A MATHEMATICAL TREATMENT OF NON-LINEAR OSCILLATIONS

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A MATHEMATICAL TREATMENT OF NON-LINEAR OSCILLATIONS

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CHAPTER I

INTRODUCTION

Network Parameters. The entire field of network theory is currently receiving concentrated study encouraged by increasing general familiarity with more powerful mathematical methods. Principal reference is made to the concept of a time domain and the associated complex frequency domain, the two being rigorously related by the Laplace transform.[1] The time domain describes, as a function of time, the voltage or current response of a linear network to a given excitation, while the complex frequency expression describes the same function in terms of fixed singularities of an algebraic function of a complex variable. This latter expression is somewhat analogous to the well known Fourier series expression for a complex wave form consisting of an infinite, or in some cases finite, sum of harmonics, each having a distinct coefficient of amplitude.

The time response to a given excitation of a linear network consisting of a general configuration of resistances, capacitances and inductances can be expressed in terms of a differential equation. When this time function is transformed into the complex algebraic expression

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[1]Murray F. Gardner and John L. Barnes, Transients in Linear Systems, Vol. I (New York: John Wiley & Sons, Inc., 1942), 389 pp.

we find a polynomial in the denominator which is known as the characteristic equation of the network. The factors of this polynomial are identified with the real and imaginary parts of the exponential solution of the original differential equation taken with no excitation function.

The following simple example serves to illustrate these ideas:

$$A \frac{d^2y}{dt^2} + B \frac{dy}{dt} + C y = f(t) , \tag{1.1}$$

where

A, B, and C are constants determined by circuit parameters;

y = y(t) is a current (or voltage) response as a function of time;

f(t) is the voltage (or current) excitation as a function of time.

This differential equation can be Laplace transformed into an algebraic function of the complex variable s as follows:[2]

$$Y(s) = \frac{F(s) + y(0)(As + B) + y'(0)A}{As^2 + Bs + C} , \tag{1.2}$$

where

F(s) is the Laplace transform of the excitation function f(t);

y(0) and y'(0) are initial conditions.

The characteristic equation in this case is:

$$As^2 + Bs + C = 0 , \tag{1.3}$$

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[2] *Ibid.*, p. 170.

or, dividing through by A,

$$s^2 + \frac{B}{A}s + \frac{C}{A} = 0. \qquad (1.3\ a)$$

The factors of (1.3 a) are

$$(s - a + jb)(s - a - jb) = 0, \qquad (1.4)$$

where

$$a = -\frac{B}{2A} \quad \text{and} \quad b = \sqrt{\frac{C}{A} - a^2}\ . \qquad (1.5)$$

The roots s_1 and s_2 of (1.3) locate the poles of (1.2) in the complex s plane of Figure 1; that is, they define values of s at which Y(s) is infinite.

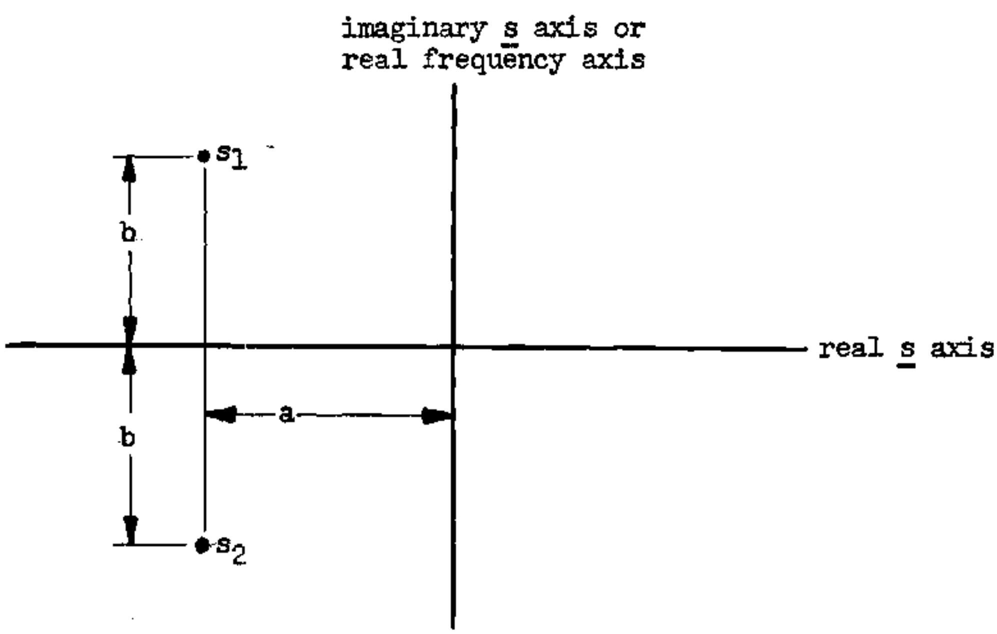

Figure 1. s PLANE

It should be noted that a and b involve the values of the circuit constants only and are independent of any initial conditions or applied excitation function. Therefore these singular points completely describe the network under consideration. In fact, modern manipulations of general network theory are carried out in the compact terminology of these complex frequency plane singularities rather than in terms of the more cumbersome differential equations of the time domain.

