PARTIALLY HYPERBOLIC SETS FROM A CO-DIMENSION ONE BIFURCATION

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ABSTRACT. We study the saddle-node bifurcation of a partially hyperbolic fixed point in a Lipschitz family of C^k diffeomorphisms on a Banach manifold (possibly infinite dimensional) in the case that the fixed point is a saddle along hyperbolic directions and has multiple curves of homoclinic orbits. We show that this bifurcation results in an invariant set which consists of a countable collection of closed invariant curves and an uncountable collection of nonclosed invariant curves which are the topological limits of the closed curves. In addition, it is shown that these curves are C^k-smooth and that this invariant set is uniformly partially hyperbolic.

1. INTRODUCTION

We study the bifurcation of a partially hyperbolic fixed point in a Lipschitz family of C^k-diffeomorphisms of an infinite dimensional manifold with a finite set $\{\Gamma_0^j\}_{j=1}^n$ of one-dimensional invariant curves of noncritical homoclinic orbits. Specifically, we will study the case where the fixed point has one simple eigenvalue equal to 1 and is a saddle along hyperbolic directions. We show that the bifurcation results in the birth of countably many closed invariant curves winding around the original curves, as well as an uncountable collection of smooth nonclosed curves. It is also shown that these curves are of class C^k and that their union forms a uniformly partially hyperbolic set (UPH-set). A bifurcation of this type was first investigated for flows in the thesis of Afraimovich [Af74]. A similar result to the present one was mentioned without proof in [AS82] in the finite dimensional case for flows.

This bifurcation can occur in the Poincaré map of a limit cycle with characteristic multiplier 1, such as in the coalesence of two limit cycles of saddle type [NPT] [AAIS]. Another possible application is to periodically or quasi-periodically perturbed differential equations. Recently, saddle-node bifurcations have attracted attention in applications because of thier role in intermittancy (See [Ta89]). For bifurcations values close to a sadle-node orbits stay close to the fixed point (or periodic trajectories) for relatively long periods of time and make an unexpected excursion away from the fixed point. Such phenomena perhaps plays a role in the onset of chaos or turbulence [PM].

In the case that the fixed point is a stable node along hyperbolic directions, there can be no more than 1 invariant curve of homoclinic orbits, because the

unstable set of the fixed point is 1 dimensional. For this case, Afraimovich and Shil'nikov [AS74] proved that if the diffeomorphism is sufficiently smooth, there exists a unique, Lipschitz (in fact C^1) invariant curve homeomorphic to a circle after the fixed point disappears. Young [Yo] proved C^k-smoothness of the invariant curve for a general C^k diffeomorphism of an infinite dimensional manifold, also for a semi-stable fixed point.

The case of one critical homoclinic curve was investigated in [Af74], [NPT], and [AAIS]. By critical we mean that the curve has tangencies to the stable foliation of the stable set. It was shown in the stable case that the resulting attractor contains dynamics conjugate to a topological Markov chain or subshift of finite type..

The result that the invariant set is a UPH-set is important from the standpoint of ergodic theory. It implies the existence of invariant Gibb's s-measures and u-measures on the set [PS], [BP]. It seems that this is the first nontrivial example of a one-dimensional UPH-set which arises from a Morse-Smale system via a codimension one bifurcation.

For this bifurcation the case $k = 2$ (or $k = 1$ with bounded variation of the derivative) is of special importance due to the Denjoy Theory [De, Ni] for diffeomorphisms of the circle, S^1. If it is known that a resulting invariant closed curve is C^2-smooth, then the restriction of the map to the curve is a C^2 map as well. This would imply that either trajectories on the invariant curve are everywhere dense or a 'resonance picture' occurs and there exist stable and unstable periodic points on this curve. This behavior is completely determined by the so called 'rotation number' of the map. In the case that the map is merely C^1 the picture can be much more complicated.

We note that in any case this bifurcation is "inaccessible". For values of the bifurcation parameter greater than the bifurcation value, the bifurcation surface cannot be approached along a path of structurally stable systems. This is because the rotation numbers of the invariant circles must approach zero, passing through infinitely many rational and irrational numbers as the parameter approaches the bifurcation value.

The main idea of the proof will be to subdivide the phase space into parts; one in a neighborhood of the disappearing fixed point and others in neighborhoods of the homoclinic curves [Sh]. The contraction mapping theorem is then applied to appropriate function spaces on certain 'fundamental regions' within the neighborhood of the fixed point. Previous applications of this method, which was developed by Afraimovich and Shil'nikov [AS74], depend heavily on the existence of an invariant foliation transversal to the center subspace of the fixed point to construct the fundamental regions. In the hyperbolic case such a foliation does not exist and we are forced to take advantage of the foliations on the center-stable and center-unstable manifolds. In addition, the problem is further complicated by the presence of multiple curves of homoclinic orbits and this has to be taken care of by multiple changes of variables. For the proof of uniform partial hyperbolicity we show that each curve is k-normally hyperbolic with uniform constants [HPS]. The invariant subbundles required for this property are constructed using the method of fundamental regions with spaces of 1-jets of curves. As in Chow and Lu [CL], we will employ two simple but powerful lemmas to construct the invariant curves and to show smoothness.

2. Results

Consider the following hypotheses.

Hyp. 1. *T_μ is a one-parameter family of C^k-diffeomorphisms of a manifold M, modeled on a Banach space X, which is Lipschitz continuous in μ in the C^k topology, $\mu \in [-\mu_0, \mu_0]$. If $dim(M) = n$ is finite, then we require $n \geq 3$.*

Hyp. 2. *For $\mu = 0$, T_μ has a nonhyperbolic fixed point $\bar{0}$ which is a uniform saddle along hyperbolic directions, that is, $DT_0(\bar{0})$ has spectrum with modulus uniformly bounded away from one, with the exception of one simple eigenvalue equal to one.*

With assumptions Hyp.1,2, on some neighborhood U of $\bar{0}$ and for μ sufficiently small, there are C^k-smooth coordinates (x, y, z), $x \in X_1$ $y \in X_2$, and $z \in R$, where $X = X_1 \oplus X_2 \oplus R$, for which T_μ has the local form Φ_μ given by:

$$
\begin{aligned}
\bar{x} &= Ax + f(x, y, z, \mu) & = \; Ax + \tilde{f}(x, y, z, \mu)x \\
\bar{y} &= By + g(x, y, z, \mu) & = \; By + \tilde{g}(x, y, z, \mu)y \\
\bar{z} &= z + R(z, \mu) + h(x, y, z, \mu) & = \; z + R(z, \mu) + h(x, y, z, \mu)
\end{aligned}
\tag{1}
$$

where the spectrum of A and B^{-1} satisfy

$$
\begin{aligned}
|\sigma(A)| &< q_0 \\
|\sigma(B^{-1})| &< q_0,
\end{aligned}
$$

for some $q_0 < 1$. Without loss of generality, we will assume hereout that

$$
\begin{aligned}
|A| &< q_1 \\
|B^{-1}| &< q_1,
\end{aligned}
$$

for $q_1 < 1$. The functions f, g and h are C^k-smooth in (x, y, z), and are Lipschitz continuous in μ. The functions f, g, h and their derivatives vanish for $(x, y, z, \mu) = (0, 0, 0, 0)$ and the functions $\tilde{f}$, $\tilde{g}$ and vanish for $(0, 0, 0, 0)$, and $h(0, 0, z, \mu) \equiv 0$. Further, all the partial derivatives of f with respect to y and z up to order k vanish identically for $x = 0$ and so for $j \leq k - 1$

$$
D^j_{(y,z)} f(x, y, z, \mu) = D^j_{(y,z)} \tilde{f}(x, y, z, \mu)x,
\tag{2}
$$

and

$$
\lim_{x \to 0} D^k_{(y,z)} f(x, y, z, \mu) = 0.
\tag{3}
$$

Similar equalities hold for $g(x, y, z, \mu)$. This form is obtained by first changing variables to rectify the local center stable manifold and then rectifying the center unstable manifold.

We note that a stronger form may be obtained in the finite case, with the restriction of C^∞-smoothness of the diffeomorphism and certain nonresonance conditions [Ta71].

We will use repeatedly the fact that for maps of the form (1), the planes $x = 0$ and $y = 0$ contain invariant foliations given by

$$
\begin{aligned}
z &= z_0 + \phi(x, z_0)x, \\
z &= z_0 + \psi(y, z_0)y,
\end{aligned}
\tag{4}
$$

respectively. Here $\phi \cdot x$ and $\psi \cdot y$ are both C^k functions [Af90] [Yo].

Hyp. 3. *For $\mu > 0$, T_μ has no fixed point in a neighborhood of 0.*

From hyp. 3 we can assume without loss of generality that R in (1) has the following form

$$(5) \qquad\qquad R(z, \mu) = \alpha(\mu) + \gamma(\mu)z^2 + o(z^2),$$

with $\alpha(0) = 0$, $\alpha'(0) \geq 0$, $\gamma(0) > 0$, and $\alpha(\mu) > 0$ for $\mu > 0$.

Hyp. 4. *The unstable set $S_{\overline{0}}^u$ of $\overline{0}$ for $\mu = 0$ intersects the stable set $S_{\overline{0}}^s$ transversally in a finite set of 1-dimensional manifolds $\{\Gamma_0^j\}_{j=1}^n$ of orbits which approach $\overline{0}$ along the nonhyperbolic direction in forward and backward time.*

By the stable set $S_{\overline{0}}^s$ we mean the set of all points whose forward orbits limit to $\overline{0}$. The unstable set $S_{\overline{0}}^u$ is the set of all points whose backward orbits limit to $\overline{0}$. It is known that these sets are in fact C^k-manifolds with boundary [NPT].

It is clear from (1) that for $z < 0$ each invariant curve Γ_0^j must be contained in the local stable set $S_0^s = \{(x, y, z) : y = 0\}$ and for $z > 0$ it must be contained in the local unstable set $S_0^u = \{(x, y, z) : x = 0\}$, which are foliated by (4).

Hyp. 5. *The curves Γ_0^j are nowhere tangent to any leaf of the invariant foliations of the local stable or unstable sets.*

With these assumptions we can state the first result. Let Ω be the set of all bi-infinite sequences on n elements.

Theorem 1. *Under hypotheses Hyp. 1-5 there exists $\mu_0 > 0$ such that for positive μ less than μ_0 and for each $\bar{\omega} \in \Omega$ there exists a C^k-smooth invariant curve $\Gamma_\mu^{\bar{\omega}}$ in a neighborhood of $\Lambda_0 = \cup_{j=1}^n \Gamma_0^j$ which winds around the curves $\{\Gamma_0^j\}$ in the order of $\bar{\omega}$. Moreover, if $\bar{\omega}$ is periodic, then $\Gamma_\mu^{\bar{\omega}}$ is closed.*

We will make the term 'winds around' precise in the proof, but the meaning is the obvious one.

In addition to the existence and smoothness of $\Gamma_\mu^{\bar{\omega}}$ we will also show that $\Lambda_\mu = \cup_{\bar{\omega} \in \Omega} \Gamma_\mu^{\bar{\omega}}$ it is unique in some sense and we shall also describe its structure in terms of the 'suspension' of Ω.

