# Pullback Attracting Inertial Manifolds for Nonautonomous Dynamical Systems

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**Abstract.** In this paper we present an abstract approach to inertial manifolds for nonautonomous dynamical systems. Our result on the existence of inertial manifolds requires only two geometrical assumptions, called cone invariance and squeezing property, and two additional technical assumptions, called boundedness and coercivity property. Moreover we give conditions which ensure that the global pullback attractor is contained in the inertial manifold.

In the second part of the paper we consider special nonautonomous dynamical systems, namely two-parameter semi-flows. As a first application of our abstract approach and for reason of comparison with known results we verify the assumptions for semilinear nonautonomous evolution equations whose linear part satisfies an exponential dichotomy condition and whose nonlinear part is globally bounded and globally Lipschitz.

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# 1 Introduction

Let us consider a dissipative nonlinear evolution equation of the form

$$\dot{u} + Au = f(u)$$

in a Banach space $\mathcal{X}$, where $A$ is a linear sectorial operator with compact resolvent and $f$ is a nonlinear function. Such an evolution equation may be an ordinary differential equation ($\mathcal{X} = \mathbb{R}^n$) or the abstract formulation of a semilinear parabolic differential equation with $\mathcal{X}$ as a suitable function space over the spatial domain. In the last case, $A$ corresponds to a linear differential operator and $f$ is a nonlinearity which may involve derivatives of lower order than $A$.

The dissipativity finds its expression in the fact that there is a bounded set $\hat{\mathcal{A}}$ in $\mathcal{X}$ which absorbs bounded subsets of the phase space after a finite time. Given a compact set $\hat{\mathcal{A}}$ which is absorbing and positively invariant with respect to the semi-flow, the $\omega$-limit set of $\hat{\mathcal{A}}$ under the semi-flow has some interesting properties: There is a compact, invariant set $\mathcal{A}$ in $\hat{\mathcal{A}}$ which attracts all trajectories. Such a set $\mathcal{A}$ is called the *global attractor* [Tem97], and it is maximal in the sense that it is the largest set with this property. Often global attractors have a finite (fractal) dimension and, therefore, they are important objects for the study of long time behavior. At the present level of understanding of dynamical systems, the attractors are expected to be very complicated objects (fractals) and their practical utilization, for instance for numerical simulations, may be difficult. Because of the finite dimensionality of the attractor, it is a natural question if the attractor could be generated by a (possibly high dimensional) system of ordinary differential equations, i.e. if the flow on the attractor could be generated by a system of ordinary differential equations. An indirect way to obtain such a system is to embed the attractor $\mathcal{A}$ into a smooth finite dimensional manifold.

Inertial manifolds are positively invariant, exponentially attracting, finite dimensional Lipschitz manifolds. They go back to D. Henry and X. Mora [Hen81, Mor83] and were first introduced and studied by P. Constantin, C. Foias, B. Nicoalenko, G.R. Sell and R. Temam [FST85, FNST85, CFNT86] for selfadjoint $A$. For the construction of inertial manifolds with $A$ being non-selfadjoint see for example [SY92] and [Tem97]. Inertial manifolds are generalizations of center-unstable manifolds and they are more convenient objects which capture the long-time behavior of dynamical systems. If such a manifold exists, then it contains the global attractor $\mathcal{A}$. Usually an inertial manifold $\mathcal{M}$ is seeked as the graph of a sufficiently smooth function $m$ on $P\mathcal{X}$, i.e.

$$\mathcal{M} = \text{graph}(m) := \{x + m(x) : \ x \in P\mathcal{X}\} ,$$

where $P$ is a finite dimensional projector. The finite dimensionality and the exponential attracting property permit the reduction of the dynamics of the infinite or high dimensional equation to the dynamics of a finite or low dimensional ordinary differential equation

$$\dot{x} + Ax = Pf(x + m(x)) \quad \text{in } P\mathcal{X}$$

called inertial form system. A stronger reduction property is the asymptotical completeness property [Rob96]: Each trajectory of the evolution equation tends exponentially to a trajectory in the inertial manifold.

Thus, one can reduce (in some sense) an infinite-dimensional physical system to a system with a finite number of degrees of freedom and use known properties of ordinary differential equations in $\mathbb{R}^n$ for the qualitative analysis of the behavior of the solutions of the original equation for large time. Another advantage of studying the inertial form system is the reduction of the dimension of the original problem which could be important for the numerical analysis. In concrete applications, the $N$-dimensional projector $P$ is the projector onto the eigenspace spanned by the first $N$ Fourier modes of $L$, i.e. by eigenvectors of $A$ belonging to the set of the $N$ smallest eigenvalues of $A$. Hence the inertial form systems describes the dynamics of the slow modes. For applications of the concept of inertial manifolds see for example [RB95, SK95, BH96, MSZ00].

There are a few ways of constructing an inertial manifold. Most of them are generalization of methods developed for the construction of unstable, center-unstable or center manifolds for ordinary differential equations. Following the introduction of [Wig94], there are at least the following two basic methods:

The Lyapunov-Perron method has been developed by A.M. Lyapunov [Lya47, Lya92] and O. Perron [Per28, Per29, Per30] for the proof of the existence of stable and unstable manifolds of hyperbolic equilibrium points. In the context of ordinary differential equations, it deals with the integral equation formulation of the differential equation and constructs the invariant manifold as a fixed point of an operator that is derived from this integral equation. This method has been used in many different situations. The book of J. Hale [Hal80] may serve as a good reference. In the infinite dimensional setting, the method is used by D. Henry [Hen81] to prove the existence of stable, unstable and center manifolds for semilinear parabolic equations. In [FST88, Tem88, CFNT89b, FST89, DG91], it has been adapted for the construction of inertial manifolds. At the moment, the Lyapunov-Perron method is the most common method for inertial manifolds.

Hadamard's method [Had01], also called graph transformation method, has been developed to prove the existence of stable and unstable manifolds of fixed points of diffeomorphisms. The graph transformation method is more geometrical in nature than the Lyapunov-Perron method. In presence of a hyperbolic fixed point, the stable and unstable manifolds are constructed as graphs over the linearized stable and unstable subspaces. For the infinite dimensional case, we refer to [BJ89]. A review on different construction methods (including Hadamard's method) for inertial manifolds can be found in [LS89, Nin93].

The above mentioned notion of inertial manifolds is translated and extended to more general classes of differential equations like nonautonomous differential equations, [GV97, WF97, LL99], retarded parabolic differential equations, [TY94, BdMCR98], or differential equations with random or stochastic perturbations, [Chu95, BF95, CL99, CS01, DLS01].

The construction of inertial manifolds often is redone for different classes of equations. Our aim is to separate the general structure of the construction from the technical estimates which vary from example to example. To achieve this goal we develop the existence result of inertial manifolds on the abstract level of nonautonomous dynamical systems and then apply it to explicit nonautonomous evolution equations under various assumptions.

In Sec. 2 we introduce the abstract notion of a nonautonomous dynamical system extending dynamical systems to the nonautonomous situation. For a nonautonomous dynamical system we introduce inertial manifolds and gloabl pullback attractors. We formulate a cone invariance, squeezing, boundedness and coercivity property which are assumed in Subsection 3.4 where the existence of an inertial manifold is proved. Moreover we give conditions which imply that the global pullback attractor is contained in the inertial manifold. The assumptions of the existence result are formulated on the general level of nonautonomous dynamical systems and are independent of a specific equation which generates the nonautonomous dynamical system.

In Sec. 3 we apply it to special nonautonomous dynamical systems, namely two-parameter semi-flows. As a first application we verify the assumptions for nonautonomous semilinear evolution equations whose linear part satisfies an exponential dichotomy condition and whose nonlinear part is globally bounded and globally Lipschitz under the assumption that a special algebraic problem is solvable. For this we develop and apply comparison theorems. The algebraic problem is solvable if the exponential rates in the exponential dichotomy condition satisfy a gap condition whose form depends on the used Lipschitz inequality and the concrete form of the exponential dichotomy condition. Finally we compare our results with known ones.

Applications to nonautonomous dynamical systems which are generated by stochastic or retarded partial differential equations are subject of a forthcoming paper.

# 2 Nonautonomous Dynamical Systems

## 2.1 Preliminaries

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space.

**Definition 2.1 (Nonautonomous Dynamical System (NDS)).** A *nonautonomous dynamical system* (NDS) on $\mathcal{X}$ is a *cocycle* $\varphi$ over a *driving system* $\theta$ on a set $\mathcal{B}$, i.e.

(i) $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ is a *dynamical system*, i.e. the family $\theta(t, \cdot) = \theta(t) : \mathcal{B} \to \mathcal{B}$ of self-mappings of $\mathcal{B}$ satisfies the *group property*

$$\theta(0) = \mathrm{id}_{\mathcal{B}}\,, \quad \theta(t+s) = \theta(t) \circ \theta(s)$$

for all $t, s \in \mathbb{R}$.

(ii) $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$ is a *cocycle*, i.e. the family $\varphi(t, b, \cdot) = \varphi(t, b) : \mathcal{X} \to \mathcal{X}$ of self-mappings of $\mathcal{X}$ satisfies the *cocycle property*

$$\varphi(0, b) = \mathrm{id}_{\mathcal{X}}\,, \quad \varphi(t + s, b) = \varphi(t, \theta(s)b) \circ \varphi(s, b)$$

for all $t, s \geq 0$ and $b \in \mathcal{B}$. Moreover $(t, x) \mapsto \varphi(t, b, x)$ is continuous.

*Remark* 2.2. (i) The set $\mathcal{B}$ is called *base* and in applications it has additional structure, e.g. it is a probability space, a topological space or a compact group and the driving system has additional regularity, e.g. it is ergodic or continuous.

(ii) The pair of mappings

$$(\theta, \varphi) : \mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X} \to \mathcal{B} \times \mathcal{X}\,, \quad (t, b, x) \mapsto (\theta(t, b), \varphi(t, b, x))$$

is a special semi-dynamical system a so-called *skew product flow* (usually one requires additionally that $(\theta, \varphi)$ is continuous). If $\mathcal{B} = \{b\}$ consists of one point then the cocycle $\varphi$ is a semi-dynamical system.

(iii) We use the abbreviations $\theta_t b$ or $\theta(t)b$ for $\theta(t, b)$ and $\varphi(t, b)x$ for $\varphi(t, b, x)$. We also say that $\varphi$ is an NDS to abbreviate the situation of Definition 2.1.

**Definition 2.3 (Nonautonomous Set).** A family $\mathcal{M} = (\mathcal{M}(b))_{b \in \mathcal{B}}$ of non-empty sets $\mathcal{M}(b) \subset \mathcal{X}$ is called a *nonautonomous set* and $\mathcal{M}(b)$ is called the *b-fiber* of $\mathcal{M}$ or the *fiber* of $\mathcal{M}$ over $b$. We say that $\mathcal{M}$ is *closed*, *open*, *bounded*, or *compact*, if every fiber has the corresponding property. For notational convenience we use the identification $\mathcal{M} \simeq \{(b, x) : b \in \mathcal{B},\ x \in \mathcal{M}(b)\} \subset \mathcal{B} \times \mathcal{X}$.

**Definition 2.4 (Invariance of Nonautonomous Set).** A nonautonomous set $\mathcal{M}$ is called *forward invariant* under the NDS $\varphi$, if $\varphi(t, b)\mathcal{M}(b) \subset \mathcal{M}(\theta_t b)$ for $t \geq 0$ and $b \in \mathcal{B}$. It is called *invariant*, if $\varphi(t, b)\mathcal{M}(b) = \mathcal{M}(\theta_t b)$ for $t \geq 0$ and $b \in \mathcal{B}$.

**Definition 2.5 (Inertial Manifold).** Let $\varphi$ be an NDS. Then a nonautonomous set $\mathcal{M}$ is called (nonautonomous) *inertial manifold* if

(i) every fiber $\mathcal{M}(b)$ is a *finite-dimensional Lipschitz manifold* in $\mathcal{X}$ of dimension $N$ for an $N \in \mathbb{N}$;

(ii) $\mathcal{M}$ is *invariant*;

(iii) $\mathcal{M}$ is *exponentially attracting*, i.e. there exists a positive constant $\eta$ such that for every $b \in \mathcal{B}$ and $x \in \mathcal{M}(b)$ there exists an $x' \in \mathcal{M}(b)$ with

$$\|\varphi(t, b)x - \varphi(t, b)x'\| \leq K\mathrm{e}^{-\eta t} \qquad \text{for } t \geq 0 \text{ and } b \in \mathcal{B}$$

and a constant $K = K(b, x, x') > 0$.

The property (iii) is also called *exponential tracking property* or *asymptotic completeness property* and $x'$ or $\varphi(\cdot, b)x'$ is said to be the *asymptotic phase* of $x$ or $\varphi(\cdot, b)x$, respectively.

Recall that if $\mathcal{D}$ and $\mathcal{A}$ are nonempty closed sets in $\mathcal{X}$, the Hausdorff semi-metric $d(\mathcal{D}|\mathcal{A})$ is defined by

$$d(\mathcal{D}|\mathcal{A}) := \sup_{x\in\mathcal{D}} d(x,\mathcal{A})\,, \quad d(x,\mathcal{A}) := \inf_{y\in\mathcal{A}} d(x,y) = \inf_{y\in\mathcal{A}} \|x-y\|\,.$$

Now we want to generalize the attractor notion of dynamical systems to nonautonomous dynamical systems. A natural generalization of convergence to a nonautonomous set $\mathcal{A}$ seems to be the *forwards running convergence* defined by

$$d(\varphi(t,b)x, \mathcal{A}(\theta_t b)) \to 0 \qquad \text{for } t\to\infty\,.$$

However, this does not ensure convergence to a specific component set $\mathcal{A}(b)$ for a fixed $b$. For that one needs to start "progressively earlier" at $\theta_{-t}b$ in order to "finish" at $b$. This leads to the concept of *pullback convergence* defined by

$$d(\varphi(t,\theta_{-t}b)x, \mathcal{A}(b)) \to 0 \qquad \text{for } t\to\infty\,.$$

Using this we can define a *pullback attractor.*

**Definition 2.6 (Pullback Attracting, Absorbing, Attractor).** Let $\varphi$ be an NDS and $\mathcal{A}$ be a nonautonomous set.

(i) $\mathcal{A}$ is called *(globally) pullback attracting* if for every bounded set $\mathcal{D}\subset\mathcal{X}$ and $b\in\mathcal{B}$

$$\lim_{t\to\infty} d\left(\varphi(t,\theta_{-t}b)\mathcal{D}|\mathcal{A}(b)\right) = 0\,.$$

(ii) $\mathcal{A}$ is called *(globally) pullback absorbing* if $\mathcal{A}$ is bounded and for every bounded set $\mathcal{D}\subset\mathcal{X}$ and $b\in\mathcal{B}$ there is a $T>0$ with

$$\varphi(t,\theta_{-t}b)\mathcal{D}\subset\mathcal{A}(b) \qquad \text{for all } t\geq T\,.$$

(iii) $\mathcal{A}$ is called *(global) pullback attractor* of $\varphi$ if $\mathcal{A}$ is compact, invariant and pullback attracting.

The concept of pullback convergence was introduced in the mid 1990s in the context of random dynamical systems (see Crauel and Flandoli [CF94], Flandoli and Schmalfuss [FS96], and Schmalfuss [Sch92]) and has been used e.g. in numerical dynamics. Note that a similar idea had already been used in the 1960s by Mark Krasnoselski [Kra68] to establish the existence of solutions that exist and remain bounded on the entire time set.

Now we define a handy notion (see Ludwig Arnold [Arn98, Definition 4.1.1(ii)]) excluding exponential growth of a function.

**Definition 2.7 (Temperedness).** A function $R : \mathcal{B} \to ]0,\infty[$ is called *tempered from above* if for every $b \in \mathcal{B}$

$$\limsup_{t\to\pm\infty} \frac{1}{|t|} \log R(\theta_t b) = 0 \,.$$

Note that the following characterization holds.

**Corollary 2.8.** *Suppose that $R : \mathcal{B} \to ]0,\infty[$ is a nonautonomous variable. Then the following statements are equivalent:*

*(i) $R$ is tempered from above.*

*(ii) For every $\varepsilon > 0$ and $b \in \mathcal{B}$ there exists a $T > 0$ such that*

$$R(\theta_t b) \le \mathrm{e}^{\varepsilon|t|} \qquad \text{for } |t| \ge T \,.$$

**Definition 2.9 (Nonautonomous Projector).** Let $\varphi$ be an NDS. A family $\pi = (\pi(b))_{b\in\mathcal{B}}$ of projectors $\pi(b) \in L(\mathcal{X},\mathcal{X})$ in $\mathcal{X}$ is called *nonautonomous projector.*

(i) $\pi$ is called *tempered from above* if $b \mapsto \|\pi(b)\|$ is tempered from above.

(ii) $\pi$ is called *$N$-dimensional* for an $N \in \mathbb{N}$ if $\dim \operatorname{im}\pi(b) = N$ for every $b \in \mathcal{B}$.

## 2.2 Inertial Manifolds for Nonautonomous Dynamical Systems

Let $\pi_1$ be an $N$-dimensional nonautonomous projector in $\mathcal{X}$. We define the *complementary projector*

$$\pi_2(b) := \mathrm{id}_{\mathcal{X}} - \pi_1(b) \qquad \text{for } b \in \mathcal{B} \,.$$

Then

$$\mathcal{X}_1(b) := \pi_1(b)\mathcal{X} \quad \text{and} \quad \mathcal{X}_2(b) := \pi_2(b)\mathcal{X} \,, \qquad b \in \mathcal{B}$$

define nonautonomous sets $\mathcal{X}_i$ consisting of complementary linear subspaces $\mathcal{X}_i(b)$ of $\mathcal{X}$, i.e. $\mathcal{X}_1(b) \oplus \mathcal{X}_2(b) = \mathcal{X}$. For this fact we also write $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{B} \times \mathcal{X}$.

