Non-self-adjoint graphs

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Based on

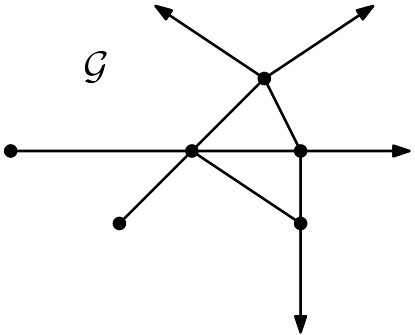
[1] A. Hussein, D. Krejčiřík and P. Siegl: *Non-self-adjoint graphs*, Transactions of the AMS 367, (2015) 2921-2957

Outline

1. Introduction and “the example”
2. Classes of boundary conditions
3. Spectral properties
4. Similarity transforms

1. Introduction and "the example"
2. Classes of boundary conditions
3. Spectral properties
4. Similarity transforms

$$-\Delta \equiv -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad + \quad \text{boundary/vertex conditions} \quad \text{in} \quad L^2(\mathcal{G})$$



$$L^2(\mathcal{G}) = \oplus_{j=1}^{N} L^2((0, a_j)), \quad a_j \in (0, +\infty], \quad N = \#\text{edges} < \infty$$

Non-self-adjoint graphs

- "fundamental non-self-adjointness":
  - non-symmetric boundary/vertex conditions, e.g. complex δ-interactions
  - no problems with too little or too many conditions

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[1] S. Albeverio, S. M. Fei, and P. Kurasov. *Lett. Math. Phys.* 59 (2002), pp. 227–242; S. Albeverio, U. Gunther, and S. Kuzhel. *J. Phys. A: Math. Theor.* 42 (2009), 105205 (22pp); S. Albeverio and S. Kuzhel. *J. Phys. A: Math. Gen.* 38 (2005), pp. 4975–4988; A. V. Kiselev. *Op. Theory: Adv. and Appl.* 186 (2008), pp. 267–283; S. Kuzhel and C. Trunk. *J. Math. Anal. Appl.* 379 (2011), pp. 272–289.

[2] A. Hussein. *J. Evol. Equ.* 14 (2014), pp. 477–497; V. Kostrykin, J. Potthoff, and R. Schrader. *J. Math. Phys.* 53 (2012), p. 095206; V. Kostrykin, J. Potthoff, and R. Schrader. *Proc. Symp. in Pure Math.* 77 (2008), pp. 423–458; V. Kostrykin, J. Potthoff, and R. Schrader. *Adventures in math. phys.* Vol. 447. Contemp. Math. AMS, Providence, 2007, pp. 175–198.

[3] P. Freitas and J. Lipovský. arXiv:1307.6377.

Non-self-adjoint graphs

- "fundamental non-self-adjointness":
  - non-symmetric boundary/vertex conditions, e.g. complex δ-interactions
  - no problems with too little or too many conditions
- motivation for complex potentials/interactions:
  - electromagnetism, optics with losses and gains
  - superconductivity, damped wave equation
  - stochastic processes
  - open quantum systems, effective models
- existing literature
  - non-self-adjoint point interactions[1]
  - m-accretive and m-dissipative graphs[2]
  - damped wave equation on graphs[3]

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Minimal and maximal operators

- minimal operator

$$\begin{aligned} \mathrm{Dom}(-\Delta_{\min}) &= W_0^{2,2}(\mathcal{G}) := \oplus_{j=1}^N W_0^{2,2}((0,a_j)) \\ (-\Delta_{\min}\psi)_j &:= -\psi_j'' \end{aligned}$$

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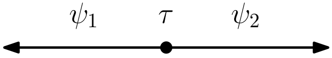
Our Laplacians

$$-\Delta_{\min} \subset -\Delta_{\mathcal{M}} \subset -\Delta_{\max}, \quad \Delta_{\mathcal{M}} \neq \Delta_{\mathcal{M}}^*$$
$$\mathrm{Dom}(-\Delta_{\mathcal{M}}) := \{\psi \in \mathrm{Dom}(-\Delta_{\max}) : [\psi] \oplus [\psi'] \in \mathcal{M} \subset \mathbb{C}^{2d}\}$$

we assume : $\dim \mathcal{M} = d$

“The example”

