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A THEORY OF GENERALIZED SPLINES WITH APPLICATIONS TO NONLINEAR BOUNDARY VALUE PROBLEMS

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CHAPTER I

INTRODUCTION

The study and application of spline functions have undergone rapid development in the last few years. There have lately appeared in the literature applications of splines to nonlinear boundary value problems, eigenvalue and eigenfunction problems, initial value problems, optimal quadrature formulae, approximation theory and stochastic processes. As another indication of the interest in spline theory, three books [4], [16], [22] have been published in the last four years devoted entirely to the study of spline functions and their applications.

While spline functions were used previously in a few isolated instances, they were first explicitly identified and developed by Schoenberg beginning in 1946 [20]. Their evolution since that time has resulted not only in a sequence of generalizations of what might be accepted as being a spline function, but also in a simplification of their development, and a strengthening of their known properties.

Splines as Solutions to a System of Linear Equations

If $a = x_0 < x_1 < \ldots < x_N = b$ is a partition of the interval $[a,b]$, one early characterization of a spline s of odd order $2n - 1$ was that s must be equal to a polynomial of degree $2n - 1$ in each interval (x_i, x_{i+1}), $0 \le i \le N-1$, and be in the overall continuity class $C^{2n-2}[a,b]$. Then the problem of finding a spline of the above type which is required to take on specified values at the node points $\{x_i\}_{i=1}^{N-1}$ along with assigned values of its first n-1 derivatives at the end points a and b, has been shown to have a unique solution for any such set of assigned values.

In the early and mid-sixties a series of generalizations of these piecewise polynomial splines appeared, based on the characterization of a polynomial of odd degree 2n-1 as being a function in the null space of the differential operator D^{2n}, where $D \equiv \frac{d}{dx}$. One such generalization, due to Ahlberg, Nilson and Walsh [2], defines the class of generalized spline functions as follows:

Let $L = \sum_{j=0}^{n} a_j(x)D^j$, where the $a_j(x) \epsilon C^j[a,b]$ and $a_n(x) \geq \omega > 0$, and let $L^* f = \sum_{j=0}^{n} (-1)^j D^j(a_j f)$ denote the formal adjoint of L. Let $a = x_0 < x_1 < \ldots < x_N = b$ be any partition of [a,b], and define the generalized L-spline interpolate of any function $f \epsilon C^{n-1}[a,b]$ to be the unique solution s of the problem

i. $L^* L[s(x)] = 0$ if $x \epsilon (x_i, x_{i+1})$, $0 \leq i \leq N-1$,

ii. $D^j s(x_i) = D^j f(x_i)$, $0 \leq j \leq p-1$, $1 \leq i \leq N-1$,

iii. $s \epsilon C^{2n-p-1}[a,b]$, and

iv. $D^j[s(x)] = D^j[f(x)]$, $0 \leq j \leq n-1$, for $x = a$ and $x = b$,

where $1 \leq p \leq n$.

This problem can be shown to have a unique solution s for all $f \epsilon W^{n,2}[a,b]$, where $W^{n,2}[a,b]$ is the Sobolev space of functions f such that $D^{n-1}f$ is absolutely continuous on [a,b], and $D^n f \epsilon L^2[a,b]$. If $L \equiv D^n$, and p = 1, this problem reduces to the earlier one.

In 1967 Schultz and Varga [24] generalized the above problem even further by associating with each interior node point x_i, a number z_i, $1 \le z_i \le n$, $1 \le i \le N-1$, and substituted for condition ii above, the condition

$$\text{ii}'. \quad D^j s(x_i) = D^j f(x_i), \quad 0 \le j \le z_i - 1, \ 1 \le i \le N-1.$$

This also necessitated a generalization of the continuity condition iii to

$$\text{iii}'. \quad D^j s(x_i^-) = D^j s(x_i^+), \quad 0 \le j \le 2n - z_i - 1, \ 1 \le i \le N-1.$$

If $z_i = p$ for all i, $1 \le i \le N-1$, this new problem reduces to the second problem. The interpolation used in ii' is referred to as being of Hermite type. Denoting $\{x_i\}_{i=0}^{N}$ and $\{z_i\}_{i=1}^{N-1}$ by π and $\bar{z}$, Schultz and Varga define $Sp(L,\pi,\bar{z})$ to be the space of all functions $s \epsilon W^{2n,2}[x_i, x_{i+1}]$, $0 \le i \le N-1$, such that s satisfies conditions i and iii'. Any $s \epsilon Sp(L,\pi,\bar{z})$ is also referred to as being an L-spline. For any $f \epsilon W^{n,2}[a,b]$ a function $s \epsilon Sp(L,\pi,\bar{z})$ is said to be an $Sp(L,\pi,\bar{z})$-interpolate of f if s satisfies conditions ii' and iv. Then they have shown that for every $f \epsilon W^{n,2}[a,b]$ there is an s which is a unique $Sp(L,\pi,\bar{z})$-interpolate of f. This generalizes the similar result obtained by

Ahlberg, Nilson and Walsh [2] referred to above.

In 1969 Lucas [17] generalized this result still further by considering an arbitrary formally self-adjoint differential operator of order 2n of the form

$$L[u(x)] = \sum_{i=0}^{n} (-1)^i D^i [a_i(x) D^i u(x)], \tag{1.0}$$

where $a_i \epsilon W^{i,2}[a,b]$, $0 \le i \le n$, and $a_n(x) \ge \omega > 0$ for all $x \epsilon [a,b]$, and replacing condition i by

i'. $L[s(x)] = 0$ if $x \epsilon (x_i, x_{i+1})$, $0 \le i \le N-1$.

Since the class of differential operators of the above form can be shown [17] to include all differential operators of the form $L_1^* L_1$, and since it is also considerably larger, the problem of finding a unique $Sp(L,\pi,\bar{z})$-interpolate of any $f \epsilon W^{n,2}[a,b]$ in the sense of conditions i', ii', iii' and iv is another generalization of the classical notion of a spline. If the partition points in π are sufficiently close together, this problem has been shown to have a unique solution [17].

We remark that in all of the above formalizations of generalized splines, if $\{u_i\}_{i=1}^{2n}$ is a basis for the null space of $L_1^* L_1$ (or L in (1.0)) then condition i (i') implies that

$$s(x) = \sum_{i=1}^{2n} a_{ij} u_i(x) \quad \text{if } x \epsilon (x_j, x_{j+1}),\ 0 \le j \le N-1,$$

which determines s as dependent on 2nN constants. For a given $f \in W^{n,2}[a,b]$, the conditions ii and iii or ii' and iii' place 2n restrictions on s at each interior node point for a total of 2n(N-1) linear equations in $\{a_{ij}\}$. The condition iv places another 2n linear constraints on $\{a_{ij}\}$, and thus the problem of finding an $Sp(L,\pi,\bar{z})$-interpolate of any $f \in W^{n,2}[a,b]$ can be reduced to solving 2nN linear equations. This fact helps one to understand the theory behind the above explicit formalizations of generalized splines, and is used in the above papers to help establish the unique existence of $Sp(L,\pi,\bar{z})$-interpolates under various conditions.

We also remark that the above papers include other more complicated formalizations which involve either replacing condition iv with other sets of 2n constraints at the end points a and b, including constraints of a periodic nature, or generalizing condition ii' still further. If $L_1 = D^n$, Schultz and Varga [24] replace the N-1 vector $\bar{z}$ with an N-1 by n matrix $E = (e_{ij})$, $1 \le i \le N-1$, $0 \le j \le n-1$, consisting solely of 0's and 1's, with at least one nonzero entry in each row. They then replace conditions ii' and iii' by

ii". $D^j s(x_i) = D^j f(x_i)$ for all (i,j) such that $e_{ij} = 1$,

iii". $D^{2n-j-1} s(x_i^+) = D^{2n-j-1} f(x_i^-)$ for all (i,j) such that $e_{ij} = 0$,

and refer to $Sp(D^n,\pi,E)$-interpolates as g-splines, and the interpolation appearing in ii" as being of Hermite-Birkhoff type. This extended work by Ahlberg and Nilson [1] and Schoenberg [21].

It now begins to be clear that the above explicit characterizations of splines require a new development every time the second condition (or the fourth condition) is changed by considering a more general class of interpolates. Thus it would be highly desirable to have a development of generalized splines which was not so highly dependent on particular forms of interpolation.

Splines as Solutions to a Minimization Problem

One result that is valid for many of the developments of generalized spline theory mentioned above is that if Λ is the family of linear functionals on the Sobolev space $W^{n,2}[a,b]$ generated by any of the different forms of conditions ii and iv (in the sense that conditions ii and iv are equivalent to $\lambda(s) = \lambda(f)$ for all $\lambda \epsilon \Lambda$), then whenever there is a unique $Sp(L,\pi,\bar{z})$ or $Sp(D^n,E)$-interpolate s for every $f \epsilon W^{n,2}[a,b]$, s is characterized by

$$(L_1 s, L_1 s) = \min\{(L_1 g, L_1 g) : g \epsilon W^{n,2}[a,b], \text{ and } \lambda(g) = \lambda(f) \text{ for all } \lambda \epsilon \Lambda\}. \quad (1.1)$$

That is, s is the unique element in $W^{n,2}[a,b]$ which minimizes the quadratic form $(L_1 g, L_1 g)$ over all Λ-interpolates of f. Equation (1.1) has been taken as a starting point for an abstract approach to interpolation by splines in a Hilbert space setting as early as 1966 by de Boor and Lynch [8] and Atteia [6].

A recent example of such a formulation has been developed by Anselone and Laurent [5]. Suppose X and Y are Hilbert spaces, T is a bounded linear operator mapping X *onto* Y, and N(T), the null space of T, is finite dimensional. Let $K = \text{span}\{k_i\}_{i=1}^k$, where the k_i are the representors of linearly independent, continuous linear functionals on X, and let $K^\perp$ denote the orthogonal complement of K in X. If $N(T)\cap K^\perp = \{0\}$, for any $x\in X$, the minimization problem

$$\text{Min}\{\|Tx\|_Y : x\in K_r^\perp\}, \quad \text{where}$$
$$K_r^\perp = \{x\in X : (x,k_i) = r_i, \quad 1\le i\le k\}, \tag{1.2}$$

has a unique solution s, characterized by

$$Ts\in(TK^\perp)^\perp.$$

In each of the formulations above, hypotheses sufficient to guarantee uniqueness of spline interpolates have been assumed, and then the existence of such interpolates has been deduced. The first researcher to separate these two properties, and indeed to show that existence can hold without uniqueness, was Golomb [11] in 1968. Golomb considered the following situation:

Let X and Y be two Hilbert spaces, let U be a flat (that is, a translate of a subspace of X) in X, not necessarily of finite codimension, and U^0 the subspace parallel to U, say $U = U^0 + z$. Let R be a bounded linear transformation of X into Y. Determine $s \in U$ by the condition

$$\|Rs\| = \inf\{\|Rx\| : x \in U\} \tag{1.3}$$

and refer to such an element s as a generalized spline, or more specifically, say that s is an R-spline interpolate of the flat U.

Golomb then proceeded to show that if

$$RU \text{ is closed} \tag{1.4}$$

or the equivalent condition that RU^0 is closed, then there exists a solution to (1.3), and that each solution satisfies the orthogonality condition

$$(Rs,Rx) = 0, \quad \text{for all } x \in U^0. \tag{1.5}$$

If in addition to (1.4), it is required that

$$N(R) \cap U^0 = \{0\} \tag{1.6}$$

where N(R) is the null space of R, then he showed s will not only exist, but s will be the unique solution to (1.3).

To develop some easily verifiable situations in which hypothesis (1.4) holds, Golomb [11], with the help of Jerome, formulated the following lemma:

<u>Lemma 1.1</u>. Suppose X and Y are Hilbert spaces, R is a bounded linear transformation with null space N(R) that maps X *onto* Y, and U^0 is a subspace of X. If $N(R) + U^0$ is closed, then RU^0 is closed.

If the additional hypothesis

$$U^0 \text{ is closed} \tag{1.7}$$

is assumed (and this will always be the case when U^0 is the null space of a family of continuous linear functionals Λ on X), and N(R) is finite dimensional, as it usually is in applications, then $N(R) + U^0$ will be closed, and hence by Lemma 1.1, hypothesis (1.4) is satisfied and the problem (1.3) will have a solution for all translates of U^0, U. Moreover that solution will be unique if and only if hypothesis (1.6) holds.

Jerome and Schumaker [15] applied Golomb's results to the Sobolev spaces $X = W^{n,2}[a,b]$ and $Y = L^2[a,b]$, where $R[f] = L[f] = \sum_{i=0}^{n} a_i D^i f$, with $a_i \epsilon C^i[a,b]$ and $a_n(x) \neq 0$ on $[a,b]$, $0 \leq i \leq n$. If $\Lambda = \{\lambda_i\}_{i=1}^N$ is a collection of continuous linear functionals which are linearly independent on $W^{n,2}[a,b]$ and $\bar{r} = (r_1, r_2, \ldots, r_N) \epsilon E^N$, then they considered the following minimization problem:

$$\begin{aligned} \|Ls\|_{L^2} &= \min_{f \epsilon U(\bar{r})} \|Lf\|_{L^2} \\ U(\bar{r}) &= \{f \epsilon W^{n,2}[a,b] : \lambda_i(f) = r_i,\ 1 \leq i \leq N\}. \end{aligned} \tag{1.8}$$

That is, they minimized $\|Lf\|_{L^2}$ over the set of all Λ-interpolates of f. They showed through the use of Lemma 1.1 that there always exists a solution s to the problem (1.8), and s is unique if and only if $N(R) \cap U(\bar{0}) = \{0\}$. They called such solutions Lg-splines, since splines associated with a general differential operator have been called L-splines, and splines associated with Hermite-Birkhoff interpolation have been called g-splines. Jerome and Schumaker then deduced as a theorom that if all of the members of Λ are of the Hermite-Birkhoff type ii", then s will satisfy both condition i, and condition iii". Also if Λ is of the type ii' then s will satisfy the continuity condition iii'. This represents a great simplification over the explicit approaches to generalized splines discussed in the previous section, such as the L-splines of Schultz and Varga. However, neither this formulation nor the results of Golomb include the generalized L-splines developed by Lucas [17].

Purpose and Preliminary Definitions

It is the purpose of this research to formulate, develop and apply a theory of generalized splines which 1) includes as special cases all of the results previously mentioned in this introduction, 2) has a foundation in Hilbert space theory, 3) includes many of the advanced results of generalized spline theory in this more general setting, 4) gives new results when this general setting is specialized to earlier studies, and 5) improves upon earlier results in some cases.

Basic to this approach is taking the orthogonality condition (1.5) as the defining characteristic of generalized splines instead of the minimization condition (1.3). This leads to the following formalization of the notion of an M-spline, where M is a continuous (not necessarily symmetric) bilinear functional on a real Hilbert space:

<u>Definition 1.1</u>. Let X be a real Hilbert space, and Λ a family of continuous linear functionals over X. Associate with Λ the linear space $N(\Lambda) = \{n \epsilon X : \lambda(n) = 0 \text{ for all } \lambda \epsilon \Lambda\}$, which we shall refer to as the null space of Λ. Let $M(x,y)$ be a continuous bilinear functional on $X \times X$ such that $M(n,n) \geq 0$ for all $n \epsilon N(\Lambda)$. A function $s \epsilon X$ is said to be an *M-spline* if $M(s,n) = 0$ for all $n \epsilon N(\Lambda)$. The class of all M-splines for a fixed Λ is denoted by *Sp(M,Λ)*.

<u>Definition 1.2</u>. Let X, Λ and M be as above, and let $x \epsilon X$. Then any $s \epsilon X$ is said to be a *Λ-interpolate of x* if $s - x \epsilon N(\Lambda)$. If s is also in $Sp(M,\Lambda)$, then s is said to be an *Sp(M,Λ)-interpolate of x*.

Note that s is a Λ-interpolate of x if and only if $\lambda(s) = \lambda(x)$ for all $\lambda \epsilon \Lambda$. Also observe that $Sp(M,\Lambda)$ is a closed linear space.

In the next chapter we shall give conditions which insure the existence of an $Sp(M,\Lambda)$-interpolate of any element in X. If for a given X, Λ and M, with $M(n,n) \geq 0$ for all $n \epsilon N(\Lambda)$, as in Definition 1.1, we define N_1 by $N_1 = \{n_1 \epsilon N(\Lambda) : M(n_1,n_1) = 0\}$, then it may easily be seen that N_1 is a closed linear subspace of X:

N_1 is clearly homogeneous. If $x,y \epsilon N_1$, let $\alpha = M(x,y) + M(y,x)$. Then $M(x-\alpha y, x-\alpha y) = -\alpha[M(x,y) + M(y,x)] = -\alpha^2 \geq 0$ since $x - \alpha y \epsilon N(\Lambda)$. Therefore $\alpha = M(x,y) + M(y,x) = 0$ for all $x,y \epsilon N_1$. Thus $M(x+y, x+y) = 0$ for any $x,y \epsilon N_1$, and N_1 is additive. By the continuity of M, N_1 is closed.

