DYNAMIC ANALYSIS OF A CLASS OF MIXED LUMPED—
DISTRIBUTED PARAMETER SYSTEMS VIA NUMERICAL TECHNIQUES

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ABSTRACT

The modeling via transfer matrices of mixed lumped-distributed parameter systems with feedback control is discussed and novel methods of obtaining information on the controlled system's dynamic response via numerical methods is presented.

These methods utilize numerical searches in the complex plane for the roots of the transcendental characteristic equation to determine the closed loop system eigenvalues. The fast Fourier transform (FFT) algorithm is utilized but in the inverse fashion (from frequency domain to time domain). The information obtained is identical to that contained in a model in modal coordinates, and can be used to construct such a model. An interactive computer program to do the required calculations is described, as well as its application to a manipulator arm design problem.

INTRODUCTION

As technology attempts higher dynamic system performance various of the approximations used in their design become questionable. Faster and more accurate performance requires that higher order dynamic models be used and in the limit the true distributed nature of many components must be recognized. This realization may be painful since most popular analysis techniques require an approximation at some level, either as a more detailed lumping or as a modal truncation, procedures which rely largely on the experience of the designer, as well as extensive analytical work with uncertain resulting complexity.

By utilizing the power of the digital computer the requirements on the designer can be shifted to the recognition of types of ideal components: distributed beams, distributed shafts, essentially rigid masses, point compliances, feedback control loops, etc. The computer program can take this definition, process it, and give to the designer information he can readily assimilate: frequency responses, eigenvalues, modal shapes, and time responses. This has been done effectively in the case of lumped parameter systems, for example the ENPORT modeling program for bond graph representations.

The topic here is the analysis of mixed lumped-distributed parameter system models via numerical methods. Well understood distributed components when connected form very complex systems which are not ammenable to analytical techniques. The numerical techniques described below are capable of providing the necessary information to evaluate the dynamic response of such systems directly and provide the information needed to construct approximate models needed in various synthesis techniques. The techniques have been used in several practical design situations with success.

RATIONALE FOR NUMERICAL MODEL DEVELOPMENT

An important class of practical engineering systems are well modeled by a combination of lumped and distributed parameter elements. Examples of distributed parameter components include long drill stems, sucker rods, hydraulic transmission lines, flexible beams, and shafts. These may be interfaced with components such as control systems, actuators, gear boxes, bearings, etc., which are well described by lumped parameter models. Demands for higher system performance may require the distributed effects in these systems to be modeled more accurately [1].

Much of the theoretical control work dealing with partial differential equations does not admit the complexity of mixed lumped and distributed parameters or of interfacing at their boundaries distributed parameter components with different parameter values or describing equations. Such complexities would be useful in practice. For example, beam vibration control studies in the time domain have been published [2,3,4], for single uniform beams. A manipulator model based on a single distributed beam described in terms of its lower modes of vibration with control optimization in the time domain has been studied by Mirro [5]. When connecting two or more distributed beams the problem of boundary conditions between the systems of partial differential equations restricts time domain models severely. Maizza-Neto [6,1] has derived in modal coordinates a model for a two beam system from Lagrange's equations but much analytical manipulation is required.

Impedance methods, which rely on transformation of the describing equations to the Laplace domain are useful here. Previous papers [7] have indicated their usefulness in arriving at the controlled response of distributed systems. Their value results largely from the ability to solve generally and exactly in terms of transformed boundry conditions the simple partial differential equations (PDE's) of the distributed components. By equating the boundary conditions of interfacing components the impedance of the composite system may be obtained [8]. These methods are especially adaptable to the above cited distributed examples and others which have only one independent space variable. Since the characteristic impedance of a distributed element may in general be a complex transcendental function, a combination of

such elements is likely to be too complex for analytical treatment. Some application of impedance techniques to the control of a single distributed element is found in the literature [9]. Theory is not adequate for dealing directly with many control problems via impedance methods.

A body of theoretical work [10,11,12,13,14] has evolved over the past several years for dealing with the control of distributed parameter systems which seems appropriate to the demands of applications mentioned above. The application of the resulting techniques is dependent however, on obtaining an appropriate model. Such models frequently involve state space equations in modal coordinates, perhaps truncated to a finite number of modes. Required for such models are the eigenvalues and the eigenfunctions or equivalently the control and observation matrices, not readily obtainable for many systems.

The techniques discussed in this paper make use of the impedance formulation of dynamic systems to obtain numerically information on the controlled response of mixed distributed-lumped parameters systems and to obtain the required information for accurate state space models. The techniques have been implemented and applied to the control of flexible manipulator arms. This application is a prime example where new applications (e.g. the 60 foot long space shuttle remote manipulator) and high performance requirements demand consideration of distributed effects.

MATHEMATICAL BASIS FOR THE PROPOSED MODEL

The linear elements we consider a system to be modeled by (e.g. beams, masses, springs) are described most fundamentally by their differential equations (ordinary or partial). Using the general form of the steady state solution (particular solution) of these equations the transfer matrices can be derived. This is a process of replacing the arbitrary constants of the assumed solution with the system boundary conditions, which remain unspecified at this point. The resulting equations can be placed in a convenient matrix form. When arranged to describe the variables at one point in the element by multiplying the vector of variables at another point of the element, the matrix is termed a transfer matrix.

The approach described has been specifically used to analyze flexible manipulator arms and thus the examples appear in that context.