Definition 1. *(See for example [Bo].) Let Ω be the set of all bi-infinite sequences on n elements. The 'suspension' of Ω over $I = [0, 1]$, denoted by $\bar{\Omega}$, is the quotient space $\Omega \times I / \sim$, where $\sim$ is the equivalence relation $(\bar{\omega}_1, 1) \sim (\bar{\omega}_2, 0)$ if $\sigma(\bar{\omega}_1) = \bar{\omega}_2$ where σ is the left shift map. We give $\bar{\Omega}$ the topology induced by the metric on $\Omega \times I$ given by*

$$d((\bar{\omega}_1, a), (\bar{\omega}_2, b)) = \sum_{i \in Z} \frac{|\omega_1^i - \omega_2^i|}{2^{|i|}} + |a - b|.$$

Definition 2. *Given an open set U and diffeomorphism T, the 'invariant set' of U is the set of all points p such that $T^i(p) \in U$ for all integers $i \in Z$.*

Theorem 2. *Given any neighborhood U of $\cup_{i=1}^n \Gamma_0^i$ there exists $\mu_0 > 0$ and a neighborhood V of $\Lambda_0 = \cup_{i=1}^n \Gamma_0^i$, $V \subset U$ such that for all $\mu_0 > \mu > 0$ the invariant set of V is $\Lambda_\mu = \cup_{\bar{\omega} \in \Omega} \Gamma_\mu^{\bar{\omega}}$. Moreover, Λ_μ is homeomorphic to $\bar{\Omega}$.*

Finally we will show that Λ_μ is a uniformly partially hyperbolic set. The following definition is the usual one for finite dimensional manfolds.

Definition 3. *An invariant set S of a diffeomorphism f is said to be a uniform partially hyperbolic set in the narrow sense if for each trajectory $\{f^n(s)\}$ in S the tangent bundle $T_{f^n(s)}M$ has an invariant splitting*

$$T_{f^n(s)}M = E^s(f^n(s)) \oplus E^0(f^n(s)) \oplus E^u(f^n(s))$$

and constants C, $\lambda > 1$ and $\nu < 1$ such that for all integers $n \geq 1$

$$|Df^n v| \leq C\nu^n |v|, \qquad\qquad v \in E^s(f^n(s))$$

$$|Df^n v| \geq C^{-1}\lambda^n |v|, \qquad\qquad v \in E^u(f^n(s))$$

$$C^{-1}\nu^n |v| \leq |Df^n v| \leq C\nu^n |v|, \qquad v \in E^0(f^n(s))$$

$$\gamma(f^n(s)) \geq constant,$$

where $\gamma(f^n(s))$ is the sum of the angles between the subspaces.

The condition on the angles is satisfied trivially if M is compact. We will drop this condition from the above defintion and use the defintion for diffeomorphisms on any infinite-dimensional manifold with a Finsler structure on the tangent bundle. This is justified since we are only interested in a neighborhood of a compact set and any comparable condition is trivially satisfied.

Since we know that Λ_μ is the union of smooth manifolds, partial hyperbolicity on Λ_μ is is implied by normal hyperbolicity on each $\Gamma_\mu^{\bar\omega}$, with constants which are uniform in $\bar\omega$. We will use here the notion of eventual, absolute, normal hyperbolicity.

Definition 4. *A diffeomorphism f is called eventually, absolutely k-normally hyperbolic at V if $f \in C^k$ and $T_V M$, the tangent bundle of M over V, has an invariant splitting*

$$T_V M = N^u \oplus TV \oplus N^s$$

and a Finsler structure on TM such that for all $p \in V$, $0 \leq l \leq k$

$$(6) \qquad \begin{aligned} \inf_{p \in V} m(Df^n|_{N_p^u}) &> \frac{\lambda^n}{C} \sup_{p \in V} |Df^n|_{T_p V}|^l \\ \sup_{p \in V} |Df^n|_{N_p^s}| &< C\nu^n \inf_{p \in V} m(Df^n|_{T_p V})^l \end{aligned}$$

for some constants $0 < \nu < 1 < \lambda < \infty$.

Here the minimum norm $m(A)$ for a linear transformation A is defined to be

$$m(A) = \inf\{|Ax| : |x| = 1\}.$$

It can be shown that this definition is equivalent to other definitions of absolute normal hyperbolicity [HPS], and so we omit the adjective 'eventual'. Again, since we will work in a neighborhood of a compact set, we may take the Finsler structure to be the one inherited from the local coordinates.

Theorem 3. *For μ sufficiently small each $\Gamma_\mu^{\bar\omega}$ is absolutely k-normally hyperbolic, with constants which do not depend on $\bar\omega$. In particular, Λ_μ is a uniformly partially hyperbolic set.*

3. Fundamental Regions

We begin by noting that $\Gamma_0^j \setminus \{\bar{0}\}$ is the transversal intersection of $S_{\bar{0}}^u$ and $S_{\bar{0}}^s$, which are known to be C^k. Therefore $\Gamma_0^j \setminus \{\bar{0}\}$ is C^k. Let $(\xi_j(z), \rho_j(z))$ be the local C^k coordinates of Γ_0^j. Note from (1) that $\xi_j(z) \equiv 0$ for $z > 0$ and $\rho_j(z) \equiv 0$ for $z < 0$. We can use the methods in [Yo] to show that

$$\lim_{z \to 0} \xi_j^{(l)}(z) = 0$$
$$\lim_{z \to 0} \rho_j^{(l)}(z) = 0,$$

for all $0 \leq l \leq k$. This implies that ξ_j and ρ_j are C^k at 0 [Ru, p.115]. In fact, the graph of $(\xi_i(z, \mu), \rho_j(z, \mu))$ is also C^k-smooth. This graph is Γ_0^i for $z \leq 0$ and Γ_0^j for $z > 0$. Thus it is a C^k-smooth local center manifold, which we will denote by Γ_0^{ij}. Note also that the planes $x = 0$ and $y = 0$ are invariant under (1).

The parameterized map $\bar{\Phi}$ of Φ_μ is the map from $X \times [-\mu_0, \mu_0]$ into itself given by $\bar{\Phi} : (m, \mu) \mapsto (\Phi_\mu(m), \mu)$, where $m \in X$. Consider the following proposition.

Proposition 1. *Suppose that Φ_μ, $-\mu_0 \leq \mu \leq \mu_0$, is a Lipschitz family of C^k diffeomorphisms defined locally on a Banach space by (1). Then the parameterized map $\bar{\Phi}$ given by $\bar{\Phi} : (m, \mu) \mapsto (\Phi_\mu(m), \mu)$ has a C^k-smooth local center unstable manifold W^{cu}, which contains the half plane $\{(0, y, z, \mu) : z \geq 0, \mu = 0\}$ as well as $\Gamma_0^j \cap X^-$, where X^- is the half space with $z < 0$. Similarly, there is a local center stable manifold W^{cs} which contains $\Gamma_0^j \cap X^+$ and $\{(x, 0, z, 0) : z \leq 0\}$.*

Outline of the proof: The proof is in two steps. First for $\mu = 0$, and for fixed j, we show that there exists a local center unstable manifold W_0^{cu} which contains $\{(0, y, z) : z \geq 0\}$ as well as $\Gamma_0^j \cap X^-$. This is accomplished by using a cutoff function outside of a neighborhood of $\bar{0}$ and repeating the steps of the proof of the center unstable manifold theorem, but including the invariant sets as boundary conditions in the function spaces. The functional map induced by the diffeomorphism preserves these boundary conditions and the resulting fixed point must contain the invariant sets as a subset.

The second step is to show that there is a local center unstable manifold W^{cu} of the parameterized system which extends W_0^{cu}. This follows by considering the parameterized system with the same cutoff function as above. The uniqueness implies that the resulting center unstable manifold must contain W^{cu} as a submanifold.

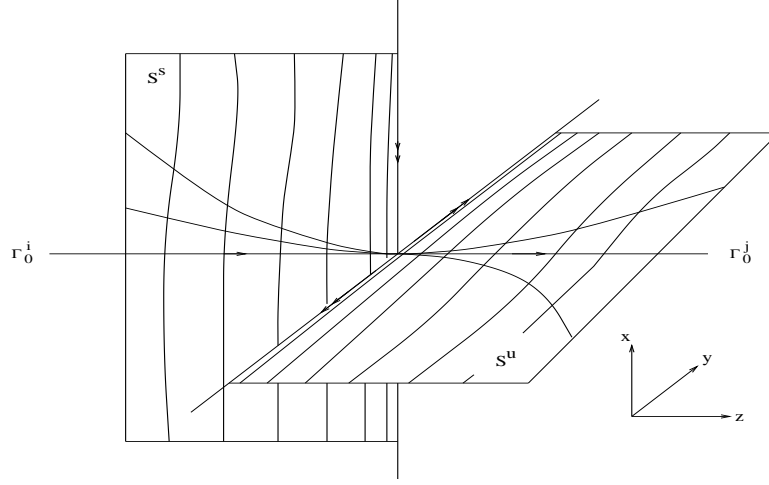These two steps may then be repeated for Φ_μ^{-1} to construct W^{cs}. $\square$

Figure 1. A partially hyperbolic fixed point after variable changes (6) and (7), showing the stable and unstable sets with their foliations, and three homoclinic curves. We have brought $\Gamma^i \cap X^-$ and $\Gamma^j \cap X^+$ to the z-axis. The stable and unstable sets are 'flat' in this coordinate system, but the foliations are not.

For each pair (i, j), $1 \leq i, j \leq n$, let W_{ij}^{cs} be the center stable manifold from Prop. 1 which contains $\Gamma_0^j \cap X^+$. Let $Y_{ij}(x, z, \mu)$ be the local coordinates of W_{ij}^{cs} and consider the change of variables

$$(7) \qquad\qquad y' = y - Y_{ij}(x, z, \mu).$$

This transformation leaves the form of the local diffeomorphism (1) unchanged. Let W_{ij}^{cu} be a center unstable manifold of the resulting system which contains $\Gamma_0^i \cap X^-$, with local coordinates $X_{ij}(y, z, \mu)$. Then the transformation

$$(8) \qquad\qquad x' = x - X_{ij}(y, z, \mu)$$

puts Γ_0^{ij} along the z-axis, but does not alter the form of the local diffeomorphism (1). We will denote by Φ_μ^{ij} the local diffeomorphism in the transformed cooordinates. Figure 1 illustrates the fixed point in the local coordinate system.

Choose $\mu_1 > 0$, $x_1 > 0$, $y_1 > 0$, and $z_0 > 0$ such that for all $\mu < \mu_1$, all pairs (i, j) and for all $(x, y, z) \in U_0 = \{(x, y, z) : |x| < x_1, |y| < y_1, |z| < z_0\}$ we have

$$(9) \qquad
\begin{aligned}
|A + \tilde{f}(x, y, z, \mu)| &< q \\
|A + f_x(x, y, z, \mu)| &< q \\
|(B + \tilde{g}(x, y, z, \mu))^{-1}| &< q \\
|(B + g_x(x, y, z, \mu))^{-1}| &< q,
\end{aligned}$$

for some $q_1 \leq q < 1$.

Note that none of the variable changes thus far have changed z or $R(z, \mu)$. Choose a_0 so that $-z_0 < a_0 < 0$. From (5) we may then choose μ_2 such that

$$a_0 + \sup_{\mu < \mu_2} R(a_0, \mu) < 0,$$

and choose z_1 and z_2 so that

$$-z_0 < z_1 < a_0 < a_0 + \sup_{\mu < \mu_2} R(a_0, \mu) < z_2 < 0.$$

Next, choose $\mu_3 > 0$, $z_3 > 0$ and $z_4 > 0$ such that

$$z_3 + \sup_{\mu < \mu_3} R(z_3, \mu) + \sup_{\mu < \mu_3} R(R(z_3, \mu), \mu) + \sup_{\mu < \mu_3} R(R(R(z_3, \mu), \mu), \mu) < z_4 < z_0.$$

Denote by μ_0 the minimum of μ_1, μ_2, and μ_3. Now define a sequence $\{a_i\}$, beginning with a_0, given by

$$(10) \qquad\qquad\qquad\qquad a_{i+1} = a_i + R(a_i, \mu),$$

and let

$$(11) \qquad\qquad\qquad\qquad n_\mu = \min\{i : a_i > z_3 + \sup_{\mu < \mu'} R(z_3, \mu)\}.$$

It is clear from (5) that $n_\mu \to \infty$ as $\mu \to 0$. We also observe that n_μ is independent of the variable changes.

