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Exact relations for two-component Fermi gases with contact interactions

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Outline

- The system
- Hamiltonian
- Energy functional
- The contact
- Other exact relations
- Other approaches & results
- Experimental tests
- Other systems

The system

N fermionic atoms populating two spin states (up and down, or blue dots and red dots).

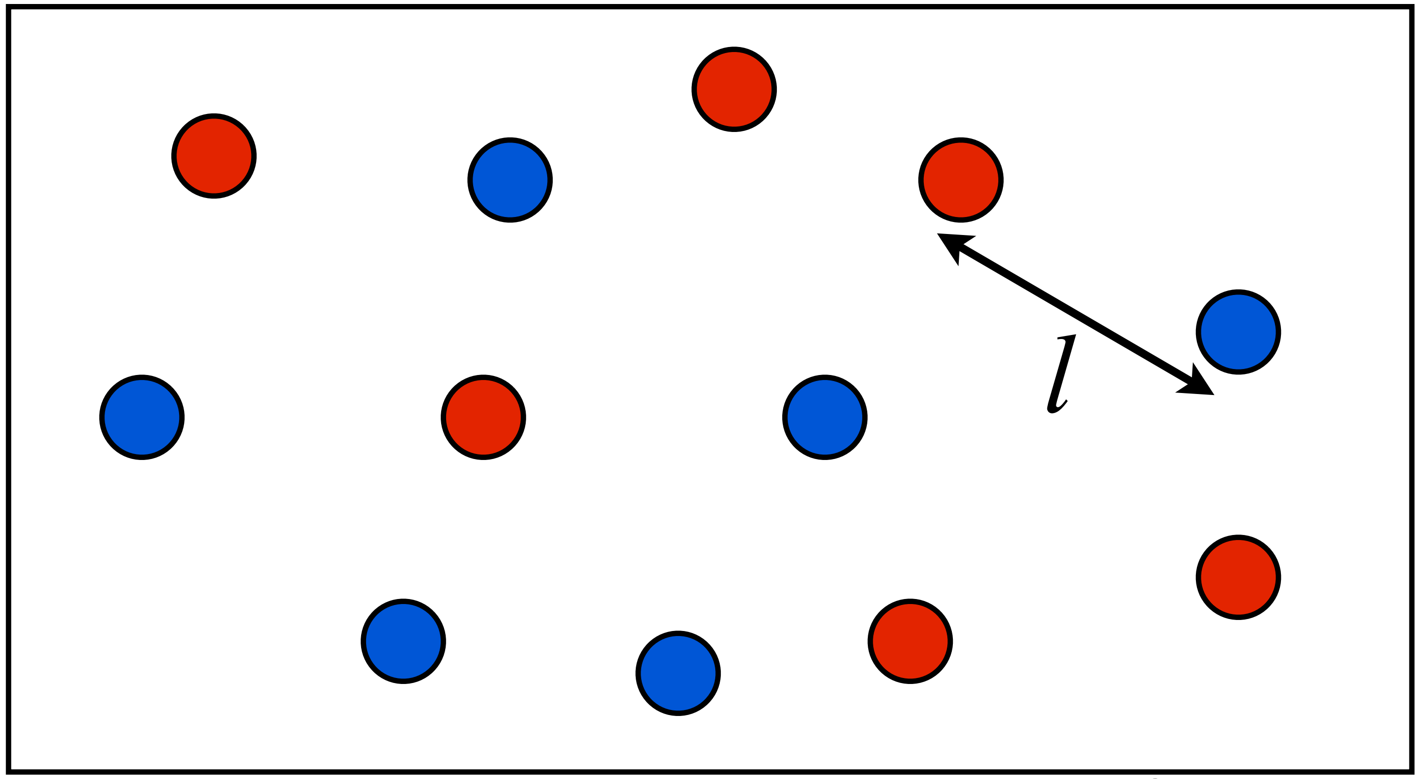
(N is arbitrary)

$r_{vdw} \sim$ a few nanometers

Ultradilute: $l \gg r_{vdw}$

Ultracold: $\lambda_{dB} \gg r_{vdw}$

Strongly interacting: $|a| \gg r_{vdw}$



Can take the idealized model: *contact interaction* $(r_{vdw} = 0)$

The Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \frac{\hbar^2 k^2}{2m} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{4\pi\hbar^2 a}{m\Omega} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} \Lambda(\mathbf{k}') c^\dagger_{\mathbf{q}/2+\mathbf{k}\uparrow} c^\dagger_{\mathbf{q}/2-\mathbf{k}\downarrow} c_{\mathbf{q}/2-\mathbf{k}'\downarrow} c_{\mathbf{q}/2+\mathbf{k}'\uparrow}$$
$$+ \int d^3r \sum_\sigma V(\mathbf{r}) \psi^\dagger_\sigma(\mathbf{r}) \psi_\sigma(\mathbf{r}).$$

