Lévy Processes in Cones of Banach Spaces

Víctor Pérez-Abreu*
Centro de Investigación en Matemáticas
CIMAT, Guanajuato, Mexico

Alfonso Rocha-Arteaga
Escuela de Ciencias Físico-Matemáticas
Universidad Autónoma de Sinaloa
Culiacan, Mexico

Abstract

Subordinators in Banach spaces are studied. The existence of the special Lévy-Khintchine representation is related to the geometry of the space and a Pettis integral with respecto to the underlying Lévy measure. Rates of growth of subordinators in a special type of Banach spaces are studied, including laws of iterated logarithm which give new results in finite dimensions bigger than one.

Key Words: subordinator, Pettis integral, law of iterated logarithm.
AMS Subject classification: 60G51, 60B05.

1 Introduction

One dimensional increasing Lévy processes (also called subordinators) have been widely studied. They have important properties and are useful in building other Lévy processes; see [4], [5] and [21]. A one dimensional subordinator $\{\sigma_t : t \geq 0\}$ is a nonnegative Lévy process characterized by the special form of the Lévy-Khintchine representation of its Fourier transform

*This work was completed while the first author was visiting the School of Mathematics of the Georgia Institute of Technology during the Fall 2003.

$$Ee^{iu\sigma_t} = \exp\left\{t\int_{(0,\infty)}\left(e^{iux}-1\right)\nu\left(dx\right)+ itu\gamma_0\right\} \quad u\in\mathbb{R}, \tag{1}$$

where there is no Gaussian part, the *drift* γ_0 is nonnegative and the Lévy measure ν is concentrated in the cone [0,∞) with the order of singularity

$$\int_{(0,1]} x\nu(dx) < \infty. \tag{2}$$

The study of subordinators in higher dimensional spaces leads to consider Lévy processes with values in cones. When a cone is proper, a Lévy process is cone-valued if and only if it is cone-increasing. Then it is natural to call these processes cone-valued subordinators or simply subordinators. In the finite dimensional case, cone-valued Lévy processes are already discussed in the classical books by Bochner [7] and Skorohod [24] and have been recently studied in [3], [18] and [19].

The purpose of this paper is to study the structure of Lévy processes with values in cones of infinite dimensional Banach spaces and investigate whether there is an intrinsic relation between probability and functional analytic aspects. As a basic point of interest one may ask whether there exists a special form of the Lévy-Khintchine representation (SLKR) similar to the one dimensional case (1). A restricted class of Banach spaces where the answer is affirmative was considered in [11]. In the present work we conduct a systematic study of cone-valued Lévy processes in more general Banach spaces and on the convergence of their non-compensated jumps. We find that the existence of the SLKR (in which case we say that the subordinator is regular) is related to the geometry of the cone. Specifically, it is shown that a cone-valued subordinator is always regular for proper normal cones containing no copy of c_0^+, the cone of nonnegative scalar sequences converging to zero. This fact is related to the existence of a Pettis integral with respect to the underlying Lévy measure.

¿From the application point of view, we consider cone-valued Lévy processes in Birkhoff-Kakutani spaces, which arises from Banach spaces where there is a continuous linear functional that agrees with the norm in the cone. They are named in this way after Birkhoff [6] and Kakutani [14] who studied Banach lattices and the so-called *AL-spaces* which form a subclass of the more general Birkhoff-Kakutani spaces. Specific examples are the Euclidean space $\mathbb{R}^d$, the space of real symmetric $d\times d$-matrices $\mathbb{M}^d$, the space $L_1(H)$ of trace class operators in a Hilbert space H and duals of C^*-algebras. In connection with Lévy processes, we prove that Birkhoff-Kakutani-valued subordinators inherit several properties from the one dimensional subordinators associated to the norm-process. As a consequence of this inheritance, sample path properties on the asymptotic behavior of one dimensional subordinators are easily transferred to this Banach space setting, such as the one-dimensional laws of iterated logarithm in [4]. These asymptotic properties give new results even for subordinators taking values in the octant $\mathbb{R}_+^d$ and in the cone $\mathbb{M}_+^d$ of nonnegative definite $d \times d$-matrices when d is bigger than one.

The paper is organized as follows. In section 2 we review basic facts about convergences in cones of Banach spaces. In Section 3 a systematic and detailed analysis of cone-valued Lévy processes is done. We first review some facts about Banach space-valued Lévy processes and give sufficient conditions to have the SLKR and being cone-valued. These conditions include the existence of a Pettis integral with respect to the Lévy measure. Then we consider the converse problem, by studying the convergence of the non-compensated sum of jumps of cone-valued subordinators and conditions on the geometry of the cone for the existence of the SLKR and the associated Pettis integral. In section 4 we concentrate in regular subordinators, including the class of LK-cones in which every subordinator is regular. We prove that the cones considered in [11] are particular cases of LK-cones. We also present concrete examples of regular subordinators in arbitrary proper cones of general Banach spaces, showing that there exist cone-valued regular subordinators even when the cone has a copy of c_0^+. Finally, in Section 5 we deal with subordinators in Birkhoff-Kakutani cones, their one dimensional similarities and the relation with the associated norm process. As an application of this intimate relation we point out laws of iterated logarithm for Birkhoff-Kakutani valued subordinators.

2 Preliminaries on Cones in Banach Spaces

Recall that a nonempty closed convex set K of B is said to be a *cone* if $\lambda \geq 0$ and $x \in K$ imply $\lambda x \in K$. Note that a cone is closed under finite sums and contains the zero element. A cone K is *generating* if $B = K - K$, that is, every $x \in B$ can be written as $x = x_1 - x_2$ for $x_1 \in K$ and $x_2 \in K$. It is *proper* if $x = 0$ whenever x and $-x$ are in K. The *dual cone* K^* of K is defined as $K^* = \{f \in B^* : f(s) \geq 0 \text{ for every } s \in K\}$. A proper cone K of a Banach space B induces a partial order on B by defining $x_1 \leq_K x_2$ whenever $x_2 - x_1 \in K$ for $x_1 \in B$ and $x_2 \in B$. Given a sequence $(x_n) = (x_n)_{n=1}^\infty$ in B, if $x_n \leq_K x_{n+1}$ (respectively $x_{n+1} \leq_K x_n$) for each $n \geq 1$, the sequence is called *K-increasing* (respectively *K-decreasing*). Likewise, a function $f : [0, \infty) \to B$ is called *K-increasing* (respectively *K-decreasing*) if $f(t_1) \leq_K f(t_2)$ (respectively $f(t_2) \leq_K f(t_1)$) for $t_1 \leq t_2$.

A sequence (x_n) in K is said to be *K-majorized* if there exists $x \in K$ with $x_n \leq_K x$, for $n \geq 1$. A cone K is said to be *regular* if every K-increasing and K-majorized sequence in K is norm convergent. A cone K is said to have *generating dual* if K^* is generating for B^*. A useful characterization in Banach spaces is that K^* is generating for B^* if and only if K is normal, *i.e.,* $0 \leq_K x \leq_K y$ for $y \in K$, implies $\|x\| \leq \lambda \|y\|$, where $\lambda > 0$ ([13, Th. 1.5.4 and

Prop. 1.5.7]).

Given two cones K_1 and K_2 of the Banach spaces B_1 and B_2, it is said that K_1 is *isomorphic to* K_2 if there is an isomorphism φ between $\overline{Span}(K_1)$ and $\overline{Span}(K_2)$ such that $\varphi(K_1) = K_2$.

