HEAT KERNEL AND GEOMETRY OF METRIC MEASURE SPACES WITH RICCI CURVATURE LOWER BOUNDS

A Dissertation
Presented to
The Academic Faculty

By

Xingyu Zhu

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Mathematics
Department of Science

Georgia Institute of Technology

May 2022

HEAT KERNEL AND GEOMETRY OF METRIC MEASURE SPACES WITH RICCI CURVATURE LOWER BOUNDS

Thesis committee:

Dr. Igor Belegradek
School of Mathematics
Georgia Institute of Technology

Dr. Michael Loss
School of Mathematics
Georgia Institute of Technology

Dr. Albert Fathi
School of Mathematics
Georgia Institute of Technology

Dr. John Etnyre
School of Mathematics
Georgia Institute of Technology

Dr. Guido De Philippis
Courant Institute of Mathematical Sciences
New York University

Date approved: April 1st, 2022

E quindi uscimmo a riveder le stelle

Dante Alighieri

To my loved ones

ACKNOWLEDGMENTS

I would like express the sincerest gratitude towards the following people.

To my advisor Prof. Igor Belegradek for introducing me to the subject of synthetic Ricci curvature lower bounds. The guidance and support he provided and insights he shared paved the way for my first steps into this research area. I am also grateful to his helpful feedback and suggestions on my paper writing. I enjoyed the readings, conversations we had over 3 years.

To my collaborators. First and foremost I thank Prof. Shouhei Honda, who always encourages me, answers all my dumb questions, and shows patience with my slow understanding. This work would not have been possible without him. His contribution towards this thesis can never be overestimated, as a large portion of this thesis is the joint work with him. If my advisor Igor is the one who shows me the vision of my current research area, then Prof. Honda is the one who leads me to dive into it. I also thank Prof. Nicola Gigli. Although we do not have frequent direct communication, his writing style has influenced me profoundly. I am also grateful to many simplifications he made to the arguments in our joint paper with Camillo Brena and Prof. Honda, which I incorporated into my thesis.

To the committee members. I thank Prof. Albert Fathi and Prof. Michael Loss for going through the basics with me in my research area. The reading materials both of them recommended have been more than helpful. I learned optimal transport from Prof. Fathi and Bakry-Émery theory of diffusion operators from Prof. Loss. Both topics have helped me build the correct intuitions into my research. I thank Prof. John Etynre for the nice notes on Hodge theory and joining my defense committee under my very late request. I also thank Prof. Guido De Philippis for all the encouragements during my job search. It is my honor to have him in my defense committee.

To all who have helped me with thesis writing. I thank Dr. David Tewodrose, Prof. Huichun Zhang, Prof. Yannick Sire, and Prof. Vitali Kapovitch, for lots of research ideas they shared with me, for always being supportive, for insightful discussions and academic guidance. Chapter 6 is based on an conversation with Vitali and Qin Deng.

To the teaching professionals. I am grateful to Klara Grodzinsky and Dr. Morag Burke for organizing the amazing instructor training workshop, and for sharing their own teaching experiences. My improvements in teaching skills are largely attributed to their efforts in teaching training.

To my peers in geometric analysis and friends at Georgia Tech. I thank Zetian Yan for accommodating me in the reading course at Penn State University. I thank Camillo Brena, who is also my collaborator, for many suggestions when we wrote paper together. I also thank my friends Tianyi Zhang, Hongyi (Hugo) Zhou, Xiaofan Yuan, Xiaonan Liu, Mengyi Tang, Haodong Sun, Hao Wu, Tao Yu, Weiwei Zhang, Ruilin Li, Yian Yao, Roy Agniva, Anubhav Mukherjee, Christina Giannitsi, Hamed Mousavi and Michail Sarantis, and many others. Surrounding by friends like you is such a relief especially during the unprecedentedly challenging time of pandemic. I cherish all the moments of joy with you.

To all the staffs and faculty members in School of Math. Their support has always been essential to my campus life. I especially thank Mitchell Everett, who saved me infinitely many times when I locked myself out of my office. I also thank Lea Marzo, Natache Sumter, Melissa Raine, Malcom Emerson, Judith Taylor, Kimberly Stanley and Chris Jankowski for various assistance they offered. I thank Prof. John Etynre, Prof. Galyna Livshyts, Prof. Michael Loss, Prof. Shahaf Nitzan and many other teachers for their intriguing lectures that provide me with a broad view into math.

To my parents Hongtao Zhu and Shoulan Wang, who always trust me in time of uncertainty. I could pursue my Ph.D. degree overseas without homesick because I always know there is a home waiting for me. To Caiwei (Iris) Tang, who is the light of my life, thank you for the suggestion on the epigraph.

Finally, I acknowledge the support of COVID disruption GRA funds, which helps me concentrate on the thesis writing.

TABLE OF CONTENTS

Summary

The thesis is a study of geometric properties of non-collapsed metric measure spaces with Ricci curvature lower bounds. We establish some characterizations of non-collapsed spaces and as a consequence, solve the De Philippis-Gigli conjecture which states that weakly non-collapsed spaces are actually non-collapsed. It is obtained as a corollary of the following equivalence, which holds under mild volume ratio condition:

- $\mathrm{tr}(\mathrm{Hess} f) = \Delta f$ on $U \subseteq \mathsf{X}$ for every f sufficiently regular,

- $\mathfrak{m} = c\mathcal{H}^n$ on $U \subseteq \mathsf{X}$ for some $c > 0$,

where $U \subseteq \mathsf{X}$ is open and X is a - possibly collapsed - RCD space of essential dimension n. The method we use is smoothing the canonical Riemannian metric by a family of metrics g_t induced by the heat kernel.

We also study the short time expansion of g_t, and show that the weakly asymptotically divergence free property of the second term of the expansion is equivalent to the metric measure space being non-collapsed, under the same volume ratio condition as above. The expansion is made explicit for weighted Riemannian manifolds.

Finally, we prove an almost everywhere convexity of the regular set $\mathcal{R}_n$ which states that for every point in the regular set of essential dimension $\mathcal{R}_n$ there is a geodesic lies completely in $\mathcal{R}_n$ joining almost every other point in $\mathcal{R}_n$. This result can also be interpreted as an almost convexity of the interior of an non-collapsed $\mathrm{RCD}(K, N)$ space.

CHAPTER 1

INTRODUCTION

In this thesis we focus on the study of metric measure spaces with Ricci curvature lower bounds. The geometric properties under investigation are being non-collapsed and convexity. In our study, especially when study the criteria for non-collapsed spaces, the heat kernel and heat flow serve as a basic tool to deal with non-smooth objects, we developed a method that uses heat kernel to mollify the original metric, which has much broader applications to be discovered in the future. For the study of convexity we adapted the one dimension localization technique, originated from optimal transport.

In the introduction, we give an account of the history of synthetic Ricci curvature bounds on metric measure spaces.

1.1 Ricci Curvature on manifolds

We start from the study Ricci curvature for smooth manifolds. For Riemannian manifolds, The Ricci curvature lower bound plays a central poly in the interplay between geometry and analysis. In geometric aspect, by comparing quantities to those in the space form of constant sectional curvature, one can find nice bounds of geometric quantities of interests. For example, the Bishop-Gromov inequality, Laplacian of the distance function, hence the mean curvature of sphere, first eigenvalue of Laplacian, first Betti number, and the growth rate of fundamental group, etc.. It evolves to a vast subject named comparison geometry, see for instance [1]. It is particularly interesting that some quantities are rigid under Ricci curvature bounds. Famous results include:

- Cheng's maximal diameter theorem [2, Theorem 3.1], which states that if the Ricci curvature of an n-dimensional manifold is bounded from below by $(n-1)$, and its diameter achieves its maximum π, then such a manifold must be the standard sphere.

- Obata theorem [3, Theorem 2], which states that if the Ricci curvature of an n-dimensional manifold is bounded from below by $(n-1)$ and the first eigenvalue of Laplacian λ achieves its minimum n, then such a manifold must be the standard sphere.

- Cheeger-Gromoll Splitting theorem [4] , which states that if the Ricci curvature of an n-dimensional manifold is bounded from below by 0 and contains an isometric image of a straight line, then it is actually a direct product of $\mathbb{R}$ and an $(n-1)$ dimensional submanifold, we also say in this case that the manifold splits an $\mathbb{R}$-factor.

In analytical aspect, under lower Ricci curvature bound, Li-Yau in [5] studied the solution of heat equation $(\Delta - \frac{\partial}{\partial t})u = 0$ and found the optimal bound for the solution u, therefore they proved a parabolic Harnack inequality. Later, Saloff-Coste in [6] proved that the volume doubling condition and L^2 Poincaré inequality is equivalent to parabolic Harnack inequality and established local and global heat kernel bounds.

On the other hand, the Ricci curvature upper bound is shown to be flexible, indeed, Lohkamp in [7] showed that every manifold of dimension $n \geq 3$ admits a complete metric with negative Ricci curvature, so there is no rigidity of Riemannian manifolds with Ricci curvature upper bounds. Nevertheless, the two-sided Ricci curvature bounds grant even more rigidity then lower Ricci curvature bound alone. Indeed, by estimating the harmonic norm, Anderson [8] showed that for $n \geq 2$, and given $\Lambda, D, R > 0$, the class of manifolds that satisfy $|\mathrm{Ric}| \leq \Lambda$, $\mathrm{diam} \leq D$ and $\mathrm{inj} \geq R$ is precompact in $C^{1,\alpha}$ topology for any $\alpha \in (0,1)$, precompact in $C^{m,\alpha}$ topology for any positive integer m if furthermore the metric is Einstein. In particular, there are only finitely many homeomorphic types of such manifolds.

1.2 Synthetic notion of Ricci Curvature lower bounds

In view of the success of describing sectional curvature bounds via purely metric terms without referring to any smooth structure, i.e., the theory of Alexandrov geometry, see for example [9], it is desirable to also develop a notion of Ricci curvature bounds without referring any smooth structure. The question was raised by Gromov [10, P.89] that calling *synthetic* a set of conditions defining a class of metric spaces without referring to any notion of smoothness, is there a synthetic notion of Ricci curvature bounds? This question has become increasingly important after Cheeger-Colding's work on the structure of Ricci limit spaces [11, 12, 13, 14], which are (pointed) Gromov-Hausdorff limit of sequences of smooth manifolds of uniform Ricci curvature lower bounds. Ricci limit spaces are potentially non-smooth metric spaces that inherit many properties of smooth manifolds with Ricci curvature lower bounds, including the almost splitting theorem [11], a quantitative version of Cheeger-Gromoll splitting theorem. They are natural candidates for notion of metric space with lower Ricci curvature bounds, however, they are only defined extrinsically.

Insights from optimal transport and Bakry-Eméry theory reveals that Ricci curvature, unlike sectional curvature, is not only a metric notion, but a metric measure notion, that is, the reference measure plays a role when consider Ricci curvature. To explain this point, we introduce some basic notions in optimal transport. Readers can refer to [15] for more details.

Let (X, d) be a complete and measurable metric space. Let $\mathcal{P}_2$ be a set of a probability measure μ, such that for some $x_0 \in \mathsf{X}$

$$\int_{\mathsf{X}} \mathsf{d}^2(x_0, \cdot) \mathrm{d}\mu < \infty.$$

For any $\mu_0, \mu_1 \in \mathcal{P}_2$, define the 2-Wasserstein distance $W_2(\mu_0, \mu_1)$ between them as:

$$W_2(\mu_0, \mu_1) := \inf_{\gamma} \int_{\mathsf{X}} \int_{\mathsf{X}} \mathsf{d}^2(x_0, x_1) \mathrm{d}\gamma,$$

where the infimum is taken over all probability measure γ on $\mathsf{X} \times \mathsf{X}$ with marginal μ_0 and μ_1. Such a γ is refered to as the *optimal coupling* between μ_0 and μ_1. Thanks to the metric structure on X, $(\mathcal{P}_2, W_2)$ is a complete metric space, and the W_2 convergence $\mu_n \to \mu$ is equivalent to the combination of weak convergence and the second moment convergence $\int_{\mathsf{X}} \mathsf{d}^2(x, x_0) \mathrm{d}\mu_n \to \int_{\mathsf{X}} \mathsf{d}^2(x, x_0) \mathrm{d}\mu$, for some $x_0 \in \mathsf{X}$. It is worth pointing out that $(\mathcal{P}_2, W_2)$ is a geodesic space if (X, d) is.