Transfer Matrix Approach

The transfer matrix approach provides a versatile method of describing the interaction between the linear components of a system when that interaction occurs at no more than two stations of the component. For beams these two stations correspond physically to the two ends of the beam. For pure rotary springs these stations correspond to the ends of the springs. For rigid body inertias these stations correspond to the points of attachment. When three or more components interact at a single station it is still possible to use the transfer matrix approach if this interaction is well defined. The transfer matrix method is well explained by Pestel and Leckie [8] and only the essentials will be discussed here.

The interaction between two components is described by means of a vector of state variables. The transfer matrix state variables should not be confused with the state variables of the state variable formulation of modern control theory. At the station where two

components are joined the value of their state variables is identical. A transfer matrix is used to describe the relation between the state variables at the two stations of each component. If the component is a static component (does not involve differentials with respect to time) such as an ideal spring, the transfer matrix is a function only of the component parameters. For dynamic components such as an ideal mass, the transfer matrix is also a function of the time derivatives of the state variables. For linear components (described by linear differential equations) it is convenient to deal with the Fourier transform of these equations which yields the steady state amplitude and phase (or complex amplitude) of the state variables under a pure sinusoidal excitation of frequency ω. The transfer functions for these components are functions of ω.

The state vector z that was used in an arm model consisted of four variables displayed in Figure 1 for a beam with flexure in the x-z plane. These four state variables are sufficient to describe most arm vibrations of interest. Arm flexure in two planes can be described if these motions are decoupled. In addition torsional compliance of beams can be accounted for when vibrations out of the plane of two beams is studied. In general the neutral axis of the undeformed arm must lie in a plane for these four state variables to sufficiently describe the arm. Two link designs which predominate arms built today (except for short wrist segments) automatically qualify. Additional arm links or beam like supports modeled as part of the arm may of course be arranged in nonplanar configurations. There is no conceptual difficulty in extending the state vector to the complete three dimensional case. The twelve state variables then needed (flexure in two planes, twisting, and compression) lengthen numerical computation disproportionately to the information obtained. State vectors with between 4 and 12 variables may give added information in some specific cases. Four state variables are all that will be considered here.

Given the transfer matrix B for a component and the state vector at one of its stations z_i, the state vector z_{i-1} at the other station is given by the matrix multiplication.

$$z_{i-1} = B\, z_i$$

It is thus a simple matter when components are connected serially (two components per station except for the end components) to find an overall transfer matrix by multiplication of the individual matrices to eliminate the intermediate state vectors. This is demonstrated for an arm model in Figure 2.

Numerical Operations with Transfer Matrices. The product of matrices such as appears in Figure 2 essentially is the implementation of the model of the arm which consists of beams, lumped masses, controlled joints and angles, joined end on end. The implementation of the model provides ways of getting useful information from that model. One possibility is to express analytically the elements of each component matrix, multiply the matrices and obtain a single matrix each element of which is a sum of products of the original matrix elements. While this in fact can usefully be done for simple cases it is not recommended for more complex cases unless the same configuration is to be used many times because of the complexity of useful elements, such as appears in Figure 3. The alternative is to evaluate each term before the matrix product is taken, then multiply the numerical values. This is a procedure which can be carried out in a straightforward

fashion by digital computer.

The advantage to numerical evaluation of this nature is that the complex functions need not be manipulated avoiding large amounts of designer time and potential for mistakes.

The disadvantages are for three types:

(1) More computer time is required to evaluate expressions which might be simplified using trigonometric and hyperbolic identities. The simplification is not apt to be great unless there are identical components or at least many identical parameters. Additional computer savings may be observed when some of the transfer matrices have many zero or unity elements. It may not be difficult for the designer to combine several simple elements into one matrix analytically if this combination is to be used frequently.

(2) Numerical errors may become significant. The larger number of calculations may cause roundoff errors to become significant especially in some cases (evaluation of determinants) which require taking the difference of two large, nearly equal numbers. This difficulty has been encountered only in rare and unusual cases and been solved by using extended precision in those cases. It is also possible to get a numerical overflow in the product of the transfer matrices.

(3) For simple cases the analytical expressions resulting from the matrix product may give the designer insight into the problem that the numerical results obscure.

Boundary Conditions and Forcing Functions. The transfer matrix, whether it describes a single component or a group them, expresses the relationship between the state variables at its two stations. In order for the transfer relation to be valid between state vectors, at most half of the state variables may be arbitrarily established. More precisely for the state vectors of physical systems only one of the complimentary variables of displacement or force and angle or moment, etc., may be arbitrarily specified. This specification may be as a simple boundary condition, as a forcing function, or as a linear combination of variables which may implicity include the other variables of the same state vector. This last case is in essence what one does when he appends another component by multiplying another transfer matrix. In this case one merely transfers the specification to another station in the extended system.

For simple boundary conditions one prescribes two non-associated variables of an arm. Figure 4 displays the physically possible combinations of zero state variables. The non-trivial solution of this case can result in solving for the natural frequency or complex eigenvalues (for damped systems). Additionally one can solve for the eigenfunctions of the system (the mode shape at the eigenvalue). The imposition of a forcing function yields the steady state forced response of the system, assuming the forcing function is a sinusoid of frequency ω. These techniques will be discussed in the following sections.

Natural Frequencies and Eigenvalues

If disturbed from the equilibrium position and then allowed to move freely after $t = 0$ (without disturbance or outside input) the state variables of a linear system will be described over time by a function of the form

(1) $\quad z_{i_k} = a_1 e^{j\omega_1 t} + a_2 e^{j\omega_2 t} + a_3 e^{j\omega_3 t} + \ldots a_j e^{j\omega_j}$

z_{i_k} = the ith element of z_k

For a lumped system there will be a finite number of these terms while a distributed system may theoretically have a countably infinite number. When the ω_i are complex they occur in conjugate pairs, and it is more common to refer to the eigenvalues $s_i = j\omega_i$.