Let c_0 denote the Banach space of real sequences $a = (a_n)$ converging to zero with norm $\|a\| = \sup_{n\geq 1} |a_n|$ and let c_0^+ denote the cone of c_0 consisting of all sequences with nonnegative terms. The cone c_0^+ plays an important role in the study of convergence of cone valued series and sequences. The following Bessaga-Pelczynski type result for convergence of series in Banach spaces follows straightforward from Theorem 5.8 in [9], assuming that the summands are cone-valued. Recall that a series $\sum_{k=1}^{\infty} x_k$ in B is weakly *unconditionally Cauchy (w.u.C.)* if for all $f \in B^*$, $\sum_{k=1}^{\infty} |f(x_k)|$ is a real convergent series.

Proposition 1 *Let K be a cone of a Banach space B. In order that any w.u.C. series $\sum_{k=1}^{\infty} x_k$, with $x_n \in K$, $n \geq 1$, be (norm) unconditionally convergent, it is necessary and sufficient that K contain no subcone isomorphic to c_0^+.*

A major result in the convergence of cone-valued sequences is given by a result in [13, Th. 12.4.5] proved in the more general setting of Fréchet spaces.

Proposition 2 *Let K be a proper normal cone. Then K is regular if and only if K contains no subcones isomorphic to c_0^+.*

In view of the main results of this work we introduce the following terminology.

Definition 3 *A proper cone K is said to be an LK-cone if it is normal and contains no copy of c_0^+ .*

Remark 4 *a) A Banach space that contains no copy of c_0 does not have cones isomorphic to c_0^+.*

b) Every Banach space has a cone with no subcones isomorphic to c_0^+.

c) There are Banach spaces containing a copy of c_0 with cones containing no subcones isomorphic to $c_0^{+\cdot}$.

There are several well known examples of Banach spaces containing no copy of c_0. Among others we mention reflexive Banach spaces, weakly sequentially complete spaces and Schur spaces for which proper cones with generating duals are LK-cones. More concretely, the space of real sequences l_p, the space of p-integrable functions in a measure space $L_p(X, \mathcal{A}, \mu)$ and the space $L_p(H)$ of p-trace class operators in a Hilbert space H, for any $1 \leq p < \infty$, are examples of spaces containing no copy of c_0 and where their corresponding proper cones l_p^+, $L_p^+(X, \mathcal{A}, \mu)$ and $L_p^+(H)$ have generating duals.

For a cone containing a copy of c_0^+ -and therefore an example of a cone that is not of LK-type- we mention the cone of positive compact selfadjoint operators $K^+(H)$ in a Hilbert space H with the operator norm. As for an example of a cone with no generating dual, we mention the subcone of c_0 defined by $K = \{(a_n) \in c_0 : a_n + a_{n+1} \geq 0\}$; see [13, Ex. 12.3.5].

3 Cone-valued Lévy Processes

Throughout this section B will denote a separable Banach space with norm $\|\cdot\|$, topological dual space B^* and dual norm $\|\cdot\|'$.

3.1 The special Lévy-Khintchine representation

Recall that a Banach space valued (B-valued) *Lévy process* $X = \{X_t : t \geq 0\}$ is a stochastic process with values in B defined on a probability space $(\Omega, \mathcal{F}, P)$ such that i) $X_0 = 0$ a.s., ii) it has independent and stationary increments, ii) it is stochastically continuous with respect to the norm $\|\cdot\|$ (for every $\varepsilon > 0$, $P(\|X_t - X_s\| > \varepsilon) \to 0$ as $s \to t$) and iii) almost surely the paths are right-continuous in $t \geq 0$ and have left-limits in $t > 0$ (*càdlàg*) with respect to the norm.

Let $D_0 = \{0 < \|x\| \leq 1\}$. The following Lévy-Khintchine representation for B-valued Lévy processes is presented in [11].

Theorem 5 *Let $\{X_t : t \geq 0\}$ be a Lévy process in a separable Banach space B. Then, its characteristic functional is such that*

$$Ee^{if(X_t)} = \exp\left\{t\left(-\frac{1}{2}(Af, f) + if(\gamma) + \psi(f, \nu)\right)\right\} \qquad f \in B^*, \tag{3}$$

where

$$\psi(f, \nu) = \int \left[e^{if(x)} - 1 - if(x)1_{D_0}(x)\right] \nu(dx), \tag{4}$$

$\gamma \in B$, A is a nonnegative selfadjoint operator from B^* to B, the Lévy measure ν on $\mathcal{B}(B \backslash \{0\})$ the Borel σ-algebra of $B \backslash \{0\}$, is such that for any $f \in B^*$

$$\int_{D_0} |f(x)|^2 \nu(dx) < \infty. \tag{5}$$

The triplet of parameters (A, ν, γ) in Theorem 5 is called the *generating triplet* of the Lévy process X and it is unique.

Remark 6 *a) For infinite dimensional Banach spaces, Lévy measures are not characterized by the condition* $\int_{D_0} \|x\|^2 \nu(dx) < \infty$*; see [1] and [16]. They are rather identified by the fact that the mapping*

$$f \longmapsto \exp\{\psi(f, \nu)\} \quad f \in B^* \tag{6}$$

is the characteristic functional of some probability measure on B*.*

b) Denote $X_{s-} := \lim_{s \uparrow t} X_s$ *and let* $C \in \mathcal{B}_0$ *the ring of Borel sets of* B *with positive distance from* 0*, then the process*

$$X_t^C = \sum_{s<t} (X_s - X_{s-}) \, 1_C \, (X_s - X_{s-}) \tag{7}$$

is well defined and represents the sum of jumps of the process X *occurred up to time* t *and took place in* C*. Its characteristic functional has the form*

$$Ee^{if(X_t^C)} = exp\left\{ t \int_C \left(e^{if(x)} - 1 \right) \nu(dx) \right\} \qquad f \in B^*. \tag{8}$$

As in the finite dimensional case, there is a one-to-one relation between Lévy processes and infinitely divisible laws in Banach spaces.

Proposition 7 *Let* μ *be an infinitely divisible probability measure with generating triplet* (A, ν, γ)*. Then there exists a Lévy process* $\{X_t : t \geq 0\}$ *with generating triplet* (A, ν, γ) *such that* X_1 *has the law* μ *and viceversa.*

The Lévy-Khintchine representation for Lévy processes of bounded variation is also derived in [11].

Proposition 8 *Let* $\{Z_t : t \geq 0\}$ *be a* B*-valued Lévy process. Then,* $\{Z_t\}$ *has bounded variation on each interval* $[0, t]$*, almost surely, if and only if, it has characteristic functional given by*

$$Ee^{if(Z_t)} = exp\left\{ t \int_B \left(e^{if(x)} - 1 \right) \nu(dx) + itf(\gamma) \right\} \qquad f \in B^*,$$

where the Lévy measure ν *satisfies*

$$\int_{D_0} \|x\| \, \nu(dx) < \infty. \tag{9}$$

A straightforward but key observation is that a Lévy process is cone-increasing if and only if it is cone-valued.

Proposition 9 *Let K be a proper cone of B and let $\{Z_t : t \geq 0\}$ be a B-valued Lévy process. Then the following are equivalent:*

a) For any fixed $t \geq 0$, Z_t is concentrated on K almost surely.

b) Almost surely, $Z_t(\omega)$ is K-increasing in t.

In view of the above result and similar to the one dimensional case, we shall say that a K-increasing Lévy process in B is a K-*subordinator* or *subordinator* if the underlying cone is well understood.

Given a Lévy measure ν on $\mathcal{B}(B\backslash\{0\})$, we shall say that an element $I_\nu \in B$ is a ν-*Pettis centering* if

$$\int_{D_0} |f(x)|\, \nu(dx) < \infty \qquad \text{for every } f \in B^* \tag{10}$$

and $f(I_\nu) = \int_{D_0} f(x)\nu(dx)$ for every $f \in B^*$. Sometimes we shall write $I_\nu = \int_{D_0} x\nu(dx)$.

Sufficient conditions on the generating triplet (A, ν, γ) of a Lévy process $\{Z_t : t \geq 0\}$ to be cone-subordinator are now presented. The only assumption on the cone K is to be proper.

