# Counting sets of integers, no $k$ of which sum to another

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Proposed Running Head: Counting sets of integers

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## Abstract

We show that for every $k \geq 3$ the number of subsets of $\{1, 2, \ldots, n\}$ containing no solution to $x_1 + x_2 + \ldots + x_k = y$, where the $x_i$ need not be distinct, is at most $c2^{\alpha n}$, where $\alpha = (k-1)/k$.

A set $S$ of positive integers is *sum-free* if $S$ contains no $x$, $y$ and $z$ (not necessarily distinct) such that $x + y = z$. Cameron and Erdös have shown [3] that the number of sum-free sets contained in $\{\frac{1}{3}n, \frac{1}{3}n + 1, \ldots, n\}$ is $c2^{\frac{n}{2}}$, and Alon [1], Calkin [2] and Erdös and Granville (personal communication) have independently shown that the number of sum-free sets contained in $\{1, 2, \ldots, n\}$ is $o(2^{n(\frac{1}{2}+\epsilon)})$ for every $\epsilon > 0$. Erdös has asked (personal communication) if the number of sets contained in $\{1, 2, \ldots, n\}$ without a solution to $x + y + z = t$ is $c2^{\frac{2n}{3}}$. In this paper, we answer this question in the affirmative and show more generally that the number of sets contained in $\{1, 2, \ldots, n\}$ with no solution to $x_1 + x_2 + \ldots + x_k = y$ (with the $x_i$ not necessarily distinct) is at most $c2^{\alpha n}$, where $\alpha = (k-1)/k$ and $k \geq 3$ . (Note that $k = 2$ corresponds to the sum-free case mentioned above. It is interesting that we get a stronger result for $k \geq 3$ than for $k = 2$, and we shall later show where the method used here fails for $k = 2$.) We know this number must be at least $c2^{\alpha n}$, since if a set $S$ has all its elements in $[n - \alpha n, n]$, then the sum of any $k$ elements of $S$ will be greater than $n$. Hence all $2^{\alpha n}$ subsets of $[n - \alpha n + 1, n]$ will be included in this number.

In what follows, we will define

$$(*)\text{-free to mean having no solution to } \textstyle\sum_{i=1}^{k} x_i = y$$

$$\mathcal{F}_n = \text{ the set of } (*)\text{-free sets in } \{1, 2, \ldots, n\}$$

$$f_n = |\mathcal{F}_n|$$

$$g_n = \text{the number of } (*)\text{-free sets in } \{1, 2, \ldots, n\} \text{ which contain less than } \epsilon q \text{ elements greater than } n - q$$

$$h_n = \text{the number of } (*)\text{-free sets in } \{1, 2, \ldots, n\} \text{ which contain at least } \epsilon q \text{ elements greater than } n - q$$

$$h_{n,l} = \text{the number of } (*)\text{-free sets in } \{1, 2, \ldots, n\} \text{ which contain at least } \epsilon q \text{ elements greater than } n - q \text{ and which have least element } l$$

**Theorem 1** *Fix $k \geq 3$, and let $\alpha = (k-1)/k$. There exists a constant $c$ such that the number of subsets of $\{1, 2, \ldots, n\}$ containing no solution to*

$$\sum_{i=1}^{k} x_i = y$$

*is at most $c2^{\alpha n}$.*

**Proof** The proof will be along the following lines: we shall split $\mathcal{F}_n$ into several parts, where each part will be determined by the number of elements each set has in $[n-q+1, n]$ and by the size of its least element $l$. The reason we consider the size of the least element in a set is that any set which contains many small elements (in relation to $n$) cannot contain many medium or large elements, and a set with many medium elements cannot contain many large elements. Hence, the $(*)$-free sets of greatest cardinality will be those with a large least element $l$. Each subset of a $(*)$-free set is clearly $(*)$-free, so most of $\mathcal{F}_n$ will be those sets with many elements in $[n-q+1, n]$ and a large least element $l$.