Definition 1.3. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$ such that

$$M(n,n) \geq 0 \text{ for all } n \in N(\Lambda). \tag{1.9}$$

If there is an $m>0$ such that

$$M(n,n) \geq m\|n\|^2 \text{ for all } n \in N(\Lambda), \tag{1.10}$$

then we shall say that *the system* $\{X,\Lambda,M,N(\Lambda)\}$ *is well-posed.* Denote by N_1 the closed linear subspace of $N(\Lambda)$,

$$N_1 = \{n_1 \in N(\Lambda) : M(n_1,n_1) = 0\}. \tag{1.11}$$

If

$$M(x,n_1) = 0 \text{ for all } x \in X,\ n_1 \in N_1, \tag{1.12}$$

if there exists a closed linear subspace of $N(\Lambda)$, N_2, such that

$$N(\Lambda) = N_1 \oplus N_2, \tag{1.13}$$

and an $m>0$ such that

$$M(n_2,n_2) \geq m\|n_2\|^2 \text{ for all } n_2 \in N_2, \tag{1.14}$$

then we shall say that *the system* $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ *is* N_1*-posed.*

Note that if $N_1 = \{0\}$, $\{X,\Lambda,M,N(\Lambda)\}$ is well-posed if and only if $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is $\{0\}$-posed.

Example 1.1. Let X and Y be real Hilbert spaces, Λ a family of continuous linear functionals on X, and T a continuous linear transformation of X onto Y, such that the dimension of the null space of T,N(T), is finite. Define the continuous bilinear functional M by

$$M(x_1,x_2) = (Tx_1,Tx_2)_Y \text{ for all } x_1,\ x_2\epsilon X.$$

Then $M(x,x)\geq 0$ for all $x\epsilon X$, and $N_1 = N(\Lambda)\cap N(T)$. If $n\epsilon N_1$, $M(x,n) = (Tx,Tn)_Y = 0$ since $n\epsilon N(T)$, so (1.12) is satisfied. Let $N_2 = (N_1)^{\perp}_{N(\Lambda)}$, the orthogonal complement of N_1 in $N(\Lambda)$. Since N_2 is closed and N(T) is finite dimensional, $N_2 + N(T)$ is closed [12, Prob. 8], and by Lemma 1.1, $T(N_2)$ is closed. Thus T maps N_2 one to one and onto the closed subspace $T(N_2)$, and therefore by the open mapping theorem, T restricted to N_2 has a continuous inverse. Thus there is an $m>0$ such that $\|Tn_2\|\geq m\|n_2\|$ for all $n_2\epsilon N_2$. But then $M(n_2,n_2) = \|Tn_2\|^2_Y\geq m^2\|n_2\|^2$ for all $n_2\epsilon N_2$ giving (1.14), and thus the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed. If $N_1 = N(\Lambda)\cap N(T) = \{0\}$ then $N(\Lambda) = N_2$, and the system $\{X,\Lambda,M,N(\Lambda)\}$ is well-posed.

CHAPTER II

M-SPLINES IN HILBERT SPACE

Existence and Uniqueness of M-Splines

The following theorem gives conditions which insure the existence and uniqueness of M-splines. Note that there is no symmetry requirement placed on M.

Theorem 2.1. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. If the system $\{X,\Lambda,M,N(\Lambda)\}$ is well-posed then for any $y \in X$ there is a unique $Sp(M,\Lambda)$-interpolate s of y, which depends continuously on y.

Proof. Since M is continuous, there is a $K>0$ such that

$$K\|x\|\|n\| \geq M(x,n) \text{ for all } x \in X,\ n \in N(\Lambda). \tag{2.1}$$

Thus for any fixed $x \in X$, $M(x,\cdot)$ is a bounded linear functional on $N(\Lambda)$. Therefore there is a $z \in N(\Lambda)$ such that $M(x,n) = (z,n)$ for all $n \in N(\Lambda)$. Let $Tx=z$. Then T is a continuous linear mapping of X into $N(\Lambda)$, such that

$$K\|n\|^2 \geq M(n,n) = (Tn,n) \geq m\|n\|^2 \text{ for all } n \in N(\Lambda). \tag{2.2}$$

Denote by T_N the restriction of T to $N(\Lambda)$. Clearly T_N is 1-1. It will now be shown that the range of T_N, $R(T_N)$, is actually equal to $N(\Lambda)$.

Suppose $\{n_i\}_{i=0}^{\infty}$ is a sequence of elements $n_i \epsilon R(T_N)$ such that $n_i \to n \epsilon N(\Lambda)$. Then there exist $x_i \epsilon N(\Lambda)$ such that $Tx_i = n_i$. From (2.2),

$$\|Tn\| \geq m\|n\| \quad \text{for all } n \epsilon N(\Lambda). \tag{2.3}$$

Since $\{n_i\}$ is a Cauchy sequence, so is $\{Tx_i\}$. But by (2.3), $\{x_i\}$ must then be Cauchy also. Let $x_i \to x \epsilon N(\Lambda)$. Then since T is continuous, $Tx_i \to Tx$, so $Tx = n$. This establishes that $R(T_N)$ is closed. Let n_1 be in the orthogonal complement of $R(T_N)$ in $N(\Lambda)$. Then

$$0 = (Tn_1, n_1) \geq m\|n_1\|^2.$$

Therefore $\|n_1\| = 0$, so $n_1 = 0$, and $R(T_N) = N(\Lambda)$. Since T_N is a 1-1 mapping of $N(\Lambda)$ onto $N(\Lambda)$, by the open mapping theorem T_N has a continuous inverse T_N^{-1}.

Now let $y \epsilon X$. Suppose $s \epsilon Sp(M,\Lambda)$ and $s = y + \bar{n}$ with $\bar{n} \epsilon N(\Lambda)$. Then $M(y+\bar{n}, n) = 0$ for all $n \epsilon N(\Lambda)$. Therefore $T(y+\bar{n}) = 0$, implying that $\bar{n} = -T_N^{-1}(Ty)$.

Thus

$$s = (I - T_N^{-1}T)y \epsilon y + N(\Lambda). \tag{2.4}$$

So if there is an $Sp(M,\Lambda)$-interpolate of y, s, then s is unique, and is given as a continuous function of y by (2.4). But for any $n \epsilon N(\Lambda)$, $M(s,n) = (Ts,n) = (Ty-Ty,n) = 0$, so (2.4) actually gives an

Sp(M,Λ)-interpolate of y, establishing the theorem. □

Corollary 2.1. Under the conditions of Theorem 2.1,

$$X = N(\Lambda) \oplus Sp(M,\Lambda).$$

Proof. If $y \in X$, $y = (y-s) + s$, where s is the unique Sp(M,Λ)-interpolate of y.

Corollary 2.2. Under the conditions of Theorem 2.1, if span (Λ) has a basis of dimension n, then dim(Sp(M,Λ)) = n.

Corollary 2.3. (Anselone and Laurent [5].) In Example 1.1, if $N(T) \cap N(\Lambda) = \{0\}$, then for every $x \in X$, there is a unique Sp(M,Λ)-interpolate s which depends continuously on x.

The following example generalizes the self-adjoint L-splines developed by Lucas [17].

Example 2.1. Let X be the Sobolev space $W^{n,2}[a,b]$ of all functions f in $C^{n-1}[a,b]$ whose n-1st derivative is absolutely continuous and $D^n f \in L^2[a,b]$, with inner product

$$(f,g)_n = \sum_{i=0}^{n} \int_a^b [D^i f(t)][D^i g(t)]dt.$$

Define a continuous bilinear functional M on X × X by

$$M(f,g) = \sum_{i=0}^{n} \int_a^b a_i(t)[D^i f(t)][D^i g(t)]dt,$$

where $a_i \epsilon W^{n,2}[a,b]$, $0 \le i \le n$, and $a_n(t) \ge \omega > 0$ for all $t \epsilon [a,b]$. Suppose Λ is a family of continuous linear functionals over X which includes functionals of the type $\lambda(f) = f(x)$, $x \epsilon [a,b]$, for all $f \epsilon X$. Denote the set of $x \epsilon [a,b]$ for which there is such a λ by Δ. Let $\bar{\Delta}$ be the greatest distance between the points into which [a,b] is thus partitioned. Theorem 1, parts i and iii, of [17] then assures us that there is an $\varepsilon > 0$ such that if $\bar{\Delta} < \varepsilon$, $m\|u\|_n^2 \le M(u,u)$ for all $u \epsilon N(\Lambda)$, where $m > 0$. Thus the system $\{W^{n,2}[a,b],\Lambda,M,N(\Lambda)\}$ is well-posed for any such Λ, and by the previous theorem, for any function $f \epsilon W^{n,2}[a,b]$ there is a unique $Sp(M,\Lambda)$-interpolate which depends continuously on f.

The next theorem separates the questions of existence and uniqueness of M-splines, generalizing Theorem 2.1.

<u>Theorem 2.2</u>. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. Suppose there is a closed subspace of $N(\Lambda)$, N_2, such that the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed, where N_1 is defined by (1.11). Then for any $y \epsilon X$ there is a unique $Sp(M,\Lambda)$-interpolate s of y in $y + N_2$, which depends continuously on y. Moreover, any other interpolate of y, $\bar{s}$, is an $Sp(M,\Lambda)$-interpolate of y if and only if $\bar{s} - s \epsilon N_1$.

<u>Proof</u>. By hypothesis there is a closed subspace N_2 of $N(\Lambda)$ such that (1.13) and (1.14) are valid. Let Λ_2 be the orthogonal complement of N_2

in X. Then Λ_2 can be considered as a family of continuous linear functionals on X whose null space, $N(\Lambda_2)$, is N_2. Thus, by (1.14) the system $\{X,\Lambda_2,M,N(\Lambda_2)\}$ is well-posed (where $N(\Lambda_2) = N_2$), so by Theorem 2.1, for every $y \in X$ there is a unique $Sp(M,\Lambda_2)$-interpolate s of y, which depends continuously on y. This gives a unique $\hat{n}_2 \in N_2$ such that $s = y + \hat{n}_2$ and $M(s,n_2) = 0$ for all $n_2 \in N_2$. By (1.12) $M(s,n_1) = 0$ for all $n_1 \in N_1$, and by (1.13) any $n \in N(\Lambda)$ is of the form $n = n_1 + n_2$ with $n_1 \in N_1$, $n_2 \in N_2$. Therefore s is a unique $Sp(M,\Lambda)$-interpolate of y in $y + N_2$.

Next it will be established that

$$M(n_1,n) = 0 \text{ for all } n_1 \in N_1,\ n \in N(\Lambda). \tag{2.5}$$

Let $n_1 \in N_1$, $n = \bar{n}_1 + \bar{n}_2 \in N(\Lambda)$ with $\bar{n}_1 \in N_1$, $\bar{n}_2 \in N_2$. Then by (1.12), $M(n_1,n) = M(n_1,\bar{n}_2)$. Consider for any real α, $M(\bar{n}_2+\alpha n_1,\bar{n}_2+\alpha n_1) = M(\bar{n}_2,\bar{n}_2) + \alpha M(n_1,\bar{n}_2) \geq 0$ by (1.12), (1.11) and (1.9). Then $M(n_1,\bar{n}_2)$ must be zero, or the above inequality could not hold for all α, establishing (2.5).

Now if s is the unique $Sp(M,\Lambda)$-interpolate of y in $y + N_2$, and $\bar{s}$ is any other $Sp(M,\Lambda)$-interpolate of y, then $\bar{s} - s \in N(\Lambda)$, and $M(\bar{s}-s,n) = 0$ for all $n \in N(\Lambda)$. Letting $n = \bar{s} - s$, we see by (1.11) that $\bar{s} = s \in N_1$. On the other hand if s is as above and $\bar{s} - s \in N_1$, then $\bar{s} = s + \bar{n}_1$ for some $\bar{n}_1 \in N_1$ and $M(\bar{s},n) = M(s,n) + M(\bar{n}_1,n) = 0$ for all $n \in N(\Lambda)$ by (2.5) and Definition 1.1, so $\bar{s}$ is an $Sp(M,\Lambda)$-interpolate of y. □

<u>Corollary 2.4</u>. Under the conditions of Theorem 2.2,

$$X = N_2 \oplus Sp(M,\Lambda).$$

Corollary 2.5. Under the conditions of Theorem 2.2, if span(Λ) has a basis of dimension r, and $\dim(N_1) = r_1$, then $\dim(Sp(M,\Lambda)) = r + r_1$.

The following corollary shows that if M is symmetric and non-negative over all of X, the orthogonality condition (1.12) is always satisfied, giving again the conclusions of Theorem 2.2.

Corollary 2.6. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous symmetric bilinear functional on $X \times X$ such that $M(x,x) \geq 0$ for all $x \in X$. Let $N_1 = \{n_1 \in N(\Lambda) : M(n_1,n_1) = 0\}$ and suppose that there is some closed subspace of $N(\Lambda)$, N_2, such that $N(\Lambda) = N_1 \oplus N_2$ and M is positive definite on N_2. Then for any $y \in X$, there is a unique $Sp(M,\Lambda)$-interpolate s of y in $y+N_2$, which depends continuously on y. Moreover, any other interpolate of y, $\bar{s}$, is an $Sp(M,\Lambda)$-interpolate of y if and only if $\bar{s} - s \in N_1$.

Proof. Except for the orthogonality condition (1.12), the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed. But (1.12) does hold, since for any $x \in X$, $n_1 \in N_1$, $M(x+\alpha n_1, x+\alpha n_1) = M(x,x) + 2\alpha M(x,n_1) \geq 0$ for all real α, implying that $M(x,n_1) = 0$. □

Corollary 2.7. (Golomb [11], Jerome and Schumaker [15].) In Example 1.1, for any $x \in X$ there exists an $Sp(M,\Lambda)$-interpolate s, and any other interpolate of x, $\bar{s}$, is an $Sp(M,\Lambda)$-interpolate of x if and only if $\bar{s} - s \in N(\Lambda) \cap N(T)$.

Sufficient Conditions for Well-Posed and N_1-Posed Systems

The next result gives a very useful condition which insures the existence of the space N_2 of Definition 1.3 and Theorem 2.2.

Theorem 2.3. Let X, Λ, M and N_1 be given as in Definition 1.3 with M and N_1 satisfying (1.9), (1.11) and (1.12). Suppose there is a closed subspace of $N(\Lambda)$, N_3, such that $N_1 + N_3$ is of finite codimension in $N(\Lambda)$, and an $m_1 > 0$ such that

$$M(n_3,n_3) \geq m_1\|n_3\|^2 \quad \text{for all } n_3 \epsilon N_3. \tag{2.6}$$

Then there exists a closed subspace of $N(\Lambda)$, N_2, containing N_3 such that the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed.

Proof. Since the codimension of $N_1 + N_3$ in $N(\Lambda)$ is finite, and $N_1 \cap N_3 = \{0\}$ by (1.11) and (2.6),

$$N(\Lambda) = N_1 \oplus N_3 \oplus N_4 \tag{2.7}$$

for some finite dimensional subspace of $N(\Lambda)$, N_4. It will now be shown that $N_2 = N_3 \oplus N_4$ satisfies (1.14) as well as (1.13), demonstrating that the system $(X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed. It will suffice to show this for the case where N_4 is one dimensional.

Suppose N_4 consists of the span of some $n_4 \epsilon N(\Lambda) - (N_1 \oplus N_3)$. Let Λ_3 be the orthogonal complement of N_3 in X. Then Λ_3 can be considered to be a family of continuous linear functionals on X, with null space

$N(\Lambda_3) = N_3$. Now let us define a continuous bilinear functional on $X \times X$, M_s, by

$$M_s(x,y) = \frac{1}{2}[M(x,y) + M(y,x)] \quad \text{for all } x, y \in X. \tag{2.8}$$

By (2.6) and (2.8),

$$M_s(n_3,n_3) = M(n_3,n_3) \geq m_1\|n_3\|^2 \quad \text{for all } n_3 \in N(\Lambda_3) = N_3. \tag{2.9}$$

By (2.9) the system $\{X,\Lambda_3,M_s,N(\Lambda_3)\}$ is well-posed, so by Theorem 2.1, there is a unique $Sp(M_s,\Lambda_3)$-interpolate of n_4, $\bar{s} \in n_4 + N_3$, satisfying

$$2M_s(\bar{s},n_3) = M(\bar{s},n_3) + M(n_3,\bar{s}) = 0 \quad \text{for all } n_3 \in N_3. \tag{2.10}$$

Let $m = \frac{1}{2}\min\{M(\bar{s},\bar{s})/\|\bar{s}\|^2, m_1\}$. Then

$$M(\alpha\bar{s}+n_3,\alpha\bar{s}+n_3) = \alpha^2 M(\bar{s},\bar{s}) + \alpha[M(\bar{s},n_3) + M(n_3,\bar{s})] + M(n_3,n_3)$$

$$\geq 2m(\alpha^2\|\bar{s}\|^2 + \|n_3\|^2)$$

$$\geq m\|\alpha\bar{s} + n_3\|^2, \tag{2.11}$$

by use of (2.10) and the parallelogram inequality. But span $\{n_4+N_3\}$ = span$\{\bar{s}+N_3\}$, so (2.11) establishes that M is positive definite on $N_4 \oplus N_3$. If $\dim(N_4)>1$ the above argument may be repeated. □

The following application of Theorems 2.1, 2.2 and 2.3 generalizes results of Anselone and Laurent [5], Golomb [11], Jerome and Schumaker [15], and Jerome and Varga [16].