Theorem 10 *Let K be a proper cone of a separable Banach space B. Let $\{Z_t : t \geq 0\}$ be a Lévy process in B with generating triplet (A, ν, γ). Assume the following three conditions*

a) $A = 0$,

b) $\nu(B\backslash K) = 0$, i.e., ν is concentrated on K and

c) there exists a ν-Pettis centering $I_\nu = \int_{D_0} x\nu(dx)$ such that $\gamma_0 := \gamma - I_\nu \in K$.

Then the process Z is a subordinator.

Observe that assumptions (a)-(c) above give the *special Lévy-Khintchine representation*

$$Ee^{if(Z_t)} = exp\left\{t\int_K \left(e^{if(x)} - 1\right)\nu(dx) + itf(\gamma_0)\right\}, \tag{11}$$

since for all $f \in B^*$, $f(\gamma_0) = f(\gamma) - \int_{D_0} f(x)\nu(dx)$. Before proving the theorem we point out the following facts.

Remark 11 *a) The existence of a ν-Pettis centering is related to the ν-integrability of the function $h(x) = x1_{D_0}(x)$ in the sense of Pettis. Recall that the B-valued function h is Pettis integrable w.r.t. ν if (10) is satisfied and for any $C \in \mathcal{B}(B\backslash\{0\})$ there exists and element I^C in B such that $f(I^C) = \int_{D_0} f(x)1_C(x)\nu(dx)$ for every $f \in B^*$. The element I^C is called* *the Pettis integral and we have that* $I_\nu = I^U$, *where* $U = \{x \in B : x \in D_0\}$. *We refer to [17] for a recent survey on the Pettis integral.*

b) There are subordinators in Banach spaces whose Lévy measure satisfies the stronger integrability condition (9) for the norm (see Examples 23 and 24). This yields the existence of $I_\nu = \int_{D_0} x\nu(dx)$ *as a Bochner integral. This is always the case for subordinators in Birkhoff-Kakutani cones of Section 5.*

The Laplace transform of a subordinator with Fourier transform (11) is obtained by standard analytic continuation.

Proposition 12 *Let* K *be a proper cone of a separable Banach space* B *and let* $\{Z_t : t \geq 0\}$ *be a Lévy process with the special representation (11). Let* S *be a* K*-valued infinitely divisible random variable with the same law as* Z_1. *Then*

a) The Laplace transform of S *is of the form*

$$Ee^{-f(S)} = exp\{-\Phi(f)\} \qquad f \in K^* \tag{12}$$

with Laplace exponent

$$\Phi(f) = \int_K \left(1 - e^{-f(x)}\right)\nu(dx) + f(\gamma_0). \tag{13}$$

b) The Laplace transform of Z_t *is given by*

$$Ee^{-f(Z_t)} = exp\{-t\Phi(f)\} \qquad f \in K^*. \tag{14}$$

Proof of Theorem 10. We have to show that Z_t takes values in K almost surely. For each $\varepsilon > 0$, consider the jumps sum process $Z_t^{\triangle_\varepsilon}$ defined by (7) where $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\}$. We observe that $Z_t^{\triangle_\varepsilon} \in K$ almost surely. Indeed, suppose that there exists $C \in \mathcal{B}_0$ contained in $B\backslash K$ such that the process in (7) satisfies $Z_t^C \neq 0$ with positive probability, then $0 < P\left(Z_t^C \neq 0\right) \leq 1 - e^{-t\nu(C)}$ which is not possible since $\nu(C) = 0$. Thus $Z_t^{\triangle_\varepsilon} \in K$ a.s.

Similar to the one-dimensional case, since $\gamma_0 \in K$ it is enough to prove that $J_t = Z_t - t\gamma_0 \in K$ almost surely. Notice that J_t and Z_t have the same jumps and therefore $J_t^{\triangle_\varepsilon} = Z_t^{\triangle_\varepsilon}$ a.s. Hence $\left\{J_t^{\triangle_\varepsilon}\right\}$ is a K-valued process and its characteristic functional is given by (8) on the Borel set $K \cap \{x : \|x\| > \varepsilon\}$. Since K is convex and closed then $K = \cap_{k=1}^\infty \{x : g_k(x) \geq 0\}$ for a sequence of continuous linear functionals g_k. Then, for $u \geq 0$,

$$Ee^{-ug_k(J_t^{\triangle_\varepsilon})} = exp\left\{t\int_{K\cap\{x:\|x\|>\varepsilon\}} \left(e^{-ug_k(x)} - 1\right)\nu(dx)\right\}.$$

Letting $\varepsilon \downarrow 0$ we get

$$Ee^{-u \lim_{\varepsilon\downarrow 0} g_k(J_t^{\triangle\varepsilon})} = exp\left\{t\int_K \left(e^{-ug_k(x)}-1\right)\nu(dx)\right\},$$

where the right hand side is finite by (10) and tends to 1 as u decreases to zero. Therefore $\lim_{\varepsilon\downarrow 0} g_k(J_t^{\triangle\varepsilon})$ exists a.s. and it is nonnegative for each k. From (11) and the fact that $Z_t^{\triangle\varepsilon}$ and $Z_t - Z_t^{\triangle\varepsilon}$ are independent

$$Ee^{ig_k(J_t-J_t^{\triangle\varepsilon})} = exp\left\{t\int_{K\cap\{x:\|x\|\leq\varepsilon\}} \left(e^{ig_k(x)}-1\right)\nu(dx)\right\}$$

where the right hand side tends to 1 as $\varepsilon \downarrow 0$ and therefore $g_k(J_t) = \lim_{\varepsilon\downarrow 0} g_k(J_t^{\triangle\varepsilon})$ almost surely. Since $g_k\left(J_t^{\triangle\varepsilon}\right)$ is nonnegative and increasing as $\varepsilon \downarrow 0$ for all k, then $g_k(J_t - J_t^{\triangle\varepsilon}) \geq 0$. Hence $J_t - J_t^{\triangle\varepsilon}$ and $J_t^{\triangle\varepsilon}$ are in K and therefore $J_t \in K$. ■

Proposition 8 and Theorem 10 yield a stronger result.

Corollary 13 *Let K be a proper cone of B. Let $\{Z_t : t \geq 0\}$ be a B-valued Lévy process with generating triplet (A, ν, γ) satisfying (a), (b), (c) in Theorem 10 as well as the additional condition (9). Then $\int_{D_0} x\nu(dx)$ is a Bochner integral and the process Z is a subordinator of bounded variation.*

While in finite dimensions every subordinator is of bounded variation, for infinite dimensional Banach spaces there are subordinators of unbounded variation. This is the case when the ν-Pettis centering I_ν is not Bochner integrable. For example, let $\sigma_n, n \geq 1$, be a sequence of one-dimensional independent subordinators, where for each $n \geq 1$, σ_n has generating triplet $(0, \nu_n, n^{-1})$ with Lévy measure $\nu_n = n^{-2}\delta_{\{n\}}$. Then $Z_t = (\sigma_1(t), \sigma_2(t), ...)$ is a subordinator in c_0^+ with drift $\gamma_0 = 0$ and Lévy measure $\nu(C) = \sum_{n=1}^{\infty} n^{-2} 1_C(ne_n)$, where $\{e_n\}_{n\geq 1}$ is the sequence of unit vectors in c_0^+. Then

$$\int_{D_0} \|x\|\,\nu(dx) = \sum_{n=1}^{\infty} n^{-1} = \infty,$$

but $I_\nu = \sum_{n=1}^{\infty} n^{-1} e_n \in c_0^+$ and for any $f \in l_1 = c_0^*$, $f = (f_{1,}, f_{2,}...)$,

$$\int_{D_0} |f(x)|\,\nu(dx) = \sum_{n=1}^{\infty} n^{-1}\,|f_n| < \infty.$$

¿From Theorem 10 we have that Z has the special Lévy-Khintchine representation and therefore from (9) in Proposition 8, Z cannot have bounded variation.

