity is posted upfront (<http://www.ecost.com/ecost/shop/countdown/>). Before it was acquired by eBay, FairMarket used to provide a “AutoMarkdown” solution (for big retail businesses such as Dell, J.C.Penney, and Sam’s Club) where the future price schedule as well as the on hand inventory was posted in advance.

A natural consequence of the model setting is that both the seller and the buyers have to make their decisions under incomplete information. The distinction between the levels of information available to the seller and the buyers is very crucial in investigating their optimal decisions. The buyers have a slight advantage over the seller as they know their own demand realization. As a result, while the buyers try to position themselves in response to the uncertain demand of the other buyer, the seller has to deal with two different sources of uncertainty: D_1 and D_2 . In Section 3.3.1 we list preliminary results and eliminate the trivial cases. Then, we analyze the bidding behavior of the customers in Section 3.3.2 and the seller's pricing problem in Section 3.3.3. Performance comparison of the two-step mechanism vs. single price via numerical examples in Section 3.4 is followed by managerial insights.

3.3.1 Preliminaries

We consider the seller's problem of maximizing revenue from the sale of K units in a market consisting of two customers with random demands D_i , which are known to be drawn from distributions with support on $[L_i, H_i]$, $i \in \{1, 2\}$. The seller wants to determine the prices for a two-step mechanism with the goal of inducing the high-valuation customer to buy at the higher price.

The seller does not need to use a mechanism with more than 2 steps to maximize her revenue. By contradiction, assume there is an m -step mechanism ($m > 2$) with prices $v_1 \geq p_1 > \dots > p_k > \dots > p_m$ where $p_k \geq v_2 > p_{k+1}$. Some of these steps will have prices in $(v_2, v_1]$, and the rest will be less than or equal to v_2 . (Recall that neither customer can afford prices higher than v_1 .) Since there is only one customer in the market that can afford those prices, there will be a positive bid in at most one price step, p , such that $v_1 \geq p > v_2$, (The unit surplus is constant for each additional bid at any price step so customer 1 will bid all his demand at one price step to maximize his surplus.) As a result, the seller can keep just that price and eliminate all other prices higher than v_2 . On the other hand for the remaining set of prices, the seller cannot do any worse by eliminating all but the highest one of these steps. Her revenue will not decrease since both customers can afford the remaining

step. As a result we can argue that the optimal revenue can be achieved with a two-step mechanism where $v_1 \geq p_1 > v_2$ and $v_2 \geq p_2$.

Observation 12. *In the optimal two-step mechanism:*

- a. *Second step price, $p_2 = v_2$.*
- b. *Customer 2 bids his entire demand at the second step of the mechanism.*
- c. *At any step, customers do not bid more than their demand realization.*

If $p_2 > v_2$, then customer 2 cannot afford to buy at the second step and consequently customer 1 has no incentive to bid at step 1. On the other hand, if $p_2 < v_2$, the seller can increase p_2 up to v_2 and increase her revenue.

From Observation 12(a), the optimal step 2 price is v_2 . So, the only option of customer 2 is to bid his entire demand at step 2.

Under IV, the seller had the option of adjusting scarcity by picking a p_2 in $[\bar{v}_2, v_2]$. In certain settings, using this as a lever, she was able to charge a higher p_1 while keeping customer 1 bidding at step 1. However, in ID setting the seller has no control on the scarcity at step 2 and has to cope with the built in uncertainty in the D_2 distribution without any ability to adjust or alter the scarcity.

Bidding higher than the realized demand does not increase the chances of acquiring the product for the customers, but may result in higher payments. The random allocation rule, which is used when the total bids exceed available supply, prevents customers from inflating their bid quantities. As a result, customer bids are always bounded above by their demand realization.

Observation 12 identifies the value of p_2 , and the seller's decision reduces to p_1 .

We would like to compare the performance of the two-step mechanism to the optimal single price. In a market with two customers where the valuations are common knowledge, the optimal single price is equal to either v_1 or v_2 , whichever yields the higher expected revenue for the seller. We denote the former with SP1 and the latter with SP2.

Two pricing mechanisms are considered to be *equivalent* if the mechanisms yield the same surplus (for the seller and the buyers). A mechanism is said to *dominate* another

mechanism if it generates higher (expected) surplus for the seller compared to the other mechanism.

Observation 13. *Single price revenue with SP2 constitutes a lower bound on the two-step revenue.*

If p_1 is so high that customer 1 always prefers to bid at p_2 for every realization of D_1 , then nothing is sold at step 1, and the two-step revenue is equivalent to the single price revenue with SP2.

Observation 14. *If $K \leq E[D_1]$, then the optimal single price SP1 generates higher expected revenue than a two-step mechanism.*

Observation 15. *If $K \geq H_1 + H_2$, then there is no possible scarcity of supply at any price step. In this case nothing is sold at step 1, and the two-step revenue is equivalent to the single price revenue with SP2.*

Proposition 14. *If SP2 is the optimal single price mechanism, then a two-step mechanism dominates the optimal single price if it induces some customer 1 type to bid at step 1, otherwise SP2 and the two-step mechanism are equivalent.*

This result is quite intuitive: If both customers prefer to bid at step 2, From Observation 13, we know that the two-step revenue is equal to the single price revenue with SP2. The amount sold is equal to the maximum of the total demand realization and the available supply. On the other hand if some type of customer 1 prefers to bid at $p_1 > v_2$, there is a positive probability that there will be sales at step 1, and hence, the expected revenue from the two-step mechanism exceeds the SP2 revenue.

In Observations 14 and 15 we have identified the trivial settings in which a two-step mechanism fails to dominate the optimal single price. Hence, we focus our attention to settings which satisfy the following non-triviality condition:

$$\mathbf{A-NT.} \quad E[D_1] < K < H_1 + H_2.$$

Note that this range neither implies nor eliminates the possibility that $K > D_1 + D_2$. Recall that under CI and IV settings in chapter 2 we were able to restrict our attention

to $D_1 < K < D_1 + D_2$. Essentially, lacking the exact demand information, the seller can potentially benefit from a markdown mechanism for a wider range of K values. If we can show that the optimal markdown mechanism can perform better than the optimal single price for any K value in A-NT, we can argue that the incomplete demand information creates an advantage for the markdown pricing mechanism over the optimal single price.

A-NT excludes the supply ranges for which the seller does not have any hope of increasing her revenue above what she can achieve with a single price mechanism. However, under incomplete demand information, even under A-NT, there is no guarantee that the seller will be able to set two prices p_1 and p_2 and sell some of the supply at the higher price. Without having complete demand information, sometimes the best she can hope to achieve is to induce the high valuation customer to buy at the high price for some realizations of D_1 and get a higher *expected* revenue than a single price mechanism. Hence, to be able to solve the seller's pricing problem, first we need to identify how customer 1 will bid given p_1 and p_2 .

If the two-step mechanism, (which we interchangeably refer to as "markdown") induces customer 1 to bid D_1 at step 1 for *some* $D_1 \in [L_1, H_1]$, we call it *separating*. Essentially, there is a positive probability that under this markdown, customer 1 bids at step 1 while customer 2 bids at step 2. If this probability is zero, i.e., the markdown cannot induce customer 1 to bid at step 1 for any D_1 , then we call it *pooling*. As a further refinement, if a markdown induces customer 1 to bid D_1 at step 1 for *any* $D_1 \in [L_1, H_1]$, we call it a totally separating (TS) markdown, and refer to the remaining cases as partially separating (PS). In the next section, we examine how each of these markdown types lead to a different bidding behavior.

3.3.2 Bidding Behavior of Customer 1

Observation 12(b) stated that the only possible action for customer 2 is to bid his entire demand at step 2 in an optimal two-step mechanism. However, it is not obvious how customer 1 will bid. The key factor that impacts the decision of customer 1 is how many units will be allocated to him as a result of his bid.

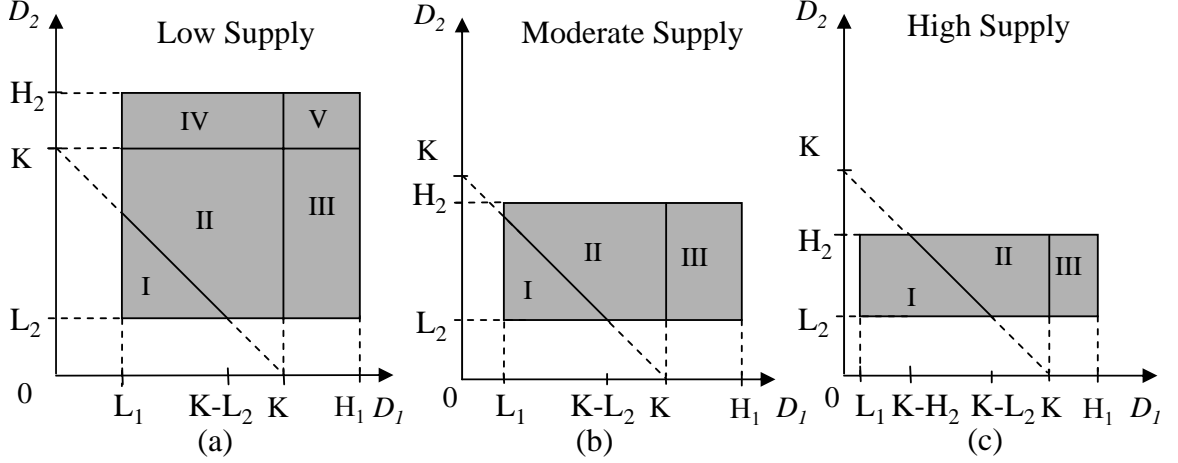


Figure 5: Relation of K to demand ranges of the customers in low, moderate and high supply scenarios.

In Theorem 15, we show that all-or-nothing bidding is optimal under incomplete demand information.

Theorem 15. *Under a two-step markdown mechanism, it is a dominant strategy for customer 1 to submit an all-or-nothing bid, i.e., to submit either all or none of his demand at a price step.*

The proof, which is presented in the Appendix B, is by showing that the expected surplus of customer 1 as a function of his bid at the first price step, q_{11} , is a (piecewise linear) convex function. Hence the expected surplus is maximized at one of the extreme points, namely at $q_{11} = 0$ or $q_{11} = D_1$. This sort of customer behavior is identical to the one we observe in case of incomplete information on valuations.

Table 5: Allocation to customer 1 if he bids D_1 at step 2.

	Region	Customer 1 is selected first	Customer 2 is selected first
I	$D_1 < K, D_1 + D_2 < K$	D_1	D_1
II	$\max\{D_1, D_2\} < K, D_1 + D_2 \geq K$	D_1	$K - D_2$
III	$D_1 \geq K, D_2 < K$	K	$K - D_2$
IV	$D_1 < K, D_2 \geq K$	D_1	0
V	$D_1 \geq K, D_2 \geq K$	K	0

In order to prove Theorem 15 and understand the bidding behavior we need to understand how many units will be allocated to customer 1 if he bids at step 2.

Recall that the seller and the customers are working with different information settings. Customer 1 knows the realization of D_1 , but the seller only knows that $D_1 \in [L_1, H_1]$. Hence the analysis involves finding answers to the following two questions:

1. How do D_1 and D_2 relate to K ? This determines the actual allocation.

If customer 1 bids $0 < q_{11} \leq D_1$ at p_1 , then he is allocated $\min\{q_{11}, K\}$. On the other hand, the expected number of units allocated to customer 1 at step 2 varies depending on the realization of D_1 and D_2 relative to K . Depending on which customer has the allocation priority at step 2, different allocations for customer 1 are given in Table 5.

2. How do L_1 , H_1 , L_2 and H_2 compare to K ? This determines which allocations will have a positive probability of occurrence.

For example, if $H_2 < K$ then IV and V, which require $D_2 > K$, will be irrelevant for that instance. Similarly, if $L_1 < K - H_2$, then for some realizations of D_1 , specifically when $L_1 \leq D_1 < K - H_2$, customer 1 can always get a guaranteed allocation of D_1 regardless of which price step he bids (See Figure 5).

If customer 1 bids at step 2, how many units are allocated to him depends on how K relates to the demand parameters (and the demand realization) of both customers. Regardless of the demand realization of customer 1, the relationship of K to H_2 and $L_1 + H_2$ is important since this determines if customer 1 might get none, some or all of his demand if he bids at step 2 and is *not* selected first. We define the following three supply levels accordingly.

Low supply (L): $E[D_1] < K < H_2$,

Moderate supply (M): $H_2 \leq K < L_1 + H_2$ and

High supply (H): $L_1 + H_2 \leq K$.

These supply levels are valid for any customer 1 demand realization in $[L_1, H_1]$. For example, if $D_1 > K$, then the allocation to customer 1 at step 2 is bounded by K when

Table 6: For each supply level, relevant allocation settings for customer 1 if he bids D_1 at step 2.

	Allocations
Low Supply (L)	I, II, III, IV and V
Moderate Supply (M)	I, II and III
High Supply (H)	I, II and III

he bids at step 2 and is selected first. Similarly if $D_1 > K - L_2$, then the allocation to customer 1 at step 2 is bounded by the leftover from customer 2 when he bids at step 2 and is *not* selected first.

Table 5 and Figure 5 present the different allocations when customer 1 bids at step 2. In Table 6, we present a complete list of allocations that are relevant in each supply level. Note that although the relevant allocations are identical in Moderate and High supply levels, the expected allocations are quite different since the expectation incorporates the probabilities associated with each allocation setting. While setting I is the only possibility for some D_1 realizations under High supply (specifically for $D_1 < K - H_2$, Figure 5(c)), settings I and II are both possible for any D_1 under Moderate supply (Figure 5(b)). Later, we show that this subtle difference plays a significant role in how the bidding behavior of customer 1 is differs under Moderate and High supply scenarios: The seller cannot achieve a TS markdown under High supply.

Next, we present a theorem that summarizes the bidding behavior of customer 1 for any relation of K to L_i and H_i .

Theorem 16. *Customer 1 with demand D_1 bids at step 1 if and only if p_1 is less than or equal to the threshold price, w_j^x where $x \in \{i, ii\}$ and $j \in \{L, H, M\}$. The closed form expression for w_j^x 's corresponding to D_1 ranges are presented in Figure 6.*

We provide a detailed discussion of the results on the bidding behavior of customer 1 under Low (L) supply scenario and present the results for Moderate (M) and High (H) supply in Appendix B.

We made the assumption that $K < H_1$ holds when presenting the results in Figure 6.

(Assume $K < H_1$)		D_1 vs. K (Actual Allocation)		
K vs. L_1, H_1, L_2, H_2 (Expected Allocation)		$\max\{L_1, K-H_2\} \leq D_1 \leq K-L_2$	$\max\{L_1, K-L_2\} \leq D_1 \leq K$	$K < D_1 \leq H_1$
H	$\max\{E[D_1], L_1+H_2\} < K < H_1+H_2$	$w_H^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K-H_2}{2D_1} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right]$	$w_H^ii(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K}{2D_1} + \frac{E[D_1]}{2D_1} \right]$	$w_H^iii(K)$
M	$\max\{E[D_1], H_2\} < K < L_1+H_2$	$w_M^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K-H_2}{2D_1} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right]$	$w_M^ii(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K}{2D_1} + \frac{E[D_1]}{2D_1} \right]$	$w_M^iii(K)$
L	$E[D_1] < K < H_2$	$w_L^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{K-D_1}^K F_2(x_2) dx_2 \right]$	$w_L^ii(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{L_1}^K F_2(x_2) dx_2 \right]$	$w_L^iii(K)$

Figure 6: Customer 1 with demand D_1 bids at step 1 unless p_1 exceeds the corresponding threshold price.

This enables us to present the most general version that includes all possible threshold function variations. Note that if $H_1 \leq K < H_1 + L_2$, then the case with $K < D_1 \leq H_1$ is impossible, hence $w_j^{ii}(K), j \in \{L, M, H\}$ will not be applicable. Similarly, if $H_1 + L_2 \leq K$ then $K - L_2 < D_1 < K$ is impossible and we need to consider $w_j^i(D_1), j \in \{L, M, H\}$ as the only threshold function over the entire D_1 range. In the rest of the chapter, we keep the assumption that $K < H_1$ holds in order to present the most general case.

3.3.2.1 Bidding Behavior of Customer 1 in Low Supply ($E[D_1] < K < H_2$)

With Theorem 15, we established that customer 1 submits all-or-nothing bids. Next, we identify the conditions under which customer 1 bids D_1 at step 1, depending on the realization of D_1 . Table 5 summarizes the step 2 allocations for customer 1 if he bids his entire demand at p_2 . In Figure 5, we also indicate how these different allocations relate to the supply levels L, M and H.

We identify the equilibrium bidding strategy of customer 1 for each possible demand realization D_1 subject to A-NT.

Proposition 17. *In a two-step markdown mechanism with prices $p_1 > p_2 = v_2$ and $K \leq \min\{H_2, H_1\}$, customer 1 bids D_1 at step 1 (and 0 at step 2) if*

PSi. $p_1 \leq w_L^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{K-D_1}^K F_2(x_2) dx_2 \right]$ and $L_1 \leq D_1 \leq K - L_2$, or

PSii. $p_1 \leq w_L^{ii}(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{L_2}^K F_2(x_2) dx_2 \right]$ and $K - L_2 < D_1 \leq K$, or

TS. $p_1 \leq w_L^{iii}(K) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2K} \int_{L_2}^K F_2(x_2) dx_2 \right]$ and $K < D_1 \leq H_1$,

and bids 0 at step 1 (and D_1 at step 2) otherwise.

Here, $w_L^i(D_1)$ and $w_L^{ii}(D_1)$ correspond to the maximum (threshold) step 1 price that would induce customer 1 of type D_1 to bid his entire demand at step 1 (separating). We use labels TS, PSi and PSii to indicate the form of markdown corresponding to each price range. The only difference between PSi and PSii is in the sensitivity of the threshold price to the changes in the value of D_1 in the corresponding demand ranges. The terms that follow $(v_1 - v_2)$ in each equation correspond to the ratio of the expected unmet demand of customer 1 to his demand, if he bids his entire demand at step 2.

In the proof, which is presented in Appendix B, we compare the expected surplus of customer 1 from bidding his entire demand at step 2 to his surplus from bidding at step 1, and derive the threshold price as a function of D_1 . As long as $D_1 < K$, the threshold price is a function of D_1 . If $K \leq D_1$, then the threshold price is constant due to Observation 12(c).

Note that $K \leq H_1$ is not a requirement for Proposition 17 to hold. When Figure 6 was presented we noted that it was for the most general case where all variations of the threshold price expressions are valid. Only a subset of the threshold functions will be valid for different D_1 ranges when $H_1 < K$. The practical implication of all this is that if the initial supply is relatively high, it is less likely for the seller to set up a mechanism where a possible demand realization of the customers would lead to a totally separating outcome.

Proposition 18. *The threshold step 1 prices, $w_j^x(D_1)$ where $x \in \{i, ii\}$ and $j \in \{L, H, M\}$ are non-decreasing functions of D_1 .*

As D_1 increases, the expected allocation for customer 1 at step 2 also increases, but the expected allocation is less than D_1 . Hence, the ratio of allocation at step 2 to step 1 decreases with D_1 . Thus, for higher values of p_1 , customer 1 still prefers to bid his entire demand at step 1. When $D_1 \leq K - L_2$, the demand by customer 2 is so low that customer 1 can get his entire demand at step 2 with positive probability. On the other hand, for $K - L_2 < D_1 \leq K$, if customer 1 does not get the allocation priority, he always gets the remaining units from customer 2 at the second step, which is less than his bid quantity. When D_1 goes above K , step 1 and step 2 allocations do not increase with D_1 anymore and

the threshold price level remains constant. Notice that this threshold is independent of D_1 , i.e., there is a constant threshold step 1 price below which customer 1 prefers to bid D_1 at step 1 when $D_1 \geq K$. In all three scenarios, for $K \leq D_1 \leq H_1$, the threshold step 1 price is a function of K and does not depend on D_1 .

A common result of these propositions is the following bidding behavior: A given p_1 makes a customer type $\Delta \in [L_1, H_1]$ indifferent between bidding in step 1 or step 2. If customer 1 of type Δ prefers to bid at step 1 then any higher type also prefers to bid at step 1. In other words, any given step 1 price separates the set of customer types into two disjoint and continuous subsets, $[L_1, \Delta)$ and $[\Delta, H_1]$, where Δ is the lowest customer 1 type that prefers to bid at step 1. Note that either one of these subsets might be empty, i.e., all customer 1 types may prefer to bid at step 1 or step 2. However, if all types prefer to bid at step 2 for a given step 1 price, then the markdown is pooling and it is equivalent to a single price mechanism with $p = v_2$.

3.3.3 Seller's Revenue

Now that we have identified the customers' bidding behavior for any given value of D_1 and p_1 , we can investigate the seller's pricing problem. The seller will set the step 1 price at a level which maximizes her surplus in expectation of the customer demands.

In our analysis of the customers' bidding behavior, we referred to a customer with demand realization of D_1 . However, when we analyze the seller's decision, we need to explicitly consider that customer 1 could be one of many alternative types, where type is defined by the realization of D_1 . Since the seller has incomplete information on D_1 , she assumes a customer 1 with demand that can be anywhere in $[L_1, H_1]$, with a known probability distribution. Any choice of p_1 corresponds to a threshold demand realization of Δ , where any demand realization at or above the level of Δ would lead to customer 1 bidding at step 1.

Before proceeding with the detailed analysis, we revisit the pooling/separating markdowns distinction, and provide results on under what conditions each type would exist. This helps us later on when we analyze whether the seller can achieve the same revenue with a

single price: A pooling markdown can never outperform a single price, since everything is sold at a single price level in the markdown, the seller can fix the price and still sell the same amount with a single price. Hence, a markdown may yield higher revenue than a single price only if it is separating.

First, we show that a separating markdown always exists and identify a special setting where all separating markdowns are TS. Then we identify conditions under which a TS markdown exists.

Proposition 19. *For any given instance that satisfies A-NT, a PS markown exists.*

The proof is by comparing the expected revenues of customer 1 from bidding D_1 in step 1 vs. step 2. If customer 1 bids D_1 in step 2, the expected allocation to him in step 2 takes a value between 0 and D_1 . Hence, the proportion of his demand satisfied by the allocation is in $[0,1]$. Recall from Observation 12(a) that step 2 price is v_2 . So, customer 1's expected surplus from bidding D_1 at step 2 takes a value between 0 and $D_1(v_1 - v_2)$. It is always possible to find a step 1 price, p_1 in $(v_2, v_1]$ such that, for some $D_1 \in [L_1, H_1]$, the surplus to customer 1 from bidding D_1 at step 1, $D_1(v_1 - p_1)$, is at least as good as the expected surplus from bidding D_1 at step 2.

If A-NT is satisfied, we can always determine a customer 1 type D_1 , for which the optimal mechanism would be separating. Thus, in her search for the optimal mechanism, the seller cannot eliminate the two-step mechanism without thoroughly analyzing the probability of occurrence for the separating outcome and comparing the expected revenue to the optimal single price.

Under CI, with D_1 and D_2 being common knowledge, the seller can determine exact bidding behavior and tell for sure whether the mechanism would be separating or pooling for any p_1 value. However, without the complete demand information, the best she can do is to consider the expected values and try to increase the possibility of getting a separating outcome.