τ-interaction[4]



$$\psi_1(0) = e^{\mathrm{i}\tau}\psi_2(0), \quad \psi_1'(0) = -e^{-\mathrm{i}\tau}\psi_2'(0), \quad \tau \in [0, \pi/2]$$

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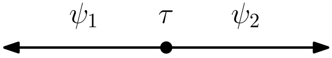

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  - $-\Delta_{\mathcal{M}} = -\Delta_{\mathcal{M}}^* = -\Delta_{\mathbb{R}}$ with $\sigma(-\Delta_{\mathcal{M}}) = [0, +\infty)$
  - $Q_{\mathcal{M}}[\psi] = \|\psi'\|^2_{L^2(\mathcal{G})}$

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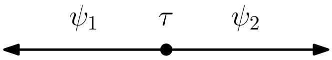

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  - in fact: $-\Delta_{\mathcal{M}} \sim -\Delta_{\mathbb{R}}$: $\Phi^{-1}(-\Delta_{\mathcal{M}})\Phi = -\Delta_{\mathbb{R}}$ $\qquad \Phi, \Phi^{-1} \in \mathscr{B}(L^2(\mathcal{G}))$
  - $Q_{\mathcal{M}}[\psi] = \|\psi'\|^2_{L^2(\mathcal{G})} + (1 - e^{2i\tau})\psi_2(0)\,\overline{\psi_2'(0)}$
  - cannot be defined through sectorial forms: $\mathrm{Num}(-\Delta_{\mathcal{M}}) = \mathbb{C}$

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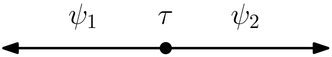

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  - cannot be defined through sectorial forms: $\mathrm{Num}(-\Delta_{\mathcal{M}}) = \mathbb{C}$
- $\tau = \pi/2$:
  - $-\Delta_{\mathcal{M}} \neq -\Delta_{\mathcal{M}}^*$ with $\sigma(-\Delta_{\mathcal{M}}) = [0, +\infty) \cup \mathbb{C} \setminus [0, +\infty) = \mathbb{C}$
  - no sectorial forms: $\mathrm{Num}(-\Delta_{\mathcal{M}}) = \mathbb{C}$

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Boundary conditions

- subspaces $\mathcal{M}$ parametrized by matrices $A, B \in \mathbb{C}^{d\times d}$, $\mathcal{M} = \mathcal{M}(A, B)$

$$\mathrm{Dom}(-\Delta(A,B)) = \left\{\psi \in \mathrm{Dom}(-\Delta_{\max}) : A[\psi] + B[\psi'] = 0\right\}$$

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$\Longleftrightarrow$ (A,B) parametrization[5]: $AB^* = BA^*$

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$\Longleftrightarrow$ Cayley transform: $\mathfrak{S} \equiv U$ unitary

$$\mathfrak{S} := -\left(A + \mathrm{i}kB\right)^{-1}\left(A - \mathrm{i}kB\right), \quad k > 0$$

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$\Longleftrightarrow$ m-sectorial parametrization[6]: $(A,B) \simeq (L + P, P^{\perp})$

$$Q_{\mathcal{M}}[\psi] = \|\psi'\|^2_{L^2(\mathcal{G})} - \langle LP^{\perp}[\psi], P^{\perp}[\psi]\rangle_{\mathbb{C}^d}$$

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Regular boundary conditions

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Irregular boundary conditions

- BC defined by A, B are irregular if
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$\Leftrightarrow$ $\mathrm{Ker} A \cap \mathrm{Ker} B \neq \{0\}$

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m-sectorial boundary conditions

