SMALL TORSION GENERATING SETS FOR MAPPING CLASS GROUPS

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SMALL TORSION GENERATING SETS FOR MAPPING CLASS GROUPS

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Nourish beginnings, let us nourish beginnings.

Not all things are blessed, but the seeds of all things are blessed.

The blessing is in the seed.

Muriel Rukeyser

To my parents.

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SUMMARY

A surface of genus g has many symmetries. These form the surface's mapping class group $\mathrm{Mod}(S_g)$, which is finitely generated. The most commonly used generating sets for $\mathrm{Mod}(S_g)$ are comprised of infinite order elements called Dehn twists; however, a number of authors have shown that torsion generating sets are also possible. For example, Brendle and Farb showed that $\mathrm{Mod}(S_g)$ is generated by six involutions for $g \geq 3$. We will discuss our extension of these results to elements of arbitrary order: for $k > 5$ and g sufficiently large, $\mathrm{Mod}(S_g)$ is generated by three elements of order k.

Generalizing this idea, in joint work with Margalit we showed that for $g \geq 3$ every non-trivial periodic element that is not a hyperelliptic involution normally generates $\mathrm{Mod}(S_g)$. This result raises a question: does there exist an N, independent of g, so that if f is a periodic normal generator of $\mathrm{Mod}(S_g)$, then $\mathrm{Mod}(S_g)$ is generated by N conjugates of f? We show that in general there does not exist such an N, but that there do exist universal bounds when additional conditions are placed on f.

In Chapter 1 we give an introduction and overview of our work. In Chapter 2 we show how to construct elements of order k in $\mathrm{Mod}(S_g)$ for sufficiently large values of g. In Chapter 3 we show how to write a Dehn twist as a product in elements of order k. In Chapters 4 and 5 we construct generating sets for $\mathrm{Mod}(S_g)$ that are comprised of elements of fixed finite order. In Chapter 6 we show that that in general that there is no universal upper bound on the number of conjugates of a periodic normal generator required to generate $\mathrm{Mod}(S_g)$, although universal bounds do exist when additional conditions are placed on f.

CHAPTER 1

INTRODUCTION

Let S_g be a closed, connected, and orientable surface of genus g. The mapping class group $\text{Mod}(S_g)$ is the group of homotopy classes of orientation-preserving homeomorphisms of S_g. In this paper, we construct small generating sets for $\text{Mod}(S_g)$ where all of the generators have the same finite order.

Theorem 1.1. *Let $k \geq 6$ and $g \geq (k-1)^2 + 1$. Then $\text{Mod}(S_g)$ is generated by three elements of order k. Also, $\text{Mod}(S_g)$ is generated by four elements of order 5 when $g \geq 8$.*

Theorem 1.1 follows from a stronger but more technical result that we prove as Theorem 4.1. Our generating sets for $\text{Mod}(S_g)$ are constructed explicitly. In addition, the elements in any particular generating set are all conjugate to each other. Of course, attempting to construct generating sets consisting of elements of a fixed order k only makes sense if $\text{Mod}(S_g)$ contains elements of order k in the first place. A construction of Tucker [1] guarantees an element of any fixed order k in $\text{Mod}(S_g)$ whenever g is sufficiently large, as described in Section 2.

Later in the introduction we describe prior work by several authors on generating $\text{Mod}(S_g)$ with elements of fixed finite orders 2, 3, 4, and 6. In each case, the authors show that the number of generators required is independent of g. Set alongside this prior work, a new phenomenon that emerges in our results is that the sizes of our generating sets for $\text{Mod}(S_g)$ are not only independent of the genus of the surface, but they are also independent of the order of the elements.

Our generators are all finite-order elements that can be realized by rotations of S_g embedded in $\mathbb{R}^3$. There are values of g and k where there exist elements of order k in $\text{Mod}(S_g)$, but where these cannot be realized as rotations of S_g embedded in $\mathbb{R}^3$. For instance, there are elements of order 7 in $\text{Mod}(S_3)$ that cannot be realized in this way.

Problem 1.2. *Extend Theorem 4.1 to cases where elements of order k exist in $\mathrm{Mod}(S_g)$ but cannot be realized as rotations of S_g embedded in $\mathbb{R}^3$.*

Our results in Chapter 6 provide an answer to this problem.

We can also seek smaller generating sets for $\mathrm{Mod}(S_g)$ consisting of elements of order k. We note that any such sharpening of Theorem 4.1 would seem to demand a new approach. Our proofs hinge on applications of the lantern relation, and a lantern has only a limited number of symmetries.

Problem 1.3. *For fixed $k \geq 3$ and any $g \geq 3$ where elements of order k exist, can $\mathrm{Mod}(S_g)$ be generated by two elements of order k? What about three elements for orders 4 and 5?*

Background and prior results

The most commonly-used generating sets for $\mathrm{Mod}(S_g)$ consist of Dehn twists, which have infinite order. Dehn [2] showed that $2g(g-1)$ Dehn twists generate $\mathrm{Mod}(S_g)$, and Lickorish [3] showed that $3g + 1$ Dehn twists suffice. Humphries [4] showed that only $2g + 1$ Dehn twists are needed, and he also showed that no smaller set of Dehn twists can generate $\mathrm{Mod}(S_g)$. The curves for these Dehn twists are depicted in Figure 1.1.

There have also been many investigations into constructing generating sets for $\mathrm{Mod}(S_g)$ that include or even consist entirely of periodic elements. For instance, Maclachlan [5] showed that $\mathrm{Mod}(S_g)$ is normally generated by a set of two periodic elements that have orders $2g + 2$ and $4g + 2$, and McCarthy and Papadopoulos [6] showed that $\mathrm{Mod}(S_g)$ is normally generated by a single involution (element of order 2) for $g \geq 3$. Korkmaz [7] showed that $\mathrm{Mod}(S_g)$ is generated by two elements of order $4g + 2$ for $g \geq 3$.

Luo [8] explicitly constructed a finite generating set for $\mathrm{Mod}(S_g)$ consisting of $6(2g+1)$ involutions, given that $g \geq 3$. Luo asked whether there exists a universal upper bound (that is, independent of g) for the number of involutions required to generate $\mathrm{Mod}(S_g)$. Brendle and Farb [9] showed that six involutions suffice to generate $\mathrm{Mod}(S_g)$, again for $g \geq 3$. Kassabov [10] sharpened this result by showing that only five involutions are needed for

$g \geq 5$ and only four are needed for $g \geq 7$. Monden [11] showed that $\mathrm{Mod}(S_g)$ can be generated by three elements of order 3 and by four elements of order 4, each for $g \geq 3$. Recently Yoshihara [12] has shown that $\mathrm{Mod}(S_g)$ can be generated by three elements of order 6 when $g \geq 10$ and by four elements of order 6 when $g \geq 5$.
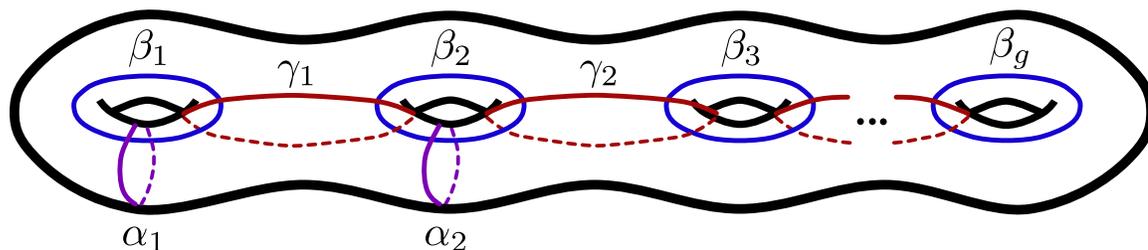
Figure 1.1: The $2g + 1$ Humphries curves in S_g.

Much work has been done to establish when elements of a particular finite order exist in $\mathrm{Mod}(S_g)$. In his paper, Monden noted that for all $g \geq 1$, $\mathrm{Mod}(S_g)$ contains elements of orders 2, 3, and 4. Aside from order 6, elements of larger orders do not always exist in $\mathrm{Mod}(S_g)$. For example, $\mathrm{Mod}(S_3)$ contains no element of order 5 and $\mathrm{Mod}(S_4)$ contains no element of order 7.

Determining the orders of the periodic elements in $\mathrm{Mod}(S_g)$ for any particular g is a solved problem, at least implicitly. In fact, this is even true for determining the conjugacy classes of periodic elements in $\mathrm{Mod}(S_g)$. Ashikaga and Ishizaka [13] listed necessary and sufficient criteria for determining the conjugacy classes in $\mathrm{Mod}(S_g)$ for any particular g. The criteria are number theoretic and consist of the Riemann–Hurwitz formula, an upper bound of $4g + 2$ on the order of elements due to Wiman, an integer-sum condition on the valencies of the ramification points due to Nielsen, and several conditions on the least common multiple of the ramification indices that are due to Harvey.

Ashikaga and Ishizaka also gave lists of the conjugacy classes of periodic elements in $\mathrm{Mod}(S_1)$, $\mathrm{Mod}(S_2)$, and $\mathrm{Mod}(S_3)$. Hirose [14] gave a list of the conjugacy classes of periodic elements in $\mathrm{Mod}(S_4)$. Broughton [15] listed criteria for determining actions of finite groups on S_g, and hence for determining conjugacy classes of finite subgroups of

$\mathrm{Mod}(S_g)$. Broughton also gave a complete classification of actions of finite groups on S_2 and S_3. Kirmura [16] gave a complete classification for S_4.

Several results have been proved about guaranteeing the existence of elements of order k in $\mathrm{Mod}(S_g)$ for sufficiently large g. Harvey [17] showed that $\mathrm{Mod}(S_g)$ contains an element of order k whenever $g \geq (k^2 - 1)/2$. Glover and Mislin [18] showed that $\mathrm{Mod}(S_g)$ contains an element of order k whenever $g > (2k)^2$. A fundamental result in this direction was shown by Kulkarni [19]: for any finite group G, the g for which G acts faithfully on S_g all fall in some infinite arithmetic progression; and further, all but finitely many values in the arithmetic progression are admissible g.

Tucker [1] gave necessary and sufficient conditions for the existence of an element of order k in $\mathrm{Mod}(S_g)$ that can be realized as a rotation of S_g embedded in $\mathbb{R}^3$. Using this characterization, Tucker showed that for any k and for sufficiently large g, $\mathrm{Mod}(S_g)$ contains an element of order k that is realizable by a rotation of S_g embedded in $\mathbb{R}^3$. We give a proof of this fact in Lemma 2.1.

CHAPTER 2

CONSTRUCTING ELEMENTS OF A GIVEN ORDER IN $\mathrm{Mod}(S_g)$

In this section we construct elements of order k in $\mathrm{Mod}(S_g)$ whenever g is sufficiently large. We will use elements that are conjugate to these elements when we build our generating sets.

The following result gives sufficient conditions for the existence of elements of order k in $\mathrm{Mod}(S_g)$ that can be realized by a rotation of S_g embedded in $\mathbb{R}^3$. The result was proved by Tucker [1], who additionally showed that these sufficient conditions are in fact necessary. We include a proof of the result in order to establish conventions about the geometric realizations of the elements it guarantees, as these geometric realizations will be important in proving our main result.

Lemma 2.1. *Let $k \geq 2$. Then $\mathrm{Mod}(S_g)$ contains an element of order k that can be realized as a rotation of S_g embedded in $\mathbb{R}^3$ whenever $g > 0$ can be written as $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or as $ak + 1$ with $a \in \mathbb{Z}_{>0}$.*

Proof. In Figure 2.1 we depict two ways of embedding a surface in $\mathbb{R}^3$ so that it has k-fold rotational symmetry. First, we can embed a surface of genus k in $\mathbb{R}^3$ so that it has a rotational symmetry of order k by evenly spacing k handles about a central sphere. We can also embed a surface of genus $k - 1$ in $\mathbb{R}^3$ so that it has a rotational symmetry of order k, as follows. Arrange two spheres along an axis of rotation and remove k disks from each sphere, evenly spaced along the equator of each. Then connect pairs of boundary components, one from each sphere, with a cylinder. This can be done symmetrically so that a rotation by $2\pi/k$ permutes the cylinders cyclically.