Proposition 2. *There exist x_2 and y_2 and a collection of invariant surfaces $\{\Sigma_i\}_{i=0}^{n_\mu+1}$, transversal to the z-axis given by functions $\{\chi_i(x, y, \mu)\}$ such that*

$$(12) \qquad\qquad\qquad z_1 < \chi_0(x, y, \mu) < \chi_1(x, y, \mu) < z_2,$$

$$(13) \qquad\qquad \chi_{n_\mu-3}(x, y, \mu) < z_3 < \chi_{n_\mu}(x, y, \mu) < \chi_{n_\mu+1}(x, y, \mu) < z_4,$$

for all $1 \le i, j \le n$, $|x| < x_2$ and $|y| < y_2$.

Proof.   Let

$$\chi_0(x, y, \mu) = a_0 + \phi(y, a_0, \mu)y + \psi(x, a_0, \mu)x$$

with ϕ and ψ as in (4). Denote by Σ_0 the codimension 1 surface which is the graph of $z = \chi_0(x, y, \mu)$. Define surfaces $\{\Sigma_i\}$ as the iterates of Σ_0 under the local map Φ_μ. Note that each Σ_i intersects the z-axis at the points a_i given by (10) and the planes $x = 0$ and $y = 0$ along the foliations

$$(14) \qquad\qquad \begin{aligned} z &= a_i + \phi(y, a_i)y \\ z &= a_i + \psi(x, a_i)x. \end{aligned}$$

Assume that locally near $(x, y) = (0, 0)$ the surface Σ_i may be written as the graph of a function χ_i. The coordinates $(\bar{x}, \bar{y}, \bar{z})$ on Σ_{i+1} must satisfy

$$\begin{aligned} \bar{x} &= Ax + f(x, y, \chi_i(x, y, \mu), \mu) \\ \bar{y} &= By + g(x, y, \chi_i(x, y, \mu), \mu) \\ \bar{z} &= \chi_i(x, y, \mu) + R(\chi_i(x, y, \mu), \mu) + h(x, y, \chi_i(x, y, \mu), \mu). \end{aligned}$$

We may apply the Implicit Function Theorem to the first two equations in a neighborhood of $(x, y) = (0, 0)$ to show that for (x, y) small, x and y may be expressed as functions of $\bar{x}$ and $\bar{y}$, and consequently, Σ_{i+1} must be the graph of a C^k function $\bar{z} = \chi_{i+1}(\bar{x}, \bar{y})$. By Taylor's theorem

$$\chi_i(x, y, \mu) = z_i + D\chi_i(0, 0)(x, y) + o(|(x, y)|).$$

Since we have (14), we know that the Gateaux derivatives of χ_i along the x and y directions are $d_x\chi_i = \psi(0, a_i, \mu)$ and $d_y\chi_i = \phi(0, a_i, \mu)$. Thus the Frechet derivative is given by

$$D\chi_i(0, 0, \mu) = (\psi(0, a_i, \mu), \phi(0, a_i, \mu)).$$

The functions ψ and ϕ along with their derivatives vanish identically for $(x, y, z, \mu) = (0, 0, 0, 0)$ so for x and y small, $|\chi_i - z_i|$ is arbitrarily small. Thus we may choose x_2 and y_2 such that (12) and (13) hold. $\square$

Let $x_0 = \min\{x_1, x_2\}$, $y_0 = \min\{y_1, y_2\}$ and

$$U = \{(x, y, z) : |x| < x_0, |y| < y_0, |z| < z_0\}.$$

For each set of local coordinates indexed by (i, j), let D_0^{ij} be the closed region bounded by the surfaces $|x| = x_0$, $|y| = q^{-n_\mu}y_0$, Σ_0^{ij}, and Σ_1^{ij}, and let $D_{n_\mu}^{ij}$ be the region bounded by $|x| = q^{n_\mu}x_0$, $|y| = y_0$, $\Sigma_{n_\mu}^{ij}$ and $\Sigma_{n_\mu+1}^{ij}$. Suppress the index (i, j) and define a sequence of regions $\{D_i\}_{i=0}^{n_\mu-1}$ by $D_{i+1} = \Phi_\mu(D_i) \cap U$. Since Φ_μ is a diffeomorphism, the boundary of each D_i is C^k-peicewise smooth and consists of two disjoint simply connected components which are subsets of σ_i and σ_{i+1}, along with an annular type region which is the image of the boundaries $|x| = x_0$ and $|y| = q^{-n_\mu}y_0$ of D_0. Since we have uniform contraction in the x direction and expansion in the y direction the collection $\{D_i\}$ resembles a 'staircase'. Note that the component of the boundary of D_i which is a subset of σ_i is a subset of the boundary of D_{i-1}. Also note from (9) that $\Phi_\mu(D_{n_\mu-1})$ intersects D_{n_μ} away from $|x| = q^{n_\mu}x_0$.

We can define sets E_μ^+ and E_μ^- in $X_1 \times X_2 \times R$ by

$$E_\mu^+ = \{(x, y, z) : |x| \le x_0, |y| \le q^{-n_\mu}y_0, z_1 \le z \le z_2|\}$$

$$E_\mu^- = \{(x, y, z) : |x| \le q^{n_\mu}x_0, |y| \le y_0, z_3 \le z \le z_4|\}.$$

Here, $D_0^{ij} \subset E^+$ and $D_{n_\mu}^{ij} \subset E^-$. See figure 2. Also denote

$$R_\mu^+ = \{(y, z) : |y| \le q^{-n_\mu}y_0, z_1 \le z \le z_2|\}$$

$$R^- = \{(y, z) : |y| \le y_0, z_3 \le z \le z_4|\}.$$

Let $\zeta_0 \in C^k(R_\mu^+, X_1)$ be the function whose graph is $S_0^u \cap E^+$. Now define function spaces

$$
\begin{aligned}
H_\delta^{ij} \;=\; & \{\zeta \in C^{k-1,1}(R_\mu^+, X_1) : \|\zeta - \zeta_0\|_{k-1,1} \le \delta, \\
& \text{\textit{the graph of }} \zeta \text{ \textit{is invariant in }} E_\mu^+ \text{ \textit{under} } \Phi_\mu^{ij}\}
\end{aligned}
$$

$$
\begin{aligned}
J_\epsilon^{ij} \;=\; & \{\eta \in C^{k-1,1}(R^-, X_1) : \|\eta\|_{k-1,1} \le \epsilon, \\
& \text{\textit{the graph of }} \eta \text{ \textit{is invariant in }} E_\mu^- \text{ \textit{under} } \Phi_\mu^{ij}\}
\end{aligned}
$$

Here $C^{r,1}$ is the space of r-th continuously differentiable functions with Lipschitz r-th derivative. The norms $\| \cdot \|_{r,1}$ will be the usual norms on those spaces. It can be easily shown that H_δ^{ij} and J_ϵ^{ij} are closed subsets of $C^{k-1,1}(R_\mu^+, X_1)$ and $C^{k-1,1}(R^-, X_1)$, respectively. And so, from Lemma 1 they are closed, bounded subsets of $C^0(R_\mu^+, X_1)$ and $C^0(R^-, X_1)$ under the usual uniform topology.

Because we made the change of variables (8) last, we observe the following

Proposition 3. *There exists $\mu_0 > 0$ such that for each $\mu \le \mu_0$ and for each pair $(i,j) : i \ne j$ there exists a map $\mathcal{F}_{cv}^{ij} : J_\epsilon^{ij} \to J_{2\epsilon}^{jj}$ induced by the change of variables (8).*
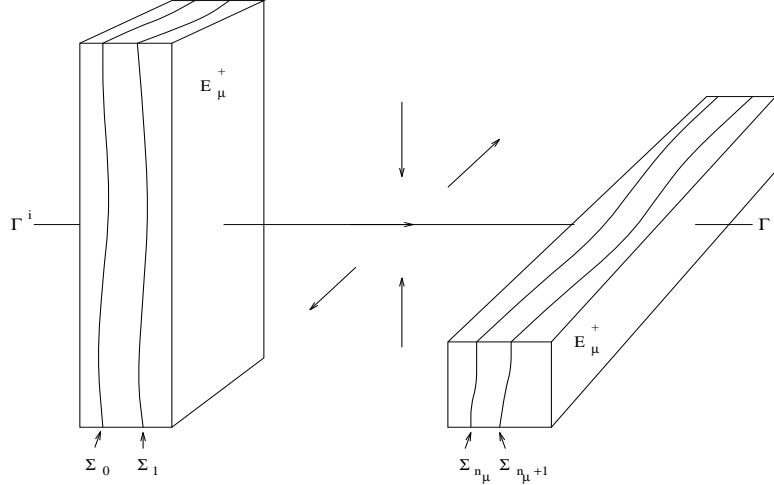

Figure 2. Fundamental regions E^+ and E^-. The region in E_μ^+ bounded by the surfaces Σ_0 and Σ_1 is D_0^{ij} and $D_{n_\mu+1}^{ij}$, which is the image under $\Phi_\mu^{n_\mu}$ of D_0^{ij} intersected with E_μ^-, is contained in the region between Σ_{n_μ} and Σ_{n_μ}. In previous applications of the method of fundamental regions D_0^{ij} and $D_{n_\mu+1}^{ij}$ would be rectified (made rectangular) in a smooth way and used as the fundamental regions. When the fixed point is not semi-stable (or semi-unstable) this is not possible.

4. Global Map

Fix Γ_0^j and make the changes of variables (7), (8) with $i = j$. Fix $\eta \in J_\epsilon^{jj}$. Provided that μ and ϵ are sufficiently small, there is a surface ς in E_μ^+ which is the image under a finite number of iterations of the graph of η. We use here the assumption that T_μ is a diffeomorphism and the assumption that S_0^u is transversal to S_0^s. So, for each $(\bar{y}, \bar{z}) \in R^+$ there exists a minimal integer m_μ^j and $(y, z) \in R^-$ such that

$$T^{m_\mu^j}(\eta(y,z), y, z) = (\bar{x}, \bar{y}, \bar{z}).$$

Since T_μ is a diffeomorphism and has no fixed points other than $\bar{0}$ in a neighborhood of Γ_0, m_μ^j is bounded above. Let ζ be the function $\zeta(\bar{y}, \bar{z}) = \bar{x}$ and define $\zeta = \mathcal{F}_{gl}^j(\eta)$.

Given $\delta > 0$, since T_μ is Lipschitz with respect to μ in the C^k topology, there is an $\epsilon > 0$ such that

$$\|\zeta - \zeta_0\|_{k-1,1} \le \delta$$

provided that $\|\eta\|_{k-1,1} \le \epsilon$.

Further, ζ satisfies the other conditions of H_δ^{jj} by the construction.