Next, we deal with complete separable metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, where $\mathfrak{m}$ is a Borel regular measure. Let $\mathcal{P}_2^a(\mathsf{X}) \subseteq \mathcal{P}_2$ be set of measures in $\mathcal{P}_2$ and absolutely continuous w.r.t to $\mathfrak{m}$. It follow that for any $\mu \in \mathcal{P}_2^a(\mathsf{X})$ there exists $\rho \in L^1_{loc}(\mathsf{X}, \mathfrak{m})$, called the density of μ so that $\mathrm{d}\mu = \rho \mathrm{d}\mathfrak{m}$. McCann [16] defined the *displacement convexity* for functionals on $\mathcal{P}_2^a(\mathsf{X})$:

Definition 1.2.1. Given $K \in \mathbb{R}$, a functional $F : \mathcal{P}_2^a(\mathsf{X}) \to \mathbb{R} \cup \{+\infty\}$ is called K-displacement convex if for any W_2 geodesic $\{\mu_t\}_{t\in[0,1]}$ in $\mathcal{P}_2^a(\mathsf{X})$,

$$F(\mu_t) \leq (1-t)F(\mu_0) + tF(\mu_1) + \frac{Kt(1-t)}{2} W_2(\mu_0, \mu_1) \qquad \forall t \in [0,1]$$

An cornerstone characterization of Ricci curvature lower bound made by Sturm-Renesse in [17, Theorem 1] is the following

Theorem 1.2.2. *Let (M, g, vol_g) be a connected Riemannian manifold, and $K \in \mathbb{R}$. The following 2 properties are equivalence*

1. $\mathrm{Ric_g} \geq Kg$

2. *The entropy functional*

$$\mathrm{Ent}_{\mathrm{vol}_g}(\mu) := \int_M \frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{vol}_g} \log \frac{\mathrm{d}\mu}{\mathrm{d}\,\mathrm{vol}_g} \mathrm{d}\,\mathrm{vol}_g$$

is K-displacement convex.

It is worth pointing out that the condition (2) make sense in complete separable metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfying that for some $x \in \mathsf{X}$ there exists constant $C > 0$ such that

$$\mathfrak{m}(B_r(x)) \leq Ce^{Cr^2}$$

This condition ensures the integrability of the negative part of $\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}} \log \frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}$. Such an observation linking displacement convexity of entropy functional and Ricci curvature leads to the idea of $\mathrm{CD}(K, \infty)$ condition. However, the K-displacement convexity is too strong for metric measure space since it requires convexity along every W_2 geodesic. A relaxed condition would be only requiring this kind of convexity along one geodesic, this relaxed condition is sometimes referred to as weak $\mathrm{CD}(K, N)$ condition. We are now ready to define the $\mathrm{CD}(K, N)$ condition which is the desired synthetic definition of Ricci curvature lower bounds, here CD stands for curvature-dimension. First, we take a look at $\mathrm{CD}(K, \infty)$ condition:

Definition 1.2.3. A complete separable metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{CD}(K, \infty)$ condition for $K \in \mathbb{R}$ if for any $\mu_0, \mu_1 \in \mathcal{P}_2^a(\mathsf{X})$, there exists a W_2 geodesic $\{\mu_t\})t \in [0, 1]$, such that

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \leq (1 - t)\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\mathrm{Ent}_{\mathfrak{m}}(\mu_1) + \frac{Kt(1 - t)}{2} W_2(\mu_0, \mu_1).$$

Pushing this idea further, one can define the $\mathrm{CD}(K, N)$ condition for finite $N \in [1, \infty)$. In this case, the distorsion of volume element along geodesic need to be taken into account.

For $\theta \geq 0$, define the distorsion coefficient as

$$\sigma^t_{K,N}(\theta) := \begin{cases} \frac{\sin\left(t\theta\sqrt{\frac{K}{N}}\right)}{\sin\left(\theta\sqrt{\frac{K}{N}}\right)} & K > 0 \\ t & K = 0 \text{ or } N = 1 \\ \frac{\sinh\left(t\theta\sqrt{\frac{-K}{N}}\right)}{\sinh\left(\theta\sqrt{\frac{-K}{N}}\right)} & K > 0 \end{cases} \tag{1.2.1}$$

And the modified distorsion coefficient as

$$\tau^t_{K,N}(\theta) := \begin{cases} \infty & K > 0, N = 1 \\ t^{\frac{1}{N}} \sigma^t_{K,N-1}(\theta)^{1-\frac{1}{N}} & \text{otherwise} \end{cases} \tag{1.2.2}$$

Also, consider the Renyí entropy defined as

$$S_{N,\mathfrak{m}}(\mu) = -\int_{\mathsf{X}} \left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}}\right)^{1-\frac{1}{N}} \mathrm{d}\mathfrak{m} \qquad \forall \mu \in \mathcal{P}^a_2(\mathsf{X})$$

Definition 1.2.4. A complete separable metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfies $\mathrm{CD}(K, N)$ condition for $K \in \mathbb{R}$ and $N \in [1, \infty)$, if for every $\mu_0, \mu_1 \in \mathcal{P}^a_2(\mathsf{X})$, there exists a W_2 geodesic $\{\mu_t\}_{t\in[0,1]}$ and a optimal coupling γ such that

$$S_{N,\mathfrak{m}}(\mu_t) \leq -\int_{\mathsf{X}} \left[\tau^t_{K,N}(\mathsf{d}(x,y))\rho_0(x)^{-\frac{1}{N}} + \tau^{1-t}_{K,N}(\mathsf{d}(x,y))\rho_1(y)^{-\frac{1}{N}}\right] \mathrm{d}\mathfrak{m} \tag{1.2.3}$$

where $\rho_i := \frac{\mathrm{d}\mu_i}{\mathrm{d}\mathfrak{m}}$, $i = 0, 1$.

$\mathrm{CD}(K, N)$ should be morally understood as a metric space with Ricci curvature lower bound K, and dimension upper bound N. Indeed, Sturm showed in [18] that $\dim_{\mathcal{H}}(\mathsf{X}) \leq N$ if $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is $\mathrm{CD}(K, N)$.

The stability of $\mathrm{CD}(K, N)$ condition under pointed measured Gromov-Hausdorff convergence of metric measure spaces is discussed by Lott-Villani, see [15, Theorem 29.25], which confirms that Ricci limit spaces satisfies $\mathrm{CD}(K, N)$ condition. Also Sturm proved the stability of $\mathrm{CD}(K, N)$ condition under his D-convergence, see the discussion in Section 2.3.1. Although $\mathrm{CD}(K, N)$ condition defines a class of metric measure spaces with synthetic Ricci curvature lower bounds. This class is too large. In particular, it includes Finsler manifolds with Ricci curvature lower bounds ([19, Theorem 2]) , making some extension of theorems for Riemmanian manifolds to non-smooth setting impossible. For instance, in general no isometric splitting is possible for Finsler manifolds, even though diffeomorphic and measure preserving splitting is possible, see [20]. This implies that the $\mathrm{CD}(K, N)$ condition should be further refined to retain Riemannian structure.

1.3 Riemannian curvature-dimension condition

In pursuit of Riemnnian structure in non-smooth setting, Ambrosio-Gigli-Savaré in [21, Theorem 5.1] proposed the $\mathrm{RCD}(K, \infty)$ condition by requiring the heat flow to be linear or the Cheeger energy (see Definition 2.1.1) to be quadratic combined with strong $\mathrm{CD}(K, \infty)$ condition, meaning K-displacement convexity is satisfied (for every W_2 geodesic). As expected, $\mathrm{RCD}(K, \infty)$ condition excludes the Finsler manifolds for $N = \infty$, and one can also consider $\mathrm{RCD}(K, N)$ for finite $N \in [1, \infty)$ by imposing $\mathrm{CD}(K, N)$ and $\mathrm{RCD}(K, \infty)$ at the same time. A more intrinsic condition was proposed by Gigli in [22, Definition 4.19] named Infinitesimally Hilbertian. A metric measure space is infinitesimally Hilbertian if the Sobolev space $H^{1,2}$ defined by Cheeger energy is Hilbert, meaning the Sobolev norm satisfies the parallelogram rule. Also infinitesimal Hilbertianity is equivalent to the fact that the Cheeger energy is quadratic.

A different approach is to generalize the Bochner inequality via Γ-Calculus. Erbar-Kuwada-Sturm [23] took this approach and in this thesis, we will adapt the definition of $\mathrm{RCD}(K, N)$ space from this perspective as our working definition, and it does not require optimal transport as a prerequisite, see definition 2.1.2.

The stability of $\mathrm{RCD}(K, N)$ condition under pointed measured Gromov-Hausdorff convergence is shown in [24], which further confirms that Ricci limit spaces satisfies suitable RCD condition, moreover, the classical rigidity results on Riemannian manifolds admit natural extension to $\mathrm{RCD}(K,N)$ spaces. For example the maximal diameter theorem [25], Obata's eigenvalue rigidity theorem [26], and the Cheeger-Gromoll's isometric splitting theorem [27].

1.4 Local-to-global and essentially non-branching

The local-to-global question asks if a metric measure space locally satisfies $\mathrm{CD}(K,N)$ condition, does it globally satisfy $\mathrm{CD}(K,N)$ condition. More precisely, for a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ if there is a countable partition of $\mathsf{X}=\sqcup_i \mathsf{X}_i$ such that $\mathfrak{m}(\mathsf{X}_i)>0$, $(\mathsf{X}_i,\mathsf{d}|_{\mathsf{X}_i\times\mathsf{X}_1},\mathfrak{m}\llcorner X_i)$ satisfies $\mathrm{CD}(K',N')$ for some $K'\geq K$ and $N'\geq N$, does $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies $\mathrm{CD}(K,N)$? Note that for Riemannian manifolds, the curvature, hence its lower bound, is defined pointwise, so local-to-global property holds trivially on Riemannian manifolds. The same question is also asked for Alexandrov spaces, and for Alexandrov spaces with synthetic lower sectional curvature bounds, the question is affirmatively answered by Toponogov, see [9, Theorem 10.3.1]. The local-to-global question for $\mathrm{CD}(K,N)$ condition was firstly answered by Sturm [28, Theorem 4.17] for and Villani [15, Theorem 30.37] for non-branching, compact $\mathrm{CD}(0,N)$ and $\mathrm{CD}(K,\infty)$ spaces. However, Rajala in [29] constructed a counterexample that is locally $\mathrm{CD}(0,4)$ but not globally $\mathrm{CD}(K,N)$ for any $K\in\mathbb{R}$ and $N\in[1,\infty]$. This construction is based on the observation that $\mathbb{R}^n$ with L^∞ norm and Lebesgue measure satisfies $\mathrm{CD}(0,n)$ condition, but it is highly branching. To bypass the local-to-global failure, Bacher-Sturm in [30] introduced a reduced curvature dimension condition, namely, the $\mathrm{CD}^*(K,N)$ condition, replacing the coefficients τ_t in the definition by σ_t in Definition 1.2.4, recall (1.2.1), (1.2.2). $\mathrm{CD}^*(K,N)$ is weaker than $\mathrm{CD}(K,N)$, but implies $\mathrm{CD}(\frac{N-1}{N}K,N)$ for $K>0$, and the point is that $\mathrm{CD}^*(K,N)$ spaces satisfy local-to-global property. Finally, with the correct notion, essentially non-branching spaces, proposed by Rajala-Sturm in [31], Cavalletti-Milman in [32] proved that essentially non-branching $\mathrm{CD}(K,N)$ spaces with finite total mass satisfy the local-to-global property.

It is also expected that the local-to-global property to hold for σ-finite measures.

In particular, $\mathrm{RCD}(K, N)$ spaces are essentially non-branching, so $\mathrm{RCD}(K, N)$ spaces with finite mass are equivalent to $\mathrm{RCD}^*(K, N)$ spaces. In fact, a recent result of Deng [33] on the Hölder continuity of tangent cones reveals that $\mathrm{RCD}(K, N)$ spaces are actually non-branching for finite N. We will discuss another consequence of Deng's work in Chapter 6. However, an essentially non-branching but branching example of $\mathrm{CD}(K, N)$ space is constructed by Ohta.

1.5 Two-sided curvature bounds

As pointed out at the first section, although Ricci curvature upper bound has no implication for a Riemannian manifold, a two-sided Ricci curvature bound can imply better regularity than Ricci curvature lower bound alone. For Ricci limit spaces, a famous example is that, Cheeger-Colding proved that the singular set of a non-collapsed Ricci limit space has Hausdorff codimension at least 2, see [12, Chapter 6], and they conjectured that a non-collapsing sequence (see Definition 2.5.1) of manifolds with two-sided Ricci curvature bound converges to a (non-collapsed) Ricci limit space whose singular set (see section 6.2) is of Hausdorff codimension at least 4. This conjecture is finally confirmed by Cheeger-Naber [34], some better regularity results coming from two-sided Ricci curvature bounds are also discussed in [35], in particular, it is shown there that the regular set (see Theorem 2.4.4) of a limit space with two-sided Ricci curvature bound is geodesically convex.

Given the achievements made for Ricci limit spaces coming from sequence with two-sided Ricci curvature bounds, it is also interesting to find a synthetic notion of Ricci curvature upper bound. Naber [36, Definition 15.1] made an attempt to define a two-sided Ricci curvature bound by Bakry-Émery type of inequality on path space. The author is informed by Yifan Guo that Sturm also studied the notion of synthetic Ricci curvature upper bound [37, Definition 1.1]. The point of view is that for Ricci curvature lower bound K it holds that

$$W_2(\mathrm{h}_t\delta_x, \mathrm{h}_t\delta_y) \le e^{-Kt}\mathsf{d}(x,y).$$

This suggests that the quantity

$$\theta^+(x,y) := -\liminf_{t\to 0} \frac{1}{t}\log\left(\frac{W_2(\mathrm{h}_t\delta_x, \mathrm{h}_t\delta_y)}{\mathsf{d}(x,y)}\right)$$

behaves like the Ricci curvature. And the Ricci curvature upper bound $K \in \mathbb{R}$ of a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is defined by requiring for each $x \in \mathsf{X}$,

$$\theta^*(x) := \limsup_{y,z\to x} \theta^+(y,z) \le K.$$

Under this Ricci curvature bound. Yifan Guo [38] was able to partially extend a result of Lohkamp [7], which roughly states that negative Ricci curvature implies finite isometry group.