We can use these two types of embeddings to construct embeddings of surfaces of higher genus that also have rotational symmetry of order k. Whenever $g = ak + b(k-1)$,

Figure 2.1: Embeddings of S_5 and S_4 with rotational symmetry of order 5.
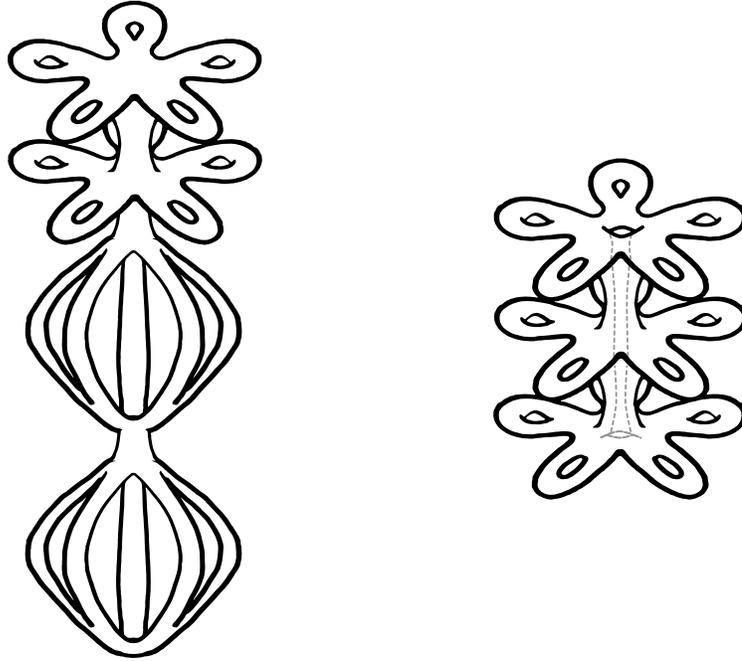
Figure 2.2: Embeddings of S_{18} and S_{16} with rotational symmetry of order 5.

we can construct an embedding of S_g in $\mathbb{R}^3$ by taking a connected sum of surfaces of genus k and $k-1$ along their axis of rotational symmetry. See the left of Figure 2.2 for an example. Rotating a surface embedded in this way by $2\pi/k$ produces an element of $\mathrm{Mod}(S_g)$ of order k for any genus $g = ak + b(k-1)$. That an element so formed does not have order less than k can be seen by the element's action on homology.

In order to produce elements of order k in the case where $g = ak + 1$, we first construct a surface of genus ak with k-fold rotational symmetry by the above construction. We can modify this surface to increase its genus by 1 while preserving its symmetry as follows. See the right of Figure 2.2. The axis of a genus ak surface intersects the surface

at two points—at the top and the bottom. Removing an invariant disk around each of these points creates two boundary components. Connecting the two boundary components with a cylinder yields an embedding of a surface of genus $ak + 1$ with k-fold symmetry. $\square$

By way of some elementary number theory, we show that all sufficiently large integers have either the form $ak + b(k - 1)$ or the form $ak + 1$.

Lemma 2.2. *If $k \geq 5$ and $g \geq (k - 1)(k - 3)$, then g can either be written in the form $ak + b(k - 1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or in the form $ak + 1$ with $a \in \mathbb{Z}_{>0}$.*

Proof. All integers at least $(k - 1)(k - 2)$ can be written in the form $ak + b(k - 1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ by the solution to the Frobenius coin problem. Further, every number from $(k - 1)(k - 3)$ to $k(k - 3)$ can also be written as a sum of k's and $k - 1$'s. Start with $k - 3$ copies of $k - 1$ and replace the $k - 1$'s one at a time by k's. Finally, $k(k - 3) + 1 = (k - 1)(k - 2) - 1$ is of the form $ak + 1$. $\square$

In addition to producing elements of order k in the stable range $g \geq (k - 1)(k - 3)$, we note that the construction given in Lemma 2.1 is also valid for approximately half of the values of g less than $(k - 3)(k - 1)$. Specifically, $(k^2 - 3k - 4)/2$ of these $k^2 - 4k + 2$ smaller values of g are either of the form $ak + b(k - 1)$ or $ak + 1$. This amount is simply $\sum_{i=3}^{k-2} i$, since $\{k - 1, k, k + 1\}$ is the first run of numbers of the given forms and $\{(k - 4)(k - 1), ..., (k - 4)k + 1)\}$ is the last run less than $(k - 1)(k - 3)$.

Also, note that Lemma 2.1 includes the cases where k is 2, 3, or 4. However, the construction we use to create the generating sets of Theorem 4.1 does not work for these small orders. However, these values of k are those already treated by Luo, Brendle and Farb, Kassabov, and Monden in their work on generating sets for $\mathrm{Mod}(S_g)$ consisting of elements of fixed finite order.

CHAPTER 3

BUILDING A DEHN TWIST

In this section, we show that a Dehn twist in $\text{Mod}(S_g)$ about a nonseparating curve may be written as a product in four elements whenever these elements act on a small collection of curves in a specified way. In fact, even fewer than four elements will suffice as long as products in these elements act on the collection of curves as specified. In our proof, we follow the argument that Luo [8] gave for writing a Dehn twist as a product of involutions, as well as the pair swap argument made by Brendle and Farb [9].

We write T_c for the (left) Dehn twist about the curve c. Recall the *lantern relation* that holds among Dehn twists about seven curves arranged in a sphere with four boundary components, called a *lantern*. In the left of Figure 3.1 we depict a lantern L that is a subsurface of S_g. Singling out this particular lantern is convenient for our proof of Theorem 4.1. Note that $S_g \setminus L$ is connected. Seven curves lie in L in a lantern arrangement, and several of these are Humphries curves. We will call these seven curves *lantern curves*. We have the following lantern relation:

$$T_{\alpha_1} T_{\alpha_2} T_{x_1} T_{\gamma_2} = T_{\gamma_1} T_{x_3} T_{x_2}.$$

Recall also that for a Dehn twist T_c and a mapping class f, we have $f T_c f^{-1} = T_{f(c)}$.

Lemma 3.1. *Suppose we are given the subsurface L in S_g and elements f, g, and h in* $\text{Mod}(S_g)$ *such that*

$$f(\gamma_1) = \gamma_2$$

$$g(x_3, x_1) = (\gamma_1, \gamma_2)$$

$$h(x_2, \alpha_2) = (\gamma_1, \gamma_2).$$

Then the Dehn twist T_{α_1} may be written as a product in f, g, h, an element conjugate to f, and their inverses.

While this lemma is stated for a specific Dehn twist by way of a specific lantern, the result holds for other Dehn twists by the *change of coordinates principle*: if two collections of curves on a surface S_g are given by the same topological data, then there exists a homeomorphism of S_g to itself that maps the first collection of curves to the second. Details are given by Farb and Margalit [20].

Proof. Since Dehn twists about nonintersecting curves commute, one form of the lantern relation for L is

$$T_{\alpha_1} = (T_{\gamma_1}T_{\gamma_2}^{-1})(T_{x_3}T_{x_1}^{-1})(T_{x_2}T_{\alpha_2}^{-1}).$$

Applying the assumptions on the elements g and h yields

$$T_{\alpha_1} = (T_{\gamma_1}T_{\gamma_2}^{-1})(g^{-1}(T_{\gamma_1}T_{\gamma_2}^{-1})g)(h^{-1}(T_{\gamma_1}T_{\gamma_2}^{-1})h).$$

Applying the assumption on the element f and regrouping yields

$$T_{\alpha_1} = ((f^{-1}T_{\gamma_2}f)T_{\gamma_2}^{-1})(g^{-1}(f^{-1}T_{\gamma_2}f)T_{\gamma_2}^{-1}g)(h^{-1}(f^{-1}T_{\gamma_2}f)T_{\gamma_2}^{-1}h)$$
$$= (f^{-1}(T_{\gamma_2}fT_{\gamma_2}^{-1}))(g^{-1}f^{-1}(T_{\gamma_2}fT_{\gamma_2}^{-1})g)(h^{-1}f^{-1}(T_{\gamma_2}fT_{\gamma_2}^{-1})h).$$

We have written T_{α_1} as a product in f, g, h, $T_{\gamma_2}fT_{\gamma_2}^{-1}$, and their inverses. □

Note that if f has order k, then so does $T_{\gamma_2}fT_{\gamma_2}^{-1}$ since it is a conjugate of f. Finally, notice that we required very little of f, g, and h in this argument—only that they map one specific curve or one specific pair of curves to another. We will take advantage of this flexibility in the proof of Theorem 4.1.
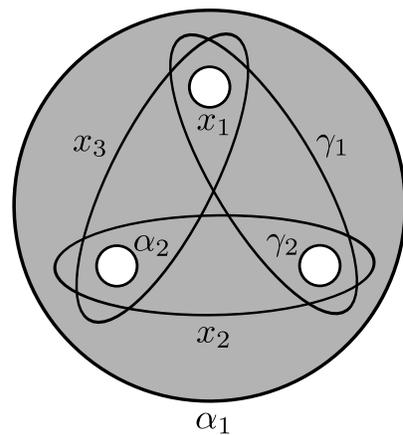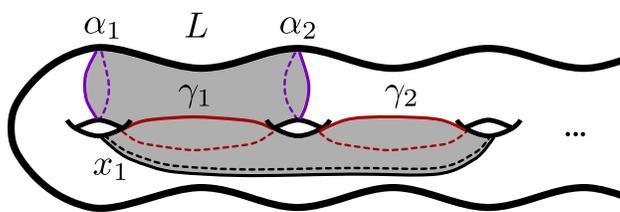
Figure 3.1: On the left, the subsurface L and five of the lantern curves. Two lantern curves are omitted for clarity. On the right, the subsurface L and all seven lantern curves.

CHAPTER 4

GENERATING $\mathrm{Mod}(S_g)$ WITH FOUR ELEMENTS OF ORDER k

In this section and in the following section we prove the two parts of the following theorem, which is our main technical result.

Theorem 4.1. *(1) Let $k \geq 5$ and let $g > 0$ be of the form $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$ or of the form $ak+1$ with $a \in \mathbb{Z}_{>0}$. Then $\mathrm{Mod}(S_g)$ is generated by four elements of order k.*

(2) Let $k \geq 8$ or $k = 6$ and let $g > 0$ be of the form $ak + b(k-1)$ with $a, b \in \mathbb{Z}_{\geq 0}$. Then $\mathrm{Mod}(S_g)$ is generated by three elements of order k. If instead $k = 7$ and g is of the form $7 + 7a + 6b$ with $a, b \in \mathbb{Z}_{\geq 0}$, then $\mathrm{Mod}(S_g)$ is generated by three elements of order 7.

Theorem 1.1 in the introduction follows directly from Theorem 4.1 along with Lemma 2.2 (for the case $g = 5$) and the observation that any $g \geq (k-1)^2 + 1$ may be written as a sum of k's and $(k-1)$'s with at least one summand equal to k.

In this section we prove the first part of Theorem 4.1 about generating with four elements of fixed finite order. In order to illustrate our construction, we depict the particular case $k = 5$ and $g = 18$ in Figure 4.3. In what follows, a *chain* of curves on a surface is a sequence of curves $c_1, \ldots, c_t$ such that pairs of consecutive curves in the sequence intersect exactly once and each other pair of curves is disjoint.

Proof of Theorem 4.1, (1). We begin with the case where $g = ak + b(k-1)$ and treat the case where $g = ak + 1$ with a small modification at the end of the proof. Since $g = ak + b(k-1)$, we have a k-fold symmetric embedding of S_g in $\mathbb{R}^3$ as constructed in Lemma 2.1. Call this embedded surface Σ_g and let it be comprised of a surfaces of genus k followed by b surfaces of genus $k - 1$. Let σ_1 through σ_{a+b} denote these k-symmetric subsurfaces of Σ_g. Let r be a rotation of Σ_g by $2\pi/k$ about its axis.

We will construct our desired elements by mapping S_g to Σ_g, performing a rotation r, and then mapping back to S_g. In doing so we will specify how individual curves map over and back again, and so control how curves are permuted among themselves. In order to construct these maps, it is convenient to label curves on S_g and Σ_g as follows. Take on the one hand the usual embedding of the Humphries curves in S_g as shown in Figure 1.1 and the upper-left of Figure 4.3. We will refer to the α_i, β_i and γ_i curves as α *curves*, β *curves*, and γ *curves*, respectively. The Humphries curves consist of a chain of $2g - 1$ curves that alternate between β curves and γ curves as well as two additional α curves.

Similarly, take the k-fold symmetric embedded surface Σ_g and embed in each σ_i a chain of curves of length $2g_i - 1$, where g_i is the genus of σ_i. See Figure 4.1 and the upper-left of Figure 4.3. We label the curves in these chains also as β and γ curves and note that they are embedded so that $r(\beta_i) = \beta_{i+1}$, $1 \le i \le g_i - 1$, and $r(\gamma_i) = \gamma_{i+1}$, $1 \le i \le g_i - 2$. We will use these labels as "local coordinates"—saying, for instance, "the β_2 curve in σ_3." In σ_1 we additionally embed two α curves, α_1 and α_2, such that each respectively intersects β_1 and β_2 once, intersects no other curves, and $r(\alpha_1) = \alpha_2$.