Thus we have the following result

Proposition 4. *Given $\delta > 0$, there exist $\epsilon_0 > 0$ and $\mu_0 > 0$ such that $\mathcal{F}_{gl} : J_\epsilon^{jj} \to H_\delta^{jj}$ for all $1 \leq j \leq n$, $\mu < \mu_0$ and $\epsilon < \epsilon_0$.*

Since the map is a diffeomorphism the image ζ of the function η under $\mathcal{F}_{gl}^j$ is given locally by

$$
\begin{array}{rcl}
\zeta(\bar{y}, \bar{z}) &=& F^j(\eta(y,z), y, z, \mu) \\
\bar{y} &=& G^j(\eta(y,z), y, z, \mu) \\
\bar{z} &=& H^j(\eta(y,z), y, z, \mu).
\end{array}
$$
(15)

5. Local Map

We will show here only the cases $k \geq 3$. It is possible to show that the cases $k = 1, 2$ are similar. Fix (i, j) and make the changes (7), and (8). Fix $\zeta \in H_\delta^{ij}$ and let $\eta_0 = \zeta|_{P_{23}(D_0 \cap \{graph\zeta\})}$, where P_{23} is the projection onto $X_2 \oplus R$. We have that

$$\|\eta_0\|_{k-1,1} \leq \|\zeta_0\|_{k-1,1} + \delta.$$

Consider the surface ς_1, which is the image under Φ_μ of the graph ς_0 of η_0. From (1) the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of ς_1 satisfy

$$
\begin{array}{rcl}
\bar{x} &=& A\eta_0(y,z) + f(\eta_0(y,z), y, z, \mu) \\
\bar{y} &=& By + g(\eta_0(y,z), y, z, \mu) \\
\bar{z} &=& z + R(z, \mu) + h(\eta_0(y,z), y, z, \mu).
\end{array}
$$

We may apply the Implicit Function Theorem in a neighborhood of a point on ς_1 to invert the last two equations and obtain locally a function $\bar{x} = \eta_1(\bar{y}, \bar{z})$

Differentiating with respect to (y, z) we obtain

$$D\eta(\bar{y}, \bar{z}) \cdot D(\bar{y}, \bar{z}) = (A + f_x(\eta_0(y,z), y, z, \mu))D\eta_0(y,z) + f_z(\eta_0(y,z), y, z, \mu)$$

where

$$D(\bar{y}, \bar{z}) = \frac{\partial(\bar{y}, \bar{z})}{\partial(y, z)}(\eta_0(y,z), y, z, \mu).$$

Now from (1)

(16)
$$
\begin{array}{rcl}
\partial\bar{y}/\partial y &=& B + g_x(\eta_0(y,z), y, z, \mu)D_y\eta_0(y,z) + g_y(\eta_0(y,z), y, z, \mu) \\
\partial\bar{y}/\partial z &=& g_x(\eta_0(y,z), y, z, \mu)D_z\eta_0(y,z) + g_z(\eta_0(y,z), y, z, \mu) \\
\partial\bar{z}/\partial y &=& h_x(\eta_0(y,z), y, z, \mu)D_y\eta_0(y,z) + h_y(\eta_0(y,z), y, z, \mu) \\
\partial\bar{z}/\partial z &=& 1 + R_z(z, \mu) + h_x(\eta_0(y,z), y, z, \mu)D_z\eta_0(y,z) + h_z(\eta_0(y,z), y, z, \mu).
\end{array}
$$

Since $D\eta_0$ is bounded and Dg and Dh are small for μ small, $D(\bar{y}, \bar{z})$ is invertible, with

$$D(\bar{y}, \bar{z})^{-1} = \begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix} + O(|\mu| + |(x, y, z)|).$$

Therefore, for μ small enough

$$|D\eta_1| \leq q|D\eta_0| + |\tilde{f}_z|\|\eta_0\| < K < \infty.$$

This implies that ς_1 is the graph of a function $\eta_1(\bar{y}, \bar{z})$ over an appropriate domain in $P_{23}D_1$.

Assuming *a priori* that $|D\eta_i|$ remains bounded, by induction we can define a sequence of functions:

$$\{\eta_0, \eta_1, \eta_2, \ldots, \eta_{n_\mu-1}, \eta_{n_\mu}\}$$

with domains in $P_{23}D_i$ whose graphs are iterates under Φ_μ of the surface ς_0. By (1) these functions must satisfy

(17)
$$\begin{aligned}
\eta_{i+1}(\bar{y}, \bar{z}) &= A\eta_i(y, z) + f(\eta_i(y, z), y, z, \mu) \\
\bar{y} &= By + g(\eta_i(y, z), y, z, \mu) \\
\bar{z} &= z + R(z, \mu) + h(\eta_i(y, z), y, z, \mu).
\end{aligned}$$

Therefore, from (9)

$$\begin{aligned}
|\eta_{i+1}| &\leq |A + \tilde{f}||\eta_i| \\
&\leq q|\eta_i|
\end{aligned}$$

so that

(18)
$$\|\eta_i\|_0 \leq q^i \|\eta_0\|_0.$$

Differentiating we find

(19) $D\eta_{i+1}D(\bar{y}, \bar{z}) = (A + f_x(\eta_i(y, z), y, z, \mu))D\eta_i(y, z) + f_z(\eta_i(y, z), y, z, \mu).$

Here

$$D(\bar{y}, \bar{z}) = \frac{\partial(\bar{y}, \bar{z})}{\partial(y, z)}(\eta_i(y, z), y, z, \mu).$$

Provided that $\{D\eta_i\}$ remains bounded, $D(\bar{y}, \bar{z})$ is invertible for μ small, and there exists q such that

$$|A + f_x||D(\bar{y}, \bar{z})^{-1}| \leq q < 1,$$

and from (2) there exists a constant Q_1 so that

$$|f_z(\eta_i(y, z), y, z, \mu)||D(\bar{y}, \bar{z})^{-1}| \leq qQ_1|\eta_i|.$$

Therefore we obtain

$$\begin{aligned}
|D\eta_{i+1}| &\leq q|D\eta_i| + qQ_1|\eta_i| \\
&\leq q|D\eta_i| + q^{i+1}Q_1|\eta_0|,
\end{aligned}$$

and, after induction, employing (18)

(20)
$$\|D\eta_i\| \leq q^i \|D\eta_0\|_0 + iq^i Q_1 \|\eta_0\|_0.$$

For $2 \leq j \leq k - 1$, differentiating j times we find

(21)
$$\begin{aligned}
D^j\eta_{i+1}(\bar{y}, \bar{z})(D(\bar{y}, \bar{z}))^j &= (A + f_x)D^j\eta_i(y, z) - D\eta_{i+1}(\bar{y}, \bar{z})D^j(\bar{y}, \bar{z}) \\
&\quad + D^j_{y,z}f(\eta_i(y, z), y, z, \mu) + \sum_{l=1}^{j-1} P_l(y, z)D^l\eta_i(y, z)
\end{aligned}$$

where P_l is a polynomial with constant coefficients on the variables $D^\alpha(\bar{y}, \bar{z})$, $D^\beta \tilde{f}$ and $D^\gamma \eta_i$ with $1 \leq \alpha \leq j - l$, $1 \leq \beta \leq j - l + 1$ and $0 \leq \gamma \leq l - 1$. For μ small enough

$$|A + f_x||D(\bar{y}, \bar{z})^{-1}|^j \leq q < 1,$$

and from (2) we obtain

$$|D_{y,z}^j f||D(\bar{y},\bar{z})^{-1}|^j \le \bar{Q}_j|\eta_i|.$$

Further, from (16)

$$D(\bar{y},\bar{z}) = \begin{pmatrix} g_x \\ h_x \end{pmatrix} D^j\eta_i(y,z) + \begin{pmatrix} B + g_y & g_z \\ h_y & 1 + R_z + h_z \end{pmatrix},$$

so that

$$D^j(\bar{y},\bar{z}) = \begin{pmatrix} g_x \\ h_x \end{pmatrix} D^j\eta_i(y,z) + R(y,z),$$

where $R(y,z)$ is a polynomial in lower order derivatives of η_i. Therefore, using (20)

$$|D^j\eta_{i+1}| \le q|D^j\eta_i| + qQ_j\|\eta_i\|_{j-1},$$

where Q_j is independent of i. After induction, employing (18)

$$\|D^j\eta_i\| \le q^i\|D^j\eta_0\| + p_j(i)q^iQ_j\|\eta_0\|_{j-1},$$

where $p_j(i)$ is a j-th degree polynomial in i. In particular,

$$(22) \qquad \|\eta\|_{k-1} \le p_{k-1}(i)q^i\|\eta_0\|_{k-1}.$$

To show that $D^{k-1}\eta_{i+1}$ is a Lipschitz function, assume that $D^{k-1}\eta_i$ is Lipschitz and consider

$$
\begin{aligned}
&|D^{k-1}\eta_{i+1}(\overline{y},\overline{z}) - D^{k-1}\eta_{i+1}(\overline{\hat{y}},\overline{\hat{z}})| \\
&\le \quad |A||D^{k-1}\eta_i(y,z)(D(\bar{y},\bar{z})(y,z)^{-1})^k - D^{k-1}\eta_i(\hat{y},\hat{z})(D(\bar{y},\bar{z})(\hat{y},\hat{z})^{-1})^k| \\
&\quad + |f_x(\eta_i(y,z),y,z,\mu)D^{k-1}\eta_i(y,z)(D(\bar{y},\bar{z})(y,z)^{-1})^k \\
&\quad -f_x(\eta_i(\hat{y},\hat{z}),\hat{y},\hat{z},\mu)D^{k-1}\eta_i(\hat{y},\hat{z})(D(\bar{y},\bar{z})(\hat{y},\hat{z})^{-1})^k| \\
&\quad + \sum_{l=0}^{k-2}|P_l(y,z)D^l\eta_i(y,z) - P_l(\hat{y},\hat{z})D^l\eta_i(\hat{y},\hat{z})| \\
&\quad +|D_{y,z}^k f(\eta_i(y,z),y,z,\mu)(D(\bar{y},\bar{z})(y,z)^{-1})^k \\
&\quad -D_{y,z}^k f(\eta_i(\hat{y},\hat{z}),\hat{y},\hat{z},\mu)(D(\bar{y},\bar{z})(\hat{y},\hat{z})^{-1})^k|.
\end{aligned}
$$

Noting that $P_l D^l\eta_i$ is Lipschitz for $l \le k-2$, with Lipschitz constant proportional to $\|\eta_i\|_{l+1}$, we obtain

$$
\begin{aligned}
&|D^{k-1}\eta_{i+1}(\bar{y},\bar{z}) - D^{k-1}\eta_{i+1}(\overline{\hat{y}},\overline{\hat{z}})| \\
&\le \quad [((|A + f_x|)(D(\bar{y},\bar{z})^{-1})^k Lip(D^{k-1}\eta_i)) \\
&\quad + Lip((A + f_x(\eta_i(y,z),y,z,\mu))(D(\bar{y},\bar{z})(y,z)^{-1})^k)\|D^{k-1}\eta_i\| \\
&\quad + \sum_{l=0}^{k-2}\hat{Q}_l\|\eta_i\|_{l+1}\|D(\bar{y},\bar{z})^{-1}\|^k \\
&\quad +|D_{y,z}^k f(\eta_i(y,z),y,z,\mu)\|D(\bar{y},\bar{z})^{-1}|^k]|(y,z) - (\hat{y},\hat{z})| \\
&\le \quad (qLip(D^{k-1}\eta_i) + Q_k\|\eta_i\|_{k-1} + |D_{y,z}^k f(\eta_i(y,z),y,z,\mu)\|D(\bar{y},\bar{z})^{-1}|^k) \\
&\quad \cdot|(y,z) - (\hat{y},\hat{z})|.
\end{aligned}
$$

Again, for μ small
$$|A + f_x||D(\bar{y}, \bar{z})(y, z)^{-1}|^k \le q < 1.$$
Thus, since $D^{k-1}\eta_0$ is Lipschitz, $D^{k-1}\eta_{i+1}$ is Lipschitz, with constant
$$Lip(D^{k-1}\eta_{i+1}) \le q Lip(D^{k-1}\eta_i) + q Q_k \|\eta_i\|_{k-1} + \|D_{y,z}^k f(\eta_i)\|\|D(\bar{y}, \bar{z})^{-1}\|^k.$$
After induction it can be shown that