On the other hand, Kapovitch-Ketterer [39] suggested a mixed curvature bound, that is a $\mathrm{CD}(K,N)$ condition combined with $\mathrm{CAT}(\kappa)$ condition on a metric measure space, which should be thought of as having Ricci curvature lower bound K and sectional curvature upper bound κ. They were able to show that $\mathrm{CD}(K,N)$ condition along with $\mathrm{CAT}(\kappa)$ condition forces the space to be $\mathrm{RCD}(K,N)$ and if the space is non-collapsed, it is in fact an Alexandrov space with lower sectional curvature bound. They further studied the fine structure of such spaces and proved the conjecture of De Philippis-Gigli (see Conjecture 4.1.1) under this extra $\mathrm{CAT}(\kappa)$ assumption.

1.6 Smoothing metric g_t

We now introduce the main object we study in this thesis. We present the results of [40] and [41] in Chapter 3, 4, 5, and note that we provide a new proof to Theorem 5.1.1 in the thesis. The object we look at can trace back to Bérard-Besson-Gallot's work [42].

They studied a family of embedding $\{\Phi_t\}_{t\geq 0}$ from a closed Riemannian manifold (M,g) to $\ell^2(M)$ by expansion of the heat kernel involving eigenfunctions φ_i of the Laplacian and the corresponding to eigenvalues λ_i [42, Definition 4]:

$$\begin{aligned} \Phi_t : M &\to \ell^2(M) \\ x &\mapsto \left\{e^{-\lambda_i t}\varphi_i\right\}_{i\geq 0}, \end{aligned} \tag{1.6.1}$$

Note that the exponential term is slight different from the original version. Then they defined a pull-back metric from $\ell^2(M)$ by the embedding Φ_t:

$$g_t = \sum_{i\geq 0} e^{-2\lambda_i t} \mathrm{d}\varphi_i \otimes \mathrm{d}\varphi_i$$

and showed that as $t \to 0$, the follow asymptotic formula holds

$$t^{(n+2)/2} g_t = c_n \left((g - \frac{2t}{3} G_g + O(t^2) \right)$$

where $G_g := \frac{1}{2}\mathrm{Scal}_g g - \mathrm{Ric}_g$ is the Einstein tensor. Ambrosio-Honda-Portegies-Tewodrose in [43] considered the family of metrics g_t on compact $\mathrm{RCD}(K,N)$. where

$$g_t = \int \mathrm{d}_x p_{y,t} \otimes \mathrm{d}_x p_{y,t}$$

and $p(x,y,t) = p_{y,t}(x)$ is the heat kernel. In fact their formulation makes sense in non-compact setting as well. They proved in the L^2 sense the first term in the expansion still holds for some canonical Riemannian metric g (see Proposition 2.4.8) on an $\mathrm{RCD}(K,N)$ space. More rigorously, they showed that $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t$ L^2 strongly converges to $c_n g$ as $t \to 0$. We will point out in Chapter 3 that their proof actually works in non-compact setting with all the L^p convergence replaced by L^p_{loc} convergence for $p \in [1,\infty)$, and only very minor modification of the proof is needed.

Through the work of Ambrosio-Honda-Tewodrose [44], and Honda [45], which largely enlightened main contents to be presented in this thesis, g_t can be understood as a smoothing of the canonical Riemannian metric g. We will take this point of view and apply the convergence results of g_t to achieve 2 objectives.

Let us first point out some common feature of the 2 objectives. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space of essential dimension n. Assume that volume ratio has a lower bound, i.e. ,

$$\inf_{r\in(0,1),x\in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0,$$

where $A \subseteq \mathsf{X}$ is a compact subset. We can also study the convergence of $t^{(n+2)/2}g_t$ in place of $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t$, the assumption ensures that

$$\theta(x) = \lim_{r\to 0} \frac{r^n}{\mathfrak{m}(B_r(x))} \in L^\infty_{loc}(\mathsf{X}, \mathfrak{m}),$$

and $t^{(n+2)/2}g_t$ converges L^2-strongly to $c_n\theta g$, this convergence will be combined with the formula of $\nabla^* g_t$ first derived in [45],

$$\nabla^* g_t = -\frac{1}{4}\mathrm{d}\Delta p(x, x, 2t).$$

This formula is extended to non-compact setting, see Theorem 3.3.5.

The first objective is to extend Honda's proof of Conjecture 4.1.1 to non-compact setting. Having the formula of $\nabla^* g_t$ and and the L^2 convergence of $t^{(n+2)/2}g_t$ at our disposal the proof is essential the same as the compact case. We derived a integrate-by-part formula w.r.t Hausdorff measure, Theorem 4.2.1, and an important equivalence:

- $\mathrm{tr}(\mathrm{Hess} f) = \Delta f$ on $U \subseteq \mathsf{X}$ for every f sufficiently regular,

- $\mathfrak{m} = c\mathcal{H}^n$ on $U \subseteq \mathsf{X}$ for some $c > 0$,

The second objective is to study the second term of the asymptotic expansion of g_t in

$\mathrm{RCD}(K,N)$ context, which is

$$\frac{c_n t^{(n+2)/2} g_t - \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} g}{t}$$

as $t \to 0$. In comparison, this term for closed Riemannian manifold is G_g the Einstein tensor. It has the divergence free property, i.e. , $\nabla^* G_g = 0$. Although we cannot expect the limit of the above quotient to exist in any L^p sense, we can compute its divergence before taking the limit $t \to 0$. By testing regular enough 1-forms, what we will see is that in the RCD context, the (weakly asymptotically) divergence free property of the quotient is equivalent to the space being non-collapsed. See precise statement in Theorem 5.1.1. To make this result more explicit, we computed the corresponding expansion of g_t for a closed weighted Riemannian manifold as well.

1.7 Rest of topics in the thesis

In last chapter, we use a direct corollary of Deng's Hölder continuity and the 1D localization technique of Cavalletti and Mondino to improve a known result of convexity of the regular set at essential dimension, i.e. , Theorem 6.1.1. We highlight that such convexity can also be interpreted as interior convexity and it is connected to the notion of boundary of non-collapsed $\mathrm{RCD}(K,N)$ spaces.

In the Appendices, we present a useful fact that the essential dimension is independent of the reference measure stated in [46], and a Rellich type compactness theorem for 1-forms on compact $\mathrm{RCD}(K,N)$ space stated in [40]. As a consequence, we obtain the spectrum decomposition of Hodge Laplacian.

CHAPTER 2

PRELIMINARIES

We make some standing assumptions at beginning, throughout this thesis,

- by *metric measure space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ we always mean a complete and separable metric space equipped with a non-negative Borel measure finite on bounded sets such that $\mathrm{supp}(\mathfrak{m}) = \mathsf{X}$;

- C denotes a positive constant, that may vary from step to step. Occasionally we may emphasize the parameters on which the constant depends, so that, say, $C(K, N)$ denotes a positive constant depending only on K and N;

- $\mathrm{Lip}(\mathsf{X}, \mathsf{d})$ (resp. $\mathrm{Lip}_b(\mathsf{X}, \mathsf{d})$, resp. $\mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$) denotes the set of all Lipschitz (resp. bounded Lipschitz, resp. Lipschitz with bounded support) functions on a metric space (X, d);

- We denote by $\mathrm{lip}\, f : \mathsf{X} \to [0, \infty]$ the *local Lipschitz constant* of the function $f : \mathsf{X} \to \mathbb{R}$ defined by
$$\mathrm{lip}\, f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\mathsf{d}(y, x)}$$
if x is not isolated and has to be understood as 0 if x is isolated;

- L^p_{loc} means that the restriction (for functions, tensors and so on) to any compact subset of the domain is L^p.

2.1 Riemannian Curvature-Dimension Conditions

2.1.1 Definitions

In this section we introduce our working definition of $\mathrm{RCD}(K,N)$ spaces. Fix a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$. The Cheeger energy, originally introduced by Cheeger in [47], $\mathsf{Ch} : L^2(\mathsf{X},\mathfrak{m}) \to [0,\infty]$ is defined by

$$\mathsf{Ch}(f) := \inf_{\|f_i - f\|_{L^2}\to 0} \left\{ \liminf_{i\to\infty} \int_{\mathsf{X}} (\mathrm{lip}\, f_i)^2 \mathrm{d}\mathfrak{m} : \ f_i \in \mathrm{Lip}_b(\mathsf{X},\mathsf{d}) \cap L^2(\mathsf{X},\mathfrak{m}) \right\}. \qquad (2.1.1)$$

Then, the Sobolev space $H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ is defined as the finiteness domain of Ch endowed with the norm $\|f\|_{H^{1,2}} = \mathsf{Ch}(f)$. It follows from the definition of Cheeger energy that it is convex and lower semi-continuous. By looking at the optimal sequence in (2.1.1) one can identify a canonical object $|\mathrm{D}f|$, called the minimal relaxed slope, which is local on Borel sets (i.e. $|\mathrm{D}f_1| = |\mathrm{D}f_2|$ $\mathfrak{m}$-a.e. on $\{f_1 = f_2\}$) and provides integral representation to Ch, namely

$$\mathsf{Ch}(f) = \int_{\mathsf{X}} |\mathrm{D}f|^2 \mathrm{d}\mathfrak{m} \qquad \forall f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m}).$$

The Cheeger energy makes sense in metric measure spaces that are locally volume doubling and support $(1,2)$-poincaré inequality, but it is not in general quadratic, hence does not always induce a Dirichlet form. Partially in light of this fact, Ch is quadratic if and only if it satisfies the following parallelogram rule:

$$\mathsf{Ch}(f+g) + \mathsf{Ch}(f-g) = 2\mathsf{Ch}(f) + 2\mathsf{Ch}(g) \quad \forall f,g \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$$

We recall the notion of infinitesimally Hilbertian metric measure spaces.

Definition 2.1.1. A complete separable metric space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if $H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ endowed with the norm $\|\cdot\|_{H^{1,2}}$

Thus, we have that

$$\mathrm{RCD}(K,N) = \mathrm{CD}(K,N) + \text{Infinitesimally Hilbertian}$$

Although originally the $\mathrm{CD}(K,N)$ condition is defined by optimal transport tools, for $\mathrm{RCD}(K,N)$ spaces thanks to the infinitesimal Hilbertianity and the work of Erbar-Kuwada-Sturm[23], we can now define $\mathrm{RCD}(K,N)$ condition by Sobolev-to-Lipschitz property and generalized Bochner's inequality through Γ- calculus:

Definition 2.1.2 ($\mathrm{RCD}(K,N)$ space)**.** For any $K \in \mathbb{R}$ and any $N \in [1,\infty]$, a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is said to be an $\mathrm{RCD}(K,N)$ space if the following four conditions are satisfied.

1. There exist $x \in X$ and $C > 1$ such that $\mathfrak{m}(B_r(x)) \le Ce^{Cr^2}$ holds for any $r > 0$.
2. Ch is a quadratic form. In this case for $f_i \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})(i=1,2)$ we put

$$\langle \nabla f_1, \nabla f_2\rangle := \lim_{\epsilon\to 0} \frac{|\mathrm{D}(f_1+\epsilon f_2)|^2 - |\mathrm{D}f_1|^2}{2\epsilon} \in L^1(\mathsf{X},\mathfrak{m}).$$

3. Any $f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ with $|\mathrm{D}f| \le 1$ for $\mathfrak{m}$-a.e. has a 1-Lipschitz representative.
4. For any $f \in D(\Delta)$ with $\Delta f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ we have

$$\frac{1}{2}\int_{\mathsf{X}} |\mathrm{D}f|^2 \Delta\varphi \mathrm{d}\mathfrak{m} \ge \int_{\mathsf{X}} \varphi \left(\frac{(\Delta f)^2}{N} + \langle \nabla \Delta f, \nabla f\rangle + K|\mathrm{D}f|^2 \right) \mathrm{d}\mathfrak{m} \tag{2.1.2}$$

for any $\varphi \in D(\Delta) \cap L^\infty(\mathsf{X},\mathfrak{m})$ with $\varphi \ge 0$, $\Delta\varphi \in L^\infty(\mathsf{X},\mathfrak{m})$, where

$$D(\Delta) := \Big\{ f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m}) : \exists h \in L^2(\mathsf{X},\mathfrak{m}) \text{ s.t. } \int_{\mathsf{X}} \langle \nabla f, \nabla \varphi\rangle \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} h\varphi \mathrm{d}\mathfrak{m},\ \forall \varphi \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m}) \Big\}$$

and $\Delta f := h$ for any $f \in D(\Delta)$.

In what follows, when writing $\mathrm{RCD}(K, N)$ we implicitly assume that $N < \infty$, we will only write $\mathrm{RCD}(K, \infty)$ for infinite dimension upper bound.