The values of ω_i for an arm system model depend only on the parameters of, and the boundary conditions on, the system. They are independent of initial conditions, independent of which state variable is observed, and independent of the point in the system at which it is being observed. The values a_i depend on all of these quantities.

The transfer matrix technique allows one to simultaneously consider all the components and the boundary conditions on the system and thus determine the ω_i of interest. Multiplying transfer matrices eliminates the intermediate state variables at the interface between components and expresses state variables at one end of an arm directly in terms of the other end. Imposing two boundary conditions at each end restricts the values ω_i can assume for a non-trivial solution of the remaining state variables. These ω_i are the same ω_i appearing in Eq. 1. The restriction is developed in Fig. 5 for specific boundary conditions on a specific arm model. In general for a system represented by a matrix product U such that

$$z_0 = \begin{bmatrix} z_{10} \\ z_{20} \\ z_{30} \\ z_{40} \end{bmatrix} = U z_n = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \begin{bmatrix} z_{1n} \\ z_{2n} \\ z_{3n} \\ z_{4n} \end{bmatrix}.$$

With the boundary conditions

$z_{i0} = z_{j0} = 0$ at station 0

and

$z_{kn} \neq 0, z_{\ell n} \neq 0$ at station n

(implying the remaining two variables at station n are zero) requires for a nontrivial solution that the frequency determinant

(2) $\quad d = \begin{bmatrix} u_{ik} & u_{i\ell} \\ u_{jk} & u_{j\ell} \end{bmatrix}_{\omega = \omega_i} = (u_{ik} \ u_{j\ell} - u_{jk} \ u_{i\ell})\Big|_{\omega = \omega_i} = 0$

The elements of U and thus the terms of the frequency determinant are generally complex functions of ω. This being the case one must numerically search for values of ω where $d = 0$. When dealing with systems with no damping one can restrict the search to real values of ω or imaginary values of the eigenvalue s. In general however, $s = s_i$, where both the real and imaginary parts of d are zero. [1] In order to use conventional search routines one can search for minimum values of

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1. The assumption here is that the ω_i are distinct. For physical systems this is always true if one cares to look at the values with enough accuracy. The more general case $\omega_i = \omega_k$ does not restrict the results presented.

$|d| = \sqrt{(\text{Im } d)^2 + (\text{Re } d)^2}$, then check to see if $d = 0 + j0$ for the values of s returned. This topic will be discussed in more detail in a later section.

For single beams and variable impedance terminations Vande Vegte [3] found solutions to complex eigenvalues and displayed them as root loci. He reversed the procedure that was used in this paper. The real and imaginary parts of the eigenvalues were specified, as well as some of the termination impedances. The remainder of impedances were then solved for by requiring the frequency determinant to be zero. This procedure is acceptable for analysis but might be cumberson for design, since it assumes that an eigenvalue can exist at a given position, and that eigenvalue is the one desired (i.e. that it corresponds to the mode of interest and not some higher mode). This would be difficult to determine for complex arms.

Modal Shapes. Associated with each eigenvalue is an arm shape called the modal shape which describes the relative amplitude of all points of the arm when vibrating at that frequency. Looking at the problem from another perspective, there is an arm shape which when it constitutes the initial condition will result in arm vibration described by a single eigenvalue, to the exclusion of all other system eigenvalues.

The transfer matrix method can be employed to find the mode shape after the eigenvalue has been found. Consider the boundary conditions resulting in Equation 2. The homogeneous equations which precede the frequency determinant are

$$\begin{bmatrix} u_{ik} & u_{i\ell} \\ u_{jk} & u_{j\ell} \end{bmatrix} \begin{bmatrix} z_{in} \\ z_{jn} \end{bmatrix} = \underline{0}$$

If z_{in} is required to equal one, the solution for z_{jn} is

$$z_{jn} = \frac{-u_{ik}}{u_{i\ell}} = \frac{-u_{jk}}{u_{j\ell}}$$

Selecting the appropriate 2 x 2 submatrix from U will enable one to solve for the unspecified state variables at station 0.

In order to visualize the modal shape values of the state variables at intermediate points are helpful. For this one must refer to the transfer matrices of the separate components. For distributed beams, for example, the trigonometric and hyperbolic functions describing the shape within the component are far from obvious. For plotting these functions one can essentially divide the component into smaller components thus creating additional intermediate state vectors (for purposes of plotting only, not for finding eigenvalues). When plotted versus distance along the axis of the arm the state variables indicate the shape, angle, moment and shear amplitudes along the arm.

Steady State Frequency Response

Another way to specify the system boundary conditions is to impose sinusoidal forcing functions of frequency Ω and arbitrary but constant amplitude on form one to four of the state variables as for boundary conditions indicated in Figure 4. The procedure here is actually more straightforward than for finding eigenvalues.

Consider once again an arm model with describing transfer matrix U.