But first we must choose $\epsilon$ and $q$ in an appropriate way. We will pick $d$ such that $d > \frac{1}{2\alpha - 1}$ and then choose $\epsilon$ and $q$ such that

$$\binom{q}{\epsilon q} \epsilon q < \frac{1}{2} 2^{\alpha q}$$

and such that any set of $\epsilon q$ elements in $\{1, 2, \ldots, q\}$ contains an arithmetic progression of length at least $2d+1$. We are guaranteed the ability to do this by [4].

We shall first consider the sets which have density less than $\epsilon$ in the largest $q$ elements of $\{1, 2, \ldots, n\}$; that is, they have less than $\epsilon q$ elements in $[n-q+1, n]$. The number of ways to get less than $\epsilon q$ elements in $[n-q+1, n]$ is less than

$$\binom{q}{\epsilon q} \epsilon q$$

and this is less than $\frac{1}{2} 2^{\alpha q}$, by our choice of $\epsilon$ and $q$. We multiply this by the number of $(*)$-free sets in $\{1, 2, \ldots, n-q\}$ and we see that the number $g_n$ of $(*)$-free sets in $\{1, 2, \ldots, n\}$ having fewer than $\epsilon q$ elements in $[n-q+1, n]$ is at most

$$\binom{q}{\epsilon q} \epsilon q f_{n-q} < \frac{1}{2} 2^{\alpha q} f_{n-q}.$$

We shall now prove that the number of sets in $\mathcal{F}_n$ having at least $\epsilon q$ elements in $[n-q+1, n]$ is at most $c 2^{\alpha n}$, where $c$ is independent of $n$, and the result will then follow by induction. First we shall state two lemmas due to Calkin [2].

**Lemma 1** *The number of binary sequences of length $b$ without any pairs of 1s at distance exactly* $1, 3, 5, 7, \ldots, 2d-1$, *is at most* $2^{\frac{d+1}{2d}(b+2d)}$.

**Proof** The number of sequences of length $2d$ without pairs of 1s at an odd distance is exactly $2^{d+1} - 1$. Thus the number of sequences of length $b$ without pairs of 1s at an odd distance less than $2d$ is at most

$$(2^{d+1} - 1)^{\lceil \frac{b}{2d} \rceil} < (2^{d+1})^{\frac{b}{2d} + 1} = 2^{\frac{d+1}{2d}(b+2d)}$$

as required.

**Lemma 2** *Given an arithmetic progression* $b-da, b-(d-1)a, \ldots, b+da$*, the number of subsets of* $\{1, 2, \ldots, b-1\}$ *having no pairs* $x, y$ *such that* $x+y$ *is an element of the progression, is at most*

$$2^{\frac{d+1}{2d}(b+a(2d+1))}.$$

**Proof** Write the elements of $\{1, 2, \ldots, b-1\}$ in the following $a$ sequences:

$$A_1 = \{1, b-1, 1+a, b-1-a, 1+2a, b-1-2a, \ldots\},$$

$$A_2 = \{2, b-2, 2+a, b-2-a, 2+2a, b-2-2a, \ldots\},$$

$$\vdots$$

$$A_a = \{a, b-a, 2a, b-2a, 3a, b-3a, \ldots\},$$

where each sequence has either $\lceil \frac{b}{a} \rceil$ or $\lfloor \frac{b}{a} \rfloor$ elements, and every element of $\{1, 2, \ldots, b\}$ occurs in exactly one such sequence. Then, for any set $S$ which has no pair of elements summing to a member of the arithmetic progression, the characteristic sequence of $S$ is such that when written as $a$ binary sequences in the order given by $A_1, \ldots, A_a$, each of these binary sequences has the property that there are no 1s at distance exactly 1, 3, 5, 7, …, $2d-1$. The number of ways of choosing such a set S is thus at most the number of ways of choosing $a$ sequences of length $\frac{b}{a}+1$, without 1s at an odd distance less than $2d$. This is at most

$$2^{\frac{d+1}{2d}(\frac{b}{a}+1+2d)a} = 2^{\frac{d+1}{2d}(b+a(2d+1))}$$

as desired.

Now we shall place an upper bound on $h_n$.