3.2 Convergence of non-compensated jumps

Whether the converse of Theorem 10 is true or not, relies in the analysis of the sums of the non-compensated jumps of the subordinator falling into the cone. While in the finite dimensional case these sums are always convergent ([24, Th. 3.21]), for infinite dimensional Banach spaces a more detailed analysis is needed.

Fix $t > 0$ and consider for each $\varepsilon > 0$, the sum of non-compensated jumps of size bigger than ε, defined in (7),

$$Z_t^{\triangle_\varepsilon} = \sum_{s<t} \Delta Z_s 1_{\triangle_\varepsilon}(\Delta Z_s) \tag{15}$$

where $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\}$ and $\Delta Z_s = Z_s - Z_{s^-}$. Observe that for any sequence $\varepsilon_n \downarrow 0$, one has the alternative representation of (15) as sums of independent random elements in K

$$Z_t^{\triangle_{\varepsilon_n}} = \sum_{k=1}^{n} \xi_k \tag{16}$$

where $\xi_1 = Z_t^{\triangle_{\varepsilon_1}}, \xi_k = \sum_{s<t} \Delta Z_s 1_{\triangle_{\varepsilon_k,\varepsilon_{k-1}}}(\Delta Z_s), k \geq 2$, with $\triangle_{\varepsilon_k,\varepsilon_{k-1}} = \{x : \varepsilon_k < \|x\| \leq \varepsilon_{k-1}\}$.

We first prepare a technical lemma on the one-dimensional processes $f(Z_t)$, $f \in B^*$.

Lemma 14 *Let K be a proper cone of B and let $\{Z_t : t \geq 0\}$ be a K-valued Lévy process. For any $f \in K^*$, the one dimensional family $f(Z_t^{\triangle_\varepsilon})$ is nonnegative, increasing as $\varepsilon \downarrow 0$ and bounded by $f(Z_t)$, almost surely.*

Proof. The process $\{f(Z_t) : t \geq 0\}$ is a one dimensional subordinator since $\{Z_t\}$ is a K-valued Lévy process and $f \in K^*$. Its corresponding sum of non-compensated jumps

$$[f(Z_t)]^{\triangle_\varepsilon^f} := \sum_{s<t} (f(Z_s) - f(Z_{s-})) 1_{\{f(x):|f(x)|>\varepsilon\}} (f(Z_s) - f(Z_{s-}))$$

is nonnegative, increasing as $\varepsilon \downarrow 0$ and bounded by $f(Z_t)$. Hence $lim_{\varepsilon\downarrow 0} [f(Z_t)]^{\triangle_\varepsilon^f}$ exists.

Let ε_n be any decreasing sequence to 0 and let $\varepsilon_n(f) = \varepsilon_n \|f\|^{'}$. Assume, without loss of generality, that $\|f\|^{'} > 0$. If $f(\Delta Z_s) > \varepsilon_n(f)$ then $\|\Delta Z_s\| > \varepsilon_n$. Therefore $[f(Z_t)]^{\triangle^f_{\varepsilon_n(f)}} \leq f(Z_t^{\triangle_{\varepsilon_n}})$ almost surely for every $n \geq 1$ and hence $f(Z_t^{\triangle_{\varepsilon_n}})$ is nonnegative almost surely. Next, when $\varepsilon_2 < \varepsilon_1$ we have that $\triangle_{\varepsilon_1} \subset \triangle_{\varepsilon_2}$ and hence $f(Z_t^{\triangle_{\varepsilon_2}} - Z_t^{\triangle_{\varepsilon_1}}) = \sum_{s<t} f(\Delta Z_s) 1_{\{x:\varepsilon_2<\|x\|\leq\varepsilon_1\}}(\Delta Z_s) \geq 0$, proving the increasingness. Finally, let $\varepsilon > 0$ and note that $f(Z_t^{\triangle_\varepsilon}) = \sum_{s<t} f(\Delta Z_s) 1_{\{x:\|x\|>\varepsilon\}}(\Delta Z_s)$ is bounded by $f(Z_t)$ almost surely since it represents a finite number of jumps of the one-dimensional subordinator $\{f(Z_t)\}$. ∎

For each $t > 0$ the jumps sum $Z_t^{\triangle\varepsilon}$ is K-increasing as function of ε and it is K-majorized by Z_t.

Lemma 15 $Z_t^{\triangle\varepsilon_2} - Z_t^{\triangle\varepsilon_1} \in K$ *for* $\varepsilon_1 < \varepsilon_2$ *and* $Z_t - Z_t^{\triangle\varepsilon} \in K$ *for* $\varepsilon > 0$.

Proof. If $\varepsilon_1 < \varepsilon_2$ then $Z_t^{\triangle\varepsilon_2} - Z_t^{\triangle\varepsilon_1} = \sum_{s<t} \Delta Z_s 1_{\{x:\varepsilon_1<\|x\|\le\varepsilon_2\}}(\Delta Z_s) \in K$. This proves the first assertion. If $0 \le s < t$ then $Z_{t-} - Z_s = \lim_{\varepsilon\downarrow 0} Z_{t-\varepsilon} - Z_s \in K$. Hence, if $0 < s_1 < s_2 < ... < s_n \le t$

$$
\begin{aligned}
Z_t - \sum_{k=1}^{n} (Z_{s_k} - Z_{s_k-}) &= Z_t + \sum_{k=1}^{n} (Z_{s_k-} - Z_{s_k}) \\
&= Z_t - Z_{s_n} + (Z_{s_2-} - Z_{s_1}) + ... + (Z_{s_n-} - Z_{s_{n-1}}) + Z_{s_1-}
\end{aligned}
$$

which belongs to K. Thus $Z_t - Z_t^{\triangle\varepsilon} \in K$ for all $\varepsilon > 0$. ∎

In order to get more insight into the structure of cone-valued Lévy processes on infinite dimensional Banach spaces, we require additional assumptions on the cone. The following *weak* result always holds for subordinators with values in normal cones.

Theorem 16 *Let K be a proper normal cone of B and let $\{Z_t : t \ge 0\}$ be a K-valued Lévy processes. Then the jumps sum $Z_t^{\triangle\varepsilon_n}$ is w.u.C. almost surely for any sequence $\varepsilon_n \downarrow 0$.*

Proof. By assumption, any continuous linear functional f on B can be decomposed into $f = f^+ - f^-$ where $f^+ \in K^*$ and $f^- \in K^*$. It is enough to prove the assertion for any positive (with respect to K) linear functional since $|f| = f^+ + f^-$. Let $f \in K^*$. ¿From (16) $\sum_{k=1}^{n} f(\xi_k) = f(Z_t^{\triangle\varepsilon_n})$ which is increasing as function of ε_n and bounded, by Lemma 14. Therefore $\sum_{k=1}^{n} f(\xi_k)$ converges a.s. ∎

Under the additional condition on the norm convergence of the jumps sum, the converse of Theorem 10 is true for proper normal cones.

Theorem 17 *Let K be a proper normal cone of a separable Banach space B and let $\{Z_t : t \ge 0\}$ be a K-subordinator. If the jumps sum $Z_t^{\triangle\varepsilon}$ converges a.s. in norm as $\varepsilon \downarrow 0$, then the characteristic functional of Z has the special Lévy-Khintchine representation (11) and its generating triplet satisfies conditions (a)-(c) in Theorem 10. In particular, there exists a ν-Pettis centering I_ν such that $\gamma_0 = \gamma - I_\nu$.*

Proof. Step 1. We first prove (b), that is, ν is concentrated on K. For some sequence of continuous linear functionals g_k we have $K = \cap_{k=1}^{\infty} \{x : g_k(x) \geq 0\}$. Then for each $k \geq 1$, the one dimensional subordinator $\{g_k(Z_t)\}$ has only nonnegative jumps since it is nonnegative. So if C is contained in $\cup_{k=1}^{\infty} \{x : g_k(x) < 0\}$ then $\nu(C) = 0$. Thus ν is concentrated on K.