2.1.2 Examples

We provide a list of examples of $\mathrm{CD}(K, N)$ or $\mathrm{RCD}(K, N)$ spaces.

- The Ricci limit spaces obtained from a sequence of Riemannian manifolds of dimension N and uniform Ricci curvature lower bound K are $\mathrm{RCD}(K, N)$ spaces. This is a consequence of the stability of RCD condition under pointed-measured-Gromov-Hausdorff convergence.

- ([48, Main Theorem]) An N dimensional Alexandrov space with sectional curvature bounded from below by K is a (non-collapsed, see Definition 2.5.2) $\mathrm{RCD}(K, N)$ space. For example, the boundary of any convex body in $\mathbb{R}^n$ is a possibly non-smooth Alexandrov space with positive sectional curvature bound.

- ([23, Proposition 4.21]) An n dimensional weighted manifold $(M^n, g, e^{-f}\mathrm{dvol}_g)$, is an $\mathrm{RCD}(K, N)$ for some $N \geq n$ if the following Bakry-Émery N-Ricci tensor

$$\mathrm{Ric}_N = \begin{cases} \mathrm{Ric}_g + \mathrm{Hess}_f - \frac{\mathrm{d}f \otimes \mathrm{d}f}{N-n} & N > n; \\ \mathrm{Ric}_g & N = n \\ -\infty & \text{otherwise} \end{cases} \tag{2.1.3}$$

  satisfies $\mathrm{Ric}_N \geq Kg$ for some $K \in \mathbb{R}$, where $f \in C^\infty(M)$, vol_g is the Riemannian volume measure, which is also the Hausdorff measure $\mathcal{H}^N$ of metric space (M, d_g), Ric_g is the standard Ricci curvature induced by the metric tensor g and defined as trace of the Riemann curvature tensor. See also [49, 50]. When $N = n$, f must be a constant. It is worth pointing out that the weighted Laplacian on M, defined as $\Delta_f = \mathrm{trHess}_f - g(\nabla f, \cdot)$, is not in general the trace of Hessian.

- ([25, Theorem 1.1]) A metric measure cone over a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a metric measure space $C(\mathsf{X}) := [0, \infty) \times \mathsf{X}$ with metric

$$\mathsf{d}_{\mathrm{Con}}((t, x), (r, y)) = \sqrt{t^2 + s^2 - 2ts\cos(\mathsf{d}(x, y) \wedge \pi)},$$

and measure $\mathfrak{m}_{\mathrm{Con}} = r^N \mathrm{d}r \otimes \mathrm{d}\mathfrak{m}$. If in addition, $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is $\mathrm{RCD}(N-1, N)$, then $(C(\mathsf{X}), \mathsf{d}_{\mathrm{Con}}, \mathfrak{m}_{\mathrm{Con}})$ is an $\mathrm{RCD}(0, N+1)$ space. In fact, a more general result for (K, N)-cones (see [25, Definition 5.1]) including spherical suspensions holds.

- ([51, Theorem A]) A compact stratified space (X, g) of dimension less than or equal to N, equipped with the length metric and measure induced by g is $\mathrm{RCD}(K, N)$ if and only if the Ricci tensor in the top stratum $\mathsf{X}^{\mathrm{reg}}$ is bouned below by Kg, and the angle along codimension 2 stratum Σ^{n-2} is smaller than or equal to 2π. See relevant definitions in [51, Definition 1.1, 1.4, 1.6, 1.10]. In particular, this class of spaces includes Riemannian orbifolds with Ricci curvature lower bounds.

- ([52]) The non-collapsed Ricci limit space with boundary is defined as the pGH limit of sequence of Riemannian manifolds with boundary, having uniform lower volume and Ricci curvature bounds. These are also RCD spaces (with boundary). In contrast, Cheeger-Colding in [11] only consider manifolds without boundary and show that the limit space has no boundary, i.e., no tangent cone at any point is the half space $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$.

We introduce a criterion of $\mathrm{CD}(K, N)$ condition for one dimensional spaces following [32, Appendix A], for this and also for latter use we first recall the notion of $\mathrm{CD}(K, N)$ density.

Definition 2.1.3. A nonnegative function h defined on an interval $I \subseteq \mathbb{R}$ is called a

$\mathrm{CD}(K,N)$ density for $k \in \mathbb{R}$ and $N \in (1,\infty]$ if for $x_0, x_1 \in I$ and $t \in [0,1]$

$$\begin{cases} h(tx_1+(1-t)x_0)^{\frac{1}{N-1}} \geq \sigma_{K,N}^{(t)}(d)h(x_1)^{\frac{1}{N-1}} + \sigma_{K,N}^{(1-t)}(d)h(x_0)^{\frac{1}{N-1}} & N \in (1,\infty) \\ \log(h(tx_1+(1-t)x_0)) \geq t\log h(x_1) + (1-t)\log h(x_0) - \frac{K}{2}t(1-t)d^2 & N = \infty \end{cases}$$

where $d = |x_1 - x_0|$.

The criterion is the following:

Theorem 2.1.4. *If h is a $\mathrm{CD}(K,N)$ density on an interval, then $(I, |\cdot|, h(t)\mathrm{d}t)$ is a $\mathrm{CD}(K,N)$ space. If $(I, |\cdot|, \mu)$ is a $\mathrm{CD}(K,N)$ space and $I = \mathrm{supp}(\mu)$ is not a single point, then $\mu \ll \mathcal{L}^1$ and $h := \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}$ is a $\mathrm{CD}(K,N)$ density.*

If $h \in C^2$, there is a neat inequality to replace inequalities in Definition 2.1.3 and is easier to check, see [32, Theorem A 3].

There are some examples to keep in mind and can be used as test cases to various problems.

- ([15, Example 29.16]) $(\mathbb{R}^n, \|\cdot\|_{L^\infty}, \mathcal{L}^n)$ is $\mathrm{CD}(0,N)$ but not $\mathrm{RCD}(0,N)$, and it is branching, so it does not satisfy splitting theorem.

- The Gaussian space $(\mathbb{R}^n, |\cdot|, e^{-|x|^2}\mathrm{d}x)$ is $\mathrm{RCD}(1,\infty)$ but not $\mathrm{RCD}(K,N)$ for any finite $N > 1$ and $K \in \mathbb{R}$.

- ([53]) The metric measure cone (see definition in the previous list) over 2-dimensional real projective space $C(\mathbb{R}P^2)$ is $\mathrm{ncRCD}(0,3)$ but it is not a non-collapsed Ricci limit space. That is, it cannot be the pGH or pmGH limit of a sequence of 3-dimensional connected complete manifolds with uniform lower volume bound and lower Ricci curvature bound.

- $([0,\pi], |\cdot|, \sin^{N-1})\mathrm{d}t)$ is an $\mathrm{RCD}(N-1,N)$ space for any $N \geq 1$ of essential dimension 1.

2.2 Calculus on $\mathrm{RCD}(K,N)$ spaces

2.2.1 Differential structure for general metric measure spaces

Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. We briefly review the notion of a normed module, introduced in [54], inspired by the theory developed in [55]. Refer to [54], [56] for a comprehensive treatise of this subject.

A L^0-normed module is a topological vector space $\mathscr{M}$ that is also a module over the commutative ring with unity $L^0(\mathsf{X},\mathfrak{m})$, possessing a *pointwise norm*, i.e. a map $|\cdot|:\mathscr{M}\to L^0(\mathsf{X},\mathfrak{m})$ such that

$$|fv+gw|\le|f||v|+|g||w| \qquad \mathfrak{m}\text{-a.e., } \forall v,w\in\mathscr{M}, \forall f,g\in L^0(\mathsf{X},\mathfrak{m}),$$

and such that the distance

$$\mathsf{d}_{\mathscr{M}}(v,w):=\int_{\mathsf{X}}1\wedge|v-w|\,\mathrm{d}\mathfrak{m}' \tag{2.2.1}$$

is complete and induces the topology of $\mathscr{M}$, where here $\mathfrak{m}'$ is a Borel probability measure such that $\mathfrak{m}\ll\mathfrak{m}'\ll\mathfrak{m}$ (the actual choice of $\mathfrak{m}'$ affects the distance but not the topology nor completeness).

$\mathscr{M}$ is said to be a Hilbert module provided

$$|v+w|^2+|v-w|^2=2(|v|^2+|w|^2) \qquad \mathfrak{m}\text{-a.e., } \forall v,w\in\mathscr{M}$$

and in this case by polarization we can define a pointwise scalar product as

$$\langle v,w\rangle:=\tfrac{1}{2}(|v+w|^2-|v|^2-|w|^2) \qquad \mathfrak{m}\text{-a.e., } \forall v,w\in\mathscr{M}$$

that turns out to be L^0-bilinear and continuous. The tensor product of two Hilbert modules

$\mathscr{M}_1, \mathscr{M}_2$ is defined as the completion of the algebraic tensor product as L^0-modules w.r.t. the distance induced by the pointwise norm that in turn is induced by the pointwise scalar product characterized by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\mathsf{HS}} := \langle v_1, v_2 \rangle_1 \langle w_1, w_2 \rangle_2.$$

The pointwise norm and scalar product on a tensor product will often be denoted with the subscript HS, standing for *Hilbert-Schmidt*. The dual $\mathscr{M}^*$ of $\mathscr{M}$ is defined as the collection of L^0-linear and continuous maps $L : \mathscr{M} \to L^0(\mathsf{X}, \mathfrak{m})$, is equipped with the natural multiplication by L^0 functions $(f \cdot L(v) := L(fv))$ and the pointwise norm

$$|L|_* := \operatorname*{ess\,sup}_{v:|v|\leq 1 \ \mathfrak{m}\text{-a.e.}} L(v).$$

It is then easy to check that $\mathscr{M}^*$ equipped with the topology induced by the distance defined as in (2.2.1) is a L^0-normed module. If $\mathscr{M}$ is Hilbert, then so is $\mathscr{M}^*$ and the map sending $v \in \mathscr{M}$ to $(w \mapsto \langle v, w \rangle) \in \mathscr{M}^*$ is an isomorphism of L^0-modules, called Riesz isomorphism.

The kind of differential calculus on metric measure spaces we are going to use is based around the following result, that defines both the *cotangent module* and the *differential* of Sobolev functions:

Theorem 2.2.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then there is a unique, up to unique isomorphism, couple* $(L^0(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})), \mathrm{d})$ *such that* $L^0(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ *is a* L^0-*normed module,* $\mathrm{d} : H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \to L^0(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ *is linear and such that:*

1) $|\mathrm{d}f| = |Df|$ $\mathfrak{m}$-*a.e. for every* $f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$*,*

2) L^0-*linear combinations of elements of the kind* $\mathrm{d}f$ *for* $f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *are dense in* $L^0(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$.

The dual of $L^0(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is denoted $L^0(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ and called *tangent module*. Elements of $L^0(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ are called 1-forms and elements of $L^0(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ are called vector fields on X.

In this case we shall denote by $\nabla f \in L^0(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ the image of $\mathrm{d}f$ under the Riesz isomorphism.

L^0-modules can be described by local basis and consequently a local dimension of L^0-module can be defined. We first recall few definitions. Fix a borel set $A \subseteq \mathsf{X}$ with $\mathfrak{m}(A) > 0$. We say a finite family $v_1, v_2 \ldots, v_n \in \mathscr{M}$ is *independent* on A provided that the identity

$$\sum_{i=1}^{n} f_i v_i = 0 \qquad \mathfrak{m}\text{-a.e. on } A$$

holds only if $f_i = 0$ $\mathfrak{m}$-a.e. on A, $i = 1, 2, \ldots, n$. Let $V \subseteq \mathscr{M}$. The span of V on A, denoted by $\mathrm{Span}_A(V)$ is the subset of $\mathscr{M}$ consisting of vectors v concentrated on V, i.e. $v = 0$ $\mathfrak{m}$-a.e. on A^c, with the following property: there exist disjoint Borel sets A_n, $n \in \mathbb{N}$, such that $A = \cup_n A_n$ and for every m_n elements $v_{1,n}, \ldots, v_{m_n,n}$, and $f_{1,n}, \ldots, f_{m_n,n}$ such that

$$\chi_{A_n} v = \sum_{i=1}^{m_n} f_{i,n} v_{i,n}.$$

With the notion local spanning and local independence, we can define the following:

Definition 2.2.2. Let A be a Borel set of X. We say a finite family $v_1, \ldots, v_n \in \mathscr{M}$ is a basis on A provided it is independent and $\mathrm{Span}_A(\{v_1, \ldots, v_n\}) = \mathscr{M}_{|A}$, i.e., the submodule of $\mathscr{M}$ consisting of elements that are $\mathfrak{m}$-a.e. zero on A^c.

If $\mathscr{M}$ admits a basis of cardinality n on A, we say that it has local dimension n on A. In particular if $A = \mathsf{X}$, we say that $\mathscr{M}$ has local dimension n.

It is also notable that we have the following characterization of Hilbert module in connection with infinitesimal Hilbertian property, see [54, Theorem 2.3.17].

Theorem 2.2.3. *A complete separable metric measure space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *is Infinitesimally Hilbertian if and only if both* $L^2(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$. *and* $L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ *are Hilbert Modules.*

We make some conventions on the notation. The tensor product of $L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with itself will be denoted $L^0(T^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$, similarly for $L^0(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$. Notice that, $L^0(T^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and $L^0((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ are one the dual of each other, in a natural way, see [54, Section 2.3.2].