We are now prepared to define three homeomorphisms $\hat{f}$, $\hat{g}$, and $\hat{h}$ from S_g to Σ_g. We will use these maps to define three homeomorphisms of the form $\hat{f}^{-1} r \hat{f}$ and will show that the corresponding mapping classes (1) have order k, (2) satisfy Lemma 3.1, and (3) generate a subgroup that puts the Humphries curves into the same orbit.

We first construct a homeomorphism $\hat{f}$. The β and γ Humphries curves in S_g form a chain of length $2g - 1$. By removing some of the γ curves from this chain, we form $a + b$ smaller chains. The first a chains will be $2k - 1$ curves long and the last b chains will be $2k - 3$ curves long. We accomplish this by removing every kth γ curve up to γ_{ak}, and then every $(k - 1)$st γ curve thereafter. We call these the *excluded* γ curves. Call the resulting chains F_i, keeping their sequential order. We add to F_1 the curves α_1 and α_2.

Note that the curves in each F_i form a chain of simple closed curves in S_g and the union of the F_i is nonseparating. (Note that F_1 is not quite a chain because of the α_2 curve.) By
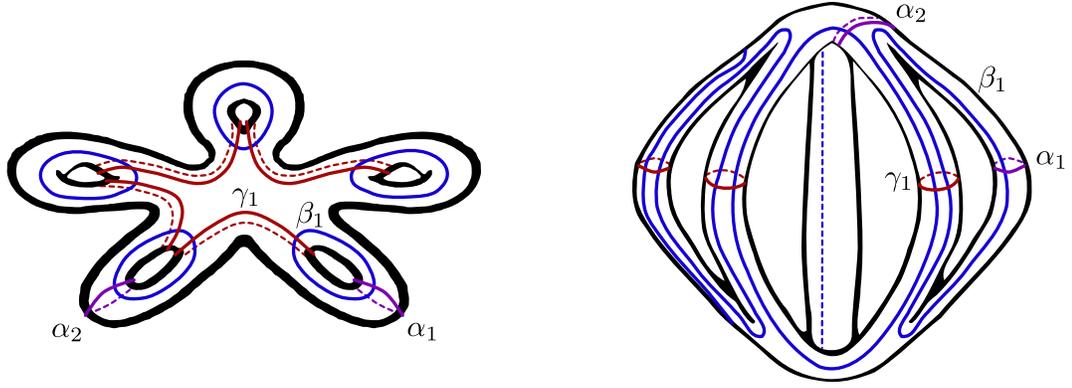
Figure 4.1: The embeddings of chains of curves in the σ_i. The α curves are only included in the subsurface σ_1.

the change of coordinates principle, there is a homeomorphism $\hat{f}$ that takes curves in the F_i to the curves in the chains in σ_i as specified above, as these chains of curves have the same length. (Recall that in σ_1 we have two additional curves that correspond to the α curves of S_g.) Let f be the mapping class of $\hat{f}^{-1}r\hat{f}$. Then f has order k and maps γ_1 to γ_2 as required by Lemma 3.1.

We now construct $\hat{g}$. We form triples of curves G_i, $2 \le i \le a + b$. To form each G_i, we take the second-to-last β curve in F_{i-1}, the excluded γ curve falling between F_{i-1} and F_i, and the second β curve in F_i.

Note that the curves in $\cup_i G_i$ are in the complement of L, that they are nonseparating simple closed curves, that they are disjoint, and that their union is nonseparating. By the change of coordinates principle, there is a homeomorphism $\hat{g}$ that maps the curves in L and the curves $\cup_i G_i$ to a collection of curves of the same topological type in Σ_g as follows:

$$\hat{g} \colon S_g \longrightarrow \Sigma_g$$

$$(x_3, x_1, \gamma_1, \gamma_2) \longmapsto (a, b, c, d) \text{ in } \sigma_1 \text{ as in Figure 4.2}$$

$$G_i \longmapsto (\beta_1, \beta_2, \beta_3) \text{ in } \sigma_i, \ 2 \le i \le a + b$$

Note that the specified image curves are of the same topological type as the four curves in L and the curves in $\cup_i G_i$. Note also that the embedding of the lantern curves depends on

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whether the genus of σ_1 is k or $k-1$; see Figure 4.2. In both embeddings, the image of L is again nonseparating. Let g be the mapping class of $\hat{g}^{-1}r\hat{g}$. Then g has order k and maps the pair (x_3, x_1) to the pair (γ_1, γ_2) as required in Lemma 3.1.
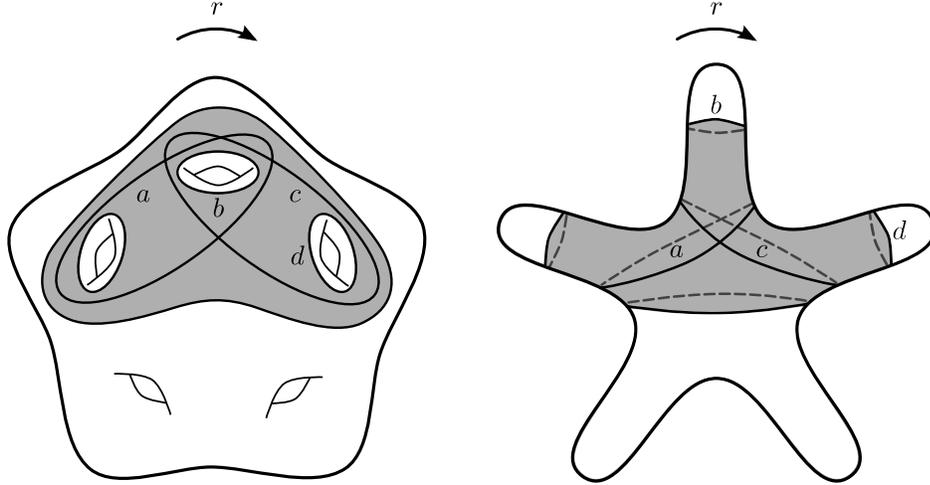
Figure 4.2: The important curves of the subsurface L as embedded by $\hat{g}$ and $\hat{h}$ in σ_1 when the genus of σ_1 is k and when it is $k-1$. The latter is depicted as seen from above.

Finally, we construct $\hat{h}$. We form pairs of curves H_i, $2 \le i \le a+b$. Each H_i consists of the first β curve and the second γ curve in F_i. The map $\hat{h}$ also specifies the mapping of the Humphries curve β_4. Let $\hat{h}$ be a homeomorphism that maps curves as follows:

$$\hat{h}\colon S_g \longrightarrow \Sigma_g$$

$$(\gamma_1, \gamma_2, x_2, \alpha_2) \longmapsto (a, b, c, d) \text{ in } \sigma_1 \text{ as in Figure 4.2}$$

$$\beta_4 \longmapsto r(d) \text{ in } \sigma_1$$

$$H_i \longmapsto (\beta_1, \beta_2) \text{ in } \sigma_i, \ 2 \le i \le a+b$$

Let h be the mapping class of $\hat{h}^{-1}r\hat{h}$. Then h has order k and h^{-1} maps the pair (x_2, α_2) to the pair (γ_1, γ_2) as required by Lemma 3.1. (We use h^{-1} here because we want all of our generators to be conjugate and because the lantern curves are in a fixed cyclic order.)

We now show that the Humphries curves are in the same orbit under $\langle f, g, h \rangle$. Refer to Figure 4.4. First, note that every β and γ Humphries curve in S_g is in some F_i or G_i.
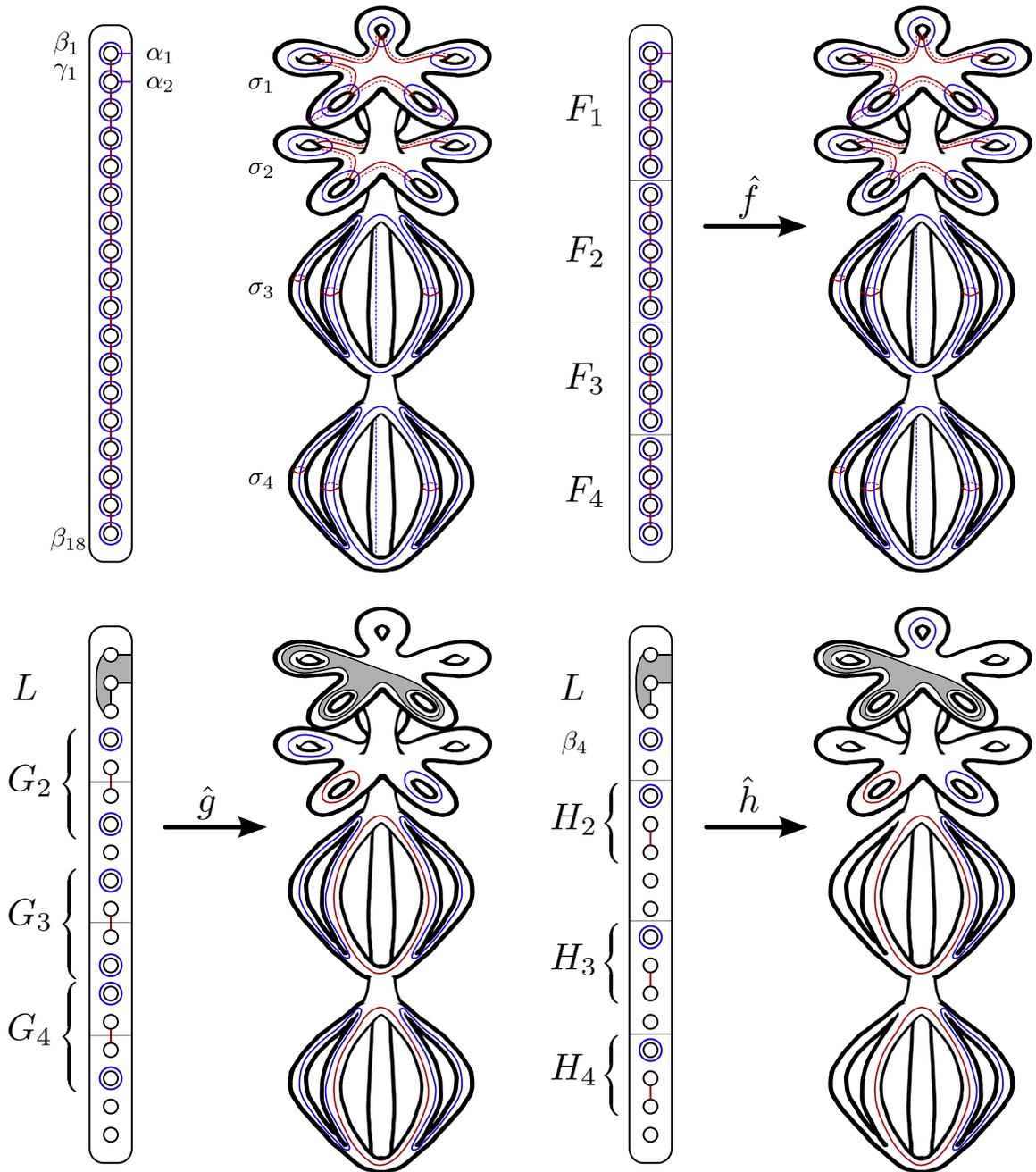
Figure 4.3: The Humphries curves in S_{18}. Σ_{18} with "local coordinate" curves in each σ_i. The curves in the F_i, G_i, H_i, and the subsurface L, along with their images under $\hat{f}$, $\hat{g}$, and $\hat{h}$.