Step 2. We now show that the Gaussian part is zero. Let $\triangle_\varepsilon = \{x : \|x\| > \varepsilon\} \cap K$. By assumption $Z_t^{\triangle_\varepsilon}$ converges strongly to some Z_t^0 as $\varepsilon \downarrow 0$ almost surely. Therefore the process $\{Z_t - Z_t^0\}$ is continuous almost surely and

$$Ee^{if(Z_t - Z_t^0)} = exp\left\{-\frac{1}{2}(Af, f) + if(\gamma_0)\right\}. \tag{17}$$

Since $Z_t - Z_t^{\triangle_\varepsilon} \in K$ for all $\varepsilon > 0$ (Lemma 15) then $Z_t - Z_t^0 \in K$ by closedness of K. Hence for every $f^+ \in K^*$ the process $\{f^+(Z_t - Z_t^0)\}$ is nonnegative, continuous and Gaussian, therefore $var\,(f^+(Z_t - Z_t^0)) = (Af^+, f^+) = 0$. This gives $var\,(f(Z_t - Z_t^0)) = 0$ for any $f \in B^*$, which shows that the covariance operator A is null.

Step 3. We check the required form of the drift. The fact that $\gamma_0 \in K$ follows since $f(Z_t - Z_t^0) \geq 0$ for every $f \in K^*$ and from (17) we get $\gamma_0 \in K$. Moreover, using $A = 0$, (17) and the fact that (see (8))

$$Ee^{if(Z_t^0)} = \lim_{\varepsilon \downarrow 0} Ee^{if(Z_t^{\triangle_\varepsilon})} = exp\left\{t \int_K \left(e^{if(x)} - 1\right) \nu(dx)\right\}, \tag{18}$$

we obtain (11).

Next, let $f^+ \in K^*$. From (3), (18) and

$$\begin{aligned} Ee^{if^+(Z_t^0)} = \lim_{\varepsilon \downarrow 0} exp\Bigg\{t \int_{\{x \in K: \varepsilon < \|x\| \leq 1\}} \left[e^{if^+(x)} - 1 - if^+(x)\right] \nu(dx) \\ + t \int_{\{x \in K: \|x\| > 1\}} \left(e^{if^+(x)} - 1\right) \nu(dx) + it \int_{\{x \in K: \varepsilon < \|x\| \leq 1\}} f^+(x) \nu(dx)\Bigg\} \end{aligned}$$

we have that $exp\left\{it \int_{\varepsilon < \|x\| \leq 1} f^+(x) \nu(dx)\right\}$ converges as $\varepsilon \downarrow 0$. This implies the convergence of the degenerate distribution at point $t \int_{\varepsilon < \|x\| \leq 1} f^+(x) \nu(dx)$ and consequently $\int_{\varepsilon < \|x\| \leq 1} f^+(x) \nu(dx) \to \int_{D_0} f^+(x) \nu(dx)$ as $\varepsilon \downarrow 0$. Since ν is concentrated on K and K^* is a generating cone of B^*, (10) holds for every $f \in B^*$.

Finally, $I_\nu = \gamma - \gamma_0 \in B$ is a well defined ν-Pettis centering, since (10) holds and $\int_{D_0} f(x) \nu(dx) = f(\gamma - \gamma_0)$ for any $f \in B^*$ by the uniqueness of the generating triplet of the process Z. ∎

Lemma 15 and Theorem 17 yield the following result for subordinators with values in regular cones.

Theorem 18 *Let K be a proper regular cone of a separable Banach space B and let $\{Z_t : t \geq 0\}$ be a K-subordinator. Then*

a) The jump process $Z_t^{\triangle_\varepsilon}$ in (15) is always norm convergent a.s.

b) If K is normal, Z has the special Lévy-Khintchine representation.

4 Regular subordinators

Given a proper cone K of a separable Banach space B, a K-subordinator $\{Z_t : t \geq 0\}$ is called a *regular subordinator* in K or a *K-valued regular subordinator* if its has the special Lévy-Khintchine representation (11).

4.1 Subordinators in LK-cones

In the case of LK-cones every subordinator is a regular subordinator.

Theorem 19 *Let K be an LK-cone of a separable Banach space B. A B-valued Lévy process is a K-subordinator if and only if it is regular.*

Proof. Assume the special Lévy-Khintchine representation, then by Theorem 10 the process Z is K-valued. Conversely, if Z is a K-valued Lévy process then the assertion follows from Proposition 2 and Theorem 18 (b). ∎

As a special case of Theorem 19 we recover a result formulated in [11, Cor. p. 278], that considered a restricted class of cones. Here we give a rigorous proof of this result by proving that the underlying cone is an LK-cone.

Proposition 20 *Let B be a separable Banach space with a proper cone K such that there is a continuous linear functional f_0 satisfying $k_0 = \inf_{x\in K,\|x\|=1} f_0(x) > 0$.*

A B-valued Lévy process has the special Lévy-Khintchine representation if and only if it is a K-valued process.

Proof. We first observe that for $0 \neq x \in K$, $0 < k_0 \leq f_0(x/\|x\|)$ and therefore

$$\|x\| \leq k_0^{-1} f_0(x), \qquad x \in K. \tag{19}$$

For any nonzero continuous linear functional f on B, define $f_1(x) = f(x) + \|f\|' k_0^{-1} f_0(x)$ and $f_2(x) = \|f\|' k_0^{-1} f_0(x)$ for all $x \in B$. Then f_1, $f_2 \in K^*$ and $f_1 - f_2 = f$. Indeed, observe that f_2 is nonnegative on K since $f_0(x) > 0$ for $x \in K$ and using (19) f_1 is also nonnegative on K. Then K^* is a generating cone for B^*.

Let $\sum_{k=1}^{\infty} x_k$ be any w.u.C. series of elements in K and let $s_n = \sum_{k=1}^{n} x_k$. Then $\sum_{k=1}^{\infty} |f_0(x_k)|$ is finite and for $n > m$, $s_n - s_m \in K$ and therefore $\|s_n - s_m\| \leq k_0^{-1} f_0(s_n - s_m)$. Then (s_n) is norm convergent and hence from Proposition 1, K does not contain any copy of c_0^+. Then, K is an LK-cone and the result then follows from Theorem 19. ■

As a consequence of Propositions 7 and 12, infinitely divisible random elements in LK-cones are characterized by a special form of their Laplace transform. The next result may be thought as the Banach space analogue of the well known characterization of the Laplace transform of a real nonnegative infinitely divisible random variable in [10, Th. 13.7.2]. It is proved in [8] for normal and regular cones of general ordered vector spaces.

Corollary 21 *Let K be an LK-cone of a separable Banach space B. In order that a K-valued random variable S have an infinitely divisible law it is necessary and sufficient that S have the Laplace transform (12).*

Proof. Suppose S is an infinitely divisible random variable in K and let $\{Z_t : t \geq 0\}$ be the associated K-subordinator such that S and Z_1 have the same law. From Theorem 19 the process Z is a K-regular subordinator. Then from Proposition 12 (b) Z has the Laplace transform (12). The converse is immediate. ■

Remark 22 *a) It remains the open question whether the above result is a characterization of LK-cones. That is, given a proper normal cone, is it true that every K-subordinator is a K-regular subordinator if and only if K is a LK-cone (which contains no isomorphic copy of c_0^+)? We believe the answer is positive and that the characterization is related to the existence of a Pettis integral.*

b) It is well known that the convergence of sums of independent symmetric random variables in a Banach Space is related to the geometry of the space. For example, [12] and [15] characterize the class of separable Banach spaces B for which the boundedness of the partial sums of independent random variables implies the convergence of the series, as those containing no copy of c_0. In this direction we point out that the non-compensated sum of jumps has non symmetric elements, and that a symmetric argument does not lead us to the characterization.