Under IV, existence of TS and PS mechanisms depended on a measure of scarcity. A markdown could only lead to a pooling outcome if $\frac{\bar{v}_2 - v_2}{\bar{v}_1 - v_2} > \frac{D_1 + D_2 - K}{2D_1}$, and the seller did

not have any hope of achieving a revenue higher than the optimal single price revenue in this case. With proposition 19, such a result becomes impossible under ID, increasing the chances that incomplete information on demands provides an advantage for markdown pricing over the optimal single price.

Corollary 20. *Given an instance that satisfies A-NT; if $L_2 \geq K$, then all separating markdowns are TS.*

This follows immediately from Proposition 19 since the expected allocation to customer 1 from bidding D_1 at step 2 is $D_1/2$ when $L_2 \geq K$. Hence for any $D_1 \in [L_1, H_1]$, customer 1 gets half of his maximum possible surplus, $D_1/2(v_1 - v_2)$, if he bids at step 2. Since bidding D_1 at step 1 yields $D_1(v_1 - p_1)$, the actual decision of the seller is independent of D_1 and all prices that induce customer 1 to bid at step 1 for some D_1 , induce him to bid at step 1 for all D_1 .

If supply is scarce enough (to be exact, if the minimum possible demand of customer 2 would deplete the entire supply) then the outcome would be separating regardless of the demand realization of customer 1.

Proposition 19 and Corollary 20 together suggest that if A-NT and $L_2 \geq K$ both hold, then there is always a step 1 price, p_1 , for which customer 1 bids at step 1. The analysis is much simpler in this case since the seller can assume that an amount equal to the expected demand of customer 1 would always sell at step 1.

Proposition 21. *For any given instance that satisfies A-NT, a TS markown exists if and only if $K \leq L_1 + H_2$.*

If $L_1 + H_2 < K$, a TS markdown fails to exist since for $D_1 < K - H_2$, the expected allocation to customer 1 from bidding D_1 at step 2 is equal to D_1 . In other words, for some D_1 realizations, customer 1 gets his entire demand at step 2 and no $p_1 > v_2$ can induce him to bid at step 1.

Proposition 21 directly implies that the seller cannot get a TS markdown in the High supply scenario. This result is similar to results 1 and 2 (in Appendix A) in the IV setting, suggesting that if scarcity is low, it is more likely to see a pooling outcome.

Customer 1 decides whether to bid D_1 at step 1 or step 2 based on p_1 . Hence, the seller's pricing decision is closely related to the bidding behavior of customer 1. Any particular selection of p_1 will lead to a continuous subset of customer 1 types, $[\Delta, H_1]$, bidding at step 1 while the rest $[L_1, \Delta]$ prefer to bid at step 2. Given the one-to-one relationship with p_1 and the minimum D_1 that would be induced to bid at step 1 given p_1 , the seller is essentially deciding on which customer 1 type to make indifferent between bidding in step 1 versus bidding in step 2.

Looking at the threshold price for customer 1 provided in Theorem 16, we can examine the seller's choice of step 1 price in four distinct intervals, which are presented in the Theorem 22 for each possible scenario.

Theorem 22. *Given a supply scenario $j \in \{L, M, H\}$ and corresponding p_1 ranges as defined in Table 7; if one of the following holds, then customer 1 prefers to bid D_1 at step 1.*

i. $p_1 \in jTS$

ii. $p_1 \in jPS(ii)$ and $p_1 \leq w_j^{ii}(D_1)$

iii. $p_1 \in jPS(i)$ and $p_1 \leq w_j^i(D_1)$

Otherwise; i.e, if if one of the following holds

i. $p_1 \in jPS(ii)$ and $w_j^{ii}(D_1) < p_1$

ii. $p_1 \in jPS(i)$ and $w_j^i(D_1) < p_1$

iii. $p_1 \in jP,$

then customer 1 prefers to bid D_1 at step 2.

This result follows from Theorem 16 by substituting the end points of the ranges in the corresponding threshold price expression for each case. In order to find p_1^* , we need to solve for the best price over each range and compare the corresponding optimal seller profits in these four price ranges and determine which one of these prices maximizes her profit.

Table 7: p_1 ranges corresponding PS, TS, and Pooling results in Low, Moderate and High Supply

Low Supply	Moderate Supply	High Supply
$LP: (w_L^{ii}(K), v_1]$	$MP: (w_M^{ii}(K), v_1]$	$HP: (w_H^{ii}(K), v_1]$
$LPS(ii): (w_L^{ii}(K - L_2), w_L^{ii}(K)]$	$MPS(ii): (w_M^{ii}(K - L_2), w_M^{ii}(K)]$	$HPS(ii): (w_H^{ii}(K - L_2), w_H^{ii}(K)]$
$LPS(i): (w_L^i(L_1), w_L^{ii}(K - L_2)]$	$MPS(i): (w_M^i(L_1), w_M^{ii}(K - L_2)]$	$HPS(i): (v_2, w_H^{ii}(K - L_2)]$
$LTS: (v_2, w_L^i(L_1)]$	$MTS: (v_2, w_M^i(L_1)]$	$HTS: \emptyset$

We provide the detailed result for the seller's expected revenue in the low supply scenario and identify the optimal step 1 price for each interval.

The practical interpretation of Theorem 22 is as follows: For a given instance, the seller needs to consider at most four possible ranges for p_1 separately. Of these four, the lowest three ranges differ by the rate of change in the premium seller can charge to customer 1 in step 1 per unit change in his demand realization, D_1 . For the highest range of p_1 values, the seller cannot induce customer 1 to bid at step 1, so there is no premium to speak of. Specifically, these correspond to the p_1 ranges for which the resulting mechanism becomes pooling, partially separating or totally separating. Recall that for the partially separating p_1 values, the increase in expected allocation to customer 1 for a unit increase in D_1 is governed by two different expressions, depending on whether D_1 exceeds $K - L_2$ or not. Hence, we have to analyze the two p_1 ranges separately for partially separating mechanism, in order to account for the different rates at which the expected allocation to customer 1 at step 2 changes with D_1 .

Another takeaway from Theorem 22 is; as supply increases, it gets harder for the seller to keep customer 1 bidding at step 1 and extract more of his surplus.

3.3.3.1 Seller's Revenue in the Low Supply Scenario

Figure 7 summarizes the result of Proposition 17 and displays how customer 1 bids for different values of D_1 and p_1 when K is low relative to H_2 . In Figure 7, we partition the original p_1 range into four: LP corresponds to the p_1 range for which customer 1 always bids at step 2. LPS(ii) and LPS(i) are the ranges in which customer 1 bids at step 1 if and only if $p_1 < w_L^{ii}(D_1)$ and $p_1 < w_L^i(D_1)$, respectively. Finally, LTS is the p_1 range which

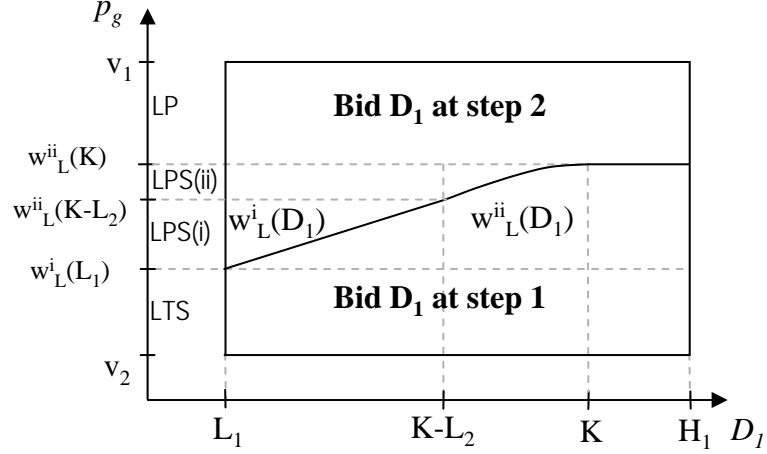


Figure 7: Customer 1's optimal bid in relation to D_1 and p_1 in low supply scenario when D_2 is uniformly distributed.

induces customer 1 to always bid at step 1. By substituting the endpoints in the bidding thresholds, we formally define the four price ranges described above.

$$\text{LP. } v_1 \geq p_1 > w_L^{ii}(K) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2K} \int_{L_2}^K F_2(x_2) dx_2 \right]$$

$$\text{LPS(ii). } w_L^{ii}(K) \geq p_1 > w_L^{ii}(K - L_2) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2(K-L_2)} \int_{L_2}^K F_2(x_2) dx_2 \right]$$

$$\text{LPS(i). } w_L^{ii}(K - L_2) \geq p_1 > w_L^i(L_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2L_1} \int_{K-L_1}^K F_2(x_2) dx_2 \right]$$

$$\text{LTS. } w_L^i(L_1) \geq p_1 > v_2$$

We need to solve for the optimal step 1 price p_1^* over each range and compare the corresponding optimal seller profits and determine which one of these prices maximizes her profit.

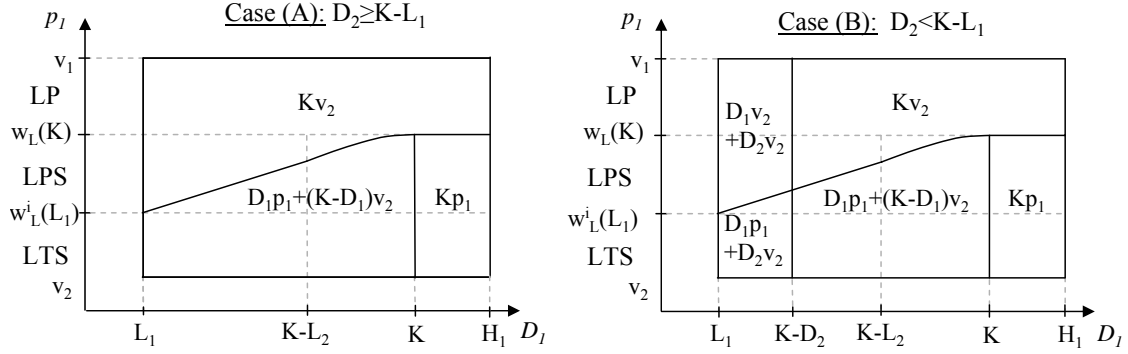


Figure 8: Seller's revenue function in relation to D_1 and p_1 for a fixed D_2 in the low supply scenario.

Figure 8 displays two representative cases of how the seller's revenue function changes with respect to the random variables D_1 and D_2 , and the decision variable p_1 . In Case (A) D_2 is "high", so the total demand exceeds the available supply even when $D_1 = L_1$. Alternatively, Case (B) corresponds to "low" D_2 , and it is possible that the total demand may not exceed the supply for some realizations of D_1 .

If D_2 is "low", the seller faces the risk of not selling K units for a range of D_1 realizations (specifically $[L_1, K - D_2]$). However the additional revenue from a separating outcome constitutes a higher expected revenue increase. (D_1 constitutes a larger portion of total sales $D_1 + D_2$ compared to K , when $D_1 + D_2 < K$)

How these two opposing effects compare determines whether the expected revenue would increase or decrease for an increase in the step 1 price p_1 .

If p_1 is within the range LP, both customers prefer to bid at step 2. From Observation 13, we know that the markdown revenue is equal to the single price revenue with SP2. The amount sold is equal to the maximum of the total demand realization and the available supply. Since nothing is sold at step 1, the step 1 price does not affect the seller's surplus, so she is indifferent between all the prices in the range (LP) $v_1 \geq p_1 > w_L^{ii}(K)$.

Proposition 23. *In the low supply scenario*

- i. the seller is indifferent between the p_1 values in range LP since customer 1 prefers to bid at step 2.*

ii. the value of p_1 in range LTS that maximizes the expected surplus of the seller is $p_1^{*(LTS)} = w_L^i(L_1)$.

The practical meaning of Proposition 23 is that the seller need not consider p_1 values outside $[w_L^i(L_1), w_L^{ii}(K)]$ when searching for the optimal p_1 .

Recall that $w_L^i(D_1)$ and $w_L^{ii}(D_1)$ are the bidding threshold functions that relate D_1 to the maximum step 1 price that would induce customer 1 with demand realization D_1 to bid his entire demand at step 1. Conversely, let us define $\Delta_L^i(p_1)$ and $\Delta_L^{ii}(p_1)$ as the demand realization thresholds, which relate the step 1 price p_1 to the minimum demand realization that would induce customer 1 to bid his entire demand at step 1 when p_1 is picked from the corresponding range. Given a specific distribution function for D_1 and D_2 , the seller can determine the corresponding threshold demand realization for any p_1 , and identify a closed form expression for her expected revenue. However, unless we assume a specific distribution, it is impossible to find closed form expressions for the optimal step 1 price and the optimal revenue. In Proposition 14, we stated that the markdown revenue is always higher than single price if the optimal single price mechanism is SP2. However since we do not have closed form solutions for the markdown prices and revenue, it is not possible derive any conclusions as to whether a markdown mechanism can outperform the single price of type SP1. Numerical examples will be used to understand how the parameter changes and combinations impact the performance of the optimal markdown mechanism compared to SP1.

We provide the derivations for the bidding behavior of the customers in moderate and high supply scenarios in Appendix B. The following propositions summarize the analogous analytical results for the seller's decision on step 1 price:

Proposition 24. *In the moderate supply scenario*

i. the seller is indifferent between the p_1 values in range MP since customer 1 prefers to bid at step 2.

ii. the value of p_1 in range MTS that maximizes the expected surplus of the seller is $p_1^{*(MTS)} = w_M^i(L_1)$.

Proposition 25. *In the high supply scenario, the seller is indifferent between the p_1 values in range HP since customer 1 prefers to bid at step 2.*

Extensions: The theoretical results we presented so far were derived assuming there are two customers in the market. We argued that a two-step mechanism would be optimal when the seller faces two customers. We can generalize the same idea to characterize the properties of the optimal mechanism when the seller faces $N > 2$ customers.

With N customers in the market, the optimal mechanism need not have more than one price p_h , where $p_h > v_t$ such that $\sum_{j=1}^{t-1} H_j \leq K$ and $\sum_{j=1}^t H_j > K$. Notice that there is no scarcity risk for all other higher prices, hence the customers do not have any incentive to submit positive bids at any price higher than p_h .

Similarly, the seller would need at most one price p_l , where $p_l \leq v_u$ such that $\sum_{j=1}^{u-1} L_j \leq K$ and $\sum_{j=1}^u L_j > K$. The reasoning behind this observation is that the seller can eliminate all but one of such prices without risking reduction of units sold.

As a generalization of the two-customer case, we state the following observation in case of N customers:

Observation 16. *Under ID with $N > 2$ customers, the optimal markdown mechanism has at most one price p_h such that $p_h > v_t$ and at most one price p_l such that $p_l \leq v_u$. There can be an arbitrary number of steps with prices between v_t and v_u .*

The optimal mechanism would one price at which there is no risk of scarcity ($p_h > v_t$) and one price at which there is a positive probability of having leftover supply ($p_l \leq v_u$). However, unlike the two-customer setting, where these two were the only necessary prices, there can be an arbitrary number of steps with prices between v_t and v_u , each of which have different availability/scarcity levels. Furthermore, we cannot say anything about the bidding behavior of customers at any of these price steps.

Under CI and IV settings with $N > 2$ customers, we were able to identify theoretical limits on the number with theorem 1 and observation 9. The fact that there can be an arbitrary number of price steps under ID is in stark contrast with these results. This observation suggests that incomplete demand information may be one of the reasons why

we observe multiple price steps in online applications of markdown pricing mechanisms.

While we do not provide a result for optimality of a two-step mechanism under ID, we believe this would be a reasonable framework for analysis given the common use in practice ([20]), and the theoretical ([6], [22]) and empirical results ([20], [42]).

Our main results hold when a discount factor is present (or customer valuations decline over time). Assume that customer j 's valuation drops to δv_j in step 2, $0 < \delta \leq 1$. We can still show that customer 1 makes all-or-nothing bids. With discounting, customer 1 bids D_1 at step 1 (and 0 at step 2) if

$$\begin{aligned} \text{PSi. } p_1 &\leq w_L^i(D_1) = v_1 - \delta(v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{K-D_1}^K F_2(x_2) dx_2 \right] \text{ and } L_1 \leq D_1 \leq K - L_2, \text{ or} \\ \text{PSii. } p_1 &\leq w_L^{ii}(D_1) = v_1 - \delta(v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{L_2}^K F_2(x_2) dx_2 \right] \text{ and } K - L_2 < D_1 \leq K, \text{ or} \\ \text{TS. } p_1 &\leq w_L^{ii}(K) = v_1 - \delta(v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2K} \int_{L_2}^K F_2(x_2) dx_2 \right] \text{ and } K < D_1 \leq H_1, \end{aligned}$$

and bids 0 at step 1 (and D_1 at step 2) otherwise.

If there is no discounting, $\delta = 1$, and the conditions above are identical to the conditions in Theorem 17. At the other extreme, $\delta = 0$, the customers valuations at the second step reduce to zero and the threshold prices all converge to v_1 . Note that the threshold price is decreasing in δ , implying that if the valuation of customer 1 is less in the second step, he agrees to pay a higher price in the first step. If customers have constant valuations but they are risk-averse such that they undervalue surplus from probabilistic outcomes by a factor of δ , the surplus generated in the second step will have lower value due to the random allocation and the impact on the threshold price will be similar to the case of discounting.

So far, we assumed that the seller has fixed capacity K . Next, we briefly comment on how the seller's revenue is affected by an increase in the available supply. As K gets higher we assume that the seller appropriately adjusts the prices of the optimal single price and two-step mechanisms.

Note that the range of interesting K values are such that $L_1 < K < H_1 + H_2$. SP1 revenue increases at a decreasing rate up to $K = H_1$ and remains constant beyond this threshold since SP1 only sells to customer 1. SP2 revenue increases v_2 per unit until

$K = L_1 + L_2$, and continues to increase at a decreasing rate up to $K = H_1 + H_2$. Similarly, the revenue from the two-step mechanism has a diminishing rate of increase over K values satisfying A-NT. Looking at each of these mechanisms separately and assuming that c (the unit cost if increasing K) is positive and non-decreasing, it is obvious that the seller would prefer not to increase K beyond a specific threshold where the marginal revenue increase matches the marginal cost of increasing K . However, if we look at the best revenue for the seller among all three possible mechanisms at any supply level, the conclusion is not straightforward.

For instances where SP1 revenue stays higher than SP2 revenue as K increases, we still observe that the increase in K benefits the seller up to a threshold. Nonetheless, in the next section we present an instance, for which the overall best revenue of the seller initially increases with K , stays constant up to certain threshold and increases further with K beyond that. This example shows that the seller's decision of K is not an easy one, and requires an involved analysis.

3.4 Results and Observations

So far, we have identified A-NT as the range of K values for which the seller can possibly exceed optimal single price revenues using a markdown mechanism. With proposition 19 we have shown that the seller can come up with at least a partially separating mechanism for all K values satisfying A-NT. We also noted that, in expectation, A-NT covers a wider range of values compared to the IV and CI settings. We are one step away from claiming that a markdown mechanism benefits the seller for a richer set of parameter values under ID setting. We will complete the last step if we can show that the seller can achieve higher revenues compared to optimal single price in all supply scenarios.

In order to accomplish this and answer other research questions, we resort to numerical examples since we do not have a closed form solution to the optimal prices and seller's revenue with an optimal two-step mechanism:

- What is the optimal step 1 price and the corresponding expected revenue of the seller in an optimal two-step mechanism?

- How do the changes in supply level (K), minimum and maximum demand parameters of the customers (L_1, H_1, L_2 and H_2), and customer valuations (v_1 vs. v_2) impact the optimal prices and/or revenues of the optimal single price and two-step mechanisms?

- How does the optimal markdown revenue compare to the optimal single price?

- What is the effect of reducing the variability of the demand distributions on the seller's revenue?

We also demonstrate through a numerical example that the seller's highest overall revenue from single price and two-step mechanisms is non-monotonous in K .

First we present theoretical results on how the single price revenue changes with K, L_i, H_i and $v_i, i \in \{1, 2\}$. Note that SP1 revenue is independent of the demand and valuation of customer 2. In Appendix B, we provide the proofs for both results, for which we assume that the customer demand is uniformly distributed over the support intervals.

Proposition 26. *Seller's revenue from SP1 increases with L_1, H_1, v_1 , and also increases with K up to $K = H_1$ and remains constant as K exceeds H_1 .*

Proposition 27. *Seller's revenue from SP2 is non-decreasing in L_1, H_1, L_2, H_2 ; increasing in v_2 . It also increases with K at the rate of v_2 per unit up to $K = L_1 + L_2$ and continues to increase at a decreasing rate up to $K = H_1 + H_2$.*

Next, we present the numerical examples focusing on the structure of the optimal markdown mechanism and its performance compared to the optimal single price. We vary the supply, K , and the demand parameters of the customers and observe how the seller's revenue from the optimal markdown and the optimal single price changes.

We generate instances in six groups depending on how D_1 and D_2 ranges may compare. We introduce the first five next (see Figure 9 for a visualization) and save the last one for later:

- G1. $D_1 \gg D_2$: ($H_1 > L_1 \geq H_2 > L_2$) The minimum possible demand of customer 1 is greater than the maximum demand of customer 2.
- G2. $E[D_1] > E[D_2]$: ($H_1 > H_2, L_1 > L_2$) Even though $D_1 > D_2$ or $D_2 > D_1$ are both possible, in expectation demand of customer 1 exceeds that of customer 2.

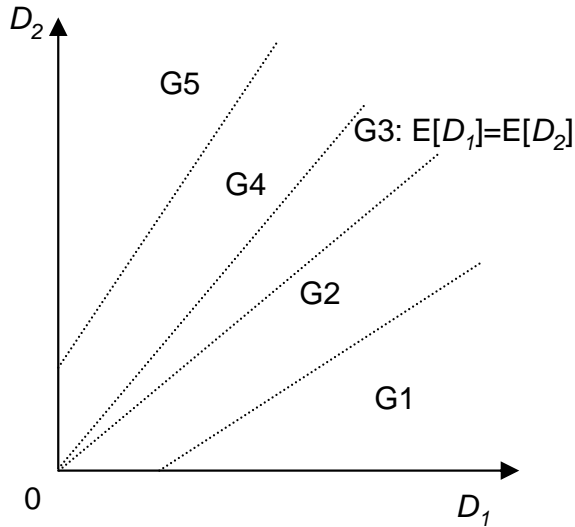


Figure 9: Instance groups for numerical examples.

- G3. $D_1 = D_2$: ($H_1 = H_2 > L_1 = L_2$) Customer demands are identically and independently distributed.
- G4. $E[D_2] > E[D_1]$: ($H_2 > H_1, L_2 > L_1$) This is in contrast with the second setting above. The expected demand by customer 2 exceeds expected demand by customer 1, but $D_1 > D_2$ is possible.
- G5. $D_2 \gg D_1$: ($H_2 > L_2 \geq H_1 > L_1$) This is the opposite of setting 1 above. The minimum demand of customer 2 exceeds maximum demand of customer 1.

We create instances by changing H_1 from 20 to 60 in increments of 10, and for each value of H_1 , we set L_1 to take values from 10 to $H_2 - 10$, again in increments of 10. This yields 15 different D_1 distributions from $[10, 20]$ to $[10, 60]$ and $[50, 60]$. For each D_1 distribution, we generate D_2 distributions corresponding to six instance groups.