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$$
\begin{array}{ccc}
G_2 & G_3 & \\
\uparrow{\scriptstyle g} & \uparrow{\scriptstyle g} & \\
F_1\beta \xleftrightarrow{\;g\;} F_2\beta \xleftrightarrow{\;g\;} F_3\beta \xleftrightarrow{\;g\;} \dots \\
\uparrow{\scriptstyle h} \quad \uparrow{\scriptstyle h} \quad \uparrow{\scriptstyle h} \\
F_1\gamma \qquad F_2\gamma \qquad F_3\gamma \\
\end{array}
$$

$$
f(a{=}0)\uparrow \qquad \nwarrow{\;g\;}
$$

$$
\alpha_1 \xleftarrow[f(a>0)]{} \alpha_2
$$
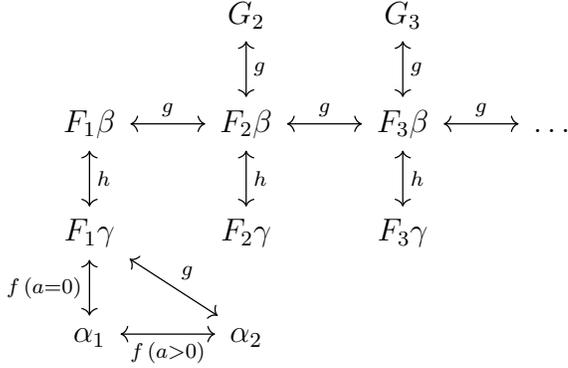
Figure 4.4: Each node is a collection of curves that are in the same orbit under the subgroup generated by a single element. Each arrow indicates when a power of an element maps a curve in one collection to a curve in another. Since every Humphries curve is in at least one of the collections, all Humphries curves are in the same orbit under the subgroup $\langle f, g, h \rangle$.

Additionally, powers of f map any β curve in F_i to any other β curve in the same F_i, and likewise for γ curves. Call these orbits of curves $F_i\beta$ and $F_i\gamma$. In the same way, a power of g maps any curve in G_i to any other curve in the same G_i. Thus at most we have the following orbits of the Humphries curves under $\langle f, g, h \rangle$: the $F_i\beta$, the $F_i\gamma$, the G_i, α_1, and α_2. We will show that these are all in fact a single orbit under $\langle f, g, h \rangle$.

The element f maps α_1 to α_2 when σ_1 has genus k and maps α_1 to γ_1 when σ_1 has genus $k-1$. The element g maps the lantern curve γ_2 to the lantern curve α_2. Thus each of α_1 and α_2 is in the same orbit as some γ curve.

A power of g takes a curve in $F_i\beta$ to a curve in $F_{i-1}\beta$ as well as to a curve in G_i, $2 \leq i \leq a+b$. Additionally, a power of h takes a curve in $F_i\beta$ to a curve in $F_i\gamma$, $1 \leq i \leq a+b$. (Note that in the case of F_1, we have $h^2(\gamma_2) = \beta_4$.) Thus all Humphries curves are in a single orbit under $\langle f, g, h \rangle$. By Lemma 3.1, the Dehn twist about α_1 may be written as a product in f, g, h, and $T_{\gamma_2} f T_{\gamma_2}^{-1}$. Thus all Dehn twists about the Humphries curves may be written as products in our four elements of order k, and so they generate $\mathrm{Mod}(S_g)$.

In the case where $g = ak + 1$, we may modify the construction to show that $\mathrm{Mod}(S_g)$ is again generated by four elements of order k. Take a connect sum of a surfaces of genus k and insert one further handle along the axis of rotation, as in Lemma 2.1. The element r is a rotation of this embedded surface by $2\pi/k$ and f is defined as above by ignoring

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the final two Humphries curves β_g and γ_{g-1}. We must modify our other elements of order k so that they place these two additional Humphries curves into the same orbit as all of the other Humphries curves under the subgroup $\langle f, g, h \rangle$. Modify $\hat{g}$ so that it additionally maps β_g to $r(d)$ in σ_1 and modify $\hat{h}$ so that it additionally maps γ_{g-1} to $r^2(d)$ in σ_1. These modifications preserve the fact that the curves involved are disjoint and that their union is nonseparating. The elements g and h now put the curves β_g and γ_{g-1} into the same orbit as the other Humphries curves. Hence $\mathrm{Mod}(S_g)$ is also generated by four elements of order k when $g = ak + 1$. $\qquad\square$

CHAPTER 5

SHARPENING TO THREE ELEMENTS

In this section we prove the second part of Theorem 4.1.

Proof of Theorem 4.1, (2). We first provide the construction for the cases $k \geq 8$ and then afterwards give the constructions for $k = 7$ and $k = 6$. Let $k \geq 8$. By assumption we may write g in the form $ak + b(k-1)$. We construct the homeomorphism $\hat{f} : S_g \to \Sigma_g$ as in the proof of the first part of theorem, except with the modification that it additionally maps the α curve that intersects the final β curve in F_1 (called α_ℓ) to the curve $r^{-1}\hat{f}(\alpha_1)$. See Figure 5.1. We again let f be the mapping class of $\hat{f}^{-1}r\hat{f}$, and f has order k.

We now construct $\hat{g}$. Let G_2 consist of α_ℓ, the excluded γ curve falling between F_1 and F_2, the first γ curve in F_2, and the third β curve in F_2. For $2 < i \leq a + b$, let G_i be the last γ curve in F_{i-1}, the excluded γ curve between F_{i-1} and F_i, the first γ curve in F_i, and the third β curve in F_i. See Figure 5.2. Let $\hat{g}$ be a homeomorphism that maps the specified curves as follows:

$$\hat{g} \colon S_g \longrightarrow \Sigma_g$$

$$(x_3, x_1, \gamma_1, \gamma_2, x_2, \alpha_2) \longmapsto (a, b, c, d, e, f) \text{ as in Figure 5.3}$$

$$(\gamma_3, \gamma_4) \longmapsto (r^3(e), r^3(f))$$

$$\beta_6 \longmapsto r^4(f) \text{ in } \sigma_1$$

$$G_i \longmapsto (\beta_1, \beta_2, \beta_3, \beta_4) \text{ in } \sigma_i, \ 2 \leq i \leq a + b$$

Let g be the mapping class of $\hat{g}^{-1}r\hat{g}$. Then g has order k and maps the pair (x_3, x_1) to the pair (γ_1, γ_2) as required by Lemma 3.1. Additionally, $g^3(x_2, \alpha_2) = (\gamma_3, \gamma_4)$ and $f^{-2}(\gamma_3, \gamma_4) = (\gamma_1, \gamma_2)$. We may therefore define $h = f^{-2}g^3$ so that h satisfies the hypoth-
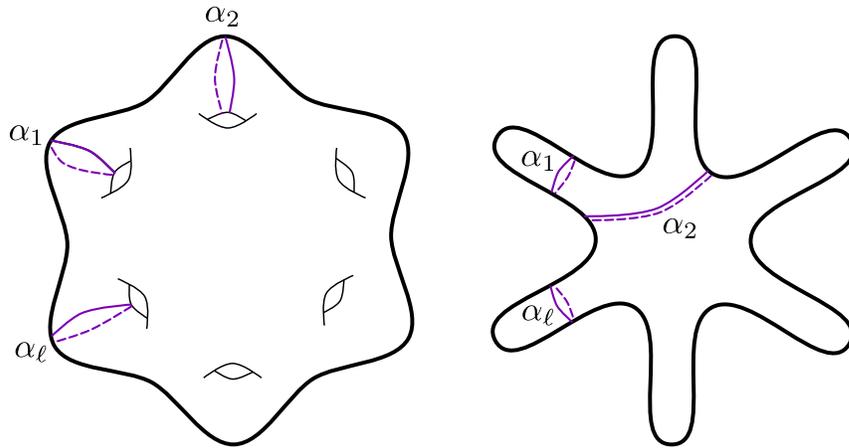
Figure 5.1: The images of α curves embedded in σ_1 by $\hat{f}$. With this embedding, the rotation r maps α_ℓ to α_1.
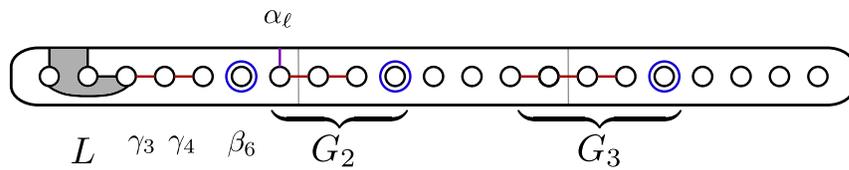
Figure 5.2: The curves mapped by $\hat{g}$ in the case $k = 8$, $g = 21$. This is a worst case example where k has the smallest possible value and all of the σ_i have genus $k - 1$.
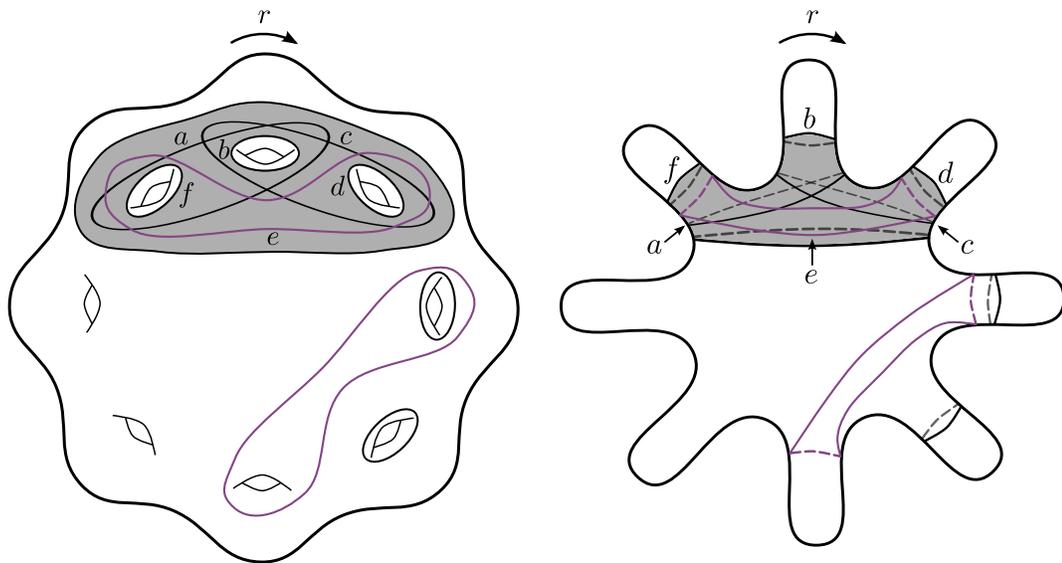
Figure 5.3: The embedding of the subsurface L in σ_1 when the genus of σ_1 is k and when it is $k - 1$. Also, the embedding of γ_3, γ_4, and β_6. These diagrams depict the case $k = 8$.

esis of Lemma 3.1, as $h(x_2, \alpha_2) = (\gamma_1, \gamma_2)$. Thus the Dehn twist about α_1 may be written as a product in f, g, and $T_{\gamma_2} f T_{\gamma_2}^{-1}$.

$$
\begin{array}{ccc}
& G_2 & G_3 \\
& \Big\updownarrow g & \Big\updownarrow g \\
F_1\beta & F_2\beta & F_3\beta \\
\Big\updownarrow g & \Big\updownarrow g & \Big\updownarrow g \\
F_1\gamma & F_2\gamma \xleftrightarrow{\ g\ } F_3\gamma \xleftrightarrow{\ g\ } \dots \\
\end{array}
$$

$h=f^{-2}g^3 \quad \Big\updownarrow f \,(a=0) \quad \Big\updownarrow g$

$\alpha_2 \xleftrightarrow[f\,(a>0)]{} \alpha_1 \xleftrightarrow{\ f\ } \alpha_l$
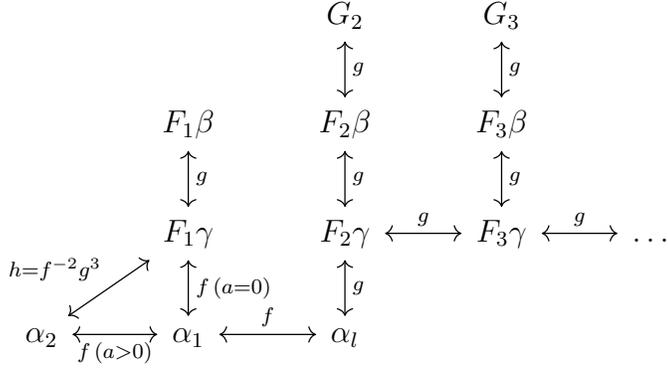
Figure 5.4: Again, each node is a collection of curves that are in the same orbit under the subgroup generated by a single element. Each arrow indicates when a power of an element maps a curve in one collection to a curve in another. Since every Humphries curve is in at least one of the collections, all Humphries curves are in the same orbit under $\langle f, g \rangle$.