Now

$$\begin{bmatrix} -w \\ \psi \\ M \\ V \end{bmatrix}_0 = U \begin{bmatrix} -w \\ \psi \\ M \\ V \end{bmatrix}_n$$

Assume the rows and columns of U are rearranged to form $\tilde{U}$ such that the first two state variables of the rearranged state vectors $\tilde{z}_0$ and $\tilde{z}_0$ are forced with a sinusoid of arbitrary but constant complex amplitude. Then

$$\tilde{z}_0 = \tilde{U}\, \tilde{z}_n$$

Let us partition this matrix expression such that

$$\tilde{z}_0 = \begin{bmatrix} \tilde{f}_0 \\ \tilde{r}_0 \end{bmatrix} = \begin{bmatrix} \tilde{\tilde{U}}_{11} & \tilde{\tilde{U}}_{12} \\ \tilde{\tilde{U}}_{21} & \tilde{\tilde{U}}_{22} \end{bmatrix} \begin{bmatrix} \tilde{f}_n \\ \tilde{r}_n \end{bmatrix} = \tilde{\tilde{U}}\, \tilde{\tilde{z}}_n$$

f_0 and f_n contain the forced state variables and r_0 and r_n contain the remaining state variables, termed the response variables.

$$(3) \quad \begin{bmatrix} \underline{r}_0 \\ \underline{r}_n \end{bmatrix} = \left[\begin{array}{c|c} \tilde{\tilde{U}}_{22}\tilde{\tilde{U}}_{12}^{-1} & \tilde{\tilde{U}}_{21} - \tilde{\tilde{U}}_{22}\tilde{\tilde{U}}_{12}^{-1}\tilde{\tilde{U}}_{11} \\ \hline \tilde{\tilde{U}}_{12}^{-1} & -\tilde{\tilde{U}}_{12}^{-1}\tilde{\tilde{U}}_{11} \end{array} \right] \begin{bmatrix} \underline{f}_0 \\ \underline{f}_n \end{bmatrix}$$

Assuming $\tilde{\tilde{U}}_{12}^{-1}$ exists.

Equation (3) expresses the four response state variables in terms of the four forced state variables. The value of ω at which the transfer matrices will be evaluated will be the forcing frequency Ω. Complex amplitudes can be used to represent a phase shift between the various forced state variables r_0 and r_n will contain the amplitudes of responses. In practice it is seldom informative to force more than one state variable simultaneously.

Once again it is usually preferable to numerically evaluate the individual component transfer matrices prior to multiplication, enabling straightforward implementation on the digital computer.

Impulse Time Response. It is well known that the steady state frequency response is equivalent to the Fourier transform of the impulse time response for linear systems. Thus if one desires the impulse response of a given state variable to an impulse disturbance at another state variable one first uses Equation (3) to solve for the frequency response of that variable. One then uses the inverse Fourier transform given by

$$(4) \quad z_{hi}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z_{hi}(j\omega) e^{j\omega t}\, d\omega$$

h = response state variable index

i = response station index

to determine the time inpulse response.

In practice we desire to evaluate this response digitally using discrete samples of both the time and frequency responses. This approach to visualizing the impulse response of distributed systems was described by Kadymov et al [15,16] for certain

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practical systems. It is one of few methods for directly obtaining the response of mixed systems and can be efficiently obtained via a Fast Fourier Transform (FFT) algorithm.

Inverse Fourier Transform via Digital Methods

The calculations of the small motion response of the system can be carried out as described in the frequency domain. This information is related to the time domain response via an almost symmetrical pair of transformations--the Fourier transform and its inverse.

A periodic time function $f(t)$ may be represented in terms of its Fourier series

(5a)
$$f(t) = \frac{1}{T} \sum_{i=-\infty}^{\infty} F_i \exp (j\Omega t), i = \ldots -2,-1,0,1,2,\ldots$$

Where $j = \sqrt{-1}$

Ω = the fundamental frequency in rad/sec

(5b)
$$F_i = \int_{-T/2}^{T/.2} f(t) \exp (-ji\Omega t) \, dt$$

Where $T = 2\pi/\Omega$ = Period of one cycle in sec.

The F_i are the Fourier coefficients.

It is usually possible to truncate the series at a finite number of harmonics, which depends on the accuracy desired.

An aperiodic function may be described as a signal with infinite period, or infinitesimal fundamental frequency. In fact in the limit

(6a)
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \exp (j\omega t) \, d\omega$$

(6b)
$$F(j\omega) = \int_{-\infty}^{\infty} f(t) \exp (-j\omega t) \, dt$$

$F(j\omega)$ is the Fourier integral and is equivalent to the Fourier transform of the time signal. The transformation is defined when the integral in (6b) converges.

Starting in either the frequency or time domain, one can at least formally transform to the other domain via Equations (6). It is also possible to approximate the transform by replacing the integration with summation and replacing the differential $d\omega$ with the increment $\Delta\omega$ resulting in

(7)
$$f(t) \simeq \hat{f}(t) = \frac{\Delta\omega}{2\pi} \sum_{i=-n}^{n} F(ji\Delta\omega) \exp (ji \, \Delta\omega t)$$

By comparing this with Equation (5) one realizes that this approximation must be periodic and of period $2\pi/\Delta\omega$, indicating that distortion of some sort has occurred.

Certain steps must be taken to assure that the distortions resulting from the discrete approximations do not render the approximate response $\hat{f}(t)$ useless. For $\hat{f}(t)$ to result from a stable physical system the response for $t<0$ (before any input) must be zero. Recalling the periodic nature of $\hat{f}(t)$ one can correctly conclude that the results of Equation (7) must be at least approximately zero at the end of each period. This requires sufficient time for the impulse response to settle out, placing a lower constraint on the period $2\pi/\Delta\omega$ (or upper constraint on $\Delta\omega$).

It is not readily apparent in the frequency domain what an adequately small value of $\Delta\omega$ is. If one knows the location of the system eigenvalues he can estimate the settling time of each exponential. For complex conjugate root pairs for example

$$t_s \simeq \frac{4\zeta}{\omega_n}$$

Where ζ = damping ratio of the root pair

ω_n = distance from origin of the pair

t_s = time to point where response remains within 2% of steady state

One may make $\Delta\omega$ smaller by either increasing the number of samples or by limiting the upper value of the frequency Ω_f that will be considered. This removes the components of the time response with frequency greater that Ω_f. By looking at the frequency response for the system one can estimate Ω_f. Resonant peaks that are not down at least 20 db. from the magnitude at $\omega = 0$ should be included if feasible. Thus for given settling times and given frequency Ω_f one arrives at a minimum sampling interval in the frequency domain analogous to the Nyquist sampling interval in the time domain. Just as small time intervals between samples are required to evaluate the frequency response for large frequency, small frequency intervals are required to evaluate the time response for long times.