Figures 13, 14, 15 in Appendix B tabulate the results all instance groups. In each instance group, for all 15 D_1 ranges, we provide the optimal revenue from SP1, SP2, and two-step mechanism (denoted with M) as well as the optimal step 1 price for low, moderate and high supply scenarios. We further distinguish K values less than $L_1 + L_2$. If no feasible K value exists in a given scenario, we leave the cells blank. The demand and supply

combinations for which the optimal two-step revenue is higher than the optimal single price revenue are identified by a shaded background color. Next we present a summary of our observations from these set of instances.

We observe that SP1, SP2 and markdown revenues increase (or stay the same) with K , keeping all else constant. However, SP1 benefits the from the increase as long as $K < H_1$, SP2 and optimal markdown revenues increase until K exceeds $H_1 + H_2$. The optimal step 1 price, p_1^* decreases with K , since the decrease in relative scarcity at step 2 forces the seller to charge a lower step 1 price that would induce customer 1 to still bid at step 1. We see that the optimal markdown becomes more likely to perform better than the optimal single price as K increases. We observe the same behavior in all instance groups regarding the impact of varying K .

An increase in L_1 leads to an increase in p_1^* , SP1, SP2 and the optimal markdown revenue. Similarly, as H_1 increases SP1, SP2 and the optimal markdown revenues increase. On the other hand p_1^* may increase or stay the same. If the change in H_1 affects the price ranges PS(i) and PS(ii) such that the previous p_1^* is no longer feasible, then the new p_1^* takes a higher value, otherwise the previous value is still optimal.

If L_1 or H_1 increases together with K by the same percentage, then we see an increase in the optimal single price and the optimal markdown revenues but a decrease in p_1^* . The intuition behind this is the fact that $E[D_2]$ stays at the same level so the customers perceive a decrease in relative scarcity at step 2. This forces the seller to decrease the step 1 price to make sure that the required customer 1 types prefer to bid at step 1 at optimality.

When L_2 or H_2 increases we see that SP1 revenues stay at the same level as expected, while SP2 revenues increase. The optimal markdown revenue and p_1^* increase as the relative scarcity at step 2 increases.

Regarding the differences among the instance groups, we observe that markdown revenue exceeds the optimal single price revenue for most of the instances in G5, some in G4 and only 1 in G3. The higher the demand of the low valuation customer compared to the demand of the high valuation customer, the more likely is the optimal markdown mechanism to dominate the optimal single price. Basically, it is much harder for the seller to ignore the

low valuation customer and just sell to the high type in such a case. Hence, a higher D_2 is more likely to lead to SP2 being the optimal single price, in which case we know that the optimal markdown revenue is higher.

Note that, we have been able to identify an instance under all supply scenarios for which the markdown revenue exceeds the optimal single price revenue except for the case when $K < L_1 + L_2$ under Low supply. However, this is possibly due to the relatively large valuation difference.

To investigate whether this can be overcome by reducing valuation difference between the customer, we present an additional set of examples. With these examples, we also investigate the impact of the changes in the valuations of the customers, v_1 and v_2 . For six of the original fifteen D_1 ranges considered, we reduced v_2 from 20 to 12 and calculated the optimal revenues from the single price and the two step mechanisms. See Figure 16 in Appendix B for tabulated results. We observe that the optimal markdown is more likely to dominate the optimal single price when the customer valuations are close. As the valuations of the customers gets closer we observe that SP1 revenues decrease relative to SP2.

We can also see that for all supply scenarios, even when supply is less than the minimum total demand in the market, ($K < L_1 + L_2$), the seller can achieve higher revenues using a markdown mechanism compared to the optimal single price. This completes the final step, and we can claim that the ID setting indeed provides an advantage for the markdown mechanism to perform better than the optimal single price for , in expectation, a wider range of supply values.

Under complete information we had identified that the optimal markdown is more likely to dominate when D_2 is high relative to D_1 , and/or when the customer valuations are close. Our observations from the numerical examples are in line with our theoretical findings in chapter 2.

In addition to these observations regarding the performance of the optimal markdown, we also look at how the variability of the demand distribution affects the seller's revenue. We introduce an additional group of instances for this purpose:

- G6. $D_1 \subset D_2$: ($H_2 > H_1 > L_1 > L_2$) where $E[D_1] = E[D_2]$. Even though the

expected demand is the same for both customers, the variance is larger for customer 2.

We address the following questions:

1. What is the impact of the the variability of D_2 on the seller's revenue?
2. Does the seller benefit more from an equivalent reduction in D_1 or D_2 ?

A reduction in the variability of D_2 also helps customer 1, since he can make a more informed bid. So, it is not clear whether a decrease in the spread of D_2 distribution would improve the seller's revenue. Taking the identically and independently distributed demand setting as the base case, we increase L_2 and decrease H_2 simultaneously by the same amount. We observe that the decrease in variability helps the seller in this setting.

To answer the second question, we create a base set of examples and we first increase the variability of D_1 , then D_2 and look at how the seller's revenue changes as a result. (See Figure 17 in Appendix B for tabulated results. Note that the last column corresponds to instances 1, 4, 6, 11, 13 and 15 for G6 in Figure 15) We observe that reducing the variability of D_1 increases the seller's revenue more compared to an equivalent reduction in the variability of D_2 . The intuition behind this observation is the fact that a reduction in the variability of D_2 helps and the both customer 1 and the seller. Since customer 1 can also improve his bidding strategy thanks to more precise information, the seller's benefit from the additional information is set back by customer 1. On the other hand, the decrease in variability of D_1 solely benefits the seller.

The results we observe from these examples about the impact of reducing variability of D_2 do not necessarily indicate general trend for all demand and supply parameters. In fact, in chapter 4 we provide an example for which reducing D_2 variability by a certain amount hurts the seller's revenue.

While discussing the impact of increasing K on the seller's overall revenue, we hinted to an interesting instance. Specifically for one of the instances in G4, with $D_1 \in [30, 40]$, $D_2 \in [35, 45]$, when $K = 38$ the best revenue for the seller is 696 from SP1. As K increases to 45, SP1 still generates the highest revenue of all mechanisms with 700, a 4 unit increase over $K = 38$. With $K = 65$, SP1 with 700 still remains the best revenue but as K becomes

75, the seller's best overall revenue increase further up to 737.2, generated by a two-step mechanism with $p_1^* = 10.29, p_2 = 10$. The overall best revenue the seller can get for K units is not monotonous in K , initially increases, remains constant for a while and then increases further with K . We conclude that determining the optimal value of K for the seller requires detailed examination of possible values.

3.5 Summary of Results and Managerial Insights

We have analyzed a priority pricing mechanism, which can also be considered as a two-step markdown mechanism under incomplete demand information with two customers and identified the bidding behavior of the customers and how the seller should set prices to achieve the best revenue from this mechanism.

We showed that optimal step 2 price is the valuation of the low-type customer. Given any markdown with prices p_1 and $p_2 = v_2$, customer 2 can only buy at step 2. The bidding behavior of customer 1 can be identified by an indifferent type Δ , which corresponds to a D_1 realization in $[L_1, H_1]$. If Customer 1 of type Δ is indifferent between buying at steps 1 and 2, all higher types in $[\Delta, H_1]$ prefers to buy at step 1 while the rest prefers step 2.

The seller's action space for $p_1, [v_1, v_2)$ is partitioned into three regions based on the purchasing decision of customer 1: The highest region corresponds to pooling, the middle range corresponds to partially separating while the lowest corresponds to totally separating outcomes. The seller sets the price in the PS and TS ranges to maximize her revenue.

One of the interesting results from our analysis is that an optimal mechanism for $N > 2$ customers may have an arbitrary number of price steps. We cannot identify a theoretical limit on the number of prices under ID.

When the seller does not have complete information on customer demands, she can benefit from a markdown mechanism compared to a single price for, in expectation, a larger set of supply ranges, and depending on the values of the other parameters, can hope to achieve revenues higher than that can be achieved with a single price mechanism. This is in spite of the fact that she does not have any controls to manipulate scarcity at the second step.

The theoretical model in this paper is motivated by the after-sales service contracts of OEMs. We rephrase the theoretical and numerical results from our model in this context and link some of them to real-world examples from pricing of service contracts.

Our model setting corresponds to a service provider facing two customers in a market with known heterogeneous valuations v_i and randomly distributed demands D_i , drawn from commonly known distributions. The seller has the option of offering one or both of the following contracts:

- Priority contract (service capacity allocated before non-priority buyers)
- Non-priority contract (remaining capacity is rationed among buyers)

Offering both contracts is analogous to the two-step markdown, while each contract by itself corresponds to SP1 and SP2.

If the expected demand from the high valuation customer exceeds the available supply, then offering the non-priority contract at a unit price of p_1 maximizes the service revenue. If the market demand for service is too high ($K < E[D_1]$) or too low ($H_1 + H_2 < K$) compared to the service capacity, then the service provider cannot increase revenues by offering both type of contracts. Essentially, she would choose to just serve the high types and intentionally leave low type customers out of the market.

Offering both type of contracts with prices such that all customer types prefer the non-priority contract does not bring any higher revenue than just offering only that contract type. The larger the portion of the demand comes from the low type, the more likely it is that offering both contracts will improve the service revenue. This is also the case when the valuations of the customer types are close. Both of these cases constitute a setting where the OEM cannot ignore the low types since the potential revenue coming from them is significant.

[31] classifies service strategies for different product segments based on the fixed and variable costs incurred by the customers due to product failure. When discussing significant shifts in cost structures, PCs and printers identified as prominent examples. These costs are significant determinants for the valuations customers have for the after-sales service.

When PCs were first introduced to the market they were scarcely available, purchasing

and ownership costs were high, and they were hard to replace if any failure occurred. High percentage of the customers had high valuations for the service and they usually bought expensive on-site maintenance contracts with service guarantees. As microprocessor technology advanced, prices declined drastically. As PCs became abundant, failure of a single PC started not to matter significantly in the continuity of operations for the businesses. Consequently a very large percentage of the customers ended up having low valuations for the service that comes with computers. Now, the manufacturers offer mostly extended warranties to the PC owners as opposed to an on-site maintenance contract. This example shows how the change in the valuation difference for the two types of service leads to industry-wide shift to discontinue with the availability of a certain type of after-sales contract in the market.

The following example, also from [31], is similar in that sense, except that the forces in play include a change in the high and low valuation demand proportions in the market in addition to the changes in valuation difference between the two types.

In the printer industry, not only did the valuations of the high segment for printer service decline but the demand from the low valuation segment also shrunk as low-end and small-office printers have become reliable and cheap enough to make service contracts unnecessary. Small office copiers became almost disposable as the lower prices no longer justify major repair costs.

We have mentioned Paris Metro as an example where the same service is offered at two different prices and the customers are allowed to self-select whether they would like to pay the premium for higher probability of available seats. From our model, we have found out that if the availability difference between the two offers is not significant, the seller would not be much worse off if she offered only one price. [36] mentions that one nice feature of the Paris Metro class pricing is the self regulating nature of the system. If the overall ridership was too low, then the 2nd class carts had ample room and there was no incentive to pay the premium to ride the 1st class. Conversely, if the 1st class cars were too full such that seat availability was not any different from the second class, then there was no incentive for the new passengers to pay the 1st class fare. This confirms our observation

that there needs to be a significant availability difference between the two price offerings for the multi-price mechanism to bring higher revenues. [1] mentions that Paris Regional Transport Authority has recommended that subway system switch from two to only one class of car by the end of 1991. The reason for the recommendation is cited as the decline in the ticket checks and abundance of turnstile-jumpers and second-class passengers riding in the first class cars. Out of 120 million tickets sold in 1990, only 21000 for were for first class, but each train had a first class car. This is another example confirming that, if the scarcity difference between the two price offers cannot be maintained, then the seller can do just as good with offering only one price. The other alternative is to create some other type of difference in the service or products offered at different prices. In the case of French railroads, they are known to have tried various methods ranging from offering champagne and delicacies to first class passengers to ripping off the roof of a rail car and billing it as “third class” and offer it at a discount to the second class [2].

CHAPTER IV

ON THE VALUE OF INFORMATION AND INFORMATION SHARING WHEN SELLING TO STRATEGIC CUSTOMERS

4.1 Introduction

The plethora of information businesses have been accumulating in the recent years makes understanding the potential value of information and strategically exploiting it more crucial than ever. The challenge is not only in storing, categorizing and interpreting the transaction data but also doing it strategically by tackling the portions with higher potential benefits with higher priority.

The value of information has been of special interest in the field of operations management. The real world phenomena are usually stochastic in nature, while the models used to study them are either deterministic, or incorporate a simplified version of the variability in stochastic models and assume complete information. Hence, exploring the value of having complete information in various contexts has been a popular research topic.

There are quite a few examples in inventory theory where researchers tried to analyze the robustness of models under incomplete information settings. In inventory management context, value of information was studied using a newsvendor model in [39]. Value of information has also been a popular in the context of supply contracts (recently [12]).

The attention on analysis of customer behavior in revenue management and pricing research has been increasing in the past decade. However, the value of information and information sharing in the context of pricing with strategic customers is mostly unexplored territory. See [40] for a recent review of the literature on customer behavior modeling in revenue management and auctions.

We focus on a specific pricing problem where a seller is using a priority pricing mechanism to sell a limited amount of a product or service to two strategic customers with multi-unit demands. The seller has complete information on customer valuations, but only

knows the distribution of their demand. The seller cannot tell which customer the demand is coming from so she cannot exercise first or third degree price discrimination. Instead, she uses a priority pricing mechanism with two prices promising different availability/scarcity levels. Purchase requests at the higher price are satisfied before the requests at the lower price. If the total requests exceed the available supply, the seller picks a customer at random and satisfies his request, and allocates the remainder to the other customer.

The customers know how many units the seller has, the prices associated with both priority levels and the valuation and demand distribution of the other customer. Each customer also knows his exact demand realization and tries to maximize the surplus ((valuation - price paid) * expected number of units allocated). By employing a priority pricing mechanism, the seller is trying to get the high-valuation customer to buy at the higher price. This is not a very easy task, as the seller has incomplete demand information and she has hardly any tools to establish a scarcity threat at the lower price.

In this setting, we investigate the benefit to the seller from improving the quality of demand information about the customers' demands.

Specifically, we answer the following questions:

- Does the seller always benefit from reduced demand variability of the customers?
- If she can get access to the private valuation information of the customers, would she benefit from sharing this information with the customers?

In [45], a newsvendor model is used as a basis for exploring the value of inventory information when selling to strategic customers. The seller, facing a random market demand from customers with search costs, sets an observable price and an unobservable stocking quantity. Consumers anticipate the likelihood of stockout and determine whether to visit the seller. They show that the seller can improve profits by sharing the stocking quantity information with the market or by promising to compensate consumers in the event of stockout.

Our approach is different since the initial inventory of the seller is public information in our model. We complement the previous work by analyzing the value of demand information and the seller's option of sharing this information with the market.

The rest of this section is organized as follows: We summarize the model and the properties of the optimal pricing mechanisms. We present the numerical examples, analyze the impact of reducing variance of demand and list our observations. Directions for future research conclude the chapter.

4.2 *Model*

A seller operates in a market consisting of two customers with a fixed starting inventory, K , and does not know the customers' exact demands. The customers have constant marginal valuations, v_1 and v_2 , for up to a certain number of units of the product. The valuations of the customers, the initial supply level are common knowledge. Each customer's private demand is drawn from a commonly known atomless continuous distribution with support over the interval $[L_i, H_i]$. Each customer shares this belief with the seller regarding the demand of the other customer but knows his own exact demand realization.

The customers and the seller are assumed to be risk-neutral, so they maximize their expected surplus. The seller employs a two-step mechanism, which has an initial higher price only the high type can afford, and a lower price both customers can afford. While the low-valuation customer can only buy at the lower price, the high-valuation customer chooses between the two prices, based on his expected surplus.

The high-type customer faces a tradeoff, and buys at the high price if and only if $(v_1 - p_1)D_1 \geq (v_1 - p_2)(\text{expected allocation at } p_2)$. The expectation is due to two factors; incomplete information about the demand of customer 2 and the use a random allocation rule by the seller. If the total bids exceed the available supply at the lower price, the seller randomly chooses one of the customers and satisfies his request completely and allocates the remaining units to the other customer.

Lacking the exact demand information, the seller cannot guarantee that customer 1 will buy at step 1. Using the distribution information on both demands, she can identify what exact D_1 realization would make customer 1 better off by buying for a particular step 1 price, p_1 . Accordingly, the optimal step 1 price is the one that maximizes the expected revenue calculated over all possible demand distributions of the customers.

We categorize a two-step mechanism that induces customer 1 to bid D_1 at step 1 for some $D_1 \in [L_1, H_1]$ as *separating*. If customer 1 bids at step 1 for all $D_1 \in [L_1, H_1]$ then we use the term *totally separating*. Conversely, if the mechanism cannot induce customer 1 to bid at step 1 for any D_1 , then we call it *pooling*.

With access to complete demand information, the seller can identify a clearing price and come up with the optimal set of prices that guarantee high-valuation customer bids at the higher price. However, under incomplete demand information there is no closed form solution to the pricing problem of the seller. Hence we resort to numerical examples for our analysis.

Specifically, restricting our attention to mean-preserving variance reductions of the demand distributions, we seek to answer the following questions: Does the seller benefit from reducing variability of D_1 or D_2 ? If the seller could achieve equivalent reduction in the variance of D_1 or D_2 , which one provides more benefit to the seller?

The purchasing decision of the high-valuation customer depends on the perceived scarcity at step 2, which is a function of the demand distribution of the low valuation customer. This interaction motivates a follow-up question: Would the seller benefit from sharing the improved demand information about customer 2 with customer 1?

Intuitively, reducing D_1 variability can never hurt the seller since this information only factors into the seller's decision. However it is hard to predict the effect of reducing D_2 variability. Our main finding is that the answer is not always straightforward and simple; it depends on the instance. We introduce two examples, present the optimal mechanisms before and after the variance reduction and arrive at opposite conclusions about the impact of reducing variance of D_2 . We also show that sometimes reducing neither D_1 nor D_2 variability benefits the seller.

As for the follow-up question, we demonstrate via numerical examples that the seller can be made better or worse off by sharing the D_2 information with customer 1.

In the next section, we introduce two numerical examples, and investigate the impact of reducing the variation of D_1 or D_2 on the buying decision of the high-valuation customer and the expected revenue of the seller.

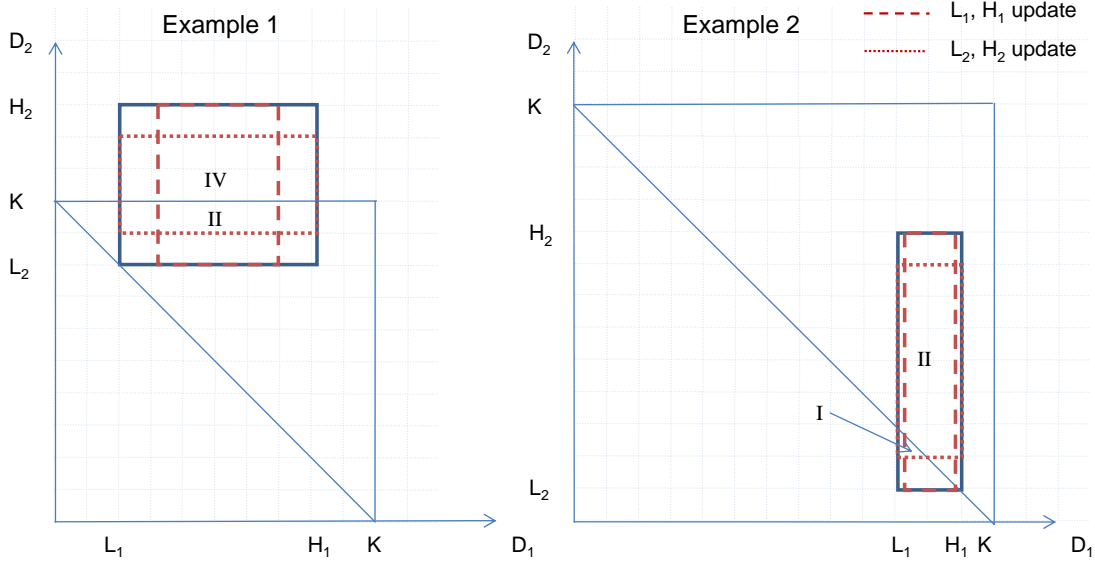


Figure 10: Relation of K to the demand ranges of customers in the numerical examples.

4.3 Numerical Examples

We present two numerical examples, modify both examples by reducing the variance of D_1 and D_2 distributions and examine how the seller's revenue is impacted by either change in both examples (Figure 10). Then we investigate if the seller would be better off if she does not share the improved information on D_2 distribution with customer 2.

To simplify the analysis in both examples, we assume that the customer demands are uniformly distributed over their respective support intervals, $[L_i, H_i]$. As we investigate the impact of reduced variance, we consider equal and opposite changes in the bounds of the support interval, hence the changes are mean-preserving. Commonly known customer valuations are $v_1 = 20$, $v_2 = 10$ and the optimal step 2 price, $p_2^* = v_2 = 10$ in both examples.

Example 1: $L_1 = 10$, $H_1 = 40$, $L_2 = 40$, $H_2 = 65$ and $K = 50$.

In the two-step mechanism that maximizes the revenue of the seller, $p_1^* = 14.169$ and $p_2 = 10$. Faced with these prices, the high-type customer prefers to buy at step 1 as long as his exact demand is at least 11.69. Hence, the seller cannot guarantee a separating mechanism and ends up taking the risk of selling all existing supply at the lower price for

the opportunity of selling D_1 at p_1 in the cases when D_1 exceeds 11.69, with an expected revenue of 601.11. Note that, if the seller were to set $p_1 = 14$, customer 1 would prefer to bid at step 1 for all demand realizations in $[10, 40]$. However, the expected revenue is less in this case.

First, we consider the case where D_1 range shrinks by 40% to $[16, 34]$. With this change, if the seller sets step 1 price as high as $p_1 = 14.375$, customer 1 chooses to buy at step 1 for any D_1 realization in $[16, 34]$. Note that this p_1 value is higher than the optimal step 1 price prior to the variance reduction. The mechanism is guaranteed to be separating and leads to a higher expected revenue level at 609.75.

Next, we analyze the impact of reducing D_2 variance by shrinking its support interval by 40% to $[45, 60]$. With this change the probability that $D_2 \geq K$, which was originally 0.6, increases to 0.67. With a higher probability, demand of customer 2 depletes the existing supply, effectively increasing the scarcity risk perceived by customer 1. Hence, the seller can charge a higher step 1 price, $p_1^* = 14.583$, and still manage to induce customer 1 for all D_1 realizations in $[10, 40]$. Benefiting from the increased scarcity risk at step 2, the seller increases her expected revenue to 614.583.

In this example, we observe that the seller benefits more from an equivalent reduction in variance of D_2 , since it boosts the scarcity risk.

Example 2: $L_1 = 50$, $H_1 = 60$, $L_2 = 5$, $H_2 = 55$ and $K = 65$.

The optimal step 1 price is $p_1^* = 11.531$. The high-valuation customer buys at step 1 for all D_1 realizations in $[50, 60]$. The expected revenue of the seller from the two-step mechanism for this example is 734.607.