For $p \in [1,\infty]$, the collection of 1-forms ω with $|\omega| \in L^p(\mathsf{X},\mathfrak{m})$ (resp. $L^p_{loc}(\mathsf{X},\mathfrak{m})$) will be denoted $L^p(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ (resp. $L^p_{loc}(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$). Similarly for vector fields and other tensors.

To end this section, we review the necessary notion to define gradient flow, in particular heat flow, in metric measure spaces. First we recall the notion of absolutely continuous curves, see e.g. [57].

Definition 2.2.4. Let (X,d) be a metric space. We say a curve $\gamma : [a,b] \to \mathsf{X}$ is absolutely continuous, if there exists non-negative $g \in L^1([a,b])$ such that for all $x, y \in [a,b]$,$y \leq x$, we have

$$\gamma(x) - \gamma(y) \leq \int_y^x g(t)dt.$$

We denote the space of all absolutely continuous curves from $[a,b]$ to X by $\mathrm{AC}([a,b];\mathsf{X})$. It is clear that Lipschitz continuous curves are absolutely continuous. The absolute continuous curves possess a weak notion of derivative, called the metric derivative.

Definition 2.2.5. For $\gamma \in \mathrm{AC}([a,b];\mathsf{X})$, the limit

$$\lim_{h\to 0} \frac{\mathsf{d}(\gamma(t+h),\gamma(t))}{|h|} = |\gamma|'(t)$$

exists for $\mathcal{L}^1$- a.e. $t \in [a,b]$. Up to $\mathcal{L}^1$-negligible sets, $|\gamma|'(t)$ coincides with the minimal g in the Definition 2.2.4. This $|\gamma|'$ is called the metric derivative of γ.

All this for general metric measure spaces. Now we narrow down our consideration to $\mathrm{RCD}(K,\infty)$ spaces.

2.2.2 Differential Operators on $\mathrm{RCD}(K,N)$ spaces

In the $\mathrm{RCD}(K,\infty)$ case we now recall the definition of the set of *test* functions (introduced in [58]):

$$\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m}) := \left\{ f \in \mathrm{Lip}(\mathsf{X},\mathsf{d}) \cap D(\Delta) \cap L^\infty(\mathsf{X},\mathfrak{m}) : \Delta f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m}) \right\}$$

which is an algebra. It is known from [58] that $|\nabla f|^2 \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ for any $f \in \mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$, that $\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$ is dense in $(D(\Delta), \|\cdot\|_D)$, where $\|f\|_D^2 := \|f\|_{H^{1,2}}^2 + \|\Delta f\|_{L^2}^2$. The following result is proved in [54, Theorem 3.3.8, Corollary 3.3.9].

Theorem 2.2.6. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,\infty)$ *space. For any* $f \in \mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$ *there exists a unique* $T \in L^2((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ *such that for all* $f_i \in \mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$,

$$T(\nabla f_1, \nabla f_2) = \frac{1}{2}\left(\langle \nabla f_1, \nabla\langle \nabla f_2, \nabla f\rangle\rangle + \langle \nabla f_2, \nabla\langle \nabla f_1, \nabla f\rangle\rangle - \langle f, \nabla\langle \nabla f_1, \nabla f_2\rangle\rangle\right) \tag{2.2.2}$$

holds for $\mathfrak{m}$*-a.e.* $x \in \mathsf{X}$*. Since such* T *is unique, we denote it by* $\mathrm{Hess} f$ *and call it the Hessian of* f*. Moreover for any* $f \in \mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$ *and any* $\varphi \in D(\Delta) \cap L^\infty(\mathsf{X},\mathfrak{m})$ *with* $\Delta\varphi \in L^\infty(\mathsf{X},\mathfrak{m})$ *and* $\varphi \geq 0$*, we have*

$$\int_{\mathsf{X}} \varphi |\mathrm{Hess} f|_{\mathsf{HS}}^2 \mathrm{d}\mathfrak{m} \leq \int_{\mathsf{X}} \frac{1}{2}\Delta\varphi \cdot |\nabla f|^2 - \varphi\langle \nabla\Delta f, \nabla f\rangle - K\varphi|\nabla f|^2 \mathrm{d}\mathfrak{m} \tag{2.2.3}$$

and

$$\int_{\mathsf{X}} |\mathrm{Hess} f|_{\mathsf{HS}}^2 \mathrm{d}\mathfrak{m} \leq \int_{\mathsf{X}} (\Delta f)^2 - K|\nabla f|^2 \mathrm{d}\mathfrak{m}. \tag{2.2.4}$$

Thanks to (2.2.3) with the density of $\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$ in $D(\Delta)$, for any $f \in D(\Delta)$ we can also define $\mathrm{Hess} f \in L^2((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with the equality (2.2.2), where $\langle \nabla f, \nabla f_i\rangle \in H^{1,1}(\mathsf{X},\mathsf{d},\mathfrak{m})$.

Definition 2.2.7 (Divergence div)**.** Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space. Denote by $D(\mathrm{div})$ (resp. $D_{loc}(\mathrm{div})$) the set of all $V \in L^2(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ (resp. $V \in L^2_{loc}(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$)

for which there exists $f \in L^2(\mathsf{X}, \mathfrak{m})$ (resp. $f \in L^2_{loc}(\mathsf{X}, \mathfrak{m})$) such that

$$\int_{\mathsf{X}} \langle V, \nabla h \rangle \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} fh \, \mathrm{d}\mathfrak{m} \qquad \forall h \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d}).$$

Since such f is unique (because $\mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$ is dense in $L^2(\mathsf{X}, \mathfrak{m})$), we define $\mathrm{div} V := f$.

Note that for any $f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, $f \in D(\Delta)$ if and only if $\nabla f \in D(\mathrm{div})$. Moreover if $f \in D(\Delta)$, then for any $\varphi \in \mathrm{Lip}_b(\mathsf{X}, \mathsf{d})$ we have $\varphi \nabla f \in D(\mathrm{div})$ with

$$\mathrm{div}(\varphi \nabla f) = \langle \nabla \varphi, \nabla f \rangle + \varphi \Delta f.$$

Recalling that the covariant derivative of $f\mathrm{d}h$ is given by $\mathrm{d}f \otimes \mathrm{d}h + f\mathrm{Hess}h$, the following definition is justified:

Definition 2.2.8 (Adjoint operator ∇^*)**.** Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space. Denote by $D(\nabla^*)$ (resp. $D_{loc}(\nabla^*)$) the set of all $T \in L^2((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ (resp. $T \in L^2_{loc}((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$) for which there exists $\eta \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ (resp. $\eta \in L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$) such that

$$\int_{\mathsf{X}} \langle T, \mathrm{d}f \otimes \mathrm{d}h + f \, \mathrm{Hess}h \rangle_{\mathsf{HS}} \, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \langle \eta, f\mathrm{d}h \rangle \mathrm{d}\mathfrak{m} \qquad \forall f \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d}), \forall h \in D(\Delta).$$

Since such η is unique (because objects of the kind $f\mathrm{d}h$ generate $L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$), we denote it by $\nabla^* T$.

It follows from a direct calculation that the following holds. See [59, Proposition 2.18] for the proof.

Proposition 2.2.9. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and let* $f \in \mathrm{Test}F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. *Then we have* $\mathrm{d}f \otimes \mathrm{d}f \in D(\nabla^*)$ *with*

$$\nabla^*(\mathrm{d}f \otimes \mathrm{d}f) = -\Delta f \mathrm{d}f - \frac{1}{2}\mathrm{d}|\mathrm{d}f|^2. \tag{2.2.5}$$

Definition 2.2.10 (Adjoint operator δ)**.** Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,\infty)$ space. Denote by $D(\delta)$ the set of $\omega\in L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ such that there exists unique $f\in L^2(X,\mathfrak{m})$ such that

$$\int_{\mathsf{X}}\langle\omega,\mathrm{d}h\rangle\mathrm{d}\mathfrak{m}=\int_{\mathsf{X}}fh\mathrm{d}\mathfrak{m},\quad\forall h\in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m}) \tag{2.2.6}$$

holds. We denote this unique f by $\delta\omega$.

Similar to the space of test functions, let us define the space of test 1-forms:

$$\mathrm{Test}T^*(\mathsf{X},\mathsf{d},\mathfrak{m}):=\left\{\sum_{i=1}^{l}f_{0,i}\mathrm{d}f_{1,i};l\in\mathbb{N},f_{j,i}\in\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})\right\}. \tag{2.2.7}$$

It is proved in [54, Prop.3.5.12] that $\mathrm{Test}T^*(\mathsf{X},\mathsf{d},\mathfrak{m})\subseteq D(\delta)$ holds with

$$\delta(f_1\mathrm{d}f_2)=-\langle\nabla f_1,\nabla f_2\rangle-f_1\Delta f_2,\quad\forall f_i\in\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m}). \tag{2.2.8}$$

Definition 2.2.11 (Sobolev space $W^{1,2}_C$)**.** Let us denote by $W^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ the set of all $\omega\in L^2(T^*(X,\mathsf{d},\mathfrak{m}))$ such that there exists unqiue $T\in L^2((T^*)^{\otimes 2}(X,\mathsf{d},\mathfrak{m}))$ such that

$$\int_{\mathsf{X}}\langle T,f_0\mathrm{d}f_1\otimes\mathrm{d}f_2\rangle\mathrm{d}\mathfrak{m}=\int_{\mathsf{X}}\left(-\langle\omega,\mathrm{d}f_2\rangle\delta(f_0\mathrm{d}f_1)-f_0\langle\mathrm{Hess}_{f_2},\omega\otimes\mathrm{d}f_1\rangle\right)\mathrm{d}\mathfrak{m} \tag{2.2.9}$$

holds. We denote this unique T by $\nabla\omega$.

It is proved in [54, Thm.3.4.2] that $\mathrm{Test}T^*(\mathsf{X},\mathsf{d},\mathfrak{m})\subseteq W^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ holds with

$$\nabla(f_1\mathrm{d}f_2)=\mathrm{d}f_1\otimes\mathrm{d}f_2+f_1\mathrm{Hess}_{f_2},\quad\forall f_i\in\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m}). \tag{2.2.10}$$

Definition 2.2.12 (Sobolev space $H^{1,2}_C$)**.** Let us denote by $H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ the closure of $\mathrm{Test}T^*(\mathsf{X},\mathsf{d},\mathfrak{m})$ in $W^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$.

Definition 2.2.13 (Exterior derivative d)**.** Let us denote by $W^{1,2}_{\mathrm{d}}(T^*(X\,\mathsf{d},\mathfrak{m}))$ the set of all

$\omega \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ such that there exists unique $\eta \in L^2(\bigwedge^2 T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ such that

$$\int_{\mathsf{X}} \langle \eta, \alpha_0 \otimes \alpha_1 \rangle \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \left(\langle \omega, \alpha_0 \rangle \delta\alpha_1 - \langle \omega, \alpha_1 \rangle \delta\alpha_0 \right) \mathrm{d}\mathfrak{m}, \quad \forall \alpha \in \mathrm{Test}T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \tag{2.2.11}$$

holds. We denote this unique by $\mathrm{d}\omega$.

It is proved in [54, Thm.3.5.2] that $\mathrm{Test}T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \subseteq W_{\mathrm{d}}^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ holds.

Definition 2.2.14 (Sobolev space $H_H^{1,2}$)**.** Let us denote by $H_H^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ the completion of $\mathrm{Test}T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with respect to the norm:

$$\|\omega\|_{H_H^{1,2}}^2 := \|\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\mathrm{d}\omega\|_{L^2}^2. \tag{2.2.12}$$

Definition 2.2.15 (Hodge Laplacian $\Delta_{H,1}$)**.** Let us denote by $D(\Delta_{H,1})$ the set of all $\omega \in H_H^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ such that there exists unique $\eta \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ such that

$$\int_{\mathsf{X}} \left(\langle \mathrm{d}\omega, \mathrm{d}\alpha \rangle + \delta\omega \cdot \delta\alpha \right) \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \langle \eta, \alpha \rangle \mathrm{d}\mathfrak{m}, \quad \forall \alpha \in H_H^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})) \tag{2.2.13}$$

holds. We denote this unique η by $\Delta_{H,1}\omega$.