Finally, we show that all of the Humphries curves are in the same orbit under $\langle f, g \rangle$, as can be seen in Figure 5.4. Again, every β and γ Humphries curve in S_g is in some $F_i\beta$, $F_i\gamma$, or G_i. Therefore we have at most the following orbits of the Humphries curves under $\langle f, g \rangle$: the $F_i\beta$, the $F_i\gamma$, the G_i, α_1, and α_2. We will show that these are all in fact a single orbit under $\langle f, g \rangle$. For $i > 2$, powers of g put the curves in G_i, $F_i\beta$, $F_i\gamma$, and $F_{i-1}\gamma$ in the same orbit. Note that powers of g map β_6 to γ_2 and a γ curve in F_2 to α_ℓ, while the product h carries α_2 to γ_2. Also, f maps α_ℓ to α_1 and maps α_1 either to α_2 or γ_1, depending on the genus of σ_1. Considering this, all of the curves are in the same orbit under the subgroup $\langle f, g \rangle$. Therefore the Dehn twist about each of the Humphries curves may be written as a product in the three elements f, g, and $T_{\gamma_2} f T_{\gamma_2}^{-1}$, and so they generate $\mathrm{Mod}(S_g)$.

In the case where $k = 7$, the same construction as above goes through as long as the genus of σ_1 is 7. As illustrated in Figure 5.2, under this assumption there is enough room to configure all of the required curves in the construction of $\hat{g}$. The hypotheses of the theorem in this case exactly demand that the genus of σ_1 be 7.

In the case where $k = 6$, we use the same construction as above for f and construct the
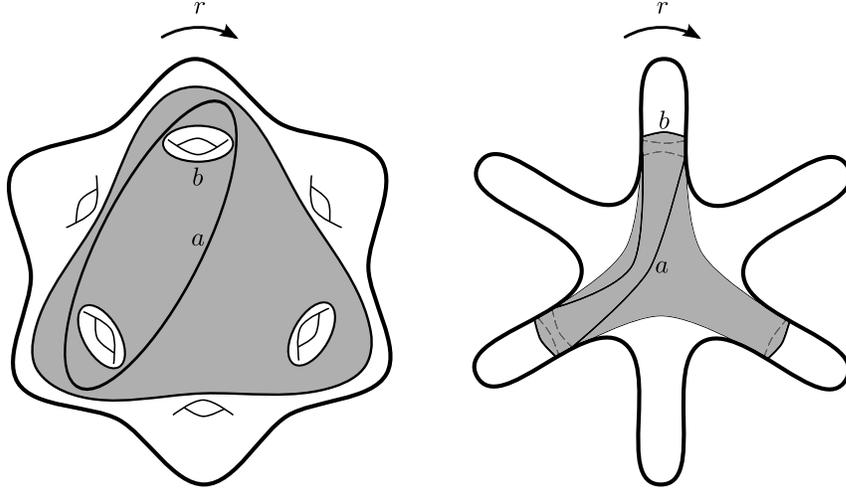
Figure 5.5: The embeddings of the subsurface L in σ_1 when $k = 6$.

element g as follows, exploiting the three-fold symmetry of a lantern. See Figure 5.5. Let $\hat{g}$ be a homeomorphism that maps the specified curves as follows:

$$\hat{g} \colon S_g \longrightarrow \Sigma_g$$

$$(x_3, x_1, \gamma_1, \gamma_2, x_2, \alpha_2) \longmapsto (a, b, r^2(a), r^2(b), r^4(a), r^4(b)) \text{ as in Figure 5.5}$$

$$\beta_4 \longmapsto r(b) \text{ in } \sigma_1$$

$$G_i \longmapsto (\beta_1, \beta_2, \beta_3, \beta_4) \text{ in } \sigma_i, \ 2 \leq i \leq a + b$$

In this case, g^2 and g^4 play the roles of g and h in Lemma 3.1, and all Humphries curves are again in the same orbit. □

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CHAPTER 6

UNIVERSAL BOUNDS FOR TORSION GENERATING SETS OF $\mathrm{Mod}(S_g)$

6.1 Introduction

The author and Margalit proved the following theorem.

Theorem 6.1. *[21, Theorem 1.1] For every $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates $\mathrm{Mod}(S_g)$.*

Based on this result and other corroborating evidence, the author and Margalit raised the following question, which was also recorded as Problem 4.3 by Margalit in his problems paper [22].

Question 6.2. *[21, Question 3.4] Is there a number N, independent of g, so that if f is a periodic normal generator of $\mathrm{Mod}(S_g)$ then $\mathrm{Mod}(S_g)$ is generated by N conjugates of f?*

It will sometimes be convenient for us to use the notation $N(f)$ for the number of conjugates of f required to generate $\mathrm{Mod}(S_g)$. Note that it is not difficult to give an upper bound on $N(f)$ for periodic normal generators that is linear in g. A soft bound is $N(f) \leq 24g + 12$, which follows from our Lemma 6.6 and the fact that $\mathrm{Mod}(S_g)$ is generated by $2g + 1$ Dehn twists about non-separating curves.

The results in this paper resolve Question 6.2.

For the class of involutions—and therefore for the case of general periodic normal generators—we show that there is no universal bound on the number of conjugates of f required to generate $\mathrm{Mod}(S_g)$.

Theorem 6.3. *There does not exist a number N, independent of g, so that if f is a periodic normal generator of $\mathrm{Mod}(S_g)$ with $|f| = 2$ then $\mathrm{Mod}(S_g)$ is generated by N conjugates of f.*

Alternatively, this may be written as:

$$\sup_{g \geq 3} \{N(f) \mid f \in \mathrm{Mod}(S_g) \text{ an involution normal generator}\} = \infty$$

This result should be viewed in contrast with the results by Brendle–Farb, Kassabov, Korkmaz, and Yildiz giving involution generating sets for $\mathrm{Mod}(S_g)$ of universally bounded size [9, 10, 23, 24]. For these results, the involutions in a given generating set need not all be conjugate, and the conjugacy classes used are hand-picked, rather than arbitrary.

On the other hand, we show that involutions are the exceptional case: there does exist a universal upper bound N under the assumption that $|f| \geq 3$.

Theorem 6.4. *There a number N, independent of g, so that if f is a periodic normal generator of $\mathrm{Mod}(S_g)$ with $|f| \geq 3$ then $\mathrm{Mod}(S_g)$ is generated by N conjugates of f.*

Our proof of this theorem shows that N may be taken to be 60. This can be rephrased as: for $g \geq 3$ and f periodic with $|f| \geq 3$, $N(f) \leq 60$. We can of course ask what the sharp universal upper bound may be under the hypothesis $|f| \geq 3$; we know of no obstacle to it being $N = 2$.

Prior results

There have been many results about generating mapping class groups with torsion; an overview is given in an earlier paper of the author [25]. In terms of giving upper bounds on the sizes of torsion generating sets for $\mathrm{Mod}(S_g)$ consisting of conjugate elements, the following results were previously known. Korkmaz showed that two conjugate elements of order $4g+2$ generate $\mathrm{Mod}(S_g)$ for $g \geq 3$. The author showed that three conjugate elements of order $k \geq 6$ generate $\mathrm{Mod}(S_g)$ for $g \geq (k-1)+1$ [25]; all of the elements used in the constructions of that paper can be realized as rotations of an embedding of S_g in $\mathbb{R}^3$.

In Section 2 we prove our results about generating sets for $\mathrm{Mod}(S_g)$ comprised of conjugates of an involution. In Section 3 we describe our proof strategy for Theorem 6.4, prove several preliminary technical lemmas, and then apply these to prove Theorem 6.4.

6.2 Involution generating sets

Every pair (r, s) of non-negative integers determines a conjugacy class of involution homeomorphisms on S_g where $g = 2r + s$; we denote a representative of this class by $i_{r,s}$. The involution $i_{r,s}$ rotates r pairs of handles to swap them and "skewers" s handles, as illustrated in Figure 6.1. Pairs (r, s) with $g = 2r + s$ in fact parametrize conjugacy classes of involutions in $\mathrm{Mod}(S_g)$ for $g \geq 2$, a classification that goes back to the work of Klein; see, for instance, the survey by Dugger [26]. An involution $i_{r,s}$ induces an action on $H_1(S_g; \mathbb{Q})$ that preserves $2s$ subspaces of dimension 1 and induces involutions on r pairs of subspaces of dimension 2.

The intersection of the preserved subspaces of the generators of a group action is also preserved by the group. Consequently, any generating set for $\mathrm{Mod}(S_g)$ comprised of conjugates of $i_{r,s}$ must contain least $(2r + s)/2r$ generators, since $\mathrm{Mod}(S_g)$ acts transitively on 1-dimensional subspaces of $H_1(S_g; \mathbb{Q})$. The quantity $(2r + s)/2r$ is arbitrarily large whenever we let s be sufficiently large compared to r.

Note that $i_{0,g}$ is a hyperelliptic involution in $\mathrm{Mod}(S_g)$ and it is not a normal generator of $\mathrm{Mod}(S_g)$; for $i_{0,g}$, the quantity $(2r + s)/2r$ is undefined.
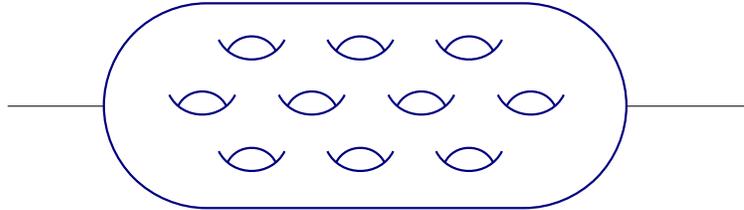
Figure 6.1: A schematic for the involution $i_{r,s}$, where an involution $i_{3,4}$ on S_{10} is illustrated.

While $(2r + s)/2r$ is a lower bound on the number of conjugates of $i_{r,s}$ required to generate $\mathrm{Mod}(S_g)$, we can also give an easy upper bound. This will also show that with the constraint $(2r + s)/2r < n$, there exists a universal bound $N(n)$ on the number of conjugates of $i_{r,s}$ required to generate $\mathrm{Mod}(S_g)$. Our argument additionally serves as a warm-up for our proof of Theorem 6.4.

Let H_g be the set of curves $\{a_1, \ldots, a_{2g}, b\}$ corresponding to the Humphries generating set, where the curves a_i form a chain of length $2g$; the curves H_g are depicted in Figure 6.2. Each generator that is a conjugate of $i_{r,s}$ puts $2r$ disjoint non-separating curves into r 2-cycles, and these $2r$ curves are collectively non-separating. Therefore in a subgroup generated by $k = \lceil g/2r \rceil = \lceil (2r + s)/2r \rceil$ generators, we can ensure that each of g disjoint curves is in a 2-cycle, and that these are collectively non-separating. In the subgroup generated by k additional generators, the same holds true for all $2g$ curves in $H_g - \{b\}$. We may add a further $2k - 1$ generators so that the curves in $H_g - \{b\}$ are all in the same orbit under the generated subgroup, and adding one further generator ensures that b is also in the same orbit. So far this is a total of $4k$ generators. Since T_b can be written as a product in at most an additional 6 generators by the lantern relation trick (see Lemma 6.6), we have that all Dehn twists about the curves in H_g are in the generated subgroup, which therefore equals $\mathrm{Mod}(S_g)$. Summing up, we have that $N(i_{r,s}) \leq 4k + 6 = 4 \cdot \lceil (2r + s)/2r \rceil + 6$.

We record the results of this section in the following theorem.

Theorem 6.5. *Let $g \geq 3$ and let $i_{r,s}$ be an involution in $\mathrm{Mod}(S_g)$ that is not a hyperelliptic involution. Any generating set for $\mathrm{Mod}(S_g)$ consisting of conjugates of the involution $i_{r,s}$ contains at least $(2r + s)/2r$ generators. Further, $4 \cdot \lceil (2r + s)/2r \rceil + 6$ conjugates of $i_{r,s}$ suffice to generate $\mathrm{Mod}(S_g)$.*

This result immediately implies Theorem 6.3, stated in the introduction.
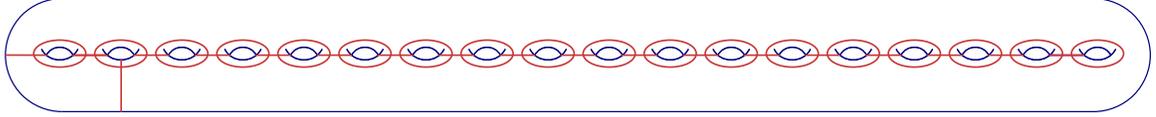
Figure 6.2: The set of curves $H_g = \{a_1, \ldots, a_{2g}, b\}$ corresponding to the Humphries generating set.

6.3 A universal bound for non-involution torsion generating sets

In this section we prove our main result, Theorem 6.4. We begin by outlining our proof strategy, which breaks up into three steps. We then prove the lemmas that carry out the first two steps and then conclude with our proof of Theorem 6.4, which carries out the final step.