(23)
$$\begin{aligned} Lip(D^{k-1}\eta_i) &\le q^i Lip(D^{k-1}\eta_0) + q^i p_k(i) Q_k \|\eta_0\|_{k-1} \\ &+ \sum_{l=0}^{i} q^l \|D_{y,z}^k f(\eta_{i-l})\|\|D(\bar{y}, \bar{z})^{-1}\|^k. \end{aligned}$$

Using (18) and (3) choose i_0 so that
$$\|D_{y,z}^k f(\eta_{i_0})\|\|D(\bar{y}, \bar{z})^{-1}\|^k \le \frac{\epsilon}{2}(1 - q).$$

Then

$$\sum_{l=0}^{i} q^l |D_{y,z}^k f(\eta_{i-l}(y, z), y, z, \mu)||D(\bar{y}, \bar{z})(y, z)^{-1}|^k$$

$$\le \sum_{l=0}^{i-i_0} q^l \|D_{y,z}^k f(\eta_{i-l})\|\|D(\bar{y}, \bar{z})^{-1}\|^k$$

$$+ \sum_{l=i-i_0+1}^{i} q^l \|D_{y,z}^k f(\eta_{i-l})\|\|D(\bar{y}, \bar{z})^{-1}\|^k$$

$$\le \sum_{l=0}^{i-i_0} q^l \frac{\epsilon}{2}(1 - q)$$

$$+ \sum_{l=i-i_0+1}^{i} q^l \max_{0 \le l \le i_0 - 1} \|D_{y,z}^k f(\eta_l)\|\|D(\bar{y}, \bar{z})^{-1}\|^k$$

$$\le \frac{\epsilon}{2} + (i_0 - 1) q^{i-i_0+1} \max_{0 \le l \le i_0 - 1} \|D_{y,z}^k (\eta_l)\|\|D(\bar{y}, \bar{z})^{-1}\|^k,$$

so that the last term of (23) may be made arbitrarily small. Let η be the extension of η_{n_μ} onto the domain R^- through iteration of Φ_μ and Φ_μ^{-1} and define $\mathcal{F}_{loc}^{ij}(\zeta) = \eta$. Now (13), (18), (22) and (23) imply that if μ is sufficiently small, then $\|\eta\|_{k-1,1} < \epsilon$, since $n_\mu - 2$ is large.

Finally, η satisfies the invariance condition for H_ϵ^{ij} by composition. Thus $\eta \in J_\epsilon^{ij}$ and we have proved the following

Proposition 5. *Given $\epsilon > 0$ and $\delta > 0$, there exists $\mu_0 > 0$ such that $\mathcal{F}_{loc}^{ij} : H_\delta^{ij} \to J_\epsilon^{ij}$, for all $1 \le i, j \le n$ and $0 < \mu < \mu_0$.*

6. Existence of Invariant Curves

To show the existence and $C^{k-1,1}$-smoothness of the curve $\Gamma_\mu^{\bar{\omega}}$ we will use the following [CL].

Lemma 1. *Let X and Y be Banach spaces and U an open subset of X. Then a closed bounded ball in $C^{k,1}(U, Y)$, $(k = 1, 2, 3, \ldots)$, is a closed bounded subset in $C^0(U, Y)$.*

Let μ be sufficiently small that the map $\mathcal{F}^{ij} : J_\epsilon^{ii} \to J_\epsilon^{jj}$ given by

$$\mathcal{F}^{ij} = \mathcal{F}_{cv}^{ij} \circ \mathcal{F}_{loc}^{ij} \circ \mathcal{F}_{gl}^i,$$

is well-defined for all ϵ sufficiently small and for all $1 \leq i, j \leq n$. Now given $\bar{\omega} = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$, consider the infinite sequence of maps

$$(24) \qquad \cdots \longrightarrow J_\epsilon^{\omega_{i-1}\omega_{i-1}} \xrightarrow{\mathcal{F}^{\omega_{i-1}\omega_i}} J_\epsilon^{\omega_i\omega_i} \xrightarrow{\mathcal{F}^{\omega_i\omega_{i+1}}} J_\epsilon^{\omega_{i+1}\omega_{i+1}} \longrightarrow \cdots.$$

This is equivalent to a map $F : J \to J$, where

$$J = \cdots \times J_\epsilon^{\omega_{i-1}\omega_{i-1}} \times J_\epsilon^{\omega_i\omega_i} \times J_\epsilon^{\omega_{i+1}\omega_{i+1}} \times \cdots,$$

supplied with the sup norm $\|\eta\|_J = \sup_{i\in Z} \|\eta_i\|_0$. By propositions 3, 4 and 5, for μ sufficiently small

$$|\mathcal{F}^{\omega_i\omega_{i+1}}(\hat{\eta}) - \mathcal{F}^{\omega_i\omega_{i+1}}(\eta)| \leq c|\hat{\eta} - \eta|,$$

where $c < 1$ independent of ω_i. Thus F is a contraction on the complete metric space J and has a unique fixed point $\eta_* = (\dots, \eta_*^{i-1}, \eta_*^i, \eta_*^{i+1}, \dots)$. In other words,

$$\cdots \longmapsto \eta_*^{i-1} \xrightarrow{\mathcal{F}^{\omega_{i-1}\omega_i}} \eta_*^i \xrightarrow{\mathcal{F}^{\omega_i\omega_{i+1}}} \eta_*^{i+1} \longmapsto \cdots.$$

Let

$$W_{\bar{\omega}\mu}^u = \bigcup_{j\in Z}\{(\bigcup_{i=0}^{m_\mu^{\omega_j}} T_\mu^i(S^j)) \bigcup (\bigcup_{i=m_\mu^{\omega_j}+1}^{m_\mu^{\omega_j}+n_\mu} (T_\mu^i(S^j) \cap U))\},$$

where S^j is the preimage in the original coordinate system of the graph of η_*^j. The expansion in the y-direction insures that $W_{\bar{\omega}\mu}^u$ is negatively invariant. Similarly, we construct a manifold $W_{\bar{\omega}\mu}^s$ which is positively invariant by constructing an invariant sequence of functions $\nu_* = (\dots, \nu_*^{-1}, \nu_*^0, \nu_*^1, \dots)$ with $\nu_* \in C^{k-1,1}(I_{x_0} \times [z_3, z_4], X_2)$, where $I_{x_0} = \{x \in X_1 : |x| \leq x_0\}$

Now fix the change of variables by (ω_0, ω_1) and let $I_{y_0} = \{y \in X_2 : |y| \leq y_0\}$ and let

$$G : I_{x_0} \times I_{y_0} \times (z_3, z_4) \times (0, \mu_0) \to I_{x_0} \times I_{y_0}$$

be the function defined by

$$G(x, y, z, \mu) = (\eta_*^0(y, z, \mu), \nu_*^0(x, z, \mu)).$$

It is easy to show that G is a uniform contraction with respect to (z, μ) and so for each (z, μ) it has glldea fixed point $\lambda_*(z, \mu)$, which depends $C^{k-1,1}$-smoothly on z and Lipschitz continuously on μ. Let γ_μ^j be the graph of λ_*. Now by the invariance properties of $W_{\bar{\omega}\mu}^u$ and $W_{\bar{\omega}\mu}^s$ the images of γ_μ^j are the unique invariant intersection of these two manifolds and since T_μ is a C^k diffeomorphism this intersection is $C^{k-1,1}$.

Now suppose that $\bar{\omega}$ is periodic, with minimal period N. Let $(\omega_1, \omega_2, \dots, \omega_N)$ be the smallest periodic part of $\bar{\omega}$ and consider the map

$$(25) \quad \mathcal{F}^{\omega_N\omega_1} \circ \mathcal{F}^{\omega_{N-1}\omega_N} \circ \mathcal{F}^{\omega_{N-2}\omega_{N-1}} \circ \cdots \circ \mathcal{F}^{\omega_2\omega_3} \circ \mathcal{F}^{\omega_1\omega_2} : J_\epsilon^{\omega_1\omega_1} \to J_\epsilon^{\omega_1\omega_1}.$$

Because of periodicity of the map (24) and the uniqueness of the fixed point η_*, the first component, η_*^1 of η_* is the unique fixed point of the map (25). A similar result holds for ν_*, and so the intersection is periodic.

The proof of Theorem 1 will be complete after we show in the next section that the curves are C^k-smooth.

7. C^k-SMOOTHNESS OF THE INVARIANT CURVES

We will use the following lemma to prove the final degree of differentiability of $\Gamma_\mu^{\bar\omega}$.

Lemma 2. *(D. Henry) Let X and Y be Banach spaces and U be an open subset of X. Assume that $h : U \to Y$ is locally Lipschitz continuous. Then h is continuously differentiable if and only if for every $x_0 \in U$,*

$$\|h(x + \Delta) - h(x) - h(x_0 + \Delta) + h(x_0)\| = o(\|\Delta\|)$$

as $(x, \Delta) \to (x_0, 0)$.

For a proof of this result see Chow and Lu [CL].

Fix $\bar\omega$. Here again we show only the cases $k \geq 3$. The cases $k = 1, 2$ are similar. Let $\zeta = \mathcal{F}_{gl}^{\omega_j}(\eta_*^j)$. From (15) for $k \geq 3$

$$\begin{aligned} D^{k-1}\zeta(\bar y, \bar z) &= [(F_x - D\zeta(\bar y, \bar z)(G_x, H_x)^t)D^{k-1}\eta_*^j(y, z) + R_1(z)] \\ &\quad \cdot (D(\bar y, \bar z)(y, z)^{-1})^{k-1}, \end{aligned}$$

where R is a function of $D^\alpha F$, $D^\beta \eta_*^j$ and $D^\gamma(\bar y, \bar z)$ with $1 \leq \alpha \leq k-1$, $1 \leq \beta \leq k-2$ and $1 \leq \gamma \leq k-2$. Thus R is continuously differentiable as a function of z, and so, using Lemma 2 we have

$$\begin{aligned} & D^{k-1}\zeta(\overline{y + \Xi}, \overline{z + \Delta}) - D^{k-1}\zeta(\bar y, \bar z) - D^{k-1}\zeta(\overline{y_0 + \Xi}, \overline{z_0 + \Delta}) + D^{k-1}\zeta(\bar y_0, \bar z_0) \\ &= [F_x - D\zeta(\bar y, \bar z)(G_x, H_x)^t] \\ &\quad \cdot [D^{k-1}\eta_*^j(y + \Xi, z + \Delta) - D^{k-1}\eta_*^j(y, z) - D^{k-1}\eta_*^j(y_0 + \Xi, z_0 + \Delta) \\ &\quad + D^{k-1}\eta_*^j(y_0, z_0)] \cdot (D(\bar y, \bar z)(y_0, z_0)^{-1})^{k-1} + o(|\Delta|) \end{aligned}$$

as $(y, z, \Xi, \Delta) \to (y_0, z_0, 0, 0)$. Define

$$(26) \qquad\qquad (c, d) \equiv \overline{(y_0 + \Xi, z_0 + \Delta)} - \overline{(y_0, z_0)}.$$