The terms a and b of (1.5) also appear in the exponential solution of (1.1) with no excitation.

$$A \frac{d^2y}{dt^2} + B \frac{dy}{dt} + C y = 0. \tag{1.1 a}$$

The solution of (1.1 a) is

$$y = K_1 e^{(a + jb)t} + K_2 e^{(a - jb)t}, \tag{1.6}$$

where K_1 and K_2 are constants which depend upon initial conditions.

The real part of the exponent, a, is referred to as the logarithmic decrement or damping constant, and the imaginary part, b, as the angular frequency. A physical interpretation of these terms will be discussed later.

Network Classifications. Electrical (and mechanical) networks fall into four main classifications: linear passive, linear active, non-linear passive, and non-linear active. A linear circuit contains only elements whose properties are not functions of the dependent variable, while a non-linear circuit may include such elements. A passive network contains no current or voltage generators while an active network may contain such elements.

The determination of the roots or poles of linear passive networks and the interpretation of these singularities in the complex frequency plane is fairly well understood and practiced to advantage by design engineers handling problems concerning both analysis and synthesis. Linear active network theory is understood to a somewhat lesser degree. However, problems concerning non-linear circuits and in particular those concerning non-linear active circuits still confound the theorists. This latter category includes all networks having the property of sustaining oscillations. In fact, relatively little is understood about a most commonly used physical device, the oscillator. The theory of oscillators has been carried up to a certain point, however, by assuming linear conditions.

Linear Oscillators. Consider a linear series circuit consisting of a purely reactive inductance L, a purely reactive capacitance C, and a non-reactive resistance R. With appropriate substitution of coefficients and dependent variable, (1.1 a) describes the time variation of charge q in this network.

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0. \qquad (1.7)$$

By direct analogue to (1.1 a), the roots of this network are:

$$s_1, s_2 = +a \pm jb = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad . \qquad (1.8)$$

We wish to examine the behavior of these roots as R is varied over a wide range while the values of L and C are held constant.[3] To make the plotting easier a normalizing resistance term r is employed as a variable. Let

$$R = r\sqrt{\frac{4L}{C}},$$

and (1.9)

$$w = \frac{1}{\sqrt{LC}}.$$

Then, from (1.8):

$$s_1,\ s_2 = -\frac{r}{\sqrt{LC}} \pm \sqrt{\frac{1}{LC}(r^2 - 1)} = w(-r \pm \sqrt{r^2 - 1}) \ . \quad (1.10)$$

Figure 2 shows a plot of the normalized roots,

$$\frac{s_1}{w}\ \frac{s_2}{w} = \frac{a}{w} \pm j\,\frac{b}{w} = -r \pm j\sqrt{1 - r^2}\ , \qquad (1.11)$$

in the complex plane as a function of resistance factor r which is now equal to the previously defined decrement divided by w. If r is exactly zero, the network will sustain sinusoidal oscillations of frequency $\frac{b}{w} = 1$, the amplitude being dependent only upon the magnitude of the initial excitation. This is illustrated by the wave form immediately above the

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[3] W. A. Edson, Vernon R. Widerquist, Frederick Dixon, and Catherine Yoe, The Keying Properties of Quartz Crystal Oscillators (Final Report, Contract #W36-039-sc-32100. Atlanta, Georgia: State Engineering Experiment Station, Georgia Institute of Technology, 1947), pp. 6-57.