(also good model for dilute neutron matter)

a : scattering length

Ω : large volume

$V(\mathbf{r})$: external potential

$\Lambda(\mathbf{k}) = 1$ for all finite k, BUT $\int d^3k \frac{\Lambda(\mathbf{k})}{k^2} \equiv 0$,

a spinoff of the Huang-Yang pseudopotential,

$$U(r) = \frac{4\pi\hbar^2 a}{m} \delta(\mathbf{r}) \frac{\partial}{\partial r} r$$

The pseudopotential

$$H_{\rm int} = \frac{4\pi\hbar^2 a}{m}\int d^3r_1 d^3r_2\, \psi^\dagger_\uparrow(\mathbf{r}_1)\psi^\dagger_\downarrow(\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_2)\frac{\partial}{\partial r_{12}}\left[r_{12}\psi_\downarrow(\mathbf{r}_2)\psi_\uparrow(\mathbf{r}_1)\right]$$

mimicing Eq. (38) of *K. Huang and C. N. Yang, Phys. Rev.* ***105****, 767 (1957)*

$\delta(\mathbf{r})\frac{\partial}{\partial r}r$ behaves almost like the delta function, except $\delta(\mathbf{r})\frac{\partial}{\partial r}r\frac{1}{r}=0$

So I introduced a generalized function $\lambda(\mathbf{r})$ satisfying

$$\lambda(\mathbf{r})=0,\quad \mathbf{r}\neq 0. \qquad \int d^3r\lambda(\mathbf{r})=1, \qquad \int d^3r\lambda(\mathbf{r})\frac{1}{r}=0$$

Its Fourier transform $\Lambda(\mathbf{k})\equiv\int d^3r\,\lambda(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}$ satisfies

$$\Lambda(\mathbf{k})=1,\quad k<\infty;\quad \text{but} \int\frac{d^3k}{(2\pi)^3}\Lambda(\mathbf{k})\frac{4\pi}{k^2}=0$$

so $H_{\rm int} = \frac{4\pi\hbar^2 a}{m\Omega}\sum_{\mathbf{qkk'}} c^\dagger_{\mathbf{q}/2+\mathbf{k}\uparrow}c^\dagger_{\mathbf{q}/2-\mathbf{k}\downarrow}c_{\mathbf{q}/2-\mathbf{k}'\downarrow}c_{\mathbf{q}/2+\mathbf{k}'\uparrow}\Lambda(\mathbf{k}')$

Tan, Ann. of Phys. ***323*** *(2008), 2952*

$$H = \sum_{\mathbf{k}\sigma} \frac{\hbar^2 k^2}{2m} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \frac{4\pi\hbar^2 a}{m\Omega} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} \Lambda(\mathbf{k}') c^\dagger_{\mathbf{q}/2+\mathbf{k}\uparrow} c^\dagger_{\mathbf{q}/2-\mathbf{k}\downarrow} c_{\mathbf{q}/2-\mathbf{k}'\downarrow} c_{\mathbf{q}/2+\mathbf{k}'\uparrow}$$
$$+ \int d^3r \sum_\sigma V(\mathbf{r}) \psi^\dagger_\sigma(\mathbf{r}) \psi_\sigma(\mathbf{r})$$

implies the *Bethe-Peierls boundary condition*:

$$\{N\text{-body wave function}\} \propto \frac{1}{r} - \frac{1}{a} + O(r), \quad r \to 0$$

r : distance between a spin-up fermion & a spin-down fermion

This $1/r$ singularity implies that $n_{\mathbf{k}\sigma} \to \frac{C}{k^4}, \quad k \to \infty$

$\sigma = \uparrow, \downarrow$

Energy functional

Taking the expectation value of H,
and using the Bethe-Peierls boundary condition, we get

$$E = \frac{\hbar^2 \Omega C}{4\pi a m} + \int \frac{\Omega d^3 k}{(2\pi)^3} \sum_\sigma \frac{\hbar^2 k^2}{2m} \left(n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \int d^3 r \sum_\sigma V(\mathbf{r}) \rho_\sigma(\mathbf{r})$$

$$n_{\mathbf{k}\sigma} \equiv \langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \rangle \qquad C \equiv \lim_{k\to\infty} k^4 n_{\mathbf{k}\sigma}$$