When D_1 support range is reduced by 20% to $[51, 59]$, with the optimal 2-step mechanism the seller can charge a higher step 1 price at $p_1 = 11.648$. The motivation behind this change is that she does not need to worry about keeping a potential customer 1 with $D_1 = 50$ from bidding at step 2, while setting a step 1 price based on the tradeoff of customer 1. With the higher step 1 price, the seller's expected revenue increases to 737.609.

On the other hand, when the coefficient of variation for D_2 is reduced by 20% by shrinking the support interval to $[10, 50]$, the outcome is detrimental to the seller's revenue.

$K - L_2$ was greater than H_1 prior to the reduction, suggesting that if customer 2 had the allocation priority at step 2, the allocation to customer 1 would always depend on the demand realization of customer 2. However, after the variance reduction we have $K - L_2 = 55$, and for 50% of his demand realizations ($D_1 \in [50, 55]$), he gets a positive remainder from customer 2, no matter what the realization of D_2 is. Hence the sensitivity of the perceived scarcity risk for customer 1 to his demand realization is diminished. As a result, the seller cannot charge as high a step 1 price while keeping customer 1 buying at step 1 for all $D_1 \in [50, 60]$. With $p_1 = 11.531$, the two-step mechanism is totally separating and the seller ends up with an expected revenue of 733.698, which is slightly less than the original value.

As predicted, a reduction in D_1 variance does not lead to a decrease in the revenue of the seller. However, we have shown two examples where the impact of reducing D_2 variance can be negative or positive on the seller's expected revenue. Our examples demonstrate that this is due to the indirect effect a variance reduction on D_2 has on the seller's revenue through the risk of scarcity perceived by the high-valuation customer.

Even though this has not been exhibited by the two examples we presented, we can argue that it is possible for the seller to not benefit from a variance reduction of D_1 : When $L_1 > K$, allocation to customer 1 at step 2 is independent from the variance of D_1 . Since the allocation to customer 1 is K if he is picked first and 0 otherwise, it does not decrease the seller's revenue either in such instances.

Recall our follow-up question regarding the decision of the seller to share the improved information on D_2 with customer 1. The observations so far suggest that if the seller has the option to share information with customer 1, he may indeed choose not to.

Note that in the case of example 1, the seller benefits from the impact of reduced variance on the buying decision of customer 1. Hence, having acquired this information, the seller does not benefit from withholding it from the high-valuation customer. On the other hand, looking closely at example 2, we can see that if the seller keeps the updated D_2 distribution to herself, the high-valuation customer's behavior is not altered, and the seller is isolated from the negative impact on the expected revenue.

4.4 *Research Directions*

The decisions of the seller regarding the pricing mechanism if she chooses to keep D_2 information private is beyond the scope of our analysis. The notion that the customers are rational and would be able to deduce from any non-optimal action that the seller might have additional or incorrect information, brings about questions about commitment and gaming the market and opens doors for using other methods for analysis. It exceeds the purpose we set out with; answering questions about the value of information in the context of pricing under incomplete demand information when selling to strategic customers.

We believe that the ability to identify instances in which improved information accuracy benefits the agent who is allocating limited resources to analysis of virtually unlimited data is crucial to the decision of choosing which data to analyze and how to perform the analysis. Although we focused on numerical examples based on a stylized model, we were able to demonstrate that the answer is not straightforward.

We believe that our model captures the essence of the dilemma in the motivating real-world settings as it incorporates the incomplete demand information the seller has to work with when pricing goods or services in a market with strategic customers: The changes in information about a customer's parameters may benefit the seller directly but hurt her cause indirectly through the response of other customers in the market.

Nevertheless, we think future research incorporating heterogeneity in the strategic nature the customers, wherein some subset of the demand may be coming from myopic or non-rational customers may yield to interesting insights.

Another interesting direction for extension is incorporating repeated interaction of the seller and the buyers in the market whereby the buying decisions are not only a result of the current prices but also depend on the pricing policy decisions of the seller in previous iterations.

CHAPTER V

CONCLUSION

Motivated by the recent popularity of non-static pricing mechanisms and increased sophistication of the customers due to advances in computing and information technologies, we studied the design of pricing mechanisms in the presence of rational customers with multi-unit demands.

In chapter 2, we focused on the optimal design of pre-announced markdown mechanisms. Under complete information, we showed that the optimal mechanism need not have more than 2 steps and identified the optimal bidding behavior of the customers and the pricing decision of the seller in a 2-customer setting. For a more general, N -customer setting we devised an algorithm to find the optimal prices, and showed via numerical examples how the seller's revenue changes as the valuation distribution of the customers and ratio of available to the total market demand changes.

Then, we modified the problem setting to an incomplete valuation information setting, characterized the bidding behavior of the customers and provided guidelines as to how the seller can identify the optimal prices when facing two customers.

Common results from these settings include, all-or-nothing bidding behavior of customers and the form of the first step price, which charges a premium over the second step price proportional to the scarcity perceived by the customer if he bids at the second step.

In chapter 3, we investigated the design of a priority pricing mechanism for a seller operating with incomplete demand information in a setting similar to the one in the previous chapter. In contrast with the results in the previous chapter, we showed that the number of prices in the optimal mechanism can not be limited to a small number when there are multiple customers in the market. For the two-customer setting, we identified the bidding behavior of the customers is still all-or-nothing and narrowed the K range for which the seller can hope to achieve higher revenue than the optimal single price as A-NT $E[D_1] <$

$K < H_1 + H_2$. In expectation, this supply range is larger than the $D_1 < K < D_1 + D_2$ range, which was identified under complete demand information settings considered in the previous chapter. Via numerical examples, we showed that the seller can indeed achieve a higher revenue using a two-step mechanism for all supply scenarios within A-NT.

In chapter 4, we focused on the benefit to the seller from improved demand information about the customers building on the incomplete demand information model in chapter 3. We showed that the seller may or may not benefit from reduced variability of demand distribution of the low-valuation customer, since this may also signal a reduced scarcity to the high-valuation customer altering his purchasing decision at step 1. Hence, we also identified an instance for which the seller would be better off by not sharing information with the market.

Our results and insights shed light to interesting dynamics involved in designing pricing policies in a market with rational customers. Some of our results can be used to explain the interactions in real-world settings motivating the model settings we study

One of the limitations of our analysis is assuming that all customers in the market are rational and all of them are present in the market throughout the sale period. The result that limits the number of price steps are likely to not hold if customers arrive in time or if they do not stay until the end.

Another area for future work is the analysis of pricing mechanisms where the number of units available for sale, K , is not public information. This is usually the case with retail stores, and some online merchants, as the customers have no visibility to the entire inventory of the seller.

We believe the exploring these directions will improve understanding of the potential buyers and seller who are likely to be faced with or employ such mechanisms.

Appendix A

Examples of Markdowns with Pre-Announced Prices from Sam’s Club (Figure 11) and Filene’s Basement

“Filene developed a revolutionary way to price merchandise called the ‘Automatic Mark Down System.’ The price tag on each item was marked with the date it hit the selling floor. The longer an item remained unsold, the more the price would automatically be reduced, first 25%, then 50% and finally 75%.” <http://www.filenesbasement.com/master.html>



Figure 11: An example of a markdown mechanism with pre-announced prices from the website of Sam’s Club, under the name of Plunging Prices. Sam’s Club logo reproduced with permission. Microsoft product screen shot reprinted with permission from Microsoft Corporation

Example for the Two-Customer Case (Section 2.3.1)

To illustrate the results of Section 2.3.1, we present a numerical example with two customers. We consider an instance where the customers are willing to buy up to 10 and

18 units ($D_1=10$, $D_2=18$) and are willing to pay $v_1=20$ and $v_2=10$ per unit, respectively. We assume that the seller has an exogenously determined initial inventory of 20 units for sale ($K=20$).

For this instance, the optimal single price is not unique. The seller can either set the price equal to 20 and only sell to Customer 1, or set the price equal to 10 and sell the entire supply. With either one of these prices, the optimal single-price revenue of the seller is 200.

In the optimal two-step markdown, the optimal step 2 price is $p_2 = v_2 = 10$ (Observation 3). Optimal step 1 price, $p_1^* = 20 - \frac{10+20-18}{20}(20 - 10) = 14$ (Proposition 4). Customer 1 buys 10 units at step 1, and the remaining 10 are sold to Customer 2 at step 2. The revenue of the seller is 240. The markdown mechanism with prices (p_1^*, p_2) dominates the single price as stated in Observation 6.

Figures 12(a) and 12(b) illustrate how the seller's revenue under the markdown and the optimal single-price mechanisms changes as a function of (D_1) and (v_1) , respectively. In Figure 12(a), for $D_1 \leq v_2K/v_1 = 10$, markdown pricing dominates the optimal single price, as stated in Observation 6. For $D_1 > v_2K/v_1 = 10$, markdown pricing dominates the optimal single price only if $(v_1 + v_2)D_1 < (v_1 - v_2)D_2 + K(3v_2 - v_1)$ (as stated in Observation 7), that is, if $D_1 < 12.7$. In Figure 12(b), we observe a similar dominance between the two pricing mechanisms as we keep v_2 fixed at 10, but vary v_1 . From Figures 12(a) and 12(b) we also make the following observation: The revenues under the optimal single price remain constant and then start increasing after either D_1 or v_1 reach a threshold where it becomes more profitable to sell only to high-valuation customers. However, revenues under markdown pricing monotonically increase with D_1 and v_1 .

Figures 12(c) and 12(d) reaffirm the conditions stated in Observation 7, under which the optimal markdown mechanism dominates the optimal single price: Markdown pricing dominates the optimal single price when (i) D_2 is large relative to D_1 (Figure 12(c)), (ii) K is close to the total quantity demanded by both customers ($D_1 + D_2$) (Figure 12(c)) and (iii) the valuations of the two customers are close to each other (Figure 12(d)).

Proof of Theorem 2 Given the bid quantities (q_{i1}, q_{i2}) of all customers $i \neq j$, we want to find the best response (optimal bid quantities) of customer j . If $v_j < p_2$, customer j bids

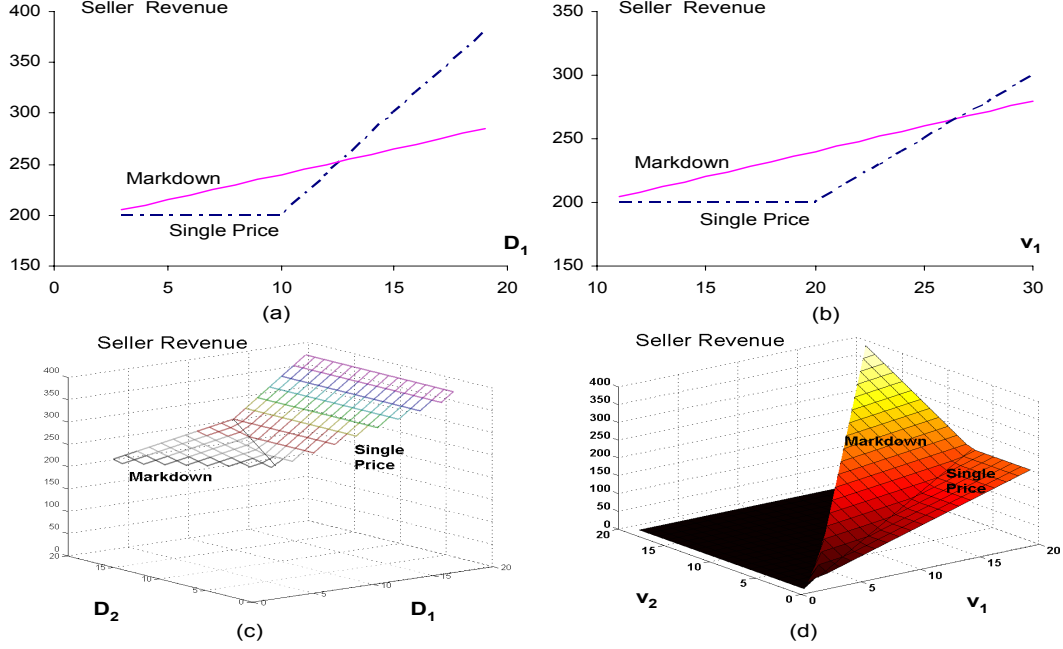


Figure 12: Seller's revenue under the optimal markdown mechanism and comparison of the markdown mechanism with the optimal single price.

zero at both price steps. If $p_1 > v_j \geq p_2$, then from Observation 1 customer j bids his entire demand at step 2. If $v_j \geq p_1$, customer j 's expected profit from bidding ($q_{j1}, q_{j2} = D_j - q_{1j}$) can be expressed as follows

$$\Pi_j = (v_j - p_1)q_{j1} + (v_j - p_2)E[A_j]$$

where $E[A_j]$ denotes the expected quantity allocated to customer j in step 2. Note that since we consider effective markdowns, customer j is guaranteed to receive all his bid quantity at step 1, i.e., his profit at step 1 is $(v_j - p_1)q_{j1}$. Customer j 's expected profit from bidding at step 2 depends on the (expected) quantity that will be allocated to that customer at step 2. Recall that when the total bid quantity exceeds supply at a given step, the seller uses the random allocation rule. This is equivalent to choosing a permutation of N customers randomly and satisfying the customers' demand in sequence based on their position in this permutation. There are $N!$ distinct permutations of N customers and the seller will choose any of these permutations with equal probability $1/N!$. Let us denote the set of all permutations by U and let B_π^j denote the set of customers whose bid quantities are satisfied

before customer j in some permutation $\pi \in U$. $B_\pi^j = \emptyset$ indicates that customer j 's demand is satisfied first.

$$E[A_j] = \frac{1}{N!} \sum_{\forall \pi \in U} \min\left\{ \left(K - \sum_{i \in B_\pi^j} D_i - \sum_{i \notin B_\pi^j} q_{i1} - q_{j1} \right)^+, (D_j - q_{j1}) \right\},$$

We will show that the profit function Π_j is convex in q_{j1} by analyzing its slope $\Pi'_j = \frac{\partial \Pi_j}{\partial q_{j1}} = (v_j - p_1) + (v_j - p_2)(E[A_j])'$, where $(E[A_j])' = \frac{\partial E[A_j]}{\partial q_{j1}}$ evaluates to $-\frac{x}{N!}$ for some positive real x . The higher the value of q_{j1} , the higher is the number of permutations for which $(K - \sum_{i \in B_\pi^j} D_i - \sum_{i \notin B_\pi^j} q_{i1} - q_{j1})^+$ is equal to zero, i.e., the lower is the value x . Hence we observe that as q_{j1} increases, the slope of the profit function either increases (possibly changing from negative to positive) or remains the same.

Based on these observations, we conclude that the profit function is convex, and will be maximized at one of the extreme points $\{0, D_j\}$, implying an all-or-nothing bidding strategy at each step. \square

Proof of Proposition 4 Since $p_2 = v_2$, using the equilibrium bidding strategies from Table 3 we can write the total profit of the seller as follows:

$$\Pi_S(p_1) = \begin{cases} p_1 D_1 + v_2(K - D_1) & \text{if } p_1 \leq \hat{p}_{CI}(v_1, v_2) \\ v_2 K & \text{if } p_1 > \hat{p}_{CI}(v_1, v_2) \end{cases}$$

Π_S is strictly monotonically increasing in p_1 up to $\hat{p}_{CI}(v_1, p_2)$, after which it sharply drops to $v_2 K$. Hence, this break-point $\hat{p}_{CI}(v_1, p_2) = p_1^*$ is the optimal price. At p_1^* , customer 1 is indifferent between buying at step 1 or 2. By assumption (A2) customer 1 bids his entire demand at step 1. \square

Proof of Theorem 5 Suppose there exists some partition $\{S_1, S_2\}$, which satisfies conditions (C1) and (C2) simultaneously. We will demonstrate that the proposed bidding strategies constitute an equilibrium by considering any profitable deviations. Suppose that all buyers, except buyer k , bid according to ES. From Theorem 2, we know that it is always optimal for customer k to bid all of his potential demand in one step.

For $k \in S_1$, if buyer k bids according to ES, his profit is given by $\Pi_{k1} = D_k(v_k - p_1)$. On the other hand, if customer k deviates from ES and bids D_k at step 2, he will compete with

the other customers in S_2 for the allocation of the remaining units; in this case customer k 's expected profit is $\Pi_{k2} = (v_k - p_2)E[A_k]$ where $E[A_k]$ is customer k 's expected allocation at step 2.

We know that $\Pi_{k1} = D_k(v_k - p_1) > (v_k - p_2)E[A_k] = \Pi_{k2}$ (by the definition of the partition $\{S_1, S_2\}$ and condition (C1)). Therefore, given all other buyers bid according to ES, it is an optimal response for buyer $k \in S_1$ to do the same.

Similarly, for $k \in S_2$, if buyer k bids according to ES, his profit is given by $\Pi_{k2} = (v_k - p_2)E[A_k]$. If buyer k deviates and bids D_k in step 1, his profit is given by $\Pi_{k1} = D_k(v_k - p_1)$. From the definition of $\{S_1, S_2\}$ and condition (C2), we know that $\Pi_{k1} = D_k(v_k - p_1) < (v_k - p_2)E[A_k] = \Pi_{k2}$. Therefore, given all other buyers bid according to ES, it is an optimal response for buyer $k \in S_2$ to do the same. \square

Proof of Proposition 6 We can rewrite condition (C1) as follows:

$$\frac{v_k - p_1}{v_k - p_2} \geq \frac{E[A_k]}{D_k} \quad (C1)$$

As we increase D_k by one unit, the denominator of $\frac{E[A_k]}{D_k}$ increases by one unit, but the numerator increases by at most one unit, i.e., as D_k increases, $\frac{E[A_k]}{D_k}$ decreases (or remains the same). On the other hand, $\frac{v_k - p_1}{v_k - p_2}$ is increasing in v_k . Then we have

$$\frac{v_j - p_1}{v_j - p_2} > \frac{v_k - p_1}{v_k - p_2} > \frac{E[A_k]}{D_k} \geq \frac{E[A_j]}{D_j}$$

for any $j < k$ since $v_j > v_k$ and $D_j \geq D_k$. This implies that if condition (C1) holds for customer k for a given partition $\{S_1, S_2\}$, then it should hold for all customers $j < k$. \square

Proof of Theorem 7 Let q_{jt} denote customer j 's bid quantity at step t , $t \in \{1, \dots, m\}$. Note that for any customer j who can afford to bid at p_m , we have $q_{jm} = D_j - \sum_{t < m} q_{jt}$.

The expected profit of customer j can be expressed as:

$$\Pi_j(q_{j1}, \dots, q_{jm}) = \sum_{t=1}^m (v_j - p_t)E[A_{jt}]$$

where $E[A_{jt}]$ is the expected allocation to customer j at step t . Recall that when the total bid quantity exceeds supply at a given step, the seller uses a random allocation rule and

this is equivalent to choosing a permutation of N customers randomly and satisfying the demands based on their sequence in this permutation. The seller will choose each of $N!$ permutations with equal probability $\frac{1}{N!}$. We denote the set of all permutations with U and define B_π^j as the set of customers whose bids are satisfied before customer j in some permutation $\pi \in U$. Using this notation, we can define the expected allocation of customer j at step t as follows:

$$E[A_{jt}] = \frac{1}{N!} \sum_{\forall \pi \in U} \min\{K_t^+, q_{jt}\}$$

where

$$\begin{aligned} K_1 &= K - \sum_{\forall i \in B_\pi^j} q_{i1}, & K_m &= K - \sum_{\forall i \in B_\pi^j} D_i - \sum_{\forall i \notin B_\pi^j} \sum_{k < m} q_{ik} - \sum_{k < m} q_{jk} \\ K_t &= K - \sum_{\forall i \in B_\pi^j} \sum_{k \leq t} q_{ik} - \sum_{\forall i \notin B_\pi^j} \sum_{k < t} q_{ik} - \sum_{k < t} q_{jk}, & m > t > 1 \end{aligned}$$

We look at the derivative of Π_j with respect to q_{jk} and show that the profit is maximized at the end-points of the range for q_{jk} .

$$\frac{\partial \Pi_j(q_{j1}, \dots, q_{jm})}{\partial q_{jk}} = \sum_{t=1}^m (v_j - p_t) \frac{\partial E[A_{jt}]}{\partial q_{jk}}$$

$\frac{\partial E[A_{jt}]}{\partial q_{jk}}$ evaluates to zero for $k > t$. For $k < t$, $\frac{\partial E[A_{jt}]}{\partial q_{jk}}$ evaluates to $-\frac{x_k}{N_t!}$ for some x_k . The higher the value of q_{jk} , the higher is the number of permutations for which K_t^+ is equal to zero, i.e., the lower is the value of x_t . For $k = t$, $\frac{\partial E[A_{jt}]}{\partial q_{jt}}$ evaluates to $\frac{w_t}{N_t!}$ for some w_t . Note that K_t^+ is constant with respect to q_{jt} , hence, as q_{jt} increases, the number of permutations for which $\min\{K_t^+, q_{jt}\} = q_{jt}$ decreases (or remains the same), implying that $\frac{\partial E[A_{jt}]}{\partial q_{jt}}$ is non-decreasing in q_{jt} . In summary, as q_{jk} increases, $\frac{\partial E[A_{jt}]}{\partial q_{jk}}$ increases or remains the same. From this observation, it follows that the slope of the profit function in the direction of q_{jk} is non-decreasing in q_{jk} implying that customer j 's profit is maximized at one of the end-points 0 or $D_j - \sum_{l < t} q_{jl}$.

Since the optimal bid quantity at step t is 0 or $D_j - \sum_{l < t} q_{jl}$, customer j submits 0 or D_j at step 1. By induction, if the customer submits 0 or D_j at steps $1, \dots, t$, $t < m$, then he will submit 0 or D_j at step $t + 1$. Hence, it is optimal for customer j to submit all-or-nothing bids at any step. \square

Proof of Observation 8: Consider a markdown where $p_1 > p_2 > \dots > p_j > \bar{v}_k > p_{j+1} > \dots$ where $p_c \in [\underline{v}_k, \bar{v}_k]$.

(i) Since the total demand is less than K for any price higher than \bar{v}_k , customers will not purchase but simply wait until the markdown reaches p_j , which is the lowest price above \bar{v}_k . Hence, one can eliminate p_1, p_2, \dots, p_{j-1} from the markdown without affecting the expected profit of the seller.

(ii) Suppose $p_1 > \dots > \underline{v}_k > p_l > \dots > p_m$, i.e, p_l is the highest price which is smaller than \underline{v}_k . We claim that the seller would be better off by eliminating the last price step p_m . Since $D[p_i] > K \forall p_i \leq p_l$, eliminating p_m does not decrease the number of units sold. We also need to show that the seller is guaranteed to sell all K units at the same or higher prices than with the m -step markdown. Customers originally bidding at step m may bid at higher price steps or they may not bid at all in the new markdown with fewer steps. If they do not bid in the new mechanism, then the competition at the higher price levels is unaffected. On the other hand, if they bid at higher price steps, then the competition can only increase since there would be more customers bidding at a price step. As a result, at any price step $j < m$, the competition either increases or remains the same, preventing the seller's revenue from decreasing. Hence, the revenue of the seller is not made any worse by eliminating the lowest price step from the markdown. Using these arguments repeatedly, we can eliminate all price steps (strictly) smaller than p_l .