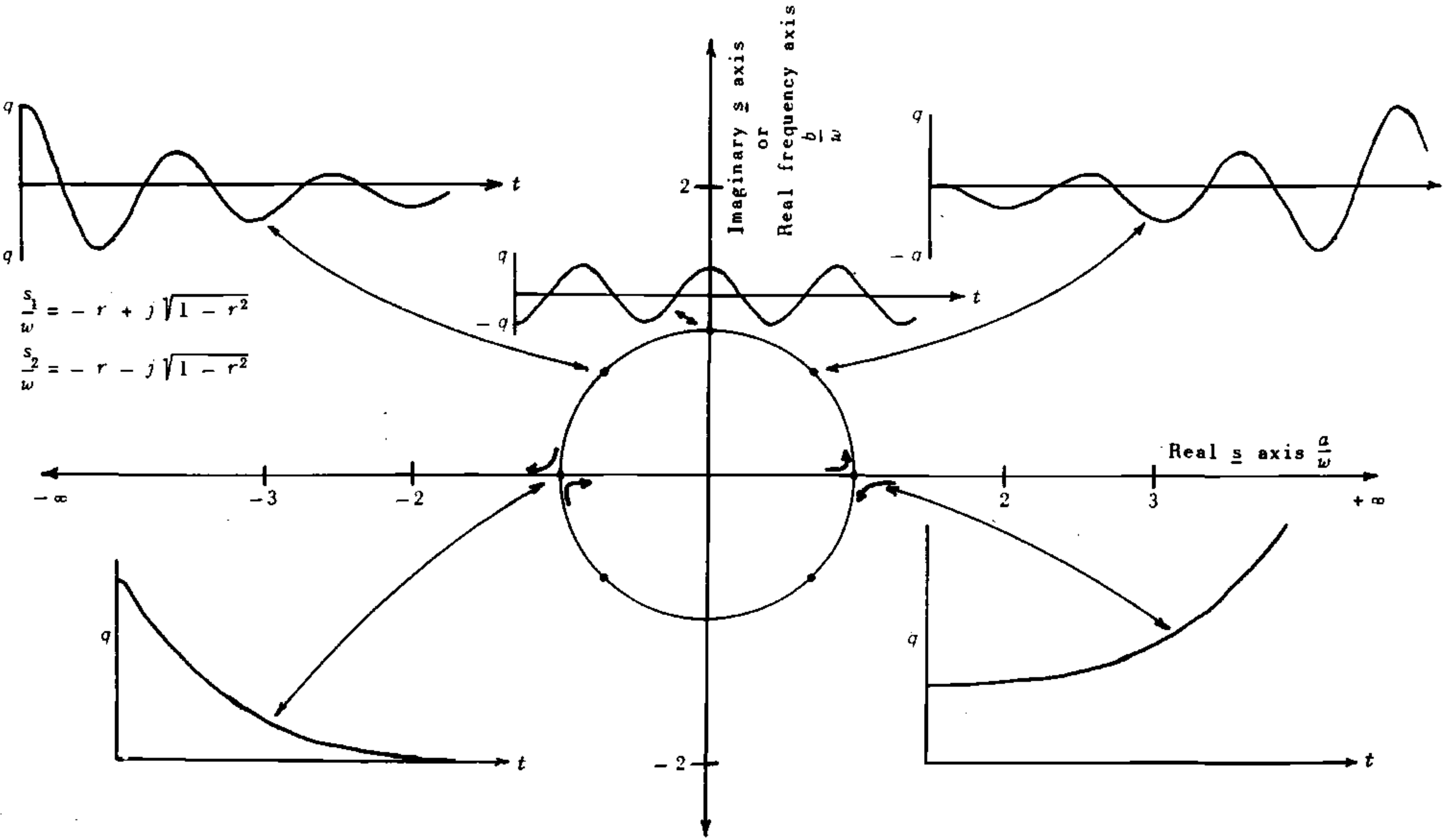

FIGURE 2. PLOT OF NORMALIZED s ROOTS IN THE COMPLEX PLANE *vs.* RESISTANCE FACTOR r FOR SERIES RLC CIRCUIT, WITH TIME RESPONSES

circle in Figure 2. If r is positive, but less than 1, oscillations of frequency $b/w = \sqrt{1 - r^2}$ will die out exponentially with time regardless of the magnitude of the initial excitation. This is shown by the upper left wave form in Figure 2. If r is negative, but less than 1, oscillations of frequency $b/w = \sqrt{1 - r^2}$ will increase exponentially in magnitude if the system is subjected to even the slightest initial excitation. This is illustrated by the wave form to the upper right of Figure 2. If $|r|$ is equal to or greater than 1, a non-oscillatory (monotonic) expanding or decreasing wave form will follow any initial shock. This is shown by the two lower wave forms of Figure 2. Dr. W. A. Edson of the Georgia Institute of Technology chooses to identify this latter case with relaxation oscillations, the condition associated with hyperbolic frequencies.

Inadequacy of Linear Theory. That linear theory is inadequate to explain sustained oscillation can be appreciated from the following:

1. Surely the resistance cannot be identically zero as this defines a point of conditional stability. Deviation of r from this value, however slight, will cause the oscillations to die out or build up indefinitely depending on whether the resistance becomes positive or negative, respectively.

2. Even if the resistance term were identically zero, the amplitude of oscillation would depend only on initial shock conditions and a periodic wave form of unlimited amplitude could exist in the system.

3. Experiment has shown that even the Meacham oscillator, generally referred to as "essentially linear," can be synchronized to an

external signal which differs slightly in frequency, a process solely dependent on the property of non-linearity,[4,5] in this case, the non-linearity of the volt-ampere characteristic of the stabilizing lamp.

It will be shown in this paper that these inadequacies of linear theory may be surmounted by extending the definition of fixed singularities which are permitted to move about in the plane as a function of time.

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[4] L. A. Meacham, "The Bridge Stabilized Oscillator," Proceedings of the Institute of Radio Engineers, 26:1278-1294, 1938.

[5] R. D. Huntoon and A. Weiss, "Synchronization of Oscillators," Proceedings of the Institute of Radio Engineers, 35:1415-1423, 1947.