We want to construct a nonautonomous inertial manifold

$$\mathcal{M} = (\mathcal{M}(b))_{b\in\mathcal{B}}$$

consisting of manifolds $\mathcal{M}(b)$ which are trivial in the sense that each of them can be described by a single chart, i.e.

$$\mathcal{M}(b) = \mathrm{graph}(m(b,\cdot)) := \{x_1 + m(b,x_1) : \; x_1 \in \mathcal{X}_1(b)\}$$

with $m(b,\cdot) = m(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b)$.

For a positive constant $L$ we introduce the nonautonomous set

$$\mathcal{C}_L := \{(b,x) \in \mathcal{B} \times \mathcal{X} : \; \|\pi_2(b)x\| \le L\|\pi_1(b)x\|\} \,.$$

Since the fibers $\mathcal{C}_L(b)$ are cones it is called *(nonautonomous) cone.*

**Definition 2.10 (Cone Invariance).** The NDS $\varphi$ satisfies the (nonautonomous) *cone invariance property* for a cone $\mathcal{C}_L$ if there is a $T_0 \geq 0$ such that for $b \in \mathcal{B}$ and $x, y \in \mathcal{X}$,

$$x - y \in \mathcal{C}_L(b)$$

implies

$$\varphi(t, b)x - \varphi(t, b)y \in \mathcal{C}_L(\theta_t b) \qquad \text{for } t \geq T_0 \,.$$

Now we define a property of a cocycle $\varphi$ which describes a kind of squeezing outside a given cone.

**Definition 2.11 (Squeezing Property).** The NDS $\varphi$ satisfies the (nonautonomous) *squeezing property* for a cone $\mathcal{C}_L$ if there exist positive constants $K_1$, $K_2$ and $\eta$ such that for every $b \in \mathcal{B}$, $x, y \in \mathcal{X}$ and $T > 0$ the identity

$$\pi_1(\theta_T b)\varphi(T, b)x = \pi_1(\theta_T b)\varphi(T, b)y$$

implies for all $x' \in \mathcal{X}$ with $\pi_1(b)x' = \pi_1(b)x$ and $x' - y \in \mathcal{C}_L(b)$ the estimates

$$\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)y]\| \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)[x - x']\| \,, \quad i = 1, 2 \,,$$

for $t \in [0, T]$.

*Remark* 2.12. We consider the special case that $\mathcal{B} = \{b\}$ is a singleton. Then the NDS $\varphi$ defines a semiflow $S$ on $\mathcal{X}$ by $S^t x = \varphi(t, b)x$ for $(t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{X}$ and we may replace $\pi_i(b)$ by $\pi_i$. As a special case of the cone invariance property we obtain the

> *Autonomous Cone Invariance Property:* There exists $T_0 \geq 0$ such that $x, y \in \mathcal{X}$ with
>
> $$\|\pi_2[x - y]\| \leq L\|\pi_1[x - y]\|$$
>
> implies
>
> $$\|\pi_2[S^t x - S^t y]\| \leq L\|\pi_1[S^t x - S^t y]\| \qquad \text{for } t \geq T_0 \,.$$

The squeezing property in the autonomous case becomes the following

> *Autonomous Squeezing Property:* There exist positive constants $K_1$, $K_2$, $\eta$ such that for every $x, y \in \mathcal{X}$ and $T > 0$ the identity
>
> $$\pi_1 S^T x = \pi_1 S^T y$$
>
> implies for all $x' \in \mathcal{X}$ with $\pi_1 x' = \pi_1 x$ and $\|\pi_2[x' - y]\| \leq L\|\pi_1[x' - y]\|$ the estimates
>
> $$\|\pi_i[S^t x - S^t y]\| \leq K_i \mathrm{e}^{-\eta t} \|\pi_2[x - x']\| \,, \quad i = 1, 2 \,,$$
>
> for $t \in [0, T]$.

A combination of cone invariance and squeezing properties for evolution equations, sometimes called strong squeezing property, was first introduced for the Kuramoto-Sivashinsky equations in [FNST85, FNST88], an abstract version of it was developed in [FST89], another formulation of it can be found for example in [Tem88, FST88, CFNT89a, Rob93, JT96]. Essentially, a strong squeezing property states that if the difference of two solutions of the evolution equation belongs to a special cone then it remains in the cone for all further times (that is the cone invariance property); otherwise the distance between the solutions decays exponentially (that is the squeezing property).

Our autonomous cone invariance property with $T_0 = 0$ corresponds to these cone invariance properties. Our autonomous squeezing property is a modification of the usual squeezing properties, however, at least for semilinear parabolic evolution equations, a strong squeezing property and an additional cone invariance property imply our autonomous squeezing property.

**Definition 2.13 (Boundedness Property).** The NDS $\varphi$ satisfies the (nonautonomous) *boundedness property* if for all $t \geq 0$, $b \in \mathcal{B}$ and all $M_1 \geq 0$ there exists a $M_2 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_2(b)x\| \leq M_1$ the estimate

$$\|\pi_2(\theta_t b)\varphi(t, b)x\| \leq M_2$$

holds.

**Definition 2.14 (Coercivity Property).** The NDS $\varphi$ satisfies the (nonautonomous) *coercivity property* if for all $t \geq 0$, $b \in \mathcal{B}$ and all $M_3 \geq 0$ there exists an $M_4 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_1(b)x\| \geq M_4$ the estimate

$$\|\pi_1(\theta_t b)\varphi(t, b)x\| \geq M_3$$

holds.

The coercivity property will ensure the existence of global homeomorphisms used for the definition of the graph transformation mapping. With the boundedness property we will ensure that the graph transformation mapping maps bounded graphs on bounded graphs.

*Remark* 2.15. As we will show later in Sec. 3.4, for evolution equations the coercivity and boundedness property of $\varphi$ follows from the boundedness of the nonlinearity and exponential dichotomy properties of the linear part.

We say that $\varphi$ satisfies an *inverse Lipschitz inequality* if for all $t \geq 0$, $b \in \mathcal{B}$ there exists a $K > 0$ such that

$$\|\pi_1(b)x - \pi_1(b)y\| \leq K\|\pi_1(\theta_t b)\varphi(t, b)x - \pi_1(\theta_t b)\varphi(t, b)y\|$$

holds for $x, y \in \mathcal{X}$. Obviously, the inverse Lipschitz property of $\varphi$ implies the coercivity property of $\varphi$. For evolution equations, the inverse Lipschitz property follows from a uniform Lipschitz property of the nonlinearity and exponential dichotomy properties of the linear part.

**Theorem 2.16 (Existence of Inertial Manifold).** *Let $\varphi$ be an NDS on a Banach space $\mathcal{X}$ over a driving system $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ on a set $\mathcal{B}$ and assume that $\varphi$ satisfies the cone invariance, squeezing, coercivity and boundedness property. Then there exists an inertial manifold $\mathcal{M} = (\mathcal{M}(b))_{b\in\mathcal{B}}$ of $\varphi$ with the following properties:*

*(i) $\mathcal{M}(b)$ is a graph in $\mathcal{X}_1(b) \oplus \mathcal{X}_2(b)$,*

$$\mathcal{M}(b) = \{x_1 + m(b, x_1) : \ x_1 \in \mathcal{X}_1(b)\}$$

*with a mapping $m(b, \cdot) = m(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b)$ which is globally Lipschitz continuous*

$$\|m(b, x_1) - m(b, y_1)\| \leq L\|x_1 - y_1\| \,.$$

*(ii) $\mathcal{M}$ is exponentially attracting, more precisely*

$$\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)x']\| \leq K_i \mathrm{e}^{-\eta t}\|\pi_2(b)x - m(b, \pi_1(b)x)\|$$

*for $t \geq 0$, $i = 1, 2$, with an asymptotic phase $x' = x'(b, x) \in \mathcal{M}(b)$ of $x$ and the constants $K_1, K_2 > 0$ from the squeezing property which are independent of $b$ and $x$.*

*(iii) If in addition $\pi_1$ is tempered from above, then $\mathcal{M}$ is pullback attracting, more precisely, there is a $T \geq 0$ such that for each bounded set $\mathcal{D} \subset \mathcal{X}$*

$$d(\varphi(t, \theta_{-t}b)\mathcal{D}|\mathcal{M}(b)) \leq \mathrm{e}^{-\eta t/2} d(\mathcal{D}|\mathcal{M}(\theta_{-t}b)) \qquad \text{for } t \geq T \,. \tag{1}$$

*Proof.* As the proof is rather involved we split it into four parts. First we define the graph transformation mapping. In the second step we show that it has a unique fixed point $m(b)$ which gives rise to a nonautonomous invariant set $\mathcal{M}$ of Lipschitz manifolds $\mathcal{M}(b)$. In the third step the exponential tracking property is proved and in the fourth step we investigate the pullback attractivity of $\mathcal{M}$.

**Step 1: Definition of graph transformation mapping**

We construct the manifolds $\mathcal{M}(b) = \mathrm{graph}(m(b))$ as the fixed point of the cocycle $\varphi$ acting on a certain class of functions $g$ with

$$g(b, \cdot) = g(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b) \,, \qquad b \in \mathcal{B} \,.$$

The set $\mathcal{G}$ of mappings of the form

$$\mathcal{X}_1 \ni (b, x_1) \mapsto (b, g(b, x_1)) \in \mathcal{X}_2 \,,$$

such that $g$ is bounded and $g(b, \cdot)$ is continuous for every $b \in \mathcal{B}$ is a Banach space with the norm

$$\|g\|_{\mathcal{G}} = \sup_{(b,x_1)\in\mathcal{X}_1} \|g(b, x_1)\| \,.$$

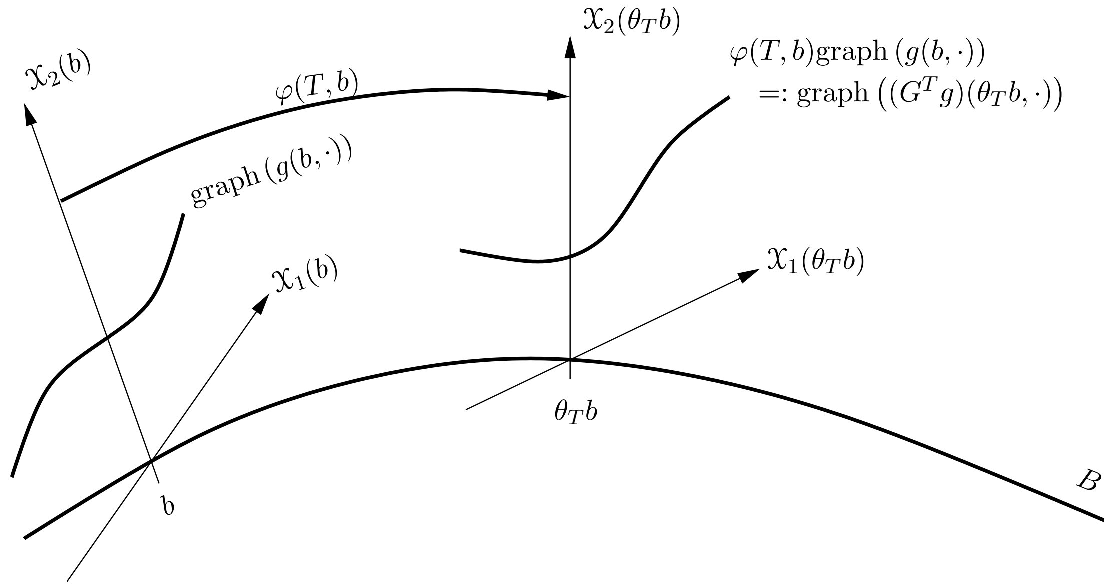


Figure 1: Graph transformation mapping $G^T$

Moreover let $\mathcal{G}_L$ denote the subset of $\mathcal{G}$ containing all mappings which satisfy the global Lipschitz condition

$$\|g(b,x_1) - g(b,y_1)\| \le L\|x_1 - y_1\|$$

for $(b,x_1),(b,y_1) \in \mathcal{X}_1$ with $L$ from the cone invariance property. Note that both $\mathcal{G}$ and $\mathcal{G}_L \subset \mathcal{G}$ are complete metric spaces.

Let $T > 0$ be arbitrary, but fixed. We wish to define the *graph transformation mapping* $G^T : \mathcal{G}_L \to \mathcal{G}$ such that

$$\mathrm{graph}((G^T g)(\theta_T b,\cdot)) = \varphi(T,b)\mathrm{graph}(g(b,\cdot))$$

(see Fig. 2.2) and this means that $\tilde{x} \in \mathrm{graph}((G^T g)(\theta_T b,\cdot))$ equals $\varphi(T,b)x$ for some $x \in \mathrm{graph}(g(b,\cdot))$. More precisely, for an $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$ we want to define $(G^T g)(\theta_T b,\tilde{x}_1) = \tilde{x}_2 \in \mathcal{X}_2(\theta_T b)$ by determining an $x \in \mathrm{graph}(g(b,\cdot))$ such that

$$\pi_1(\theta_T b)\varphi(T,b)x = \tilde{x}_1 \quad \text{and} \quad \pi_2(\theta_T b)\varphi(T,b)x =: (G^T g)(\theta_T b,\tilde{x}_1)\ .$$

To this end we show that for each $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, $g \in \mathcal{G}_L$, the boundary value problem

$$x \in \mathrm{graph}(g(b,\cdot))\ , \quad \pi_1(\theta_T b)\varphi(T,b)x = \tilde{x}_1 \tag{2}$$

has a unique solution $x = \beta(T,b,\tilde{x}_1,g)$.

*Uniqueness* of a solution of (2). Assume there exist $x$ and $y$ with

$$x,y \in \mathrm{graph}(g(b,\cdot))\ , \quad \pi_1(\theta_T b)\varphi(T,b)x = \pi_1(\theta_T b)\varphi(T,b)y = \tilde{x}_1\ .$$

We get $x - y \in \mathcal{C}_L(b)$ and the squeezing property (with $x' = x$) implies the identity

$$\varphi(t,b)x = \varphi(t,b)y \qquad \text{for } t \in [0,T] .$$

Hence $x = y$ and there is at most one solution $x = \beta(T, b, \tilde{x}_1, g)$ of (2).

*Existence* of a solution of (2). Let $T > 0$, $b \in \mathcal{B}$, $g \in \mathcal{G}_L$ be fixed and define $H\colon \mathcal{X}_1(b) \to \mathcal{X}_1(\theta_T b)$ by

$$H(x_1) := \pi_1(\theta_T b)\varphi(T,b)(x_1 + g(b,x_1)) .$$

By the continuity of $\varphi(T,b)$ and $g(b,\cdot)$, $H$ is continuous. For $\tilde{x}_1 \in H\mathcal{X}_1(b) \subset \mathcal{X}_1(\theta_T b)$, any $x_1$ in the preimage $H^{-1}(\tilde{x}_1) = \{x_1 \in \mathcal{X}_1(b) : H(x_1) = \tilde{x}_1\}$ gives rise to a solution $x = x_1 + g(b, x_1)$ of the boundary value problem (2). As we have already seen, there exists at most one solution denoted by $\beta(T, b, \tilde{x}_1, g)$ and therefore the inverse $H^{-1}$ of $H$ is given by

$$H^{-1}(\tilde{x}_1) = \pi_1(b)\beta(T, b, \tilde{x}_1, g) \qquad \text{on} \quad H\mathcal{X}_1(\theta_T b) .$$

Now we show the continuity of $H^{-1} : H\mathcal{X}_1(b) \to \mathcal{X}_1(b)$. Suppose, there is a $\tilde{\xi} \in H\mathcal{X}_1(b) \subset \mathcal{X}_1(\theta_T b)$ such that $H^{-1}$ is not continuous at $\tilde{\xi}$. Then there are $\varepsilon > 0$ and a sequence $(\tilde{\xi}_k)_{k\in\mathbb{N}}$ in $\mathcal{X}_1(\theta_T b)$ such that $\tilde{\xi}_k \to \tilde{\xi}_0$ as $k \to \infty$ and

$$\|\xi_k - \xi_0\| \geq \varepsilon \qquad \text{for all } k \in \mathbb{N} \tag{3}$$

where $\xi_k := \pi_1(b)\beta(T, b, \tilde{\xi}_k, g)$ for $k = 0, 1, \ldots$.

First, we suppose that there is a subsequence of $(\xi_k)_{k\in\mathbb{N}}$, denoted for shortness again by $(\xi_k)_{k\in\mathbb{N}}$, with $\|\xi_k\| \to \infty$ as $k \to \infty$. Then the coercivity property of $\varphi$ implies

$$\|\tilde{\xi}_k\| = \|H(\xi_k)\| = \|\pi_1(\theta_T b)\varphi(T,b)(\xi_k + g(b,\xi_k))\| \to \infty \qquad \text{as } k \to \infty ,$$

but this contradicts $\tilde{\xi}_k \to \tilde{\xi}_0$.

Therefore we have proved that $(\xi_k)_{k\in\mathbb{N}}$ is bounded. Since $\mathcal{X}_1(b)$ is a finite-dimensional space, there is a convergent subsequence, denoted for shortness again by $(\xi_k)_{k\in\mathbb{N}}$, with a limit

$$\xi_\infty = \lim_{k\to\infty} \xi_k \in \mathcal{X}_1(b) . \tag{4}$$

By the continuity of $H$, we have $H(\xi_k) \to H(\xi_\infty)$. Since also $H(\xi_k) = \tilde{\xi}_k \to \tilde{\xi}_0 = H(\xi_0)$ we get $\xi_0 = \xi_\infty$ in contradiction to (3) and (4). Therefore, $H$ and $H^{-1}$ are continuous.