(iii) By contradiction, suppose $p_m \in [\bar{v}_{j+1}, \underline{v}_j]$ for some $j < N$. By setting $p = \underline{v}_j$, the seller does not decrease the demand at any price step, hence, the competition at each price step remains the same. However, due to the increase in p_m , some customers who were previously bidding at p_m may now choose to bid at higher price steps, potentially increasing the seller's revenues. \square

Proof of Observation 10: Since $p_c \in [\underline{v}_N, \bar{v}_N]$, from Observation 8, there is at most one price, p_1 , exceeding \bar{v}_N . By the definition of an INT markdown, only one price can be chosen from $[\underline{v}_N, \bar{v}_N]$. \square

Proof of Proposition 9: If customer 1 bids D_1 at p_1 , his surplus is $\Pi_{11} = (v_1 - p_1)D_1$.

Alternatively, if he bids D_1 at p_2 his expected surplus is

$$Pi_{12} = (v_1 - p_2) \left\{ F_2(p_2)D_1 + [1 - F_2(p_2)] \frac{D_1 + K - D_2}{2} \right\}$$

Hence, customer 1 bids at p_1 if and only if $\Pi_{11} \geq \Pi_{12}$. Rearranging terms, we get Equations (4) and (5). \square

To identify the range of p_2 values for which TS or PS markdowns are feasible, first we find the p_2 values which satisfy $\hat{p}_{IV}(v_1, p_2) = \bar{v}_2$ (the lower bound on allowable prices for p_1 under an INT markdown), and find two solutions, namely, \bar{v}_2 and $p_2(v_1)$ where

$$p_2(v_1) = v_1 - \frac{2D_1}{D_1 + D_2 - K}(\bar{v}_2 - \underline{v}_2) \quad (6)$$

(Note that when customer valuations are uniformly distributed, the threshold step 1 price, $\hat{p}_{IV}(v_1, p_2)$, is a quadratic convex function of p_2 .) To induce *some* customer 1 types to bid at p_1 , we need $p_2 \leq p_2(\bar{v}_1)$. In addition, $p_2 < \bar{v}_2$ from A4-IV, hence, we need $p_2 \leq \min\{\bar{v}_2, p_2(\bar{v}_1)\}$. Combining this with the result of Corollary 12, we get the following condition for a feasible PS markdown.

Observation 17. *A feasible PS markdown satisfies the following:*

$$\underline{v}_2 \leq p_2 < \min\{\bar{v}_2, p_2(\bar{v}_1)\} \quad \text{and} \quad \max\{\hat{p}_{IV}(\underline{v}_1, p_2), \bar{v}_2\} < p_1 \leq \hat{p}_{IV}(\bar{v}_1, p_2) \quad (7)$$

Similarly, we can identify conditions for the existence of a TS markdown.

Observation 18. *A feasible TS markdown satisfies the following:*

$$\underline{v}_2 \leq p_2 < \min\{\bar{v}_2, p_2(\underline{v}_1)\} \quad \text{and} \quad \bar{v}_2 < p_1 \leq \hat{p}_{IV}(\underline{v}_1, p_2) \quad (8)$$

Note that since $\hat{p}_{IV}(\bar{v}_1, p_2) > \hat{p}_{IV}(\underline{v}_1, p_2)$, a PS markdown exists whenever a TS markdown exists.

Results 1 and 2 spell out conditions (7) and (8) (presented in Observations 17 and 18) for uniformly distributed valuations, and are used in the proofs of some of the theorems that follow.

Result 1. *When the valuations are uniformly distributed, there exists a PS markdown with price p_2 if and only if one of the following conditions holds:*

- i. for $p_2 \in [v_2, p_2(\bar{v}_1)]$ if $\frac{\bar{v}_2 - v_2}{\bar{v}_1 - v_2} \leq \frac{D_1 + D_2 - K}{2D_1} < \frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2}$ or equivalently if $v_2 \leq p_2(\bar{v}_1) < \bar{v}_2$*
- ii. for all $p_2 \in [v_2, \bar{v}_2)$ if $\frac{\bar{v}_2 - v_2}{\bar{v}_1 - \bar{v}_2} \leq \frac{D_1 + D_2 - K}{2D_1}$ or equivalently if $\bar{v}_2 \leq p_2(\bar{v}_1)$*

The proof of Result 1 follows immediately by inserting the appropriate values for the uniform distribution into the conditions (7) of Observation 17. We have a similar result for the TS markdown.

Result 2. *When the valuations are uniformly distributed, there exists a TS markdown with price p_2 if and only if one of the following conditions holds:*

- i. for $p_2 \in [v_2, p_2(v_1)]$ if $\frac{\bar{v}_2 - v_2}{v_1 - v_2} \leq \frac{D_1 + D_2 - K}{2D_1} < \frac{\bar{v}_2 - v_2}{v_1 - \bar{v}_2}$ or equivalently if $v_2 \leq p_2(v_1) < \bar{v}_2$*
- ii. for all $p_2 \in [v_2, \bar{v}_2)$ if $\frac{\bar{v}_2 - v_2}{v_1 - \bar{v}_2} \leq \frac{D_1 + D_2 - K}{2D_1}$ or equivalently if $\bar{v}_2 \leq p_2(v_1)$*

In the following, whenever there is a strict inequality $a < b$ as a constraint, this should be perceived as $a \leq b - \epsilon$, where $\epsilon \approx 0$. To simplify the notation, we use $a < b$. ϵ can be thought of as the minimum possible price increment.

Figure 4 shows how the existence of PS and TS markdowns depends on the scarcity of supply measured by $\frac{D_1 + D_2 - K}{2D_1}$ (the ratio of customer 1's expected unmet demand to his entire demand, if both customers bid at step 2): if scarcity is low, it is more likely to see a pooling rather than a separating outcome. (Figure 4 is based on Results 1 and 2 presented above.)

The seller can determine the optimal PS markdown by maximizing Π_S^{PS} subject to (7), where:

$$\Pi_S^{PS} = [1 - F_1(\hat{v}_1(p_1, p_2))]p_1D_1 + F_1(\hat{v}_1(p_1, p_2))p_2D_1 + (K - D_1)[1 - F_2(p_2)]p_2 \quad (9)$$

Similarly, the seller can determine the optimal TS markdown by maximizing Π_S^{TS} subject to (8), where:

$$\Pi_S^{TS} = p_1D_1 + (K - D_1)[1 - F_2(p_2)]p_2 \quad (10)$$

Define \mathcal{P}_{PS} and \mathcal{P}_{TS} to be the set of p_2 values for which a PS and a TS markdown exists, respectively. Note that since $\mathcal{P}_{TS} \subseteq \mathcal{P}_{PS}$, we have $\mathcal{P}_{TS} \cap \mathcal{P}_{PS} = \mathcal{P}_{TS}$. In what follows, we characterize the optimal PS and TS markdowns. Based on those results, we are able to derive sufficient conditions for a PS or a TS markdown to be optimal. Define $p_1^{*M}(p_2)$ to be the optimal first step price given a second price step of p_2 under markdown type $M = TS, PS$.

Characterization of the Optimal TS Markdown

To characterize the optimal TS markdown, we find $p_1^{*TS}(p_2)$ and then demonstrate some properties of the seller's revenue function.

Theorem 28. *The optimal TS markdown has the following properties:*

(i) *For a given second step price p_2 , the optimal first price step is $\hat{p}_{IV}(\underline{v}_1, p_2)$.*

(ii) $\Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, p_2), p_2)$ *is convex in p_2 if $K < D_1 + D_2/3$, and it is concave in p_2 if $K \geq D_1 + D_2/3$.*

(iii) *If $\Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, p_2), p_2)$ is concave in p_2 , then the optimal step 2 price is:*

$$p_2^{*TS} = \begin{cases} \underline{v}_2 & \text{if } p_2^{TS} \leq \underline{v}_2 \\ p_2^{TS} & \text{if } \underline{v}_2 < p_2^{TS} \leq \min\{p_2(\underline{v}_1), \bar{v}_2\} \\ p_2(\underline{v}_1) & \text{if } p_2(\underline{v}_1) < \min\{p_2^{TS}, \bar{v}_2\} \\ \bar{v}_2 - \epsilon & \text{if } \bar{v}_2 \leq \min\{p_2^{TS}, p_2(\underline{v}_1)\} \end{cases}$$

(iv) *If $\Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, p_2), p_2)$ is convex in p_2 , then the optimal step 2 price is:*

$$p_2^{*TS} = \begin{cases} \underline{v}_2 & \text{if } \frac{D_1 + D_2 - K}{2D_1} \geq \frac{D_1 \bar{v}_2 - K \underline{v}_2}{D_1(\underline{v}_1 - \underline{v}_2)} + \frac{(K - D_1)(\bar{v}_2 - p_2(\underline{v}_1))}{D_1} \frac{p_2(\underline{v}_1)}{(\bar{v}_2 - \underline{v}_2)(\underline{v}_1 - \underline{v}_2)} \\ p_2(\underline{v}_1) & \text{otherwise} \end{cases}$$

where $p_2^{TS} = \frac{\bar{v}_2(3K - D_1 - D_2) - 2D_1 \underline{v}_2 - (D_1 + D_2 - K)\underline{v}_1}{2(3K - 3D_1 - D_2)}$.

Proof of Theorem 28:

(i) The seller's profit is increasing in p_1 (since $\frac{\partial \Pi_S^{TS}}{\partial p_1} = D_1 > 0$). Therefore, for a given p_2 , the seller would prefer to set p_1 to its upper bound, which is $\hat{p}_{IV}(\underline{v}_1, p_2)$.

(ii) The first derivative of seller's revenue with respect to p_2 after substituting $p_1 = \hat{p}_{IV}(\underline{v}_1, p_2)$ and uniform CDF and pdf for $F_i(p_i)$ and $f_i(p_i)$ is as follows:

$$\frac{\partial \Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, p_2), p_2)}{\partial p_2} = D_1 - \frac{\underline{v}_1}{(\bar{v}_2 - \underline{v}_2)} \frac{(D_1 + D_2 - K)}{2} + \frac{(\bar{v}_2 - 2p_2)(3K - 3D_1 - D_2)}{(\bar{v}_2 - \underline{v}_2)2}$$

Taking the second derivative with respect to p_2 yields:

$$\frac{\partial^2 \Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, p_2), p_2)}{\partial p_2^2} = (3D_1 + D_2 - 3K) \frac{1}{\bar{v}_2 - \underline{v}_2}$$

Since $\bar{v}_2 > \underline{v}_2$ we obtain a sufficient and necessary condition for the function to be concave (convex) in p_2 as $K \geq D_1 + D_2/3$ ($K < D_1 + D_2/3$).

(iii) When the revenue function is concave in p_2 , its maximizer is p_2^{*TS} . If $p_2^{*TS} \in [\underline{v}_2, \min\{p_2(\underline{v}_1), \bar{v}_2\}]$, then it is the optimal step 2 price. Otherwise, the optimal p_2 is one of the boundary values of the feasible p_2 range.

(iv) When the revenue is a convex function of p_2 , one of the boundary points will be the optimal step 2 price. First we show that if a TS markdown exists for all $p_2 \in [\underline{v}_2, \bar{v}_2)$, then $p_2^{*TS} = \underline{v}_2$. For \underline{v}_2 to be the optimal step 2 price, revenue at this price should be higher than the revenue at $p_2 = \bar{v}_2$, i.e.,

$$\Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, \underline{v}_2), \underline{v}_2) \geq \Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, \bar{v}_2), \bar{v}_2)$$

$$\rightarrow K\underline{v}_2 + (\underline{v}_1 - \underline{v}_2) \frac{D_1 + D_2 - K}{2} \geq D_1 \bar{v}_2 \rightarrow \frac{D_1 + D_2 - K}{2} \geq \frac{D_1 \bar{v}_2 - K \underline{v}_2}{\underline{v}_1 - \underline{v}_2}$$

From Result 2(ii), we have $\frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \bar{v}_2} \leq \frac{D_1 + D_2 - K}{2D_1}$ as the condition for a TS markdown to exist for all p_2 , hence we conclude that \underline{v}_2 is always optimal provided that a TS markdown exists for all p_2 as a result of the following series of inequalities,

$$\frac{D_1 + D_2 - K}{2D_1} \geq \frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \bar{v}_2} > \frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \underline{v}_2} > \frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \underline{v}_2} - \frac{K - D_1}{D_1} \frac{\underline{v}_2}{(\underline{v}_1 - \underline{v}_2)} = \frac{D_1 \bar{v}_2 - K \underline{v}_2}{D_1(\underline{v}_1 - \underline{v}_2)}$$

If a TS markdown exists for only $p_2 \in [\underline{v}_2, p_2(\underline{v}_1)]$, where $p_2(\underline{v}_1) < \bar{v}_2$, then either $p_2^{*TS} = p_2(\underline{v}_1)$ or $p_2^{*TS} = \underline{v}_2$. We compare the corresponding revenues to find conditions under which either one is optimal. For $p_2^{*TS} = \underline{v}_2$, we need $\Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, \underline{v}_2), \underline{v}_2) \geq \Pi_S^{TS}(\hat{p}_{IV}(\underline{v}_1, p_2(\underline{v}_1)), p_2(\underline{v}_1)) = \Pi_S^{TS}(\bar{v}_2, p_2(\underline{v}_1))$, i.e.,

$$\frac{D_1 + D_2 - K}{2D_1} \geq \frac{D_1 \bar{v}_2 - K \underline{v}_2}{D_1(\underline{v}_1 - \underline{v}_2)} + \frac{(K - D_1)(\bar{v}_2 - p_2(\underline{v}_1))}{D_1} \frac{p_2(\underline{v}_1)}{(\bar{v}_2 - \underline{v}_2)(\underline{v}_1 - \underline{v}_2)}$$

From Result 2 we know that a TS markdown exists for $p_2 \in [\underline{v}_2, p_2(\underline{v}_1)]$ when $\frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \bar{v}_2} < \frac{D_1 + D_2 - K}{2D_1} < \frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \bar{v}_2}$. Hence $p_2^{*TS} = \underline{v}_2$ if $\frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \bar{v}_2} > \frac{D_1 + D_2 - K}{2D_1} > \frac{\bar{v}_2 - \underline{v}_2}{\underline{v}_1 - \bar{v}_2} \geq \frac{D_1 \bar{v}_2 - K \underline{v}_2}{D_1(\underline{v}_1 - \underline{v}_2)} + \frac{(K - D_1)(\bar{v}_2 - p_2(\underline{v}_1))}{D_1} \frac{p_2(\underline{v}_1)}{(\bar{v}_2 - \underline{v}_2)(\underline{v}_1 - \underline{v}_2)}$. Combining the ranges that yield the same p_2^{*TS} we get the desired conditions. \square

Characterization of the Optimal PS Markdown

To characterize the optimal PS markdown, we first show some properties of the seller's revenue function in and then derive $p_1^{*PS}(p_2)$ for a given p_2 .

Theorem 29. *The optimal PS markdown has the following properties*

(i) Π_S^{PS} is concave in p_1 and $p_1^{PS}(p_2) = p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)]\frac{D_1+D_2-K}{4D_1}$ maximizes the unconstrained Π_S^{PS} .

(ii) The optimal step 1 price is the following:

$$p_1^{*PS}(p_2) = \begin{cases} p_1^{PS}(p_2) & \text{if } \underline{v}_1 < \frac{\bar{v}_1+p_2}{2} \text{ and } p_2 < \tilde{p}_2 = \bar{v}_1 - \frac{4D_1}{D_1+D_2-K}(\bar{v}_2 - \underline{v}_2) \\ \max\{\hat{p}_{IV}(\underline{v}_1, p_2), \bar{v}_2\} + \epsilon & \text{otherwise} \end{cases}$$

Proof of Theorem 29:

(i) First we show that the revenue function is concave in p_1 from the second order condition, and find the maximizer from the first order condition. The first and second order partial derivatives of $\Pi_S^{PS}(p_1, p_2)$ in Equation (9) with respect to p_1 are:

$$\frac{\partial \Pi_S^{PS}}{\partial p_1} = [1 - F_1(\hat{v}_1(p_1, p_2))]D_1 - \frac{\partial \hat{v}_1(p_1, p_2)}{\partial p_1} f_1(\hat{v}_1(p_1, p_2))(p_1 - p_2)D_1$$

$$\frac{\partial^2 \Pi_S^{PS}}{\partial p_1^2} = -2f_1(\hat{v}_1(p_1, p_2))\frac{\partial(\hat{v}_1(p_1, p_2))}{\partial p_1}D_1$$

Since $\frac{\partial(\hat{v}_1(p_1, p_2))}{\partial p_1} = \frac{1}{[1-F_2(p_2)]}\frac{2D_1}{D_1+D_2-K}$, and $f_1(\cdot)$ are both positive and independent of p_1 under uniform distribution, the revenue function is concave in p_1 for all p_2 .

Substituting $\hat{v}_1(p_1, p_2)$ from Equation (5), $\frac{\partial(\hat{v}_1(p_1, p_2))}{\partial p_1} = \frac{1}{[1-F_2(p_2)]}\frac{2D_1}{D_1+D_2-K}$, and uniform CDF and pdf for $F_1(\cdot)$ and $f_1(\cdot)$ in the first derivative and setting equal to zero, we get the first order condition:

$$\frac{\bar{v}_1 - p_2}{\bar{v}_1 - \underline{v}_1}D_1 - \frac{1}{\bar{v}_1 - \underline{v}_1}\frac{(p_1 - p_2)}{[1 - F_2(p_2)]}\frac{4D_1^2}{D_1 + D_2 - K} = 0$$

We solve for the p_1 value that satisfies the first order condition and get $p_1^{PS}(p_2) = p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)]\frac{D_1+D_2-K}{4D_1}$.

(ii) To find out the optimal p_1 for a given p_2 , we need to understand the conditions under which $p_1^{PS}(p_2) \in (\max\{\hat{p}_{IV}(\underline{v}_1, p_2), \bar{v}_2\}, \hat{p}_{IV}(\bar{v}_1, p_2)]$ as stated in constraint (7). It

is easy to see that $p_1^{PS}(p_2) < \hat{p}_{IV}(\bar{v}_1, p_2)$ always, hence, we only need to check the lower bounds. Next, we show that $p_1^{PS}(p_2) \in (\max\{\hat{p}_{IV}(\underline{v}_1, p_2), \bar{v}_2\}, \hat{p}_{IV}(\bar{v}_1, p_2)]$ if and only if $\underline{v}_1 < \frac{\bar{v}_1 + p_2}{2}$ and $p_2 < \tilde{p}_2$

(ii.a) $p_1^{PS}(p_2) > \hat{p}_{IV}(\underline{v}_1, p_2)$ if and only if $\underline{v}_1 < \frac{\bar{v}_1 + p_2}{2}$.

We substitute $v_1 = \underline{v}_1$ in Equation (4), and compare with $p_1^{PS}(p_2)$.

$$p_1^{PS}(p_2) = p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{4D_1} > p_2 + (\underline{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{2D_1}$$

Simplifying the expression, we get $\underline{v}_1 < \frac{\bar{v}_1 + p_2}{2}$ as the equivalent condition.

(ii.b) $p_1^{PS}(p_2) > \bar{v}_2$ if and only if $p_2 < \tilde{p}_2 = \bar{v}_1 - \frac{4D_1}{D_1 + D_2 - K}(\bar{v}_2 - \underline{v}_2)$.

In order for $p_1^{PS}(p_2)$ to exceed \bar{v}_2 , we get:

$$p_2 + (\bar{v}_1 - p_2)[1 - F_2(p_2)] \frac{D_1 + D_2 - K}{4D_1} > \bar{v}_2$$

Substituting uniform CDF for F_2 and rearranging we get $p_2 < \tilde{p}_2$ as the equivalent condition.

□

Plugging $p_1 = p_1^{*PS}(p_2)$ in Equation (9), now we can determine the optimal step 2 price, p_2^{*PS} , corresponding to each case.

Result 3. When $p_1^{*PS}(p_2) = p_1^{PS}(p_2)$, the optimal step 2 price for a PS markdown is:

$$p_2^{*PS} = \begin{cases} \underline{v}_2 & \text{if } p_2^{PS} \leq \underline{v}_2 \\ p_2^{PS} & \text{if } \underline{v}_2 < p_2^{PS} < \min\{\bar{v}_2, p_2(\bar{v}_1)\} \\ p_2(\bar{v}_1) & \text{if } p_2(\bar{v}_1) \leq p_2^{PS} \text{ and } p_2(\bar{v}_1) < \bar{v}_2 \\ \bar{v}_2 - \epsilon & \text{otherwise} \end{cases}$$

Proof of Result 3: Substituting $p_1 = p_1^{PS}(p_2)$ into Equation (9), we get:

$$\Pi_S^{PS}(p_1^{PS}(p_2), p_2) = p_2 D_1 + [1 - F_2(p_2)] \left(\frac{(\bar{v}_1 - p_2)^2 (D_1 + D_2 - K)}{8(\bar{v}_1 - \underline{v}_1)} + (K - D_1)p_2 \right)$$

The first derivative with respect to p_2 yields:

$$\begin{aligned} \frac{\partial \Pi_S^{PS}}{\partial p_2} &= D_1 - [1 - F_2(p_2)] \left(\frac{(\bar{v}_1 - p_2)(D_1 + D_2 - K)}{4(\bar{v}_1 - \underline{v}_1)} + (K - D_1) \right) \\ &\quad - f_2(p_2) \left(\frac{(\bar{v}_1 - p_2)^2 (D_1 + D_2 - K)}{8(\bar{v}_1 - \underline{v}_1)} + (K - D_1)p_2 \right) \end{aligned}$$

The second derivative with respect to p_2 after substituting uniform distribution for F_2 yields:

$$\frac{\partial^2 \Pi_S^{PS}}{\partial p_2^2} = \frac{(\bar{v}_1 - p_2)(D_1 + D_2 - K)}{2(\bar{v}_1 - \underline{v}_1)(\bar{v}_2 - \underline{v}_2)} - \frac{2(K - D_1)}{(\bar{v}_2 - \underline{v}_2)} + \frac{(\bar{v}_2 - p_2)(D_1 + D_2 - K)}{4(\bar{v}_1 - \underline{v}_1)(\bar{v}_2 - \underline{v}_2)}$$

Solving for p_2 from the first order condition $\frac{\partial \Pi_S^{PS}}{\partial p_2} = 0$, we get $p_2 = \frac{A \pm \sqrt{A^2 - B \cdot C}}{C}$, where $A = (\bar{v}_2 + 2\bar{v}_1)(D_1 + D_2 - K) - 8(K - D_1)(\bar{v}_1 - \underline{v}_1)$, $B = 8(\underline{v}_2 D_1 - \bar{v}_2 K)(\bar{v}_1 - \underline{v}_1) + \bar{v}_1(\bar{v}_1 + 2\bar{v}_2)(D_1 + D_2 - K)$ and $C = 3(D_1 + D_2 - K)$.

The second order condition required for p_2 to be a maximizer yields $p_2 > \frac{A}{C}$, hence we get $p_2^{PS} = \frac{A + \sqrt{A^2 - B \cdot C}}{C}$ as the unique maximizer of the unconstrained revenue function. By comparing p_2^{PS} with boundary points of the feasible p_2 range, $[\underline{v}_2, \min\{p_2(\bar{v}_1), \bar{v}_2\}]$, we determine the optimal step 2 price. \square

In order not to clutter the presentation in the proofs in this document, we will drop ϵ when substituting arguments with ϵ in other expressions and use $\lim_{\epsilon \rightarrow 0}$ in expressions that involve $=$.