CHAPTER II

NON-LINEAR OSCILLATORS

Previous Work on Non-Linear Theory. Lord Rayleigh was the first to recognize the necessity of a non-linear element in systems producing sustained oscillations.[6] The next most notable contribution to the problem was made by Balth. van der Pol when he demonstrated a graphical method for the solution of the problem.[7] This work was based on publications of H. Poincare[8] and A. Lienard.[9] In 1935 Ph. le Corbeiller assimilated and extended this work in a very readable publication.[10] A brief abstract of this latter paper is included here to facilitate the interpretation of our results. More recent publications are not concerned directly with the method of attack described in this paper, although occasional mention is made of decrement and associated frequency.[11,12]

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[6]Lord Rayleigh, "On Maintained Vibrations," Philosophical Magazine, 15:229, 1883; also "Theory of Sound," 2nd ed., 1:par. 68(a).

[7]B. van der Pol, "On Relaxation Oscillations," Philosophical Magazine, 2:978, 1926.

[8]H. Poincare, "Memoir on the Curves defined by a Differential Equation," Journal de Mathematiques, 8:251, 1882.

[9]A. Lienard, "Study of Maintained Oscillations," Revue Generale de l'Electricite, 23:901,946, 1928.

[10]Ph. le Corbeiller, "The Non-Linear Theory of the Maintenance of Oscillations," Proceedings of the Institute of Radio Engineers, 23:361-378, 1935.

[11]N. Kryloff and N. Bogoliuboff, Introduction to Non-Linear Mechanics (Princeton: Princeton University Press, 1947), 105 pp.

[12]A. A. Andronow and C. E. Chaikin, Theory of Oscillations (Princeton: Princeton University Press, 1949), 358 pp.

<u>Abstract of le Corbeiller's Paper</u>. The linear series network described by (1.7) will be modified by the addition of a non-linear negative resistance. For simplicity it is assumed that this non-linear element has a simple cubic current-voltage characteristic,

$$v = -mi + \frac{n}{3} i^3, \tag{2.1}$$

although the method which follows is by no means limited to any such simple function of non-linearity.

The differential equation representative of the above circuit is

$$L \frac{d^2q}{dt^2} + (R - m) \frac{dq}{dt} + \frac{n}{3} \left(\frac{dq}{dt}\right)^3 + \frac{q}{C} = 0. \tag{2.2}$$

It will be convenient to change the physical variables of (2.2) to non-dimensional ones.

Let $\sqrt{LC}$ be the unit of normalized time <u>T</u>. Then

$$t = \sqrt{LC}\, T , \tag{2.3}$$

and (2.2) becomes

$$\frac{1}{C} \frac{dq^2}{dT^2} + (R - m) \frac{1}{\sqrt{LC}} \frac{dq}{dT} + \frac{n}{3(LC)^{3/2}} \left(\frac{dq}{dT}\right)^3 + \frac{q}{C} = 0. \tag{2.4}$$

Let <u>p</u> be the unit of normalized charge. Then

$$q = p\, X. \tag{2.5}$$

Now let $\dot{X}$ represent $\frac{dX}{dT}$ and $\ddot{X}$ represent $\frac{d^2X}{dT^2}$ (Newtonian notation).

Then (2.4) becomes

$$\frac{p}{C}\ddot{X} - \frac{(m - R)p}{\sqrt{LC}}\dot{X} + \frac{p^3 n}{(LC)^{3/2}}\frac{\dot{X}^3}{3} + \frac{p}{C}X = 0. \qquad (2.6)$$

Let

$$p = \sqrt{\frac{(m - R)LC}{n}}. \qquad (2.7)$$

Then (2.6) becomes

$$\sqrt{\frac{(m - R)L}{nC}}\ddot{X} - \sqrt{\frac{(m - R)^3}{n}}\dot{X} + \sqrt{\frac{(m - R)^3}{n}}\frac{\dot{X}^3}{3} + \sqrt{\frac{(m - R)L}{nC}}X = 0. \qquad (2.8\text{ a})$$

Dividing through by $\sqrt{\frac{(m - R)L}{nC}}$,

$$\ddot{X} - \sqrt{\frac{(m - R)^2 C}{L}}\left(\dot{X} - \frac{\dot{X}^3}{3}\right) + X = 0. \qquad (2.8\text{ b})$$

Then, by letting

$$g = \sqrt{\frac{(m - R)^2 C}{L}}, \qquad (2.9)$$

the original expression (2.2) then reduces to

$$\ddot{X} - g\left(\dot{X} - \frac{\dot{X}^3}{3}\right) + X = 0. \qquad (2.10)$$

A graphical relationship between X and $\dot{X}$ can be obtained by the method of isoclines as follows:

Let $\dot{X} = y$. Then (2.10) becomes

$$y\frac{dy}{dX} - g(y - \frac{y^3}{3}) + X = 0. \qquad (2.11)$$

The curve $X = f(\dot{X}) = f(y) = g(y - \frac{y^3}{3})$ is shown on the Xy coordinate system of Figure 3 with g equal to one.

