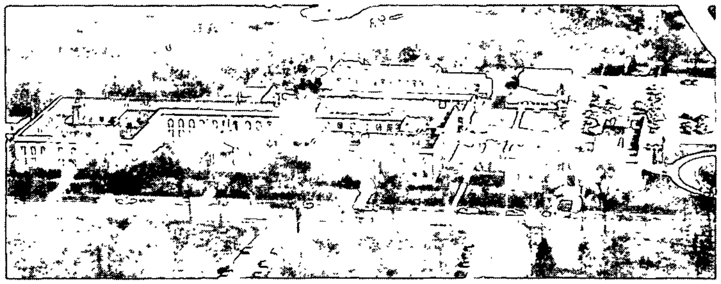

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ENUMERATING PARTITIONS OF N

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INTRODUCTION

The problem of distributing items into a variable number of categories (called partitioning) is sometimes encountered. It is usually necessary to evaluate the result of distributing the items in the various ways possible and select the "best" of the distributions.

Computer applications, in which the problem appears, must be able to produce all of the partitions. An algorithm for enumerating all partitions of N (the number of items) into K parts (the number of categories) is shown. The algorithm is in the form of a subroutine program in the FORTRAN IV language.

Enumerating Partitions of N

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A partition of N is defined as a collection of integers without regard to order which sum to N. The partition may be restricted to K integers, where K has the range $1 < K \leqslant N$.

The problem is, given N and K, to enumerate all partitions P = [A(1), A(2),...,A(K)] which satisfy the requirements

$$A(1) + A(2) + \ldots + A(K) = N \tag{1}$$

$$A(1) \leqslant A(2) \leqslant \ldots \leqslant A(K) \tag{2}$$

$$A(I) > 0 \tag{3}$$

$$I < K \leqslant N \tag{4}$$

For example, given N = 5 and K = 3 the partitions are $P_1 = (1,1,3)$ and $P_2 = (1,2,2)$. When requirement (2) is removed, the collection of integers is called a composition. [The integers must be in the range $1 \leqslant A(I) \leqslant L(I)$ for them to satisfy the requirements for partitions. The L(I) are given by the solutions to

$$L(I) = (N - I + 1)/(K - I + 1), \tag{5}$$

where I = 1,2,...,K-1. The integer A(K) is obtained by subtraction.]

The algorithm gives the partitions one at a time to the calling program. The subroutine is first called with the value of IND = -1. This establishes the L(I), places the first partition in the array IV, and returns with IND = 0. Repeated calls of the subroutine produce other partitions in the array IV. When the subroutine returns a value of IND = 1, all partitions of N into K integers have been enumerated.

References:

Riordan, J. An Introduction to Combinatorial Analysis, New York, J. Wiley and Sons, 1958.

Hall, M. Combinatorial Theory, Waltham, Mass., Blaisdell Publishing Co., 1967.

```
      SUBROUTINE PARTN ( IV, N, K, IND )
C  IV = THE PARTITION OF 'N'
C  N  = THE INTEGER TO BE PARTITIONED
C  K  = THE NUMBER OF INTEGERS 'N' IS TO BE PARTITIONED INTO
C  IND= AN INDICATOR FOR STARTING, CONTINUING, AND STOPPING
C        THE GENERATION OF PARTITIONS.
C     = -1 TO START THE PROCESS
C     =  0 WHEN ANOTHER PARTITION HAS BEEN GENERATED
C     = +1 WHEN ALL PARTITIONS HAVE BEEN GENERATED
      DIMENSION IV(1),LV(50)
      L = K - 1
C.......    IF IND = -1, INITIALIZE THE LIMITING VALUES
      IF ( IND )  1,3,3
    1 DO 2  I = 1,L
      IV(I) = 1
    2 LV(I) = (N - I + 1)/(K - I + 1)
      IV(K) = N - L
      IND = 0
      RETURN
    3 IND = 0
C.......    INCREMENT THE INTEGER  'IV(K-1)'
    4 IF ( L )  12,12,5
    5 IV(L) = IV(L) + 1
      IF ( IV(L) - LV(L) )  6,6,9
C.......    TEST FOR SUMMATION TO 'N',REQUIREMENT NO. 1
    6 ITOT = 0
      J = K - 1
      DO 7 I = 1,J
      ITOT = ITOT + IV(I)
      IF ( ITOT - N )  7,9,9
    7 CONTINUE
      IV(K) = N - ITOT
C.......    TEST FOR NON-DECREASING SEQUENCE, REQUIREMENT NO. 2
      IF ( IV(K) - IV(K-1) )  9,8,8
    8 RETURN
C.......    THE INTEGERS ARE NOT IN SEQUENCE, RESET
    9 IF ( L - 1 )  12,12,10
   10 J = K - 1
      DO 11 I = L,J
   11 IV(I) = IV(L-1) + 1
      L = L - 1
      GO TO 4
C.......    SET IND = 1, ENUMERATION OF PARTITIONS COMPLETE
   12 IND = 1
      RETURN
      END
```