Result 4. When $p_1^{*PS}(p_2) = \bar{v}_2 + \epsilon$, then in a PS markdown we have:

(i) $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is convex in p_2 if and only if $\frac{(K - D_1)}{\bar{v}_2 - \underline{v}_2} < \frac{D_1}{\bar{v}_1 - \underline{v}_1}$, and it is concave in p_2 otherwise.

(ii) If $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is concave in p_2 , then the optimal step 2 price is:

$$p_2^{*PS}(\bar{v}_2) = \begin{cases} \underline{v}_2 & \text{if } p_2^{PS}(\bar{v}_2) \leq \underline{v}_2 \\ p_2^{PS}(\bar{v}_2) & \text{if } \underline{v}_2 < p_2^{PS}(\bar{v}_2) \leq p_2(\bar{v}_1) \\ p_2(\bar{v}_1) & \text{if } p_2(\bar{v}_1) < p_2^{PS}(\bar{v}_2) \end{cases}$$

where $p_2^{PS}(\bar{v}_2)$ be the value of p_2 that solves $\frac{\partial \Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)}{\partial p_2} = 0$.

(iii) If $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is convex in p_2 , then the optimal step 2 price is:

$$p_2^{*PS}(\bar{v}_2) = \begin{cases} \bar{v}_2 - \epsilon & \text{if } p_2(\underline{v}_1) \geq \left(1 - \frac{(K - D_1)(\bar{v}_1 - \underline{v}_1)}{D_1(\bar{v}_2 - \underline{v}_2)}\right) \underline{v}_2 \text{ and } p_2(\bar{v}_1) > \bar{v}_2 \\ p_2(\bar{v}_1) & \text{if } p_2(\bar{v}_1) \left(D_1 + (K - D_1) \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2}\right) \geq \Pi_S^{PS}(\bar{v}_2 + \epsilon, \underline{v}_2) \text{ and } p_2(\bar{v}_1) \leq \bar{v}_2 \\ \underline{v}_2 & \text{otherwise} \end{cases}$$

where $\lim_{\epsilon \rightarrow 0} \Pi_S^{PS}(\bar{v}_2 + \epsilon, \underline{v}_2) = K\underline{v}_2 + D_1(\bar{v}_2 - \underline{v}_2) \left(\frac{\bar{v}_1 - \underline{v}_2}{\bar{v}_1 - \underline{v}_1} - \frac{\bar{v}_2 - \underline{v}_2}{\bar{v}_1 - \underline{v}_1} \frac{2D_1}{D_1 + D_2 - K}\right)$.

Proof of Result 4: (i) We substitute $p_1 = \bar{v}_2 + \epsilon$ in the seller's revenue given in (9) and uniform CDF for $F_i(\cdot)$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2) &= \left(\frac{\bar{v}_1 - p_2}{\bar{v}_1 - \underline{v}_1} - \frac{\bar{v}_2 - \underline{v}_2}{(\bar{v}_1 - \underline{v}_1)(D_1 + D_2 - K)} \right) \bar{v}_2 D_1 + \\ &\left(-\frac{\bar{v}_1 - p_2}{\bar{v}_1 - \underline{v}_1} + \frac{\bar{v}_2 - \underline{v}_2}{(\bar{v}_1 - \underline{v}_1)(D_1 + D_2 - K)} \right) p_2 D_1 + (K - D_1) p_2 \frac{\bar{v}_2 - p_2}{\bar{v}_2 - \underline{v}_2} \end{aligned}$$

The first derivative with respect to p_2 is:

$$\begin{aligned} \frac{\partial \Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)}{\partial p_2} &= \\ -\frac{\bar{v}_2 D_1}{\bar{v}_1 - \underline{v}_1} + \frac{p_2 D_1}{\bar{v}_1 - \underline{v}_1} + \left(-\frac{\bar{v}_1 - p_2}{\bar{v}_1 - \underline{v}_1} + \frac{\bar{v}_2 - \underline{v}_2}{(\bar{v}_1 - \underline{v}_1)(D_1 + D_2 - K)} \right) D_1 + (K - D_1) \frac{\bar{v}_2 - 2p_2}{\bar{v}_2 - \underline{v}_2} \end{aligned}$$

Taking the second derivative with respect to p_2 , we get:

$$\frac{\partial^2 \Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)}{\partial p_2^2} = \frac{2D_1}{\bar{v}_1 - \underline{v}_1} - \frac{2(K - D_1)}{\bar{v}_2 - \underline{v}_2}$$

Hence the function is concave if $\frac{(K - D_1)}{\bar{v}_2 - \underline{v}_2} \geq \frac{D_1}{\bar{v}_1 - \underline{v}_1}$ and concave otherwise.

(ii) Given that $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_1)$ is concave in p_2 , the optimal step 2 price is $p_2^{PS}(\bar{v}_2) = \frac{\bar{v}_2}{2} + \frac{\frac{p_2(\bar{v}_1)D_1}{\bar{v}_1 - \underline{v}_1}}{2\left[\frac{D_1}{\bar{v}_1 - \underline{v}_1} - \frac{K - D_1}{\bar{v}_2 - \underline{v}_2}\right]}$, provided that $p_2^{PS}(\bar{v}_2)$ is in the p_2 range for which a PS markdown exists.

We first show that $p_2^{PS}(\bar{v}_2)$ is less than \bar{v}_2 . Since $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is concave, the denominator of the second term in $p_2^{PS}(\bar{v}_2)$ expression above is negative, hence $p_2^{PS}(\bar{v}_2) < \frac{\bar{v}_2}{2} < \bar{v}_2$.

If $p_2^{PS}(\bar{v}_2) < \underline{v}_2$, we conclude that the revenue function is decreasing in p_2 over the entire range and \underline{v}_2 is optimal. If $p_2^{PS}(\bar{v}_2) > \underline{v}_2$, then the revenue function is increasing in p_2 at $p_2 = \underline{v}_2$, and we have to compare $p_2^{PS}(\bar{v}_2)$ with the maximum p_2 for which a PS markdown exists in order to find the optimal step 2 price. From Observation 17 we know that a PS markdown exists for all $p_2 < \min\{p_2(\bar{v}_1), \bar{v}_2\}$. If $p_2^{PS}(\bar{v}_2) < p_2(\bar{v}_1)$, then the markdown with prices $(\bar{v}_2, p_2^{PS}(\bar{v}_2))$ is optimal. If $p_2^{PS}(\bar{v}_2) \geq p_2(\bar{v}_1)$, then $p_2(\bar{v}_1)$ is optimal.

(iii) When $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ is convex in p_2 , one of the endpoints of the p_2 range will be the optimal step 2 price. If $p_2(\bar{v}_1) < \bar{v}_2$, then the p_2 range for which a PS markdown exists is $[\underline{v}_2, \bar{v}_2]$. Otherwise, the p_2 range is $[\underline{v}_2, p_2(\bar{v}_1)]$.

In each case, we identify which endpoint maximizes the seller's revenue by evaluating $\Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2)$ at each endpoint and comparing the resulting revenues.

For $p_2 \in [\underline{v}_2, \bar{v}_2)$, we show that $p_2^{*PS}(\bar{v}_2) = \bar{v}_2 - \epsilon$ if $\Pi_S^{PS}(\bar{v}_2 + \epsilon, \underline{v}_2) \leq \Pi_S^{PS}(\bar{v}_2 + \epsilon, \bar{v}_2 - \epsilon)$. By substituting the p_1 and p_2 values in (9), we get the condition for $p_2^{*PS}(\bar{v}_2) = \bar{v}_2 - \epsilon$:

$\lim_{\epsilon \rightarrow 0} \Pi_S^{PS}(\bar{v}_2 + \epsilon, \bar{v}_2 - \epsilon) = \bar{v}_2 D_1$, hence we require $p_2(\bar{v}_1) \left(D_1 + (K - D_1) \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2} \right) \geq \bar{v}_2 D_1$. Rearranging terms and simplifying we get, $p_2(\underline{v}_1) \geq \left(1 - \frac{(K - D_1)(\bar{v}_1 - \underline{v}_1)}{D_1(\bar{v}_2 - \underline{v}_2)} \right) \underline{v}_2$.

Similarly, for $p_2 \in [\underline{v}_2, p_2(\bar{v}_1)]$, we show that $p_2^{*PS}(\bar{v}_2) = p_2(\bar{v}_1)$ if $\Pi_S^{PS}(\bar{v}_2 + \epsilon, \underline{v}_2) \leq \Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2(\bar{v}_1))$.

$\lim_{\epsilon \rightarrow 0} \Pi_S^{PS}(\bar{v}_2 + \epsilon, p_2(\bar{v}_1)) = p_2(\bar{v}_1) \left(D_1 + (K - D_1) \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2} \right)$, and $\lim_{\epsilon \rightarrow 0} \Pi_S^{PS}(\bar{v}_2 + \epsilon, \underline{v}_2) = K \underline{v}_2 + D_1(\bar{v}_2 - \underline{v}_2) \left(\frac{\bar{v}_1 - \underline{v}_2}{\bar{v}_1 - \underline{v}_1} - \frac{\bar{v}_2 - \underline{v}_2}{\bar{v}_1 - \underline{v}_1} \frac{2D_1}{D_1 + D_2 - K} \right)$. Hence, we have the following as the condition for $p_2^{*PS}(\bar{v}_2) = p_2(\bar{v}_1)$.

$$p_2(\bar{v}_1) \left(D_1 + (K - D_1) \frac{\bar{v}_1 - \underline{v}_1}{\bar{v}_2 - \underline{v}_2} \right) \geq K \underline{v}_2 + D_1(\bar{v}_2 - \underline{v}_2) \left(\frac{\bar{v}_1 - \underline{v}_2}{\bar{v}_1 - \underline{v}_1} - \frac{\bar{v}_2 - \underline{v}_2}{\bar{v}_1 - \underline{v}_1} \frac{2D_1}{D_1 + D_2 - K} \right).$$

For both p_2 ranges, the other alternative optimal is to have p_2 at the lower bound, which is \underline{v}_2 , hence we get the conditions given in the result. \square

Comparing PS and TS Markdowns

Proposition 30. *When $\mathcal{P}_{TS} \neq \emptyset$,*

- (i) *the optimal markdown is PS if $\underline{v}_1 < \frac{\bar{v}_1 + \underline{v}_2}{2}$.*
- (ii) *the optimal markdown is TS if $\underline{v}_1 \geq \frac{\bar{v}_1 + \underline{v}_2}{2}$.*

Proposition 30(i) implies that it is optimal for the seller to only partially separate the high types when the range of the high types is large relative to the low valuation range. Proposition 30(ii) implies that the seller is best served by inducing all of the high types to purchase at the first price step when the 'high' and 'low' types are fairly far apart.

Proof of Proposition 30: (i) From Theorem 29 we know that for a given p_2 , the PS revenue function is maximized at $p_1^{*PS}(p_2) = p_1^{PS}(p_2)$ if $\underline{v}_1 < \frac{\bar{v}_1 + p_2}{2}$ (i.e., $p_1^{PS}(p_2) > \hat{p}_{IV}(\underline{v}_1, p_2)$) and $p_2 < \tilde{p}_2$ (i.e., $p_1^{PS}(p_2) > \bar{v}_2$). Since Proposition 30 is stated only for the case when TS is feasible, from Equation 8, we have $\bar{v}_2 < \hat{p}_{IV}(\underline{v}_1, p_2)$. Hence, the condition $\underline{v}_1 < \frac{\bar{v}_1 + p_2}{2}$ combined with Equation 8 automatically implies that $p_1^{PS}(p_2) > \bar{v}_2$ and therefore $p_1^{*PS}(p_2)$ is optimal, i.e., the PS markdown's optimal step 1 price is not at the boundaries.

Table 8: Optimal markdown and single price performance for different values of D_1 .

$D_1 = 3$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
2	2.67	42.67	SP2	none exists			5.00	2.5	45.42
3	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
4	2.88	44.08	SP2	5.58	2.29	55.42	$5.58+\epsilon$	2.29	
5	12.00	60.00	SP1	5.89	2.15	60.10	$5.89+\epsilon$	2.15	
6	12.00	72.00	SP1	6.17	2.00	65.00	$6.17+\epsilon$	2.00	
7	12.00	84.00	SP1	6.29	2.00	70.00	$6.29+\epsilon$	2.00	
8	12.00	96.00	SP1	6.38	2.00	75.00	$6.38+\epsilon$	2.00	
9	12.00	108.00	SP1	6.44	2.00	80.00	$6.44+\epsilon$	2.00	
10	12.00	120.00	SP1	6.50	2.00	85.00	$6.50+\epsilon$	2.00	
	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
20	12.00	240.00	SP1	6.75	2.00	135.00	$6.75+\epsilon$	2.00	

Since the PS revenue is concave (and the boundaries approach the TS revenue), the PS markdown dominates the TS markdown.

(ii) From Theorem 29, we know that for a given p_2 , the PS revenue function is maximized at $p_1^{*PS}(p_2) = \max\{\hat{p}_{IV}(\underline{v}_1, p_2), \bar{v}_2\} + \epsilon$ if $\underline{v}_1 \geq \frac{\bar{v}_1 + p_2}{2}$. As $\epsilon \rightarrow 0$, the markdown converges to a TS markdown, and hence the TS markdown revenue exceeds the optimal PS markdown revenue in this case. If $\underline{v}_1 \geq \frac{\bar{v}_1 + \bar{v}_2}{2}$, then $\underline{v}_1 \geq \frac{\bar{v}_1 + p_2}{2}$ for all $p_2 \in [\underline{v}_2, \bar{v}_2)$ since the right-hand-side is maximized at $p_2 = \bar{v}_2$. \square

Tabulated Results for Numerical Examples Section (Section 2.4.3)

Each table corresponds to one parameter being changed while keeping everything else the same. First column of the header identifies the parameter that is altered and the original value. Each row corresponds to a different instance and the new value of the parameter is given in the first column in that row. Whenever the PS markdown converges to a TS markdown, ϵ is used to differentiate step 1 prices of PS and TS markdowns. In this case, the corresponding PS markdown revenue is slightly less than the TS markdown revenue, hence the corresponding cell has been left blank.

Table 9: Optimal markdown and single price performance for different values of D_2 .

$D_2 = 19$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
18	2.76	43.31	SP2	none exists			none exists		
19	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
20	2.76	43.31	SP2	6.76	2.21	55.23	$6.76+\epsilon$	2.21	

Table 10: Optimal markdown and single price performance for different values of K .

$K = 20$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
19	2.78	41.26	SP2	6.19	2.78	53.17	$6.19+\epsilon$	2.78	
20	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
21	2.75	45.38	SP2	none exists			none exists		

Table 11: Optimal markdown and single price performance for different values of v_2 .

$v_2 = 2$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
1	12.00	36.00	SP1	none exists			5.00	2.50	38.44
2	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
3	3.00	60.00	SP2	6.00	3.00	69.00	$6.00+\epsilon$	3.00	
4	4.00	80.00	SP2	6.67	3.00	88.00	$6.67+\epsilon$	3.00	

Table 12: Optimal markdown and single price performance for different values of \bar{v}_2 .

$\bar{v}_2 = 5$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
3	2.00	40.00	SP2	5.33	2.00	50.00	$5.33+\epsilon$	2.00	
4	2.18	40.27	SP2	5.33	2.00	50.00	$5.33+\epsilon$	2.00	
5	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
6	3.35	47.78	SP2	none exists			6.00	3.00	51.75
7	3.94	52.81	SP2	none exists			7.00	3.00	49.80
8	4.53	58.13	SP2	none exists			none exists		

Table 13: Optimal markdown and single price performance for different values of \underline{v}_1 .

$\underline{v}_1 = 12$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
6	2.76	43.31	SP2	none exists			5.00	2.57	47.01
7	2.76	43.31	SP2	none exists			5.00	2.55	47.36
8	2.76	43.31	SP2	none exists			5.00	2.53	47.80
9	2.76	43.31	SP2	none exists			5.00	2.50	48.33
10	2.76	43.31	SP2	none exists			5.00	2.47	49.02
11	2.76	43.31	SP2	5.00	2.00	49.00	5.00	2.42	49.92
12	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
13	2.76	43.31	SP2	5.47	2.38	51.75	$5.47+\epsilon$	2.38	
14	2.76	43.31	SP2	5.78	2.34	52.63	$5.78+\epsilon$	2.34	
15	15.00	45.00	SP1	6.10	2.31	53.52	$6.10+\epsilon$	2.31	
16	16.00	48.00	SP1	6.43	2.28	54.42	$6.43+\epsilon$	2.28	
17	17.00	51.00	SP1	6.76	2.25	55.33	$6.76+\epsilon$	2.25	

Table 14: Optimal markdown and single price performance for different values of \bar{v}_1 .

$\bar{v}_1 = 18$	Single Price			TS Markdown			PS Markdown		
	p^*	Revenue	Type	p_1^{*TS}	p_2^{*TS}	Revenue	p_1^{*PS}	p_2^{*PS}	Revenue
13	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
14	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
15	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
16	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
17	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
18	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
19	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
20	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
21	2.76	43.31	SP2	5.17	2.41	50.88	$5.17+\epsilon$	2.41	
22	2.76	43.31	SP2	5.17	2.41	50.88	5.23	2.41	50.89
23	2.76	43.31	SP2	5.17	2.41	50.88	5.37	2.41	50.92
24	2.76	43.31	SP2	5.17	2.41	50.88	5.52	2.41	50.98
25	2.76	43.31	SP2	5.17	2.41	50.88	5.66	2.41	51.07
41	20.5	43.37	SP1	5.17	2.41	50.88	8.07	2.33	53.72
64	32.0	59.08	SP1	5.17	2.41	50.88	11.89	2.17	58.65

Appendix B

Proof of Theorem 15: We show that the expected surplus of customer 1 as a function of his bid at the first price step, q_{11} , is a (piecewise linear) convex function.

Customer 1 has to trade-off between bidding at steps 1 and 2. Regardless of his bid at step 1, it is optimal for customer 1 to bid all his remaining demand, $D_1 - q_{11}$, at step 2. He does not benefit from withholding any of his demand at the last price step.

In order to identify the bidding behavior of customer 1, we consider three alternative scenarios regarding how K compares to the customer demand parameters. Specifically we consider three distinct scenarios: low ($E[D_1] < K < H_2$), moderate ($H_2 \leq K < L_1 + H_2$) and high supply ($L_1 + H_2 \leq K < H_1 + H_2$). For each scenario, given D_1 , we identify his expected surplus and show that the second derivative with respect to q_{11} is positive, hence the function is convex.

Low Supply ($E[D_1] < K < H_2$)

We examine three cases depending on how D_1 compares to the supply of the seller. While presenting the expected surplus of customer 1, we will refer to the regions in Figure 5(a)(b) and (c).

Case (i): $K \leq D_1 \leq H_1$ (Regions III and V)

If the bid of customer 1 at p_1 is greater than the available supply, then there is nothing left for the second step; otherwise, the total bids exceed the amount available at step 2. Due to the random allocation rule, with probability $\frac{1}{2}$, customer 1 has the allocation priority and gets $K - q_{11}$, and with the remaining probability, he gets any leftover from the other customer (if the demand realization is in region III). In region V, there is no leftover from customer 2 since his demand exceeds the available supply.

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} \int_{L_2}^{K - q_{11}} f_2(x_2)(K - q_{11} - x_2) dx_2 \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K \\ K(v_1 - p_1) & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{K - q_{11}} f_2(x_2) dx_2 \right] & \text{if } 0 \leq q_{11} < K \\ 0 & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

	G1				L ($E[D_1] < K < H_2$)				M ($H_2 < K < L_1 + H_2$)				H ($L_1 + H_2 < K < H_1 + H_2$)			
	L_1	H_1	L_2	H_2	SP1	SP2	2-step	p_1^*	SP1	SP2	2-step	p_1^*	SP1	SP2	2-step	p_1^*
1	10	20	5	10					296.0	179.1	192.0	11.5	300.0	220.8	221.8	10.3
2	10	30	5	10									375.5	221.9	235.5	11.4
3	20	30	5	10					496.0	279.1	291.3	10.9	500.0	320.8	321.8	10.2
4	10	40	5	10									466.7	273.6	286.1	11.3
5	20	40	5	10									575.5	321.9	335.3	11.0
6	30	40	5	10					696.0	379.1	391.0	10.7	700.0	420.8	421.8	10.1
7	10	50	5	10									600.0	374.0	374.2	10.1
8	20	50	5	10									691.7	398.6	405.3	10.7
9	30	50	5	10									787.5	435.4	445.5	10.7
10	40	50	5	10					896.0	479.1	490.9	10.5	900.0	520.8	521.8	10.1
11	10	60	5	10									620.0	349.2	364.2	10.9
12	20	60	5	10									727.8	399.7	415.6	10.9
13	30	60	5	10									866.7	473.6	486.1	10.8
14	40	60	5	10									987.5	535.4	545.4	10.6
15	50	60	5	10					1096.0	579.1	590.8	10.4	1100.0	620.8	621.8	10.1

	G2				L ($E[D_1] < K < H_2$)				M ($H_2 < K < L_1 + H_2$)				H ($L_1 + H_2 < K < H_1 + H_2$)			
	L_1	H_1	L_2	H_2	SP1	SP2	2-step	p_1^*	SP1	SP2	2-step	p_1^*	SP1	SP2	2-step	p_1^*
1	10	20	5	15					296.0	179.6	201.4	12.0	300.0	233.3	234.3	10.3
2	10	30	5	15					375.5	225.7	245.5	11.8	400.0	291.7	293.6	10.4
3	20	30	5	25					496.0	279.8	324.6	11.8	500.0	391.7	393.6	10.2
4	10	40	5	15									466.7	281.9	299.0	11.5
5	20	40	5	25					575.5	327.9	362.2	11.7	600.0	449.5	449.6	10.0
6	30	40	5	35					696.0	379.9	450.3	12.0	700.0	544.4	545.7	10.1
7	10	50	5	15									600.0	395.8	396.8	10.2
8	20	50	5	25									691.7	427.8	446.2	11.1
9	30	50	5	35					787.5	447.2	493.8	11.4	800.0	590.6	592.8	10.2
10	40	50	5	45					896.0	479.9	515.7	12.1	900.0	695.8	696.8	10.1
11	10	60	5	15									620.0	359.2	379.2	11.3
12	20	60	5	25					727.8	417.9	450.2	11.6	800.0	543.0	544.6	10.2
13	30	60	5	35					866.7	493.8	535.6	11.5	900.0	599.8	599.9	10.0
14	40	60	5	45					987.5	547.9	617.4	11.4	1000.0	616.7	650.8	10.9
15	50	60	5	55					1100.0	646.7	734.7	11.6	1100.0	846.7	847.4	10.0

Figure 13: Numerical results for groups 1 and 2.