Example 2.2. Let X and $\{Y_i\}_{i=1}^n$ be real Hilbert spaces; $T_i:X\to Y_i$ be a continuous linear transformation from X onto Y_i, such that the dimension of $N(T_i)$, the null space of T_i, is finite, $1\le i\le n$; and Λ be a family of continuous linear functionals on X with null space $N(\Lambda)$. Let M be the continuous bilinear functional defined on $X \times X$ by

$$M(x_1,x_2) = \sum_{i=1}^{n} (T_i x_1, T_i x_2)_{Y_i} \text{ for all } x_1,x_2\in X.$$

For any i, $1\le i\le n$, let $N_1^{(i)} = N(\Lambda)\cap N(T_i) = \{n\in N(\Lambda):(T_i n,T_i n) = 0\}$, and let $N_2^{(i)}$ be the orthogonal complement of $N_1^{(i)}$ in $N(\Lambda)$. Then it was shown in Example 1.1 that T_i restricted to $N_2^{(i)}$ had a continuous inverse on $T_i(N_2^{(i)})$. Thus for some $m_i>0$, $\|T_i n\| \ge m_i\|n\|$ for all $n\in N_2^{(i)}$. Certainly $N_1 = \{n\in N(\Lambda):M(n,n) = 0\}$ is equal to $(\bigcap_{i=1}^{n} N(T_i))\cap N(\Lambda)$. Since

$$\operatorname{codim}(N_2^{(i)}) \text{ in } N(\Lambda) = \dim(N_1^{(i)}) \le \dim(N(T_i))<\infty,$$

the codimension of $N_3 = \bigcap_{i=1}^{n} N_2^{(i)}$ in $N(\Lambda)$ is finite, and so $N_1 + N_3$ is of finite codimension in $N(\Lambda)$. Note that if $n_3\in N_3$,

$$M(n_3,n_3) = \sum_{i=1}^{n} \|T_i n_3\|^2 \ge \sum_{i=1}^{n} m_i^2\|n_3\|^2 = m\|n_3\|^2$$

where $m = \sum_{i=1}^{n} m_i^2$, and if $n_1 \in N_1$, $T_i n_1 = 0$, $1 \le i \le n$, so for all $x \in X$ $M(x,n_1) = \sum_{i=1}^{n} (T_i x, T_i n_1)_{Y_i} = 0$, showing that M, N_1 and N_3 satisfy (1.12) and (2.6). Therefore by Theorem 2.3, there exists a closed subspace N_2 of $N(\Lambda)$ such that the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed. By Theorem 2.2, for any $x \in X$ there is a unique $Sp(M,\Lambda)$-interpolate s of x in $x + N_2$ which depends continuously on x. Moreover, any other interpolate of x, $\bar{s}$, is an $Sp(M,\Lambda)$-interpolate of x if and only if $\bar{s} - s \in N_1$. Finally, s will be unique if $N_1 = (\bigcap_{i=1}^{n} N(T_i)) \cap N(\Lambda) = \{0\}$.

<u>Theorem 2.4</u>. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. Suppose there is a closed subspace of $N(\Lambda)$, N_2, such that the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed. If $\Lambda_1 \supset \Lambda$ is another family of continuous linear functionals on X, and if the codimension (codim) of $N(\Lambda_1)$ in $N(\Lambda)$ is finite, then there exists a closed subspace $N_2^{(1)}$ such that the system $\{X,\Lambda_1,M,N(\Lambda_1),N_1^{(1)},N_2^{(1)}\}$ is $N_1^{(1)}$-posed.

<u>Proof</u>. Let $N_1^- = N_1 \cap N(\Lambda_1)$ and $N_3 = N_2 \cap N(\Lambda_1)$. Then with $N_1^{(1)} = \{n \in N(\Lambda_1): M(n,n) = 0\}$, $N_1^{(1)} = N_1^-$. Since $\text{codim}(N(\Lambda_1))$ in $N(\Lambda)$ is finite it follows that $\text{codim}(N_1^{(1)})$ in N_1 is finite and $\text{codim}(N_3)$ in N_2 is finite. Thus since $N(\Lambda) = N_1 \oplus N_2$, and $N_1^{(1)} \subset N_1$, $N_3 \subset N_2$, $N_1^{(1)} + N_3$ must be of finite codimension in $N(\Lambda_1)$, and M is positive definite on N_3. Therefore by Theorem 2.3, there exists a closed subspace of $N(\Lambda_1)$, $N_2^{(1)}$, containing N_3 such that the system $\{X,\Lambda_1,M,N(\Lambda_1),N_1^{(1)},N_2^{(1)}\}$ is $N_1^{(1)}$-posed. □

Corollary 2.8. Let X be a real Hilbert space, $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals such that $\Lambda_{i+1} \supset \Lambda_i$ and codim($N(\Lambda_{i+1})$) in $N(\Lambda_i)$ is finite for $i \geq 0$, and let M be a continuous bilinear functional on $X \times X$ such that there is a closed subspace of $N(\Lambda_0)$, $N_2^{(0)}$, such that the system $\{X,\Lambda_0,M,N(\Lambda_0),N_1^{(0)},N_2^{(0)}\}$ is $N_1^{(0)}$-posed. Then for all $i \geq 0$ there is a closed subspace of $N(\Lambda_i)$, $N_2^{(i)}$, such that the system $\{X,\Lambda_i,M,N(\Lambda_i),N_1^{(i)},N_2^{(i)}\}$ is $N_1^{(i)}$-posed, where $N_1^{(i)} = \{n \in N(\Lambda_i): M(n,n) = 0\}$. Moreover, if for any $i_0 \geq 0$, $N_1^{(i_0)} = \{0\}$, then for all $i \geq i_0$, the system $\{X,\Lambda_i,M,N(\Lambda_i)\}$ is well-posed, and no restriction need be placed on codim($N(\Lambda_{i+1})$) in $N(\Lambda_{i_o})$.

An important application of this corollary is the situation where the system $\{X,\Lambda_0,M,N(\Lambda_0),N_1^{(0)},N_2^{(0)}\}$ is $N_1^{(0)}$-posed and Λ_{i+1} is formed from Λ_i by augmenting Λ_i with one continuous linear functional not in the span of Λ_i, $i \geq 0$. Then the system $\{X,\Lambda_i,M,N(\Lambda_i),N_1^{(i)},N_2^{(i)}\}$ is $N_1^{(i)}$-posed for all $i \geq 0$. Note that in this application it does not matter whether or not the dimension of the span of Λ_0 is finite, and if $N_1^{(i_0)} = \{0\}$, then Λ_{i+1} may be formed from Λ_i by augmenting Λ_i with any set of continuous linear functionals, for all $i \geq i_0$.

Extremal Results

For a given real Hilbert space X, and continuous bilinear functional on $X \times X$, M, it will be useful to associate another continuous bilinear functional on $X \times X$, M_s, defined by

$$M_s(x,y) = \frac{1}{2}\,[M(x,y) + M(y,x)] \text{ for all } x,\ y \in X. \tag{2.12}$$

Then $M_s(x,x) = M(x,x)$ for all $x \in X$, and if M is symmetric, then $M_s(x,y) = M(x,y)$ for all $x, y \in X$. The following lemma generalizes a result sometimes referred to in the literature as the "first integral relation" (cf. [24, Th.4]).

Lemma 2.1. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$ such that $M(n,n) \geq 0$ for all $n \in N(\Lambda)$. Then for any $x \in X$ if s is an $Sp(M_s,\Lambda)$-interpolate of x,

$$M(x,x) = M(x-s,x-s) + M(s,s). \tag{2.13}$$

Proof. Since s is an $Sp(M_s,\Lambda)$-interpolate of x, $M_s(s,x-s) = 0$, so

$$\begin{aligned} M(x,x) &= M(x-s,x-s) + 2M_s(s,x-s) + M(s,s) \\ &= M(x-s,x-s) + M(s,s). \quad \square \end{aligned}$$

The next theorem gives an extremal result for nonsymmetric bilinear functionals, generalizing [17, Th.7].

Theorem 2.5. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$. Suppose $M(n,n) \geq 0$ for all $n \in N(\Lambda)$, and that there is a closed subspace N_2 of $N(\Lambda)$ such that the system $\{X,\Lambda,M_s,N(\Lambda),N_1,N_2\}$ is N_1-posed where $N_1 = \{n_1 \in N(\Lambda): M_s(n_1,n_1) = 0\}$. Then for any $y \in X$:

i. there is at least one $Sp(M_s,\Lambda)$-interpolate s of y,

$$\text{ii. } M(s,s) = \operatorname*{Min}_{x\in X}\{M(x,x): x \text{ is a } \Lambda\text{-interpolate of } y\}, \tag{2.14}$$

iii. if x is a Λ-interpolate of y, and $M(x,x) = M(s,s)$, then $x - s\in N_1$ and x is also an $Sp(M_s,\Lambda)$-interpolate of y.

Proof. Property i follows immediately from Theorem 2.2. Suppose x is any Λ-interpolate of y. Then s is also an $Sp(M_s,\Lambda)$-interpolate of x, and since $x - s\in N(\Lambda)$, $M(x-s,x-s) \geq 0$ and (2.13) of the lemma implies that $M(x,x) \geq M(s,s)$ giving property ii. If in addition $M(x,x) = M(s,s)$, then (2.13) of the lemma implies that $M_s(x-s,x-s) = 0$, so $x - s\in N_1$, and by Theorem 2.2, $x\in Sp(M_s,\Lambda)$. □

Corollary 2.9. Under the hypothesis of Theorem 2.5, if $N_1 = \{0\}$ then the $Sp(M_s,\Lambda)$-interpolate, s, of y gives the unique solution to the extremal problem (2.14).

Corollary 2.10. Let X be a real Hilbert space, Λ a family of continuous linear functionals on X, and M a continuous bilinear functional on $X \times X$ such that $M(x,x) \geq 0$ for all $x \in X$. Suppose there are closed subspaces of $N(\Lambda)$, N_1 and N_2, such that $N(\Lambda) = N_1 \oplus N_2$, $M(n_1,n_1) = 0$ for all $n_1\in N_1$, and $M(n_2,n_2) \geq m\|n_2\|^2$ for all $n_2\in N_2$. Then for any $y\in X$, there is at least one $Sp(M_s,\Lambda)$-interpolate s of y, and the extremal problem (2.14) is solved by s. Moreover, if x is any other Λ-interpolate of y which minimizes M as in (2.14), then $x\in Sp(M_s,\Lambda)$ and $x - s\in N_1$.

Proof. Equations (1.9), (1.11), (1.13) and (1.14) are explicitly satisfied by M_s, N_1 and N_2. Just as in the proof of Corollary 2.6,

$$M(x,x) = M_s(x,x) \geq 0 \text{ for all } x \in X \text{ implies that } M_s(x,n_1) = 0$$

for all $x \in X$, $n_1 \in N_1$. Therefore the orthogonality condition (1.12) also holds and the system $\{X,\Lambda,M_s,N(\Lambda),N_1,N_2\}$ is N_1-posed. Thus Theorem 2.5 applies. □

Now consider applying Theorem 2.5 and Corollary 2.9 to Example 2.1, where $M_s = M$, $M(f,f) \geq 0$ for all $f \in W^{n,2}[a,b]$, and the system $\{W^{n,2}[a,b],\Lambda,M,N(\Lambda)\}$ is well-posed. We see from Theorem 7 of [17], that if s is the unique $Sp(M,\Lambda)$-interpolate of $f \in W^{n,2}[a,b]$, where Λ consists solely of the Hermite type functionals considered in [17], then s must also be an L-spline, since s uniquely minimizes M over Λ-interpolates of f. Theorem 2.5 also can be applied to Example 1.1, where $M_s = M$, to demonstrate that the R-splines of Golomb [11] and the Lg-splines of Jerome and Schumaker [15] are special cases of M-splines.

The next result generalizes Theorem 6 of [17], and also implicitly offers a generalization of the property P used in that paper.

<u>Theorem 2.6</u>. Let X be a real Hilbert space and $\{\Lambda_i : i \geq 0\}$ be a nested sequence of families of continuous linear functionals on X, such that $\Lambda_{i+1} \supset \Lambda_i$ and the codimension of $N(\Lambda_{i+1})$ in $N(\Lambda_i)$ is finite, for all $i \geq 0$. Suppose M is a continuous bilinear functional on $X \times X$ such that $M(x,x) \geq 0$ for all $x \in N(\Lambda_0)$, and the system $\{X,\Lambda_0,M_s,N(\Lambda_0),N_1^{(0)},N_2^{(0)}\}$ is $N_1^{(0)}$-posed. Then for any $x \in X$, and all $i \geq 0$:

i. the system $\{X,\Lambda_i,M_s,N(\Lambda_i),N_1^{(i)},N_2^{(i)}\}$ is $N_1^{(i)}$-posed,

ii. there is at least one $Sp(M_s,\Lambda_i)$-interpolate s_i of x,

iii. $M(x-s_i,x-s_i) = \min\{M(x-\bar{s},x-\bar{s}): \bar{s}\in Sp(M_s,\Lambda_i)\cap U\}$,

where $U = \{y: y$ is a Λ_0-interpolate of $x\}$.

If $M(y,y) \geq 0$ for all $y\in X$, then

iv. $M(x-s_i,x-s_i) = \min\{M(x-\bar{s},x-\bar{s}): \bar{s}\in Sp(M_s,\Lambda_i)\}$.

Proof. Property i follows immediately from Corollary 2.8, and property ii follows from property i and Theorem 2.2. Suppose $\bar{s}\in Sp(M_s,\Lambda_i)$. Then $s_i - \bar{s}$ is an $Sp(M_s,\Lambda_i)$-interpolate of $x - \bar{s}$, so substituting $x - \bar{s}$ for x and $s_i - \bar{s}$ for s in (2.13) of Lemma 2.1 gives

$$M(x-\bar{s},x-\bar{s}) = M(x-s_i,x-s_i) + M(s_i-\bar{s},s_i-\bar{s}). \tag{2.15}$$

If s is also in U, then $s_i - \bar{s}\in N(\Lambda_0)$ so $M(s_i-\bar{s},s_i-\bar{s}) \geq 0$, and from (2.15), $M(x-\bar{s},x-\bar{s}) \geq M(x-s_i,x-s_i)$ establishing property iii. Similarly if $M(y,y) \geq 0$ for all $y\in X$, then again $M(s_i-\bar{s},s_i-\bar{s}) \geq 0$ giving property iv from (2.15).

Convergence of M-Splines

The next theorem generalizes a result of Golomb [11, Corollary 7.1].

Theorem 2.7. Let X be a real Hilbert space, $\{\Lambda_i\}_{i=0}^{\infty}$ a nested sequence of families of continuous linear functionals on X with $\Lambda_{i+1}\supset\Lambda_i$, $i\geq 0$, and M a continuous symmetric bilinear functional on $X \times X$. Suppose the system $\{X,\Lambda_0,M,N(\Lambda_0)\}$ is well-posed, and let $\Lambda_\infty = \bigcup_{i=0}^{\infty}\Lambda_i$. Then for all $i\geq 0$ and every $x\in X$, there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of x, and a unique $Sp(M,\Lambda_\infty)$-interpolate of x, s_∞, where $\{s_i\}$ and s_∞ depend continuously on x. Moreover, $\lim_{i\to\infty} s_i = s_\infty$ and $\lim_{i\to\infty} M(s_i,s_i) = M(s_\infty,s_\infty)$.

Proof. The system $\{X,\Lambda_0,M,N(\Lambda_0)\}$ is well-posed, so there exists an $m>0$ such that

$$M(n,n) \geq m\|n\|^2 \quad \text{for all } n\epsilon N(\Lambda_0). \tag{2.16}$$

Hence (2.16) also holds for all $n\epsilon N(\Lambda_\infty)$ and all $n\epsilon N(\Lambda_i)$, and thus the systems $\{X,\Lambda_\infty,M,N(\Lambda_\infty)\}$ and $\{X,\Lambda_i,M,N(\Lambda_i)\}$ for all $i\geq 0$ are well-posed, and the first part of the theorem follows from Theorem 2.1. Since s_∞ is a Λ_j-interpolate of $s_j\epsilon Sp(M,\Lambda_j)$ for all $j\geq 0$, and if $j\geq i\geq 0$, s_j is a Λ_i-interpolate of $s_i\epsilon Sp(M,\Lambda_i)$, property ii of Theorem 2.5 gives

$$M(s_\infty,s_\infty) \geq M(s_j,s_j) \geq M(s_i,s_i) \text{ for all } j\geq i\geq 0. \tag{2.17}$$

Lemma 2.1 then gives,

$$M(s_j-s_i,s_j-s_i) = M(s_j,s_j) - M(s_i,s_i), \tag{2.18}$$

since s_i is an $Sp(M,\Lambda_i)$-interpolate of s_j. From (2.17) and (2.18) it follows that $\lim_{i,j\to\infty} M(s_j-s_i,s_j-s_i) = 0$, and by (2.16), $\{s_i\}_{i=0}^{\infty}$ is a Cauchy sequence. Let $s^\infty = \lim_{i\to\infty} s_i$. Then it may easily be seen that $s_\infty = s^\infty$, and the result follows. □

As an application of this theorem, consider a variation of Example 2.1 with $\{\Lambda_i\}_{i=0}^{\infty}$ being a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$, $n\geq 1$, and $\{\bar{\Delta}_i\}_{i=0}^{\infty}$ the associated partition norms. Suppose $\bar{\Delta}_i\to 0$. Then there is an $i_0\geq 0$ such that the system $\{W^{n,2}[a,b],\Lambda_{i_0},M,N(\Lambda_{i_0})\}$ is well-posed, and for every $f\in W^{n,2}[a,b]$ there is a unique $Sp(M,\Lambda_i)$-interpolate s_i for all $i\geq i_0$. Let s_∞ be the $Sp(M,\bigcup_{i=0}^{\infty}\Lambda_i)$-interpolate of f, and let $x\in[a,b]$. Since $\bar{\Delta}_i\to 0$, there is a sequence $\{x_i\}$ with $x_i\in\Delta_i$ such that $x_i\to x$. But $f(x_i) = s_i(x_i) = s_\infty(x_i)$, so $f(x) = s_\infty(x)$ by the continuity of f and s_∞. Therefore $f \equiv s_\infty$, and by Theorem 2.7,

$$\lim_{i\to\infty}\|f-s_i\|_n = 0. \tag{2.19}$$

The convergence in the Sobolev norm given by (2.19) may be shown by a Rolle's theorem argument [cf. 17, Th. 1] to imply uniform convergence of $D^j s_i$ to $D^j f$ on $[a,b]$ for $0\leq j\leq n-1$, and L^2 convergence when $j = n$.