It is proved in [54] that $H_H^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})) \subseteq H_C^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ holds with

$$\int_{\mathsf{X}} |\nabla\omega|^2 \mathrm{d}\mathfrak{m} \leq \int_{\mathsf{X}} (|\mathrm{d}\omega|^2 + |\delta\omega|^2 - K|\omega|^2) \mathrm{d}\mathfrak{m}, \quad \forall \omega \in H_H^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})). \tag{2.2.14}$$

On the other hand, it follows from Definition 2.2.11 and Definition 2.2.13 that for any $\omega \in H_C^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$,

$$\mathrm{d}\omega(V_1, V_2) = (\nabla_{V_1}\omega)(V_2) - (\nabla_{V_2}\omega)(V_1), \quad \forall V_i \in L^\infty(T(\mathsf{X}, \mathsf{d}, \mathfrak{m})) \tag{2.2.15}$$

holds, where $\nabla_{V_1}\omega := \nabla\omega(\cdot, V_1)$. In particular, we see that $H_C^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is a subset of $H_{\mathrm{d}}^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$, where $H_{\mathrm{d}}^{1,2}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ denotes the $W_{\mathrm{d}}^{1,2}$-closure of $\mathrm{Test}T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, with

$$|\mathrm{d}\omega|^2 \le 2|\nabla\omega|^2, \quad \mathfrak{m} - a.e.\, x \in \mathsf{X} \tag{2.2.16}$$

2.3 Convergence of Meatric Measure Spaces and stability

In this section we collect the basics of Gromov-Hausdorff convergence, and explain why the CD and RCD conditions are stable under the pointed measured Gromov-Hausdorff convergence. Then we introduce the notion of convergence on varying spaces of functions and tensors along with the convergence of spaces, following [24] and [60]. All the convergence notions of functions and tensors to be mentioned also make sense for local spaces.

2.3.1 Gromov-Hausdorff Convergence

Different notions of convergence of spaces arise for different applications. For example, the Cheeger-Gromov-Hamilton convergence for $C^{1,\alpha}$ Riemmanian metrics, the d_p convergence considered in [61] by Lee-Naber-Neumayer for their attempt to study the rigidity of lower scalar curvature bounds. The notion of convergence that is suitable for metric measure spaces with Ricci curvature lower bound is the (measured) Gromov-Hausdorff convergence, we will see shortly that RCD condition is preserved under such convergence. Let us start with the definition of Gromov-Hausdorff distance between compact metric spaces.

Definition 2.3.1. Let $(\mathsf{X}, \mathsf{d}_\mathsf{X})$, $(Y, \mathsf{d}_\mathsf{Y})$ be compact metric spaces. The *Gromov-Hausdorff* distance between $(\mathsf{X}, \mathsf{d}_\mathsf{X}), (\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ is

$$\mathsf{d}_{\mathrm{GH}}(\mathsf{X}, \mathsf{Y}) = \inf\{\mathsf{d}_{H,\mathsf{Z}}(i_\mathsf{X}(\mathsf{X}), i_\mathsf{Y}(\mathsf{Y})) : (\mathsf{Z}, \mathsf{d}_Z), i_\mathsf{X}, i_\mathsf{Y}\}$$

where the infimum is taken over all complete metric spaces $(\mathsf{Z}, \mathsf{d}_Z)$ and isometric embeddings $i_\mathsf{X} : \mathsf{X} \to \mathsf{Z}$, $i_\mathsf{Y} : \mathsf{Y} \to \mathsf{Z}$, and $\mathsf{d}_{H,Z}$ is the Hausdorff distance on Z.

We then define the Gromov-Hausdorff (GH for short) convergence by the convergence under this distance d_{GH} in the space of all compact metric spaces. This is an extrinsic approach to define the GH convergence. We can also take the intrinsic approach, without appealing to isometric embeddings into ambient metric space, instead, we use ε-isometry. We say sequence of compact metric spaces $(\mathsf{X}_i, \mathsf{d}_{\mathsf{X}_i})$ GH converges to (X, d) if there exists $\varepsilon_i \searrow 0$, so that there exists function $\varphi_i : \mathsf{X}_i \to \mathsf{X}$ such that

- $|\mathsf{d}_X(\varphi_i(x), \varphi_i(y)) - \mathsf{d}_{\mathsf{X}_i}(x, y)| < \varepsilon_i$ for any $x, y \in \mathsf{X}_i$ (almost isometric)
- $B_{\varepsilon_i}(\varphi_i(\mathsf{X}_i)) = \mathsf{X}$ (almost surjective)

where $B_{\varepsilon_i}(\varphi_i(\mathsf{X}_i))$ is the ε_i neighborhood of $\varphi_i(\mathsf{X}_i)$.

In the non-compact setting, one needs to in addition take the reference points into consideration. More precisely, we consider *pointed* metric spaces $(\mathsf{X}, \mathsf{d}, x)$, where $x \in \mathsf{X}$. a sequence of pointed metric spaces $(\mathsf{X}_i, \mathsf{d}_{\mathsf{X}_i}, x_i)$ pointed Gromov-Haudorff (pGH for short) converges to $(\mathsf{X}, \mathsf{d}_X, x)$ if there exists $\varepsilon_i \searrow 0$ and $r_i \nearrow \infty$, so that there exists function $\varphi_i : \mathsf{X}_i \to \mathsf{X}$ such that

- $|\mathsf{d}_X(\varphi_i(x), \varphi_i(y)) - \mathsf{d}_{\mathsf{X}_i}(x, y)| < \varepsilon_i$ for any $x, y \in \mathsf{X}_i$ (almost isometric)
- $\varphi_i(x_i) = \varphi(x)$ and $B_{r_i - \varepsilon_i}(x) \subseteq B_{\varepsilon_i}(\varphi_i(B_{r_i}(x_i)))$ (almost surjective)

If we not only consider metric spaces but also metric measure spaces, a notion of convergence of measure along with GH convergence is needed. Fukaya proposed the notion of measured Gromov-Hausdorff (mGH for short) convergence. It also admits a generalization to the non-compact setting by taking the reference points into consideration, then we arrive at the definition of pointed measured Gromov-Hausdorff (pmGH for short) convergence. A sequence of pointed metric measure spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ pmGH converges to $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$ if $(\mathsf{X}_i, \mathsf{d}_i, x_i)$ pGH converges to $(\mathsf{X}, \mathsf{d}, x)$, φ_i given by pGH convergence is Borel, and for every $r > 0$, $(\varphi_i)_\sharp(\mathfrak{m}_i \llcorner B_r(x_i)) \rightharpoonup \mathfrak{m} \llcorner B_r(x)$ in duality with $C_{\mathrm{bs}}(\mathsf{X})$, the space of continuous function with bounded support in X.

Let $(\mathcal{M}, \mathsf{d}_{\mathrm{GH}})$ be the space of isometric classes of compact complete metric spaces equipped with GH distance. It is a complete metric space. The following Gromov precompactness theorem is a fundamental result, see also [62, Chapter 5], [63, Section 11.1]:

Theorem 2.3.2. *Let $n \geq 2$ be an integer, $k \in \mathbb{R}$ and $D \in (0, \infty)$. Then*

1. *The collection of all (isometric classes) of closed n-dimensional Riemannian manifolds (M, g) with $\mathrm{Ric}_g \geq Kg$ and $\mathrm{diam} \leq D$ is precompact in* GH *topology.*

2. *The collection of all pointed complete n-dimensional Riemannian manifolds (M, g, x) with $\mathrm{Ric}_g \geq Kg$ is precompact in* pGH *topology.*

3. *The collection of all pointed complete n-dimensional Riemannian manifolds with renormalized measure $(M, g, \mathrm{vol}^{-1}(B_1(x)) \mathrm{vol}_g, x)$ is precompact in* pmGH *topology.*

In particular, the Gromov precompactness theorem grants the existence of Ricci limit spaces. It is desirable to know if $\mathrm{CD}(K, N)$ and $\mathrm{RCD}(K, N)$ conditions are stable under pmGH convergence as well. It is shown in [15, Theorem 29.25] that $\mathrm{CD}(K, N)$ condition is stable under pmGH convergence. Historically, Sturm's $\mathbb{D}$-convergence [28] (for compact case), and its variant [24, Definition 3.11] (for non-compact case), and also the pointed measured Gromov (pmG for short)convergence are are also introduced to tackle the stability of $\mathrm{CD}(K, N)$ and $\mathrm{RCD}(K, N)$ conditions. For example, Sturm([28, Theorem 4.20], [18, Theorem 3.1]) showed that $\mathrm{CD}(K, N)$ condition is stable under $\mathbb{D}$-convergence. The variant of $\mathbb{D}$ convergence and pmG convergence are shown to be equivalent to pmGH convergence for $\mathrm{RCD}(K, N)$ spaces [24, Theorem 3.15]. Through various different notions of convergence, the stability of CD and RCD conditions under pmGH convergence is proved. In summary, we have the following precompactness theorem for $\mathrm{RCD}(K, N)$ spaces:

Theorem 2.3.3. *If a sequence of pointed $\mathrm{RCD}(K, N)$ spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$,$n \in \mathbb{N}$, satisfies*

$$0 < \liminf_{n \to \infty} \mathfrak{m}_i(B_1(x_i)) \leq \limsup_{n \to \infty} \mathfrak{m}_i(B_1(x_i)) < \infty,$$

then the sequence has a subsequence $(\mathsf{X}_{n_j}, \mathsf{d}_{n_j}, \mathfrak{m}_{n_j}, x_{n_j})$ pmGH converging to a pointed $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$.

2.3.2 Convergence of Functions and Tensors

Throughout this subsection we assume that the sequence of pointed $\mathrm{RCD}(K,\infty)$ spaces $(\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ pmGH converges to $(\mathsf{X},\mathsf{d},\mathfrak{m},x)$ and $(\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathfrak{m}_\mathsf{Y})$ be the ambient space all X_i and X embedded into in the pmGH convergence.

Let us first recall the L^2 convergence of functions.

Definition 2.3.4. We say that $f_i \in L^2(\mathsf{X}_i,\mathfrak{m}_i)$ L^2-weakly converges to $f \in L^2(\mathsf{X},\mathfrak{m})$ if $\sup_i \|f\|_{L^2} < \infty$ and $f_i\mathfrak{m}_i \rightharpoonup f\mathfrak{m}$ in duality with $C_c(\mathsf{Y})$, L^2-strongly converges to $f \in L^2(\mathsf{X},\mathfrak{m})$ if f_i L^2-weakly converges to f and $\limsup_i \|f_i\|_{L^2} \le \|f\|_{L2}$.

Note that $\liminf_i \|f_i\|_{L^2} \ge \|f\|_{L^2}$ is implied in the weak convergence. The seemingly natural way to define strong L^2-convergence from f_i to f may be taking ε_i-isometry φ_i, so that $\varepsilon_i \nearrow 0$ and $\|f-\varphi_i\circ f_i\|_{L^2}\to 0$. But this does not work. Take $\mathsf{X}_i=\mathsf{X}=[0,1]$ with Euclidean distance and Lebesgue measure, then take $f_i=1$, $f=\chi_\mathbb{Q}$, $\mathbb{Q}$ being the set of rationals in $[0,1]$. Now if we take ε_i-isometry φ_i taking values in $\mathbb{Q}$, then $\|f-\varphi_i\circ f_i\|_{L^2}\to 0$, however, $\|f_i\|_{L^2}=1$ whereas $\|f\|_{L^2}=0$. So this is not the right way to generalize L^2 convergence to the case of varying spaces.

Next, we recall the $H^{1,2}$ convergence.

Definition 2.3.5. We say that $f_i \in H^{1,2}(\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i)$ $H^{1,2}$-weakly converges to $f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ if f_i L^2-weakly converges to f and $\sup_i \mathsf{Ch}_i(f_i)$ is finite, $H^{1,2}$-strongly converges to $f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ if f_i L^2-strongly converges to f and $\mathsf{Ch}(f)=\lim_i \mathsf{Ch}_i(f_i)$.

All this prepares us to state:

Theorem 2.3.6. *[Stability of Laplacian] Let $f_i \in D(\Delta)$ and $f \in L^2(\mathsf{X},\mathfrak{m})$ the strong L^2-limit of f_i. If*

$$\sup_i \|f_i\|_{L^2(\mathsf{X}_i,\mathfrak{m}_i)} + \|\Delta f_i\|_{L^2(\mathsf{X}_i,\mathfrak{m}_i)} < \infty,$$

then $f \in D(\Delta)$ and that f_i $H^{1,2}$strongly converge to f, Δf_i L^2 weakly converges to Δf.

See also [60, Theorem 4.4] for a local version.

Now we introduce the notion of convergence of 2-tensors. These are very similar to those of functions, but now we fix the space, which is enough for the purpose of the thesis.

Definition 2.3.7. We say a sequence of symmetric 2-tensors $T_i \in L^2((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}.\mathfrak{m}))$ L^2-weakly converges to symmetric $T \in L^2((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ if $T_i(V, V)$ L^2 weakly converges to $T(V, V)$ as functions for any $V \in L^\infty(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ and $\sup_i \||T_i|_{\mathrm{HS}}\|)_{L^2} < \infty$, T_i converges to T strongly if $\||T_i - T|_{\mathrm{HS}}\|_{L^2} \to 0$ as $i \to \infty$.

Since for a symmetric tensor T, $|T|_{\mathrm{HS}}$ is completely determined by $T(V, V)$, we see that similar to the case of weak L^2 convergence of functions, the weak L^2 convergence of T_i to T implies that $\liminf_i \||T_i|_{\mathrm{HS}}\| \geq \||T|_{\mathrm{HS}}\|$. The following criterion is useful when proving L^2 strong convergence.