G3		K<L ₁ +L ₂																					
		L' (E[D ₁] ₁ <K<H ₂)					M' (H ₂ <K<L ₁ +H ₂)																
L ₁	H ₁	L ₂	H ₂	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *				
1	10	20	10	20	296.0	180.0	230.3	13.4				300.0	200.0	238.1	12.7	300.0	247.9	262.7	11.5	300.0	297.9	298.4	10.1
2	10	30	10	30								375.5	229.9	287.4	13.4	400.0	335.9	353.5	11.4	400.0	395.8	396.8	10.2
3	20	30	20	30	496.0	280.0	384.2	14.2	500.0	350.0	412.5	12.5				500.0	447.9	461.7	10.9	500.0	483.3	487.3	10.4
4	10	40	10	40								466.7	298.1	360.4	13.1	500.0	421.1	442.8	11.4	500.0	498.2	498.6	10.1
5	20	40	20	40	575.5	330.0	443.5	13.9				600.0	400.0	476.1	12.5	600.0	495.8	525.4	11.5	600.0	595.8	596.8	10.1
6	30	40	30	40	696.0	380.0	535.4	14.5	700.0	450.0	566.7	13.3				700.0	647.9	661.3	10.7	700.0	697.9	698.4	10.1
7	10	50	10	50								543.8	346.5	419.7	13.1	600.0	505.3	531.6	11.4	600.0	583.7	587.7	10.4
8	20	50	20	50	652.0	380.0	499.0	13.7				691.7	449.8	539.7	12.9	700.0	621.1	641.6	11.1	700.0	685.2	688.7	10.3
9	30	50	30	50	787.5	450.0	610.0	14.1	800.0	550.0	650.0	12.5				800.0	695.8	724.2	11.2	800.0	835.9	839.2	10.3
10	40	50	40	50	896.0	480.0	686.1	14.6	900.0	700.0	784.4	11.9				900.0	847.9	861.1	10.5	900.0	897.9	898.4	10.1
11	10	60	10	60								620.0	394.7	478.9	13.1	700.0	589.3	620.1	11.4	700.0	657.3	668.3	10.7
12	20	60	20	60								727.8	429.6	553.9	13.6	800.0	671.9	707.0	11.4	800.0	764.3	773.2	10.5
13	30	60	30	60	866.7	500.0	668.5	13.9				900.0	600.0	714.2	12.7	900.0	850.0	862.3	10.6	900.0	885.2	888.7	10.3
14	40	60	40	60	987.5	550.0	762.2	14.3	1000.0	650.0	806.3	13.1				1000.0	895.8	923.4	10.9	1000.0	995.8	996.8	10.1
15	50	60	50	60	1096.0	580.0	836.5	14.7	1100.0	650.0	870.0	14.0				1100.0	1047.9	1061.0	10.4	1100.0	1097.9	1098.4	10.0

G4		K<L ₁ +L ₂																					
		L' (E[D ₁] ₁ <K<H ₂)					M' (H ₂ <K<L ₁ +H ₂)																
L ₁	H ₁	L ₂	H ₂	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *				
1	10	20	15	25	296.0	180.0	250.7	14.8				300.0	250.0	288.1	12.7	300.0	297.9	312.7	11.5	300.0	333.3	337.4	10.6
2	10	30	15	35	375.5	230.0	308.9	14.2				400.0	299.5	352.3	13.0	400.0	345.8	378.3	12.2	400.0	435.9	439.4	10.5
3	20	30	25	35	496.0	280.0	401.2	14.9	500.0	350.0	443.8	13.8				500.0	450.0	481.3	11.3	500.0	495.8	509.6	10.9
4	10	40	15	45								466.7	299.8	379.5	13.7	500.0	471.1	492.8	11.4	500.0	543.8	545.2	10.2
5	20	40	25	45	575.5	330.0	462.4	14.6				600.0	450.0	526.1	12.7	600.0	585.9	602.4	11.0	600.0	635.9	639.3	10.3
6	30	40	35	45	696.0	380.0	551.4	14.9	700.0	450.0	595.8	14.2				700.0	650.0	681.5	11.1	700.0	733.3	737.2	10.3
7	10	50	15	55								543.8	349.0	437.4	13.6	600.0	555.3	581.6	11.4	600.0	621.9	629.0	10.6
8	20	50	25	55	652.0	380.0	520.0	14.3				691.7	450.0	565.7	13.4	700.0	635.2	666.8	11.4	700.0	721.1	728.2	10.5
9	30	50	35	55	787.5	450.0	630.5	14.6	800.0	550.0	683.3	13.3				800.0	699.5	745.5	11.6	800.0	835.9	839.2	10.3
10	40	50	45	55	896.0	480.0	701.5	14.9	900.0	700.0	812.5	12.5				900.0	850.0	880.7	10.8	900.0	933.3	964.7	10.5
11	10	60	15	65								620.0	397.8	495.8	13.5	700.0	639.3	670.1	11.4	700.0	689.3	705.2	10.9
12	20	60	25	65	727.8	430.0	575.1	14.0				775.0	499.9	623.4	13.5	800.0	683.7	732.0	11.8	800.0	805.3	816.5	10.7
13	30	60	35	65	866.7	500.0	689.6	14.4				900.0	650.0	764.2	12.7	900.0	871.1	891.0	10.9	900.0	921.1	921.0	10.4
14	40	60	45	65	987.5	550.0	781.4	14.7	1000.0	650.0	837.5	13.8				1000.0	899.5	943.9	11.2	1000.0	1035.9	1039.2	10.2
15	50	60	55	65	1096.0	580.0	851.5	15.0	1100.0	650.0	897.5	14.5				1100.0	1050.0	1080.2	10.7	1100.0	1133.3	1137.1	10.2

Figure 14: Numerical results for groups 3 and 4.

G5		K<L ₁ +L ₂																					
		L' (E[D ₁] ₁ <K<H ₂)					M' (H ₂ <K<L ₁ +H ₂)																
L ₁	H ₁	L ₂	H ₂	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *								
1	10	20	20	65	296.0	180.0	255.0	15.0				300.0	349.5	407.9	13.9	300.0	563.0	566.2	10.3	300.0	574.5	574.6	10.0
2	10	30	30	65	375.5	230.0	330.0	15.0				400.0	497.6	554.8	12.9	400.0	613.1	631.7	11.2	400.0	656.0	660.8	10.5
3	20	30	30	65	496.0	280.0	405.0	15.0				500.0	500.0	589.3	13.6	500.0	634.5	670.2	11.4	500.0	720.2	721.4	10.1
4	10	40	40	65	466.7	300.0	425.0	15.0				500.0	500.0	601.1	14.2	500.0	682.2	708.3	11.7	500.0	767.5	769.3	10.3
5	20	40	40	65	575.5	330.0	480.0	15.0				600.0	600.0	690.0	13.0	600.0	696.7	735.8	11.7	600.0	813.8	816.4	10.3
6	30	40	40	65	696.0	380.0	555.0	15.0				700.0	650.0	752.1	12.9	700.0	749.2	795.8	11.3	700.0	874.2	874.4	10.0
7	10	50	50	65	543.8	350.0	500.0	15.0				600.0	600.0	711.1	14.0	600.0	697.2	756.7	12.7	600.0	740.6	782.5	12.2
8	20	50	50	65	652.0	380.0	555.0	15.0				700.0	650.0	760.8	13.3	700.0	796.3	834.3	11.8	700.0	870.8	885.0	11.0
9	30	50	50	65	787.5	450.0	641.4	14.9				800.0	700.0	816.7	12.9	800.0	894.4	915.7	11.0	800.0	956.3	975.9	10.8
10	40	50	50	65	896.0	480.0	703.3	15.0				900.0	850.0	920.3	11.6	900.0	948.6	969.9	10.7	900.0	1013.9	1016.4	10.2
11	10	60	60	65	700.0	400.0	575.0	15.0				700.0	650.0	791.2	14.3	700.0	700.0	804.1	13.5	700.0	794.2	853.5	12.5
12	20	60	60	65	727.5	430.0	630.0	15.0				800.0	700.0	830.2	13.5	800.0	800.0	874.2	12.5	800.0	892.7	932.3	11.7
13	30	60	60	65	866.7	500.0	725.0	15.0				900.0	850.0	856.3	14.6	900.0	900.0	952.8	11.7	900.0	990.3	1013.9	11.1
14	40	60	60	65	987.5	550.0	800.0	15.0				1000.0	650.0	884.4	14.7	1000.0	1000.0	1035.4	11.1	1000.0	1085.4	1095.8	10.6
15	50	60	60	65	1096.0	580.0	855.0	15.0				1100.0	650.0	911.3	14.8	1100.0	1100.0	1120.7	10.6	1100.0	1145.8	1153.1	10.3

G6		K<L ₁ +L ₂																					
		L' (E[D ₁] ₁ <K<H ₂)					M' (H ₂ <K<L ₁ +H ₂)																
L ₁	H ₁	L ₂	H ₂	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *								
1	10	20	5	25								296.0	179.8	224.2	13.0	300.0	241.7	260.9	11.4	300.0	291.7	293.7	10.3
2	10	30	5	35								375.5	228.6	281.8	12.9	400.0	327.8	349.4	11.4	400.0	390.6	392.9	10.3
3	20	30	15	35	496.0	280.0	377.8	13.9				500.0	350.0	412.5	12.5	500.0	441.7	458.8	10.9	500.0	471.9	478.8	10.5
4	10	40	5	45								466.7	295.3	355.1	12.8	500.0	440.6	457.0	11.1	500.0	495.3	499.8	10.4
5	20	40	15	45	575.5	330.0	435.0	13.6				600.0	399.7	474.7	12.5	600.0	556.9	567.9	10.7	600.0	560.6	592.9	10.2
6	30	40	25	45	696.0	380.0	529.5	14.3				700.0	450.0	566.7	13.3	700.0	641.7	658.0	10.6	700.0	691.7	693.6	10.1
7	10	50	5	55								543.8	343.3	414.1	12.9	600.0	496.7	526.7	11.4	600.0	577.5	580.2	10.3
8	20	50	15	55	787.5	450.0	603.1	13.9				691.7	448.6	535.8	12.7	700.0	612.5	636.2	11.1	700.0	678.3	683.6	10.4
9	30	50	25	55	896.0	480.0	680.3	14.5				800.0	550.0	650.0	12.5	800.0	690.6	721.5	11.1	800.0	790.6	792.8	10.2
10	40	50	35	55	996.0	480.0	680.3	14.5				900.0	700.0	784.4	11.9	900.0	841.7	857.7	10.5	900.0	891.7	893.6	10.1
11	10	60	5	65								620.0	391.3	473.0	12.9	700.0	607.6	633.4	11.2	700.0	649.4	662.7	10.8
12	20	60	15	65								727.8	429.6	546.2	13.3	800.0	664.3	702.9	11.4	800.0	764.3	773.2	10.5
13	30	60	25	65	866.7	500.0	660.2	13.7				900.0	599.8	712.6	12.6	900.0	840.6	855.5	10.7	900.0	878.3	883.5	10.3
14	40	60	35	65	987.5	550.0	755.7	14.2				1000.0	650.0	806.3	13.1	1000.0	890.6	920.5	10.9	1000.0	990.6	992.8	10.1
15	50	60	45	65	1096.0	580.0	830.8	14.6				1100.0	650.0	870.0	14.0	1100.0	1041.7	1057.4	10.4	1100.0	1091.7	1093.6	10.1

Figure 15: Numerical results for groups 5 and 6.

K < L ₁ + L ₂																								
				L' (E[D ₁] < K < H ₂)				M' (H ₂ < K < L ₁ + H ₂)				L (E[D ₁] < K < H ₂)				M (H ₂ < K < L ₁ + H ₂)				H (L ₁ + H ₂ < K < L ₁ + H ₂)				
L ₁	H ₁	L ₂	H ₂	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	SP1	SP2	2-step	p ₁ *	
G1																								
1	10	60	5	10												177.6	179.1	181.7	10.3	180.0	220.8	221.0	10.1	
4	10	40	5	10																280.0	273.6	276.1	10.3	
6	30	40	5	10												417.6	379.1	381.5	10.1	420.0	420.8	421.0	10.0	
11	10	60	5	10																372.0	349.2	352.2	10.2	
13	30	60	5	10																520.0	473.6	476.1	10.2	
15	50	60	5	10												657.6	579.1	581.4	10.1	660.0	620.8	621.0	10.0	
G2																								
1	10	60	5	15												177.6	179.6	183.9	10.4	180.0	233.3	234.2	10.1	
4	10	40	5	15																280.0	281.9	285.4	10.3	
6	30	40	5	35												417.6	379.9	393.9	10.4	420.0	544.4	544.7	10.0	
11	10	60	5	15																372.0	359.2	363.2	10.3	
13	30	60	5	35												520.0	493.8	502.1	10.3	540.0	599.8	599.8	10.0	
15	50	60	5	55												660.0	646.7	664.3	11.6	660.0	846.7	846.8	10.0	
G3																								
1	10	60	10	60	177.6	180.0	190.1	10.7					180.0	200.0	207.6	10.5	180.0	247.9	250.9	10.3	180.0	297.9	298.0	10.0
4	10	40	10	40									280.0	298.1	310.6	10.6	300.0	421.1	425.4	10.3	300.0	498.2	498.2	10.0
6	30	40	30	40	417.6	380.0	411.1	14.5	420.0	450.0	473.3	10.7					420.0	647.9	650.6	10.1	420.0	697.9	698.0	10.0
11	10	60	10	60									372.0	394.7	411.5	10.6	420.0	589.3	595.4	10.3	420.0	657.3	659.5	10.1
13	30	60	30	60	520.0	500.0	533.7	10.8					540.0	600.0	714.2	12.7	540.0	850.0	852.5	10.1	540.0	885.2	885.9	10.1
15	50	60	50	60	657.6	580.0	631.3	10.9	660.0	650.0	694.0	10.8					660.0	1047.9	1050.5	10.1	660.0	1097.9	1098.0	10.0
G4																								
1	10	60	15	65	177.6	180.0	194.1	11.0					180.0	250.0	257.6	10.5	180.0	297.9	300.9	10.3	180.0	333.3	334.2	10.1
4	10	40	15	45									280.0	299.8	315.7	10.7	300.0	471.1	475.4	10.3	300.0	543.8	544.0	10.0
6	30	40	35	45	417.6	380.0	414.3	11.0	420.0	450.0	479.2	10.8					420.0	650.0	656.3	10.2	420.0	733.3	734.1	10.1
11	10	60	15	65									372.0	397.8	417.4	10.7	420.0	639.3	645.4	10.3	420.0	689.3	692.4	10.2
13	30	60	35	65	520.0	500.0	537.9	10.9					540.0	650.0	672.8	10.5	540.0	871.1	875.0	10.2	540.0	921.1	922.5	10.1
15	50	60	55	65	657.6	580.0	634.3	11.0	660.0	650.0	699.5	10.9					660.0	1050.0	1056.0	10.1	660.0	1133.3	1134.1	10.0
G5																								
1	10	60	60	65	177.6	180.0	195.0	11.0					180.0	349.5	361.2	10.8	180.0	563.0	563.6	10.1	180.0	574.5	574.6	10.1
4	10	40	40	65	280.0	300.0	325.0	11.0					300.0	500.0	520.2	10.8	300.0	682.2	687.4	10.3	300.0	767.5	767.9	10.1
6	30	40	40	65	417.6	380.0	415.0	11.0	420.0	650.0	670.4	10.6					420.0	749.2	758.5	10.3	420.0	874.2	874.2	10.0
11	10	60	60	65	372.0	400.0	435.0	11.0	420.0	650.0	678.2	10.9					420.0	700.0	720.8	10.7	420.0	794.2	806.0	10.5
13	30	60	60	65	520.0	500.0	545.0	11.0	540.0	650.0	691.2	10.9					540.0	900.0	910.6	10.3	540.0	990.3	995.0	10.2
15	50	60	60	65	657.6	580.0	635.0	11.0	660.0	650.0	702.3	11.0					660.0	1100.0	1104.1	10.1	660.0	1145.8	1147.3	10.3
G6																								
1	10	60	5	65									177.6	179.8	188.7	10.6	180.0	241.7	245.5	10.3	180.0	291.7	292.1	10.1
4	10	40	5	45									280.0	295.3	307.3	10.6	300.0	440.6	457.0	10.2	300.0	495.3	496.5	10.1
6	30	40	25	45	417.6	380.0	409.9	10.9	420.0	450.0	473.3	10.7					420.0	641.7	644.9	10.1	420.0	691.7	692.1	10.1
11	10	60	5	65									372.0	391.3	407.7	10.6	420.0	607.6	612.8	10.2	420.0	649.4	652.0	10.2
13	30	60	25	65	520.0	500.0	532.0	10.7					540.0	599.8	622.4	10.5	540.0	840.6	843.6	10.1	540.0	878.3	879.3	10.1
15	50	60	45	65	657.6	580.0	630.2	10.9	660.0	650.0	694.0	10.8					660.0	1041.7	1044.8	10.1	660.0	1091.7	1092.0	10.0

Figure 16: Numerical results with reduced customer 1 valuation.

						BASE			Decrease D_2 [L_2+5, H_2-5]			Decrease D_1 [L_1+5, H_1-5]		
	L_1	H_1	L_2	H_2	K	SP1	SP2	Markdown	SP1	SP2	Markdown	SP1	SP2	Markdown
1	5	25	10	20	18	275.5	177.9	213.6	275.5	179.8	220.3	296.0	179.8	224.2
1	5	25	10	20	25	300.0	235.9	254.7	300.0	241.7	259.0	300.0	241.7	260.9
1	5	25	10	20	30	300.0	266.7	275.6	300.0	272.9	280.0	300.0	291.7	293.7
4	5	45	10	40	30	443.8	291.7	347.2	443.8	295.3	353.8	466.7	295.3	355.1
4	5	45	10	40	50	500.0	433.3	451.7	500.0	440.6	456.4	500.0	440.6	457.0
4	5	45	10	40	70	500.0	491.7	493.7	500.0	495.3	496.4	500.0	495.3	499.8
6	25	45	30	40	38	675.5	380.0	520.3	675.5	380.0	527.3	696.0	380.0	529.5
6	25	45	30	40	45	700.0	450.0	555.0	700.0	450.0	555.0	700.0	450.0	566.7
6	25	45	30	40	65	700.0	635.9	652.1	700.0	641.7	657.2	700.0	641.7	658.0
6	25	45	30	40	75	700.0	685.9	689.3	700.0	691.7	693.6	700.0	691.7	693.6
11	5	65	10	60	40	595.8	387.5	465.4	595.8	391.3	472.0	620.0	391.3	473.0
11	5	65	10	60	70	700.0	600.0	627.9	700.0	607.6	632.7	700.0	607.6	633.4
11	5	65	10	60	80	700.0	642.1	657.4	700.0	649.4	662.4	700.0	649.4	662.7
13	25	65	30	60	50	843.8	500.0	645.3	843.8	500.0	654.7	866.7	500.0	660.2
13	25	65	30	60	60	893.8	599.0	703.6	893.8	599.8	706.2	900.0	599.8	712.6
13	25	65	30	60	90	900.0	833.3	850.2	900.0	840.6	855.4	900.0	840.6	855.5
13	25	65	30	60	100	900.0	871.9	878.7	900.0	878.3	883.5	900.0	878.3	883.5
15	45	65	50	60	58	1075.5	580.0	823.6	1075.5	580.0	829.8	1096.0	580.0	830.8
15	45	65	50	60	65	1100.0	650.0	863.9	1100.0	650.0	863.9	1100.0	650.0	870.0
15	45	65	50	60	105	1100.0	1035.9	1051.5	1100.0	1041.7	1056.7	1100.0	1041.7	1057.4
15	45	65	50	60	115	1100.0	1085.9	1089.2	1100.0	1091.7	1093.6	1100.0	1091.7	1093.6

Figure 17: Numerical results with reduced demand distribution.

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K \\ 0 & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

Case (ii): $K - L_2 \leq D_1 < K$ (Regions II and IV)

In step 2, with probability $\frac{1}{2}$, customer 1 has the allocation priority and gets $D_1 - q_{11}$, and with probability $\frac{1}{2}$, he gets the leftover from the other customer (if any). Since $K - L_2 \leq D_1$, leftover from the other customer will always be less than his own remaining demand.

$$\Pi_1(q_{11}) = q_{11}(v_1 - p_1) + \left[\frac{D_1 - q_{11}}{2} + \frac{1}{2} \left(\int_{L_2}^{K - q_{11}} f_2(x_2)(D_1 - q_{11}) dx_2 \right) \right] (v_1 - v_2)$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = (v_1 - p_1) - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2} \int_{L_2}^{K - q_{11}} f_2(x_2) dx_2 \right]$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0$$

Case (iii): $L_1 \leq D_1 < K - L_2$ (Region I)

In step 2, with probability $\frac{1}{2}$, customer 1 has the allocation priority and gets $D_1 - q_{11}$, and with probability $\frac{1}{2}$, he gets the maximum of the leftover from the other customer (if any) and his own remaining demand.

$$\begin{aligned} \Pi_1(q_{11}) = & q_{11}(v_1 - p_1) + \left[\frac{D_1 - q_{11}}{2} + \frac{1}{2} \left(\int_{L_2}^{K-D_1} f_2(x_2)(D_1 - q_{11})dx_2 \right. \right. \\ & \left. \left. + \int_{K-D_1}^{K-q_{11}} f_2(x_2)(K - q_{11} - x_2)dx_2 \right) \right] (v_1 - v_2) \end{aligned}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = (v_1 - p_1) - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2} \int_{L_2}^{K-D_1} f_2(x_2)dx_2 + \frac{1}{2} \int_{K-D_1}^{K-q_{11}} f_2(x_2)dx_2 \right]$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \frac{1}{2} f_2(K - q_{11})(v_1 - v_2) > 0$$

Next we consider the moderate supply scenario.

Moderate Supply ($H_2 \leq K < L_1 + H_2$)

The only difference from low supply is due to $H_2 \leq K$. Since $K - q_{11} > H_2$ is possible in this scenario, when $K - H_2 > q_{11}$, there will always be some leftover from customer 2 if customer does not get the allocation priority at the second price step. We calculate the expectation on demand realization of customer 2 incorporating this observation. Accordingly, in each case we have an additional expression which is valid for $0 \leq q_{11} < K - H_2$, while the expression for $K - H_2 \leq q_{11} < K$ remains as in the low supply scenario.