CHAPTER III

LOWER ORDER CONVERGENCE RESULTS IN $W^{n,2}[a,b]$

In this chapter we shall develop existence, uniqueness and lower order convergence results for M-splines in the Sobolev space $W^{n,2}[a,b]$ of all functions $f \in C^{n-1}[a,b]$ such that $D^{n-1}f$ is absolutely continuous and $D^n f \in L^2[a,b]$, with

$$(f,g)_n = \sum_{i=0}^{n} \int_a^b [D^i f(x)][D^i g(x)]dx, \tag{3.1}$$

and where M is of the following form.

<u>Definition 3.1</u>. Let M denote the continuous bilinear functional defined on $W^{n,2}[a,b] \times W^{n,2}[a,b]$ by

$$M(f,g) = \sum_{i,j=0}^{n} \int_a^b b_{ij}(x)[D^i f(x)][D^j g(x)]dx,$$

where $b_{nn}(x) \geq \omega$, $a \leq x \leq b$ for some $\omega > 0$, and where the b_{ij} are bounded, real-valued measurable functions on $[a,b]$.

Definition 3.1 defines a *continuous* bilinear functional, since if $K_1 = \max_{0 \leq i,j \leq n} \|b_{ij}(x)\|_{L^\infty[a,b]}$,

$$|M(f,g)| \leq K_1 \sum_{i,j=0}^{n} \|D^i f\|_{L^2} \|D^j g\|_{L^2}$$

$$= K_1\left(\sum_{i=0}^{n} \|D^i f\|_{L^2}\right)\left(\sum_{j=0}^{n} \|D^j g\|_{L^2}\right)$$

$$\le K\|f\|_n \|g\|_n, \quad \text{for all } f,g\epsilon W^{n,2}[a,b], \tag{3.2}$$

where $K = (n+1)K_1$, since $\|f\|_n = \left(\sum_{i=0}^{n} \|D^i f\|_{L^2}^2\right)^{1/2} \ge (n+1)^{-1/2} \sum_{i=0}^{n} \|D^i f\|_{L^2}$. The notation in the following definition is due to Jerome and Varga [16].

Definition 3.2 Let Λ be a family of continuous linear functionals over $W^{n,2}[a,b]$, and let $\bar{\Lambda}$ denote the closure of the span of Λ. Associate with Λ the subset Δ of $[a,b]$ consisting of all $x\varepsilon[a,b]$ such that there exists a $\lambda\epsilon\bar{\Lambda}$ satisfying $\lambda(f) = f(x)$ for all $f\epsilon W^{n,2}[a,b]$. Refer to Δ as the *partition* of $[a,b]$ induced by Λ. If Δ is not empty, denote by $\bar{\Delta}$ the maximum length of the subintervals into which $[a,b]$ is decomposed by the points of Δ. Also if Δ is not empty, then for every $x\varepsilon\Delta$, let $i(x)$ be the maximum positive integer such that there exists a $\lambda_k\varepsilon\bar{\Lambda}$ for which

$$\lambda_k(f) = D^k f(x),\ f\epsilon W^{n,2}[a,b],\ 0\le k\le i(x)-1.$$

Finally, let

$$\gamma(\Delta) = \sum_{x\varepsilon\Delta} i(x)$$

if Δ is not empty. Otherwise, let $\gamma(\Delta) = 0$.

Notice that i(x) is simply the number of consecutive derivative point functionals associated with the point $x \varepsilon \Delta$. Thus Definition 3.2 associates with every family of continuous linear functionals Λ with non-empty Δ, a system $\{\Lambda, \Delta, \gamma(\Delta), \bar{\Delta}\}$, and for every $x \varepsilon \Delta$, a number i(x). Note also that $N(\Lambda) = N(\bar{\Lambda})$.

The argument used in the following lemma is an extension of that used in Theorem 1 of the author's paper [17].

Lemma 3.1. Let Λ be a family of continuous linear functionals over $W^{n,2}[a,b]$, such that $\gamma(\Delta) \geq n$. Let $g \varepsilon N(\Lambda) = \{f \varepsilon W^{n,2}[a,b] : \lambda(f) = 0$ for all $\lambda \varepsilon \Lambda\}$. Then the following inequalities hold:

$$\text{i.} \quad \|D^j g\|_{L^2[a,b]} \leq M_{1,j}(\bar{\Delta})^{n-j} \|D^n g\|_{L^2[a,b]}, \quad 0 \leq j \leq n \tag{3.3}$$

$$\text{ii.} \quad \|D^j g\|_{L^\infty[a,b]} \leq M_{2,j}(\bar{\Delta})^{n-j-\frac{1}{2}} \|D^n g\|_{L^2[a,b]}, \quad 0 \leq j \leq n-1 \tag{3.4}$$

where $\{M_{1,j}\}$ and $\{M_{2,j}\}$ are independent of f and Λ.

Proof. Let $\Delta = \{x_0, x_1, \cdots, x_N\}$. Then $D^i g(x_k) = 0$, $0 \leq i \leq i(x_k)-1$, $0 \leq k \leq N$. By repeated applications of Rolle's theorem, there exist points $\{\xi_\ell^{(k)}\}_{\ell=0}^{N_k}$, $0 \leq k \leq n-1$, where $N_0 = N$ and for $n-1 \geq k \geq 1$, $N_k = N_{k-1} - 1 + n(k)$, n(k) being the number of times $i(x) \geq k$ for $x \varepsilon \Delta$ and

A. $D^k g(\xi_\ell^{(k)}) = 0$, $0 \leq \ell \leq N_k$, $0 \leq k \leq n-1$,

B. $a \leq \xi_0^{(k)} < \xi_1^{(k)} < \cdots < \xi_{N_k}^{(k)} \leq b$, $0 \leq k \leq n-1$,

C. $\xi^{(k+1)}_{\hat{\ell}-1} \le \xi^{(k)}_{\ell} < \xi^{(k+1)}_{\hat{\ell}} < \xi^{(k)}_{\ell+1} \le \xi^{(k+1)}_{\hat{\ell}+1}$

for some $\hat{\ell}, \ell \le \hat{\ell} \le n(k+1)+\ell$, $0 \le \ell \le N_k - 1$, $0 \le k \le n-2$. Define also $\xi^{(k)}_{-1} = a$ and $\xi^{(k)}_{N_k+1} = b$, $0 \le k \le n-1$. By a Wirtinger-type inequality [13, p. 184],

$$\int_{\xi^{(k)}_{\ell}}^{\xi^{(k)}_{\ell+1}} (D^k g(x))^2 dx \le \left(\frac{2(k+1)\bar{\Delta}}{\pi}\right)^2 \int_{\xi^{(k)}_{\ell}}^{\xi^{(k)}_{\ell+1}} (D^{k+1} g(x))^2 dx$$

for $-1 \le \ell \le N_k$, $0 \le k \le n-1$, since $\xi^{(k)}_{\ell+1} - \xi^{(k)}_{\ell} \le (k+1)\bar{\Delta}$.

Summing both sides above from $\ell = -1$ to N_k, gives

$$\int_a^b (D^k g(x))^2 dx \le \left(\frac{2(k+1)\bar{\Delta}}{\pi}\right)^2 \int_a^b (D^{k+1} g(x))^2 dx. \qquad (3.5)$$

Letting k range from j to n-1, and using (3.5) repeatedly, yields

$$\int_a^b (D^j g(x))^2 dx \le \left[\frac{2^{n-j} n!}{j!(\pi)^{n-j}}\right]^2 \bar{\Delta}^{2(n-j)} \int_a^b (D^n g(x))^2 dx.$$

Now let $M_{1,j} = (2/\pi)^{n-j} n!/j!$, $0 \le j \le n$, giving (3.3).

For any j, $0 \le j \le n-1$, there is a $y_j \epsilon [a,b]$ such that $\|D^j g\|_{L^\infty} = |D^j g(y_j)|$. Then

$$\|D^j g\|_{L^\infty} = \left|\int_{\xi^{(j)}_{\hat{\ell}}}^{y_j} D^{j+1} g(x) dx\right| \le (j+1)^{1/2} (\bar{\Delta})^{1/2} \|D^{j+1} g\|_{L^2},$$

by the Schwarz inequality, where $\xi_{\hat{\ell}}^{(j)}$ is the closest of $\{\xi_{\ell}^{(j)}: 0\le\ell\le N_j\}$ to y_j, and hence $|y_j - \xi_{\hat{\ell}}^{(j)}| \le (j+1)\bar{\Delta}$, $0\le j\le n-1$. But by (i), $\|D^{j+1}g\|_{L^2} \le M_{1,j+1}(\bar{\Delta})^{n-j-1}\|D^n g\|_{L^2}$, so letting $M_{2,j} = (j+1)^{1/2} M_{1,j+1}$, $0\le j\le n-1$, (3.4) follows. □

<u>Theorem 3.1</u>. Let M be given by Definition 3.1, let Λ be a family of continuous linear functionals on $W^{n,2}[a,b]$, and suppose there is a positive constant m such that

$$M(g,g) \ge m\|g\|_n^2 \quad \text{for all } g\in N(\Lambda). \tag{3.6}$$

Then for every $f\in W^{n,2}[a,b]$ there is a unique Sp(M,Λ)-interpolate s, which depends continuously on f. If $\gamma(\Delta)\ge n$, then the following error bounds exist for f-s:

$$\text{i.}\quad \|D^j(f-s)\|_{L^2[a,b]} \le M_{3,j}(\bar{\Delta})^{n-j}\{M(f-s,f-s)\}^{1/2},$$

$$\le M_{4,j}(\bar{\Delta})^{n-j}\|f\|_n,\ 0\le j\le n, \tag{3.7}$$

$$\text{ii.}\quad \|D^j(f-s)\|_{L^\infty[a,b]} \le M_{5,j}(\bar{\Delta})^{n-j-\frac{1}{2}}\{M(f-s,f-s)\}^{1/2}$$

$$\le M_{6,j}(\bar{\Delta})^{n-j-\frac{1}{2}}\|f\|_n,\ 0\le j\le n-1, \tag{3.8}$$

where the constants $\{M_{3,j}\}$, $\{M_{4,j}\}$, $\{M_{5,j}\}$ and $\{M_{6,j}\}$ are independent of Λ and f, but dependent on m.

<u>Proof.</u> Since the system $\{W^{n,2}[a,b],\Lambda,M,N(\Lambda)\}$ is well-posed by (3.6), the unique existence of an $Sp(M,\Lambda)$-interpolate s for any $f\epsilon W^{n,2}[a,b]$ follows from Theorem 2.1. From (3.1) and (3.6),

$$\|D^n(f-s)\|_{L^2[a,b]} \leq \|f-s\|_n \leq m^{-\frac{1}{2}}\{M(f-s,f-s)\}^{\frac{1}{2}}, \tag{3.9}$$

since $f-s\epsilon N(\Lambda)$, so the first half of (3.7) and (3.8) follows from Lemma 3.1 by letting $M_{3,j} = m^{-\frac{1}{2}}M_{1,j}$, $0\leq j\leq n$, and $M_{5,j} = m^{-\frac{1}{2}}M_{2,j}$, $0\leq j\leq n-1$. From (3.2) and the fact that $s\epsilon Sp(M,\Lambda)$,

$$M(f-s,f-s) = M(f,f-s) \leq K\|f\|_n\|f-s\|_n. \tag{3.10}$$

Combining (3.9) and (3.10), gives

$$\{M(f-s,f-s)\}^{\frac{1}{2}} \leq Km^{-\frac{1}{2}}\|f\|_n. \tag{3.11}$$

Thus the second half of (3.7) and (3.8) comes from the first half by letting $M_{4,j} = Km^{-\frac{1}{2}}M_{3,j}$, $0\leq j\leq n$, and $M_{6,j} = Km^{-\frac{1}{2}}M_{5,j}$, $0\leq j\leq n-1$. □

Note that in Theorem 3.1, if Λ_1 is a family of continuous linear functionals on $W^{n,2}[a,b]$, such that $\Lambda_1\supset\Lambda$, then since $N(\Lambda_1)\subset N(\Lambda)$,

$$M(g,g) \geq m\|g\|_n^2 \quad \text{for all } g\epsilon N(\Lambda_1)$$

so there will be a unique $Sp(M,\Lambda_1)$-interpolate s_1 of f, and the error bounds (3.7) and (3.8) will apply to $f-s_1$ with $\bar{\Delta}_1 \leq \bar{\Delta}$. A similar statement holds for any nested sequence of such families, $\{\Lambda_i\}_{i=1}^{\infty}$, $\Lambda_{i+1} \supset \Lambda_i$, $i \geq 1$. The next theorem gives a simple set of conditions for which the hypotheses of Theorem 3.1 are satisfied.

Theorem 3.2. Let M be the continuous bilinear functional of Definition 3.1. Then there exist positive constants ε and m, depending only on M, such that if Λ is any family of continuous linear functionals on $W^{n,2}[a,b]$ with $\bar{\Delta}<\varepsilon$, and $\gamma(\Delta)\geq n$, then

$$M(g,g) \geq m\|g\|_n^2 \quad \text{for all } g\in N(\Lambda). \tag{3.12}$$

Proof. Let I be the index set $\{(i,j)\colon 0\leq i,\ j\leq n \text{ and } (i,j) \neq (n,n)\}$, and consider the following identity:

$$\int_a^b b_{nn}(D^n g)^2 dx = M(g,g) - \sum_{(i,j)\in I} \int_a^b b_{ij} D^i g D^j g dx.$$

Then

$$\omega\|D^n g\|_{L^2}^2 \leq M(g,g) + \sum_{(i,j)\in I} \|b_{ij}\|_{L^\infty} \|D^i g\|_{L^2} \|D^j g\|_{L^2}. \tag{3.13}$$

Since $\gamma(\Delta)\geq n$, (3.3) of Lemma 3.1 may be applied to (3.13), giving for all $g\in N(\Lambda)$,

$$\omega\|D^n g\|_{L^2}^2 \leq M(g,g) + \sum_{(i,j)\in I} \|b_{ij}\|_{L^\infty} M_{1,i} M_{1,j} (\bar{\Delta})^{2n-i-j} \|D^n g\|_{L^2}^2. \tag{3.14}$$

Let $\eta(t) = \omega - \sum_{(i,j)\in I} \|b_{ij}\|_{L^\infty} M_{1,i} M_{1,j} (t)^{2n-i-j}$. Since $2n-i-j\geq 1$ for all

$(i,j)\in I$, $\eta(t)\to\omega$ as $t\to 0$, and there exists an $\varepsilon>0$ such that $\eta(t)\geq\omega/2$ for all $t<\varepsilon$. Therefore if $\bar{\Delta}<\varepsilon$, by (3.14)

$$\omega/2\|D^n g\|^2_{L^2} \leq \eta(\bar{\Delta})\|D^n g\|^2_{L^2} \leq M(g,g) \text{ for all } g\in N(\Lambda). \tag{3.15}$$

Applying (3.15) to (3.3) gives

$$\|g\|^2_n = \sum_{j=0}^{n} \|D^j g\|^2_{L^2} \leq \sum_{j=0}^{n} M^2_{1,j}(\bar{\Delta})^{2(n-j)} \frac{2}{\omega} M(g,g)$$

$$\leq 1/m\; M(g,g) \text{ for all } g\in N(\Lambda) \tag{3.16}$$

where $1/m = \frac{2}{\omega}\sum_{j=0}^{n} M^2_{1,j}(\varepsilon)^{2(n-j)}$. □

Theorem 3.3. Let M be the continuous bilinear functional of Definition 3.1, and let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of continuous linear functionals on $W^{n,2}[a,b]$, with $\Lambda_{i+1}\supset\Lambda_i$, $i\geq 0$, such that $\bar{\Delta}_i\to 0$. Then there exists an $i_0\geq 0$ such that for all $i\geq i_0$, and any $f\in W^{n,2}[a,b]$, there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of f which depends continuously on f, and the following error bounds apply to $f-s_i$:

$$\text{i.}\quad \|D^j(f-s_i)\|_{L^2[a,b]} \leq M_{3,j}(\bar{\Delta}_i)^{n-j}\{M(f-s_i,f-s_i)\}^{1/2},$$

$$\leq M_{4,j}(\bar{\Delta}_i)^{n-j}\|f\|_n,\quad 0\leq j\leq n, \tag{3.17}$$

$$\text{ii.}\quad \|D^j(f-s_i)\|_{L^\infty[a,b]} \le M_{5,j}(\bar{\Delta}_i)^{n-j-\frac{1}{2}}\{M(f-s_i,f-s_i)\}^{1/2}$$

$$\le M_{6,j}(\bar{\Delta}_i)^{n-j-\frac{1}{2}}\|f\|_n,\ 0\le j\le n-1, \qquad (3.18)$$

where the constants $\{M_{3,j}\}$, $\{M_{4,j}\}$, $\{M_{5,j}\}$ and $\{M_{6,j}\}$ are independent of $\{\Lambda_i\}_{i=0}^{\infty}$ and f, but dependent on M. Moreover, $M(f-s_i,f-s_i)\to 0$ as $i\to\infty$.

Proof. Let ε and m be the positive constants determined by M in Theorem 3.2. Then choose i_0 so that $\bar{\Delta}_{i_0} < \varepsilon$ and $\gamma(\Delta_{i_0}) \ge n$, and apply Theorems 3.2 and 3.1. Finally, $s_i\to f$ by Theorem 2.7 and the discussion in the application which followed Theorem 2.7, so $M(f-s_i,f-s_i)\to 0$. □

Theorem 3.4. Let M be the continuous bilinear functional of Definition 3.1. If Λ is any family of continuous linear functionals on $W^{n,2}[a,b]$ such that $M(g,g)\ge 0$ for all $g\in N(\Lambda)$ and $N_1 = \{g\in N(\Lambda)\colon M(g,g) = 0\}$, then N_1 is finite dimensional. If in addition $M(f,g) = 0$ for all $f\in W^{n,2}[a,b]$, and $g\in N_1$, then there exists a closed subspace of $N(\Lambda)$, N_2, such that the system $\{W^{n,2}[a,b],\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed.