Now we show that $H$ is onto, i.e. satisfies

$$H\mathcal{X}_1(b) = \mathcal{X}_1(\theta_T b) . \tag{5}$$

By the coercivity of $\varphi$, we have the norm coercivity $\|H(\xi)\| \to \infty$ for $\|\xi\| \to \infty$ of $H$. Since $\mathcal{X}_1(b)$ is finite-dimensional, $H$ is a sequentially compact mapping. By [Rhe69, Theorem 3.7], $H$ is a homeomorphism from $\mathcal{X}_1(b)$ onto $\mathcal{X}_1(\theta_T b)$ and hence we have (5).

Thus we have unique solvability of (2), and we can define the graph transformation mapping $G^T$ by

$$(G^T g)(\theta_T b, \tilde{x}_1) = \pi_2(\theta_T b)\varphi(T,b)\beta(T,b,\tilde{x}_1,g)$$

for $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, and $g \in \mathcal{G}_L$. Note that we have

$$\operatorname{graph}((G^T g)(\theta_T b, \cdot)) = \varphi(T,b)\operatorname{graph}(g(b,\cdot)) .$$

Since $H^{-1}$, $g(b,\cdot)$ and $x \mapsto \pi_1(b)\varphi(T,b)x$ are continuous, and

$$\beta(T,b,\tilde{x}_1,g) = H^{-1}(\tilde{x}_1) + g(b, H^{-1}(\tilde{x}_1))$$

holds, the mapping $(G^T g)(\theta_T b, \cdot)$ is also continuous.

It remains to show that

$$\|G^T g\|_{\mathcal{G}} < \infty .$$

Since $g \in \mathcal{G}$ there is a $M_1$ with $\|\pi_2(b)x\| \leq M_1$ for all $b \in \mathcal{B}$ and all $x \in \operatorname{graph}(g(b,\cdot))$. By the boundedness property of $\varphi$ there is a $M_2$ with $\|\pi_2(\theta_T b)\varphi(t,b)x\| \leq M_2$ for all $b \in \mathcal{B}$ and all $x \in \mathcal{X}$ with $\|\pi_2(b)x\| \leq M_1$. Let $b \in \mathcal{B}_0$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$ be arbitrary. Then $\beta(T,b,\tilde{x}_1,g) \in \operatorname{graph}(g(b,\cdot))$, hence $\|\pi_2(b)\beta(T,b,\tilde{x}_1,g)\| \leq M_1$, and therefore

$$\|(G^T g)(\theta_T b, \tilde{x}_1)\| = \|\pi_2(\theta_T b)\varphi(T,b)\beta(T,b,\tilde{x}_1,g)\| \leq M_2$$

proving that $\|G^T g\|_{\mathcal{G}} < \infty$, i.e. $G^T g \in \mathcal{G}$.

**Step 2: Unique fixed-point of graph transformation mappings**

Let $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1, \tilde{x}_2 \in \mathcal{X}_1(\theta_T b)$, $g \in \mathcal{G}_L$ be arbitrary. Since $\beta(T,b,\tilde{x}_1,g)$, $\beta(T,b,\tilde{x}_2,g) \in \operatorname{graph}(g(b,\cdot))$, we get

$$\beta(T,b,\tilde{x}_1,g) - \beta(T,b,\tilde{x}_2,g) \in \mathcal{C}_L(b)$$

and the cone invariance property implies for $T \geq T_0$ a Lipschitz estimate for $G^T g$,

$$\begin{aligned}
&\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T g)(\theta_T b, \tilde{x}_2)\| \\
&\quad \leq L\|\pi_1(\theta_T b)(\varphi(T,b)\beta(T,b,\tilde{x}_1,g) - \varphi(T,b)\beta(T,b,\tilde{x}_2,g))\| \\
&\quad = L\|\tilde{x}_1 - \tilde{x}_2\| ,
\end{aligned}$$

i.e. $G^T$ maps $\mathcal{G}_L$ into $\mathcal{G}$ for every $T \geq 0$, and it is self-mapping for $T \geq T_0$.

Now let $T \geq 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, $g, h \in \mathcal{G}$, and $x \in \operatorname{graph}(g(b,\cdot))$, $y \in \operatorname{graph}(h(b,\cdot))$ with

$$\pi_1(\theta_T b)\varphi(T,b)x = \pi_1(\theta_T b)\varphi(T,b)y = \tilde{x}_1 .$$

Define

$$x' := \pi_1(b)x + h(b, \pi_1(b)x) .$$

Then $x' - y \in \mathcal{C}_L$, and the squeezing property implies

$$\|\pi_2(\theta_T b)[\varphi(T,b)x - \varphi(T,b)y]\| \le K_2 \mathrm{e}^{-\eta T} \|g(b,\pi_1(b)x) - h(b,\pi_1(b)x)\|$$

for $T \ge 0$. Thus

$$\begin{aligned} &\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T h)(\theta_T b, \tilde{x}_1)\| \\ &\qquad \le K_2 \mathrm{e}^{-\eta T} \|g(b, \pi_1(b)\beta(T,b,\tilde{x}_1,g)) - h(b,\pi_1(b)\beta(T,b,\tilde{x}_1,g))\| , \end{aligned}$$

and passing to the sup over all $(\theta_T b, \tilde{x}_1) \in \mathcal{X}_1$ we get

$$\|G^T g - G^T h\|_{\mathcal{G}} \le \kappa(T) \|g - h\|_{\mathcal{G}}$$

for all $T > 0$, $g, h \in \mathcal{G}_L$, where

$$\kappa(T) := K_2 \mathrm{e}^{-\eta T} .$$

Since $\eta > 0$, there is a positive $T_1 \ge T_0$ with $\kappa(T) < 1$ for $T \ge T_1$. Thus, for $T \ge T_1$, $G^T$ is a contractive self-mapping on the complete metric space $\mathcal{G}_L$. Now choose and fix an arbitrary $\tilde{T} \ge T_1$ and let $m$ denote the unique fixed-point of $G^{\tilde{T}}$ in $\mathcal{G}_L$. We show that $m$ is the unique fixed-point of $G^T$ for every $T \ge 0$.

For every $T \ge 0$ the mapping $G^T m \in \mathcal{G}$ is uniquely determined by the graphs

$$\mathrm{graph}((G^T m)(\theta_T b, \cdot)) = \varphi(T,b)\mathrm{graph}(m(b,\cdot)) , \qquad b \in \mathcal{B} .$$

For $T \ge 0$ and $b \in \mathcal{B}$ we have the identity

$$\begin{aligned} \mathrm{graph}((G^{T+\tilde{T}} m)(\theta_{T+\tilde{T}} b, \cdot)) &= \varphi(T+\tilde{T}, b)\mathrm{graph}(m(b,\cdot)) \\ &= \varphi(T, \theta_{\tilde{T}} b)\varphi(\tilde{T}, b)\mathrm{graph}(m(b,\cdot)) \\ &= \varphi(T, \theta_{\tilde{T}} b)\mathrm{graph}(m(\theta_{\tilde{T}} b,\cdot)) \\ &= \mathrm{graph}((G^T m)(\theta_{T+\tilde{T}} b, \cdot)) \end{aligned}$$

and therefore $G^T m = G^{T+\tilde{T}} m \in \mathcal{G}_L$. Hence the composition $G^T G^{T'} m$ makes sense for $T, T' \ge 0$ and we get

$$\begin{aligned} \mathrm{graph}((G^T G^{T'} m)(\theta_{T+T'} b, \cdot)) &= \varphi(T, \theta_{T'} b)\mathrm{graph}(G^{T'} m(\theta_{T'} b, \cdot)) \\ &= \varphi(T, \theta_{T'} b)\varphi(T', b)\mathrm{graph}(m(b,\cdot)) \\ &= \varphi(T+T', b)\mathrm{graph}(m(b,\cdot)) \\ &= \mathrm{graph}((G^{T+T'} m)(\theta_{T+T'} b, \cdot)) \end{aligned}$$

and therefore $G^T G^{T'} m = G^{T'} G^T m = G^{T+T'} m$ for $T, T' \ge 0$. We get

$$G^{\tilde{T}}(G^T m) = G^T(G^{\tilde{T}} m) = G^T m .$$

Thus $G^T m$ equals the unique fixed-point $m$ of $G^{\tilde{T}}$ and we have

$$G^T m = m \qquad \text{for } T \geq 0 \, .$$

To prove the uniqueness of the fixed-point $m$ of $G^T$, assume that $m^*$ is another fixed-point. But then $m$ and $m^*$ are both fixed-points of $G^{kT}$ for every $k \in \mathbb{N}$. Choosing $k$ large enough such that $kT \geq T_1$ we know that $G^{kT}$ has a unique fixed-point and this implies $m = m^*$.

Thus $m$ is the unique mapping in $\mathcal{G}_L$ with the invariance property

$$\varphi(t,b)\mathrm{graph}(m(b,\cdot)) = \mathrm{graph}(m(\theta_t b,\cdot)) \qquad \text{for } t \geq 0 \text{ and } b \in \mathcal{B} \, .$$

We define $\mathcal{M}(b) := \mathrm{graph}(m(b,\cdot))$ for $b \in \mathcal{B}$.

**Step 3: Existence of asymptotic phase**

Let $b \in \mathcal{B}$ and $x \in \mathcal{X}$ be arbitrary and let $(t_k)_{k\in\mathbb{N}}$ be a monotonously increasing sequence of positive real numbers $t_k$ with $t_k \to \infty$ for $k \to \infty$. Define $y' := \pi_1(b)x + m(b, \pi_1(b)x) \in \mathrm{graph}(m(b,\cdot))$ and

$$x_k := \beta(t_k, b, \pi_1(\theta_{t_k} b)\varphi(t_k, b)x, m) \in \mathrm{graph}(m(b,\cdot)) \, .$$

We get $y' - x_k \in \mathcal{C}_L(b)$ and the squeezing property implies for $i = 1, 2$, $t \in [0, t_k]$

$$\|\pi_i(\theta_t b)[\varphi(t,b)x - \varphi(t,b)x_k]\| \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\| \, .$$

In particular, we find for $i = 1$ and $t = 0$

$$\begin{aligned}
\|\pi_1(b)x_k\| &\leq \|\pi_1(b)x\| + \|\pi_1(b)[x - x_k]\| \\
&\leq \|\pi_1(b)x\| + K_1 \|\pi_2(b)x - m(b, \pi_1(b)x)\| \, .
\end{aligned}$$

Therefore we have proved that $(\pi_1(b)x_k)_{k\in\mathbb{N}} \subset \mathcal{X}_1(b)$ is bounded. Since $\mathcal{X}_1(b)$ is a finite-dimensional space, there is a convergent subsequence, denoted again by $(\pi_1(b)x_k)_{k\in\mathbb{N}}$. Since

$$x_k = \pi_1(b)x_k + m(b, \pi_1(b)x_k)$$

and $m(b,\cdot)$ is continuous, also $(x_k)_{k\in\mathbb{N}}$ is converging to some

$$x' \in \mathrm{graph}(m(b,\cdot)) \, ,$$

Then we get for $i = 1, 2$

$$\begin{aligned}
&\|\pi_i(\theta_t b)[\varphi(t,b)x - \varphi(t,b)x']\| \\
&\qquad \leq \|\pi_i(\theta_t b)[\varphi(t,b)x - \varphi(t,b)x_k]\| + \|\pi_i(\theta_t b)[\varphi(t,b)x_k - \varphi(t,b)x']\| \\
&\qquad \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\| + \|\pi_i(\theta_t b)[\varphi(t,b)x_k - \varphi(t,b)x']\|
\end{aligned}$$

for all $T > 0$, $t \in [0,T]$ and all $k \in \mathbb{N}_{>0}$ with $t_k \geq T$. By the continuity of $\varphi(t,b)$, and because of $x_k \to x'$, the second term can be made arbitrary small for each fixed $t \in [0,T]$ choosing $k$ large enough. Therefore,

$$\|\pi_i(\theta_t b)[\varphi(t,b)x - \varphi(t,b)x']\| \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\|$$

for $t \geq 0$, i.e. $x' \in \mathcal{M}(b)$ is an asymptotic phase of $x$.

**Step 4: Pullback attractivity**

Note that with $\pi_1$ also the complementary projector $\pi_2$ is tempered from above. Since $\varphi(t,b)x' \in \mathcal{M}(\theta_t b)$ for every $x \in \mathcal{X}$, $t \geq 0$, $b \in \mathcal{B}$, Step 3 implies

$$d(\varphi(t,b)x, \mathcal{M}(\theta_t b)) \leq K \mathrm{e}^{-\eta t} \|\pi_2(b)[x - x']\| \qquad \text{for } t \geq 0$$

with $x' = \pi_1(b)x + m(b, \pi_1(b)x)$. Let $z \in \mathcal{M}(b)$ be arbitrary. Then, because of $z - x' \in \mathcal{C}_L(b)$ and $\pi_1(b)x = \pi_1(b)x'$,

$$\begin{aligned} \|\pi_2(b)[x - x']\| &\leq \|\pi_2(b)[x - z]\| + \|\pi_2(b)[z - x']\| \\ &= (\|\pi_2(b)\| + L\|\pi_1(b)\|) \|x - z\| \,. \end{aligned}$$

Hence

$$\|\pi_2(b)[x - x']\| \leq (\|\pi_2(b)\| + L\|\pi_1(b)\|)\, d(x, \mathcal{M}(b)) \,.$$

Replacing $b$ by $\theta_{-t}b$, choosing a $T > 0$ such that

$$K \cdot (\|\pi_2(\theta_{-t}b)\| + L\|\pi_1(\theta_{-t}b)\|) \leq \mathrm{e}^{\frac{\eta}{2}t} \qquad \text{for } t \geq T$$

(Corollary 2.8), we obtain

$$d(\varphi(t, \theta_{-t}b)x, \mathcal{M}(b)) \leq \mathrm{e}^{-\frac{\eta}{2}t} d(x, \mathcal{M}(\theta_{-t}b)) \leq \mathrm{e}^{-\frac{\eta}{2}t} d(\mathcal{D}, \mathcal{M}(\theta_{-t}b))$$

for $t \geq T$ proving the pullback attractivity of $\mathcal{M}$ with $T$ independent of $\mathcal{D}$. $\square$

**Corollary 2.17.** *If $\mathcal{B}$ is a metric space and if $\varphi \in C(\mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X}, \mathcal{X})$ then $m \in C(\mathcal{X}_1, \mathcal{X})$ with $\mathcal{X}_1 := \{(b,x) \in \mathcal{B} \times \mathcal{X} : \ x \in \mathcal{X}_1(b)\}$.*

*Proof.* Consider the graph transformation mappings $G^T : \mathcal{G}_L \subset \mathcal{G} \to \mathcal{G}$ on the Banach space $(\mathcal{G}, \|\cdot\|_{\mathcal{G}})$ of the continuous mappings $g \in C(\mathcal{X}_1, \mathcal{X})$,

$$g(b, \cdot) = g(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b) \,, \qquad b \in \mathcal{B}$$

with bounded norm

$$\|g\|_{\mathcal{G}} := \sup_{(b,x_1) \in \mathcal{X}_1} \|g(b, x_1)\| \,,$$

where $\mathcal{G}_L$ denotes the closed subset of $\mathcal{G}$ of all functions $g$ satisfying the Lipschitz property

$$\|g(b,x_1) - g(b,y_1)\| \leq L\|x_1 - y_1\| \qquad \text{for } (b,x_1), (b,y_1) \in \mathcal{X}_1 \,.$$

Then the fixed-points of the graph transformation mappings are continuous functions from $\mathcal{X}_1$ into $\mathcal{X}$ which are Lipschitz in the second argument. $\square$

## 2.3 Pullback Attractors for Nonautonomous Dynamical Systems

The following theorem shows that we can easily construct a global pullback attractor once we have a compact, forward invariant, globally pullback absorbing set.

**Theorem 2.18.** *[Sch92, CF94, FS96, Sch99] Suppose that $\hat{\mathcal{A}}$ is a compact, forward invariant, globally pullback absorbing set. Then $\mathcal{A}$,*

$$\mathcal{A}(b) := \bigcap_{t\geq 0} \varphi(t,\theta_{-t}b)\hat{\mathcal{A}}(\theta_{-t}b) \qquad \textit{for } b \in \mathcal{B}\,,$$

*is a global pullback attractor.*

*Proof.* Since $\hat{\mathcal{A}}$ is forward invariant, $\mathcal{A}(b)$ is the intersection of a decreasing sequence of compact sets contained in $\hat{\mathcal{A}}$, hence is non-void and compact, moreover

$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}b)\hat{\mathcal{A}}(\theta_{-t}b)|\mathcal{A}(b)) = 0\,. \tag{6}$$

Using the elementary fact that if $(\mathcal{D}_n)$ is a decreasing sequence of compact sets and $f$ is continuous then $\bigcap_n f(\mathcal{D}_n) = f(\bigcap_n \mathcal{D}_n)$, the cocycle property and the monotonicity of $(\varphi(t,\theta_{-t}b)\hat{\mathcal{A}}(\theta_{-t}b))$ we obtain for any $T \in \mathbb{R}$

$$\begin{aligned}
\varphi(T,b)\mathcal{A}(b) &= \bigcap_{t\geq 0} \varphi(T,b)\varphi(t,\theta_{-t}b)\hat{\mathcal{A}}(\theta_{-t}b)\\
&= \bigcap_{t\geq 0} \varphi(T+t,\theta_{-t-T}(\theta_T b))\hat{\mathcal{A}}(\theta_{-t-T}(\theta_T b))\\
&= \bigcap_{t\geq T} \varphi(T,\theta_{-t}(\theta_T b))\hat{\mathcal{A}}(\theta_{-t}(\theta_T b))\\
&= \bigcap_{t\geq 0} \varphi(T,\theta_{-t}(\theta_T b))\hat{\mathcal{A}}(\theta_{-t}(\theta_T b)) = \mathcal{A}(\theta_T b)\,,
\end{aligned}$$

proving the invariance of $\mathcal{A}$.