Proposition 2.3.8. *A sequence of symmetric* 2*-tensors* T_i L^2*-strongly converges to symmetric* T *if and only if*

$$\lim_{i\to\infty} \int_{\mathsf{X}} T_i(V, V) = \int_{\mathsf{X}} g(V, V) \mathrm{d}\mathfrak{m} \qquad \forall V \in L^\infty(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$$

and

$$\limsup_{i\to\infty} \int_{\mathsf{X}} |T_i|_{\mathrm{HS}} \leq \int_{\mathsf{X}} |T|_{\mathrm{HS}} \mathrm{d}\mathfrak{m} < \infty$$

2.4 Structure of $\mathrm{RCD}(K, N)$ spaces

Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space for some $K \in \mathbb{R}$ and some $N \in [1, \infty)$. The main purpose of this section is to provide a more detailed metric measure structure theory of $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$.

First let us recall the locally volume doubling metric measure space.

Definition 2.4.1. A metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is called locally volume doubling if For

$R > 0$ and all r so that $2r < R$, there exists a constant C_R such that

$$\mathfrak{m}(B_{2r}(x)) \leq C_R \mathfrak{m}(B_r(x)) \quad \forall x \in \mathsf{X}.$$

It is called (globally) volume doubling if the constant C_R is in fact independent of R.

Similar as the Riemannian manifolds with lower Ricci curvature bounds, the Bishop-Gromov inequality holds for $\mathrm{CD}(K,N)$ hence $\mathrm{RCD}(K,N)$ spaces, to state the theorem we define the volume of geodesic balls of radius R in the space form of constant sectional curvature K as

$$\mathrm{vol}_{K,N}(R) = \begin{cases} \omega_N \int_0^R \sin\Big(t\sqrt{\frac{K}{N-1}}\Big)^{N-1} \mathrm{d}t & K > 0, \\ \omega_N R^N & K = 0, \\ \omega_N \int_0^R \sinh\Big(t\sqrt{\frac{-K}{N-1}}\Big)^{N-1} \mathrm{d}t & K < 0, \end{cases} \tag{2.4.1}$$

where $\omega_N = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$ and $\Gamma(z)$ denotes the usual Gamma function for $\mathrm{Re}(z) > 0$. The Bishop-Gromov inequality (which is also valid for $\mathrm{CD}(K,N)$ spaces states that: (see [64, Theorem 5.31], [18, Theorem 2.3])

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))} \leq \frac{\mathrm{vol}_{K,N}(R)}{\mathrm{vol}_{K,N}(r)} \qquad \forall x \in \mathsf{X}, \forall r < R, \tag{2.4.2}$$

where, in the case $K \leq 0, N = 1$, $\sinh\Big(t\sqrt{\frac{-K}{N-1}}\Big)^{N-1}$ has to be interpreted as 1. It then follows from (2.4.2) that

$$\frac{\mathfrak{m}(B_R(x))}{\mathfrak{m}(B_r(x))} \leq C(K,N) \exp\left(C(K,N)\frac{R}{r}\right) \qquad \forall x \in \mathsf{X}, \forall r < R \tag{2.4.3}$$

and

$$\frac{\mathfrak{m}(B_r(x))}{\mathfrak{m}(B_r(y))} \leq C(K,N) \exp\left(C(K,N)\frac{\mathsf{d}(x,y)}{r}\right) \qquad \forall x, y \in \mathsf{X}, \forall r > 0 \tag{2.4.4}$$

are satisfied. This is to say that $\mathrm{RCD}(K,N)$ spaces are locally volume doubling. Moreover, if $K \geq 0$ then they are (globally) volume doubling. It is well-known that from the Bishop-Gromov inequality it follows that the metric structure (X,d) is proper, hence geodesic, with (X,d) being a length space. The length space property of RCD spaces follows quite easily from the so called *Sobolev to Lipschitz* property, namely item 3) of Definition 2.1.2 (e.g. [50, Theorem 3.10] and references therein).

The local $(1,1)$-Poincaré inequality holds for $f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ and $r > 0$:

$$\int_{B_r(x)} \left| f - \frac{1}{\mathfrak{m}(B_r(x))} \int_{B_r(x)} f \mathrm{d}\mathfrak{m} \right| \mathrm{d}\mathfrak{m} \leq 4e^{|K|r^2} r \int_{B_{2r}(x)} |\mathrm{D}f| \mathrm{d}\mathfrak{m} \tag{2.4.5}$$

and it is also valid for $\mathrm{CD}(K,\infty)$ spaces. See [65] for the detail. The local doubling condition along with $(1,1)$- Poincaré can deduce the $(2,2)$ Poincaré inequality on $\mathrm{CD}(K,N)$ spaces, see[66, Theorem 5.1].

Now we move to the fine structure of $\mathrm{RCD}(K,N)$ spaces by looking at them infinitesimally, to this end we introduce the notion of *tangent cones*, which are non-smooth generalization of tangent spaces. Structural results for $\mathrm{RCD}(K,N)$ spaces are strongly motivated and influenced by those of Ricci limit spaces. For example, [11, 12, 13, 14].

Definition 2.4.2 (Tangent cones)**.** Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. For $x \in \mathsf{X}$, we denote by $\mathrm{Tan}(\mathsf{X},\mathsf{d},\mathfrak{m},x)$ the set of tangent cones to $(\mathsf{X},\mathsf{d},\mathfrak{m})$ at x: the collection of all isomorphism classes of pointed metric measure spaces $(\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathfrak{m}_\mathsf{Y},y)$ such that, as $i \to \infty$, one has

$$\left(\mathsf{X}, \frac{1}{r_i}\mathsf{d}, \frac{1}{\mathfrak{m}(B_{r_i}(x))}\mathfrak{m}, x\right) \overset{\mathrm{pmGH}}{\to} (\mathsf{Y},\mathsf{d}_\mathsf{Y},\mathfrak{m}_\mathsf{Y},y) \tag{2.4.6}$$

for some $r_i \to 0^+$.

Note that Theorem 2.3.3 proves $\mathrm{Tan}(\mathsf{X},\mathsf{d},\mathfrak{m},x) \neq \emptyset$ for any $x \in \mathsf{X}$, but in general there is no uniqueness, which means there could be non-isometric tangent cones at some point. Moreover, there is an interesting example of Ricci limit space constructed by Colding-Naber with a point of which tangent cones can even be non-homeomorphic, see [67].

We are now in position to introduce the key notions of *regular sets* and the *essential dimension* as follows.

Definition 2.4.3. [Regular set $\mathcal{R}_k$] Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. For any $k \geq 1$, we denote by $\mathcal{R}_k$ the k-dimensional regular set of $(X, \mathsf{d}, \mathfrak{m})$, namely the set of points $x \in \mathsf{X}$ such that

$$\mathrm{Tan}(X, \mathsf{d}, \mathfrak{m}, x) = \left\{ \left(\mathbb{R}^k, \mathsf{d}_{\mathbb{R}^k}, (\omega_k)^{-1} \mathcal{H}^k, 0_k \right) \right\},$$

where ω_k is the k-dimensional volume of the unit ball in $\mathbb{R}^k$ with respect to the k-dimensional Hausdorff measure $\mathcal{H}^k$.

The following result is proved in [68, Theorem 0.1].

Theorem 2.4.4 (Essential dimension)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then there exists a unique integer* $n \in [1, N]$*, called the essential dimension of* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$*, denoted by* $\mathrm{essdim}(\mathsf{X})$*, such that*

$$\mathfrak{m}(X \setminus \mathcal{R}_n) = 0.$$

Remark 2.4.5. The essential dimension is a purely metric concept, actually it is equal to the maximal number $n \in \mathbb{N}$ satisfying

$$\left(\mathsf{X}, \frac{1}{r_i} \mathsf{d}, x \right) \overset{\mathrm{pGH}}{\to} (\mathbb{R}^n, \mathsf{d}_{\mathbb{R}^n}, 0_n)$$

for some $x \in \mathsf{X}$ and some $r_i \to 0^+$ because of the splitting theorem [27, Theorem 1.4] and the phenomenon of *propagation of regularity*. See [46, Remark 4.3], [69, Proposition 2.4] and [70]. We summarize the references above and provide a proof in Appendix A. ■

Next we remark on the link between local dimension of tangent module (bundle) and the essential dimension. In [71] rectifiability of $\mathrm{RCD}(K, N)$ spaces is shown, it in particular implies $\mathfrak{m}(\mathsf{X} \setminus \cup_{k=1}^N \mathcal{R}_k) = 0$. In [72] it is further proved that on $\mathcal{R}_k$, $\mathfrak{m} \ll \mathcal{H}^k$. Finally [73] proves the tangent module (bundle) defined through normed module is isomorphic to the Gromov-Hausdorff tangent bundle defined by "gluing" $\mathcal{R}_k \times \mathbb{R}^k$, now by the definition essential dimension we see that $\mathfrak{m}$-a.e. "fibers" of the Gromov Hausdorff tangent bundle has dimension the same as the essential dimension. Through the isomorphism between the two notions of tangent bundles, we know that the local dimension of M is equal to the essential dimension for $\mathrm{RCD}(K, N)$ spaces.

Theorem belowe gives refined relation between $\mathfrak{m}$ and the Hausdorff measure of the essential dimension. See [44, 74, 75, 72] for the detail.

Theorem 2.4.6. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space and let* n *be its essential dimension. Then* $\mathfrak{m} \ll \mathcal{H}^n \llcorner \mathcal{R}_n$. *Also, letting* $\mathfrak{m} = \theta \mathcal{H}^n \llcorner \mathcal{R}_n$ *and*

$$\mathcal{R}_n^* := \left\{ x \in \mathcal{R}_n : \, \exists \lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} \in (0, \infty) \right\} \tag{2.4.7}$$

we have that $\mathfrak{m}(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$, $\mathfrak{m} \llcorner \mathcal{R}_n^*$ *and* $\mathcal{H}^n \llcorner \mathcal{R}_n^*$ *are mutually absolutely continuous and*

$$\lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_n r^n} = \theta(x) \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \in \mathcal{R}_n^*.$$

Moreover $\mathcal{H}^n(\mathcal{R}_n \setminus \mathcal{R}_n^*) = 0$ *if* $n = N$.

A more general and classical result concerning densities, that we shall use later on, is the following (see e.g. [76, Theorem 2.4.3] for a proof):

Lemma 2.4.7. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space,* $\alpha \geq 0$ *and* $A \subseteq \mathsf{X}$ *a Borel subset such that*

$$\limsup_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{r^\alpha} > 0 \qquad \forall x \in A.$$

Then $\mathcal{H}^\alpha \llcorner A$ *is a Radon measure absolutely continuous w.r.t.* $\mathfrak{m}$.

The fact that $L^0(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is a Hilbert module is an indication of the existence of some (weak) Riemannian metric on X. This statement can easily be made more explicit by building upon the fact that such module has local dimension equal to the essential dimension of X as discussed above (see [73]):

Proposition 2.4.8. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n*. Then there is a unique* $g \in L^0((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ *such that*

$$g(V_1 \otimes V_2) = \langle V_1, V_2 \rangle \qquad \mathfrak{m}\text{-a.e., } \forall V_1, V_2 \in L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m})).$$

Moreover, g *satisfies*

$$|g|_{\mathrm{HS}} = \sqrt{n}, \qquad \mathfrak{m}\text{-a.e.}. \tag{2.4.8}$$

We can use this 'metric tensor' to define the *trace* of any $T \in L^0((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ by

$$\mathrm{tr}(T) := \langle T, g \rangle_{\mathsf{HS}} \in L^0(\mathsf{X},\mathfrak{m}).$$

Notice that by (2.4.8) and the Cauchy-Schwarz inequality it follows that

$$T \in L^p_{loc}((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m})) \Rightarrow \mathrm{tr}(T) \in L^p_{loc}(\mathsf{X},\mathsf{d},\mathfrak{m}).$$

Finally let us end this section by recalling the lower semicontinuity of the essential dimensions with respect to pmGH convergence proved in [77, Theorem 1.5]. This can also be understood as a consequence of L^2_{loc}-weak convergence of Riemannian metrics (see [43, Remark 5.20]), and an alternative proof of the theorem below can be based on Remark 2.4.5.

Theorem 2.4.9. *Let*

$$(\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i,x_i) \overset{\mathrm{pmGH}}{\rightarrow} (\mathsf{X},\mathsf{d},\mathfrak{m},x)$$

be a pmGH *convergent sequence of pointed* $\mathrm{RCD}(K,N)$ *spaces. Then*

$$\liminf_{i\to\infty} \mathrm{essdim}(\mathsf{X}_i) \geq \mathrm{essdim}(\mathsf{X}).$$

2.5 Weakly Non-collapsed and (Strongly) Non-collapsed spaces

Historically, the notion of non-collapsed spaces originated from non-collapsed Ricci limit spaces, which are systematically studied by Cheeger-Colding in [12]. Let us recall their definition:

Definition 2.5.1. Let (M_i^n, g_i, x_i) be a sequence of complete, connected Riemannian manifolds with $\mathrm{Ric}_{g_i} \geq K g_i$ for some $K \in \mathbb{R}$. This sequence is called a non-collapsing consequence if $\mathrm{vol}_{g_i}(B_1(x_i)) \geq \nu > 0$. The corresponding Gromov-Hausdorff limit space (X, d) is called a non-collapsed Ricci limit space. On the other hand, if $\liminf \mathrm{vol}_{g_i}(B_1(x_i)) \to 0$, the sequence is called a collapsing sequence and the Ricci limit space obtain from the sequence is called a collapsed space.