Proof strategy

The proof of Theorem 6.4 follows the same basic strategy taken by the author in his article [25], which gives a sharper version of Theorem 6.4 for a certain class of periodic elements (ones that can be represented by rotations of S_g embedded in $\mathbb{R}^3$) under the further assumption that $|f| \geq 5$. It is also the strategy used in the proof of Theorem 6.5 above.

To prove Theorem 6.4, it suffices to show that each of the generators in the Humphries generating set of $2g+1$ Dehn twists about nonseparating curves lies in a subgroup generated by 60 conjugates of an arbitrary periodic element f with $|f| \geq 3$. Again, call the set of Humphries curves H_g; it consists of a chain of curves $a_1, \ldots, a_{2g}$ and an additional curve b. See Figure 6.2. By passing to powers, it suffices to consider f where $|f|$ is either 4 or an odd prime.

The proof then consists of three steps. First, using the lantern relation trick, we show that a single Dehn twist about a non-separating curve can be written as a product in at most 12 distinct conjugates of f. We show this as Lemma 6.6.

It remains to show that the curves in H_g lie in a single orbit under the action of a subgroup generated by at most 48 conjugates of f. As the second step, we show that there

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exists a maximal non-separating chain C of $2g$ curves in S_g such that f acts on a subset C^* of C of size at least $g/4$ where every orbit under $\langle f \rangle$ that contains an element of C^* in fact contains at least two elements of C^*. That is, we show that f shuffles a nice collection of curves, of size a definite fraction of g, in a nice way. This is shown for irreducible elements in Lemma 6.8 and for reducible elements in Lemma 6.9. We conclude by proving the main result, which involves showing that at most 48 conjugates of f generate a subgroup of $\mathrm{Mod}(S_g)$ that acts transitively on H_g; along with the 12 conjugates of f that generate a subgroup that contains T_b, this yields a generating set for $\mathrm{Mod}(S_g)$ consisting of at most 60 conjugates of f.

Step 1

Our first lemma is a straightforward consequence of the work in Sections 2 and 3 of the author's paper with Margalit [21].

Lemma 6.6. *Let $g \geq 3$ and let $f \in \mathrm{Mod}(S_g)$ be a periodic mapping class with $|f| \geq 3$. Then the Dehn twist about any fixed non-separating curve in S_g can be written as a product in at most 12 distinct conjugates of f.*

Proof. By the proof of Theorem 1.1 of [21], there exists a non-trivial power f^k of f and a curve c in S_g such that

1. c is non-separating and c and $f^k(c)$ are disjoint and non-homologous,

2. c is non-separating and c and $f^k(c)$ intersect exactly once, or

3. c is separating and c and $f^k(c)$ are disjoint.

The third case implies the existence of a non-separating curve d such that d and $f^k(d)$ are disjoint and non-homologous, and so reduces to the first case. The second case implies that there is a conjugate g of f such that c and $g^k f^k(c)$ are disjoint and non-homologous, and so reduces to the first case at the cost of doubling the number of conjugates of f

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required. Finally, using the lantern relation trick, the first case shows that a single Dehn twist about a non-separating curve can be written as a product of 6 conjugates of f. (See for instance [20, Theorem 7.16].) So considering all cases, a single Dehn twist about a given non-separating curve can be written as a product in 12 conjugates of f. □

Step 2

We now proceed to the main technical work of the paper, where we guarantee that a definite fraction, independent of g, of the $2g + 1$ curves in H_g can be shuffled around by a periodic element f with $|f| \geq 3$. We begin by classifying the irreducible mapping classes of odd prime order and of order 4.

Lemma 6.7. *Let p be either an odd prime or 4. Then there exists a unique $g(p) \geq 1$ such that $\mathrm{Mod}(S_{g(p)})$ contains an irreducible element of order p. Up to conjugacy and powers, $\mathrm{Mod}(S_{g(p)})$ contains exactly 1 such element when $p = 4$, which has signature $(4, 0; (1, 2), (1, 4), (1, 4))$; and finitely many such elements when p is an odd prime, each with signature of the form $(p, 0; (c_1, p), (c_2, p), (c_3, p))$, with $0 < c_i < p$.*

Proof. Let f be an irreducible periodic mapping class of order p, with p equaling either an odd prime or 4. By a result of Gilman [27], any irreducible periodic mapping class has as its quotient orbifold a sphere with three marked points. Since the index of each marked point is greater than 1 and must divide p, each index must equal p when p is prime and must equal 2 or 4 when p is 4. By the Riemann–Hurwitz formula, we have when p is an odd prime

$$2g - 2 = p(0 - 2) + 3(p - 1)$$

And so $g(p) = \frac{p-1}{2}$.

Similarly, when $p = 4$, the possible triples of indices are $\{2, 2, 4\}$, $\{2, 4, 4\}$, and $\{4, 4, 4\}$, since the LCM of the indices must be 4. We have, respectively,

$$2g - 2 = 4(0 - 2) + 7$$

$$2g - 2 = 4(0 - 2) + 8$$

$$2g - 2 = 4(0 - 2) + 9$$

Only the second equation, corresponding to $\{2, 4, 4\}$, yields a natural number for g, and so we have $g(4) = 1$.

The conjugacy class of a periodic element (that is not a free action) is determined by its signature $(n, g_0; (c_1, n_1), \ldots, (c_\ell, n_\ell))$. For our irreducible element of order p, we have either

$$(p, 0; (c_1, p), (c_2, p), (c_3, p))$$

when p is an odd prime and

$$(4, 0; (c_1, 2), (c_2, 2), (c_3, 4))$$

when $p = 4$, with $0 < c_i < p$. In each case, there is a requirement that $\sum_i c_i = 0 \pmod{p}$. We see that there are finitely many combinations. For $p = 4$, we have that there are two conjugacy classes: $(4, 0; (1, 2), (1, 4), (1, 4))$ and $(4, 0; (1, 2), (1, 4), (1, 4))$. These are powers of each other. □

The next two lemmas form the main technical result of this paper. Under the hypothesis that $|f|$ is an odd prime or 4, we give a topological decomposition of f and use this to show that f shuffles a definite fraction of the curves in a maximal non-separating chain in S_g. Our decomposition applies work of Gilman [27] and follows the outline of a geometric decomposition for general periodic elements given by Parsad–Rajeevsarathy–Sanki [28];

see especially their Theorem 2.27. Our decomposition also resembles the construction of periodic elements of prime order in the thesis of Chrisman, although Chrisman's goal is to produce periodic homeomorphisms that realize any given number of branch points [29, Chapter 3]. (That is, Chrisman gives a construction realizing each possible number of branch points for prime order elements, but he does not show that every prime order conjugacy class can be built in his manner.)

Lemma 6.8. *Let p be either an odd prime or 4. Let $g = g(p)$ and let $f \in \text{Mod}(S_g)$ be an irreducible element of order p. Then there exists a set of curves C^* in S_g such that*

- *C^* is a subset of a maximal non-separating chain of curves in S_g,*

- *$|C^*| \geq 2g/5$, (alternatively, $|C^*| \geq p/4 - 1$),*

- *every orbit of curves under the action of $\langle f \rangle$ that contains a curve in C^* contains at least two curves in C^*, and*

- *each subset of curves in C^* that lies in the same orbit under $\langle f \rangle$ forms either a chain of length at least 2 or else consists of disjoint curves.*

In particular, when p is 3, 4, 5, 7, or 11, we may take C^* so that $|C^*|$ is 2, 2, 2, 3, or 3, respectively.

Proof. We first treat the case $p = 4$. Then S_g is the torus T and f can be realized by a rotation of a square torus by $\pi/2$ or $3\pi/2$. In either case, f exchanges the meridian and longitude of T, which form a chain of length 2. We take this chain to be C^*. We have that $|C^*| = 2 > 1 = g$.

Next we let $p \geq 13$ be an odd prime; we treat the cases $p \in \{3, 5, 7, 11\}$ afterwards. Since f has a fixed point, by a result of Kulkarni, f can be represented as a rotation of a polygon with an appropriate side pairing [30, Theorem 2]. Parsad–Rajeevsarathy–Sanki give an explicit construction for Kulkarni's existence result [28, Theorem 2.10]. In particular, they show that f can be realized by a rotation of a $2p$-gon D by $2\pi m/p$, where

$(m, p) = 1$. The sides d_i of D are identified in pairs (in an orientation-preserving way) according to the formula $d_i \sim d_{i+k}$, where indices are taken mod $2p$ and where k is a constant that depends on the signature for f. In other words, each pair of sides that is identified is at a fixed distance k apart.

Consider three cases, depending on whether k is 0, 1, or 2 (mod 3). In each case we will construct a set of curves C^* so that $|C^*| \geq 2g/5$. In each case, we will select curves from an orbit of a curves c_i under $\langle f \rangle$, $1 \leq i \leq p$, where each curve is formed by the segments connecting midpoints of two pairs of edges: d_i and d_{i+2}, and d_{i+k} and d_{i+k+2}. Each of these curves is non-separating, and we will take a collection so that they form a subset of a chain. Further, these curves are all in the same orbit, and so satisfy the third and fourth properties of C^* in the statement.

When $k = 0$ (mod 3), we may take $\lfloor 2p/6 \rfloor$ of the c_i so that they are disjoint and together do not separate. This case is illustrated in Figure 6.3 in a case where $p = 13$ and $\lfloor 2p/6 \rfloor = 4$.

When $k = 1$ (mod 3), we may take $\lfloor 2p/7 \rfloor$ of the c_i so that they are disjoint and together do not separate.

When $k = 2$ (mod 3), we may take $\lfloor 2p/8 \rfloor$ of the c_i so that they are disjoint and together do not separate.

In each case, we have that $|C^*| = \lfloor 2p/8 \rfloor = \lfloor (2g+1)/4 \rfloor \geq 2g/5$ and also $|C^*| \geq p/4 - 1$. For all odd primes $p \geq 13$ we have that $|C^*| \geq 2$ and that all of the curves in C^* belong to the same orbit, as desired.

For $p \in \{3, 5, 7, 11\}$, a more careful analysis is required. In these cases the bound $|C^*| \geq p/4 - 1$ is insufficient, since we require $|C^*| \geq 2$. We give a modified construction for C^* in each case.

When $p = 3$, $g(p) = 1$. Up to conjugacy and powers there is a single element to consider, corresponding to the signature $(3, 0; (1, 3), (1, 3), (1, 3))$. We have that D is a hexagon with opposite sides identified, and f is a rotation by $2\pi/3$. Then we may take for

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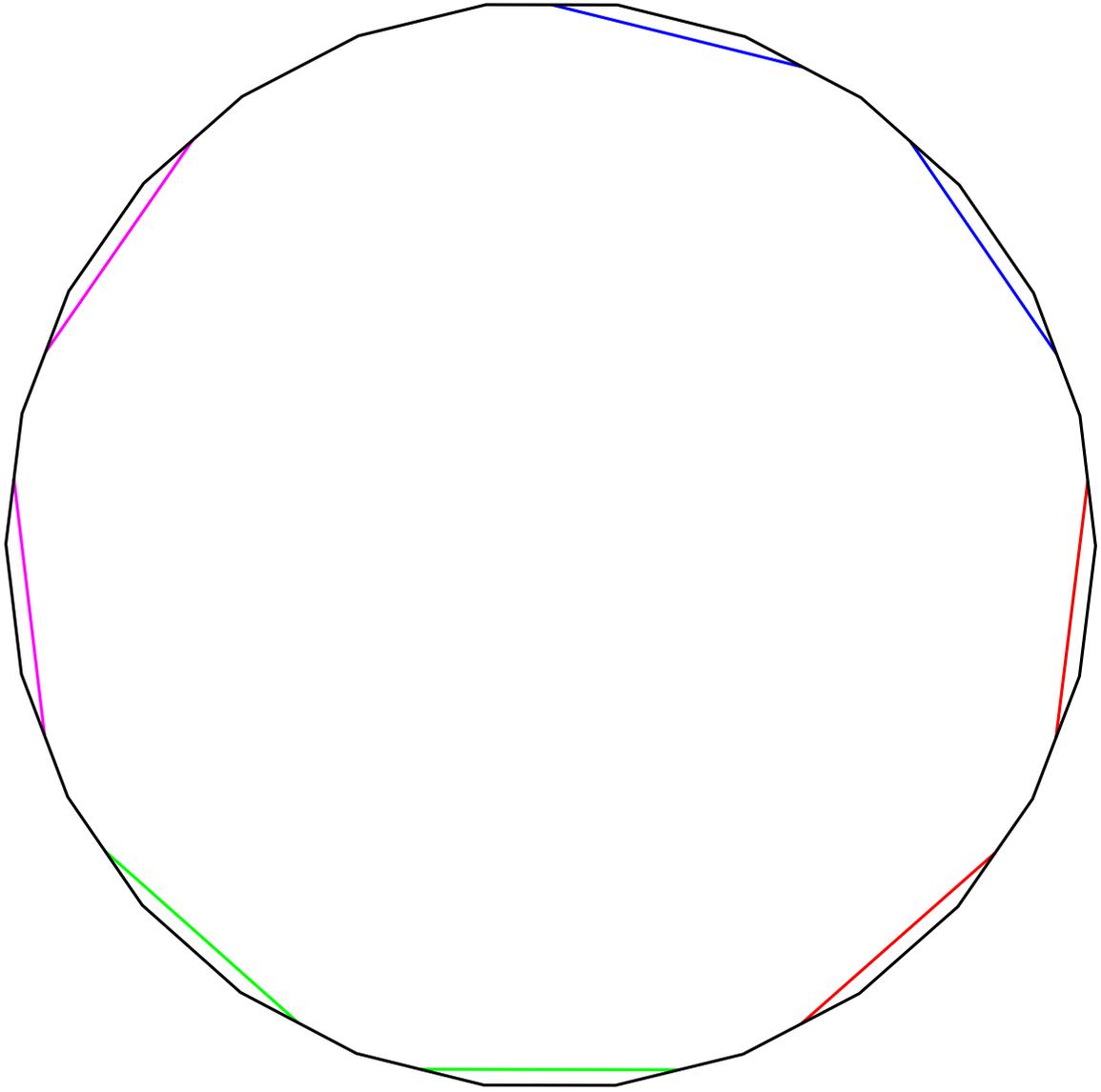
$$(13,0;(1,13),(1,13),(11,13))$$

Figure 6.3: The collections of curves C^* for an irreducible periodic mapping classes of orders 13 with skip number 3 (=0 (mod 3)).