NUMERICAL IMPLEMENTATION

The above described modeling method has been implemented on a digital computer. This section will describe briefly the program and how the user may interact with it. Also described are numerical difficulties that have arisen in use and how they have been handled. A listing of the computer programs is found in reference [17].

Host Computer

Although it was originally intended to include all the programs in one package for interactive use by the designer, this goal was abandoned initially due to the modest core storage of the machine used. The Interdata Model 70, with 40K 16 bit words of core storage at the M.I.T. Joint Civil-Mechanical Engineering Computer Facility was initially used. This is a mini computer handling interactive graphics and console input which is very useful in a design situation. An Interdata Model 80 with 32 K 16 bit words of storage was used for some of the more extensive eigenvalue searches due to a lower price structure and a faster CPU. The program as it presently exists is divided into several compatible packages. Table 1. classifies the programs in seven more or less related categories. At this writing the program is being implemented at the Georgia Institute of Technology on a CDC Cyber 70 with a DEC PDP 11 minicomputer and graphics terminal handling the interactive graphics. Categories I, II, and III utilize descriptions of the arm and subroutines in VI and VII. Categories IV and V use output from II and III to obtain additional information.

Nature of User Input

This section will briefly describe the nature of the user input to the system to indicate the designer effort required.

TABLE 1
PROGRAM CATEGORY OUTLINE FOR
NUMERICAL IMPLEMENTATION

I. Natural frequency calculation (no damping)

   A. Single Precision
   B. Double Precision

II. Eigenvalue search (two dimensional search, systems with damping)

   A. Card and console input only
   B. Interactive graphics implementation

III. Frequency response calculation

   A. Logarithmic frequency scale

      1. Bode plot
      2. Polar (nyquist) plot
      3. Modified polar plot
   B. Linear frequency scale (equal increment in frequency for FFT input)

IV. Mode shape calculation and display

V. Fast Fourier transform

VI. Component transfer matrix calculation

   A. Distributed Beam

      1. Bernoulli-Euler model
      2. Timoshenko model (includes shear and rotary inertia)

   B. Rigid Body

      1. General
      2. Uniform cross section·

   C. Angle in the arm shape

      1. In the plane of vibration
      2. Perpendicular to the plane of vibration

   D. Controlled Rotary Joint

      1. Transfer function control
      2. With flexible shaft and gear reduction

   E. Parallel elements (combines certain elements in parallel by clamping them at each end).

   F. Discontinuity in one state variable (with its associated variable equal to zero e.g. pinned connection between beams or pinned connection to ground)

VII. Search Routines

   A. Pattern search--2 dimensional
   B. IBM SSP root finding algorithm

   Arm Description. The description of the arm to be modeled is in terms of the arm elements or components selected from Category VI of Table 1. Each element requires one and sometimes more data cards giving its parameters and are arranged in the order of occurrence on the arm. In certain cases there are restrictions as to what combinations of elements can be used. For instance angles in and out of the plane of vibration would result in a nonplanar arm which cannot be handled by four state variables.

   Calculation Description. Presently the description of the desired calculation for natural frequencies and frequency response requires one card for describing arm boundary conditions, calculation type, number of increments, and extreme frequencies considered.

A number of selections of output alternatives and extended calculations are available by data switch and console input at run time.

   For eigenvalue calculations additional starting points for the two dimensional search can be read in from card or console in one implementation, or input graphically via joystick and crosshairs in another implementation.

   Mode shape calculations require the input of the eigenvalues for the system for the mode for which the shapes are to be calculated, boundary conditions, and the number of points at which the shape is to be calculated for each beam element.

   To calculate the inverse Fourier transform to obtain the time impulse response, the frequency response must be calculated with equal spacing between calculations (Table 1, IIIB). The specification of the total number of points to be used and the frequency range they cover is input by card.

Nature of Program Output

   The program output is of the nature described in Table 1. In cases where there are arrays of data (such as the values of frequency response) this data can either be plotted or plotted and printed to allow more precise comparisons. In this case there is also a selection of which endpoint state variables are to be plotted. The three types of frequency plots are the Bode diagram, Polar plot of magnitude and phase and the modified polar plot which can be used in stability analysis for certain nonlinear arm elements. The graphics assisted eigenvalue search program also yields a plot of the roots as they are found.

Numerical Difficulties

   The arms modeled to date have resulted in few numerical difficulties. The difficulties encountered and ways of dealing with them are described below.

   False Roots in Eigenvalue Search. For a damped arm system the frequency determinant is a complex number. Evaluated at the system eigenvalue both real and imaginary parts should be zero. The procedure is to minimize the complex modulus via a pattern search program developed by Prof. D. E. Whitney. The global minimums must be zero, but the search routine may return false eigenvalues corresponding to local minimums. To resolve this ambiguity one can look at the real and imaginary parts of the determinant. If an actual system root has been returned, and it is a single distinct root, the real and imaginary parts must both change signs in the neighborhood of the root. In normal operation where the change in roots with design changes is being observed the user confidence in root positions is high, and only when unexpected root locations are returned is the doubt sufficient to check the sign change.