From (15)

$$\begin{aligned} c &= G(\eta_*^j(y_0 + \Xi, z_0 + \Delta), y_0 + \Xi, z_0 + \Delta, \mu) - G(\eta_*^j(y_0, z_0), y_0, z_0, \mu) \\ &= D(\bar y)(y_0, z_0)(\Xi, \Delta) + o(|(\Xi, \Delta)|), \end{aligned}$$

so that

$$|D(\bar y, \bar z)^{-1}||(c, d)| = |(\Xi, \Delta)| + o(|(\Xi, \Delta)|).$$

Also, since G is continuously differentiable

$$\begin{aligned} \overline{y + \Xi} - (\bar y + c) &= G(\eta_*^j(y + \Xi, z + \Delta), y + \Xi, z + \Delta, \mu) - G(\eta_*^j(y, z), y, z, \mu) \\ &\quad - G(\eta_*^j(y_0 + \Xi, z_0 + \Delta), y_0 + \Xi, z_0 + \Delta, \mu) \\ &\quad + G(\eta_*^j(y_0, z_0), y_0, z_0, \mu) \\ &= o(|(\Xi, \Delta)|), \end{aligned}$$

as $(y, z, \Xi, \Delta) \to (y_0, z_0, 0, 0)$. Thus

$$\overline{y + \Xi} - (\bar y + c) = o(|(c, d)|),$$

as $(\bar{y}, \bar{z}, c, d) \to (\bar{y}_0, \bar{z}_0, 0, 0)$. A similar result holds for $\overline{z + \Delta} - (\bar{z} + d)$. Define

$$(27) \qquad \lambda(h, x_0) = \limsup_{(x, \Delta) \to (x_0, 0)} \frac{|h(x + \Delta) - h(x) - h(x_0 + \Delta) + h(x_0)|}{|\Delta|}.$$

(see Lemma 2) Then

$$\frac{1}{|(c,d)|} |D^{k-1}\zeta(\bar{y} + c, \bar{z} + d) - D^{k-1}\zeta(\bar{y}, \bar{z}) - D^{k-1}\zeta(\bar{y}_0 + c, \bar{z}_0 + d) + D^{k-1}\zeta(\bar{y}_0, \bar{z}_0)|$$

$$\leq \frac{|(\Xi, \Delta)|}{|(c,d)|} \max |F_x - D\zeta(G_x, H_x)^t| |D(\bar{y}, \bar{z})(y, z)^{-1}|^k \lambda(D^{k-1}\eta_*^j, y_0, z_0) + o(1),$$

and so there is a constant L such that

$$(28) \qquad \lambda(D^{k-1}\zeta, \bar{y}_0, \bar{z}_0) \leq L\lambda(D^{k-1}\eta_*^j, y_0, z_0).$$

Since there are only a finite number of curves Γ_0^j we may choose L independent of $\bar{\omega}$ and j.

Next let $\eta_0 = \zeta$ and let $\{\eta_i\}$ be the sequence of functions determined by (17). From (21), noting that the last two terms are continuously differentiable,

$$D^{k-1}\eta_{i+1}(\overline{y + \Xi}, \overline{z + \Delta}) - D^{k-1}\eta_{i+1}(\bar{y}, \bar{z}) - D^{k-1}\eta_{i+1}(\overline{y + \Xi}, \overline{z_0 + \Delta}) + D^{k-1}\eta_{i+1}(\bar{y}_0, \bar{z}_0)$$

$$= e_i(y_0 z_0)$$
$$\cdot [D^{k-1}\eta_i(y + \Xi, z + \Delta) - D^{k-1}\eta_i(y, z) - D^{k-1}\eta_i(y_0 + \Xi, z_0 + \Delta) + D^{k-1}\eta_i(y, z)]$$
$$\cdot (D(\bar{y}, \bar{z})(y, z)^{-1}(y_0, z_0))^{k-1} + o(|(\Xi, \Delta)|),$$

as $(y, z, \Xi, \Delta) \to (y_0, z_0, 0, 0)$, where

$$e_i(z) = A + f_x(\eta_i(y, z), y, z, \mu) - D\eta_i(\bar{y}, \bar{z})(g_x, h_x)^t.$$

Define

$$(s, t) = \overline{(y_0 + \Xi, z_0 + \Delta)} - \overline{(y_0, z_0)}.$$

From (17)

$$\begin{aligned} s &= g(y_0 + \Xi, z_0 + \Delta, \mu) - g(y_0, z_0, \mu) \\ &= D(\bar{y})(y_0, z_0)(\Xi, \Delta) + o(|(\Xi, \Delta)|). \end{aligned}$$

A similar result holds for t. Thus

$$(29) \qquad |(s, t)| \geq |(\Xi, \Delta)|(1 + o(1)).$$

Also

$$\begin{aligned} &\overline{(y + \Xi)} - (\bar{y} + s) \\ &= g(y + \Xi, z + \Delta, \mu) - g(y, z, \mu) - g(y_0 + \Xi, z_0 + \Delta, \mu) + g(y_0, z_0, \mu) \\ &= o(|(\Xi, \Delta)|) \end{aligned}$$

as $(y, z, \Xi, \Delta) \to (y_0, z_0, 0, 0)$. This implies with (29) that

$$(30) \qquad \overline{y + \Xi} - (\bar{y} - s) = o(|(s, t)|)$$

as $(y, z, s, t) \to (y_0, z_0, 0, 0)$, and similarly for $\overline{z + \Delta} - (\bar{z} + t)$. Now using (29) and (30) in definition (27) and noting that the functions $\{e_i\}$ are uniformly bounded we have

$$
\begin{aligned}
\lambda(D^{k-1}\eta_{i+1}, \bar{y}_0, \bar{z}_0) &\leq \frac{|(\Xi, \Delta)|}{|(s, t)|} \|e\| \|D(\bar{y}, \bar{z})(y, z)^{-1}\|^{k-1} \lambda(D^{k-1}\eta_i, y_0, z_0) \\
&\leq q\lambda(D^{k-1}\eta_i, y_0, z_0)
\end{aligned}
$$

where $q < 1$ for μ sufficiently small. Employing (13),

$$
(31) \qquad\qquad \lambda(D^{k-1}\eta_*^{j+1}, \bar{y}_0, \bar{z}_0) \leq q^{n_\mu - 2}\lambda(D^{k-1}\zeta, y_0, z_0).
$$

Now if we apply (28) and (31) to (25) we find

$$
\begin{aligned}
\lambda(D^{k-1}\eta_*^{j+1}, \bar{y}_0, \bar{z}_0) &\leq q^{n_\mu - 2} L\lambda(D^{k-1}\eta_*^j, y_0, z_0) \\
&\leq c\lambda(D^{k-1}\eta_*^j, y_0, z_0)
\end{aligned}
$$

where $c < 1$ for μ sufficiently small, independent of j. Thus, using the fact $\|\eta_*^j\|_{k-1,1} \leq \epsilon$, for all j, we have

$$
\sup_{(\bar{y}_0, \bar{z}_0) \in R^-} \lambda(D^{k-1}\eta_*^j, \bar{y}_0, \bar{z}_0) \leq c^n \sup_{(y_0, z_0) \in R^-} \lambda(D^{k-1}\eta_*^{j-n}, y_0, z_0) < 2\epsilon,
$$

which implies

$$
\lambda(D^{k-1}\eta_*^j, y_0, z_0) = 0
$$

for all $(y_0, z_0) \in R^-$. Therefore by Lemma 2 we have that $\eta_*^j \in C^k$ and so $\Gamma_\mu^{\bar{\omega}}$ is C^k-smooth for all μ sufficiently small. $\square$

8. Topological Structure of the Invariant Set

Choose ϵ_0 sufficiently small for Prop. 1-5 to hold. Fix j and change variables by (7). Define regions

$$
\bar{E}_\mu^{ij} = \{(x, y, z) : |x - X^{ij}(y, z, \mu)| \leq \epsilon_0, |y| \leq y_0, z_3 \leq z \leq z_4\},
$$

where X^{ij} is as in (8). The set $\bar{E}_\mu^{ij}$ is a 'slab' of positive thickness in the x-direction. Since $X^{ij}(y, z, 0) \equiv 0$ for $z > 0$ the intersection

$$
\hat{E}_\mu^j = \bigcap_{i=1}^n \hat{E}_\mu^{ij}
$$

must also be a slab of positive thickness, for all μ small enough. Repeat this process for each $1 \leq j \leq n$ and consider the sets E_μ^j which are the set $\hat{E}_\mu^j$ in the original coordinate system. We may require the collection $\{E_\mu^j\}$ to be pair-wise disjoint by taking ϵ_0 and y_0 sufficiently small.

Let V_δ be a δ neighborhood of Λ_0 with δ small enough that V_δ contains no fixed points of T_μ for $\mu > 0$ and so that V_δ intersects each ∂E_μ^j only at the front and back sides s_3 and s_4, where $s_i = \{(x, y, z) : z = z_i\}$. Since V_δ contains no fixed points, if $p \in Inv(V_\delta)$ then the orbit $\{T_\mu^i(p)\}_{i=-\infty}^{+\infty}$ must intersect the sets $\{E_\mu^j\}$ infinitely many times. Let $\bar{\omega}'(p)$ be the bi-infinite sequence defined by

$$
\omega_i(p) = \begin{cases} j & if \ T_\mu^i(p) \in E_\mu^j \\ 0 & otherwise. \end{cases}
$$

Then let $\bar{\omega}(p)$ be the sequence obtained from $\bar{\omega}'(p)$ by first deleting repeated symbols and then deleting all zeros.

We will now prove that p is on the curve $\Gamma_\mu^{\bar{\omega}(p)}$. By Propositions 3, 4, and 5, each slab E_μ^i is mapped by T_μ into each slab E_μ^j as the preimage of a region with piecewise smooth boundary. Since the map is a diffeomorphism, the images in E_μ^j of the collection $\{E_\mu^i\}_{i=1}^n$ must be pairwise disjoint. By the same reasoning, T_μ^{-1} maps each slab E_μ^i into a slab E_μ^j as the preimage of a piecewise smooth region. Let $\bar{\mathcal{F}}^{ij}$ be the set function induced by $\mathcal{F}^{ij}$ which takes regions in E_μ^i to regions in E_μ^j. Define

$$E^{\omega_{-1}\omega_0\omega_1} = \bar{\mathcal{F}}^{\omega_{-1}\omega_0}(E_\mu^{-1}) \bigcap E^{\omega_0} \bigcap (\bar{\mathcal{F}}^{\omega_0\omega_1})^{-1}(E^{\omega_1}),$$

and inductively, for each subsequence $\bar{\omega}_N$ of length $2N+1$, centered at index 0,

$$E^{\bar{\omega}_N} = E^{\bar{\omega}_{N-1}} \bigcap \bar{\mathcal{F}}^{\omega_{-1}\omega_0} \circ \bar{\mathcal{F}}^{\omega_{-2}\omega_{-1}} \circ \ldots \circ \bar{\mathcal{F}}^{\omega_{-N}\omega_{-N+1}}(E^{\omega_{-N}})$$
$$\bigcap (\bar{\mathcal{F}}^{\omega_0\omega_1})^{-1} \circ \ldots \circ (\bar{\mathcal{F}}^{\omega_{N-1}\omega_N})^{-1}(E^{\omega_N}),$$

where $\bar{\omega}_{N-1}$ is the sequence of length $2N-1$ obtained from ω_N by deleting the initial and final elements. By Propositions 3, 4, and 5, $E^{\bar{\omega}_N}$ is bounded by the graphs of functions η^-, η^+, ν^-, and ν^+, which are the graphs are the images and preimages of the boundaries of the sets $\{E_\mu^j\}$. Finally let

$$E^{\bar{\omega}} = \bigcap_N E^{\bar{\omega}_N}.$$

The orbit of p must be contained in $E^{\bar{\omega}_N}$ for any subsequence $\bar{\omega}_N$ and so it must be contained in the intersection. For each z the cross section of $E^{\bar{\omega}_N}$ is bounded by the graphs of η^-, η^+, ν^-, and ν^+. By Prop. 5 the diameter of these cross sections is decreasing to zero as $N \to \infty$, thus $E^{\bar{\omega}}$ consists of a single curve in $E_\mu^{\omega_0}$, which must be the intersection of η_*^1 and ν_*^1. Therefore, the invariant set $\Lambda_\mu \equiv \cup_{\bar{\omega}\in\bar{\Omega}} \Gamma_\mu^{\bar{\omega}}$.