Case (i): $K \leq D_1 \leq H_1$ (Region III)

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} \int_{L_2}^{H_2} f_2(x_2)(K - q_{11} - x_2)dx_2 \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K - H_2 \\ q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} \int_{L_2}^{K - q_{11}} f_2(x_2)(K - q_{11} - x_2)dx_2 \right] (v_1 - v_2) & \text{if } K - H_2 \leq q_{11} < K \\ K(v_1 - p_1) & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2)\frac{1}{2} \left[1 + \int_{L_2}^{H_2} f_2(x_2) dx_2 \right] & \text{if } 0 \leq q_{11} < K - H_2 \\ (v_1 - p_1) - (v_1 - v_2)\frac{1}{2} \left[1 + \int_{L_2}^{K-q_{11}} f_2(x_2) dx_2 \right] & \text{if } K - H_2 \leq q_{11} < K \\ 0 & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K - H_2 \\ \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } K - H_2 \leq q_{11} < K \\ 0 & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

Case (ii): $K - L_2 \leq D_1 < K$ (Region II)

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{K-q_{11}}{2} + \frac{1}{2} \int_{L_2}^{H_2} f_2(x_2)(K - q_{11} - x_2) dx_2 \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K - H_2 \\ q_{11}(v_1 - p_1) + \left[\frac{K-q_{11}}{2} + \frac{1}{2} \int_{L_2}^{K-q_{11}} f_2(x_2)(K - q_{11} - x_2) dx_2 \right] (v_1 - v_2) & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2)\frac{1}{2} \left[1 + \int_{L_2}^{H_2} f_2(x_2) dx_2 \right] & \text{if } 0 \leq q_{11} < K - H_2 \\ (v_1 - p_1) - (v_1 - v_2)\frac{1}{2} \left[1 + \int_{L_2}^{K-q_{11}} f_2(x_2) dx_2 \right] & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K - H_2 \\ \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

Case (iii): $L_1 \leq D_1 < K - L_2$ (Region I)

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{D_1-q_{11}}{2} + \frac{1}{2} \left(\int_{L_2}^{K-D_1} f_2(x_2)(D_1 - q_{11}) dx_2 + \int_{K-D_1}^{H_2} f_2(x_2)(K - q_{11} - x_2) dx_2 \right) \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K - H_2 \\ q_{11}(v_1 - p_1) + \left[\frac{D_1-q_{11}}{2} + \frac{1}{2} \left(\int_{L_2}^{K-D_1} f_2(x_2)(D_1 - q_{11}) dx_2 + \int_{K-D_1}^{K-q_{11}} f_2(x_2)(K - q_{11} - x_2) dx_2 \right) \right] (v_1 - v_2) & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2)\frac{1}{2} \left[1 + \int_{L_2}^{H_2} f_2(x_2) dx_2 \right] & \text{if } 0 \leq q_{11} < K - H_2 \\ (v_1 - p_1) - (v_1 - v_2)\frac{1}{2} \left[1 + \int_{L_2}^{K-q_{11}} f_2(x_2) dx_2 \right] & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K - H_2 \\ \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

The final scenario we consider is high supply.

High Supply ($L_1 + H_2 \leq K < H_1 + H_2$)

The only difference from moderate supply is due to $L_1 + H_2 \leq K$, which allows for $D_1 < K - H_2$. In other words, the supply level is so high that the leftover from customer 2 is enough to satisfy the entire demand by customer 1. Hence we have an additional case, namely case (iv), in which customer 1 is allocated his bid quantity with certainty at both price steps.

Case (i): $K \leq D_1 \leq H_1$ (Region III)

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} \int_{L_2}^{H_2} f_2(x_2)(K - q_{11} - x_2)dx_2 \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K - H_2 \\ q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} \int_{L_2}^{K - q_{11}} f_2(x_2)(K - q_{11} - x_2)dx_2 \right] (v_1 - v_2) & \text{if } K - H_2 \leq q_{11} < K \\ K(v_1 - p_1) & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{H_2} f_2(x_2)dx_2 \right] & \text{if } 0 \leq q_{11} < K - H_2 \\ (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{K - q_{11}} f_2(x_2)dx_2 \right] & \text{if } K - H_2 \leq q_{11} < K \\ 0 & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K - H_2 \\ \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } K - H_2 \leq q_{11} < K \\ 0 & \text{if } K \leq q_{11} \leq D_1 \end{cases}$$

Case (ii): $K - L_2 \leq D_1 < K$ (Region II)

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} \int_{L_2}^{H_2} f_2(x_2)(K - q_{11} - x_2) dx_2 \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K - H_2 \\ q_{11}(v_1 - p_1) + \left[\frac{K - q_{11}}{2} + \frac{1}{2} + \int_{L_2}^{K - q_{11}} f_2(x_2)(K - q_{11} - x_2) dx_2 \right] (v_1 - v_2) & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{H_2} f_2(x_2) dx_2 \right] & \text{if } 0 \leq q_{11} < K - H_2 \\ (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{K - q_{11}} f_2(x_2) dx_2 \right] & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K - H_2 \\ \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

Case (iii): $K - H_2 \leq D_1 < K - L_2$ (Region I, $K - H_2 \leq D_1$)

$$\Pi_1(q_{11}) = \begin{cases} q_{11}(v_1 - p_1) + \left[\frac{D_1 - q_{11}}{2} + \frac{1}{2} \left(\int_{L_2}^{K - D_1} f_2(x_2)(D_1 - q_{11}) dx_2 + \int_{K - D_1}^{H_2} f_2(x_2)(K - q_{11} - x_2) dx_2 \right) \right] (v_1 - v_2) & \text{if } 0 \leq q_{11} < K - H_2 \\ q_{11}(v_1 - p_1) + \left[\frac{D_1 - q_{11}}{2} + \frac{1}{2} \left(\int_{L_2}^{K - D_1} f_2(x_2)(D_1 - q_{11}) dx_2 + \int_{K - D_1}^{K - q_{11}} f_2(x_2)(K - q_{11} - x_2) dx_2 \right) \right] (v_1 - v_2) & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = \begin{cases} (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{H_2} f_2(x_2) dx_2 \right] & \text{if } 0 \leq q_{11} < K - H_2 \\ (v_1 - p_1) - (v_1 - v_2) \frac{1}{2} \left[1 + \int_{L_2}^{K - q_{11}} f_2(x_2) dx_2 \right] & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = \begin{cases} \frac{1}{2}(v_1 - v_2) > 0 & \text{if } 0 \leq q_{11} < K - H_2 \\ \frac{1}{2}f_2(K - q_{11})(v_1 - v_2) > 0 & \text{if } K - H_2 \leq q_{11} < D_1 \end{cases}$$

Case (iv): $L_1 \leq D_1 < K - H_2$ (Region I, $K - H_2 > D_1$)

$$\Pi_1(q_{11}) = q_{11}(v_1 - p_1) + (D_1 - q_{11})(v_1 - v_2)$$

$$\frac{d\Pi_1(q_{11})}{dq_{11}} = v_2 - p_1$$

$$\frac{d^2\Pi_1(q_{11})}{dq_{11}^2} = 0$$

Since the second derivative is non-negative in all cases considered, the expected revenue is maximized at one of the extreme points (0 or D_1), i.e., all-or-nothing bidding is the dominant strategy for customer 1. \square

Proof of Proposition 17: To identify the bidding behavior of customer 1, we compare his expected surplus from bidding D_1 at step 2 to his surplus from bidding D_1 at step 1. Let Π_{1k} denote the expected surplus of customer 1 if he bids D_1 at step k .

We consider three ranges that D_1 can be in: (i) $L_1 \leq D_1 \leq K - L_2$, (ii) $K - L_2 < D_1 < K$ and (iii) $K \leq D_1 \leq H_1$.

Case (i): $L_1 \leq D_1 \leq K - L_2$

$$\begin{aligned}\Pi_{11} &= (v_1 - p_1)D_1 \\ \Pi_{12} &= (v_1 - v_2) \left[\int_{L_2}^{K-D_1} f_2(x_2)D_1 dx_2 + \int_{K-D_1}^K f_2(x_2) \left(\frac{1}{2}D_1 + \frac{1}{2}(K - x_2) \right) dx_2 \right. \\ &\quad \left. + \int_K^{H_2} f_2(x_2) \frac{1}{2}D_1 dx_2 \right] \\ \Pi_{12} &= (v_1 - v_2) \left[D_1(F_2(K - D_1) - F_2(L_2)) + \frac{D_1 + K}{2}(F_2(K) - F_2(K - D_1)) \right. \\ &\quad \left. - \frac{1}{2} \int_{K-D_1}^K F_2(x_2) dx_2 + \frac{D_1}{2}(F_2(H_2) - F_2(K)) \right] \\ \Pi_{12} &= (v_1 - v_2) \left[\frac{1}{2}D_1 + \frac{1}{2} \int_{K-D_1}^K x_2 F_2(x_2) dx_2 \right]\end{aligned}$$

Customer 1 prefers to bid D_1 at step 1 if $\Pi_{11} \geq \Pi_{12}$, i.e. $\frac{v_1 - p_1}{v_1 - v_2} \geq \left[\frac{1}{2} + \frac{1}{2D_1} \int_{K-D_1}^K x_2 F_2(x_2) dx_2 \right]$.

We get the condition on the step 1 price as $p_1 \leq w_L^i(D_1)$.

Case (ii): $K - L_2 < D_1 \leq K$

$$\begin{aligned}\Pi_{11} &= (v_1 - p_1)D_1 \\ \Pi_{12} &= (v_1 - v_2) \left[\int_{L_2}^K f_2(x_2) \left(\frac{1}{2}D_1 + \frac{1}{2}(K - x_2) \right) dx_2 + \int_K^{H_2} f_2(x_2) \frac{1}{2}D_1 dx_2 \right] \\ \Pi_{12} &= (v_1 - v_2) \left[\left(\frac{D_1}{2} + \frac{K}{2} \right) (F_2(K) - F_2(L_2)) + \frac{D_1}{2}(F_2(H_2) - F_2(K)) \right]\end{aligned}$$

$$-\frac{1}{2} \left(KF_2(K) - L_2F_2(L_2) - \int_{L_2}^K F_2(x_2)dx_2 \right) \Bigg]$$

$$\Pi_{12} = (v_1 - v_2) \left[\frac{D_1}{2} + \frac{1}{2} \int_{L_2}^K F_2(x_2)dx_2 \right]$$

Customer 1 prefers to bid D_1 at step 1 if

$\Pi_{11} \geq \Pi_{12}$, i.e. $\frac{v_1 - p_1}{v_1 - v_2} \geq \left[\frac{1}{2} + \frac{1}{2D_1} \int_{L_2}^K F_2(x_2)dx_2 \right]$. We get the condition on the step 1 price as $p_1 \leq w_L^i(D_1)$.

Case (iii): $K < D_1 \leq H_1$

$$\Pi_{11} = (v_1 - p_1)K$$

$$\Pi_{12} = (v_1 - v_2) \left[\int_{L_2}^K f_2(x_2) \left(\frac{1}{2}K + \frac{1}{2}(K - x_2) \right) dx_2 + \int_K^{H_2} f_2(x_2) \frac{1}{2}K dx_2 \right]$$

$$\begin{aligned} \Pi_{12} &= (v_1 - v_2) \left[K(F_2(K) - F_2(L_2)) + \frac{K}{2}(F_2(H_2) - F_2(K)) \right. \\ &\quad \left. - \frac{1}{2} \left(KF_2(K) - L_2F_2(L_2) - \int_{L_2}^K F_2(x_2)dx_2 \right) \right] \end{aligned}$$

$$\Pi_{12} = (v_1 - v_2) \left[\frac{K}{2} + \frac{1}{2} \int_{L_2}^K F_2(x_2)dx_2 \right]$$

Customer 1 prefers to bid D_1 at step 1 if $\Pi_{11} \geq \Pi_{12}$, i.e. $\frac{v_1 - p_1}{v_1 - v_2} \geq \left[\frac{1}{2} + \frac{1}{2K} \int_{L_2}^K F_2(x_2)dx_2 \right]$.

Rearranging terms, we get the condition on step 1 price as $p_1 < w_L^{ii}(K)$. \square

Bidding Behavior of Customer 1 in Moderate Supply ($H_2 \leq K < L_1 + H_2$)

In this level, supply is higher than the maximum demand of customer 2, so we only have to consider regions I, II and III from the original set in Table 5 as given in Figure 5(b).

The next proposition identifies the optimal bid of customer 1 for any possible realization of his demand.

Proposition 31. *In a two-step markdown mechanism with prices $p_1, p_2^* = v_2, H_2 \leq K < L_1 + H_2$ and $K \leq H_1$, customer 1 bids D_1 at step 1 (and 0 at step 2) if*

PSi. $p_1 \leq w_M^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K-H_2}{2D_1} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right]$ and $L_1 \leq D_1 \leq K - L_2$, or

PSii. $p_1 \leq w_M^{ii}(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K}{2D_1} - \frac{E[D_2]}{2D_1} \right]$ and $K - L_2 < D_1 \leq K$, or

TS. $p_1 \leq w_M^{ii}(K) = v_1 - (v_1 - v_2) \left[1 - \frac{E[D_2]}{2K} \right]$ and $K < D_1 \leq H_1$,

and bids 0 at step 1 (and D_1 at step 2) otherwise.

Proof of Proposition 31: To identify the bidding behavior of customer 1, we compare his expected surplus from bidding D_1 at step 2 to his surplus from bidding D_1 at step 1. We consider cases (i) $L_1 \leq D_1 \leq K - L_2$, (ii) $K - L_2 < D_1 < K$ and (iii) $K \leq D_1 \leq H_1$.

i) $L_1 \leq D_1 \leq K - L_2$

$$\begin{aligned} \Pi_{11} &= (v_1 - p_1)D_1 \\ \Pi_{12} &= (v_1 - v_2) \left[\int_{L_2}^{K-D_1} f_2(x_2) D_1 dx_2 + \int_{K-D_1}^{H_2} f_2(x_2) \left(\frac{1}{2} D_1 + \frac{1}{2} (K - x_2) \right) dx_2 \right] \\ &\quad \vdots \\ \Pi_{12} &= (v_1 - v_2) \left[\frac{D_1}{2} - \frac{K - H_2}{2} + \frac{1}{2} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right] \end{aligned}$$

Customer 1 prefers to bid D_1 at step 1 if $\Pi_{11} \geq \Pi_{12}$, i.e.,

$$p_1 \leq w_M^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} - \frac{K-H_2}{2D_1} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right].$$

ii) $K - L_2 < D_1 \leq K$

$$\begin{aligned} \Pi_{11} &= (v_1 - p_1)D_1 \\ \Pi_{12} &= (v_1 - v_2) \left[\int_{L_2}^{H_2} f_2(x_2) \left(\frac{1}{2} D_1 + \frac{1}{2} (K - x_2) \right) dx_2 \right] \\ &\quad \vdots \\ \Pi_{12} &= (v_1 - v_2) \left[\frac{D_1}{2} + \frac{K}{2} - \frac{1}{2} E[D_2] \right] \end{aligned}$$

Customer 1 prefers to bid D_1 at step 1 if $\Pi_{11} \geq \Pi_{12}$, i.e.,

$$p_1 \leq w_M^{ii}(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K}{2D_1} - \frac{E[D_2]}{2D_1} \right].$$

iii) $K < D_1 \leq H_1$

$$\Pi_{11} = (v_1 - p_1)K$$

$$\Pi_{12} = (v_1 - v_2) \left[\int_{L_2}^{H_2} f_2(x_2) \left(\frac{1}{2}K + \frac{1}{2}(K - x_2) \right) dx_2 \right]$$

⋮

$$\Pi_{12} = (v_1 - v_2) \left[K - \frac{1}{2}E[D_2] \right]$$

Customer 1 prefers to bid D_1 at step 1 if $\Pi_{11} \geq \Pi_{12}$, i.e., $p_1 \leq w_M(K) = v_1 - (v_1 - v_2) \left[1 - \frac{E[D_2]}{2K} \right]$.

Combining the results from all three cases, we get the necessary condition for customer 1 to bid his entire demand at step 1. \square

Here $w_M^i(\cdot)$ and $w_M^{ii}(\cdot)$ are threshold prices, which give us the highest step 1 price in the corresponding range that would induce customer 1 to bid his entire demand at step 1.

The proof follows the same idea as Proposition 17. Once again, as long as $D_1 < K$, the threshold price is a function of D_1 . If $K \leq D_1$, then the threshold price is constant due to Observation 12(c). However, compared to low supply, we observe that the allocation to customer 1 at step 2 is always positive, hence, compared to low supply, the increase in the threshold price is less steep as D_1 increases. Since the increase in D_1 does not translate into as much unmet demand as in low supply, the step 1 price that makes customer 1 indifferent increases less rapidly. This only holds up to $D_1 = K - L_2$ though. When D_1 exceeds this value, the guaranteed allocation of D_1 to customer 1 at both steps is no longer a possibility, hence the proportional changes in allocations in both steps result in similar behavior under low and moderate supply levels.

Bidding Behavior of Customer 1 in High Supply ($L_1 + H_2 < K < H_1 + H_2$)
 If $D_1 \leq K - H_2$, customer 1 will never bid at p_1 for any p_1 , since there is no scarcity at p_2 . Hence, we only need to consider the case $K - H_2 < D_1 < H_1$ in detail. Fortunately, for this range the bidding behavior of customer 1 is identical to his bidding behavior under moderate supply, hence the following corollary describes the bidding behavior.

Corollary 32. *In a two-step markdown mechanism with prices $p_1, p_2^* = v_2$ and $L_1 + H_2 < K < H_1 + H_2$, customer 1 bids D_1 at step 1 (and 0 at step 2) if*

$$PSi. p_1 \leq w_H^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K-H_2}{2D_1} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right] \text{ and } K - H_2 \leq D_1 \leq K - L_2, \text{ or}$$

PSii. $p_1 \leq w_H^{ii}(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K}{2D_1} - \frac{E[D_2]}{2D_1} \right]$ and $K - L_2 < D_1 \leq K$, or

TS. $p_1 \leq w_H^{ii}(K) = v_1 - (v_1 - v_2) \left[1 - \frac{E[D_2]}{2K} \right]$ and $K < D_1 \leq H_1$,

and bids 0 at step 1 (and D_1 at step 2) otherwise.

Proof of Proposition 18: We will evaluate the first derivative of the threshold functions with respect to D_1 in each scenario and show that they are non-negative.

$$w_L^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{K-D_1}^K F_2(x_2) dx_2 \right]$$

$$\frac{\partial w_L^i(D_1)}{\partial D_1} = (v_1 - v_2) \left(\frac{1}{2D_1^2} \int_{K-D_1}^K F_2(x_2) dx_2 + \frac{1}{2D_1} F_2(K - D_1) \right)$$

$$w_L^{ii}(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{1}{2D_1} \int_{L_2}^K F_2(x_2) dx_2 \right]$$

$$\frac{\partial w_L^{ii}(D_1)}{\partial D_1} = (v_1 - v_2) \left(\frac{1}{2D_1^2} \int_{L_2}^K F_2(x_2) dx_2 \right)$$

$$w_M^i(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K-H_2}{2D_1} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 \right]$$

$$\frac{\partial w_M^i(D_1)}{\partial D_1} = (v_1 - v_2) \left(\frac{K-H_2}{2D_1^2} + \frac{1}{2D_1} \int_{K-D_1}^{H_2} F_2(x_2) dx_2 + \frac{1}{2D_1} F_2(K - D_1) \right)$$

$$w_M^{ii}(D_1) = v_1 - (v_1 - v_2) \left[\frac{1}{2} + \frac{K}{2D_1} - \frac{E[D_2]}{2D_1} \right]$$

$$\frac{\partial w_M^{ii}(D_1)}{\partial D_1} = (v_1 - v_2) \left(\frac{K}{2D_1^2} - \frac{E[D_2]}{2D_1^2} \right)$$

$$\frac{\partial w_M^i(D_1)}{\partial D_1} \geq 0, \quad \frac{\partial w_M^{ii}(D_1)}{\partial D_1} \geq 0 \quad \text{and} \quad \frac{\partial w_L^{ii}(D_1)}{\partial D_1} \geq 0$$

follow from the observation that all the terms in the expressions are non-negative. $\frac{\partial w_L^{ii}(D_1)}{\partial D_1} \geq 0$ holds since $E[D_2] < H_2 < K$ (by definition, moderate and high supply scenarios require $H_2 < K$). \square

Proof of Proposition 23:

When p_1 is set in this range, customer 1 always prefers to bid at step 1 for any realization of his demand. This bidding behavior is dominant for all prices in this range, so the seller's revenue is maximized by setting p_1 as high as possible in this range, which is $w_L^i(L_1)$. \square

Proof of Proposition 26:

The seller's revenue resulting from (SP1), which we denote with Π_S^{SP1} is given below:

$$\Pi_S^{SP1} = \begin{cases} v_1 \left(\int_{L_1}^K x_1 f_1(x_1) dx_1 + \int_K^{H_1} K f_1(x_1) dx_1 \right) & L_1 \leq K < H_1 \\ v_1 E[D_1] & H_1 \leq K \end{cases}$$

Substituting uniform distribution for f_1 and f_2 and taking the first derivative with respect to K , L_1 , H_1 and v_1 :

$$\frac{\partial \Pi_S^{SP1}}{\partial K} = \begin{cases} v_1 \left(\frac{H_1 - K}{H_1 - L_1} \right) > 0 & L_1 \leq K < H_1 \\ 0 & H_1 \leq K \end{cases}$$

$$\frac{\partial \Pi_S^{SP1}}{\partial L_1} = \begin{cases} v_1 \frac{K^2 - L_1^2 + 2(K - L_1)(H_1 - K) + 2L_1(K - L_1)}{2(H_1 - L_1)^2} > 0 & L_1 \leq K < H_1 \\ \frac{v_1}{2} > 0 & H_1 \leq K \end{cases}$$

$$\frac{\partial \Pi_S^{SP1}}{\partial H_1} = \begin{cases} v_1 \frac{(K - L_1)^2}{2(H_1 - L_1)^2} > 0 & L_1 \leq K < H_1 \\ \frac{v_1}{2} > 0 & H_1 \leq K \end{cases}$$

$$\frac{\partial \Pi_S^{SP1}}{\partial v_1} = \begin{cases} \frac{K^2 - L^2 + 2K(H_1 - K)}{2(H_1 - L_1)} > 0 & L_1 \leq K < H_1 \\ E[D_1] > 0 & H_1 \leq K \end{cases}$$

□

Proof of Proposition 27:

The seller's revenue resulting from (SP2), which we denote with Π_S^{SP2} is given below:

$$\Pi_S^{SP2} = \begin{cases} v_2 K & 0 < K \leq L_1 + L_2 \\ v_2 ((D_1 + D_2) \cdot \text{prob}\{D_1 + D_2 < K\} \\ + K \cdot \text{prob}\{D_1 + D_2 \geq K\}) & L_1 + L_2 < K \leq H_1 + H_2 \end{cases}$$

An increase in v_2 increases the seller's revenue since all terms multiplied by v_2 are positive.

Seller's revenue either remains constant (if $L_1 + L_2 > K$), or increases with L_1, H_1, L_2 or H_2 . If $L_1 + L_2 < K \leq H_1 + H_2$, an increase in one of these terms leads to an increase in probability of $D_1 + D_2 \geq K$, conversely decreasing the probability that $D_1 + D_2 < K$. Hence, the seller sells all K units for a higher percentage of the demand realizations, collecting a higher revenue.

Seller's revenue increases with K up to $K = H_1 + H_2$ and remains constant beyond this level. It is obvious that the revenue increases with K if $K < L_1 + L_2$. For the remaining cases, assume that seller's revenue increases from K to K' .

$$\Pi_S^{SP2}(K') = v_2 \left((D_1 + D_2) \cdot \text{prob}\{D_1 + D_2 < K'\} + K' \text{prob}\{D_1 + D_2 \geq K'\} \right)$$

$$\begin{aligned}
&= v_2 \left((D_1 + D_2) \cdot \text{prob}\{D_1 + D_2 < K\} + (D_1 + D_2) \text{prob}\{K < D_1 + D_2 < K'\} \right. \\
&\quad \left. + K' \cdot \text{prob}\{D_1 + D_2 \geq K'\} \right)
\end{aligned}$$

Since $D_1 + D_2 > K$ and $K' > K$ holds for the second and third terms above, we conclude that an increase in K (up to $K = H_1 + H_2$) leads to a strictly greater expected revenue for the seller.

□

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