   Convergence to the "Wrong" Eigenvalue. Since the eigenvalue search bases its actions on the shape of the determinant modulus over the complex plane, it will converge to different roots depending on where the search is begun. Thus one can repeatedly "find" the same root and not find a desired root. Under these circumstances graphical display and input becomes very helpful, allowing the user to quickly modify the starting point of the search based on the displayed results of the previous search. Search routines specifically designed to take advantage of the particular problem of complex root finding might do much better and cut down on the user interaction required.

   A more critical numerical problem arises when it is practically impossible to make the routine converge to a root that exists. This has been observed for extreme values of arm servo control parameters. It

corresponds to shapes of the determinant that change very rapidly in the region of the eigenvalues and is aggravated by two roots which are very close. Fortunately when this happens it also seems that the root changes very slowly with parameter changes so that roots determined with different parameter values can be used.

Numerical Overflow. In only one case has the problem of numerical overflow been encountered. The problem was solved for that case by using extended computer word size, which may not be practical in all cases for all computers.

USE OF THE METHOD IN A DESIGN CONTEXT

The model described above was first used in a real manipulator design context. The example model output given here was part of the evaluation of the arm structure for a manipulator under design at the Charles Stark Draper Laboratory.

The arm was to be a high performance arm slightly more than one meter in length and capable of carrying a payload of ten kilograms. An estimate was made of the weight of the hydraulic actuators and control valves which would supply the torque required. One of the design goals was to build an arm capable of responding to input frequencies of up to 10 hz. The question that the modeling exercise was to answer was, "What structure will provide the rigidity necessary to build a 10 hz arm?" The transient response desired was one with minimal overshoot. For the second order system which approximated the control of each joint minimal overshoot implied a damping ratio near one. Thus the first approach was to establish feedback gains which would yield dominant eigenvalues with a damping ratio of one and a magnitude of 10 hz for a rigid inertia equivalent to that of the structure, actuators and payload.

The model of the arm is indicated schematically in Figure 6. Nine elements are used in this model of the arm; four distributed beams, one controlled joint, and four rigid masses. A five element model was also used for some studies. That model was obtained by increasing the density of the outer link to account for the mass of the 4 kg and 8 kg masses.

Figure 7a shows the log magnitude and phase versus frequency of the frequency response of the five element model for a structure outer radius of 0.04 m. The joint feedback gains would achieve 10 hz. eigenvalues with damping ratios of 1.0 for an equivalent rigid inertia. The real and imaginary parts of this frequency response are displayed in Figure 7b, arranged for inverse Fourier transformation via the Fast Fourier Transform Algorithm. This (inverse) transformation yields the time impulse response displayed in Figure 7c.

As would be expected from the peak in the magnitude plot Figure 7a and verified by the impulse response, one does not have minimum overshoot as desired, or expected from a rigid analysis.

A better understanding of the nature of these occurrances may be obtained from Figure 8. Figure 8a gives a close up of the imaginary plane near the origin, and displays the complex roots responsible for the oscillatory behavior.

These root locations are influenced by the flexibility of the structure, and show a maximum damping with variations in velocity feedback which occurs at values of the velocity gain which are lower, not higher, than the gains first used.

Figure 8 also displays the locus of the dominant closed loop eigenvalues as the outer radius is varied for the nine element model, with feedback gains that would produce second order roots of magnitude 10 hz.

and damping ratio 1.0 in an equivalent rigid intertia. As can be seen, the oscillatory conjugate root pair that dominates for small radii moves to the left, eventually leaving a real root to dominate. That real root approaches the -10 hz. expected for a rigid system as the arm radius becomes larger. Roots of a higher, lightly damped mode displayed in Figure 8b move away from the origin and remain lightly damped.

An acceptable outer radius for the arm might in fact be 0.05 m., and the impulse response for that case is shown in Figure 9. It displays the third order nature one might expect from the root locus plots of figure 8.

Additional details on this application and other aspects of the techniques used are found in [18].

SUMMARY AND CONCLUSIONS

The modeling via transfer matrices of mixed lumped-distributed parameter systems with feedback control has been discussed and novel methods of obtaining information on the controlled system's dynamic response via numerical methods have been presented. These methods utilize numerical searches in the complex plane for the roots of the transcendental characteristic equation to determine the closed loop system eigenvalues. The fast Fourier transform (FFT) algorithm is utilized but in the inverse fashion (from frequency domain to time domain) to that usually found in the literature. An interactive computer program to do the required calculations has been described, as well as its application to a particular design problem.

It is possible to conclude from the work summarized above that the numerically based modeling procedure is a valuable tool for working with systems where distributed effects may be important. It provides one of the few methods for obtaining results on systems described by linear partial differential equations interacting through their boundary conditions at the two extremes of a single spatial variable. Furthermore the information necessary to provide a state space formulation in modal coordinates is obtained should that formulation be perferred. The alternative in many cases is a laborious manual development of equations which must be repeated for relatively minor adjustments in the system description. The techniques presented do require considerable computer time and are limited to linear systems. The increasing availability and economy of computations is decreasing the importance of the former of these limitations.