Energy is a linear functional of momentum distribution $n_{\mathbf{k}\sigma}$
(and density distribution in the presence of external potential)

Tan, Ann. of Phys. ***323*** *(2008), 2952*

Cancellation of Ultraviolet Divergences

$$E = \frac{\hbar^2 \Omega C}{4\pi a m} + \int \frac{\Omega d^3k}{(2\pi)^3} \sum_\sigma \frac{\hbar^2 k^2}{2m} \left(n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \int d^3r \sum_\sigma V(\mathbf{r}) \rho_\sigma(\mathbf{r})$$

kinetic energy $\int \frac{\Omega d^3k}{(2\pi)^3} \sum_\sigma \frac{\hbar^2 k^2}{2m} n_{\mathbf{k}\sigma} \to +\infty$

interaction energy $\to -\infty$

but
{kinetic energy} + {interaction energy} = finite

The contact

$$E = \frac{\hbar^2 \Omega C}{4\pi a m} + \int \frac{\Omega d^3k}{(2\pi)^3} \sum_\sigma \frac{\hbar^2 k^2}{2m} \left(n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \int d^3 r \sum_\sigma V(\mathbf{r}) \rho_\sigma(\mathbf{r})$$

$n_{\mathbf{k}\sigma} \to \frac{C}{k^4}, \quad k \to \infty$ due to the contact interaction

The contact: $\mathcal{I} = \Omega C = \int C(\mathbf{r}) d^3 r$

$C(\mathbf{r})$: local contact density

Physical meaning of the contact

density-density correlation function

$$\langle \rho_\uparrow(\mathbf{r}_c - \mathbf{r}/2)\rho_\downarrow(\mathbf{r}_c + \mathbf{r}/2)\rangle = \frac{C(\mathbf{r}_c)}{16\pi^2}\frac{1}{r^2} + O(1/r), \quad r \to 0$$

Contact $\mathcal{I} = \int C(\mathbf{r}_c) d^3 r_c$

So the contact is, roughly speaking,
the chance of finding two atoms close to each other

A simplified energy functional

ϵ_ν : single-particle energy levels in the external potential

n_ν : population of the νth level

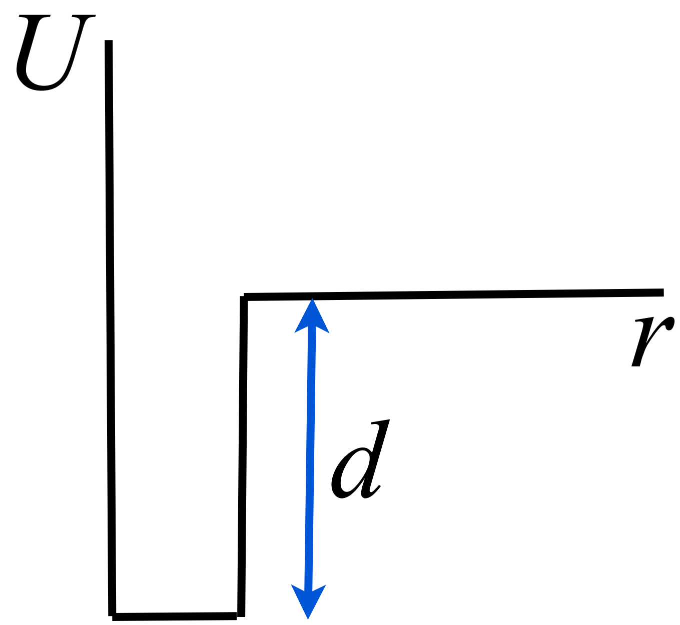
$$E = E[n_\nu] = \frac{\hbar^2 \mathcal{I}}{4\pi a m} + \lim_{\epsilon_{\max}\to\infty}\left(\sum_{\epsilon_\nu < \epsilon_{\max}} \epsilon_\nu n_\nu - \frac{\hbar \mathcal{I}}{\pi^2}\sqrt{\frac{\epsilon_{\max}}{2m}}\right)$$

valid in free space, harmonic traps, optical lattices, etc.

Tan, Phys. Rev. Lett. ***107****, 145302 (2011)*

Unlike the **Density Functional** in electronic structure, the above functional is ***explicitly known***, and is ***valid for all quantum states*** (not just the *N*-body ground state).