To see that $\mathcal{A}$ is globally pullback attracting choose a bounded set $\mathcal{D} \subset \mathcal{X}$, $b \in \mathcal{B}$ and an $\varepsilon > 0$. Due to Lemma 6 there exists a $T_0 = T_0(\varepsilon,\mathcal{D},b) \geq 0$ such that

$$d(\varphi(T_0,\theta_{-T_0}b)\hat{\mathcal{A}}(\theta_{-T_0}b)|\mathcal{A}(b)) < \varepsilon\,,$$

and since $\hat{\mathcal{A}}$ is absorbing, there exists a $T_1 = T_1(\theta_{-T_0}b,\mathcal{D}) \geq 0$ with

$$\varphi(t-T_0,\theta_{-(t-T_0)}(\theta_{-T_0}b))\mathcal{D} \subset \hat{\mathcal{A}}(\theta_{-T_0}b) \qquad \text{if } t-T_0 \geq T_1\,.$$

Applying $\varphi(T_0,\theta_{-T_0}b)$ to this inclusion, the cocycle property implies

$$\varphi(t,\theta_{-t}b)\mathcal{D} \subset \varphi(T_0,\theta_{-T_0}b)\hat{\mathcal{A}}(\theta_{-T_0}b)\,.$$

Using the fact that if $\mathcal{D}_1 \subset \mathcal{D}_2$ then $d(\mathcal{D}_1|\mathcal{A}) \leq d(\mathcal{D}_2|\mathcal{A})$, we obtain

$$d(\varphi(t,\theta_{-t}b)\mathcal{D}|\mathcal{A}(b)) < \varepsilon \qquad \text{for } t \geq T_0 + T_1 \,,$$

proving that $\mathcal{A}$ is globally pullback attracting. $\square$

The next lemma shows that we only need the existence of an absorbing set $\tilde{\mathcal{A}}$, if we have a suitable smoothing property of $\varphi$. This smoothing property is expressed by the assumption that the set $\hat{\mathcal{A}}_c$ is compact for sufficiently large $c \geq 0$. A similar property is used in [Tem97] for semiflows, and it is called there uniform compactness property.

**Lemma 2.19.** *Suppose that $\tilde{\mathcal{A}}$ is a globally pullback absorbing set. Then $\tilde{\mathcal{A}}_0$,*

$$\hat{\mathcal{A}}_0(b) := \bigcup_{s\geq 0} \varphi(s,\theta_{\cdot s}b)\tilde{\mathcal{A}}(\theta_{\cdot s}b) \qquad \textit{for } b \in \mathcal{B} \,,$$

*is globally pullback absorbing and forward invariant. If there is a $c \geq 0$ such that $\hat{\mathcal{A}}_c$,*

$$\hat{\mathcal{A}}_c(b) := \operatorname{cl}\left(\varphi(c,\theta_{-c}b)\tilde{\mathcal{A}}_0(b)\right) \qquad \textit{for } b \in \mathcal{B} \,,$$

*is compact, then $\hat{\mathcal{A}}_c$ is a compact, forward invariant, globally pullback absorbing set.*

*Proof.* For $\tau \geq 0$ let

$$\tilde{\mathcal{A}}_\tau(b) := \bigcup_{s\geq \tau} \varphi(s,\theta_{\cdot s}b)\tilde{\mathcal{A}}(\theta_{\cdot s}b) \qquad \text{for } b \in \mathcal{B} \,.$$

Let $b \in \mathcal{B}$, $t \geq 0$. Then

$$\begin{aligned}
\varphi(t,b)\tilde{\mathcal{A}}_\tau(b) &= \bigcup_{s\geq\tau} \varphi(t+s,\theta_{-s}b)\tilde{\mathcal{A}}(\theta_{-s}b) \\
&= \bigcup_{r\geq\tau+t} \varphi(r,\theta_{-r}\theta_t b)\tilde{\mathcal{A}}(\theta_{-r}\theta_t b) \\
&\subseteq \bigcup_{r\geq\tau} \varphi(r,\theta_{-r}\theta_t b)\tilde{\mathcal{A}}(\theta_{-r}\theta_t b) \\
&= \tilde{\mathcal{A}}_\tau(\theta_t b) \,.
\end{aligned}$$

Thus $\tilde{\mathcal{A}}_\tau$ is forward invariant for each $\tau \geq 0$.

Since $\tilde{\mathcal{A}}$ is globally pullback attracting, for any $b \in \mathcal{B}$ and any bounded set $\mathcal{D}$ there is a $T(b,\mathcal{D})$ with $\varphi(t,\theta_{-t}b)\mathcal{D} \subseteq \tilde{\mathcal{A}}(b)$ for $t \geq T(b,\mathcal{D})$. Since $\varphi(\tau,\theta_{-\tau}b)\tilde{\mathcal{A}}(\theta_{-\tau}b) \subseteq \tilde{\mathcal{A}}_\tau(b)$ and

$$\varphi(t,\theta_{-t}\theta_{-\tau}b)\mathcal{D} \subseteq \tilde{\mathcal{A}}(\theta_{-\tau}b) \qquad \text{for } t \geq T(\theta_{-\tau}b,\mathcal{D})$$

we have

$$\varphi(t,\theta_{-t}b)\mathcal{D} \subseteq \tilde{\mathcal{A}}_\tau(b) \qquad \text{for } t \geq \tau + T(\theta_{-\tau}b,\mathcal{D})\ .$$

Thus $\tilde{\mathcal{A}}_\tau$ is pullpack attracting for each $\tau \geq 0$. For $\tau = 0$ we obtain the first claim.

Now assume that there exists a $c \geq 0$ such that $\tilde{\mathcal{A}}_c$ is compact. Since $\varphi(c,\theta_{-c}b)\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_c(b)$, the set $\hat{\mathcal{A}}_c$ is the closure of the forward invariant, globally pullback attracting set $\tilde{\mathcal{A}}_c(b)$ and hence it is forward invariant, globally pullback attracting, too. □

Obviously, if $\tilde{\mathcal{A}}$ is compact, positively invariant and globally pullback attracting then we may choose $c = 0$ and hence $\hat{\mathcal{A}}_0 = \tilde{\mathcal{A}}$.

In Theorem 2.16 we proved the existence of an inertial manifold which is exponentially attracting in the sense of forward convergence, i.e. $\varphi(t,b)x$ is converging to its asymptotic phase $\varphi(t,b)x'$. Moreover we proved that the inertial manifold is globally pullback attracting if the projector $\pi_1$ is tempered from above.

Combining the Theorems 2.16 and 2.18, we give a sufficient condition for the existence of a global pullback attractor which is contained in the inertial manifold.

**Corollary 2.20.** *Let the assumptions of Theorem* 2.16 *be satisfied with a projector* $\pi_1$ *which is tempered from above. Moreover, suppose that* $\hat{\mathcal{A}}$ *is a compact, forward invariant, globally pullback absorbing set which is tempered from above, i.e. let the function* $b \mapsto |\hat{\mathcal{A}}(b))|$ *be tempered from above, where* $|\mathcal{D}| := \sup\{\|x\| : x \in \mathcal{D}\}$*.Then the global pullback attractor* $\mathcal{A}$*,*

$$\mathcal{A}(b) = \bigcap_{t\geq 0} \varphi(t,\theta_{-t}b)\hat{\mathcal{A}}(\theta_{-t}b) \qquad b \in \mathcal{B}\ ,$$

*is contained in the inertial manifold* $\mathcal{M}$*.*

*Proof.* By Theorem 2.16 we have the existence of the globally pullback attracting inertial manifold $\mathcal{M}$ satisfying (1). By construction there is a $K > 0$ with $|\mathcal{M}(b)| \leq K$ for all $b \in \mathcal{B}$.

Let $\hat{\mathcal{A}}$ be a compact, forward invariant, globally pullback absorbing set which is tempered from above. Theorem 2.18 implies the existence of the global pullback attractor $\mathcal{A}$ which is also tempered from above. Choosing a $T' > T$ such that $|\mathcal{A}(\theta_{-t}b)| \leq \mathrm{e}^{\frac{\eta}{4}t}$for $t \geq T'$ (Corollary 2.8) and using the invariance of $\mathcal{A}$, from (1) we get for $t \geq T'$

$$d(\mathcal{A}(b)|\mathcal{M}(b)) = d(\varphi(t,\theta_{-t}b)\mathcal{A}(\theta_{-t}b)|\mathcal{M}(b)) \leq \mathrm{e}^{-\frac{\eta}{2}t}(\mathrm{e}^{\frac{\eta}{4}t} + K)\ .$$

Since this expression tends to 0 for $t \to \infty$ we obtain $d(\mathcal{A}(b)|\mathcal{M}(b)) = 0$ and conclude that $\mathcal{A}(b) \subset \mathcal{M}(b)$ for all $b \in \mathcal{B}$. □

# 3 Nonautonomous Evolution Equations

## 3.1 Two-Parameter Semi-Flow

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space. The solutions of a nonautonomous evolution equation will not generate a semi-flow but a two-parameter semi-flow.

**Definition 3.1 (Two-parameter Semi-Flow).** A *two-parameter semi-flow* $\mu$ on $\mathcal{X}$ is a continuous mapping

$$\{(t,s,x) \in \mathbb{R} \times \mathbb{R} \times \mathcal{X} :\ t \geq s\} \ni (t,s,x) \mapsto \mu(t,s,x) \in \mathcal{X}$$

which satisfies

(i) $\mu(s,s,\cdot) = \mathrm{id}_{\mathcal{X}}$ for $s \in \mathbb{R}$;

(ii) the two-parameter semi-flow property for $t \geq \tau \geq s$, $x \in \mathcal{X}$, i.e.

$$\mu(t,\tau,\mu(\tau,s,x)) = \mu(t,s,x)\ .$$

The next lemma explains how a two-parameter semi-flow defines an NDS.

**Lemma 3.2 (Two-parameter Semi-Flow defines NDS).** *Suppose that $\mu$ is a two-parameter semi-flow. Then $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$,*

$$\varphi(t,b)x = \mu(t+b,b,x) \tag{7}$$

*is an NDS with base $\mathcal{B} = \mathbb{R}$ and driving system $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$,*

$$\theta(t)b = t + b\ .$$

*Moreover, for $t \geq s$ and $x \in \mathcal{X}$ the relation $\mu(t,s,x) = \varphi(t-s,s)x$ holds.*

*Proof.* $\theta$ is a dynamical system. We have

$$\varphi(0,b) = \mu(b,b,\cdot) = \mathrm{id}_{\mathcal{X}}\ .$$

We use the two-parameter semi-flow property of $\mu$ to obtain for $t,s \geq 0$, $b \in \mathcal{B}$

$$\begin{aligned}\varphi(t+s,b) &= \mu(t+s+b,b,\cdot)\\ &= \mu(t+s+b,s+b,\mu(s+b,b,\cdot))\\ &= \varphi(t,\theta_s b) \circ \varphi(s,b)\ ,\end{aligned}$$

proving the cocycle property of $\varphi$. The continuity of $\mu$ implies the continuity of $(t,x) \mapsto \varphi(t,b)x$. Now substitute $t$ by $t-s$ and $b$ by $s$ in (7) to see that $\mu(t,s,x) = \varphi(t-s,s)x$. $\square$

**Theorem 3.3 (Inertial Manifold for Two-parameter Semi-Flow).** *Suppose that $\mu$ is a two-parameter semi-flow on $\mathcal{X}$ and let $(\pi_i(\tau))_{\tau\in\mathbb{R}} \subset L(\mathcal{X})$, $i = 1, 2$, be two families of complementary projectors $\pi_1(\tau)$ and $\pi_2(\tau)$. Let $\mu$ satisfy the*

• cone invariance property, *i.e. there are $L > 0$ and $T_0 \geq 0$ such that for $\tau \in \mathbb{R}$ and $x, y \in \mathcal{X}$,*

$$x - y \in \mathcal{C}_L(\tau) := \{\xi : \|\pi_2(\tau)\xi\|_{\mathcal{X}} \leq L\|\pi_1(\tau)\xi\|_{\mathcal{X}}\}$$

*implies*

$$\mu(t, \tau, x) - \mu(t, \tau, y) \in \mathcal{C}_L(t) \qquad \text{for } t \geq \tau + T_0\,;$$

• squeezing property, *i.e. there exist positive constants $K_1$, $K_2$ and $\eta$ such that for every $\tau \in \mathbb{R}$, $x, y \in \mathcal{X}$ and $T > 0$ the identity*

$$\pi_1(\tau + T)\mu(\tau + T, \tau, x) = \pi_1(\tau + T)\mu(\tau + T, \tau, y)$$

*implies for all $x' \in \mathcal{X}$ with $\pi_1(\tau)x' = \pi_1(\tau)x$ and $x' - y \in \mathcal{C}_L(\tau)$ the estimates*

$$\|\pi_i(t)\,[\mu(t, \tau, x) - \mu(t, \tau, y)]\,\|_{\mathcal{X}} \leq K_i \mathrm{e}^{-\eta(t-\tau)}\|\pi_2(\tau)\,[x - x']\,\|_{\mathcal{X}}\,, \quad i = 1, 2,$$

*for $t \in [\tau, \tau + T]$;*

• boundedness property, *i.e. for all $t, \tau \in \mathbb{R}$ with $t \geq \tau$ and all $M_1 \geq 0$ there exists a $M_2 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_2(\tau)x\|_{\mathcal{X}} \leq M_1$ the estimate*

$$\|\pi_2(t)\mu(t, \tau, x)\|_{\mathcal{X}} \leq M_2$$

*holds.*

• coercivity property, *i.e. for all $t, \tau \in \mathbb{R}$ with $t \geq \tau$ and all $M_3 \geq 0$ there exists a $M_4 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_1(\tau)x\|_{\mathcal{X}} \geq M_4$ the estimate*

$$\|\pi_1(t)\mu(t, \tau, x)\|_{\mathcal{X}} \geq M_3$$

*holds.*

*Then there exists an inertial manifold $\mathcal{M} = (\mathcal{M}(\tau))_{\tau\in\mathbb{R}}$ of $\mu$ with the following properties:*

*(i) $\mathcal{M}(\tau)$ is a graph in $\pi_1(\tau)\mathcal{X} \oplus \pi_2(\tau)\mathcal{X}$,*

$$\mathcal{M}(\tau) = \{x_1 + m(\tau, x_1) :\ x_1 \in \pi_2(\tau)\mathcal{X}\}$$

*with a mapping $m(\tau, \cdot) = m(\tau) : \pi_1(\tau)\mathcal{X} \to \pi_2(\tau)\mathcal{X}$ which is globally Lipschitz continuous*

$$\|m(\tau, x_1) - m(\tau, y_1)\|_{\mathcal{X}} \leq L\|x_1 - y_1\|_{\mathcal{X}}\,.$$

*(ii) $\mathcal{M}$ is exponentially attracting, more precisely*

$$\|\pi_i(t)[\mu(t, \tau, x) - \mu(t, \tau, x')]\|_{\mathcal{X}} \leq K_i \mathrm{e}^{-\eta(t-\tau)}\|\pi_2(\tau)x - m(\tau, \pi_1(\tau)x)\|_{\mathcal{X}}$$

*for $t \geq \tau$, $i = 1, 2$, with an asymptotic phase $x' = x'(\tau, x) \in \mathcal{M}(\tau)$ of $x$ and the constants $K_1, K_2 > 0$ from the squeezing property which are independent of $\tau$ and $x$.*

*Proof.* By Lemma 3.2, the two-parameter semi-flow $\mu$ defines an NDS $\varphi$ with base $\mathcal{B} = \mathbb{R}$ and driving system $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ with $\theta(t)\tau = t + \tau$, $\tau = b \in \mathcal{B}$. Now Theorem 3.3 follows from Theorem 2.16. $\square$

## 3.2 Nonautonomous Evolution Equations

Let $\mathcal{X} \subseteq \mathcal{Y}$ be Banach spaces equipped with norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$, and let $(A(t))_{t\in\mathbb{R}}$ be a family of densely defined linear operators $A(t)$ on $\mathcal{Y}$ with domain $D(A(t))$ in $\mathcal{Y}$. We consider a nonautonomous evolution equation

$$\dot{x} + A(t)x = f(t, x) \tag{8}$$

which satisfies the following assumptions:

(A1) *Linearity $A(t)$:*

• Existence of evolution operator of the linear system: Under suitable additional assumptions on $\mathcal{X}$, $\mathcal{Y}$, $A$ and $f$ (see for example [Hen81, DKM92, Lun95]), we may define the evolution operator $\Phi : \{(t, s) \in \mathbb{R}^2 : t \geq s\} \to L(\mathcal{Y}, \mathcal{Y})$ of the linear equation

$$\dot{x} + A(t)x = 0 \tag{9}$$

in $\mathcal{Y}$ as the solution of

$$\frac{d}{dt}\Phi(t, s) + A(t)\Phi(t, s) = 0 \qquad \text{for } t > s,\ s \in \mathbb{R}$$

and

$$\Phi(\tau, \tau) = \mathrm{id}_{\mathcal{Y}} \qquad \text{for } \tau \in \mathbb{R}\,.$$

• There are constants $k_0, \dots, k_4 \geq 1$, $\beta_2 > \beta_1$, $\alpha \in [0, 1[$, a monotonously decreasing function $\psi \in C(\mathbb{R}_{>0}, \mathbb{R}_{>0})$ with $\psi(t) \leq k_0 t^{-\alpha}$, and a family $\pi_1 = (\pi_1(t))_{t\in\mathbb{R}}$ of linear, invariant projectors $\pi_1(t) : \mathcal{Y} \to \mathcal{Y}$, i.e.

$$\pi_1(t)\Phi(t, s) = \Phi(t, s)\pi_1(s) \qquad \text{for } t \geq s,$$

such that $\Phi(t, s)\pi_1(s)$ can be extended to a linear, bounded operator for $t \in \mathbb{R}$ with

$$\frac{d}{dt}\Phi(t, s)\pi_1(s) + A(t)\Phi(t, s)\pi_1(s) = 0 \qquad \text{for } t, s \in \mathbb{R}$$

and

$$\begin{aligned}
\|\Phi(t, s)\pi_1(s)\|_{L(\mathcal{X},\mathcal{X})} &\leq k_1 \mathrm{e}^{-\beta_1(t-s)} && \text{for } t \leq s\,,\\
\|\Phi(t, s)\pi_2(s)\|_{L(\mathcal{X},\mathcal{X})} &\leq k_2 \mathrm{e}^{-\beta_2(t-s)} && \text{for } t \geq s\,,\\
\|\Phi(t, s)\pi_1(s)\|_{L(\mathcal{Y},\mathcal{X})} &\leq k_3 \mathrm{e}^{-\beta_1(t-s)} && \text{for } t \leq s\,,\\
\|\Phi(t, s)\pi_2(s)\|_{L(\mathcal{Y},\mathcal{X})} &\leq k_4 \psi(t-s) \mathrm{e}^{-\beta_2(t-s)} && \text{for } t > s
\end{aligned} \tag{10}$$

with $\pi_2$, $\pi_2(t) = \mathrm{id}_{\mathcal{Y}} - \pi_1(t)$, as the complementary projector to $\pi_1$ in $\mathcal{Y}$.