Let CE be the normal to the curve satisfying (2.10) and passing through any point C in the plane. The projection ED of EC on OX is

$$ED = -y\frac{dy}{dX}. \qquad (2.12)$$

Equation (2.11) can therefore be written as

$$ED = AC - AB, \qquad (2.13)$$

which means that, given any point C in the plane, the tangent to the curve at C can be obtained by drawing CBA horizontal, BE vertical and joining EC. The required tangent is perpendicular to EC and a small segment of the curve can be so constructed with a compass. The end of this segment locates a new point C' from which, by repeating the above process, an additional segment of the curve can be constructed. Sufficient repetition of this process will enable us to start anywhere in the plane and construct the curve of y or $\dot{X}$ versus X satisfying (2.10). No matter where the construction is started the asymptotic curve of sustained oscillations will be approached as shown. The projection of this curve on the X axis is proportional to a voltage wave as a function of normalized time T since X is proportional to $q/\sqrt{C}$. This wave form could also be constructed graphically since at any point on the isocline constructed curve the value of X and its slope $\dot{X}$ are known.

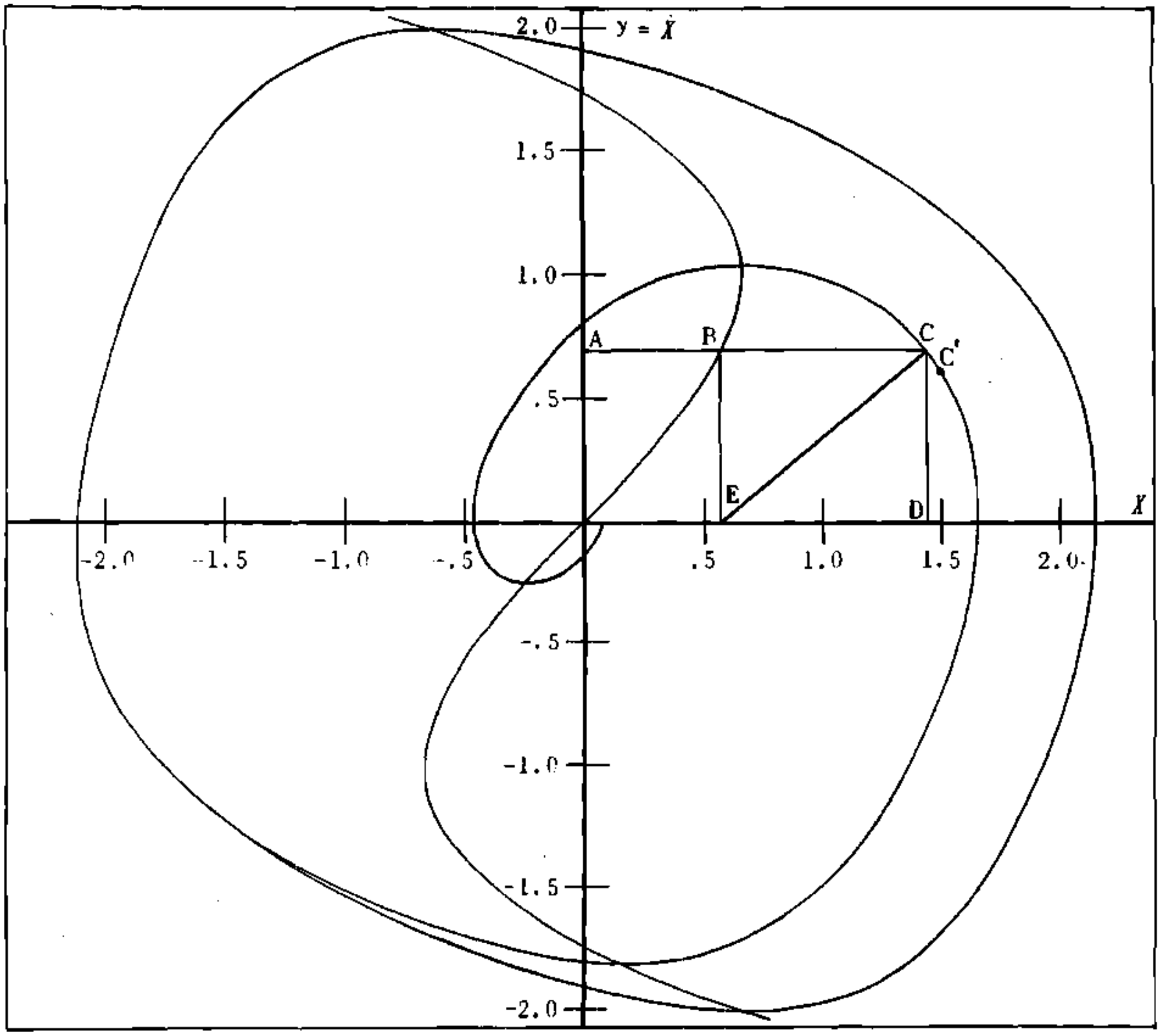

FIGURE 3. GRAPHICAL SOLUTION OF THE NON-LINEAR DIFFERENTIAL EQUATION

$$\ddot{X} - g\left(\dot{X} - \frac{\dot{X}^3}{3}\right) + X = 0 \quad \text{with } g = 1$$

Moving Roots. In the linear system studied in Chapter I the roots of the system equation were fixed by the constants of the system and were independent of time. The problem of the preceding section involves a non-linear system but the treatment does not involve the concept of the network roots. It is the purpose of the following section to explore the concept of roots which change in value as a function of time and may therefore be thought of as moving in the complex plane. Specifically, a new form of envelope and angle function[13] for expressing a system response is proposed.