What if we tune the interaction in real time?



Can fine-tune depth d to produce large a.

If a is large, $\Delta(1/a) \propto \Delta d$

{change of energy}={change of interaction}×{probability inside the range of interaction}

$$\frac{dE}{dt} = \frac{\hbar^2 \mathcal{I}(t)}{4\pi m} \frac{d(-1/a)}{dt}$$

dynamic sweep relation

Tan, Ann. of Phys. ***323*** *(2008), 2971*

Pressure relation (in free space)

$$\text{Pressure} = \frac{2}{3}\{\text{energy density}\} + \frac{\hbar^2 C}{12\pi a m}$$

Tan, Ann. of Phys. ***323*** *(2008), 2987*

Generalized virial theorm (harmonic trap)

$$E = 2E_V - \frac{\hbar^2 \mathcal{I}}{8\pi a m}$$

which generalizes the virial theorem at unitarity as found by John Thomas

Tan, Ann. of Phys. ***323*** *(2008), 2987*

Other theoretical approaches and results

- Operator product expansion (**Eric Braaten & D. Kang & L. Platter et al**); two-body decay rate (ST, E. Braaten & H.-W. Hammer et al)
- *N*-body wave functions (**F. Werner & Y. Castin & L. Tarruell, R. Combescot & F. Alzetto & X. Leyronas**)
- Thermodynamics, RF-Spectroscopy and closed-channel molecules (**S. Zhang & A.J. Leggett**)
- Theoretical analyses of the contact (H. Hu & X.J. Liu et al)
- Large-*N* expansion (T. Enss)
- few-body numerical tests (D. Blume et al)
- Quantum Monte-Carlo (J. E. Drut et al, G. Bertaina & S. Giorgini, S. Gandolfi & K.E. Schmidt & J. Carlson, K. Van Houcke & F. Werner & E. Kozik & N. Prokof'ev & B. Svistunov)
- Critical behaviors of contact (Y.Y. Chen & Y.Z. Jiang & X.W. Guan & Q. Zhou)

Other relations involving the contact

- Clock shift (M. Punk & W. Zwerger, G. Baym & C.J. Pethick & Z. Yu & M.W. Zwierlein, C. Langmack & M. Barth & W. Zwerger & E. Braaten)
- Tail of radio-frequency spectrum at large detunings (W. Schneider & V. B. Shenoy & M. Randeria, M. Punk & P.T. Dumitrescu & W. Zwerger, E. Braaten & D. Kang & L. Platter,
  P. Pieri & A. Perali & G.C. Strinati)
- Static structure factor at large q
  (Hui Hu & Xiaji Liu & P. Drummond et al)
- Dynamic structure factor at large q or large ω (D. T. Son & E. G. Thompson, E. Taylor & M. Randeria)
- Frequency dependent viscosity (E. Taylor & M. Randeria, T. Enss & R. Haussmann & W. Zwerger, W.D. Goldberger & Z.U. Khandker)
- Extension to arbitrary partial waves (M. He & S. Zhang & H.M. Chan & Q. Zhou)
- ...

Experimental verifications & measurements

- D. S. Jin group at JILA:
  tail of momentum distribution, adiabatic relation, generalized virial theorem, tail of RF spectrum; measured contact as a function of temperature at unitarity
- C. J. Vale group at Swinburne U. of Technology:
  Bragg spectroscopy, static structure factor
- R. Hulet group at Rice U. (data analyzed by F. Werner & L. Tarruell & Y. Castin):
  photoassociation, # of closed channel molecules
- C. Salomon group at ENS: contact near unitarity
- ...

Review article

Eric Braaten, *Universal Relations for Fermions with Large Scattering Length*, arXiv:1008.2922

Other systems

- 2D gases
- 1D gases (M. Barth & W. Zwerger, X. Guan)
- Bose gases: 2-body contact and **3-body contact** (E. Braaten et al, Y. Castin & F. Werner, X.J. Liu & B. Mulkerin & L. He & H. Hu)
- Bose-Fermi mixture (Z.Q. Yu & S. Zhang & H. Zhai)
- 1-component Fermi gas with p-wave resonant interaction => **p-wave contact**
(Z. Yu & J. Thywissen & S. Zhang, C. Luciuk & S. Trotzky & S. Smale & Z. Yu & S. Zhang & J.H. Thywissen, S.M. Yoshida & M. Ueda)

Summary

- Dilute ultracold gases can often be approximated by *contact-interaction models*
  => contact
  & lots of exact relations involving the contact