4.2 Examples of regular subordinators in arbitrary proper cones

Theorem 10 gives a general method for constructing regular subordinators from the original Lévy-Khintchine representation (3) in a proper cone. Below we use this technique to construct α-stable and tempered stable regular subordinators in an arbitrary proper cone K of a general separable Banach space.

Example 23 *Let B be a separable Banach space with a proper cone K. Recall from [16, Prop. 6.3.1] that a probability measure μ on B has α-stable distribution, $0 < \alpha < 2$, if and only if its characteristic functional has the form*

$$\hat{\mu}_t(f) = \exp\left\{ c_\alpha^{-1} t \int_{(0,\infty)} \int_{\partial U} \left(e^{irf(y)} - 1 - irf(y) 1_U(ry) \right) \frac{\lambda(dy)}{r^{1+\alpha}} dr + itf(\gamma) \right\}$$

for $f \in B^*$, where $\gamma \in B$, $c_\alpha > 0$ is a constant and λ is the spectral measure of μ concentrated on the unit sphere ∂U of the unit closed ball U of B. Then, from Proposition 7, there exists a Lévy process $\{Z_t : t \geq 0\}$ such that Z_t has the α-stable distribution μ. Let $0 < \alpha < 1$ and assume that λ is concentrated on $\mathcal{S}_K = \{x \in K : \|x\| = 1\}$. Let $\nu(C) = c_\alpha^{-1} \int_{(0,\infty)} \int_{\mathcal{S}_K} 1_C(ry)\lambda(dy)\frac{dr}{r^{1+\alpha}}$, for $C \in \mathcal{B}_0$, which is concentrated on K. Since

$$\int_{D_0} \|x\| \, \nu(dx) = c_\alpha^{-1} \int_{(0,1)} \int_{\mathcal{S}_K} \lambda(dy) \frac{dr}{r^\alpha} = c_\alpha^{-1} \frac{1}{1-\alpha} \lambda(\mathcal{S}_K) < \infty,$$

condition (9) is satisfied and we have that $\int_{D_0} x\nu(dx)$ and $\int_{\mathcal{S}_K} y\lambda(dy)$ are well defined Bochner integrals and $\int_{D_0} |f(x)| \, \nu(dx) < \infty$ for each $f \in B^*$. Next, choose γ_0 such that $\gamma_0 = \gamma - c_\alpha^{-1} \frac{1}{1-\alpha} t \int_{\mathcal{S}_K} y\lambda(dy)$ belongs to K. Then assumptions (a)-(c) in Theorem 10 are satisfied. Therefore the Lévy process Z is concentrated on K and has characteristic functional (11). Furthermore, using the fact that $\int_{(0,\infty)} \left(e^{-rf(y)} - 1\right) \frac{dr}{r^{1+\alpha}} = \{f(y)\}^\alpha \, \Gamma(-\alpha)$ for every $f \in K^*$, the Laplace transform of Z given by Proposition 12 becomes

$$Ee^{-f(Z_t)} = \exp\left\{ c_\alpha^{-1} \Gamma(-\alpha) \quad t \int_{\mathcal{S}_K} \{f(y)\}^\alpha \, \lambda(dy) - tf(\gamma_0) \right\} \tag{20}$$

for $f \in K^*$. The K-valued Lévy process $\{Z_t : t \geq 0\}$ is called α-stable subordinator. .

The family of finite dimensional tempered stable distributions was introduced in [20]. This class includes the case of distributions obtained by exponential tilting of positive stable distributions; see [2]. In analogy, an infinite dimensional α-tempered stable regular subordinator is constructed as follows.

Example 24 *Let B be a separable Banach space with a proper cone K. For $0 < \alpha < 2$, let υ be a σ-finite measure on $B_0 = B \backslash \{0\}$ such that $\int_{B_0} \|x\|^\alpha \, \upsilon(dx) < \infty$.*

A probability measure μ on B is tempered stable if it is infinitely divisible without Gaussian part and with Lévy measure

$$\nu(C) = \int_{B_0} \int_{(0,\infty)} 1_C(ry) \frac{e^{-r}}{r^{1+\alpha}} dr\upsilon(dy), \quad C \in \mathcal{B}_0. \tag{21}$$

Assuming that υ *is concentrated on* K*, we also have that* ν *is concentrated on* K*. Observe that for* $0 < \alpha < 1$,

$$\int_{D_0} \|x\| \, \nu(dx) \le \int_{B_0} \int_{(0,1)} \|y\|^{\alpha} \frac{e^{-r}}{r^{1+\alpha}} dr \upsilon(dy) < \infty,$$

which gives the existence of the ν*-Pettis centering* $I_\nu = k_\alpha \int_{0<\|y\|\le 1} y\upsilon(dy)$ *as a Bochner integral, where* $k_\alpha = \int_0^1 \frac{e^{-r}}{r^{1+\alpha}} dr$*. Let* $\gamma \in B$ *be such that* $\gamma_0 = \gamma - I_\nu \in K$*. Then, by Theorem 10 the associated Lévy process* $\{Z_t : t \ge 0\}$ *is concentrated on* K*. This process is called* α*-tempered subordinator. The case* $\alpha = 1/2$ *may be thought as the Banach space version of the well known one dimensional inverse Gaussian process.*

5 Subordinators in Birkhoff-Kakutani Spaces

5.1 Birkhoff-Kakutani cones

Birkhoff [6] and Kakutani [14] studied the so-called *Abstract-Lebesgue spaces* (AL-spaces) which are Banach lattices where the norm is additive in the cone of positive elements. There are more general Banach spaces where this additive property holds.

Definition 25 *Let* $(B, \|\cdot\|)$ *be a Banach space and let* K *be a proper cone of* B*. The triplet* $(B, \|\cdot\|, K)$ *is called a Birkhoff-Kakutani space if there exists a continuous linear functional* $f_0 \in B^*$ *such that* $f_0(x) = \|x\|$ *for every* $x \in K$*. In this case, it is said that* K *is a Birkhoff-Kakutani cone.*

Birkhoff-Kakutani spaces have important properties collected in the next result.

Proposition 26 *Let* $(B, \|\cdot\|, K)$ *be a Birkhoff-Kakutani space. Then*
a) K *has generating dual cone.*
b) Every weakly unconditionally Cauchy series in K *is norm convergent.*
c) K *contains no subcones isomorphic to* c_0^+*.*
d) K *is a regular cone.*
e) K *is an* LK*-cone.*

Proof. a) Assume that f_0 is a continuous linear functional on B such that $f_0(x) = \|x\|$ for all $x \in K$. Let f be any nonzero continuous linear functional on B. Define $f_1(x) = f(x) + \|f\|' f_0(x)$ and $f_2(x) = \|f\|' f_0(x)$ for all $x \in B$. Clearly f_1 and f_2 are continuous linear functionals on B which are nonnegative on K and $f_1 - f_2 = f$.

b) Let (x_n) be a sequence in K and let $s_n = \sum_{k=1}^{n} x_k$. For every $f \in B^* \sum_{k=1}^{\infty} |f(x_k)|$ is finite, in particular for f_0. If $n > m$, $s_n - s_m \in K$ and therefore $f_0(s_n - s_m) = \|s_n - s_m\|$. Then (s_n) is norm convergent.

c) Follows from Proposition 1, (d) follows using (a), (c) and Proposition 2, and finally (e) follows from (a) and (c). ■

Several interesting Banach spaces provide examples of Birkhoff-Kakutani spaces and their associated LK-cones. First we mention the AL-spaces introduced by Birkhoff [6] and characterized by Kakutani [14] which are Banach lattices whose cone of positive elements is of Birkhoff-Kakutani type; see [22]. Other concrete examples are the following.