The envelope function is closely related to the decrement in the linear system of Chapter I. For an oscillatory linear system, the solution of (1.6) is readily converted to the familiar envelope and angle form

$$y = K e^{at} \cos(bt + \phi). \qquad (2.14)$$

Considering only the envelope function,

$$u = K e^{at}, \qquad (2.15)$$

and its derivative,

$$\frac{du}{dt} = a K e^{at}, \qquad (2.16)$$

it is clear that

[13]Gardner and Barnes, loc. cit., p. 249.

$$a = \frac{1}{u}\frac{du}{dt} . \qquad (2.17)$$

This envelope function will now be taken as general. That is, a, which may be a function of time, t, is to be defined by (2.17). As a consequence,

$$a \, dt = \frac{du}{u} . \qquad (2.18)$$

Integration of both sides with respect to t yields

$$\int_0^t a \, dt = \ln u + K, \qquad (2.19)$$

where K is a constant of integration. Letting

$$K = -\ln M \qquad (2.20)$$

and transforming (2.20) to exponential form,

$$e^{\int_0^t a \, dt} = \frac{u}{M} , \qquad (2.21)$$

or

$$u = M \, e^{\int_0^t a \, dt} , \qquad (2.22)$$

where M is the value of the envelope function at time t = 0.

The angle function is obtained from a comparable process. In this case it is appropriate to recall that the constant angular velocity b of (2.14) is measured in radians per second and is the time derivative of a phase angle. That is,

$$b = \frac{d\theta}{dt}. \tag{2.23}$$

In sinusoidal waves $\underline{b}$ is a constant. However, $\underline{b}$ will now be taken as a function of time, $\underline{t}$, to be defined by (2.23). Transposing and integrating yields

$$\theta = \int_0^t b \, dt + \phi , \tag{2.24}$$

where ϕ is the value of θ at time $t = 0$.

The envelope and angle functions proposed in the previous paragraphs are combined in the form

$$X = M e^{\int_0^t a \, dt} \cos \left(\int_0^t b \, dt + \phi \right). \tag{2.25}$$

This function is capable of expressing a variety of wave forms, as shown in Appendix B. Physical reasoning indicates that it is capable of expressing the solution of le Corbeiller's equation (2.10). However, direct substitution leads to a very complex expression which has so far defied all attempts to arrive at an identity. Therefore the present discussion is limited to the proposal of (2.25) and the consideration of a few important special cases.

It should be pointed out that (2.25) is capable of representing a great variety of functions. Although the problem of non-linear oscillation is principally concerned with periodic waves it is merely necessary to set

$$b = 0 \tag{2.26}$$

to obtain a completely aperiodic expression which is still very general; in fact, the expression is sufficiently general as to raise questions of uniqueness.

Preliminary considerations indicate that (2.25) leads to a unique value of $\underline{X}$ if $\underline{a}$, $\underline{b}$, $\underline{M}$, ϕ, and $\underline{t}$ are specified, but that $\underline{a}$ and $\underline{b}$ are not uniquely determined for a specified $\underline{X}$ function of time, $\underline{t}$. However it appears that for a specified $\underline{X}$ of $\underline{t}$ and a particular $\underline{a}$ of $\underline{t}$, there is only one $\underline{b}$ of $\underline{t}$. In physical problems it is sufficient to find any expression which satisfies the original conditions, usually expressed by a differential equation. The usefulness of an expression depends upon its simplicity and the ease with which it may be translated into physical terms.

Application of Moving Roots. Because the problem of non-linear oscillation has not yet been solved in a completely satisfactory manner, it appears desirable to examine (2.25) as a possible solution. First, however, le Corbeiller's equation (2.10) will be generalized to the form

$$\ddot{X} - f(\dot{X}) + X = 0, \tag{2.27}$$

which is capable of expressing the characteristic of any form of non-linear resistance. It is assumed that $f(\dot{X})$ is analytic and substantially linear in the region of $\dot{X} = 0$, so that small oscillations expand in the simple form of (2.14). For such small oscillations, $\underline{a}$ and $\underline{b}$ are constants. Comparison with (2.18) shows that

$$a = \frac{1}{2}\frac{d}{d\dot{X}} f(\dot{X}), \tag{2.28}$$

where the derivative is evaluated for $X = 0$ and $\dot{X} = 0$.

Because a and b in (2.25) are not unique it is possible to make one further arbitrary choice. This choice is to use (2.28) as the definition of a in (2.25). The desirability of this choice is based on the fact that the dynamic resistance, h, of a non-linear resistance is defined by the slope of the characteristic $f(\dot{X})$:

$$h = \frac{d}{d\dot{X}} f(\dot{X}). \qquad (2.29)$$

That is, the constant a in (2.25) as applied to (2.27) will be chosen as equal to half the dynamic resistance, h.