Proof. Let ε and m_1 be the positive constants depending only on M given by Theorem 3.2, and let Λ_0 consist of a finite number of continuous linear functionals of the point evaluation type: $\lambda(f) = f(x)$ for all $f\in W^{n,2}[a,b]$, with $x\in[a,b]$; and suppose that Λ_0 is such that for $\Lambda_3 = \Lambda\cup\Lambda_0$, $\bar{\Delta}_3 < \varepsilon$ and $\gamma(\Delta_3) \ge n$. Let $N_3 = N(\Lambda_3)$. Then N_3 is of finite codimension in $N(\Lambda)$, and by Theorem 3.2,

$$M(g,g) \geq m_1 \|g\|_n^2 \text{ for all } g \in N_3.$$

Hence $N_3 \cap N_1 = \{0\}$, and N_1 is finite dimensional. Since $N_3 + N_1$ is of finite codimension in $N(\Lambda)$, the remainder of the theorem follows directly from Theorem 2.3. □

Corollary 3.1. Let M be the continuous bilinear functional of Definition 3.1, and suppose $M(g,g) \geq 0$ for all $g \in W^{n,2}[a,b]$. Then the linear subspace $N_1^0 = \{g \in W^{n,2}[a,b]: M(g,g) = 0\}$ is finite dimensional.

Corollary 3.2. Let M be the continuous bilinear functional of Definition 3.1, and suppose $M(g,g) \geq 0$ for all $g \in W^{n,2}[a,b]$, and $b_{ij}(x) = b_{ji}(x)$, $0 \leq i, j \leq n$, for all $x \in [a,b]$. Then for any family Λ of continuous linear functionals on $W^{n,2}[a,b]$, there exist closed subspaces N_1 and N_2 such that the system $\{X,\Lambda,M,N(\Lambda),N_1,N_2\}$ is N_1-posed.

Proof. Let $N_1 = \{g \in N(\Lambda): M(g,g) = 0\}$. Then for all real numbers α and any $f \in W^{n,2}[a,b]$, $g \in N_1$, $M(f+\alpha g, f+\alpha g) = M(f,f) + 2\alpha M(f,g) \geq 0$, so $M(f,g)$ must equal zero. Hence Theorem (3.4) applies. □

If $\{\Lambda_i\}_{i=0}^{\infty}$ is a sequence of families of continuous linear functionals such that $\Lambda_{i+1} \supset \Lambda_i$ for all $i \geq 0$, and $\bar{\Delta}_i \to 0$, then it was seen in Theorem 3.3 that there exists an $i_0 \geq 0$ such that the system $\{W^{n,2}[a,b], \Lambda_i, M, N(\Lambda_i)\}$ is well-posed for all $i \geq i_0$, and the error bounds (3.17) and (3.18) apply to $f-s_i$ where s_i is the unique $Sp(M,\Lambda_i)$-interpolate of f, $i \geq i_0$. It would be of interest to have error bounds which hold for a sequence of $Sp(M,\Lambda_i)$-interpolates of $f \in W^{n,2}[a,b]$ even when the systems

$\{W^{n,2}[a,b],\Lambda_i,M,N(\Lambda_i)\}$ are not well-posed for any i. Of course for this to be the case $\bar{\Delta}_i \geq \varepsilon$ for all i, where ε is given by Theorem 3.2.

Theorem 3.5. Let M be the continuous bilinear functional of Definition 3.1, and let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda_{i+1} \supset \Lambda_i$ for all $i\geq 0$. Suppose there exists an $i_0 \geq 0$ such that $M(g,g)\geq 0$ for all $g\in N(\Lambda_{i_0})$, and $M(f,g) = 0$ for all $f\in W^{n,2}[a,b]$ and all $g\in N_1^{(i_0)}$, where $N_1^{(i)} = \{g\in N(\Lambda_i): M(g,g) = 0\}$, $i\geq i_0$. Then there exists an $m>0$ such that for all $i\geq i_0$, there exists a closed subspace of $N(\Lambda_i)$, $N_2^{(i)}$, and

i. the system $\{W^{n,2}[a,b],\Lambda_i,M,N(\Lambda_i),N_1^{(i)},N_2^{(i)}\}$ is $N_1^{(i)}$-posed,

ii. for every $f\in W^{n,2}[a,b]$ there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of f in $f + N_2^{(i)}$, which depends continuously on f,

iii. any other Λ_i-interpolate of f, $\bar{s}_i$, is an $Sp(M,\Lambda_i)$-interpolate of f if and only if $s-\bar{s}_i\in N_1^{(i)}$,

iv. $M(g,g) \geq m\|g\|_n^2$ for all $g\in N_2^{(i)}$,

v. if for some $i_1\geq i_0$, $\gamma(\Delta_{i_1}) \geq n$, then the error bounds (3.17) and (3.18) apply to $f-s_i$, for all $i\geq i_1$, and

vi. if for any $i_2\geq i_0$, $N_1^{i_2} = \{0\}$, then for all $i\geq i_2$ the system $\{W^{n,2}[a,b],\Lambda_i,M,N(\Lambda_i)\}$ is well-posed and s_i is the *unique* $Sp(M,\Lambda_i)$-interpolate of f.

Proof. Since $N_1^{(i)} \subset N_1^{(i_0)}$ for all $i\geq i_0$, properties i, ii and iii follow from Theorem 3.4 and Theorem 2.2. Also by Theorem 3.4, the subspace $N_1^{(i_0)}$ is finite dimensional, and since $N_1^{(j)} \subset N_1^{(i)} \subset N_1^{(i_0)}$ if $j\geq i\geq i_0$, $N_1^{\infty} = \bigcap_{i=i_0}^{\infty} N_1^{(i)}$ is a finite dimensional subspace of $N_1^{(i_0)}$, and hence there exists an $i_3 \geq i_0$ such that $N_1^\infty = N_1^{(i)}$ for all $i \geq i_3$. Associate with each i, $i_0 \leq i \leq i_3$, a positive constant m_i, such that $M(g,g) \geq m_i \|g\|_n^2$ for all $g \epsilon N_2^{(i)}$. Then if the subspaces $N_2^{(i)}$, for $i > i_3$, are chosen properly, we claim that $m = \min_{i_0 \leq i \leq i_3} (m_i)$ satisfies property iv.

Denote by $N_1^\perp$ the orthogonal complement of $N_1^{(i_3)}$ in $N(\Lambda_{i_3})$. Then

$$N(\Lambda_{i_3}) = N_1^{(i_3)} \oplus N_2^{(i_3)} = N_1^{(i_3)} \oplus N_1^\perp.$$

Any $g \epsilon N_1^\perp$ is of the form $g_1 + g_2$ where $g_1 \epsilon N_1^{(i_3)}$ and $g_2 \epsilon N_2^{(i_3)}$, and hence

$$M(g,g) = M(g_2,g_2) \geq m_{i_3} \|g_2\|_n^2 = m_{i_3} \|g-g_1\|_n^2,$$

and since $\|g-g_1\|_n^2 = \|g\|_n^2 + \|g_1\|_n^2$, $M(g,g) \geq m_{i_3} \|g\|_n^2$ for all $g \epsilon N_1^\perp$. Now for all $i > i_3$, choose $N_2^{(i)}$ to be the orthogonal complement of $N_1^{(i)} = N_1^\infty$ in $N(\Lambda_i)$. Then $N_2^{(i)} \subset N_1^\perp$, and thus $M(g,g) \geq m_{i_3} \|g\|_n^2$ for all $g \epsilon N_2^{(i)}$, $i > i_3$, giving property iv. If i is restricted by $i \geq i_3$, property iv may be sharpened by letting $m = m_{i_3}$.

Suppose for some $i_1 \geq i_0$, $\gamma(\Delta_{i_1}) \geq n$. Then for any $i \geq i_1$, $\gamma(\Delta_i) \geq n$. Denote by $\bar{\Lambda}_i$ the orthogonal complement of $N_2^{(i)}$ in $W^{n,2}[a,b]$. Then $\bar{\Lambda}_i$ may be considered to be a family of continuous linear functionals on $W^{n,2}[a,b]$, and since $N(\bar{\Lambda}_i) = N_2^{(i)}$ and $M(g,g) \geq m \|g\|_n^2$ for all $g \epsilon N_2^i$, the system $\{W^{n,2}[a,b], \bar{\Lambda}_i, M, N_2^{(i)}\}$ is well-posed. Thus by Theorem 2.1, for any $f \epsilon W^{n,2}[a,b]$ there is a unique $Sp(M,\bar{\Lambda}_i)$-interpolate $\hat{s}_i$ of f, and $\gamma(\bar{\Delta}_i) \geq \gamma(\Delta) \geq n$, so by Theorem 3.1, the error bounds (3.17) and (3.18) apply to $f-\hat{s}_i$, for all $i \ge i_1$. But since s_i is an $Sp(M,\Lambda_i)$-interpolate of f in $f + N_2^{(i)}$, and $Sp(M,\Lambda_i) \subset Sp(M,\bar{\Lambda}_i)$, s_i is also an $Sp(M,\bar{\Lambda}_i)$-interpolate of f, and hence $\hat{s}_i = s$, giving property iv. If for any $i_2 \ge i_0$, $N_1^{(i_2)} = \{0\}$, then $N_1^{(i)} = \{0\}$ for all $i \ge i_2$, and property vi follows from properties i and iii. □

Corollary 3.3. Let M be given by Definition 3.1 and let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda_{i+1} \supset \Lambda_i$ for all $i \ge 0$. Suppose $M(g,g) > 0$ for all nonzero $g \in N(\Lambda_0)$. Then there is a positive constant m such that for all $i \ge 0$,

i. $M(g,g) \ge m\|g\|_n^2$ for all $g \in N(\Lambda_i)$,

ii. the system $\{W^{n,2}[a,b], \Lambda_i, M, N(\Lambda_i)\}$ is well-posed,

iii. for every $f \in W^{n,2}[a,b]$ there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of f which depends continuously on f,

iv. if for some $i_1 \ge 0$, $\gamma(\Delta_{i_1}) \ge n$, then the error bounds (3.17) and (3.18) apply to $f-s_i$ for all $i \ge i_1$.

As an application of this corollary, let the b_{ij} in Definition 3.1 be given by $b_{ij}(x) = a_i(x)a_j(x)$ for all $x \in [a,b]$, where $a_i \in C[a,b]$, $0 \le i \le n$, and $a_n(x) \ne 0$ for any $x \in [a,b]$. Then

$$M(g,f) = \sum_{i,j=0}^{n} \int_a^b a_i a_j (D^i g)(D^j f)dx$$

$$= \int_a^b L_1[g]L_1[f]dx$$

where $L_1[g] = \sum_{i=0}^{n} a_i D^i g$. Thus

$$M(g,g) = \int_a^b (L_1[g(x)])^2 dx \geq 0 \qquad \text{all } g\epsilon W^{n,2}[a,b].$$

Let $\Lambda_0 = \Lambda_c = \{\lambda:\lambda = (D^j)_c,\ 0\leq j\leq n-1\}$ for some fixed $c\epsilon[a,b]$. If $M(g,g) = 0$ for $g\epsilon N(\Lambda_0)$, $L_1[g] = 0$, and $D^j g(c) = 0$, $0\leq j\leq n-1$, so g equals zero. Hence $M(g,g)>0$ for all nonzero $g\epsilon N(\Lambda_0)$, and there is a positive constant m such that for all families of continuous linear functionals $\Lambda \supset \Lambda_0$, $M(g,g) \geq m\|g\|_n^2$ for all $g\epsilon N(\Lambda)$, and for any $f\epsilon W^{n,2}[a,b]$ there is a unique $Sp(M,\Lambda)$-interpolate s of f, and f-s satisfies the error bounds (3.7) and (3.8) by Theorem 3.1 (or Corollary 3.3). Compare this result with Theorem 6 [24] or Theorem 2 [18] in which convergence is either deduced only when $\bar{\Delta}$ is sufficiently small, or by the use of Gårdings inequality [25, p. 175] when $\Lambda_0 = \Lambda_a \cup \Lambda_b$.

CHAPTER IV

HIGHER ORDER CONVERGENCE RESULTS IN $W^{n,2}[a,b]$

If M is the continuous bilinear functional defined on $W^{n,2}[a,b] \times W^{n,2}[a,b]$ by Definition 3.1, it was shown in Chapter III that for any $f \in W^{n,2}[a,b]$ the Sp(M,Λ)-interpolate of f,s, exists and f-s satisfies the error bounds (3.7) and (3.8) for a variety of families of continuous linear functionals Λ on $W^{n,2}[a,b]$. The principle requirement on Λ was that it must *include* certain functionals of a specified type, the remainder being arbitrary. If f is also in $W^{2n,2}[a,b]$ and Λ contains only functionals of Hermite type, then the author has shown in [17] that if it is required that $D^j(f-s) = 0$ at the endpoints a and b for $0 \le j \le n-1$, then the second integral relationship holds [17, Th.4], and $\|D^j(f-s)\|_{L^\infty[a,b]} \le C(\bar{\Delta})^{2n-j-(\frac{1}{2})}$ for $0 \le j \le n-1$. Jerome and Varga [16] have developed convergence results of this order for functionals of Hermite-Birkhoff type. In both of these papers, *higher* order convergence results are obtained only for Λ consisting *strictly* of functionals of Hermite, Hermite-Birkhoff or Extended-Hermite-Birkhoff [16, p. 125] type. Jerome and Varga have remarked [16, p.125] that their *lower* order convergence results hold when all but certain functionals are *arbitrary*, as is the case in Chapter III. In this chapter, new and improved higher order convergence results will be derived for $f \in W^{p,2}[a,b]$, $n+1 \le p \le 2n$. Moreover these results, like the results for lower order convergence, will hold when Λ contains certain functionals of a specified type, the remainder being *arbitrary*. Finally, an improvement and generalization of the second integral relation will be developed.

Two Lemmas

Lemma 4.1. Let $f\epsilon W^{n+r,2}[a,b]$ where r is an integer, $0<r\leq n$, and $s\epsilon W^{n,2}[a,b]$ be any Λ^r-interpolate of f, where $\Lambda^r = \Lambda_a \cup \Lambda_b$ with

$$\Lambda_x = \{\lambda: \lambda = (D^j)_x,\ n-r\leq j\leq n-1\} \text{ for } x\epsilon[a,b]. \tag{4.1}$$

If M is given by Definition (3.1), with $b_{ij}\epsilon C^{\ell}[a,b]$ where $\ell = \max(0,j+r-n)$, then

$$M(f,f-s) = \sum_{j=0}^{n-r} \int_a^b G_{jr}(f;x)D^j(f-s)dx, \tag{4.2}$$

where

$$G_{jr}(f;x) = \begin{cases} \sum_{i=0}^{n} b_{ij}(x)D^i f(x) & \text{if } 0\leq j\leq n-r-1 \\ \sum_{\ell=n-r}^{n} (-1)^{\ell+r-n} D^{\ell+r-n}\left[\sum_{i=0}^{n} b_{i\ell}(x)D^i f(x)\right] & \text{if } j=n-r. \end{cases} \tag{4.3}$$

Proof. Since $f\epsilon W^{n+r,2}[a,b]$, repeated integration by parts gives

$$M(f,f-s) = \sum_{i=0}^{n} \sum_{j=0}^{n} \int_a^b b_{ij}(x)[D^i f(x)]D^i(f(x)-s(x))dx$$

$$= \sum_{j=0}^{n-r-1} \int_a^b [\sum_{i=0}^{n} b_{ij} D^i f] D^j (f-s)dx$$

$$+ \sum_{j=n-r}^{n} \int_a^b (-1)^{j+r-n} D^{j+r-n} [\sum_{i=0}^{n} b_{ij} D^i f] D^{n-r}(f-s)dx$$

$$+ \sum_{j=n-r+1}^{n} \sum_{\ell=n-r+1}^{j} (-1)^{j-\ell} D^{j-\ell} \sum_{i=0}^{n} b_{ij} D^j f] D^{\ell-1}(f-s) \Big|_a^b . \qquad (4.4)$$

If the last term in (4.4) can be shown to be zero, the theorem will be established. Since s is a Λ^r-interpolate of f, (4.1) gives

$$D^j f(a) = D^j s(a),\ D^j f(b) = D^j s(b),\ n-r \le j \le n-1,$$

and hence the last term in (4.4) *is* equal to zero. □

Let $C_p^\ell[a,b]$ denote the subspace $\{f \in C^\ell[a,b]: D^j f(a) = D^j f(b), 0 \le j \le \ell\}$. Then a similar result holds for the periodic case.