Now consider the section of the curve $\Gamma_\mu^{\bar{\omega}}$ from $\gamma_\mu^{\omega_0} \cap s_3$ to $\gamma_\mu^{\omega_1} \cap s_3$ corresponding to the union of the first $n_\mu + m_\mu^{\omega_0}$ images of $\gamma_\mu^{\omega_0}$ under T_μ. Let $\rho^{\bar{\omega}} : [0, l^{\bar{\omega}}] \to \gamma_\mu^{\omega_0}$ be a parameterization of this segment by arc length. Since each $\gamma^{\bar{\omega}}$ is smooth we may define this parameterization in a uniform way. Next define $\psi^{\bar{\omega}} : [0, 1] \to \gamma_\mu^{\omega_0}$ by $\psi^{\bar{\omega}}(s) = \rho^{\bar{\omega}}(s \cdot l^{\bar{\omega}})$. We then have a map $\Psi : \bar{\Omega} \to \Lambda_\mu$ defined by

$$\Psi(\bar{\omega}, s) = \psi^{\bar{\omega}}(s).$$

The function $\Psi(\bar{\omega}, s)$ is well-defined, since for each $\bar{\omega}$, $\Gamma_\mu^{\bar{\omega}}$ is unique and the parameterization by arc length is well-defined.

To show that $\Psi(\bar{\omega}, s)$ is one to one, suppose $(\bar{\omega}_1, s_1) \neq (\bar{\omega}_2, s_2)$. If $\bar{\omega}_1 \neq \bar{\omega}_2$, consider the possibility that $\Psi(\bar{\omega}_1, s_1) = \Psi(\bar{\omega}_2, s_2)$. That is $\Gamma_\mu^{\bar{\omega}_1}$ and $\Gamma_\mu^{\bar{\omega}_2}$ share a common point. This is not possible however since that point would need to have two distinct iteneraries. If however $\bar{\omega}_1 = \bar{\omega}_2$, while $s_1 \neq s_2$ and since the parameterization is one to one, $\Psi(\bar{\omega}_1, s_1) \neq \Psi(\bar{\omega}_2, s_2)$.

Since $\Lambda_\mu = \cup_{\bar{\omega}\in\bar{\Omega}} \Gamma_\mu^{\bar{\omega}}$, given $p \in \Gamma_\mu^{\bar{\omega}}$ we can shift $\bar{\omega}$ to $\bar{\omega}'$ so that

$$p \in \cup_{i=1}^{n_\mu + m_\mu^{\omega_0'}} T_\mu^i(\gamma^{\omega_0'}).$$

Then let $s = (\text{arc length from } \gamma^{\omega'_0} \cap s_3 \text{ to } p)/l^{\bar{\omega}'}$. This shows that the function $\Psi(\bar{\omega}, s)$ is onto.

To show continuity of Ψ and Ψ^{-1}, consider that points $p_1, p_2 \in \Lambda_\mu$ are close if and only if their orbits are close for a finite number of iterations. This is the case if and only if the two orbits pass through the same slabs E_μ^j for a finite number of iterations in forward and backward time. This is the case if and only if either $\bar{\omega}(p_1)$ and $\bar{\omega}(p_2)$ agree in those places or $\bar{\omega}(p_1)$ and $\sigma(\bar{\omega}(p_2))$ agree in those places. This in turn is equivalent to either $\bar{\omega}(p_1)$ and $\bar{\omega}(p_2)$ or $\bar{\omega}(p_1)$ and $\sigma(\bar{\omega}(p_2))$ being close in the topology on $\bar{\Omega}$. In the former situation $\Gamma_\mu^{\bar{\omega}_1}$ and $\Gamma_\mu^{\bar{\omega}_2}$ must be close for finite iterations of $\gamma_\mu^{\bar{\omega}_1}$ and $\gamma_\mu^{\bar{\omega}_2}$, which implies $l^{\bar{\omega}_1}$ is approximately $l^{\bar{\omega}_2}$ and p_1 is close to p_2 if and only if s_1 is close to s_2. If the case that $\bar{\omega}(p_1)$ and $\sigma(\bar{\omega}(p_2))$ agree in several places then $\bar{\omega}(T_\mu(p_1))$ must agree with $\bar{\omega}(T_\mu(p_2))$. We can then repeat the preceeding, for $T_\mu(p_1)$ and $T_\mu(p_2)$, and so, continuity of T_μ proves that p_1 is close to p_2 if and only if $(\bar{\omega}_1, s_1)$ is close to $(\bar{\omega}_2, s_2)$. $\square$

9. Uniform Partial Hyperbolicity

Since we know that Γ_μ is the union of smooth manifolds, uniform partial hyperbolicity on Γ_μ is implied by normal hyperbolicity on each $\Gamma_\mu^{\bar{\omega}}$, with constants which are uniform in $\bar{\omega}$. In order to show that f satisfies the hypotheses of Definition 4 on $\Gamma_\mu^{\bar{\omega}}$ we will use the method of fundamental regions on spaces of 1-jets on $\Gamma_\mu^{\bar{\omega}}$.

Fix $\bar{\omega}$ and a sufficiently small neighborhood U of the origin. For μ small enough $\Gamma_\mu^{\bar{\omega}} \cap U$ consists of a countable (finite in the case $\bar{\omega}$ periodic) collection $\{\gamma_\mu^i\}_{i=-\infty}^{\infty}$ of curve segments, ordered in the natural way. Each of these segments is the intersection of sections C^i and D^i of the manifolds $W_{\bar{\omega}\mu}^u$ and $W_{\bar{\omega}\mu}^s$ as in section 6. These sections are given locally as the graphs of functions $\eta^i(y, z, \mu)$ and $\nu^i(x, z, \mu)$. Note from section 6 that η^i and ν^i are uniformly bounded in the C^k norm independent of $\bar{\omega}$ and i. For each i consider the change of variables

$$(32) \qquad x' = x - \eta^i(y, z, \mu),$$

followed by

$$y' = y - \nu'^i(x', z, \mu),$$

where ν'^i is the coordinate function for D^i after (32). These changes shift the sections C^i and D^i to the hyperplanes $x = 0$ and $y = 0$ and thus γ^i is given in the new coordinates by $(x, y) = (0, 0)$.

Now consider the map restricted to the negatively invariant manifold $W_{\bar{\omega}\mu}^u$. For each i the local diffeomorphism on C_μ^i in the transformed coordinates is

$$\begin{aligned} \bar{y} &= By + \tilde{g}(y, z, \mu)y \\ \bar{z} &= z + R(z, \mu) + h(y, z, \mu). \end{aligned}$$

It has been shown that this local diffeomorphism may be transformed by a C^k-smooth change of variables to the form

$$(33) \qquad \begin{aligned} \bar{y} &= By + \tilde{g}(y, z, \mu)y \\ \bar{z} &= z + R(z, \mu) \end{aligned}$$

[Af90][Yo].

Let $(0, 0, z_0)$ denote points on γ^i. For each z_0 consider C^1-smooth surfaces on C^i with coordinates (y, z) given by

$$z = z_0 + \phi(y, z_0, \mu),$$

with $\phi(0, z_0, \mu) = 0$. We assume that ϕ is Lipschitz continuous in (z_0, μ). From (33) it can be shown using the Implicit Function Theorem that a surface given by ϕ is transformed by the local diffeomorphism to a surface given by a function $\bar{\phi}$ which must satisfy the equations

$$
\begin{aligned}
\bar{\phi}(\bar{y}, \bar{z}_0, \mu) &= (1 + \bar{R}(y, z_0, \mu))\phi(y, z_0, \mu) \\
\bar{y} &= By + g(y, z_0 + \phi(y, z_0, \mu), \mu) \\
\bar{z}_0 &= z_0 + R(z_0, \mu)
\end{aligned}
$$

where

$$\bar{R}(y, z_0, \mu)\phi(y, z_0, \mu) \equiv R(z_0 + \phi(y, z_0, \mu), \mu) - R(z_0, \mu).$$

Furthermore,

$$D_y\bar{\phi}(\bar{y}, \bar{z}_0, \mu) \cdot \frac{\partial \bar{y}}{\partial y} = (1 + \bar{R}(y, z_0, \mu))D_y\phi(y, z_0, \mu) + \bar{R}_y(y, z_0, \mu)\phi(y, z_0, \mu),$$

where

$$\frac{\partial \bar{y}}{\partial y} = B + g_y(y, z_0 + \phi, \mu) + g_z(y, z + \phi, \mu)D_y\phi(y, z_0, \mu).$$

For μ and $|y|$ small, $\frac{\partial \bar{y}}{\partial y}$ is invertible and so

$$D_y\bar{\phi}(\bar{y}, \bar{z}_0, \mu) = [(1 + \bar{R}(y, z_0, \mu))D_y\phi(y, z_0, \mu) + \bar{R}_y(y, z_0, \mu)\phi(y, z_0, \mu)](\frac{\partial \bar{y}}{\partial y})^{-1}.$$

In particular, we are interested in $y = 0$, for which

(34) $\qquad D_y\bar{\phi}(0, \bar{z}_0, \mu) = (1 + \bar{R}(0, z_0, \mu))D_y\phi(0, z_0, \mu)(B + g_y(0, z_0, \mu))^{-1}$

Now for each (z_0, μ) let

$$J_0^1(\phi) = \{\phi_1 : D\phi_1(0, z_0, \mu) = D\phi(0, z_0, \mu), \phi_1(0, z_0, \mu) = 0\}$$

This equivalence class is called the 1-jet of ϕ at $y = 0$ with target 0 [HPS]. Let J^1 denote the space of all such entities. It is clear that the local diffeomorphism maps 1-jets at (z_0, μ) into 1-jets at $(\bar{z}_0, \mu)$. Note in particular that $J_0^1(0)$ at z_0 is mapped into $J_0^1(0)$ at $\bar{z}_0$. In other words, because of the choice of local coordinates, the class of surfaces which are flat at $y = 0$ is invariant under the local diffeomorphism.

Next choose $a_0 < 0$ and $\alpha < 0$ and let

$$
\begin{aligned}
a_{i+1} &= a_i + R(a_i, \mu) \\
\beta &= \alpha + R(\alpha, \mu),
\end{aligned}
$$

and as before let

$$n_\mu = \min\{n : a_n \geq \alpha\}.$$

Set $I_\mu^+ = [a_0, a_1]$ and $I_\mu^- = [\alpha, \beta]$ and let

$$\phi_0^i \in Lip(I^- \times (0, \mu_0), C^1(X_2, R))$$

be the function which is identically 0 in the local coordinates indexed by i. If we follow the image of the surfaces defined by ϕ_0^i under the diffeomorphism T_μ, because

of the invariance of $\Gamma_\mu^{\bar\omega}$, we obtain a family of surfaces transversal to γ_μ^{i+1}. Denote by ψ_0 the function restricted to I^+ whose graph is the image of these surfaces in the $i+1$ local coordinates. Next for each μ define function spaces

$$
\begin{aligned}
K_\delta^+ \;=\; & \{ \jmath_0^1(\phi) \in Lip(I^+, J^1) : |D_y\phi - D_y\phi_0| \le \delta, \\
& D_y\phi(0, a_1) = (1 + \bar R(a_0)) D_y(0, a_0)(B + g_y(0, a_0))^{-1} \} \\
K_\epsilon^- \;=\; & \{ \jmath_0^1(\phi) \in Lip(I^-, J^1) : |D_y\phi| \le \epsilon, \\
& D_y\phi(0, \beta) = (1 + \bar R(\alpha)) D_y(0, \alpha)(B + g_y(0, \alpha))^{-1} \}.
\end{aligned}
$$

With these definitions the following propositions are almost immediate.