No method has been found to calculate the b which is required by the a of (2.28). However for the so called quasi-linear case in which $f(\dot{X})$ is small compared to one, the value of b is unlikely to differ significantly from one. That is,

$$b \doteq 1. \qquad (2.30)$$

How Non-Linear Roots Move. The definition of (2.28) together with the plausible assumption of (2.30) permits a geometrical interpretation of these quantities. Applying (2.28) to the le Corbeiller equation (2.10),

$$\ddot{X} - g\,(\dot{X} - \frac{\dot{X}^3}{3}) + X = 0 \qquad (2.31)$$

yields

$$a = \frac{g}{2}\,(1 - \dot{X}^2). \qquad (2.32)$$

For small amplitudes of oscillation $\dot{X}$ is small throughout the cycle and a is constant at the value g/2. As the amplitude increases $\dot{X}$ becomes comparable to one and a decreases in magnitude twice per cycle.

In the condition of sustained oscillations a has a time average of zero over any one cycle as given by

$$\int_{t_1}^{t_2} a \, dt = 0. \tag{2.33}$$

Therefore $\dot{X}^2$ is considerably larger than one over a substantial fraction of each cycle.

Until the build-up process has proceeded to the point where the maximum value of the non-linear term, $g\dot{X}^2/2$, becomes equal to or exceeds the value of the linear term, g/2, the roots will remain in the right half of the complex frequency plane. During this time, the network is gaining energy. When the maximum magnitude of the non-linear term becomes greater than the magnitude of the linear term, the roots are in the left half of the complex plane and the network is losing energy.

In the sustained oscillation condition, the roots oscillate across the real frequency axis at twice the fundamental wave-form frequency while the network is alternately gaining and losing energy in such a manner as to give no net change over a complete cycle. If the non-linear resistance characteristic of Figure 3 is symmetrical about the X axis, the motion of the roots will be exactly retraced twice per cycle of wave form. If the characteristic is not symmetrical, which is usually the case, the root motion will be exactly retraced only once per cycle. It is to be noted that the larger the value of the linear part of the decrement, the quicker the signal will approach the sustained oscillation condition (fast rise time). However, this can only be done at a sacrifice, as the asymptotic wave form will depart considerably from a pure sinusoid. Since the roots return to the linear position twice each cycle, the frequency will also vary, causing the wave form to be rich in harmonic content.

Conclusion. It has been shown that a logical extension of the concept of the fixed roots of a network leads to the concept of roots moving with time. By studying the motion of these roots, the mechanism of a non-linear oscillatory system may be studied in a novel and rather enlightening manner. It also appears that the general mathematical form of the moving root concept might provide an alternative to the Fourier and Laplace representations of real functions.

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APPENDIX A

APPENDIX A

THE PARALLEL NETWORK

It is a simple matter to show that the parallel circuit consisting of conductance $\underline{G}$, inductance $\underline{L}$, and capacitance $\underline{C}$ is directly analogous to the series circuit upon which the preceeding development was based.

In this case the non-linear element can be defined as a conductance through which the current is a single valued function of the terminal voltage. Supposing that the non-linearity can be expressed by the same simple cubic of (2.1),

$$i = -mV + \frac{n}{3} V^3. \tag{A.1}$$

The differential equation (2.2) represents a summation of voltages. The analogous equation for the parallel network describes a summation of currents.

$$C \frac{d^2F}{dt^2} + (G - m) \frac{dF}{dt} + \frac{n}{3} \left(\frac{dF}{dt}\right)^3 + \frac{F}{L} = 0. \tag{A.2}$$

Here the dependent variable $\underline{F}$ represents flux linkages, which is analogous to the charge $\underline{q}$ of (2.2).

This equation can be reduced to (2.10) by substitution of the following normalizing factors. Let

$$F = pX, \text{ analogous to (2.5);} \tag{A.3}$$

$$p = \sqrt{\frac{(m - G)LC}{n}}, \text{ analogous to (2.7);} \tag{A.4}$$

and

$$g = \sqrt{\frac{(m - G)^2 C}{L}}, \text{ analogous to (2.9).} \tag{A.5}$$

Then

$$\ddot{X} - g(\dot{X} - \frac{\dot{X}^3}{3}) + X = 0. \tag{A.6}$$

Now $\dot{X}$ represents a normalized voltage, the derivative of flux with respect to time. The second term of (2.10) now describes the current I flowing through the total circuit conductance. Following the reasoning developed previously for the series network, the total dynamic conductance is equal to dI/dV. Since the decrement is determined by conductance only, it follows that the roots of (2.10) above, or in the general case of (2.12), for the parallel network are, as previously,

$$a = \frac{1}{2} \frac{d}{d\dot{X}} f(\dot{X}), \tag{A.7}$$

and

$$b = \sqrt{1 - a^2}. \tag{A.8}$$

APPENDIX B

APPENDIX B

GENERAL WAVE FORMS

Equation (2.25) has some rather interesting applications to problems other than the specific oscillator network described in this paper.

$$X = M e^{\int_0^t a\, dt} \cos\left(\int_0^t b\, dt + \phi\right). \qquad (2.25)$$

With appropriate values for a and b as functions of time, the above equation can describe any conceivable wave form, periodic or non-periodic. Similarly, a and b locate, as a function of time, one pair of moving complex frequency plane singularities which describe completely this general wave form of the time domain.