Example 27 *a) The Euclidean space $\mathbb{R}^d$ with the norm $\|x\| = |x_1| + |x_2| + ... + |x_d|$ and $\mathbb{R}_+^d$ as a proper cone.*

b) The finite dimensional Banach space of real symmetric $d \times d$ matrices $M_{d\times d}$, with the trace norm $\|A\| = tr((AA^T)^{1/2})$ and the proper cone $M_{d\times d}^+$ of nonnegative definite matrices.

c) The Banach space $(L_1(H), \|\cdot\|)$ of trace class operators of a separable Hilbert space H, with trace norm $\|S\| = \sum_{n=1}^{\infty} s_n$ (where s_n are the eigenvalues of $(SS^*)^{1/2}$) and the proper cone $L_1^+(H)$ of positive trace class operators.

Examples (b) and (c) provide Birkhoff-Kakutani spaces which are not lattices. More generally, the Banach dual of any C^*-algebra is a Birkhoff-Kakutani space.

Example 28 *Recall that a Banach algebra A over $\mathbb{C}$ with involution $x \to x^*$ satisfying $\|xx^*\| = \|x\|^2$ is called a C^*-algebra. Let A^* denote the Banach dual of A and $A_{sa} = \{x \in A : x = x^*\}$ denote the selfadjoint elements of A. Let $A_+ = \{xx^* : x \in A\}$ be the (proper) cone of A_{sa} and A_+^* be the proper cone of positive linear functionals of A^*. The cone A_+^* is generating for A^* ([23, p. 270]) and the dual norm $\|\cdot\|'$ is additive in A_+^* ([23, Cor. 6.4.2]). A continuous linear functional f on A can be defined satisfying $f(\phi) = \|\phi\|$ for every $\phi \in A_+^*$. Therefore $\left(A^*, \|\cdot\|', A_+^*\right)$ is a Birkhoff-Kakutani space. Conditions for separability of duals of C^*-algebras are given in [25].*

5.2 The associated norm subordinator process

Cone-valued Lévy processes in Birkhoff-Kakutani spaces have several properties similar to the one dimensional subordinators.

Proposition 29 *Let $(B, \|\cdot\|, K)$ be a separable Birkhoff-Kakutani space.*

a) All K-valued subordinators are K-regular subordinators and the Lévy measure ν satisfies

$$\int (1 \wedge \|x\|)\, \nu(dx) < \infty \tag{22}$$

and $\gamma - \int_{0<\|x\|\leq 1} x\nu(dx) \in K$, where the last integral is in the sense of Bochner.

b) In order that a K-valued random element S has an infinitely divisible law it is necessary and sufficient that S have the Laplace transform (12), where $\gamma_0 \in K$ and the Lévy measure ν is concentrated on K satisfying the integrability condition (22).

c) Every K-valued subordinator has bounded variation on each interval $[0, t]$, $t > 0$, almost surely.

Proof. ¿From Proposition 26 K is a LK-cone. By Theorem 19, a K-subordinator is a regular subordinator. Let $f_0 \in B^*$ such that $f_0(x) = \|x\|$ for $x \in K$. Hence condition (22) follows from (10) since ν is concentrated on K. Next (b) is obtained from (a) and Proposition 12. Finally, (c) follows from (a) and Proposition 8. ■

Proposition 29 (b) answers in an affirmative way a question in [8, Remark 2] of whether there are Banach spaces other than lattices such that the condition (22) is satisfied for infinitely divisible cone valued elements.

For any cone-valued Lévy process in a Birkhoff-Kakutani space its associated norm process is a one dimensional subordinator.

Theorem 30 *Let $(B, \|\cdot\|, K)$ be a separable Birkhoff-Kakutani space and let $\{Z_t : t \geq 0\}$ be a K-valued subordinator with drift γ_0 and Lévy measure ν. Then the process $\{\|Z_t\| : t \geq 0\}$ is a one dimensional subordinator with Lévy-Khintchine representation*

$$Ee^{iu\|Z_t\|} = exp\left\{ \int_{\mathbb{R}_+} \left(e^{iur} - 1\right) \nu \circ \|\cdot\|^{-1} (dr) + iu \|\gamma_0\| \right\} \qquad u \in \mathbb{R}, \tag{23}$$

where

$$\int_{0<r\leq 1} r\nu \circ \|\cdot\|^{-1} (dr) < \infty.$$

The Laplace transform of $\{\|Z_t\| : t \geq 0\}$ is given by

$$Ee^{-u\|Z_t\|} = exp\left\{-t\Phi(u)\right\} \qquad u \in \mathbb{R}_+,$$

with Laplace exponent

$$\Phi(u) = \int_{\mathbb{R}_+} \left(1 - e^{-ur}\right) \nu_Z \circ \|\cdot\|^{-1} (dr) + u \|\gamma_0\|. \tag{24}$$

Proof. For each $u \in \mathbb{R}$ take the continuous linear functional uf_0 on B where $f_0(x) = \|x\|$ for all $x \in K$. Then, from (11) in Theorem 10 the characteristic functional of $\{Z_t\}$ evaluated in the linear functional uf_0 gives

$$Ee^{iu\|Z_t\|} = exp\left\{t\left(\int_K \left(e^{iu\|x\|} - 1\right)\nu(dx) + iu\,\|\gamma_0\|\right)\right\} \qquad u \in \mathbb{R},$$

from which (23) is obtained. Using the Laplace transform (12), a similar argument as above gives (24). ■

5.3 Norm-inheritance sample path properties

For a Birkhoff-Kakutani space $(B, \|\cdot\|, K)$, some asymptotic sample path properties of the associated one dimensional norm subordinators are inherited by the cone-valued subordinators. These results are new, even in the finite dimensional cone $\mathbb{R}_+^d$, $d > 2$, and for the cone of nonnegative definite $d \times d$ matrices.

Proposition 31 (Law of large numbers) *Let $\{Z_t : t \geq 0\}$ be K-valued subordinator with drift γ_Z^0. Then*

$$P\left(\lim_{t\to 0+}\left\|\frac{Z_t}{t} - \gamma_Z^0\right\| = 0\right) = 1.$$

Proof. Theorem 10 implies that $\{Z_t - t\gamma_Z^0 : t \geq 0\}$ is a K-valued subordinator with zero drift. Let $\{\sigma_t : t \geq 0\}$ be the corresponding norm subordinator which also has zero drift (i.e. $\sigma_t = \|Z_t - t\gamma_Z^0\|$). Then Proposition 3.8 in [4] implies $\lim_{t\to 0+}\sigma_t/t = 0$ almost surely and the result follows. ■

The following two zero-one laws for the limsup of the rate of growth of a subordinator in a Birkhoff-Kakutani space are characterized in terms of the behavior of the Lévy measure and the Laplace exponent of the associated norm subordinator.

For small times we have.

Proposition 32 *Let $\{Z_t : t \geq 0\}$ be a K-subordinator and let $\{\sigma_t : t \geq 0\}$ be its norm subordinator with Lévy measure ν_σ and Laplace exponent Φ_σ. Let $I(t) = \int_0^t \bar{\nu}_\sigma(x)dx$ where $\bar{\nu}_\sigma(x) = \nu_\sigma((x,\infty))$, $x > 0$, and suppose that*

$$\liminf_{x\to 0+} I(2x)/I(x) > 1. \tag{25}$$

Let $h : [0,\infty) \to [0,\infty)$ be an increasing function. Then the following are equivalent:

a) $P\left(\limsup_{t\to 0+}\|Z_t/h(t)\| = \infty\right) = 1$

b) $\int_0^1 \bar{\nu}_\sigma(h(t))dt = \infty$

c) $\int_0^1 \Phi_\sigma(1/h(t))dt = \infty$.