(A2) *Nonlinearity $f(t,x)$:* The nonlinear function $f \in C(\mathbb{R}\times\mathcal{X},\mathcal{Y})$ is assumed to be globally bounded and to satisfy the Lipschitz inequality

$$\|\pi_i(t)[f(t,x)-f(t,y)]\|_{\mathcal{Y}} \le \gamma_i(\|\pi_1(t)[x-y]\|_{\mathcal{X}}, \|\pi_2(t)[x-y]\|_{\mathcal{X}}) \tag{11}$$

for all $t\in\mathbb{R}$, $x,y\in\mathcal{X}$, where $\gamma_i$ are suitable norms on $\mathbb{R}^2$.

(A3) *Existence of mild solutions:* We have the existence and uniqueness of the mild solutions

$$\mu(\cdot,\tau,\xi)\in C([\tau,\infty[,\mathcal{X})$$

of (8) with initial condition $x(\tau)=\xi\in\mathcal{X}$, i.e. let $\mu$ be the continuous solution of the integral equation

$$x(t)=\Phi(t,\tau)\xi+\int_\tau^t \Phi(t,r)f(\tau,x(r))\,\mathrm{d}r \qquad \text{for } t\ge\tau\,.$$

These were the assumptions.

*Remark* 3.4. Conditions like our assumptions can be found in the literature and they are standard for ordinary differential equations and for time-independent evolution equations in the non-selfadjoint case, see for example [Tem97]. For concrete examples of the realization of these assumptions we refer to Sec. 4, where we will apply our following Theorem 3.10 on the existence of inertial manifolds in these special situations.

## 3.3 Comparison Theorems

In order to apply Theorem 3.3, we have to show the cone invariance property and the squeezing property for the two-parameter semi-flow $\mu$ with respect to the projector $\pi_1$.

For fixed $r_1,r_2\ge 0$ and $T\ge 0$, we define

$$\begin{aligned}(\Lambda^1 w)(t) &:= k_3\int_t^T \mathrm{e}_1^{-\beta_1(t-r)}\gamma_1(w(r))\,\mathrm{d}r\,,\\ (\Lambda^2 w)(t) &:= k_4\int_0^t \psi(t-r)\mathrm{e}^{-\beta_2(t-r)}\gamma_2(w(r))\,\mathrm{d}r + k_2\mathrm{e}^{-\beta_2 t}Lw^1(0)\end{aligned}$$

and

$$q(t):=\left(k_1\mathrm{e}^{-\beta_1(t-T)}r_1, k_2\mathrm{e}^{-\beta_2 t}r_2\right)$$

for $t\in[0,T]$ and $w\in C([0,T],\mathbb{R}^2)$. Then $q\in C([0,T],\mathbb{R}^2_{\ge 0})$. Because of $\psi(t)\le k_0t^{-\alpha}$ with $\alpha\in[0,1[$, $\Lambda$ is a an most weakly singular integral operator from $C([0,T],\mathbb{R}^2)$ into $C([0,T],\mathbb{R}^2)$. Moreover, $\Lambda$ is completely continuous.

**Lemma 3.5.** *Assume there are $L > 0$ and $T_0 \geq 0$ such that for each $T \geq T_0$ the inequality*

$$v^2(T) \leq L r_1 \tag{12}$$

*holds for each solution $v \in C([0,T], \mathbb{R}^2_{\geq 0})$ of*

$$v^i(t) \leq (\Lambda v)^i(t) + q^i(t) \qquad \text{for } i = 1,2,\ t \in [0,T] \tag{13}$$

*with $r_1 \geq 0$, $r_2 = 1$. Then $\mu$ possesses the cone invariance property with respect to $\pi$ with the parameters $L > 0$ and $T_0 \geq 0$.*

*Proof.* The cone invariance property is shown if

$$\|\pi_2(t)[\mu(t,\tau,x) - \mu(t,\tau,y)]\|_{\mathcal{X}} \leq L\|\pi_1(t)[\mu(t,\tau,x) - \mu(t,\tau,y)]\|_{\mathcal{X}} \tag{14}$$

for each $\tau \in \mathbb{R}$, $t \geq \tau + T_0$ and $x, y \in \mathcal{X}$ with

$$\|\pi_2(\tau)[x-y]\|_{\mathcal{X}} \leq L\|\pi_1(\tau)[x-y]\|_{\mathcal{X}}\,. \tag{15}$$

Let $\tau \in \mathbb{R}$, $T > 0$ and $x, y \in \mathcal{X}$ with (15) be fixed and let

$$\lambda(t) := \mu(t,\tau,x) - \mu(t,\tau,y)\,, \quad f_\Delta(t) := f(t,\mu(t,\tau,x)) - f(t,\mu(t,\tau,y))$$

on $[\tau, \tau+T]$. Then we have

$$\pi_1(t)\lambda(t) = \Phi(t,\tau+T)\pi_1(\tau+T)\lambda(\tau+T) + \int_{\tau+T}^{t} \Phi(t,r)\pi_1(r) f_\Delta(r)\,\mathrm{d}r$$

and

$$\pi_2(t)\lambda(t) = \Phi(t,\tau)\pi_2(\tau)[x-y] + \int_{\tau}^{t} \Phi(t,r)\pi_2(r) f_\Delta(r)\,\mathrm{d}r$$

on $[\tau, \tau+T]$. Let $v : [0,T] \to \mathbb{R}^2_{\geq 0}$ be defined by

$$v(t) = (\|\pi_1(\tau+t)\lambda(\tau+t)\|_{\mathcal{X}}, \|\pi_2(\tau+t)\lambda(\tau+t)]\|_{\mathcal{X}})\,. \tag{16}$$

Then

$$\begin{aligned}
v^1(t) &\leq \|\Phi(\tau+t,\tau+T)\pi_1(\tau+T)\lambda(\tau+T)\|_{\mathcal{X}} \\
&\quad + \int_t^T \|\Phi(\tau+t,\tau+r)\pi_1(\tau+r)\Delta f(\tau+r)\|_{\mathcal{X}}\,\mathrm{d}r \\
&\leq \|\Phi(\tau+t,\tau+T)\pi_1(\tau+T)\|_{L(\mathcal{X},\mathcal{X})}\|\pi_1(\tau+T)\lambda(\tau+T)\|_{\mathcal{X}} \\
&\quad + \int_t^T \|\Phi(\tau+t,\tau+r)\pi_1(\tau+r)\|_{L(\mathcal{Y},\mathcal{X})}\|\pi_1(\tau+r)\Delta f(\tau+r)\|_{\mathcal{Y}}\,\mathrm{d}r
\end{aligned}$$

and

$$
\begin{aligned}
v^2(t) &\le \|\Phi(\tau+t,\tau)\pi_2(\tau)\lambda(\tau)\|_{\mathcal{X}} \\
&\quad + \int_0^t \|\Phi(\tau+t,\tau+r)\pi_2(\tau+r)\Delta f(\tau+r)\|_{\mathcal{X}}\,\mathrm{d}r \\
&\le \|\Phi(\tau+t,\tau)\pi_2(\tau)\|_{L(\mathcal{X},\mathcal{X})}\|\pi_2(\tau)\lambda(\tau)\|_{\mathcal{X}} \\
&\quad + \int_0^t \|\Phi(\tau+t,\tau+r)\pi_2(\tau+r)\|_{L(\mathcal{Y},\mathcal{X})}\|\pi_2(\tau+r)\Delta f(\tau+r)\|_{\mathcal{Y}}\,\mathrm{d}r\,.
\end{aligned}
$$

The exponential dichotomy and Lipschitz assumptions (10), (11) and inequality (15) imply that the function $v$ defined by (16) satisfies (13) with

$$
r_1 := \|\pi_1(\tau)[\mu(\tau+T,\tau,x)-\mu(\tau+T,\tau,y)]\|_{\mathcal{X}}\,,\quad r_2 := 0\,.
$$

By the assumptions we have (15), i.e. (14) holds. □

**Lemma 3.6.** *Assume there are positive numbers $L$, $\eta$, $K_1$, $K_2$ such that*

$$
v^i(t) \le K_i \mathrm{e}^{-\eta t} r_2 \qquad \text{for } t\in[0,T] \tag{17}
$$

*holds for each $T>0$ and each solution $v\in C([0,T],\mathbb{R}^2_{\ge 0})$ of (13) with $r_1=0$, $r_2\ge 0$. Then $\mu$ possesses the squeezing property with respect to $\pi$ with the parameters $L$, $\eta$, $K_1$, $K_2$.*

*Proof.* The squeezing property is shown if we find $\eta>0$ and constants $K_1,K_2>0$ such that for all $\tau\in\mathbb{R}$, $x,y\in\mathcal{X}$ and $T>0$ the identity

$$
\pi_1(\tau+T)\mu(\tau+T,\tau,x) = \pi_1(\tau+T)\mu(\tau+T,\tau,y) \tag{18}
$$

implies for all $x'\in\mathcal{X}$ with

$$
\pi_1(\tau)x' = \pi_1(\tau)x \quad\text{and}\quad \|\pi_2(\tau)[x'-y]\|_{\mathcal{X}} \le L\|\pi_1(\tau)[x'-y]\|_{\mathcal{X}} \tag{19}
$$

the estimates

$$
\|\pi_i(t)[\mu(t,\tau,x)-\mu(t,\tau,y)]\|_{\mathcal{X}} \le K_i \mathrm{e}^{-\eta(t-\tau)}\|\pi_2(\tau)[x-x']\|_{\mathcal{X}} \tag{20}
$$

for $i=1,2$ and $t\in[\tau,\tau+T]$.

As in the proof of Lemma 3.5 let $\tau\in\mathbb{R}$, $T>0$ and $x,y,x'\in\mathcal{X}$ with (18), (19) be fixed and let

$$
\lambda(t) := \mu(t,\tau,x)-\mu(t,\tau,y)\,,\quad f_\Delta(t) := f(t,\mu(t,\tau,x)) - f(t,\mu(t,\tau,y))
$$

on $[\tau, \tau+T]$. Then we have

$$\pi_1(t)\lambda(t) = \Phi(t,\tau+T)\pi_1(\tau+T)\lambda(\tau+T) + \int_{\tau+T}^{t} \Phi(t,r)\pi_1(r) f_\Delta(r)\,\mathrm{d}r$$
$$= \int_{\tau+T}^{t} \Phi(t,r)\pi_1(r) f_\Delta(r)\,\mathrm{d}r$$

and

$$\pi_2(t)\lambda(t) = \Phi(t,\tau)\pi_2(\tau)[x'-y] + \Phi(t,\tau)\pi_2(\tau)[x-x']$$
$$+ \int_{\tau}^{t} \Phi(t,r)\pi_2(r) f_\Delta(r)\,\mathrm{d}r$$

on $[\tau, \tau+T]$. Similar to the proof of Lemma 3.5, the exponential dichotomy and Lipschitz assumptions (10), (11) and inequality (19) imply that the function $v : [0,T] \to \mathbb{R}^2_{\geq 0}$ defined by (16) satisfies (13) with

$$r_1 := 0\,, \quad r_2 := \|\pi_2(\tau)[x'-x]\|_{\mathfrak{X}}\,.$$

By assumption we have (17), i.e. (20) holds. $\square$

To estimate the solutions $v$ of (13), we make an excurse to the theory of monotone iterations in ordered Banach spaces, see for example [KLS89].

Let $\mathfrak{B}$ be a Banach space and let $\mathfrak{C}$ be an order cone in $\mathfrak{B}$. The order cone $\mathfrak{C}$ induces a semi-order $\leq_{\mathfrak{C}}$ in $\mathfrak{B}$ by

$$u \leq_{\mathfrak{C}} w \quad :\Longleftrightarrow \quad w - u \in \mathfrak{C}\,.$$

The norm in $\mathfrak{B}$ is called *semi-monotone* if there is a constant $c$ with $\|x\|_{\mathfrak{B}} \leq c\|y\|_{\mathfrak{B}}$ for each $x, y \in \mathfrak{B}$ with $0 \leq_{\mathfrak{C}} x \leq_{\mathfrak{C}} y$. The cone $\mathfrak{C}$ is called normal if the norm in is semi-monotone, and $\mathfrak{C}$ is called solid if $\mathfrak{C}$ contains an open ball with positive radius.

Note that $C([0,T], \mathbb{R}^N_{\geq 0})$ is a normal, solid cone in $C([0,T], \mathbb{R}^N)$.

In a Banach space $\mathfrak{B}$ with normal and solid cone $\mathfrak{C}$, we study the fixed-point problem

$$u = Pu + p \tag{21}$$

with $p \in \mathfrak{B}$ and $P : \mathfrak{B} \to \mathfrak{B}$. We assume that $P$ is completely continuous, increasing,

$$Pu \leq_{\mathfrak{C}} Pv \qquad \text{if } u \leq_{\mathfrak{C}} v\,,$$

subadditive,

$$P(u+v) \leq_{\mathfrak{C}} Pu + Pv\,, \qquad u, v \in \mathfrak{C}\,,$$

and homogeneous with respect to nonnegative factors,

$$P(\lambda u) = \lambda Pu \qquad \lambda \in \mathbb{R}_{\geq 0}\,,\ u \in \mathfrak{C}\,.$$

**Definition 3.7.** A function $w \in \mathfrak{B}$ is called *upper (lower) solution* of (21) if $Pw + p \leq_{\mathfrak{C}} w$ ($w \leq_{\mathfrak{C}} Pw + p$).

Our goal is to ensure the existence of a unique solution $w \in \mathfrak{C}$ of (21) and to estimate lower solutions $v \in \mathfrak{C}$ of (21) by solutions or upper solutions of (21).

For $x, y \in \mathfrak{B}$ with $x \leq_{\mathfrak{C}} y$, we denote by

$$[x, y]_{\mathfrak{C}} := \{z \in \mathfrak{B} : x \leq_{\mathfrak{C}} z \leq_{\mathfrak{C}} \leq y\}$$

the order interval given by $x$ and $y$.

**Lemma 3.8.** *Let $\underline{x}_0 \in \mathfrak{C}$ and $\overline{x}_0 \in \mathfrak{C}$ be a lower and an upper solution of (21) with $\underline{x}_0 \leq_{\mathfrak{C}} \leq \overline{x}_0$. Then the sequences $(\underline{x}_n)_{n\in\mathbb{N}}$ and $(\overline{x}_n)_{n\in\mathbb{N}}$ with $\underline{x}_{n+1} = P\underline{x}_n + p$ and $\overline{x}_{n+1} = P\overline{x}_n + p$ converges to solutions $\underline{x}_*$ and $\overline{x}_*$ of (21) and*

$$\underline{x}_0 \leq_{\mathfrak{C}} \underline{x}_* \leq_{\mathfrak{C}} \overline{x}_* \leq_{\mathfrak{C}} \overline{x}_0 \, . \tag{22}$$

*Proof.* We follow the proof of Theorem 3.1 in [EL75].