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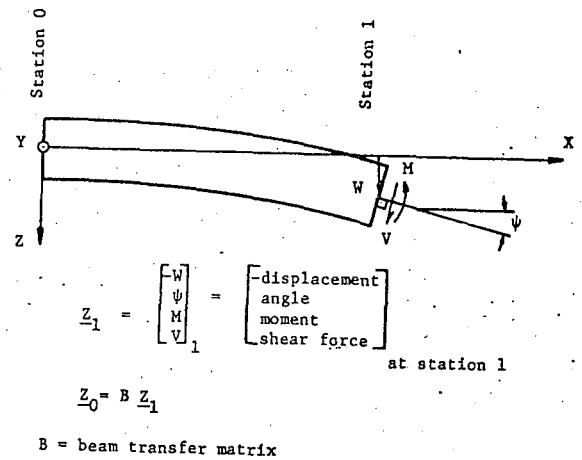   School of Mechanical Enginnering, April 1974.

$$\underline{Z}_1 = \begin{bmatrix} W \\ \psi \\ M \\ V \end{bmatrix}_1 = \begin{bmatrix} \text{displacement} \\ \text{angle} \\ \text{moment} \\ \text{shear force} \end{bmatrix}$$
at station 1

$$\underline{Z}_0 = B \, \underline{Z}_1$$

B = beam transfer matrix

Figure 1. State Vector for Arm Models

$$\begin{bmatrix} W \\ \psi \\ M \\ V \end{bmatrix}_0 = R_1 \, B_2 \, R_3 \, A_4 \, C_5 \, B_6 \, R_7 \begin{bmatrix} W \\ \psi \\ M \\ V \end{bmatrix}_7$$
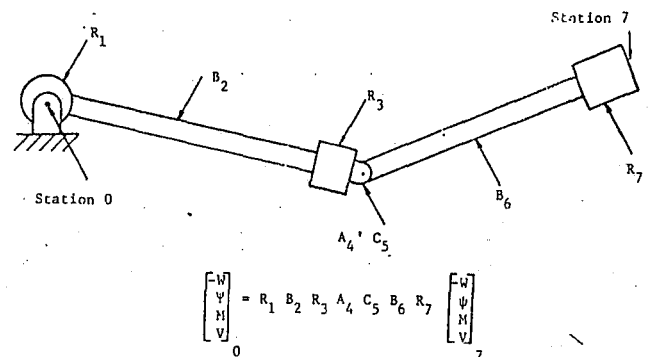
Figure 2. Transfer Matrix Representation of an Arm

$$B = \begin{bmatrix}
c_0 - \sigma c_2 & l[c_1 - (\sigma + r)c_3] & a c_2 & \frac{al}{\beta^4}[-\sigma c_1 + (\beta^4 + \sigma^2)c_3] \\
\frac{\beta^4}{l} c_3 & c_0 - r c_2 & \frac{a}{l}(c_1 - r c_3) & a c_2 \\
\frac{\beta^4}{a} c_2 & \frac{1}{a}[-r c_1 + (\beta^4 + r^2)c_3] & c_0 - r c_2 & l[c_1 - (\sigma + r)c_3] \\
\frac{\beta^4}{al}(c_1 - \sigma c_3) & \frac{\beta^4}{a} c_2 & \frac{\beta^4}{l} c_3 & c_0 - \sigma c_2
\end{bmatrix}$$

where

$$a = \frac{l^2}{EJ} \qquad c_0 = \Lambda(\lambda_2^2 \cosh \lambda_1 + \lambda_1^2 \cos \lambda_2)$$

$$\beta^4 = \frac{\mu \omega^2}{EJ} l^4 \qquad c_1 = \Lambda(\frac{\lambda_2^2}{\lambda_1} \sinh \lambda_1 + \frac{\lambda_1^2}{\lambda_2} \sin \lambda_2)$$

$$\sigma = \frac{\mu \omega^2}{GA_s} l^2 \qquad c_2 = \Lambda(\cosh \lambda_1 - \cos \lambda_2)$$

$$r = \frac{l^2}{EJ}(\mu i^2 \omega^2) \qquad c_3 = \Lambda(\frac{1}{\lambda_1} \sinh \lambda_1 - \frac{1}{\lambda_2} \sin \lambda_2)$$

$$\lambda_{1,2} = +\sqrt{\sqrt{\beta^4 + \frac{1}{4}(\sigma - r)^2} \pm \frac{1}{2}(\sigma + r)} \qquad \Lambda = \frac{1}{\lambda_1^2 + \lambda_2^2}$$

GA_s = Shear stiffness
EJ = Bending stiffness
μ = Mass per unit length
i = Radius of gyration of cross sectional area perpendicular
      to the beam's neutral axis
l = length of the beam
ω = vibration circular frequency

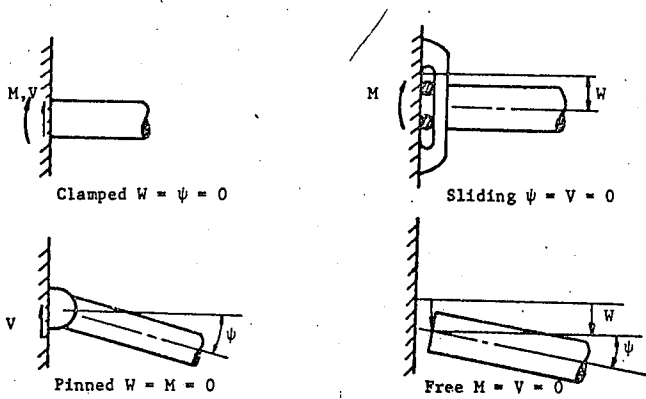Figure 3. Distributed Beam Transfer Matrix
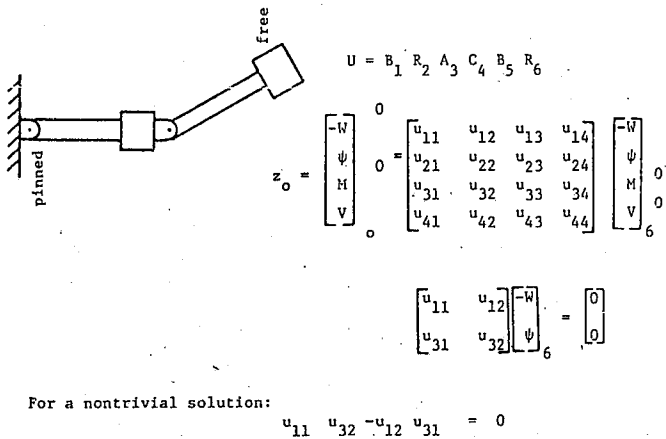(Timoshenko model)