Lemma 4.2. Let r and q be integers with $0<r\le n$, and $n-r\le q<n$, and let $f \in W^{n+r,2}[a,b] \cap C_p^{2n-q-1}[a,b]$. Suppose $s \in W^{n,2}[a,b]$ is a $\Lambda^{r,q}$-interpolate of f, where $\Lambda^{r,q} = \Lambda_{aq} \cup \Lambda_{bq} \cup \Lambda$ with

$$\Lambda_q = \{\lambda: \lambda = (D^j)_b - (D^j)_a,\ q \le j \le n-1\}, \text{ and}$$

$$(4.5)$$

$$\Lambda_{xq} = \{\lambda: \lambda = (D^j)_x,\ n-r \le j \le q-1\}, \text{ for } x \in [a,b].$$

Let M be given by Definition 3.1, and suppose the coefficients of M satisfy the conditions

$$\left.[D^{\ell} b_{ij}(x)]\right|_a^b = 0 \text{ for } 0\le\ell\le j-q-1,\ q+1\le j\le n,\ 0\le i\le n, \text{ and}$$
$$b_{ij}\epsilon C^{\ell}[a,b],\ \ell = \max(0,j+r-n),\ 0\le i\le n. \tag{4.6}$$

Then again

$$M(f,f-s) = \sum_{j=0}^{n-r} \int_a^b G_{jr}(f;x)D^j(f-s)dx, \tag{4.7}$$

where G_{jr} is defined by (4.3).

<u>Proof</u>. As in Lemma 4.1, (4.4) is valid, and (4.7) holds if the last term in (4.4) is equal to zero. Since s is a $\Lambda^{r,q}$-interpolate of f, if $n-r<q\le n$,

$$D^{\ell-1}s(a) = D^{\ell-1}f(a),\ D^{\ell-1}s(b) = D^{\ell-1}f(b),\ n-r+1\le\ell\le q,$$

and if not, $n-r+1 = q+1$. Thus in either case it suffices to show that

$$\left.\sum_{j=q+1}^{n} \sum_{\ell=q+1}^{j} (-1)^{j-\ell} D^{j-\ell}[\sum_{i=0}^{n} b_{ij}D^i f]D^{\ell-1}(f-s)\right|_a^b = 0.$$

But this follows immediately from the hypothesis on the periodicity of f and the coefficients $\{b_{ij}(x)\}$, $q+1\le j\le n$, $0\le i\le n$, and that s is a Λ_q-interpolate of f. □

The Second Integral Relationship

The next theorem generalizes and improves the second integral relationship of [2], [16], [17] and [24]. The improvement is that in the special cases considered in the above papers, the family of continuous linear functionals Λ is restricted to be exclusively of Extended Hermite-Birkhoff [15] type, while in the theorem below no such restriction is made.

Theorem 4.1. Let M be given by Definition 3.1 with $b_{ij}\epsilon C^j[a,b]$, $0\le i$, $j\le n$, and let Λ be any family of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda\supset\Lambda^r = \Lambda_a\cup\Lambda_b$, where

$$\Lambda_x = \{\lambda\colon \lambda = (D^j)_x,\ 0\le j\le n-1\},\ x\epsilon[a,b].$$

Then for all $f\epsilon W^{2n,2}[a,b]$, if s is any $Sp(M,\Lambda)$-interpolate of f,

$$M(f-s,f-s) = \int_a^b L[f(x)](f(x)-s(x))dx,$$

where

$$L[f] = \sum_{\ell=0}^{n} (-1)^{\ell} D^{\ell}[\sum_{i=0}^{n} b_{i\ell} D^i f]. \tag{4.8}$$

Proof. Since s is an $Sp(M,\Lambda)$-interpolate of f, $M(f-s,f-s) = M(f,f-s)$. Now apply Lemma 4.1 with $r = n$, getting

$$M(f,f-s) = \int_a^b G_{0,n}(f;x)(f(x)-s(x))dx.$$

The result now follows since $L[f(x)] \equiv G_{0,n}(f;x)$. □

In a similar fashion, Lemma 4.2 may be used to generalize and improve the second integral relationship for periodic functions [17, Th. 4].

Theorem 4.2. Let q be any integer, $0 \le q < n$ and let M be given by Definition 3.1, with its coefficients $\{b_{ij}\}$ satisfying the periodicity conditions (4.6), and with $b_{ij} \in C^j[a,b]$. Let Λ be any family of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda \supset \Lambda^{n,q} = \Lambda_{aq} \cup \Lambda_{bq} \cup \Lambda_q$ as in (4.5) with $r = n$. Then for all $f \in W^{2n,2}[a,b] \cap C_p^{2n-q-1}[a,b]$, if s is any $Sp(M,\Lambda)$-interpolate of f,

$$M(f-s,f-s) = \int_a^b L[f(x)](f(x)-s(x))dx,$$

where

$$L[f] = \sum_{\ell=0}^{n} (-1)^{\ell} D^{\ell}[\sum_{i=0}^{n} b_{i\ell} D^i f].$$

Higher Order Convergence Results

The results in this section improve and generalize results contained in [3], [9], [16], [17], [18] and [24]. They also generalize considerably the rather complete set of convergence results in [7].

Theorem 4.3. Let r be any integer, $0<r\leq n$, let M be given by Definition 3.1, with $b_{ij}\epsilon C^{\ell}[a,b]$, where $\ell = \max(0,j+r-n)$, $0\leq i, j\leq n$, and let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda_{i+1} \supset \Lambda_i$ for all $i\geq 0$. Suppose there exists an $i_0\geq 0$ such that $\Lambda_{i_0} \supset \Lambda^r = \Lambda_a \cup \Lambda_b$ where Λ_x is defined by (4.1), $\gamma(\Delta_{i_0}) \geq n$ and there exists an $m>0$ such that

$$M(g,g) \geq m\|g\|_n^2 \quad \text{for all } g\epsilon N(\Lambda_{i_0}). \tag{4.9}$$

Then for every $f\epsilon W^{n+r,2}[a,b]$ and $i\geq i_0$, there exists a unique $Sp(M,\Lambda_i)$-interpolate s_i, and the following error bounds exist for $f-s_i$

$$\text{i.} \quad \|D^j(f-s_i)\|_{L^2[a,b]} \leq M_{3,j}(\bar{\Delta}_i)^{n+r-j}H_0[f], \; 0\leq j\leq n, \tag{4.10}$$

$$\text{ii.} \quad \|D^j(f-s_i)\|_{L^\infty[a,b]} \leq M_{5,j}(\bar{\Delta}_i)^{n+r-j-\frac{1}{2}}H_0[f], \; 0\leq j\leq n-1, \tag{4.11}$$

with

$$H_0[f] = \sum_{j=0}^{n-r} \|G_{jr}(f;x)\|_{L^2[a,b]} M_{3,j}(\bar{\Delta}_{i_0})^{n-j-r}.$$

Proof. For any $i\geq i_0$ the hypotheses of both Theorem 3.1 and Lemma 4.1 are satisfied. Hence for any $f\epsilon W^{n+r,2}[a,b]$ there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of f for which the error bound (3.7) applies, and by (4.2)

$$0 \leq M(f-s_i,f-s_i) = M(f,f-s_i) = \sum_{j=0}^{n-r} \int_a^b G_{jr}(f;x)D^j(f-s_i)dx. \tag{4.12}$$

By inspection of (4.3), and the condition on the $\{b_{ij}\}$, $G_{jr}(f;x)\epsilon L^2[a,b]$, and applying the Schwarz inequality to (4.12) and then using (3.7),

$$M(f-s_i,f-s_i) \le \sum_{j=0}^{n-r} \|G_{jr}(f;x)\|_{L^2}\|D^j(f-s_i)\|_{L^2}$$

$$\le \sum_{j=0}^{n-r}\|G_{jr}(f;x)\|_{L^2}M_{3,j}(\bar{\Delta}_i)^{n-j}\{M(f-s_i,f-s_i\}^{\frac{1}{2}}.$$

Since $\bar{\Delta}_i \le \bar{\Delta}_{i_0}$,

$$\{M(f-s_i,f-s_i)\}^{\frac{1}{2}} \le (\bar{\Delta}_i)^r H_0[f], \qquad (4.13)$$

and (4.10) and (4.11) follow from (3.7), (3.8) and (4.13). □

A similar result for the periodic case follows by the same argument using Lemma 4.2.

Theorem 4.4. Let r and q be integers with $0<r\le n$ and $n-r\le q<n$, and let M be given by Definition 3.1 with its coefficients satisfying the periodicity condition (4.6), with $b_{ij}\epsilon C^{\ell}[a,b]$, where $\ell = \max(0,j+r-n)$, $0\le i$, $j\le n$. Let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda_{i+1} \supset \Lambda_i$ for all $i\ge 0$. Suppose there exists an $i_0\ge 0$ such that $\Lambda_{i_0} \supset \Lambda^{r,q} = \Lambda_{aq}\cup\Lambda_{bq}\cup\Lambda_q$ where Λ_{xq} and Λ_q are defined by (4.5), $\gamma(\Delta_{i_0})\ge n$, and there exists an $m>0$ such that

$$M(g,g) \ge m\|g\|_n^2 \quad \text{for all } g\epsilon N(\Lambda_{i_0}).$$

Then for every $f\epsilon W^{n+r,2}[a,b]\cap C_p^{2n-q-1}[a,b]$ and every $i\geq i_0$, there is a unique $Sp(M,\Lambda_i)$-interpolate of f, s_i, and the error bounds (4.10) and (4.11) apply to $f-s_i$.

Corollary 4.1. If r=n (if r=n, and q is such that $0\leq q<n$), and the hypotheses of Theorem 4.3 (Theorem 4.4) are satisfied by $\{\Lambda_i\}_{i=0}^{\infty}$, M and i_0, then for every $f\epsilon W^{2n,2}[a,b]$ ($f\epsilon W^{2n,2}[a,b]\cap C_p^{2n-q-1}$) and every $i\geq i_0$ there is a unique $Sp(M,\Lambda_i)$-interpolate of f, s_i, and the following error bounds exist for $f-s_i$

$$\text{i.}\quad \|D^j(f-s_i)\|_{L^2[a,b]} \leq M_{3,j}M_{3,0}(\bar{\Delta}_i)^{2n-j}\|L[f]\|_{L^2[a,b]},\quad 0\leq j\leq n, \tag{4.14}$$

$$\text{ii.}\quad \|D^j(f-s_i)\|_{L^\infty[a,b]} \leq M_{5,j}M_{3,0}(\bar{\Delta}_i)^{2n-j-\frac{1}{2}}\|L[f]\|_{L^2[a,b]},\quad 0\leq j\leq n-1, \tag{4.15}$$

where L[f] is defined by (4.8).

Theorem 4.5. Let r be any integer, $0<r\leq n$, let M be given by Definition 3.1 with $b_{ij}\epsilon C^{\ell}[a,b]$ where $\ell = \max(0,j+r-n)$, $0\leq i,\ j\leq n$, let ε be determined by Theorem 3.2, and let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda_{i+1}\supset\Lambda_i$ for all $i\geq 0$. Let $f\epsilon W^{n+r,2}[a,b]$. If there exists an $i_0\geq 0$ such that $\Lambda_{i_0}\supset\Lambda^r = \Lambda_a\cup\Lambda_b$ where Λ_x is defined by (4.1), $\gamma(\Delta_{i_0})\geq n$ and either $\bar{\Delta}_{i_0}<\varepsilon$ or $M(g,g)>0$ for all non-zero $g\epsilon N(\Lambda_{i_0})$, then for all $i\geq i_0$ there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of f, which depends continuously on f, and $f-s_i$ satisfies the error bounds (4.10) and (4.11).

Proof. The result follows by Theorem 4.3 since if either $\Delta_{i_0} < \varepsilon$ or $M(g,g)>0$ for all non-zero $g \in N(\Lambda_{i_0})$, Theorem 3.2 or Corollary 3.3, respectively, give that $M(g,g) \geq m\|g\|_n^2$ for all $g \in N(\Lambda_{i_0})$. □

The next result follows by a similar argument using Theorem 4.4.

Theorem 4.6. Let r and q be integers with $0<r\leq n$ and $n-r\leq q<n$, let M be given by Definition 3.1 with its coefficients satisfying (4.6), and with $b_{ij} \in C^{\ell}[a,b]$, where $\ell = \max(0,j+r-n)$, $0\leq i,j\leq n$, let ε be determined by Theorem 3.2, and let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals on $W^{n,2}[a,b]$ such that $\Lambda_{i+1} \supset \Lambda_i$ for all $i\geq 0$. Let $f \in W^{n+r,2}[a,b] \cap C_p^{2n-q-1}[a,b]$. If there exists an $i_0 \geq 0$ such that $\Lambda_{i_0} \supset \Lambda^{r,q} = \Lambda_{aq} \cup \Lambda_{bq} \cup \Lambda_q$ where Λ_{xq} and Λ_q are defined by (4.5), $\gamma(\Delta_{i_0}) \geq n$ and either $\bar{\Delta}_{i_0} < \varepsilon$ or $M(g,g) > 0$ for all non-zero $g \in N(\Lambda_{i_0})$, then for all $i \geq i_0$ there is a unique $Sp(M,\Lambda_i)$-interpolate s_i of f, which depends continuously on f, and $f-s_i$ satisfies the error bounds (4.10) and (4.11).

CHAPTER V

AN APPLICATION TO NONLINEAR BOUNDARY VALUE PROBLEMS

In this chapter we shall apply the theory of M-splines to study the approximate solution of a class of nonselfadjoint, nonlinear two-point boundary value problems. These results will generalize and improve results by Rose [19], Ciarlet, Schultz and Varga [9], Hulme [14], Jerome and Varga [16], Perrin, Price and Varga [18], and Schultz [23].

Consider the nonselfadjoint, nonlinear two-point boundary value problem [cf. [16]]:

$$L[u(x)] = f(x,u(x)) \quad a<x<b \tag{5.1}$$

where

$$L[u(x)] \equiv \sum_{0\le i,j\le m} (-1)^j D^j(\sigma_{ij}(x)D^i u(x)), \tag{5.2}$$

$$m\ge 1, \ D \equiv \frac{d}{dx},$$

subject to the homogeneous boundary conditions

$$D^k u(a) = D^k u(b) = 0 \qquad 0\le k\le m-1. \tag{5.3}$$

We require the following additional conditions on L and f:

i. the coefficient functions $\sigma_{ij}(x)$, $0 \le i, j \le m$, are bounded, real-valued and measurable on $[a,b]$,

ii. $f(x,u)$ is a real-valued function on $[a,b] \times R$ such that $f(x,u(x)) \in L^2[a,b]$ for any $u \in W_0^{m,2}[a,b]$, where $W_0^{m,2}[a,b]$ is the subspace of $W^{m,2}[a,b]$ of functions satisfying (5.3),

iii. there exists a positive constant c such that

$$M_L(u,u) \ge c\|u\|_m^2 \tag{5.4}$$

for all $u \in W_0^{m,2}[a,b]$, where

$$M_L(u,v) \equiv \sum_{0 \le i,j \le m} \int_a^b \sigma_{ij}(x) D^i u(x) D^j v(x) dx, \tag{5.5}$$

iv. there exists a real constant γ such that

$$\frac{f(x,u) - f(x,v)}{u - v} \le \gamma < c_1 c \tag{5.6}$$

for almost all $x \in [a,b]$ and all $-\infty < u, v < \infty$ with $u \ne v$, where

$$c_1 = \inf \left\{ \frac{(u,u)_m}{(u,u)_o} : u \in W_0^{m,2}[a,b], \quad u \ne 0 \right\} \ge 1, \tag{5.7}$$

v. for each positive real number c, there exists a positive constant M(c) such that

$$\left|\frac{f(x,u) - f(x,v)}{u - v}\right| \le M(c) \tag{5.8}$$

for almost all $x\epsilon[a,b]$, and all $u\neq v$ with $|u|,|v| \le c$.

We say that $u\epsilon W_0^{m,2}[a,b]$ is a *generalized solution* of (5.1) - (5.2) if and only if

$$a(u,v) \equiv M_L(u,v) - \int_a^b f(x,u(x))v(x)dx = 0 \tag{5.9}$$

for all $v\epsilon W_0^{m,2}[a,b]$. If S_k is any subspace of $W_0^{m,2}[a,b]$, we say that $s\epsilon S_k$ is a *Galerkin approximation* in S_k of the solution u of (5.1) - (5.2) if

$$a(s,v) = 0 \quad \text{for all } v\epsilon S_k. \tag{5.10}$$

We shall now show that the hypotheses i-v above are sufficient to insure that the boundary value problem (5.1)-(5.2) has a unique generalized solution u, and for any finite dimensional subspace of $W_0^{m,2}[a,b]$, S_k, there is a unique Galerkin approximation u_k of u. We shall then consider new M-spline subspaces of $W_0^{m,2}[a,b]$, and deduce high order error bounds on $u-u_k$, when u and the coefficients σ_{ij} are sufficiently smooth.

Lemma 5.1. Under the hypotheses i, ii, iii, iv and v above, the quasi-bilinear form a(u,v) defined by (5.9) is such that

i. there exists a continuous (nonlinear) operator T from $W_0^{m,2}[a,b]$ into $W_0^{m,2}[a,b]$ such that

$$(Tu,v)_m = a(u,v) \quad \text{for all } u,\ v \in W_0^{m,2}[a,b], \tag{5.11}$$

ii. T is *strongly monotone*, i.e., there is an $\alpha>0$ such that

$$|(Tu-Tv,u-v)_m| \geq \alpha\|u-v\|^2 \quad \text{for all } u,v \in W_0^{m,2} \tag{5.12}$$

and

iii. T is Lipschitz continuous for bounded arguments, i.e., given $K_1>0$, there exists a constant $C(K_1)$, depending only on K_1, such that

$$\|Tu-Tv\|_m \leq C(K_1)\|u-v\|_m \tag{5.13}$$

for all $u,v \in W_0^{m,2}[a,b]$ with $\|u\|_m$, $\|v\|_m \leq K_1$.