Proposition 6. *For each ϵ and μ and for each i, T_μ induces a function $\mathcal{E}_{gl}^i : K_\epsilon^- \to K_\delta^+$ which represents a map from the 1-jets of surfaces through γ_μ^i at I^- to 1-jets of surfaces through γ_μ^{i+1} at I^+.*

Proof. By the invariance of $\Gamma_\mu^{\bar\omega}$ and the definitions of I^+ and I^-, for each point $p \in \gamma_\mu^{i+1} \cap I^+ \times X_1 \times X_2$ there exists a unique point $p^* \in \gamma_\mu^i \cap I^- \times X_1 \times X_2$ such that p is the image of p^* under a minimal power m^j of the diffeomorphism. Suppose p and p^* have z coordinates z_0 and z_0^* respectively. Because T_μ is a diffeomorphism, if $\jmath_0^1(\phi^*)(y, z_0, \mu) \in K_\epsilon^-$, then a surface at z_0^* given by ϕ^* is mapped by $T_\mu^{m_j}$ to a surface at z_0 on γ_μ^{i+1}. Now by our assumptions on K_ϵ^- and K_δ^+ these surfaces are Lipschitz with respect to z_0. Further, since T_μ is a diffeomorphism, if ϵ is small the surfaces must be close to ϕ_0 in the C^1 sense. Thus they are the graph of a function $\phi(y, z_0, \mu)$. Define $\jmath_0^1(\phi) = \mathcal{E}_{gl}^i(\jmath_0^1(\phi^*))$. It is clear that $\jmath_0^1(\phi) \in K_\delta^+$ and that $\mathcal{E}_{gl}^i$ is well defined. $\square$

Proposition 7. *Given ϵ and δ there exists μ_0 such that for all i, T_μ induces a function $\mathcal{E}_{loc}^i : K_\delta^+ \to K_\epsilon^-$ which represents a map of 1-jets of cuves through γ_μ^i at I^+ to 1-jets of cuves through γ_μ^i at I^-.*

Proof. We can construct $\mathcal{E}_{loc}^i$ as we constructed previous functional maps. Define $\{\jmath(\phi_i)\}_{i=1}^{n_\mu}$ to be the sequence of 1-jets of surfaces through γ_μ^i induced by (34). From (34) for μ small enough

$$ |D_y\phi_{i+1}| \le q|D_y\phi_i| $$

where $0 \le q < 1$. If we define n_μ to be the number of iterations of transition from I^+ to I^- we have that $n_\mu \to \infty$ as $\mu \to 0$ and the result of the proposition is clear. $\square$

Using Propositions 6 and 7, let μ and ϵ be sufficiently small that for each local coordinate system indexed by $\jmath(i, i)$ the map $\mathcal{E}^i : K_\epsilon^- \to K_\epsilon^-$ given by

$$ \mathcal{E}^i(\jmath_0^1(\phi)) = \mathcal{E}_{loc}^i \circ \mathcal{E}_{loc}^i(\jmath_0^1(\phi)) $$

is well-defined. Let $E : K \to K$ be the function

$$ E = \cdots \times \mathcal{E}^{-1} \times \mathcal{E}^0 \times \mathcal{E}^1 \times \ldots $$

where

$$ K = \cdots \times K_\epsilon^- \times K_\epsilon^- \times K_\epsilon^- \ldots $$

is the complete metric space with the uniform norm

$$ \|\overline{\jmath_0^1(\phi)}\| = \sup_i \|D_y\phi^i|_{y=0}\|. $$

Proposition 8. *For μ small enough, E is a contraction on K.*

Proof. From (34)

$$
\begin{aligned}
|D_y\bar{\phi}_1(0,z_0) - D_y\bar{\phi}_2(0,z_0)| &\leq |1+\bar{R}||(B+g_y)^{-1}| \cdot |D_y\phi_1(0,z_0) - D_y\phi_2(0,z_0)| \\
&\leq q|D_y\phi_1(0,z_0) - D_y\phi_2(0,z_0)|,
\end{aligned}
$$

where $q < 1$ for μ sufficiently small. This inequality along with the boundedness $\mathcal{E}_{gl}$ and the fact that $n_\mu \to \infty$ as $\mu \to 0$ imply the proposition. $\square$

Now Proposition 8 implies that the map E has a unique fixed point. Let $j_0^1(\phi^*)$ be the zero component of this fixed point and define N^u to be the graphs of all the images and preimages of $D\phi^*$ under the tangent map. This is an invariant subbundle of $T\Gamma_\mu^{\bar{\omega}}$ which is tangent at $\Gamma_\mu^{\bar{\omega}}$ to $W^u_{\bar{\omega}\mu}$. Similarly, we may construct an invariant subbundle N^s which is tangent to $W^s_{\bar{\omega}\mu}$.

Since the functions $\eta^*_{\bar{\omega}}$ and $D\phi^*_{\bar{\omega}}$ may be made arbitraily small by taking μ small, we have for a neighborhood U of $\bar{0}$

$$
\begin{aligned}
\sup_{p\in U} |DT|_{N^s_p}| &\leq |A+f_x| + a(\mu) \\
\inf_{p\in U} m(DT|_{N^u_p}) &\geq m(B+g_y) - b(\mu) \\
\sup_{p\in U} |DT|_{T_p\Gamma_\mu^{\bar{\omega}}}| &\leq 1 + |R_z+h_z| + c(\mu) \\
\inf_{p\in U} m(DT|_{T_p\Gamma_\mu^{\bar{\omega}}}) &\geq 1 - |R_z+h_z| - c(\mu),
\end{aligned}
$$

where $a(\mu)$, $b(\mu)$, and $c(\mu)$ all limit to 0 as $\mu \to 0$. For any n,

$$
n = n_0 + m_0 + n_\mu + m_{j_1} + n_\mu + m_{j_2} + \ldots + n_\mu + m_{j_l}
$$

where, $n_0 < n_\mu$, $m_0 < \max(m_j)$ and l is the number of excursions away from U. Using this decomposition of n, if μ is small enough it can be shown that the inequalities (6) are satisfied for

$$
C = \max\left\{ \frac{(m(B+g_y) - b(\mu))^{n_\mu}|DT|^{k\overline{m}}}{m(DT)^{\overline{m}}}, \frac{|DT|^{\overline{m}}}{((A+f_x)+a(\mu))^{n_\mu} m(DT)^{k\overline{m}}} \right\}
$$

$$
\lambda = \frac{m(DT)^{\frac{\overline{m}}{n_\mu+\overline{m}}}(m(B+g_y) - b(\mu))^{\frac{n_\mu}{n_\mu+\overline{m}}}}{|DT|^{\frac{k\overline{m}}{n_\mu+\overline{m}}}(1+|R_z+h_z|+c(\mu))^{\frac{kn_\mu}{n_\mu+\overline{m}}}}
$$

$$
\nu = \frac{|DT|^{\frac{\overline{m}}{n_\mu+\overline{m}}}(|A+f_x|+a(\mu))^{\frac{n_\mu}{n_\mu+\overline{m}}}}{m(DT)^{\frac{k\overline{m}}{n_\mu+\overline{m}}}(1-|R_z+h_z|-c(\mu))^{\frac{kn_\mu}{n_\mu+\overline{m}}}}
$$

where $\overline{m} = \max\{m_j\}$ and $\underline{m} = \min\{m_j\}$. Now for μ small, the number n_μ is large whereas m^{ω_j} is fixed. Thus for μ sufficiently small $\lambda < 1$ and $\nu > 1$.

Now since the integers $\{m_j\}$ are fixed, the changes of coordinates are uniformly bounded in the C^1 sense, and n_μ is independent of the changes, we may make the preceeding estimates uniformly with respect to $\bar{\omega}$. This proves absolute k-normal hyperbolicity of each $\Gamma_\mu^{\bar{\omega}}$ with uniform constants. Thus Λ_μ is a UPH-set. In fact, for uniform partial hyperbolicity the constants may be relaxed by setting $k = 1$. $\square$

References

[Af74] V.S. Afraimovich, *Some Global Bifurcations Related to the Appearence of a Countable Set of Periodic Motions.* Dissertation, Gorky State University, 1974.

[Af90] V.S. Afraimovich. On Smooth Changes of Variables. *Selecta Mathematica Sovietica* **9** (1990) No. 3, 205-213 Birkhauser Verlag, Basel.

[AS74] V.S. Afraimovich, and L.P. Shil'nikov. On Some Global Bifurcations Connected with Disappearance of a Saddle-Node Fixed Point. *Dokl.Akad.Nauk SSSR***219** (1974) 1281-1285 [Russian]. Translated in *Soviet Math. Dokl.* **15** (1974) No. 3.

[AS82] V.S. Afraimovich and L.P. Shil'nikov. On a bifurcation of Codimension 1 Leading to a Countable Set of Tori. *Soviet Math. Dokl.* **25** (1982) No. 1, 101-105.

[AAIS] V.S. Afraimovich, V.I. Arnol'd, Y.S. Il'yashenko, and L.P. Shil'nikov. *The Theory of Bifurcations,* in *Modern Problems of Mathematics, Fundamental Directions,* Vol. 5, VINITI, Moscow, Nauka (1986) [Russian], Translated in *Encyclopedia of Mathematical Sciences* Vol. 5, Springer Verlag, 1993.

[Bo] R. Bowen. Symbolic dynamics for hyperbolic flows. *American Journal of Mathematics* **95** (1973) 429-460.

[BP] M. Brin and Y.B. Pesin. Partially hyperbolic dynamical systems. *Izv. Akad. Nauk SSSR, Ser. Mat.* **38** (1974) 170-212 [Russian] Zbl. 304.58017.

[CL] S.-N. Chow. and K. Lu. C^k centre unstable manifolds. *Proceeding of the Royal Society of Edinburgh* **108A** (1988) 303-320.

[De] A. Denjoy. Sur les courbes défines par les équations différentielles á la surfaces du tore. *J. Math. Pures Appl.* **17** (1932) 333-375.

[HPS] M. Hirsch, C.C. Pugh, and M. Shub. *Invariant Manifolds.* Lecture Notes in Mathematics, **583** Springer-Verlag 1977.

[Ni] Z. Nitecki. *Differentiable Dynamics, An Introduction to the Orbit Structure of Diffeomorphisms.* MIT Press, 1971.

[PM] Y. Pomeau and P. Manneville, Intermittent transition to turbulence in dissipative dynamical systems, *Comm. Math. Phys.* **74** (1980), 189-197.

[NPT] S. Newhouse, J. Palis, and F. Takens. Bifurcations and Stability of Families of Diffeomorphisms. *Bulletin of the AMS,* **82** (1976) 499.

[Ru] W. Rudin. *Principals of Mathematical Analysis.* McGraw–Hill, 3rd Ed. 1976.

[PS] Y.B. Pesin and Y.G. Sinai. Gibbs measures for partially hyperbolic attractors. *Ergodic Theory and Dynamical Systems* **12** (1982), 417-438.

[Sh] L.P. Shil'nikov. A case of the existence of a denumerable set of periodic motions. *Sov. Math. Dokl.,* **6** (1965) 163-166.

[Ta71] F. Takens. Partially Hyperbolic Fixed Points. *Topology* **10** (1971) 133-147.

[Ta89] F. Takens, Intermittancy; global aspects, in *Dynamical systems, Valparaiso 1986,* LNM **1331**, Springer Verlag (1989).

[Yo] T. Young. C^k-smoothness of Invariant Curves in a Global Saddle-Node Bifurcation. (1993) preprint.

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