Figure 4

Clamped $W = \psi = 0$

Sliding $\psi = V = 0$

Pinned $W = M = 0$

Free $M = V = 0$

M,V

M

W

V

W

ψ

ψ

Figure 4   Physically Possible Boundary Conditions

Figure 5

free

pinned

$U = B_1 \, R_2 \, A_3 \, C_4 \, B_5 \, R_6$

$$z_o = \begin{bmatrix} -W \\ \psi \\ M \\ V \end{bmatrix}_o \quad 0 = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \begin{bmatrix} -W \\ \psi \\ M \\ V \end{bmatrix}_6 \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{31} & u_{32} \end{bmatrix} \begin{bmatrix} -W \\ \psi \end{bmatrix}_6 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a nontrivial solution:

$$u_{11} \, u_{32} - u_{12} \, u_{31} = 0$$

Figure 5   Frequency Determinant

Figure 6

joint, 10 kg

4 kg

8 kg

10 kg

.45m.

0.1 m.

.45 m.

0.2 m.

Inner radius = .035m   Inner radius = .038m

Material: Aluminum, Density = 2705 kg/m^3, E=7.1x10^{10} nt/m^2

G=2.65x10^{10} nt/m^2

Figure 6   Example: Nine Element Arm Model

Figure 7a

Log Magnitude (m.)

Phase Angle (Degrees)
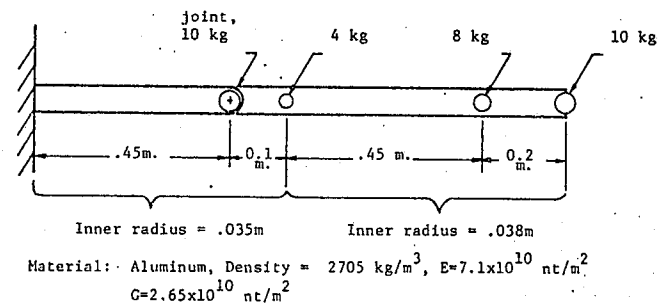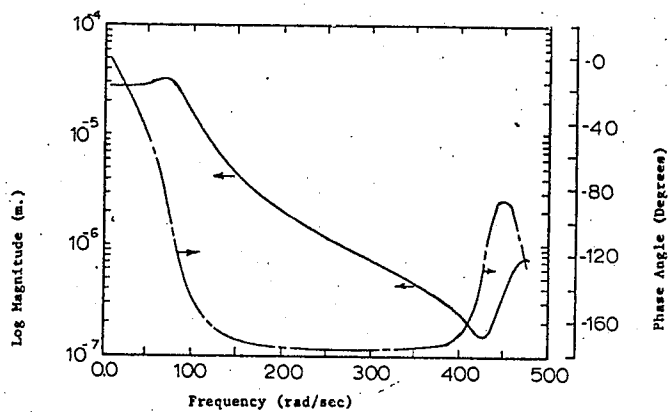
Frequency (rad/sec)

Figure 7a   Bode Diagram, Force Loading at Endpoint Joint feedback gains for equivalent inertia rigid system eigenvalues of 10hz. and 1.0 damping ratio.
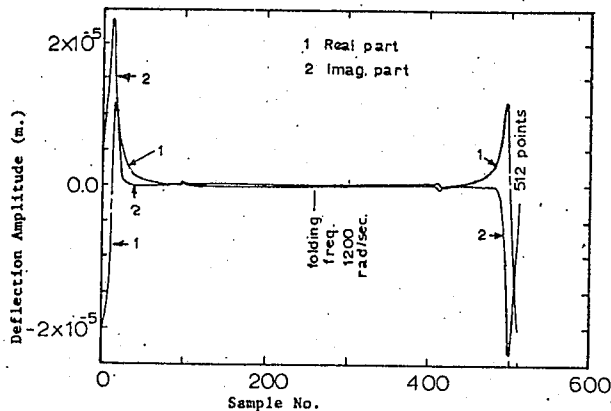
Figure 7b

1 Real part
2 Imag. part

Deflection Amplitude (m.)

folding freq. 1200 rad/sec.

512 points

Sample No.

Figure 7b   Samples of Complex Frequency Response. Arranged for FFT.

Figure 7c

Real

Imag.

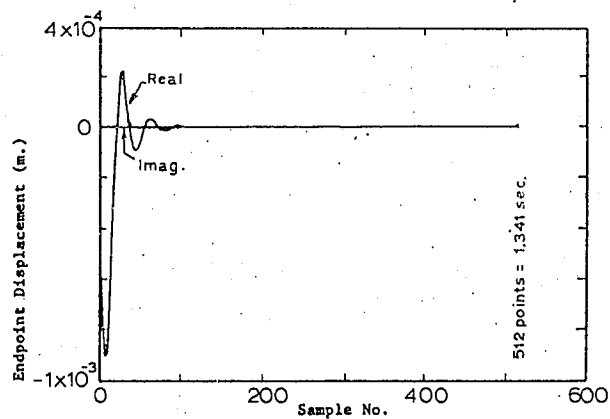Endpoint Displacement (m.)

512 points = 1.341 sec.

Sample No.

Figure 7c   Impulse Response via Inverse Fourier Transform of b.

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