Proof. It follows from the definition of a(u,v) (cf.(5.9)) that for a fixed $u \in W_0^{m,2}[a,b]$ a(u,v) is a bounded linear functional of v in $W_0^{m,2}[a,b]$, and hence there exists a (nonlinear) operator T from $W_0^{m,2}[a,b]$ into $W_0^{m,2}[a,b]$ such that (5.11) is valid. Let K be the bound on M_L given by (3.2). To show that T is continuous, it suffices to note that for any $u,u_1,v \in W_0^{m,2}[a,b]$, with $\|u\|_{L^\infty},\|u_1\|_{L^\infty} \leq c$,

$$|(Tu-Tu_1,v)_m| = |a(u,v)-a(u_1,v)|$$

$$\leq |M_L(u-u_1,v)| + |\int_a^b [f(x,u)-f(x,u_1)]vdx|$$

$$\leq K\|u-u_1\|_m\|v\|_m + M(c)\|u-u_1\|_m\|v\|_m,$$

implying that T is Lipschitz continuous, and hence continuous.

T is strongly monotone since

$$(Tu-Tv,u-v)_m = a(u,u-v) - a(v,u-v)$$

$$= M_L(u-v,u-v) - \int_a^b [f(x,u) - f(x,v)](u-v)dx$$

$$\geq c\|u-v\|_m^2 - \gamma\|u-v\|_0^2$$

$$\geq \left(c - \frac{\gamma}{c_1}\right)\|u-v\|_m^2$$

$$= \alpha\|u-v\|_m^2$$

where $\alpha = (c_1c-\gamma)c_1^{-1} > 0$. □

We now state a result from the theory of monotone operators which can be found in Ciarlet, Schultz and Varga [10].

Lemma 5.2. Let X be a real Hilbert space, and T be a (nonlinear) operator mapping X into X. If T is strongly monotone and Lipschitz continuous for bounded arguments, then the problem of determining a $u \in X$ such that

$$(Tu,v)_X = 0 \text{ for all } v \in X \tag{5.14}$$

and for any finite dimensional subspace of X, S_k, the problem of determining a $u_k \in S_k$ such that

$$(Tu_k,v)_X = 0 \quad \text{for all } v \in S_k, \tag{5.15}$$

each have a unique solution. Moreover, there exists a constant K' such that the following error bound is valid:

$$\|u_k - u\|_m \leq K' \inf\{\|w-u\| : w \in S_k\}. \tag{5.16}$$

Using Lemma 5.1 and Lemma 5.2 we can now state the following result (cf. [16]):

Theorem 5.1. With the assumptions i, ii, iii, iv and v, the two-point nonselfadjoint, nonlinear boundary value problem (5.1)-(5.3) has a *unique* generalized solution (cf. (5.9)) $u \in W_0^{m,2}[a,b]$. Moreover, if S_k is any finite dimensional subspace of $W_0^{m,2}[a,b]$, then there exists a *unique* Galerkin approximation (cf. (5.10)) u_k, and there exist positive constants K_1, K_2 and K_3, independent of S_k, such that

$$\|D^i(u_k-u)\|_{L^\infty[a,b]} \le K_1\|u_k-u\|_m$$

$$\le K_2\inf\{\|w-u\|_m\colon w\in S_k\}, \qquad (5.17a)$$

for all $0\le i\le m-1$, and

$$\|D^i(u_k-u\|_{L^2[a,b]} \le \|u_k-u\|_m \le K_3\inf\{\|w-u\|_m\colon w\in S_k\}, \quad 0\le j\le m. \qquad (5.17b)$$

Proof. By Lemma 5.2 it remains only to verify the first inequality of (5.17a). For any i, $0\le i\le m-1$, let $x_i\in[a,b]$ be such that $|D^i(u_k(x_i) - u(x_i))| = \|D^i(u_k-u)\|_{L^\infty[a,b]}$. Then if c is the closer of the endpoints a, b to x_i,

$$\|D^i(u_k-u)\|_{L^\infty[a,b]} = |\int_c^{x_i} D^{i+1}(u_k-u)dx|$$

$$\le \frac{b-a}{2}\ \|D^{i+1}(u_k-u)\|_{L^2[a,b]}$$

$$\le K_1\|u_k-u\|_m,$$

where $K_1 = (b-a)/2$. □

We now apply the L^2-interpolation results of Chapters III and IV to get high order error bounds for Galerkin approximates over subspaces of M-splines generalizing and improving [9] and [16].

<u>Theorem 5.2</u>. With the assumptions i, ii, iii, iv and v, let $u \in W_0^{n,2}[a,b]$ be the unique generalized solution to the two-point non-selfadjoint, nonlinear boundary value problem (5.1)-(5.3) and let M be the continuous bilinear functional of Definition 3.1 with $b_{ij} \in C^j[a,b]$, $0 \leq i,j \leq n$, for some $n \geq m$. Let $\{\Lambda_i\}_{i=0}^{\infty}$ be a nested sequence of families of continuous linear functionals over $W^{n,2}[a,b]$ such that $\Lambda_{i+1} \supset \Lambda_i$, and $\dim(\mathrm{span}(\Lambda_i)) < \infty$ for all $i \geq 0$. Suppose also that $\Lambda_0 \supset \Lambda_c = \{\lambda: \lambda(f) = D^j f(c),\ 0 \leq j \leq n-1\}$ for $c = a$ and $c = b$, and $\bar{\Delta}_i \to 0$. Then there exists an $i_0 \geq 0$ such that for all $i \geq i_0$, there is a unique Galerkin approximation of u, u_i, in $Sp(M,\Lambda_i)$. Moreover, if $u \in W_0^{t,2}[a,b]$, with $n \leq t \leq 2n$, there exist constants K_1, K_2 and K_3, independent of i, such that

$$\|D^j(u_i-u)\|_{L^\infty[a,b]} \leq K_1 \|u_i-u\|_m \leq K_2(\Delta_i)^{t-m}, \text{ for } 0 \leq j \leq m-1, \tag{5.18a}$$

and

$$\|D^j(u_i-u)\|_{L^2[a,b]} \leq \|u_i-u\|_m \leq K_3(\Delta_i)^{t-m}, \text{ for } 0 \leq j \leq m. \tag{5.18b}$$

<u>Proof</u>. By Theorem 3.3, there is an $i_0 \geq 0$ such that for all $i \geq i_0$ there is a unique $Sp(M,\Lambda_i)$-interpolate of every $f \in W^{n,2}[a,b]$. By Corollary 2.2, $\dim(Sp(M,\Lambda_i)) = \dim(\mathrm{span}(\Lambda_i))$, and the subspaces $Sp(M,\Lambda_i)$ exist for $i \geq i_0$. Applying Theorem 3.3 or Theorem 4.5 to (5.17a) and (5.17b) with w

set equal to the $Sp(M,\Lambda_i)$-interpolate of u, gives (5.18a) and (5.18b). □

Actually, the conditions in Theorem 5.2 could be weakened somewhat. If $u \in W_0^{n+r,2}[a,b]$, with $0 \le r < n$, the b_{ij} need only be measurable and bounded if $r = 0$, and otherwise the requirement need only be $b_{ij} \in C^{\ell}[a,b]$, $\ell = \max(0,j+r-n)$, $0 \le i, j \le n$. Also the requirement on Λ_0 could be weakened to $\Lambda_0 \supset \Lambda_c = \{\lambda: \lambda(f) = D^j f(c), 0 \le j \le m-1, n-r \le j \le n-1\}$, for $c = a$ and $c = b$. Then again equations (5.18a) and (5.18b) follow by the same argument.

We will now improve the convergence results of Theorem 5.2 for the case where the solution to the boundary value problem (5.1)-(5.3) is in $W_0^{t,2}[a,b]$ for $m \le t \le 2m$, and for a specially chosen subspace of M-splines, generalizing and improving results in [9], [10], [16], [18] and [23].

Theorem 5.3. Under the assumptions of Theorem 5.2 if the coefficients of L in (5.2) are such that $\sigma_{ij} \in C^i[a,b]$, $0 \le i, j \le m$, and M is chosen to be

$$M(f,g) = \sum_{i,j=0}^{m} \int_a^b \sigma_{ij}(x)[D^j f(x)][D^i g(x)]dx$$

$$= M_L(g,f) \qquad (5.19)$$

where M_L is given by (5.5), and the generalized solution u of (5.1)-(5.3) is such that $u \in W_0^{t,2}[a,b]$ for $m \le t \le 2m$, then for all $i \ge 0$, there is a unique Galerkin approximation of u, u_i, in $Sp(M,\Lambda_i)$ and there exist constants K_1 and K_2, independent of i, such that

$$\|D^j(u_i-u)\|_{L^\infty[a,b]} \leq K_1(\bar{\Delta}_i)^{t-j-\frac{1}{2}}H_0[u],\ 0\leq j\leq m-1, \tag{5.20}$$

and

$$\|D^j(u_i-u)\|_{L^2[a,b]} \leq K_2(\bar{\Delta}_i)^{t-j}H_0[u],\ 0\leq j\leq m, \tag{5.21}$$

where $H_0[u]$ is defined in Theorem 4.3 with $r = t - m$.

Proof. By Lemma 5.1 there exists a (nonlinear) operator T from $W_0^{m,2}[a,b]$ into $W_0^{m,2}[a,b]$ such that $(Tu,v)_m = a(u,v)$ for all $u,v\in W_0^{m,2}[a,b]$ and T is both strongly monotone and Lipschitz continuous for bounded arguments. This implies that corresponding conditions hold on the quasi-bilinear form $a(u,v)$, namely

$$|a(u,u-v) - a(v,u-v)| \geq \alpha\|u-v\|_m^2 \text{ for all } u,v\in W_0^{m,2}[a,b], \tag{5.22}$$

and for any $K>0$ there is a positive constant $C(K)$, depending only on K, such that

$$|a(u_1,v) - a(u_2,v)| \leq C(K)\|u_1-u_2\|_m\|v\|_m \tag{5.23}$$

for all $u_1,u_2\in W_0^{m,2}[a,b]$ such that $\|u_1\|_m,\|u_2\|_m \leq K$.

By (5.4) and Theorem 2.1 the i_0 of Theorem 5.2 is 0, so that for any $i\geq 0$, there is a unique Galerkin approximation of u in $Sp(M,\Lambda_i),u_i$. Let s_i be the unique $Sp(M,\Lambda_i)$-interpolate of u. Then

$$a(u,v) = 0 \quad \text{for all } v \in W_0^{m,2}[a,b], \tag{5.24}$$

$$a(u_i,v) = 0 \quad \text{for all } v \in Sp(M,\Lambda_i), \text{ and} \tag{5.25}$$

$$M_L(n,s) = M(s,n) = 0 \quad \text{for all } n \in N(\Lambda_i),\ s \in Sp(M,\Lambda_i). \tag{5.26}$$

We shall now derive a bound for $u_i - s_i$.

First we need an a priori bound for u, u_i and s_i. By (5.22),

$$\alpha\|u_i\|_m^2 \le |a(u_i,u_i) - a(0,u_i)|$$

$$= |a(0,u_i)| \le \|T(0)\|_m \|u_i\|_m,$$

so $\|u_i\|_m \le \alpha^{-1}\|T(0)\|_m$. By the same argument $\|u\|_m \le \alpha^{-1}\|T(0)\|_m$, and since $\|u_i\|_{L^\infty[a,b]} \le \frac{b-a}{2}\|u_i\|_m$ and $\|u\|_{L^\infty[a,b]} \le \frac{b-a}{2}\|u\|_m$, it follows that $2\alpha^{-1}(b-a)^{-1}\|T(0)\|_m$ is a bound for both $\|u\|_{L^\infty}$ and $\|u_i\|_{L^\infty}$. By (5.4), and Theorem 3.1, $\|u-s_i\|_{L^\infty[a,b]} \le M_{6,0}(b-a)^{m-\frac{1}{2}}\|u\|_m$, so $\|s_i\|_{L^\infty}$ is also bounded independently of i. Let C denote the maximum of these various bounds for $\|u_i\|_{L^\infty}$, $\|u\|_{L^\infty}$ and $\|s_i\|_{L^\infty}$.

It then follows from (5.22), (5.24), (5.25), (5.9), (5.26) and (5.8) that

$$\alpha\|s_i-u_i\|_m^2 \le |a(s_i,s_i-u_i) - a(u_i,s_i-u_i)|$$

$$= |a(s_i,s_i-u_i) - a(u,s_i-u_i)|$$

$$= \left| M_L(s_i-u, s_i-u_i) + \int_a^b [f(x,u(x)) - f(x,s_i(x))][s_i(x) - u_i(x)]dx \right|$$

$$= \left| \int_a^b [f(x,u(x)) - f(x,s_i(x))][s_i(x) - u_i(x)]dx \right|$$

$$\leq M(C)\|u-s_i\|_{L^2}\|s_i-u_i\|_{L^2}. \qquad (5.27)$$

Therefore by Theorem 4.3,

$$\|s_i-u_i\|_m \leq \alpha^{-1}M(C)\|u-s_i\|_{L^2}$$

$$\leq \alpha^{-1}M(C)M_{3,0}(\bar{\Delta}_i)^t H_0[u]$$

$$= K_3(\bar{\Delta}_i)^t H_0[u], \qquad (5.28)$$

where $K_3 = \alpha^{-1}M(C)M_{3,0}$.

Then using Theorem 4.3 and (5.28),

$$\|D^j(u-u_i)\|_{L^\infty} \leq \|D^j(u-s_i)\|_{L^\infty} + \|D^j(s_i-u_i)\|_{L^\infty}$$

$$\leq M_{5,j}(\bar{\Delta}_i)^{t-j-\frac{1}{2}}H_0[u] + \frac{b-a}{2}K_3(\bar{\Delta}_i)^t H_0[u]$$

$$\leq K_1(\bar{\Delta}_i)^{t-j-\frac{1}{2}}H_0[u] \qquad (5.29)$$

for $0 \leq j \leq m-1$, for some $K_1 > 0$. Also there exists a constant K_2 so that

$$\|D^j(u-u_i)\|_{L^2} \le \|D^j(u-s_i)\|_{L^2} + \|D^j(s_i-u_i)\|_{L^2}$$

$$\le M_{3,j}(\bar{\Delta}_i)^{t-j}H_0[u] + K_3(\bar{\Delta}_i)^t H_0[u]$$

$$\le K_2(\bar{\Delta}_i)^{t-j}H_0[u].$$

This establishes Theorem 5.3. □

Corollary 5.1. With the assumptions of Theorem 5.3, if $u \in W_0^{2m,2}[a,b]$, then the Galerkin approximation u_i to the problem (5.1)-(5.3) in $Sp(M,\Lambda_i)$ satisfies

$$\|D^j(u_i-u)\|_{L^\infty[a,b]} \le K_3(\bar{\Delta}_i)^{2m-j-\frac{1}{2}}\|L^*[u]\|_{L^2[a,b]}, \quad 0\le j\le m-1,$$

and,

$$\|D^j(u_i-u)\|_{L^2[a,b]} \le K_4(\bar{\Delta}_i)^{2m-j}\|L^*[u]\|_{L^2[a,b]}, \quad 0\le j\le m,$$

for constants K_3 and K_4, where L^* is the formal adjoint of L.

Corollary 5.2. With the assumptions of Theorem 5.3, suppose also that the function f in (5.1) is independent of u. Then in addition to the error bounds (5.20) and (5.21),

$$\lambda(u_i) = \lambda(u) \quad \text{for all } \lambda\in\Lambda_i, \quad i\ge 0,$$

that is, the Galerkin approximate u_i to the generalized solution u of (5.1)-(5.3) over $Sp(M,\Lambda_i)$ is a Λ_i-interpolate of u.

Proof. From the next to last line of (5.27), if f is independent of u, $\|s_i - u_i\|_m = 0$, so the Galerkin approximate to u is also the $Sp(M,\Lambda_i)$-interpolate of u. □

Corollary 5.2 generalizes a result of Rose [19] for the case where m = 1 and L is formally selfadjoint.

If the solution to (5.1)-(5.3) is sufficiently smooth, it will sometimes be possible to improve even further on Theorem 5.2. For any integer $q \geq 1$, let

$$M_q(f,g) = M_L(D^{2q}g, D^{2q}f) \quad \text{for all } f,g \in W^{m+2q,2}[a,b]. \tag{5.30}$$

Assuming (5.4) is valid,

$$M_q(u,u) = M_L(D^{2q}u, D^{2q}u) \geq c\|D^{2q}u\|_m^2 \tag{5.31}$$

for all $u \in K^{m,q}[a,b] = \{u \in W^{m+2q,2}[a,b]: D^j u(a) = D^j u(b) = 0$ for all j, $0 \leq j \leq q-1$, $2q \leq j \leq m+2q-1\}$. Let

$$c_1 = \inf_{u \neq 0} \left\{ \frac{\|D^{2q}u\|_m^2}{\|u\|_{m+2q}^2} : u \in K^{m,q}[a,b] \right\}. \tag{5.32}$$

Then by (5.31) and (5.32),

$$M_q(u,u) \geq cc_1\|u\|^2_{m+2q} \quad \text{for all } u\epsilon K^{m,q}[a,b]. \qquad (5.33)$$

Applying Corollary 3.3-i to the bilinear functional $\bar{M}(u,v) = (D^{2q}u, D^{2q}v)_m$ over $W^{m+2q,2}[a,b] \times W^{m+2q,2}[a,b]$, with Λ_0 being a family of continuous linear functionals on $W^{m+2q,2}[a,b]$ with $N(\Lambda_0) = K^{m,q}[a,b]$, it follows that $c_1 > 0$ since for any $u\epsilon K^{m,q}[a,b]$, $\bar{M}(u,u) = 0$ implies that $u = 0$.