**Lemma 3** *The number* $h_n$ *of* $(*)$*-free sets in* $\{1, 2, \ldots, n\}$ *which contain at least* $\epsilon q$ *elements greater than* $n-q$ *is less than* $2^{q+1}2^{\alpha n} + 2^{\alpha n}$.

**Proof** If a set has $l > \frac{n}{k}$, then the set is clearly $(*)$-free. Then any element of $[l, n]$ can be in the set, hence the number of sets with $l > \frac{n}{k}$ is

$$2^{n-\frac{n}{k}} = 2^{\alpha n}.$$

Now we shall consider the more interesting case where a set has $l \leq \frac{n}{k}$. We have an arithmetic progression $t-da, t-(d-1)a, \ldots, t, t+a, \ldots, t+da$, and least element $l$ in our set $S$. Let $\mathcal{K}_l$ be the family of sets with least element $l$. Then $|\mathcal{K}_l|$ is less than the number of subsets of $[1, n]$ with no solution to $x_1 + x_2 + (k-2)l = y$. Now write $x_1$ as $z_1 + l$ and $x_2$ as $z_2 + l$. Next we count the number of subsets of $[0, n-l]$ with no solution to

$$\begin{gathered}
z_1 + z_2 = t - da - kl \\
z_1 + z_2 = t - (d-1)a - kl \\
\vdots \\
z_1 + z_2 = t + da - kl
\end{gathered}$$

An upper bound for this is

$$2^{\frac{d+1}{2d}(t-kl+1+a(2d+1)}2^{(n-l)-(t-kl)+1}$$

(where the first term is obtained as in Lemma 2 and the second term allows all combinations of elements of $[(n-l)-(t-kl), n-l]$ to be chosen)

$$\begin{aligned}
&= 2^{\frac{d+1}{2d}(t-kl+1+a(2d+1))}2^{(n-t)+1}2^{(k-1)l} \\
&= 2^{\frac{d+1}{2d}(n-kl-(n-t-ad)+a(d+1)+1)}2^{(n-t)+1}2^{(k-1)l} \\
&\leq 2^{\frac{d+1}{2d}(n-kl)+\frac{(d+1)^2}{2d}a+\frac{d+1}{2d}}2^{(n-t)+1}2^{(k-1)l} \\
&= 2^{\frac{d+1}{2d}(n-kl)+\frac{da}{2}+a+\frac{a}{2d}+\frac{d+1}{2d}}2^{(n-t)+1}2^{(k-1)l} \\
&\leq 2^{\frac{d+1}{2d}(n-kl)+q}2^{(n-t)+1}2^{(k-1)l}
\end{aligned}$$

(since $t \in [n-q+1, n]$.) This is the point at which the difference between the cases of $k = 2$ and $k \geq 3$ arises. (We need $2^{\frac{d+1}{2d}n} < 2^{\alpha n}$, but if $k = 2$ this cannot happen since we have $2^{\alpha n} = 2^{\frac{1}{2}}$. ) Then, summing over $l$ from 1 to $\frac{n}{k}$, we find the number of $(*)$-free sets with least element $l \leq \frac{n}{k}$ is

$$\begin{aligned}
&2^q 2^{\frac{d+1}{2d}n}\frac{1-2^{-\frac{d+1}{2d}(n+k)}}{1-2^{-\frac{d+1}{2d}k}} \\
&\leq 2^q 2^{\alpha n} 2 \\
&= 2^{q+1}2^{\alpha n}.
\end{aligned}$$

So we have that $h_n < 2^{q+1}2^{\alpha n} + 2^{\alpha n}$ ▮

Next we shall show that we may choose $c$ independent of $n$. We know

$$f_n \leq g_n + h_n < \frac{1}{2}2^{\alpha q}f_{n-q} + 2^{q+1}2^{\alpha n} + 2^{\alpha n}$$

so let $c = 2^{q+3}$. Then if $n \leq q$,

$$f_n < c2^{\alpha n}.$$

Assume $f_r < c2^{\alpha r}$ for $r < n$. Then

$$\begin{aligned}
f_n &< (\frac{3c}{4}+1)2^{\alpha n} \\
&< c2^{\alpha n}
\end{aligned}$$

as desired ▮

## References

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