By the normality of the cone, there is a constant $c > 0$ such that $\|w_1\|_{\mathfrak{B}} \leq c\|w_2\|_{\mathfrak{B}}$ for all $w_1$, $w_2 \in \mathfrak{C}$ satisfying $w_1 \leq_{\mathfrak{C}} w_2$. Hence, the sets $[\underline{x}_0, \overline{x}_0]_{\mathfrak{C}} \subseteq [0, \overline{x}_0]_{\mathfrak{C}}$ are norm bounded by $c\|\overline{x}_0\|_{\mathfrak{B}}$ and closed. Because of the complete continuity of $P$, there is a convergent subsequence $\left(\underline{x}_{n_k}\right)_{k\in\mathbb{N}}$ of $(\underline{x}_n)_{n\in\mathbb{N}}$ in $[\underline{x}_0, \overline{x}_0]_{\mathfrak{C}}$ with limit $\underline{x}_* \in [\underline{x}_0, \overline{x}_0]_{\mathfrak{C}}$. Suppose $(\underline{x}_n)_{n\in\mathbb{N}}$ is not convergent to $\underline{x}_*$. Then there are $\delta > 0$ and a subsequence $\left(\underline{x}_{m_k}\right)_{k\in\mathbb{N}}$ of $(\underline{x}_n)_{n\in\mathbb{N}}$ satisfying

$$\|\underline{x}_{m_k} - \underline{x}_*\|_{\mathfrak{B}} > \delta \, , \quad \underline{x}_{m_k} \leq_{\mathfrak{C}} \underline{x}_* \, , \quad \underline{x}_{n_k} \leq_{\mathfrak{C}} \underline{x}_{m_k} \quad \text{for } k \in \mathbb{N} \, .$$

By the normality of the cone, we have

$$\|\underline{x}_* - \underline{x}_{m_k}\|_{\mathfrak{B}} \leq_{\mathfrak{C}} c\|\underline{x}_* - \underline{x}_{n_k}\|_{\mathfrak{B}} \to 0$$

for $k \to \infty$ and a contradiction is reached. Thus, $(\underline{x}_n)_{n\in\mathbb{N}}$ is convergent to $\underline{x}_*$.

Analogously, one can show the converges of $(\overline{x}_n)_{n\in\mathbb{N}}$.

By construction we have

$$\underline{x}_0 \leq_{\mathfrak{C}} \underline{x}_1 \leq_{\mathfrak{C}} \underline{x}_2 \leq_{\mathfrak{C}} \cdots \leq_{\mathfrak{C}} \underline{x}_* \leq_{\mathfrak{C}} \overline{x}_* \leq_{\mathfrak{C}} \cdots \leq_{\mathfrak{C}} \overline{x}_2 \leq_{\mathfrak{C}} \overline{x}_1 \leq_{\mathfrak{C}} \overline{x}_0 \, .$$

The inequalities $\underline{x}_n \leq_{\mathfrak{C}} \underline{x}_{n+1} = P\underline{x}_n + p \leq_{\mathfrak{C}} \underline{x}_*$, the convergence $\underline{x}_n \to \underline{x}_*$ and the continuity of $P$ imply

$$\underline{x}_* \leq_{\mathfrak{C}} P\underline{x}_* + p \leq_{\mathfrak{C}} \underline{x}_* \, ,$$

i.e. $\underline{x}_*$ is a solution of (21).

Similarly follows that $\overline{x}_*$ is a solution of (21), too. By construction we have (22). ∎

**Lemma 3.9.** *Assume that there are $y \in \mathrm{int}\mathfrak{C}$ and $\delta \in [0,1[$ with*

$$Py \leq_{\mathfrak{C}} \delta y\,.$$

*Then there is a unique solution $x_*$ of (21) in $\mathfrak{C}$ and*

$$\underline{x} \leq_{\mathfrak{C}} x_* \leq_{\mathfrak{C}} \overline{x} \tag{23}$$

*holds for each lower solution $\underline{x} \in \mathfrak{C}$ and each upper solution $\overline{x} \in \mathfrak{C}$ of (21).*

*Proof.* First we note that by the normality and solidity of $\mathfrak{C}$, for each $x \in \mathfrak{C}$ there is $\|x\|_y$ defined by

$$\|x\|_y := \inf\{\mu \geq 0 :\ -\mu y \leq_{\mathfrak{C}} x \leq_{\mathfrak{C}} \mu y\}\,.$$

Now we show that 0 is the unique solution of

$$x = Px \tag{24}$$

in $\mathfrak{C}$. Since $P$ is subadditive, we have $P0 = P(0+0) \leq_{\mathfrak{C}} P0 + P0$ and hence $0 \leq_{\mathfrak{C}} P0$. From $0 \leq_{\mathfrak{C}} y$ follows $0 \leq P0 \leq_{\mathfrak{C}} Py \leq_{\mathfrak{C}} \delta y$, i.e. $0 \leq_{\mathfrak{C}} P0 \leq_{\mathfrak{C}} \delta y$. Thus $0 \leq_{\mathfrak{C}} P0 \leq_{\mathfrak{C}} \delta^n y$ for all $n \in \mathbb{N}$. Hence $0 = P0$. Let $x \in \mathfrak{C}$ be a solution of (24). Then $0 \leq_{\mathfrak{C}} x \leq_{\mathfrak{C}} \mu(x)y$. By the monotony and homogeneity of $P$ we have $0 \leq_{\mathfrak{C}} x \leq_{\mathfrak{C}} \|x\|_y \delta^n y$ for all $n \in \mathbb{N}$ and hence $x = 0$. Therefore, 0 is the unique solution of (24) in $\mathfrak{C}$.

By the subadditivity of $P$ and because of $p \in \mathfrak{C}$, 0 is a lower solution of (21). Moreover $\lambda y$ with $\lambda \geq \frac{\|q\|_y}{1-\delta}$ is an upper solution of (21). By Lemma 3.8, the existence of at least one solution $x_1 \in \mathfrak{C}$ of (21) follows.

Assume there is another solution $x_2 \in \mathfrak{C}$ of (21). By Lemma 3.8, we find a solution $x_3 \in \mathfrak{C}$ of (21) with $0 \leq_{\mathfrak{C}} x_i \leq_{\mathfrak{C}} \leq x_3 \leq_{\mathfrak{C}} \lambda y$ for $i = 1, 2$ with $\lambda \geq \max\{\|x_1\|_y, \|x_2\|_y, \frac{\|q\|_y}{1-\delta}\}$. Hence, without loss of generality we may assume that $0 \leq_{\mathfrak{C}} x_1 \leq_{\mathfrak{C}} \leq x_2$ and $x_1 \neq x_2$.

Let $w := x_2 - x_1$. Then $0 \leq_{\mathfrak{C}} w$, $w \neq 0$, and, by the subadditivity of $P$,

$$w = Px_2 - Px_1 = P(x_1 + w) - Px_1 \leq_{\mathfrak{C}} Px_1 + Pw - Px_1 = Pw\,.$$

Thus, $w$ is a lower solution of (24). Since $\mu(w)y$ is an upper solution of (24) and $w \leq_{\mathfrak{C}} \|w\|_y y$, Lemma 3.8 with $p = 0$ implies the existence of a solution $v \in [w, y]_{\mathfrak{C}}$ of (24). As above shown, $v = 0$ is the unique solution of (24) in $\mathfrak{C}$. Hence $w = 0$ in contrast to the assumption $x_1 \neq x_2$. Thus (21) possesses a unique solution $x_* \in \mathfrak{C}$.

Now let $\underline{x} \in \mathfrak{C}$, $\overline{x} \in \mathfrak{C}$ be arbitrary lower and upper solutions of (21). By Lemma 3.8 and by the uniqueness of the solution $x_*$ of (21), we have $\underline{x} \leq_{\mathfrak{C}} x_* \leq_{\mathfrak{C}} \lambda y$ with $\lambda \geq \max\{\|\underline{x}\|_y, \|x_*\|_y, \frac{\|q\|_y}{1-\delta}\}$. On the other hand, with the lower solution 0 of (21), we have $0 \leq_{\mathfrak{C}} x_* \leq_{\mathfrak{C}} \overline{x}$. Thus (23) holds. $\square$

In order to apply Lemma 3.9 to our situation, we choose $\mathfrak{B} = C([0,T],\mathbb{R}^2)$ and $\mathfrak{C} = C([0,T],\mathbb{R}^2_{\geq 0})$. Then $\mathfrak{C}$ is a normal. The operator $P = \Lambda$ is increasing and completely continuous, and $p = q$ belongs to $\mathfrak{C}$. So we only have to find a function $w^*$ in the interior of $\mathfrak{C}$ with

$$\Lambda w^* \leq_{\mathfrak{C}} \varepsilon w^* \qquad \text{with some } \varepsilon \in [0,1[\,. \tag{25}$$

Further we can estimate the solutions $v$ of (13) by solutions $\bar{w} \in \mathfrak{C}$ of

$$\Lambda \bar{w} + q \leq_{\mathfrak{C}} \bar{w}\,. \tag{26}$$

## 3.4 Inertial Manifold Theorem for Nonautonomous Evolution Equations

Now we are in a position to prove the following theorem.

**Theorem 3.10 (Inertial Manifolds for Nonautonmous Evolution Equations).** *Under the general assumptions in Sec. 3.2, let $t_* \geq 0$ be fixed with*

$$\psi_* := \lim_{t\to t_*} \psi(t) < \infty\,, \quad \psi(t) > \psi_* \quad \textit{for } t < t_*\,. \tag{27}$$

*Further let*

$$k_5 \geq \delta \int_0^{t_*} \psi(r) \mathrm{e}^{-\delta r}\,\mathrm{d}r + \psi_* \lim_{t\to t_*} \mathrm{e}^{-\delta t} \quad \textit{for all } \delta \in\, ]0, \beta_2 - \beta_1[\,. \tag{28}$$

*Assume that there are positive numbers $\rho_1 < \rho_2$ with*

$$G(\rho_1) = G(\rho_2) = 0\,, \quad G\big|_{[\rho_1,\rho_2]} \neq 0 \tag{29}$$

*and*

$$k_1 k_2 \rho_1 < k_5^{-1} \psi_* \rho_2 \tag{30}$$

*where $G : \mathbb{R}_{>0} \to \mathbb{R}$ is defined by*

$$G(\rho) := \beta_2 - \beta_1 - k_3\gamma_1(1,\rho) - k_4 k_5 \rho^{-1} \gamma_2(1,\rho)\,.$$

*Then there are positive numbers $\eta_1 < \eta_2$ with*

$$\eta_i = \beta_1 + k_3\gamma_1(1,\rho_i) = \beta_2 - k_4 k_5 \rho_i^{-1} \gamma_2(1,\rho_i)\,,$$

*and the claim of Theorem 3.2 holds for the two-parameter semi-flow $\mu$ generated by* (8) *with*

$$\eta := \eta_2\,, \quad L \in\, ]k_1\rho_1, k_2^{-1} k_5^{-1} \psi_* \rho_2[\,, \tag{31}$$

*and*

$$K_1 := \frac{k_2 k_5}{\rho_2 \psi_* - k_2 k_5 L}\,, \quad K_2 := \rho_2 K_1\,. \tag{32}$$

*Remark* 3.11. Because of $\lim_{\rho\to 0} G(\rho) = \lim_{\rho\to\infty} G(\rho) = -\infty$, the existence of $\rho_* > 0$ with $G(\rho_*) > 0$ implies the existence of positive numbers $\rho_1 < \rho_2$ with (29). Since $G(\rho_1) = 0$ and $\rho_1 < \rho_*$ imply

$$\rho_1 < \frac{k_4 k_5 \rho_*^{-1} \gamma_2(1,\rho_*)}{\beta_2 - \beta_1 - k_3\gamma_1(1,\rho_*)} \,,$$

the inequality (30) holds if

$$\beta_2 - \beta_1 > k_3\gamma_1(1,\rho_*) + \frac{k_1 k_2 k_4 k_5^2}{\psi_* \rho_*}\gamma_2(1,\rho_*) \,. \tag{33}$$

Since (33) implies $G(\rho_*) > 0$, the conditions (29), (30) can be replaced by (33) for some $\rho_* > 0$.

*Proof.* (of Theorem 3.10) We show that the two-parameter semiflow $\mu$ generated by (8) satisfies the assumptions of Theorem 3.2. We split the proof into five parts: First we show that (29) implies the existence of solutions of (25) and (26). Then we verify the four required properties for the two-parameter semiflow $\mu$.

**Step 1: Determining of Solutions of (25) and (26)**

In order to find a solution $w^*$ of (25), first we look for $w \in \mathfrak{C}$ in the form

$$w(t) = \mathrm{e}^{-\eta t}(1,\rho) \tag{34}$$

with $\rho > 0$ and satisfying

$$w^1(t) \geq (\Lambda w)^1(t) + c_1 \mathrm{e}^{-\beta_1(t-T)} \tag{35}$$

and

$$w^2(t) \geq (\Lambda w)^2(t) + (c_2\rho - k_2 L)\mathrm{e}^{-\beta_2 t} \tag{36}$$

for $t \in [0,T]$ with suitable positive $c_1$ and $c_2$.

If we assume $\eta > \beta_1$ and

$$\eta \geq \beta_1 + k_3\gamma_1(1,\rho) \tag{37}$$

then, because of

$$\begin{aligned} \mathrm{e}^{\eta t}(\Lambda w)^1(t) &= k_3\gamma_1(1,\rho)\int_t^T \mathrm{e}^{(\eta-\beta_1)(t-r)}\,\mathrm{d}r \\ &= \frac{k_3\gamma_1(1,\rho)}{\eta-\beta_1}\left(1 - \mathrm{e}^{(\eta-\beta_1)(t-T)}\right), \end{aligned}$$

we may choose

$$c_1 = c_1(\rho,\eta) := \frac{k_3\gamma_1(1,\rho)}{\eta-\beta_1}\mathrm{e}^{-\eta T} \tag{38}$$

in order to satisfy (35). Remains to satisfy (36).

Inserting (34) in (36) and dividing by $\rho e^{-\eta t}$, we have to satisfy

$$1 \geq H(t, \rho, \eta) \tag{39}$$

for $t \in [0, T]$ where $H : [0, \infty[ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(t, \rho, \eta) := \frac{\gamma_2(1, \rho)}{\rho} k_4 \int_0^t \psi(r) e^{-(\beta_2 - \eta) r} \, dr + c_2 e^{-(\beta_2 - \eta) t} .$$

We choose

$$c_2 = c_2(\rho, \eta) := \frac{\gamma_2(1, \rho)}{(\beta_2 - \eta)\rho} k_4 \psi_* .$$

Because of

$$D_1 H(t, \rho, \eta) = \left( -(\beta_2 - \eta) c_2 + \frac{\gamma_2(1, \rho)}{\rho} k_4 \psi(t) \right) e^{-(\beta_2 - \eta) t}$$

and because of the monotonicity of $\psi$, the function $H(\cdot, \rho, \eta)$ is maximized at $t_*$. Hence we have

$$\begin{aligned} H(t, \rho, \eta) &\leq H(t_*, \rho, \eta) \\ &= \frac{\gamma_2(1, \rho)}{\rho} k_4 \left( \int_0^{t_*} \psi(r) e^{-(\beta_2 - \eta) r} dr + (\beta_2 - \eta)^{-1} \psi_* e^{-(\beta_2 - \eta) t_*} \right) \end{aligned}$$

for all $t \geq 0$. Because of (28), inequality (39) is satisfied if

$$\frac{\gamma_2(1, \rho)}{\rho} k_4 k_5 \leq \beta_2 - \eta . \tag{40}$$

Combining (37) with (40), we find

$$\beta_1 + k_3 \gamma_1(1, \rho) \leq \eta \leq \beta_2 - k_4 k_5 \rho^{-1} \gamma_2(1, \rho) \tag{41}$$

as a sufficient condition for (35) and (36).

By assumption there are positive numbers $\rho_1 < \rho_2$ with (29) and (30). Let

$$\eta_i := \beta_1 + k_3 \gamma_1(1, \rho_i) = \beta_2 - k_4 k_5 \rho^{-1} \gamma_2(1, \rho_i) .$$

Then $(\eta_1, \rho_1)$ and $(\eta_2, \rho_2)$ solve (41).

Moreover, $\eta_2 > \eta_1$. To show this, we note that $\eta_2 \geq \eta_1$ by the monotonicity of $\gamma_1$. Assuming $\eta_1 = \eta_2$ we find $\gamma_1(1, \rho_1) = \gamma_1(1, \rho_2)$ and $\gamma_2(\rho_1^{-1}, 1) = \gamma_2(\rho_2^{-1}, 1)$. By the convexity of the $\gamma_1$- and $\gamma_2$-balls we had $\gamma_1(1, \rho) = \gamma_1(1, \rho_1)$ and $\gamma_2(\rho^{-1}, 1) = \gamma_2(\rho_1^{-1}, 1)$ for $\rho \in [\rho_1, \rho_2]$. This would imply the constance of $G$ on $[\rho_1, \rho_2]$ in contradiction to (29).

Because of (30) we can choose

$$L \in \left]k_1\rho_1, k_2^{-1}k_5^{-1}\psi_*\rho_2\right[\,. \tag{42}$$

Now we define $w_i \in \mathfrak{C}$, $i = 1, 2$, by

$$w_i(t) := \mathrm{e}^{-\eta_i t}(1, \rho_i)\,. \tag{43}$$

Then

$$w_i^1(t) \geq (\Lambda w_i)^1(t) + \frac{k_3\gamma_1(1,\rho)}{\eta - \beta_1}\mathrm{e}^{-\eta T}\mathrm{e}^{-\beta_1(t-T)}$$

and

$$w_i^2(t) \geq (\Lambda w_i)^2(t) + \left(\rho_i k_5^{-1}\psi_* - k_2 L\right)\mathrm{e}^{-\beta_2 t}$$

on $[0, T]$ because of

$$c_1(\rho_i, \eta_i) = \frac{k_3\gamma_1(1,\rho_i)}{\eta_i - \beta_1} = 1\,, \quad c_2(\rho_i, \eta_i) = \frac{\gamma_2(1,\rho_i)k_4\psi_*}{(\beta_2 - \eta_i)\rho_i} = k_5^{-1}\psi_*\,.$$

Because of (42) we have

$$\rho_2 k_5^{-1}\psi_* > k_2 L \tag{44}$$

and inequality (25) holds for $w^* := w_2$.