- regular BC are called **m-sectorial** if $(A,B) \simeq (L + P, P^{\perp})$
- all self-adjoint BC are m-sectorial
- τ-interaction is m-sectorial iff $\tau = 0$

Totally degenerated[7] BC - irregular

- 
- BC: $\psi(0) = 0$, $\psi'(0) = 0$, $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$
- $\dim \mathcal{M}(A, B) = 2 = d$ and $A + \mathrm{i}kB$ is not invertible for any $k \in \mathbb{C}$
- spectral pathology: $\sigma(-\Delta(A, B)) = \emptyset$

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Examples

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Indefinite Laplacian[8] - irregular

- 
- BC: $\psi_1(0) = \psi_2(0), \psi_1'(0) = \psi_2'(0), \quad A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$
- spectral pathology: $\sigma(-\Delta(A,B)) = \mathbb{C}$
- $-\Delta(A,B) \simeq -\operatorname{sgn}(x)\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sgn}(x)\frac{\mathrm{d}}{\mathrm{d}x}$ in $L^2(\mathbb{R})$

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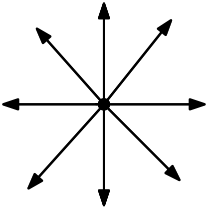
Complex δ-interaction - m-sectorial



$$\psi_1(0) = \psi_2(0) = \dots = \psi_N(0)$$

$$\sum_{i=1}^{N} \psi_i'(0) = \gamma \psi_1(0), \quad \gamma \in \mathbb{C}$$

$$A = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \dots & 1 & -1 \\ -\gamma & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

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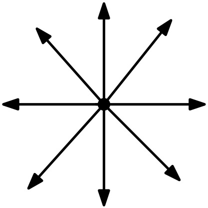

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$$\sigma(-\Delta(A,B)) = \begin{cases} \{-(\gamma/N)^2\} \cup [0,\infty), & \text{if } \operatorname{Re}\gamma < 0 \\ [0,\infty), & \text{if } \operatorname{Re}\gamma \geq 0 \end{cases}$$

Point spectrum

- $\overline{\sigma_{\mathrm{p}}(-\Delta(A,B))}$ is discrete set ($\neq \mathbb{C}$) and

$$\lambda \in \sigma_{\mathrm{p}}(-\Delta(A,B)) \setminus [0,\infty) \Longleftrightarrow \overline{\lambda} \in \sigma_{\mathrm{p}}(-\Delta(A,B)^*) \setminus [0,\infty)$$

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Residual spectrum

- $\sigma_{\mathrm{r}}(-\Delta(A,B)) \subset [0,\infty)$ (discrete subset)
- $\sigma_{\mathrm{r}}(-\Delta(A,B)) = \emptyset$ if there are no bounded/unbounded edges
- $\sigma_{\mathrm{r}}(-\Delta(A,B))$ may be non-empty!

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Essential spectrum

- warning: at least 5 different definitions of essential spectrum of nsa operators!
- σ_{e5}: complement of isolated eigenvalues of finite algebraic multiplicity

$$\sigma_{\mathrm{e5}}(-\Delta(A,B)) = \begin{cases} \emptyset & \text{if there are no unbounded edges} \\ [0,\infty) & \text{if there is an unbounded edge} \end{cases}$$

M-sectorial complex Robin BC[9]

$$\psi'(0) + \mathrm{i}\alpha\psi(0) = 0 \ \psi'(\pi) + \mathrm{i}\alpha\psi(\pi) = 0, \qquad \alpha \in \mathbb{R}$$

$$A = \begin{bmatrix} \mathrm{i}\alpha & 0 \\ 0 & -\mathrm{i}\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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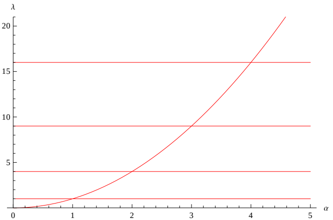