C^* a chain of 2 curves that are in the same orbit under f, namely, 2 curves corresponding to segments connecting midpoints of opposite sides of D.

When $p = 5$, $g(p) = 2$. Up to conjugacy and powers there is a single element to consider, corresponding to the signature $(5, 0; (1, 5), (1, 5), (3, 5))$. We may take C^* to be a chain of length 2, as shown in Figure 6.4.

When $p = 7$, $g(p) = 3$. Up to conjugacy and powers there are two elements to consider, corresponding to the signatures $(7, 0; (1, 7), (1, 7), (5, 7))$ and $(7, 0; (1, 7), (2, 7), (4, 7))$. In each case we may take C^* to be a chain of length 3, as shown in Figure 6.4.

When $p = 11$, $g(p) = 5$. Up to conjugacy and powers there are two elements to consider, corresponding to the signatures $(11, 0; (1, 11), (1, 11), (9, 11))$ and $(7, 0; (1, 11), (2, 11), (8, 11))$. In each case we may take C^* to be a chain of length 3; the pictures are similar to those for $p = 7$. □

Lemma 6.9. *Let p be either an odd prime or 4. Let $g \geq 3$ and let $f \in \mathrm{Mod}(S_g)$ be a reducible element of order p. Then there exists a set of curves C^* in S_g such that*

- *C^* is a subset of a maximal non-separating chain of curves in S_g,*

- *$|C^*| \geq g/4$,*

- *every orbit of curves under the action of $\langle f \rangle$ that contains a curve in C^* contains at least two curves in C^*, and*

- *each subset of curves in C^* that lies in the same orbit under $\langle f \rangle$ forms either a chain of length at least 2 or else consists of disjoint curves.*

Proof. Let p, g, and f be as in the statement. We first observe that if two sets of curves Y_1 and Y_2 are each subsets of a chain of curves in S_g and the Y_i lie in disjoint subsurfaces of S_g that are distinct in homology, then $Y_1 \cup Y_2$ is again a subset of a chain of curves in S_g. We will use this fact freely in forming our collection of curves C^*.

$(5,0;(1,5),(1,5),(3,5))$

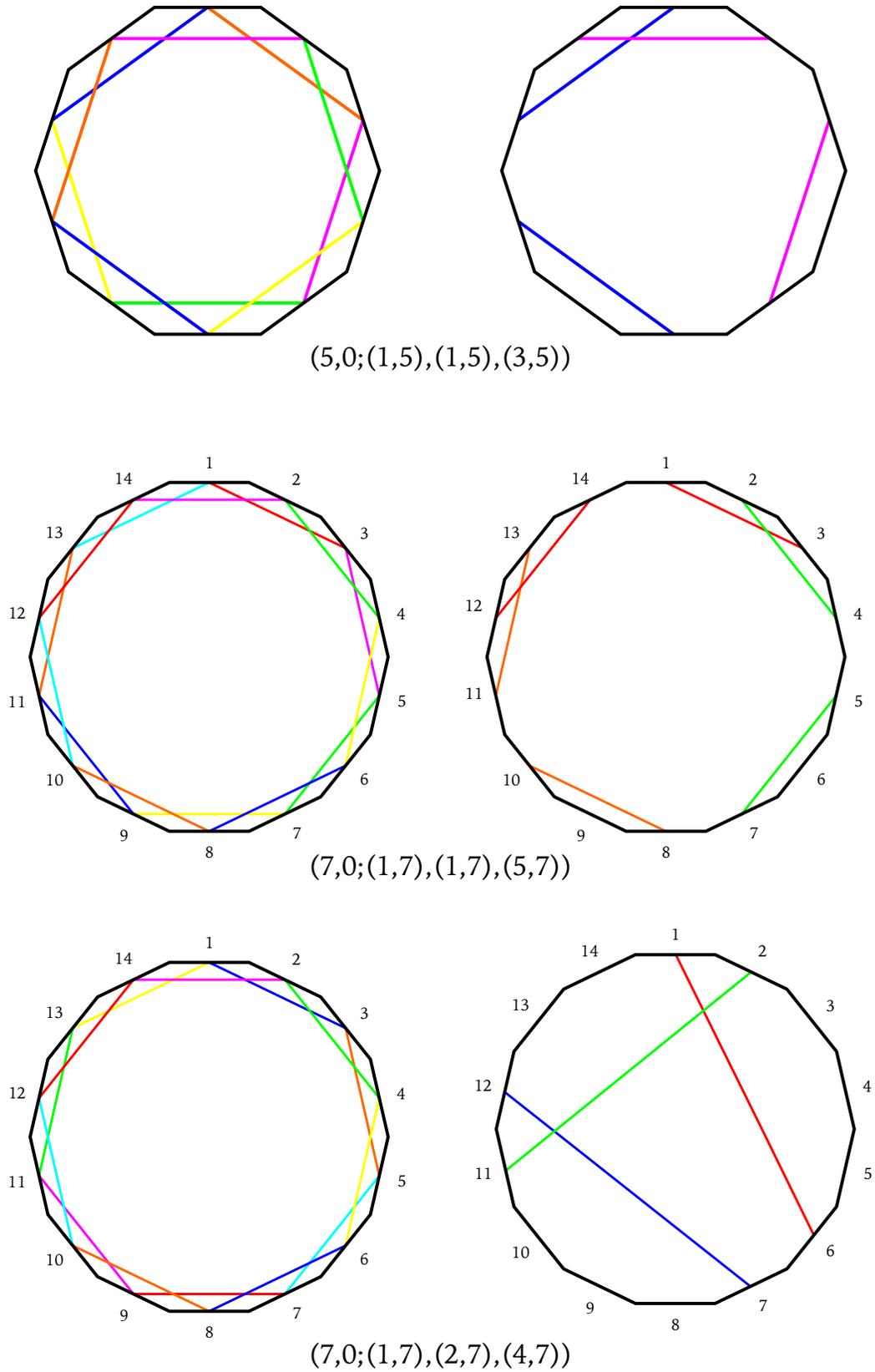$(7,0;(1,7),(1,7),(5,7))$

$(7,0;(1,7),(2,7),(4,7))$

Figure 6.4: The collections of curves C^* for the irreducible periodic mapping classes of orders 5 and 7.

We first treat the case when p is an odd prime. Since f is reducible, the quotient orbifold $Q = S_g/\langle f \rangle$ has either $g_0 > 0$ or $\ell > 3$ or both, by a result of Gilman [27]. (Here g_0 is the genus of Q and ℓ is the number of orbifold points of Q.) If $g_0 > 0$, we may take preimages under $\langle f \rangle$ of g_0 disjoint separating curves in Q, each of which cuts off a single handle from Q. In the preimage, the curves corresponding to a single separating curve in Q cut off an orbit of p handles, distinct in homology. We may take a chain of two non-separating curves in each handle and produce all together $2p$ disjoint curves, which together form a subset of a chain of curves in S_g. These free orbits of handles, then, have a curve-to-genus ratio of $2/1$. Since we are only showing that we can form C^* so that the curves-to-genus ratio is $1/4$, orbits of this type can only help the ratio.

We have therefore reduced to the case of elements f such that $g_0 = 0$. Since f is assumed reducible, again by a result of Gilman [27], f has a reduction system of disjoint essential curves C that are pairwise non-isotopic, which we may take to be maximal. Note that every orbit of curves in C under $\langle f \rangle$ has size 1 or p, since p is prime. Let $S_g(C)$ denote the (possibly disconnected) surface obtained from S_g by cutting along C and capping the resulting boundary components with disks containing a single marked point. Corresponding to f there is an action f' on $S_g(C)$.

We now consider the connected components R_i of $S_g(C)$. We first claim that f' induces the identity permutation on the R_i. Otherwise there would be an orbit of R_i of size p. If these R_i had positive genus, this would contradict the assumption of $g_0 = 0$. If they instead were spheres, they would each have at least 3 marked points in order for the curves of C to be essential and non-isotopic, and this would again contradict the assumption that $g_0 = 0$. Therefore each R_i is mapped to itself by f'.

We now analyze what the components R_i may be and how f' may act upon them. If R_i is a sphere, then f' acts on R_i by a rotation of $2m\pi/p$, with $(m, p) = 1$. The action of f' on R_i has exactly two branch points, each of order p. If instead R_i is not a sphere, f' restricts to an irreducible self-map of order p, and we classified these in Lemma 6.7. In particular,

each non-sphere R_i has genus $\frac{p-1}{2}$ and has exactly 3 branch points, each of order p.

We are now prepared to recover the action of f on S_g by reglueing annuli at pairs of marked points. For any R_i that is a sphere, it cannot only have marked points at one or both of its branch points, since the curves of $\mathcal{C}$ are essential and non-isotopic. Therefore each sphere R_i has some $r_i > 0$ orbits of p marked points, each having 0 local rotation number. Each of the p marked points in each orbit is paired with p other marked points that lie in some other R_j. (We have that $i \neq j$ by the assumption that $g_0 = 0$.) For each such orbit, we may add to C^* a collection of $p - 1$ disjoint non-separating curves. The sphere R_i may additionally have marked points at one or both of its branch points; each can either connect to a marked point on a different R_j, or else they may together form a pair. From these orbits we do not take any curves to add to C^*.

In the other case, where R_i is a surface with positive genus, R_i may have marked points at its branch points under the action of f', or at non-branch points, or at both. The surface R_i has some $r_i \geq 0$ orbits of p marked points (possibly none), each having 0 local rotation number. Each of the p marked points in each orbit is paired with p other marked points that lie in some other R_j. (We have that $i \neq j$ by the assumption that $g_0 = 0$.) For each such orbit, we may add to C^* a collection of $p - 1$ disjoint non-separating curves. The surface R_i may additionally have marked points at its branch points; each can either connect to a marked point on a different R_j, or two of these marked points may together form a pair. In fact, this last possibility does not arise, since then $c_i + c_j = 0 \pmod{p}$ but we have that $c_1 + c_2 + c_3 = 0 \pmod{p}$ and that each c_i is nonzero $\pmod{p}$. In any case, no further contribution to C^* is made for these orbits. Also, we have by Lemma 6.8 that R_i supports $p/4 - 1$ curves to contribute to C^* whenever $g \geq 13$, and either 2 or 3 curves for smaller values of p.

We are now prepared to compute bounds for each of g and $|C^*|$ in terms of the data described so far. Let a be the number of R_i that are spheres and let b be the number of R_i that are surfaces with positive genus. Then the number of branch points of the R_i totals

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$2a + 3b$. Therefore the maximal number of marked branch points is the same, and the number of fixed annuli is at most $\frac{2a+3b}{2}$. Let $k = \frac{1}{2}\sum_i r_i$ be the number of orbits of annuli of size p. Since $(a + b - 1)$ of these annuli are required so that the resulting surface is connected—and therefore do not contribute to the genus—we have the following bound for g:

$$g \le a \cdot 0 + b \cdot \frac{p-1}{2} + kp + \frac{2a+3b}{2} - (a+b-1) = \frac{bp}{2} + kp + 1$$

Note that since $g \ge 3$ by assumption, at least one of b or k is positive.