Let now $C_1 \in [0, 1]$, $C_2 > 0$ satisfy

$$\begin{aligned} C_2\left(C_1\mathrm{e}^{-\eta_1 T} + (1 - C_1)\mathrm{e}^{-\eta_2 T}\right) &\geq k_1 r_1\,, \\ C_2\left(C_1\rho_1 k_5^{-1}\psi_* + (1 - C_1)\rho_2 k_5^{-1}\psi_* - k_2 L\right) &\geq k_2 r_2\,. \end{aligned} \tag{45}$$

Then

$$\bar{w} := C_2\left(C_1 w_1 + (1 - C_1)w_2\right)$$

solves

$$\Lambda\bar{w} + q \leq_{\mathfrak{C}} \bar{w}\,,$$

and Lemma 3.9 implies

$$v \leq_{\mathfrak{C}} \bar{w}$$

for each solution $v \in \mathfrak{C}$ of (13).

**Step 2: Verification of the Cone Invariance Property**

Because of (44) we can fix

$$\tilde{\rho} \in \left]\max\{\rho_1, k_2 k_5 \psi_*^{-1} L\}, \rho_2\right[\,.$$

Let $r_2 = 0$ and $r_1 \geq 0$. Then (45) is satisfied with

$$C_1 := \frac{\rho_2 - \tilde{\rho}}{\rho_2 - \rho_1}\,, \quad C_2 := \frac{k_1 r_1}{C_1 \mathrm{e}^{-\eta_1 T} + (1 - C_1)\mathrm{e}^{-\eta_2 T}}\,.$$

Thus we find

$$v^2(t) \leq \bar{w}^2(t) = C_2 \left( C_1 w_1^2(t) + (1 - C_1) w_2^2(t) \right) = \bar{L}(\tilde{\rho}, t) r_1$$

for $t \in [0, T]$ with

$$\bar{L}(\tilde{\rho}, t) = k_1 \frac{(\rho_2 - \tilde{\rho})\rho_1 \mathrm{e}^{-\eta_1 t} + (\tilde{\rho} - \rho_1)\rho_2 \mathrm{e}^{-\eta_2 t}}{(\rho_2 - \tilde{\rho})\mathrm{e}^{-\eta_1 T} + (\tilde{\rho} - \rho_1)\mathrm{e}^{-\eta_2 T}}\,.$$

Especially we have

$$v^2(T) \leq \bar{L}(\tilde{\rho}, T) r_1\,.$$

The inequalities $\eta_2 > \eta_1 > \beta_1$ imply

$$\bar{L}(\tilde{\rho}, T) \to k_1 \rho_1 \qquad \text{as } T \to \infty\,.$$

Hence there are $T_0 \geq 0$ and $L \geq 0$ with (12), if the additional inequality

$$k_1 \rho_1 < L$$

holds, which trivially follows from (42).

By Lemma 3.5, the cone invariance property of $\mu$ as required in Theorem 3.3 is verified.

**Step 3: Verification of the Squeezing Property**

Now let $r_1 = 0$ and $r_2 \geq 0$. Then we may choose

$$C_1 := 0\,, \quad C_2 := \frac{k_2 k_5 r_2}{\rho_2 \psi_* - k_2 k_5 L}$$

in order to satisfy (45). Thus we find

$$v(t) \leq \frac{k_2 r_2}{\rho_2 \psi_* - k_2 k_5 L} \mathrm{e}^{-\eta_2 t}(1, \rho_2) \qquad \text{for } t \in [0, T]\,.$$

Hence (17) holds with

$$\eta := \eta_2\,, \quad K_1 := \frac{k_2 k_5}{\rho_2 \psi_* - k_2 k_5 L}\,, \quad K_2 := \rho_2 K_1$$

and $L$ satisfying (42). By Lemma 3.6, the squeezing property of $\mu$ as required in Theorem 3.3 is verified.

**Step 4: Verification of the Boundedness Property**

By the boundedness of $f$ there is a number $F \geq 0$ with

$$\|f(x)\|_{\mathcal{Y}} \leq F \qquad \text{for } x \in \mathcal{X} \,.$$

Thus, for $\tau \in \mathbb{R}$, $t \geq \tau$, $x \in \mathcal{X}$,

$$\pi_2(t)\mu(t,\tau,x) = \Phi(t,\tau)\pi_2(\tau)x + \int_\tau^t \Phi(t,r)\pi_2(r)f(r,\mu(r,\tau,x))\,\mathrm{d}r$$

and, by the exponential dichotomy conditions (10),

$$\begin{aligned}
\|\pi_2(t)\mu(t,\tau,x)\|_{\mathcal{X}} &\leq \|\Phi(t,\tau)\pi_2(\tau)\|_{L(\mathcal{X},\mathcal{X})}\|\pi_2(\tau)x\|_{\mathcal{X}} \\
&\qquad + \int_\tau^t \|\Phi(t,r)\pi_2(r)\|_{L(\mathcal{Y},\mathcal{X})}\|f(r,\mu(r,\tau,x))\|_{\mathcal{X}}\,\mathrm{d}r \\
&\leq k_2\mathrm{e}^{-\beta_2(t-\tau)}\|\pi_2(\tau)x\|_{\mathcal{X}} + Fk_4\int_\tau^t \psi(t-r)\mathrm{e}^{-\beta_2(t-r)}\,\mathrm{d}r \\
&= k_2\mathrm{e}^{-\beta_2(t-\tau)}\|\pi_2(\tau)x\|_{\mathcal{X}} + Fk_4\int_0^{t-\tau} \psi(r)\mathrm{e}^{-\beta_2(r)}\,\mathrm{d}r \\
&\leq k_2\mathrm{e}^{-\beta_2(t-\tau)}\|\pi_2(\tau)x\|_{\mathcal{X}} + Fk_4\int_0^{\infty} \psi(r)\mathrm{e}^{-\beta_2(r)}\,\mathrm{d}r \,.
\end{aligned}$$

Thus, for any $t,\tau$ with $t \geq \tau$ and any $M_1 \geq 0$ there is an $M_2 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_2(\tau)x\|_{\mathcal{X}} \leq M_1$ we have $\|\pi_2(t)\mu(t,\tau,x)\|_{\mathcal{X}} \leq M_2$, i.e. the two-parameter flow possesses the boundedness property of $\mu$ as required in Theorem 3.2.

**Step 5: Verification of the Coercivity Property**

For $\tau \in \mathbb{R}$, $t \in [\tau, \tau+T]$, $x \in \mathcal{X}$, we have

$$\begin{aligned}
\pi_1(t)\mu(t,\tau,x) &= \Phi(t,\tau+T)\pi_1(\tau+T)\mu(\tau+T,\tau,x) \\
&\qquad + \int_{\tau+T}^t \Phi(t,r)\pi_1(r)f(r,\mu(r,\tau,x))\,\mathrm{d}r
\end{aligned}$$

and hence

$$\begin{aligned}
\pi_1(\tau)x &= \Phi(\tau,\tau+T)\pi_1(\tau+T)\mu(\tau+T,\tau,x) \\
&\qquad + \int_{\tau+T}^\tau \Phi(\tau,r)\pi_1(r)f(r,\mu(r,\tau,x))\,\mathrm{d}r \,.
\end{aligned}$$

The exponential dichotomy conditions (10) imply

$$\begin{aligned}
\|\pi_1(\tau)x\|_{\mathcal{X}} &\le \|\Phi(\tau,\tau+T)\pi_1(\tau+T)\|_{L(\mathcal{X},\mathcal{X})}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} \\
&\quad + F\int_\tau^{\tau+T}\|\Phi(\tau,r)\pi_1(r)\|_{L(\mathcal{Y},\mathcal{X})}\,\mathrm{d}r \\
&\le k_1\mathrm{e}^{\beta_1 T}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} + Fk_3\int_\tau^{\tau+T}\mathrm{e}^{-\beta_1(\tau-r)}\,\mathrm{d}r \\
&= k_1\mathrm{e}^{\beta_1 T}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} + \frac{Fk_3}{\beta_1}(\mathrm{e}^{\beta_1 T}-1)\,.
\end{aligned}$$

Hence

$$\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} \ge \frac{1}{k_1}\|\pi_1(\tau)x\|_{\mathcal{X}} - \frac{Fk_3}{\beta_1 k_1}(1-\mathrm{e}^{-\beta_1 T})$$

for $T\ge 0$, $\tau\in\mathbb{R}$, $x\in\mathcal{X}$ which shows the coercivity property of the two-parameter flow $\mu$ as required in Theorem 3.2.

Thus all assumption of Theorem 3.2 are verified and Theorem 3.2 implies the existence of an inertial manifold with the stated properties. □

## 3.5 Pullback Attractors for Nonautonomous Evolution Equations

In addition to the assumptions of Subsection 3.2 we suppose:

- There is a Banach space $\mathcal{Z}$ which is compactly embedded in $\mathcal{X}$.
- There are $k_7$, $\gamma\in\,]0,1[$ and $\beta_0>0$ such that the evolution operator $\Phi$ of (8) maps $\mathcal{Y}$ into $\mathcal{Z}$ and satisfies

$$\begin{aligned}
&\|\Phi(t,s)\|_{L(\mathcal{X},\mathcal{X})} \le k_7\mathrm{e}^{-\beta_0(t-s)}\,, \\
&\|\Phi(t,s)\|_{L(\mathcal{Y},\mathcal{X})} \le k_7(t-s)^\alpha\mathrm{e}^{-\beta_0(t-s)}\,, \\
&\|\Phi(t,s)\|_{L(\mathcal{Y},\mathcal{Z})} \le k_7(t-s)^\gamma\mathrm{e}^{-\beta_0(t-s)} \qquad \text{for } t>s\,.
\end{aligned}$$

- There is an $\ell_0\ge 0$ with

$$\|f(t,x)\|_{\mathcal{Y}} \le \ell_0 \qquad \text{for } (t,x)\in\mathbb{R}\times\mathcal{X}\,.$$

**Theorem 3.12.** *Under the above assumptions there is a global pullback attractor $\mathcal{A} = (\mathcal{A}(\tau))_{\tau\in\mathbb{R}}$ for* (8). *If $\pi_1$ is tempered from above then $\mathcal{A}$ is contained in the inertial manifold $\mathcal{M}$.*

*Remark* 3.13. The assumption of Theorem 3.12 on the linear part $A(t)$ of (8) are easily to satisfy if $A$ is a time-independent, selfadjoint positive operator on the Hilbert space $\mathcal{Y}$. In dependence on the nonlinearity $f$, the Hilbert space $\mathcal{X}$ can be chosen as the domain $D(A^\alpha)$

of a power $A^\alpha$ of $A$ with $\alpha \in [0,1[$. Further, we may choose $\mathcal{Z} = D(A^\gamma)$ with some $\gamma \in ]\alpha, 1[$. Since $A$ is time-independent, the projector $\pi_1$ from $\mathcal{Y}$ onto the subspace spanned by the first $N$ eigenvectors $e_1, \ldots, e_N$ of $A$ belonging to the $N$ smallest eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ of $A$ is also time-independent and hence trivially tempered from above.

*Proof.* (of Theorem 3.12) Let $\mu$ denote the two-parameter semiflow generated by (8). Thus $\varphi(t,\tau)x = \mu(t+\tau,\tau,x)$ and

$$\varphi(t,\tau)x = \Phi(t+\tau,\tau)\xi + \int_0^t \Phi(t+\tau,s+\tau) f(\tau+s,\varphi(s,\tau)x)\,\mathrm{d}s\,.$$

By Lemma 2.19 it is sufficient to construct a globally pullback absorbing set $\tilde{\mathcal{A}}$ such that $\hat{\mathcal{A}}_1$,

$$\hat{\mathcal{A}}_1(\tau) := \mathrm{cl}\left(\varphi(1,\tau-1)\bigcup_{s\geq 0}\varphi(s,\tau-s)\tilde{\mathcal{A}}(\tau-s)\right) \qquad \text{for } \tau \in \mathbb{R}\,,$$

is compact.

For $t \geq 0$, $\tau \in \mathbb{R}$ and $x \in \mathcal{X}$, we estimate

$$\begin{aligned}
\|\varphi(t,\tau)x\|_{\mathcal{X}} \leq{}& \|\Phi(t+\tau,\tau)\|_{L(\mathcal{X},\mathcal{X})}\|x\|_{\mathcal{X}} \\
&+ \int_0^t \|\Phi(t+\tau,s+\tau)\|_{L(\mathcal{Y},\mathcal{X})}\|f(\tau+s,\varphi(s,\tau)x)\|_{\mathcal{Y}}\,\mathrm{d}s \\
\leq{}& k_7 \mathrm{e}^{-\beta_0 t}\|x\|_{\mathcal{X}} + \ell_0 k_7 \int_0^t t^{-\alpha}\mathrm{e}^{-\beta_0 t}\,\mathrm{d}r \\
\leq{}& k_7 \mathrm{e}^{-\beta_0 t}\|x\|_{\mathcal{X}} + \ell_0 k_7 k_8
\end{aligned}$$

with

$$k_8 := \int_0^\infty t^{-\alpha}\mathrm{e}^{-\beta_0 t}\,\mathrm{d}r\,.$$

Let $r_0 > \ell_0 k_7 k_8$ and

$$\tilde{\mathcal{A}}(\tau) := \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq r_0\} \qquad \text{for } \tau \in \mathbb{R}\,.$$

Then $\tilde{\mathcal{A}}$ is globally pullback attracting,

$$\varphi(t,\tau-t)\mathcal{D} \subseteq \tilde{\mathcal{A}}(\tau)) \qquad \text{for all } t \geq T(|D|)\,,$$

with

$$|D|_{\mathcal{Y}} := \sup_{x\in\mathcal{D}}\|x\|_{\mathcal{Y}}\,, \quad T(r) := \beta_0^{-1}\ln\left(\frac{k_7 r}{r_0 - \ell_0 k_7 k_8}\right).$$

Let $\tau \in \mathbb{R}$. We show that $\hat{\mathcal{A}}_1(\tau)$ are compact subsets of $\mathcal{X}$. First we note that, for $s \geq 0$, $\varphi(s, \tau - s)\tilde{\mathcal{A}}(\tau - s)$ is bounded in $\mathcal{X}$ by $r_1 := k_7 r_0 + \ell_0 k_7 k_8$. Thus $\tilde{\mathcal{A}}_0(\tau) := \bigcup_{s \geq 0} \varphi(s, \tau - s)\tilde{\mathcal{A}}(\tau - s)$ is bounded in $\mathcal{X}$ by $r_1$. Let $x \in \mathcal{X}$ with $\|x\|_{\mathcal{X}} \leq r_1$. Then

$$
\begin{aligned}
&\|\varphi(1, \tau - 1)x)\|_{\mathcal{Z}} \\
&\qquad \leq \|\Phi(\tau, \tau - 1)\|_{L(\mathcal{Y},\mathcal{Z})}\|x\|_{\mathcal{Y}} \\
&\qquad\qquad + \int_0^1 \|\Phi(\tau, s + \tau - 1)\|_{L(\mathcal{Y},\mathcal{Z})}\|f(s + \tau - 1, \varphi(s, \tau - 1)x)\|_{\mathcal{Y}} \,\mathrm{d}s \\
&\qquad \leq k_7 \mathrm{e}^{-\beta_0} r_1 + \ell_0 k_7 \int_0^1 (1 - s)^{-\gamma} \mathrm{e}^{-\beta_0(1-s)} \,\mathrm{d}s \\
&\qquad = k_7 \mathrm{e}^{-\beta_0} r_1 + \ell_0 k_7 \int_0^1 s \mathrm{e}^{-\beta_0 s} \,\mathrm{d}s =: r_2 \,.
\end{aligned}
$$

Hence $\varphi(1, \tau - 1)\tilde{\mathcal{A}}_0(\tau)$ is bounded in $\mathcal{Z}$ by $r_2$. Since $\mathcal{Z}$ is compactly embedded in $\mathcal{X}$, the closure $\hat{\mathcal{A}}_1(\tau)$ of $\varphi(1, \tau - 1)\tilde{\mathcal{A}}_0(\tau)$ in $\mathcal{X}$ is a compact subset of $\mathcal{X}$ for all $\tau \in \mathbb{R}$.

Remains to show that $\hat{\mathcal{A}}_1$ is tempered from above. Since $\varphi(1, \tau - 1)\tilde{\mathcal{A}}_0(\tau) \subseteq \tilde{\mathcal{A}}_0(\tau)$, the set $\hat{\mathcal{A}}_1(\tau)$ is bounded in $\mathcal{X}$ by $r_1$ for all $\tau \in \mathbb{R}$. Hence $\hat{\mathcal{A}}_1$ is trivially tempered from above. $\square$

# 4 Conclusion

Exponential dichotomy conditions of the form (10) are used, for example, in [Hen81], [Tem97], [BdMCR98], [LL99], [CS01]. There $k_3 = \beta_1^\alpha k_1$, $k_4 = \beta_2^\alpha$ with some $\alpha \in [0, 1[$ depending on the spaces $\mathcal{X}$ and $\mathcal{Y}$, and $\psi(t) = \beta_2^{-\alpha} \max\{t^{-\alpha}, 1\}$, $\psi(t) = \beta_2^{-\alpha} t^{-\alpha} + 1$, or $\psi(t) = \max\{\alpha^\alpha \beta_2^{-\alpha} t^{-\alpha}, 1\}$ where $0^0 := 1$. If $A$ is a time-independent sectorial operator, then usually $\mathcal{X}$ is the domain $D((A + a)^\alpha)$ of the power $(A + a)^\alpha$ of $A + a$ with some $\alpha \in [0, 1[$ and some $a \in \mathbb{R}$. If $\mathcal{X} = \mathcal{Y}$ then we may choose $\alpha = 0$ and $\psi = 1$.