Amplitude Modulation. Consider as a simple illustration the amplitude modulated wave:

$$X = M\left(1 + \frac{c}{100} \cos Wt\right)(\cos wt), \qquad (B.1)$$

where

- w is the radian frequency of the carrier wave;
- W is the radian frequency of the modulation wave;
- c is the percentage modulation;
- M is the maximum amplitude of the wave.

By the Laplace transform, this modulated wave is represented in the complex frequency domain as three pairs of fixed roots.[14]

From Euler's formula, (B.1) becomes

$$X = \left(1 + \frac{c}{100}\,\frac{e^{jWt} + e^{-jWt}}{2}\right) \cos wt. \tag{B.2}$$

The Laplace transform of the carrier component is,[15]

$$\mathcal{L}\left[\cos wt\right] = \frac{s}{s^2 + w^2}\,. \tag{B.3}$$

Now, multiplication by an exponential in the time domain goes over into the complex frequency domain as a translation of the function,[16]

$$\mathcal{L}\left[e^{Wt}\, f(t)\right] = F(s - W). \tag{B.4}$$

From (B.3) and (B.4), the Laplace transform of (B.2) becomes

$$\mathcal{L}\left[f(t)\right] = \frac{s}{s^2 + w^2} + \frac{c}{200}\,\frac{s - jW}{(s - jW)^2 + w^2}$$

$$+ \frac{c}{200}\,\frac{s + jW}{(s + jW)^2 + w^2}\,. \tag{B.5}$$

This expression contains three pairs of fixed conjugate roots on the real frequency axis, the unmodulated pair at $\pm$ jw, a second pair at $+j(w - W)$ and $-j(w + W)$ plus a third pair at $+j(w + W)$ and $-j(w - W)$.

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[14] *Ibid.*, p. 248.

[15] *Ibid.*, p. 120.

[16] *Ibid.*, p. 245.

It is possible, however, to determine from (2.25) a single pair of moving roots which will completely describe the same modulated wave.

The carrier frequency remains undisturbed with pure amplitude modulation. Therefore the radian frequency b of the phasor will likewise be constant and equal to w. Equating the frequency components of (2.25) and (B.1),

$$\cos \int_0^t b \, dt + \phi = \cos wt, \tag{B.6}$$

or, since the cosines are equal,

$$\int_0^t b \, dt + \phi = wt. \tag{B.6 a}$$

Differentiating both sides gives

$$b = w. \tag{B.7}$$

The decrement or logarithmic amplitude function may be determined by equating the amplitude determining parts of (2.25) and (B.1) as follows:

$$M \, e^{\int_0^t a \, dt} = M(1 + \frac{c}{100} \cos Wt). \tag{B.8}$$

Expressing (B.8) logarithmically,

$$\int_0^t a \, dt = \ln(1 + \frac{c}{100} \cos Wt). \tag{B.9}$$

Differentiating both sides,

$$a = - \frac{\frac{c}{100} W \sin Wt}{1 + \frac{c}{100} \cos Wt} . \tag{B.10}$$

Equation (B.10) gives the real part of a single pair of roots as a function of time while the imaginary part remains constant in accordance with (B.6) above.

The real part of the single pair of roots expressing <u>any</u> general amplitude modulated wave will be equal to the logarithmic derivative of the modulation coefficient, while the imaginary part, of course, will remain constant.

<u>Frequency Modulation</u>. A pure frequency modulated wave can be expressed as

$$X = M \cos(wt + B \cos Wt), \tag{B.11}$$

where

$\underline{w}$ is the radian frequency of the unmodulated carrier;

$\underline{W}$ is the modulating radian frequency;

$\underline{B}$ is the modulation index;

$\underline{M}$ is the maximum amplitude.

The decrement in this case is equal to zero since by equating the amplitude determining factors of (2.25) and (B.11),

$$M = M e^{\int_o^t a \, dt} , \tag{B.12}$$

and

$$a = 0. \tag{B.13}$$

Equating the frequency determining parts of (2.25) and (B.11) gives

$$\cos \int_0^t b\, dt + \phi = \cos(wt + B \cos Wt), \tag{B.14}$$

or, since the cosines are equal,

$$\int_0^t b\, dt + \phi = wt + B \cos Wt. \tag{B.15}$$

Differentiating both sides,

$$b = w - BW \sin Wt. \tag{B.16}$$

Equations (B.13) and (B.16) define the position with respect to time of a single pair of roots restricted to the real frequency axis completely describing the frequency modulated wave of (B.11).

Any general frequency modulated wave can be expressed in terms of a single pair of roots with position on the real frequency axis determined by the derivative of the frequency expression in the time domain.