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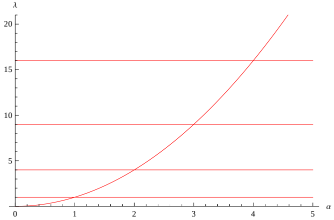

More than real spectrum

- eigenfunctions $\{\psi_n\}_{n\in\mathbb{N}_0}$ form a Riesz basis

⇒ similarity to a self-adjoint operator

$$-\Delta(A,B) \sim -\Delta_{\mathrm{N}} + \frac{\alpha^2}{\pi}\langle\cdot,1\rangle, \quad \alpha \notin \mathbb{Z}\setminus\{0\}$$

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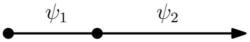
Graph with residual spectrum



$$
\begin{aligned}
-\Delta(A, B)& \\
\psi_1'(0) + \mathrm{i}\alpha\psi_1(0) &= 0 \\
\psi_1'(\pi) + \mathrm{i}\alpha\psi_1(\pi) &= 0 \\
-\psi_1(\pi) &= \psi_2'(0)
\end{aligned}
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$$
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Spectral properties

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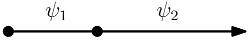

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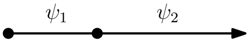

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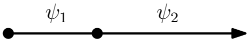

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- but for point spectra:

$$\sigma_{\mathrm{p}}(-\Delta(A,B)) = \emptyset \qquad \text{vs.} \qquad \sigma_{\mathrm{p}}(-\Delta(A,B)^*) = \{\alpha^2\} \cup \{n^2\}_{n\in\mathbb{N}}$$

$\Rightarrow$ $\sigma_{\mathrm{r}}(-\Delta(A,B)) = \{\alpha^2\} \cup \{n^2\}_{n\in\mathbb{N}}$

Recall:

$$\sigma_{\mathrm{r}}(-\Delta(A,B)) = \{\lambda \notin \sigma_{\mathrm{p}}(-\Delta(A,B)) : \overline{\lambda} \in \sigma_{\mathrm{p}}(-\Delta(A,B)^*)\}$$

Compact m-sectorial graphs

- discrete spectrum & Riesz basis of finite dimensional invariant subspaces
- in very special cases (complex Robin BC) $\Rightarrow$ similarity to normal (or self-adjoint) operator but typically not a graph Laplacian

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Similarity of graph Laplacians: assumptions

i) restriction on graphs: all bounded edges of the same length

ii) similarity of matrices A, B and A', B'

$$A' = G^{-1} A G, \quad B' = G^{-1} B G$$

where

$$G := \begin{bmatrix} G_{\text{unbdd}} & 0 & 0 \\ 0 & G_{\text{bdd}} & 0 \\ 0 & 0 & G_{\text{bdd}} \end{bmatrix}$$

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Theorem: similarity of graph Laplacians

$$-\Delta(A', B') = \Phi_G^{-1}(-\Delta(A', B'))\Phi_G$$

where

$$(\Phi_G \psi)(x_j) := \sum_{i=1}^{N} G_{ji} \psi_i(x_j)$$

Corollaries for regular BC

- sufficient to check the Cayley transform: $\mathfrak{S} = G^{-1} U G$
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Similarity transforms

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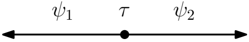

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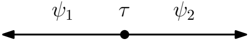

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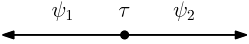

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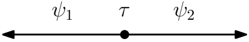

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Regular vs. irregular boundary/vertex conditions

- "usual" spectrum vs. possible pathologies $\sigma(-\Delta) = \emptyset / \mathbb{C}$
- possibly (discrete) non-empty residual spectrum in $[0, \infty)$
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[10] B. Mityagin and P. Siegl. arxiv:1608.00224. 2016.

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Outlook

??? Schrödinger operators on graphs: $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V$ on edges

??? pseudospectral analysis

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5 – 9 June 2017

Marseille, France

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