Let c be equal to the constant guaranteed by Lemma 6.8: $p/4 - 1$ when $g \ge 13$ and 2, 2, 3, or 3 when g is 3, 5, 7, or 11, respectively. As we have a contribution of c curves to C^* for each R_i of positive genus and a contribution of $p - 1$ curves from each pair of marked points with orbit of size p (corresponding to an orbit of annuli of size p), we have the following equality for $|C^*|$:

$$|C^*| = bc + k(p - 1)$$

We now apply the following three facts: (1) at least one of b or k is positive; (2) in general, $\frac{q+r}{s+t} \ge \min\{\frac{q}{s}, \frac{r}{t}\}$ for $q, r, s, t > 0$; (3) the individual inequalities

$$\frac{bc}{\frac{bp}{2} + 1} \ge \frac{1}{4} \quad \text{and} \quad \frac{k(p-1)}{kp+1} \ge \frac{1}{4}$$

hold for each $p \ge 3$ and its corresponding value of c whenever b is positive and whenever k is positive, respectively.

Applying these facts yields the desired result:

$$\frac{|C^*|}{g} \ge \frac{bc + k(p-1)}{\frac{bp}{2} + kp + 2} \ge \frac{1}{4}.$$

The case when $p = 4$ follows the same outline. $\qquad\square$
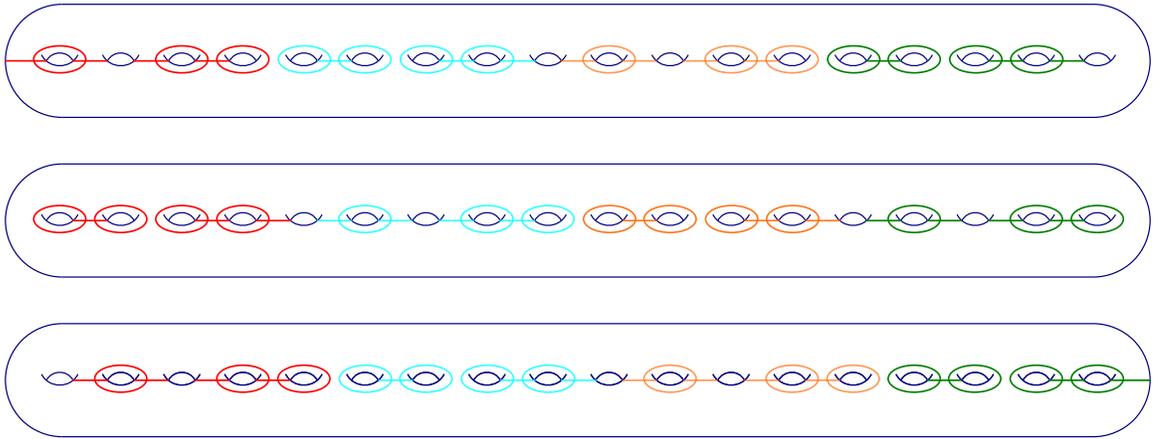
Figure 6.5: A schematic for the collections of curves $C'_1, \ldots, C'_{16}$, $C'_{17}, \ldots, C'_{32}$, and $C'_{33}, \ldots, C'_{48}$. Within a single surface, curves in different C'_i are in different colors, and within a collection C'_i the curves in the same orbit under the cyclic group $\langle f_i \rangle$ are in the same chain. Note that for the case illustrated, only four collections of curves are needed in each of the three groupings, and also that "overflow" curves are not depicted.
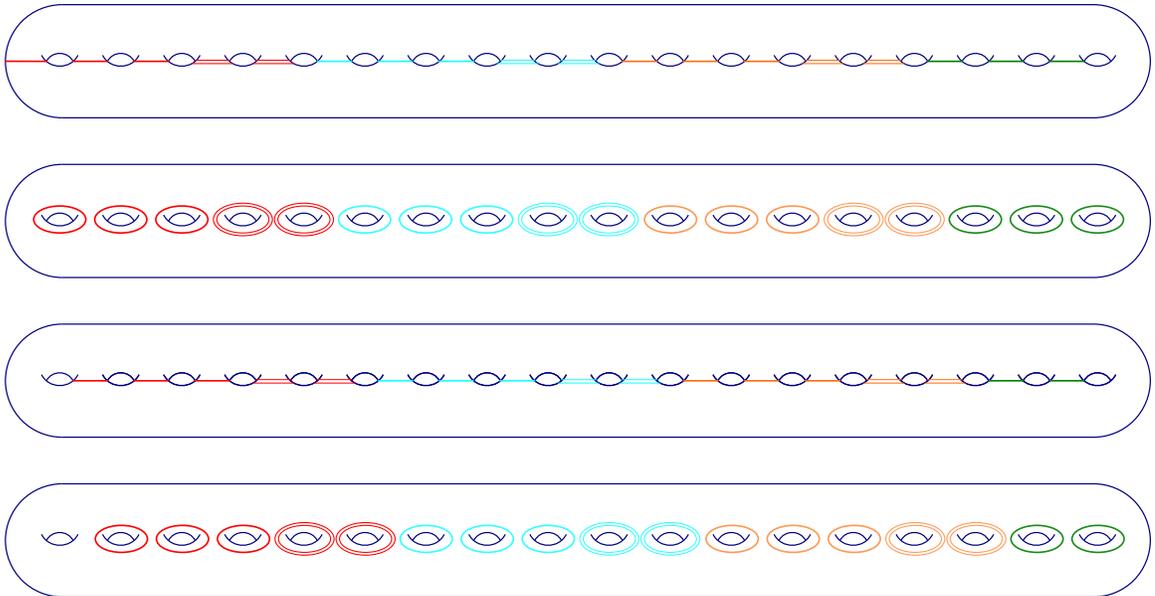
Figure 6.6: A schematic for the collections of curves C_{odd}, C_{even}, and the shifted versions of each. Within a single surface, curves in different C'_i are in different colors, and within a collection C'_i the curves in the same orbit under the cyclic group $\langle f_i \rangle$ have matching markings. Note that for the case illustrated, only four C'_i are needed in each of the four groupings, and also that "overflow" curves are not depicted.

<u>Proof of the main theorem</u>

With all of our preliminaries in hand, we are prepared to prove Theorem 6.4.

Proof of Theorem 6.4. Let $g \geq 3$ and let $H_g = \{a_1, \ldots, a_{2g}, b\}$ be a set of simple closed non-separating curves corresponding to the Humphries generating set for $\mathrm{Mod}(S_g)$. Let f be a periodic element of $\mathrm{Mod}(S_g)$ with $|f| \geq 3$. By passing to a power, we may assume without loss of generality that $|f|$ is either 4 or an odd prime. By Lemma 6.6, a Dehn twist about a non-separating curve may be written as a product in at most 12 distinct conjugates of f. Take 12 conjugates of f in which T_b is a product and let them be the start to our generating set for $\mathrm{Mod}(S_g)$.

By Lemmas 6.8 and 6.9, f acts on C^*, a subset of a maximal non-separating chain of curves in S_g, such that $|C^*| \geq g/4$ and every orbit of curves under the action of $\langle f \rangle$ that contains a curve in C^* contains at least two curves in C^*. The C^* curves that lie in a given orbit either form a chain of length at least 2, or they are all disjoint. We consider two cases, depending on whether at least half of the curves of C^* lie in orbits of the former type, or of the latter type. In either case, let this collection of curves be called C'. We have $|C'| \geq g/8$. We treat the two cases in turn, in similar fashion.

First case: chains. When the curves of C' all lie in chains of length at least 2, we form at most $16 = \frac{2g}{g/8}$ subsets of $H_g - \{b\}$, $C'_1, \ldots, C'_{16}$, that are each of the topological type of C' and that together cover $H_g - \{b\}$ except for "gaps" of size 1. See Figure 6.5. Note that fewer than 16 subsets may be required, and that it does not matter how the "overflow" curves for the last C'_i are chosen. We may then take at most 16 conjugates of f, $\{f_1, \ldots, f_{16}\}$, so that each individually acts on the corresponding C'_i in the way that f acts on C'. In particular, each curve in each C'_i is in the same orbit as the other curves belonging to the same chain in C'_i under the action of the cyclic group $\langle f_i \rangle$.

Similarly, we form $C'_{17}, \ldots, C'_{32}$ and $C'_{33}, \ldots, C'_{48}$ that are shifted one and two curves down the chain $H_g - \{b\}$ from the corresponding curves in $C'_1, \ldots, C'_{16}$. Note that it again

does not matter where the "overflow" curves are chosen, except that we ensure that b is among them. Again, see Figure 6.5. We may take 32 conjugates of f, $\{f_{17}, \ldots, f_{48}\}$, so that each acts on C'_i in the way that f acts on C', for $17 \leq i \leq 48$.

We now argue that all of the curves in H_g are in the same orbit under the subgroup generated by the 48 f_i. All but the "gap" curves are put into orbits by $\langle f_1, \ldots, f_{16} \rangle$. Each gap curve is put into the same orbit as the curve immediately "prior" to it by $\langle f_{17}, \ldots, f_{32} \rangle$ and also into the same orbit as the curve immediately "after" it by $\langle f_{33}, \ldots, f_{48} \rangle$. Thus all of the original orbits are collapsed into a single orbit "through" the gap curves. By construction we also have that b is in this orbit.

Therefore the curves in H_g are all in the same orbit under the action of a subgroup generated by 48 conjugates of f. With the additional 12 conjugates of f we have a Dehn twist about b in the subgroup, and so also the Dehn twists about all curves in H_g, and therefore the subgroup so generated is equal to $\mathrm{Mod}(S_g)$. Thus $\mathrm{Mod}(S_g)$ is generated by 60 conjugates of f, as required.

Second case: disjoint curves. When the curves of C' are all disjoint, we form at most $8 = \frac{g}{g/8}$ subsets of $H_g - \{b\}$, $C'_1, \ldots, C'_8$, that are each of the topological type of C' and that together cover the a_i curves in H_g with odd indices. Call these subsets C_{odd}. See Figure 6.6. Note that fewer than 8 subsets may be required, and that it does not matter how the "overflow" curves for the last C'_i are chosen. We may then take at most 8 conjugates of f, $\{f_1, \ldots, f_8\}$, so that each individually acts on the corresponding C'_i in the way that f acts on C'. Similarly, we form sets of curves $C'_9, \ldots, C'_{16}$ that are each of the topological type of C' and that together cover the a_i curves in H_g with even indices; we may do this by shifting all of the C_{odd} curves down the $H_g - \{b\}$ chain by one curve. Call these subsets C_{even}. We take at most 8 conjugates of f, $\{f_9, \ldots, f_{16}\}$, so that each individually acts on the corresponding C'_i in the way that f acts on C'.

We now take $C'_{17}, \ldots, C'_{24}$ and $C'_{25}, \ldots, C'_{32}$ that are shifted two and three three curves down the chain $H_g - \{b\}$ from the C_{odd} curves. (So two down from C_{odd} and two down

from C_{even}.) Note that fewer than 16 subsets may be required, and that it again does not matter where the "overflow" curves are chosen, except that we ensure that b is among them, and also that an "overflow" curve from the shifted "odd" chain is one of the a_i curves with an even index. Again, see Figure 6.6. We take at most 16 conjugates of f, $\{f_{17}, \ldots, f_{32}\}$, so that each acts on C_i' in the way that f acts on C', for $17 \leq i \leq 32$.

We now argue that all of the curves in H_g are in the same orbit under the subgroup generated by the 32 f_i. All of the a_i curves with odd index are in the same orbit, since all of their orbits under $\langle f_1, \ldots, f_8 \rangle$ are collapsed by the "shifted" conjugates. Similarly, all of the a_i curves with even index are in the same orbit under $\langle f_1, \ldots, f_{32} \rangle$. By construction, there exist an even a_i curve and an odd a_i curve that lie in the same orbit under $\langle f_1, \ldots, f_{32} \rangle$, and also b is in the same orbit as some a_i curve. Therefore the curves in H_g are all in the same orbit under the action of a subgroup generated by 32 conjugates of f.

With an additional 12 conjugates of f we have a Dehn twist about b in the subgroup, and so also the Dehn twists about all curves in H_g, and therefore the subgroup so generated is equal to $\mathrm{Mod}(S_g)$. Thus $\mathrm{Mod}(S_g)$ is generated by 44 conjugates of f, as required. $\qquad \square$

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