When the above conditions fail we have that

$$P\left(\lim_{t\to 0+}\|Z_t/h(t)\| = 0\right) = 1.$$

Proof. We apply Proposition 3.10 in [4] to the one dimensional subordinator $\sigma_t = \|Z_t\|$, $t \geq 0$, since its Lévy measure ν_σ satisfies (25). This yields the equivalence of (a), (b) and (c). ■

For large times, the following rates of growth results are also easily derived from the one-dimensional case in [4, Th. 3.13].

Proposition 33 *Let $\{Z_t : t \geq 0\}$ be a K-subordinator such that $E\|Z_1\| = \infty$. Let $\{\sigma_t : t \geq 0\}$ be its norm subordinator with Lévy measure ν_σ and Laplace exponent Φ_σ. Let $h : [0,\infty) \to [0,\infty)$ be an increasing function such that the function $t \to h(t)/t$ is increasing as well. Then the following are equivalent:*

a) $P(\limsup_{t\to\infty}\|Z_t/h(t)\| = \infty) = 1$

b) $\int_1^\infty \bar{\nu}_\sigma(h(t))dt = \infty$

c) $\int_1^\infty \left\{\Phi_\sigma(1/h(t)) - (1/h(t))\,\Phi'_\sigma(1/h(t))\right\}dt = \infty$.

If the above conditions fail then $P(\lim_{t\to\infty}\|Z_t/h(t)\| = 0) = 1$.

We finally point out how laws of iterated logarithm for cone-valued subordinators in a Birkhoff-Kakutani space $(B, \|\cdot\|, K)$ are easily transferred from those corresponding to the norm subordinator. Recall that a positive measurable function $\varphi : (0,\infty) \to (0,\infty)$ is regularly varying at ∞ (respectively, at $0+$) if for each $c > 0$, the ratio $\varphi(cx)/\varphi(x)$ converges in $(0,\infty)$ as x tends to ∞ (respectively, to $0+$). In both cases there exists $\rho > 0$ (called the *index* of φ) such that $\varphi(cx)/\varphi(x)$ converges to c^ρ. Let

$$\psi_1(t) = \frac{\log|\log(t)|}{\varphi_\sigma(t^{-1}\log|\log(t)|)}, \quad 0 < t < e^{-1}$$

and

$$\psi_2(t) = \frac{\log\log(t)}{\varphi_\sigma(t^{-1}\log\log(t))}, \quad e < t < \infty,$$

where φ_σ is the inverse function of Φ_σ.

Proposition 34 *Let* $\{Z_t : t \geq 0\}$ *be* K*-valued subordinator and* Φ_σ *the Laplace exponent of the norm-subordinator* $\{\sigma_t : t \geq 0\}$.

a) If Φ_σ *is regularly varying at* ∞ *with index* $\rho \in (0,1)$ *then*

$$P\left(\liminf_{t\to 0+}\left\|\frac{Z_t}{\psi_1(t)}\right\| = \rho(1-\rho)^{(1-\rho)/\rho}\right) = 1.$$

b) If Φ_σ *is regularly varying at* $0+$ *with index* $\rho \in (0,1)$ *then*

$$P\left(\liminf_{t\to \infty}\left\|\frac{Z_t}{\psi_2(t)}\right\| = \rho(1-\rho)^{(1-\rho)/\rho}\right) = 1.$$

Proof. a) Since the Laplace exponent Φ_σ of the one dimensional subordinator $\{\sigma_t : t \geq 0\}$ is regularly varying at ∞ with index $\rho \in (0,1)$ we can apply Theorem 3.11 in [4] to yield $\liminf_{t\to 0+} \sigma_t/\psi_1(t) = \rho(1-\rho)^{(1-\rho)/\rho}$ almost surely. The result follows from the fact that $\sigma_t = \|Z_t\|$. Similarly (b) follows using Theorem 3.14 in [4]. ∎

Acknowledgement The authors thank Maite Fernández and Ken-Iti Sato for many helpful discussions and suggestions.

References

[1] Araujo, A. and Giné, E. (1980) *The Central Limit Theorem for Real and Banach Valued Random Variables*, Wiley, New York.

[2] Barndorff-Nielsen, O. E. and Levendorskiĭ, S.Z. (2001) Feller processes of normal inverse Gaussian type, *Quantitat. Finan.* **1**, 318-331.

[3] Barndorff-Nielsen, O. E., Pedersen, J. and Sato, K. (2001) Multivariate subordination, self-decomposability and stability, *Adv. Appl. Probab.* **33**, 160–187.

[4] Bertoin, J. (1996) *Lévy Processes,* Cambridge University Press, Cambridge.

[5] Bertoin, J. (1999) Subordinators: examples and applications, *Lectures Notes in Mathematics* **1717**, Springer, Berlin, 1–91.

[6] Birkhoff, G. (1938) Dependent probabilities and the space (L), *Proc. Nat. Acad. Sci.* **24**, 154-159.

[7] Bochner, S. (1955) *Harmonic Analysis and the Theory of Probability,* University of California Press, Berkeley and Los Angeles.

[8] Dettweiler, E. (1976) Infinitely divisible measures on the cone of an ordered locally convex vector space, *Ann. Sci. Univ. Clermont* **14**, 61, 11-17.

[9] Diestel, J. (1984) *Sequences and Series in Banach Spaces*, Springer, New York.

[10] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol. II, Second Edition, Wiley, New York.

[11] Gihman, I. I. and Skorohod, A. V. (1975) *The Theory of Stochastic Processes*, Vol. II, Springer, Berlin.

[12] Hoffman-Jorgensen, J. (1974) Sums of independent Banach valued random variables, *Studia Math.* **52**, 159-186.

[13] Kamthan, P. K. and Gupta, M. (1985) *Theory of Bases and Cones*, Pitman Advances Publishing Proceedings, Research Notes in Mathematics **117**.

[14] Kakutani, S. (1941) Concrete representation of Abstract (L)-spaces and the mean Ergodic Theorem, *Ann. Math.* **42**, 523-537.

[15] Kwapien, S. (1974) On Banach spaces containing c_0, *Studia Math.* **52**, 187-188.

[16] Linde, W. (1986) *Probability in Banach Spaces -Stable and Infinitely Divisible Distributions-*, John Wiley & Sons, Berlin.

[17] Musial K. (2002) Pettis integral, *Handbook of Measure Theory, Vol I*, E. Pap (Ed.). North-Holland, Amsterdam. pp531-586.

[18] Pedersen, J. and Sato, K. (2004a) Cone-parameter convolution semigroups and their subordination, *Tokyo J. Math.* To appear.

[19] Pedersen, J. and Sato, K. (2004b) Relations between cone-parameter Lévy processes and convolution semigroups, *J. Math. Soc. Japan.* To appear.

[20] Rosinski, J. (2002) Tempered stable processes, *Proceedings 2nd MaPhySto Conference on Lévy Processes: Theory and Applications.* O. E. Barndorff-Nielsen (Ed). pp. 215-219.

[21] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions,* Cambridge University Press, Cambridge.

[22] Schaefer, H. H. (1974) *Banach Lattices and Positive Operators*, Springer, New York.

[23] Schaefer, H. H. (1999) *Topological Vector Spaces*, Springer, New York, Second edition.

[24] Skorohod, A. V. (1991) *Random Processes with Independent Increments,* Kluwer Academic Publisher, Dordrecht, Netherlands (Russian original 1986).

[25] Tomiyama, J. (1963) A characterization of C*-algebras whose conjugate space are separable, *Tôhoku Math. J.* **15**, 96-102.