Now define Λ_0^q to be the family of continuous linear functionals on $W^{m+2q,2}[a,b]$, $\Lambda_0^q = \Lambda_a \cup \Lambda_b$, where $\Lambda_c = \{\lambda\colon \lambda = (D^j)_c,\ 0 \leq j \leq m+2q-1\}$. Then $N(\Lambda_0^q) \subset K^{m,q}[a,b]$, so for any family of continuous linear functionals Λ on $W^{m+2q,2}[a,b]$ such that $\Lambda \supset \Lambda_0^q$ and $\dim(\mathrm{span}(\Lambda)) < \infty$, it follows that (5.33) holds for all $u\epsilon N(\Lambda) \subset K^{m,q}[a,b]$, and hence the system $\{W^{m+2q,2}[a,b], \Lambda, M_q, N(\Lambda)\}$ is well-posed. Also $Sp(M_q,\Lambda)$ will be finite dimensional. Let $Sp_0(M_q,\Lambda)$ denote the subspace of all $s\epsilon Sp(M_q,\Lambda)$ such that $s\epsilon K^{m,q}[a,b]$, and let $H_0^q[\Lambda]$ denote the subspace of $W_0^{m,2}[a,b]$ such that

$$H_0^q[\Lambda] = \{h\epsilon W^{m,2}[a,b]\colon h = D^{2q}s, \text{ for some } s\epsilon Sp_0(M_q,\Lambda)\}. \qquad (5.34)$$

Then there clearly exists a one to one correspondence between elements of $H_0^q[\Lambda]$ and $Sp_0(M_q,\Lambda)$.

If $\dim(\mathrm{span}(\Lambda)) = d \geq \dim(\mathrm{span}(\Lambda_0^q)) = 2m + 4q$, then by Corollary 2.2, $\dim(Sp(M_q,\Lambda)) = d$, and so $\dim(H_0^q[\Lambda]) = \dim(Sp_0(M_q,\Lambda)) = d-2m-2q \geq 2q$. We are now in a position to state:

Theorem 5.4. Suppose the boundary value problem (5.1)-(5.3) satisfies all of the assumptions of Theorem 5.2, and in addition the generalized solution u is such that $u \epsilon W^{2n,2}[a,b]$ where $n = m+q$, $q \geq 1$, and the coefficients of M_L, σ_{ij}, are such that $\sigma_{ij} \epsilon C^{i+2q}[a,b]$, $0 \leq i, j \leq m$. If $\{\Lambda_i\}_{i=0}^{\infty}$ is a nested sequence of finite dimensional families of continuous linear functionals on $W^{m+2q,2}[a,b]$ such that $\Lambda_0 \supset \Lambda_0^q = \Lambda_a \cup \Lambda_b$, where $\Lambda_c = \{\lambda: \lambda = (D^j)_c,\ 0 \leq j \leq m+2q-1\}$, then for all $i \geq 0$, there is a unique Galerkin approximate to u, $u_i \epsilon H_0^q[\Lambda_i]$, which satisfies the error bounds

$$\|D^j(u_i - u)\|_{L^\infty[a,b]} \leq K_1(\bar{\Delta}_i)^{2n-j-\frac{1}{2}} \|D^{2q}L^*[u]\|_{L^2[a,b]}, \quad 0 \leq j \leq m-1, \tag{5.35}$$

and

$$\|D^j(u_i - u)\|_{L^2[a,b]} \leq K_2(\bar{\Delta}_i)^{2n-j} \|D^{2q}L^*[u]\|_{L^2[a,b]}, \quad 0 \leq j \leq m. \tag{5.36}$$

<u>Proof</u>. The existence of u and u_i follows directly from Theorem 5.2. To establish the error bounds (5.35) and (5.36), let $\psi \epsilon W^{2n+2q,2}[a,b]$ be the unique solution to the boundary value problem

$$D^{2q}\psi = u, \quad \text{where } D^j\psi(a) = D^j\psi(b) = 0, \quad 0 \leq j \leq q-1, \qquad 5.37)$$

Then since $u \epsilon W_0^{m,2}[a,b]$, and ψ satisfies (5.37), $\psi \epsilon K^{m,q}[a,b]$. In a similar fashion let $\psi_i \epsilon K^{m,q}[a,b]$ be the unique solution to the boundary value problem

$$D^{2q}\psi_i = u_i, \; D^j\psi_i(a) = D^j\psi_i(b) = 0, \; 0 \leq j \leq q-1, \; i \geq 0. \tag{5.38}$$

Finally, let s_i be the unique $Sp(M_q,\Lambda_i)$-interpolate of ψ. Then $s_i \in Sp_0(M_q,\Lambda_i)$ since $\Lambda_i \supset \Lambda_0^q$ and $\psi \in K^{m,q}[a,b]$. Let $h_i = D^{2q}s_i$, all $i \ge 0$. Then $h_i \in H_0^q[\Lambda_i]$. Proceeding as in Theorem 5.3, we shall now determine an error bound on $\|h_i - u_i\|_m$.

It was shown in Theorem 5.3 that there exist a priori bounds for u and u_i in both $\|\cdot\|_m$ and $\|\cdot\|_{L^\infty}$. By Theorem 3.1, $\|h_i - u\|_{L^\infty} = \|D^{2q}(s_i - \psi)\|_{L^\infty}$ $\le M_{6,2q}(b-a)^{m-\frac{1}{2}}\|\psi\|_{m+2q}$, and by (5.32), $\|\psi\|_{m+2q} \le c_1^{-.5}\|u\|_m$. Therefore there exists an a priori bound C for $\|u_i\|_{L^\infty}$, $\|u\|_{L^\infty}$ and $\|h_i\|_{L^\infty}$.

Again, inequalities (5.22) and (5.23) are valid for the quasi-bilinear form a(u,v), and also

$$a(u,v) = 0 \quad \text{for all } v \in W_0^{m,2}[a,b], \tag{5.39}$$

$$a(u_i,v) = 0 \quad \text{for all } v \in H_0^q[\Lambda_i], \text{ and} \tag{5.40}$$

$$M_L(D^{2q}n, D^{2q}s) = M_q(s,n) = 0 \text{ for all } n \in N(\Lambda_i),\ s \in Sp(M_q,\Lambda_i). \tag{5.41}$$

It now follows from (5.22), (5.39), (5.40), (5.9), (5.41) and (5.8) that

$$\alpha\|h_i - u_i\|_m^2 \le |a(h_i,h_i-u_i) - a(u_i,h_i-u_i)|$$

$$= |a(h_i,h_i-u_i) - a(u,h_i-u_i)|$$

$$= \left|M_L(h_i-u,h_i-u_i) + \int_a^b [f(x,u(x)) - f(x,h_i(x))][h_i(x) - u_i(x)]dx\right|$$

$$= \Big| M_L(D^{2q}(s_i-\psi), D^{2q}(s_i-\psi_i))$$

$$+ \int_a^b [f(x,u(x)) - f(x,h_i(x))][h_i(x) - u_i(x)]dx \Big|$$

$$\le M(C)\|u-h_i\|_{L^2}\|h_i-u_i\|_{L^2}. \qquad (5.42)$$

Therefore by Corollary 4.1 (on $W^{m+2q}[a,b]$ with $r = m+2q$),

$$\|h_i-u_i\|_m \le \alpha^{-1}M(C)\|u-h_i\|_{L^2}$$

$$= \alpha^{-1}M(C)\|D^{2q}(\psi-s_i)\|_{L^2}$$

$$\le \alpha^{-1}M(C)M_{3,2q}M_{3,0}(\bar{\Delta}_i)^{2(m+2q)-2q}\|D^{2q}L^*[u]\|_{L^2[a,b]}$$

$$= K_3(\bar{\Delta}_i)^{2n}\|D^{2q}L^*[u]\|_{L^2[a,b]}, \qquad (5.43)$$

where $K_3 = \alpha^{-1}M(C)M_{3,2q}M_{3,0}$, and L^* is the formal adjoint of L (cf. (5.2)).

Since $\|D^j(u_i-h_i)\|_{L^\infty} \le \frac{(b-a)}{2}\|u_i-h_i\|_m$ for $0\le j\le m-1$, the error bounds (5.35) and (5.36) now follow directly from (5.43) and Corollary 4.1, by use of the triangle inequality:

$$\|D^j(u_i-u)\|_{L^\infty} \le \|D^j(u_i-h_i)\|_{L^\infty} + \|D^j(h_i-u)\|_{L^\infty}$$

$$\le \left(\frac{b-a}{2}\right)\|u_i-h_i\|_m + \|D^{j+2q}(s_i-\psi)\|_{L^\infty}$$

$$\leq \left[\left(\frac{b-a}{2}\right)K_3(\bar{\Delta}_i)^{2n} + M_{5,j}M_{3,0}(\bar{\Delta}_i)^{2(m+2q)-j-2q-\frac{1}{2}}\right] \|D^{2q}L^*[u]\|_{L^2}$$

$$\leq [\tfrac{1}{2} K_3(b-a)^{j+1.5} + M_{5,j}M_{3,0}][\bar{\Delta}_i)^{2n-j-\frac{1}{2}}\|D^{2q}L^*[u]\|_{L^2},$$

for $0 \leq j \leq m-1$, and hence (5.35) is valid with $K_1 = \max_{0\leq j\leq m-1} [\tfrac{1}{2}K_3(b-a)^{j+1.5} + M_{5,j}M_{3,0}]$. The error bound (5.36) follows in a similar fashion using (4.14) of Corollary 4.1. □

Corollary 5.3. In addition to the assumptions of Theorem 5.4, suppose the function f in (5.1) is independent of u. Then the error bounds (5.35) and (5.36) will hold, and also if any $\lambda \in \mathrm{span}(\Lambda_i)$ is of the form $\lambda = (D^{p+2q})_c$, with $c \in [a,b]$ and p an integer, $0 \leq p \leq m-1$, then

$$D^p u_i(c) = D^p u(c),$$

that is, the pth derivative of the Galerkin approximate u_i to the generalized solution u of (5.1)-(5.3) is equal to $D^p u$ at c.

Proof. From the next to last line in (5.42), if f is independent of u, $\|h_i - u_i\|_m = 0$. Therefore $h_i = u_i$ all $i \geq 0$, and thus $\psi_i = s_i$, $i \geq 0$. Hence ψ_i is the $Sp(M_q, \Lambda_i)$-interpolate of ψ and if $\lambda = (D^{p+2q})_c \in \mathrm{span}(\Lambda_i)$ for some $i \geq 0$, then $D^{p+2q}\psi_i(c) = D^{p+2q}\psi(c)$, and $D^p u_i(c) = D^p u(c)$. □

We remark that higher order convergence results such as are given by (5.35) and (5.36) were first found by Hulme [14] in the linear case for $L \equiv D^{2m}$ and by Perrin, Price and Varga [18] for the nonlinear case when $L \equiv L_1^* L_1$, with $L_1 \equiv \sum_{i=1}^{m} a_i D^i$. The method of relating $H_0^q[\Lambda]$ to $Sp_0(M_q,\Lambda)$ used above is a generalization of a similar approach used by Perrin, Price and Varga [18]. The next result generalizes and improves Theorem 5 of [18].

Theorem 5.5. With the assumptions i, ii, iii, iv and v, let $u \in W_0^{m,2}[a,b]$ be the unique generalized solution to the two-point non-selfadjoint, nonlinear boundary value problem (5.1)-(5.3), and suppose the bilinear from M_L of (5.5) can be expressed as

$$M_L(u,v) = M(u,v) + \sum_{(i,j)\in I_k} \int_a^b c_{i,j}(x)D^i u(x)D^j v(x)dx, \qquad (5.44)$$

where

$$M(u,v) = \sum_{0\le i,j\le m} \int_a^b b_{ij}(x)D^i u(x)D^j v(x)dx, \text{ for any } u,v\in W^{m,2}[a,b],$$

where I_k is an index set consisting of pairs (i,j) with $0\le i,j\le m$, and $i+j\le k$. Assume there is a positive constant c_2 such that $M(u,u) \ge c_2\|u\|_m^2$ for all $u\in W_0^{m,2}[a,b]$. Assume also for some $n\ge m$, $u\in W^{2n,2}[a,b]$, $b_{ij}\in C^{i+2q}[a,b]$, $0\le i,\ j\le m$, with $q = n-m$, $c_{ij}\in C^i[a,b]$, for all $(i,j)\in I_k$, and $\{\Lambda_i\}_{i=0}^{\infty}$ satisfies the hypotheses of Theorem 5.4. Let $H_0^q[\Lambda_i]$ be defined from elements $s\in Sp_0(M_q,\Lambda_i)$ by (5.34), where

$$M_q(u,v) = M(D^{2q}v,D^{2q}u) \text{ for } u,v\in W^{m+2q,2}[a,b].$$

Let u_i be the unique Galerkin approximate of u over the subspaces $H_0^q[\Lambda_i]$. Then there exist constants K_1 and K_2 such that for all $i\ge 0$,

$$\|D^j(u_i-u)\|_{L^\infty[a,b]} \le K_1(\bar{\Delta}_i)^{2n-\max(\delta,j+\frac{1}{2})},\ 0\le j\le m-1, \tag{5.45}$$

where $\delta = \max(0,k-m)$, and

$$\|D^j(u_i-u)\|_{L^2[a,b]} \le K_2(\bar{\Delta}_i)^{2n-\max(\delta,j)},\ 0\le j\le m. \tag{5.46}$$

Proof. The system $\{W^{m+2q,2}[a,b],\Lambda_i,M_q,N(\Lambda_i)\}$ is well-posed for all $i\ge 0$ since $N(\Lambda_i) \subset K^{m,q}[a,b]$ and

$$M_q(u,u) = M(D^{2q}u,D^{2q}u) \ge c_2\|D^{2q}u\|_m^2$$

$$\ge c_1c_2\|u\|_{m+2q}^2 \text{ for all } u\in K^{m,q}[a,b],$$

where c_1 is given by (5.32). As in the proof of Theorem 5.4, let ψ and ψ_i for $i\ge 0$ be given as the solutions to (5.37) and (5.38), and let s_i be the unique $Sp(M_q,\Lambda_i)$-interpolate of ψ. Then $s_i\in Sp_0(M_q,\Lambda_i)$ and $h_i = D^{2q}s_i\in H_0^q[\Lambda_i]$.

Arguing as in earlier theorems

$$\alpha\|h_i-u_i\|_m^2 \le |a(h_i,h_i-u_i) - a(u_i,h_i-u_i)|$$

$$= |a(h_i,h_i-u_i) - a(u,h_i-u_i)|$$

$$= |M_L(h_i-u,h_i-u_i)$$

$$+ \int_a^b [f(x,u(x)) - f(x,h_i(x))][h_i(x)-u_i(x)]dx|$$

$$\leq |M_q(s_i-\psi_i, s_i-\psi) + \sum_{(\ell,j)\in I_k} \int_a^b c_{\ell j} D^\ell(h_i-u)D^j(h_i-u_i)dx$$

$$+ \int_a^b [f(x,u(x)) - f(x,h_i(x)))][h_i(x) - u_i(x)]dx|. \quad (5.47)$$

Since $c_{\ell j}\epsilon C^\ell[a,b]$, we can perform the following integration by parts:

$$\int_a^b c_{\ell j} D^\ell(h_i-u)D^j(h_i-u_i)dx$$

$$= (-1)^p \int_a^b D^{\ell-p}(h_i-u)D^p(c_{\ell j}D^j(h_i-u_i))dx \quad (5.48)$$

where $p\leq\ell$ and $j+p\leq m$. Thus p can be chosen to be $\min(\ell,m-j)$ and we see that if $(\ell,j)\epsilon I_k$, $|\int_a^b c_{\ell j}D^\ell(h_i-u)D^j(h_i-u_i)dx|$ can be bounded by a sum of terms J of the form

$$J = |\int_a^b (D^{\sigma_1}(h_i-u))(D^{\sigma_2}c_{\ell j})(D^{\sigma_3}(h_i-u_i))dx|, \quad (5.49)$$

where $0\leq\sigma_1 = \ell-\min(\ell,m-j) = \max(0,\ell+j-m) \leq \delta$, $0\leq\sigma_2\leq\ell$ and $0\leq\sigma_3\leq m$. Applying Schwartz's inequality to (5.49), and using (4.14) of Corollary 4.1, we find that

$$J \leq \|D^{\sigma_2}c_{\ell j}\|_{L^\infty} \|D^{\sigma_1}(h_i-u)\|_{L^2}\|D^{\sigma_3}(h_i-u_i)\|$$

$$\leq K_3\|D^{\sigma_1+2q}(s_i-\psi)\|_{L^2} \|h_i-u_i\|_m$$

$$\leq K_4(\bar{\Delta}_i)^{2n-\delta}\|h_i-u_i\|_m$$

for constants K_3 and K_4. Applying inequalities of the above type to (5.47), and using (5.8),

$$\alpha\|h_i-u_i\|_m^2 \leq K_5(\bar{\Delta}_i)^{2n-\delta}\|h_i-u_i\|_m + M(C)K_6(\bar{\Delta}_i)^{2n}\|h_i-u_i\|_m,$$

so

$$\|h_i-u_i\|_m \leq K_7(\bar{\Delta}_i)^{2n-\delta}, \tag{5.50}$$

for constants K_5, K_6 and K_7 which are independent of i, where C is an a priori bound for $\|u\|_{L}$ and $\|h_i\|_{L^\infty}$. The error bounds (5.45) and (5.46) now follow by use of the triangle inequality, (5.50) and Corollary 4.1. □

Theorem 5.5 not only gives results of very high order accuracy, but when $k \leq m$, gives accuracy comparable to that of Theorem 5.4, losing only the interpolation properties of Corollary 5.3 for the linear case. Theorem 5.2 may be regarded in part as a special case of Theorem 5.5 with $k = 2m$, $t = 2n$ and $\delta = m$. Theorem 5.5 can give subspaces with simple bases when M contains only a few terms, such as $M(u,v) \equiv \int_a^b D^m u D^m v dx$.

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VITA

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