In the special case that $A$ is a time-independent, selfadjoint positive linear operator with compact resolvent and dense domain $D(A)$ on the Hilbert space $\mathcal{Y}$, usually one uses $\mathcal{X} = D(A^\alpha)$ with some $\alpha \in [0, 1[$. Let $\pi_1$ be the orthogonal projector from $\mathcal{Y}$ onto the linear subspace spanned by the $N$ eigenvectors of $A$ corresponding to the first $N$ eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ (counted with their multiplicity). Then we may choose $\beta_1 = \lambda_N$, $\beta_2 = \lambda_{N+1}$, $k_1 = k_2 = 1$, $k_3 = \beta_1^\alpha$, $k_4 = \beta_2^\alpha$, $\psi(t) := \max\{\alpha^\alpha \beta_2^{-\alpha} t^{-\alpha}, 1\}$, see for example [FST88, Lemma 3.1].

In [LL99] (and with $\mathcal{X} = \mathcal{Y}$), a Lipschitz inequality of the form

$$\|\pi_i(t)[f(t, x) - f(t, y)]\|_{\mathcal{X}} \leq \ell_i \max\{\|\pi_1(t)[x - y]\|_{\mathcal{X}}, \|\pi_2(t)[x - y]\|_{\mathcal{X}}\}$$

is utilized. This special form of a Lipschitz inequality is contained in our Lipschitz assumption with $\gamma_i(w) = \ell_i |w|_\infty$ and $|\cdot|_\infty$ as the maximum norm in $\mathbb{R}^2$. The standard Lipschitz inequality

$$\|f(t,x)-f(t,y)\|_{\mathcal{Y}} \le \ell \|x-y\|_{\mathcal{X}}$$

in a Hilbert space $\mathcal{Y}$ and with orthogonal projectors $\pi_1(t)$ leads to (11) with $\gamma_i(w)=\ell|w|_2$ or $\gamma_i(w)=\ell|w|_1$, where $|\cdot|_1$ denotes the sum norm and $|\cdot|_2$ denotes the euclidean norm in $\mathbb{R}^2$.

In order to compare known results with ours we verify the assumptions of Theorem 3.10 for different forms of Lipschitz estimates for $f$ and for concrete functions $\psi$ in the exponential dichotomy property.

**Corollary 4.1.** *Under the general assumptions in Sec. 3.2, let $f$ satisfy* (11) *with weighted maximum norms*

$$\gamma_i(w)=\ell_i \max\{|w^1|,|w^2|\}\,, \quad \ell_i>0 \qquad \textit{for } w\in\mathbb{R}^2\,. \tag{46}$$

*Let $t_*$, $\psi_*$ and $k_5$ with the properties as in Theorem 3.10. Then condition* (29) *and hence the claim of Theorem 3.10 hold if*

$$\beta_2-\beta_1 > \frac{k_3\ell_1+k_4k_5\ell_2}{2} + \sqrt{\frac{(k_3\ell_1-k_4k_5\ell_2)^2}{4}+\frac{k_1k_2k_3k_4k_5^2\ell_1\ell_2}{\psi_*}}\,. \tag{47}$$

*Proof.* Calculating the zeroes of $G$ with

$$G(\rho)=\beta_2-\beta_1-k_3\ell_1\max\{1,\rho\}-k_4k_5\ell_2\rho^{-1}\max\{1,\rho\}\,,$$

we find (47) as sufficient and necessary condition for (29) and (30). □

Latushkin and Layton [LL99] consider $-A$ as generator of a strongly continuous semigroup on the Banach space $\mathcal{X}=\mathcal{Y}$. Let $\mathcal{X}$ be the direct sum of two subspace $\mathcal{X}_1$ and $\mathcal{X}_2$ and let $\pi_i$ the projector from $\mathcal{X}$ onto $\mathcal{X}_i$. Assuming exponential dichotomy conditions (10) with $k_1=k_2=1$ (and $k_3=k_4=1$, $\psi=1$ because of $\mathcal{X}=\mathcal{Y}$) and $f(0)=0$ and

$$\|\pi_i[f(x)-f(y)]\|_{\mathcal{X}} \le \ell_i \max\{\|\pi_1[x-y]\|_{\mathcal{X}}, \|\pi_2[x-y]\|_{\mathcal{X}}\}$$

for the time-independent nonlinearity $f$, they found

$$\beta_2-\beta_1>\ell_1+\ell_2 \tag{48}$$

as optimal spectral gap condition. They extended this result to $(-A(t))$ as a family of linear operators on the Banach space $\mathcal{X}=\mathcal{Y}$ generating a strongly continuous semiflow, see [LL99] too. Again, assuming exponential dichotomy conditions (10) with $k_1=k_2=k_3=k_4=1$, $\psi=1$, and the Lipschitz estimate

$$\|\pi_i(t)[f(t,x)-f(t,y)]\|_{\mathcal{X}} \le \ell_i \max\{\|\pi_1(t)[x-y]\|_{\mathcal{X}}, \|\pi_2(t)[x-y]\|_{\mathcal{X}}\}$$

and $f(t,0)=0$ for $f$, they found the spectral gap condition (48) for nonautonomous inertial manifolds.

Since $\mathcal{X}=\mathcal{Y}$, we have $k_3=k_1$, $k_4=k_2$, $\psi=1$. Thus we have to choose $t_*=0$ and find $k_5=\psi_*=1$. Our condition (47) reduces to

$$\beta_2-\beta_1>k_1\ell_1+k_2\ell_2\,,$$

which in the special case of $k_1=k_2=1$ reduces to the optimal spectral gap condition (48) found by Y. Latushkin and B. Layton, [LL99].

**Corollary 4.2.** *Under the general assumptions in Sec. 3.2, let $f$ satisfy* (11) *with weighted sum norms*

$$\gamma_i(w)=\ell_{i1}|w^1|+\ell_{i2}|w^2|\,,\quad \ell_{i1},\ell_{i2}>0\qquad \text{for } w\in\mathbb{R}^2. \tag{49}$$

*Let $t_*$, $\psi_*$ and $k_5$ with the properties as in Theorem 3.10. Then condition* (29) *and hence the claim of Theorem 3.10 hold if*

$$\beta_2-\beta_1>k_3\ell_{11}+k_4k_5\ell_{22}+\frac{k_1k_2k_5+\psi_*}{\sqrt{k_1k_2\psi_*}}\sqrt{\ell_{12}\ell_{21}k_3k_4}\,. \tag{50}$$

*Proof.* Calculating the zeroes of $G$ with

$$G(\rho)=\beta_2-\beta_1-k_3\ell_{11}-k_3\ell_{12}\rho-k_4k_5\ell_{21}\rho^{-1}-k_4k_5\ell_{22}\,,$$

we find (50) as a sufficient and necessary condition for (29) and (30). $\square$

First let

$$\psi(t):=\max\{\alpha^\alpha\beta_2^{-\alpha}t^{-\alpha},1\}$$

as in [FST88, Lemma 3.1]. Here and in the following we set $0^0:=1$ in order to continuously extend the expression for $\psi$ to the limit case $\alpha=0$. We choose $t_*:=\alpha\beta_2^{-1}$ and hence we have $\psi_*=1$. To satisfy (28) we note that

$$\delta\int_0^{t_*}\psi(r)\mathrm{e}^{-\delta r}\,\mathrm{d}r+\psi_*\lim_{t\to t_*}\mathrm{e}^{-\delta t}=\delta^\alpha\alpha^\alpha\beta_2^{-\alpha}\int_0^{\delta\alpha\beta_2^{-1}}r^{-\alpha}\mathrm{e}^{-r}\,\mathrm{d}r+\mathrm{e}^{-\delta\alpha\beta_2^{-1}}\,.$$

The right hand side is monotonously increasing in $\delta>0$. Therefore, we may satisfy (28) for $0<\delta\le\beta_2-\beta_1\le\beta_2$ with

$$k_6:=\alpha^\alpha\int_0^\alpha r^{-\alpha}\mathrm{e}^{-r}\,\mathrm{d}r+\mathrm{e}^{-\alpha}-1\ge 0\,,\quad k_5:=1+\frac{(\beta_2-\beta_1)^\alpha}{\beta_2^\alpha}k_6\,.$$

If $k_3=k_1\beta_1^\alpha$, $k_4=k_2\beta_2^\alpha$ and $\ell_{11}=\ell_{12}=\ell_{21}=\ell_{22}=\ell$, condition (50) reads

$$\beta_2-\beta_1>\left(k_1\beta_1^\alpha+k_2k_5\beta_2^\alpha+(1+k_5k_1k_2)\sqrt{\beta_1^\alpha\beta_2^\alpha}\right)\ell\,. \tag{51}$$

Now we assume that $\mathcal{Y}$ is a Hilbert space, $A$ is a time-independent, selfadjoint, positive linear operator on $\mathcal{Y}$ with dense domain and compact resolvent, $f$ is a time-independent, continuous mapping from $\mathcal{X} = D(A^\alpha)$ into $\mathcal{Y}$ satisfying a global Lipschitz condition $\|f(x) - f(y)\|_{\mathcal{Y}} \le \ell \|x - y\|_{\mathcal{X}}$. Let $\lambda_1 \le \lambda_2 \le \cdots$ denote the eigenvalues of $A$ counted with their multiplicity and let $\pi_1$ be the orthogonal projector from $\mathcal{Y}$ onto the $N$-dimensional subspace spanned by the first $N$ eigenvectors of $A$. Then (10) is satisfied with $k_1 = k_2 = 1$, $\beta_1 = \lambda_N$, $\beta_2 = \lambda_{N+1}$, and we find the spectral gap condition

$$\lambda_{N+1} - \lambda_N > \left( \left( \lambda_N^{\alpha/2} + \lambda_{N+1}^{\alpha/2} \right)^2 + k_6 \frac{\lambda_N^{\alpha/2} + \lambda_{N+1}^{\alpha/2}}{\lambda_{N+1}^{\alpha/2}} (\lambda_{N+1} - \lambda_N)^\alpha \right) \ell \tag{52}$$

which holds if

$$\lambda_{N+1} - \lambda_N > 2 \left( \lambda_N^\alpha + \lambda_{N+1}^\alpha + k_6 (\lambda_{N+1} - \lambda_N)^\alpha \right) \ell \,. \tag{53}$$

Romanov [Rom94] showed that a spectral gap condition

$$\lambda_{N+1} - \lambda_N > (\lambda_{N+1}^\alpha + \lambda_N^\alpha) \ell \tag{54}$$

is sufficient for the existence of an $N$-dimesional (autonomous) inertial manifold. Note that the right hand side in (53) is at most by the factor $2(1 + k_6)$ worse than the right hand side in the sharp condition (54), where $k_6 = 0$ for $\alpha = 0$ and $k_6 \approx 0.46$ for $\alpha = \frac{1}{2}$. There are two reasons why our condition (53) is worse than (54): Firstly, Romanov used the Fourier expansion of the points in the Hilbert space $\mathcal{Y}$ and the corresponding spectral properties of $A$ instead of the exponential dichotomy condition. Secondly, he used an indefinite quadratic form along the difference of trajectories in order to show needed properties of the graph transformation mapping. This approach is more effective in Hilbert spaces than the more general approach presented here for the verification of the cone invariance and squeezing property. However, using indefinite quadratic forms too, we are also able to show that (54) is sufficient for the cone invariance and squeezing property.

Now let

$$\psi(t) = \beta_2^{-\alpha} t^{-\alpha} + 1$$

as in [Tem97]. Then $t_* = \infty$, $\psi_* = 1$ and we may choose

$$k_6 := \Gamma(1 - \alpha) \,, \quad k_5 := 1 + k_6$$

to satisfy (28). In the special case $\ell_1 = \ell_2 = \ell$, $k_3 = k_1 \beta_1^\alpha$, $k_4 = k_2 \beta_2^\alpha$, condition (50) reads now

$$\beta_2 - \beta_1 > \left( k_1 \beta_1^\alpha + (1 + k_1 k_2 (1 + k_6)) \sqrt{\beta_1^\alpha \beta_2^\alpha} + k_2 (1 + k_6) \beta_2^\alpha \right) \ell \,. \tag{55}$$

For a Banach space $\mathcal{Y}$, a time-independent, sectorial linear operator $A$ on $\mathcal{Y}$ with dense domain $D(A)$, and a time-independent, continuous mapping $f$ from $\mathcal{X} = D(A^\alpha)$ into $\mathcal{Y}$

satisfying a global Lipschitz condition $\|f(x)-f(y)\|_{\mathcal{Y}} \leq \ell\|x-y\|_{\mathcal{X}}$ where $\mathcal{X}=D((A+a)^\alpha)$ with fixed $a \in \mathbb{R}$, $\alpha \in [0,1[$ with $\Re\sigma(A)+a>0$, and under assumption (10) with $\psi(t)=\beta_2^{-\alpha}t^{-\alpha}+1$, Temam showed ([Tem97], Theorem IX.2.1) that there are constants $c_1$ and $c_2$ independent of the Lipschitz constant $\ell$ and the boundedness constant $\ell_0$ of the nonlinearity $f$, such that the spectral gap condition

$$\beta_2-\beta_1 \geq c_1(\ell_0+\ell+\ell^2)(\beta_2^\alpha+\beta_1^\alpha)\,, \quad \beta_1^{1-\alpha} \geq c_2(\ell_0+\ell) \tag{56}$$

implies the existence of an autonomous inertial manifold in the autonomous case.

Note that our condition (55) is of similar form as (56) but (55) contains only known constants and is applicable for the nonautonomous case, too. Moreover, in contrast to (56), in our condition (55), the right hand side is linear in the Lipschitz constant $\ell$.

Finally let

$$\psi(t)=\alpha^\alpha\beta_2^{-\alpha}t^{-\alpha}+1\,.$$

Then we choose $t_* := \infty$ and have $\psi_*=1$. Since

$$\begin{aligned}\delta\int_0^{t_*}\psi(\tau)\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau &= \delta\beta_2^{-\alpha}\int_0^\infty(\alpha^\alpha\tau^{-\alpha}+\beta_2^\alpha)\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau\\ &= \alpha^\alpha\delta^\alpha\beta_2^{-\alpha}\Gamma(1-\alpha)+1\,,\end{aligned}$$

for $\delta>0$, we may choose

$$k_6 := \alpha^\alpha\Gamma(1-\alpha)\,, \quad k_5 := 1+\frac{(\beta_2-\beta_1)^\alpha}{\beta_2^\alpha}k_6 \tag{57}$$

in order to satisfy (28) for $\delta \in ]0,\beta_2-\beta_1[$. In the special case $\ell_1=\ell_2=\ell$, $k_3=k_1\beta_1^\alpha$, $k_4=k_2\beta_2^\alpha$, condition (50) takes the form (51).

Whilst we are yet not in a position to deal with retardation or stochastic perturbation, we try to compare our result with that one found by L. Boutet de Monvel, I.D. Chueshov and A.V. Rezounenko in [BdMCR98] and by I.D. Chueshov, M. Scheutzow in [CS01] for the special case of a semilinear parabolic equation without perturbation and without retardation. There $\mathcal{Y}$ is a Hilbert space, $A$ is a time-independent, selfadjoint, positive linear operator on $\mathcal{Y}$ with dense domain and compact resolvent, $f$ is a continuous mapping from $\mathbb{R}\times\mathcal{X}$, $\mathcal{X}=D(A^\alpha)$, into $\mathcal{Y}$ satisfying a global Lipschitz condition $\|f(t,x)-f(t,y)\|_{\mathcal{Y}} \leq \ell\|x-y\|_{\mathcal{X}}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues of $A$ counted with their multiplicity and let $\pi_1$ be the orthogonal projector from $\mathcal{Y}$ onto the $N$-dimensional subspace spanned by the first $N$ eigenvectors of $A$. Chueshov and Scheutzow [CS01] found the spectral gap condition

$$\lambda_{N+1}-\lambda_N > 2\left(\lambda_N^\alpha+\lambda_{N+1}^\alpha+\alpha^\alpha\Gamma(1-\alpha)(\lambda_{N+1}-\lambda_N)^\alpha\right)\ell\,, \tag{58}$$

and Boutet de Monvel, Chueshov and Rezounenko found

$$\lambda_{N+1}-\lambda_N \geq 4\left(\lambda_N^\alpha+\lambda_{N+1}^\alpha+\alpha^\alpha\Gamma(1-\alpha)(\lambda_{N+1}-\lambda_N)^\alpha\right)\ell\,,$$

which is is little bit worse than (58).

In this situation the exponential dichotomy condition (10) is satisfied with $k_1 = k_2 = 1$, $\beta_1 = \lambda_N$, $\beta_2 = \lambda_{N+1}$, $k_3 = \lambda_N^\alpha$, $k_4 = \lambda_{N+1}^\alpha$, and we find again the spectral gap condition (52) but here with $k_6$ given by (57). Obviously our condition (52) is a little weaker than (58). So we have good chances to extend our result to retarded semilinear parabolic equations and, possibly, to semilinear parabolic equations with stochastic perturbation.

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