Using the notation in the above definition, Colding's convergence theorem [12, Theorem 5.9] asserts that the renormalized volume $\frac{\mathrm{vol}_{g_i}}{\mathrm{vol}_{g_i}(B_1(x_i))}$, converges to the Hausdorff measure $\mathcal{H}^n$ in (X, d) in the sense that $\mathrm{vol}_{g_i}(B_r(y_i))/\,\mathrm{vol}_{g_i}(B_1(x_i)) \to \mathcal{H}^n(B_r(y))$ for $y_i \to y$ and $r > 0$.

Inspired by Colding's work, De Philippis-Gigli [78] proposed the intrinsic definition of non-collapsed space we recall as follows:

Definition 2.5.2 (Non-collapsed $\mathrm{RCD}(K, N)$ space)**.** An $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is said to be *non-collapsed* if $\mathfrak{m} = \mathcal{H}^N$.

Similar to the non-collapsed Ricci limit spaces, non-collapsed $\mathrm{RCD}(K, N)$ spaces have better regularity than general $\mathrm{RCD}(K, N)$ spaces. For example, the tangent cones are metric cones, which follows from [79] which states that volume cones are metric cones and [78, proposition 2.8] which states that tangent cones are volume comes. Another important structure property is the stratification of *singular sets*, which we will discuss when we consider the notion of boundary of non-collapsed $\mathrm{RCD}(K, N)$ spaces in Chapter . In connection with Chapter 5, yet another fine property of non-collapsed space regarding Sobolev spaces is that

$$H^{1,2}_H(T^*(\mathsf{X},\mathsf{d},\mathcal{H}^N)) = H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathcal{H}^N)). \tag{2.5.1}$$

Finally, let us emphasize one of the properties as follows (see [78, Theorem 1.2]).

Theorem 2.5.3 (From pGH to pmGH)**.** *Let $K \in \mathbb{R}$, $N \in \mathbb{N}$ and $(\mathsf{X}_i, \mathsf{d}_i, \mathcal{H}^N, x_i)$ be a sequence of pointed non-collapsed $\mathrm{RCD}(K,N)$ spaces. Then after passing to a subsequence, there exists a pointed proper geodesic space $(\mathsf{X}, \mathsf{d}, x)$ such that*

$$(\mathsf{X}_i, \mathsf{d}_i, x_i) \overset{\mathrm{pGH}}{\to} (\mathsf{X}, \mathsf{d}, x).$$

Moreover, if $\inf_i \mathcal{H}^N(B_1(x_i)) > 0$, then $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N, x)$ is also a pointed non-collapsed $\mathrm{RCD}(K,N)$ space and the convergence of the $(\mathsf{X}_i, \mathsf{d}_i, \mathcal{H}^N, x_i)$'s to such space is in the pmGH topology.

We remark that the above theorem should be viewed as the generalization to the RCD class of non-collapsing sequence of manifolds in Definition 2.5.1, and is tightly related to the following continuity result, which is the generalization to the RCD class of the classical statement by Colding about volume convergence under lower Ricci curvature bounds [80, Theorem 0.1] (see [78, Theorem 1.3]):

Theorem 2.5.4 (Continuity of $\mathcal{H}^N$)**.** *For $K \in \mathbb{R}$, $N \in [1,\infty)$ let $\mathbb{B}(K,N)$ be the collection of (isometry classes of) open unit balls on $\mathrm{RCD}(K,N)$ spaces. Equip $\mathbb{B}(K,N)$ with the Gromov-Hausdorff distance.*

Then the map $\mathbb{B}(K,N) \ni B \mapsto \mathcal{H}^N(B) \in \mathbb{R}$ is continuous.

For technical reasons, we need a notion weaker than being non-collapsed. In order to give the precise definition, let us recall the following result which is just a collection of previously known ones:

Theorem 2.5.5. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Then the following five conditions are equivalent.*

1. *The essential dimension of* X *is* N.

2. $\mathfrak{m}$ *is absolutely continuous with respect to* $\mathcal{H}^N$.

3. $\mathfrak{m}\left(\{x \in \mathsf{X} : \theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}](x) < \infty\}\right) > 0$ *holds.*

4. $N \in \mathbb{N}$ *and the Hausdorff dimension of* (X, d) *is greater than* $N - 1$.

5. *The Hausdorff dimension of* (X, d) *is* N.

Proof. The equivalence between item 1 and item 2 is proved in [78, Theorem 1.12]. Since the implication from item 2 to item 3 is a direct consequence of Theorem 2.4.6, let us check the implication from item 3 to 1 as follows. The positivity (4.1.2) with Theorem 2.4.4 and 2.4.6 yields

$$\mathcal{H}^N(\mathcal{R}_n^*) > 0,$$

where n denote the essential dimension. In particular $N \le n$. Since the converse inequality is always satisfied by Theorem 2.4.6, we have item 1.

Notice that item 2 implies item 4, we show now that item 4 implies item 1. To see this, notice that the proof of [78, Theorem 1.4] shows that if item 4 holds, then there is an iterated tangent space isomorphic to $\mathbb{R}^N$. Since the essential dimension of the N-dimensional Euclidean space is N, the conclusion follows from Theorem 2.4.9.

If we assume item 5, then, since the Hausdorff dimension of (X, d) is at most the integer part of N (by [78, Corollary 1.5]), we see that N is an integer so that item 4 holds. Finally, if item 2 holds, then the Hausdorff dimension of (X, d) is at least N, so that we conclude by [78, Corollary 1.5] again. □

We are now in a position to introduce the notion of weakly non-collapsed $\mathrm{RCD}(K, N)$ spaces (our definition is trivially equivalent to the one in [78]):

Definition 2.5.6 (Weakly non-collapsed $\mathrm{RCD}(K, N)$ space)**.** An $\mathrm{RCD}(K, N)$ space is said to be *weakly non-collapsed* if one (and thus any) of the items in Theorem 2.5.5 is satisfied.

Note that any non-collapsed $\mathrm{RCD}(K,N)$ space is a weakly non-collapsed $\mathrm{RCD}(K,N)$ space.

We conclude this section recalling that one expects the notion of non-collapsed spaces to be related to the fact that the trace of the Hessian is the Laplacian. A first instance of this behaviour is contained in the following result, that is basically extracted from [81, Proposition 3.2] (notice that Definition 2.1.2 tells that if the stated inequality (2.5.2) holds without restrictions on the support of φ, then the space is an $\mathrm{RCD}(K,n)$ space and thus, since n is assumed to be the essential dimension, the space is weakly non-collapsed).

Theorem 2.5.7. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n *and let* $U\subseteq\mathsf{X}$ *be open. Then the following two conditions are equivalent:*

1. *For any* $f\in\mathrm{Test}F(\mathsf{X},\mathsf{d},\mathfrak{m})$ *and any* $\varphi\in D(\Delta)$ *non-negative with* $\mathrm{supp}(\varphi)\subseteq U$ *and* $\Delta\varphi\in L^\infty(\mathsf{X},\mathfrak{m})$ *we have*

$$\frac{1}{2}\int_U \Delta\varphi\,|\nabla f|^2\mathrm{d}\mathfrak{m}\geq\int_U\varphi\left(\frac{(\Delta f)^2}{n}+\langle\nabla\Delta f,\nabla f\rangle+K|\nabla f|^2\right)\mathrm{d}\mathfrak{m}. \tag{2.5.2}$$

2. *For any* $f\in D(\Delta)$ *we have*

$$\Delta f=\mathrm{tr}(\mathrm{Hess}f)\qquad \mathfrak{m}\text{-a.e. in } U. \tag{2.5.3}$$

Proof. It is easy to see the implication from item 2 to item 1 is trivial because we know

$$|\mathrm{tr}(\mathrm{Hess}f)|=|\langle\mathrm{Hess}f,g\rangle_{\mathsf{HS}}|\leq|\mathrm{Hess}f|_{\mathsf{HS}}\,|g|_{\mathsf{HS}}=|\mathrm{Hess}f|_{\mathsf{HS}}\cdot\sqrt{n}.$$

Thus item 2 gives $|\mathrm{Hess}f|_{\mathsf{HS}}^2\geq(\Delta f)^2/n$, and therefore item 1 follows directly from (2.2.3).

For the reverse implication we closely follow the proof of [81, Proposition 3.2] keeping in mind (2.5.2) and the existence, for any $A\subseteq U$ with A compact and U open, of a test function identically 1 on A and with support in U (see e.g. Theorem 2.6.3, see also

[82] or [83, Lemma 6.2.15]). In this way we easily obtain that (2.5.3) holds for any $f \in \mathrm{TestF}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. Then by the density of $\mathrm{Test}F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ in $D(\Delta)$ (see for example [45, Lemma 2.2]) (2.5.3) holds for $f \in D(\Delta)$. □

2.6 Heat Flow and Heat Kernel

2.6.1 Heat Flow

We concern about the L^2 approach to the heat flow. Fist let $(H, \| \cdot \|_H)$ be a Hilbert space. For any absolutely continuous curve $x(t)$, $\|x'(t)\|_H = |x|'(t)$ holds for $\mathcal{L}^1$ a.e. t, that is the norm of the derivative is the a.e. equal to the metric derivative. The Komura-Brezis theory asserts that for a convex and lower semicontinuous functional $F : H \to [-\infty, +\infty]$, and any initial value $x_0 \in \overline{\{F < +\infty\}}$, there exists an absolutely continuous curve $x(t)$ such that

$$\begin{cases} x'(t) & \in \partial F(x(t)), \\ \lim_{t\to 0} \|x(t) - x_0\|_H & = 0, \end{cases} \tag{2.6.1}$$

where $\partial F(x)$ is the sub-differential of F at x, defined as

$$\partial F(x) = \{y \in H : \forall z \in H F(z) \geq F(x) + \langle y, z - x\rangle\}.$$

Such $x(t)$ is called a gradient flow of F. The classical heat flow theory in Hilbert space holds for $\mathrm{RCD}(K, N)$ spaces thanks to the infinitesimal Hillbertianity condition and the fact that Ch is convex and lower semicontinuous. First let us recall the *heat flow* as the L^2-gradient flow of Cheeger energy Ch,

$$\mathrm{h}_t : L^2(\mathsf{X}, \mathfrak{m}) \to L^2(\mathsf{X}, \mathfrak{m}).$$

This family of maps is characterized by the properties: $\mathrm{h}_t f \to f$ in $L^2(\mathsf{X}, \mathfrak{m})$ as $t \to 0^+$, $\mathrm{h}_t f \in D(\Delta)$ for any $f \in L^2$, $t > 0$ and for any $t > 0$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{h}_t f = \Delta \mathrm{h}_t f \qquad \text{in } L^2(\mathsf{X}, \mathfrak{m}). \tag{2.6.2}$$

It is notable that historically, before the notion of infinitesimal Hilbertianity was introduced to the study of $\mathrm{RCD}(K, N)$ spaces, one of the definition of $\mathrm{RCD}(K, \infty)$ condition was $\mathrm{CD}(K, \infty)$ condition along with the fact that the heat flow is linear. We know have the equivalence:

Theorem 2.6.1. *A complete separable metric space is infinitesimally Hilbertian if and only if its heat flow associated to its Cheeger energy is linear.*

It will be useful to keep in mind the following a-priori estimates ([83, Remark 5.2.11]):

$$\||\mathrm{Dh}_t f|\|_{L^2} \le \frac{\|f\|_{L^2}}{\sqrt{2t}} \qquad \|\Delta \mathrm{h}_t f\|_{L^2} \le \frac{\|f\|_{L^2}}{2t} \qquad \forall f \in L^2(\mathsf{X}, \mathfrak{m}), \forall t > 0 \tag{2.6.3}$$

as well as the fact that

$$t \mapsto \|\mathrm{h}_t f\|_{L^2} \quad \text{is non-increasing for every } f \in L^2(\mathsf{X}, \mathfrak{m}). \tag{2.6.4}$$

We provide a proof of the a-priori estimates above.

Proof. We start with the computation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{h}_t f\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}} \langle \mathrm{h}_t f, \mathrm{h}_t f\rangle \mathrm{d}\mathfrak{m} = 2\int_{\mathsf{X}} \langle \Delta \mathrm{h}_t f, \mathrm{h}_t f\rangle \mathrm{d}\mathfrak{m} = -2\int_{\mathsf{X}} |\nabla \mathrm{h}_t f|^2 \mathrm{d}\mathfrak{m} \le 0,$$

which shows the L^2-norm non-increasing property, then a completely same computation shows that $t \mapsto ||\mathrm{Dh}_t f|||_{L^2}$ is also non-increasing for any $f \in L^2(\mathsf{X}, \mathfrak{m})$ and $t > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|||\mathrm{Dh}_t f|||_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}} \langle \nabla \mathrm{h}_t f, \nabla \mathrm{h}_t f\rangle \mathrm{d}\mathfrak{m} = -2\int_{\mathsf{X}} |\Delta \mathrm{h}_t f|^2 \mathrm{d}\mathfrak{m} \leq 0$$

We can use this L^2-norm non-increasing property to see that

$$\begin{aligned} t\|\mathrm{Dh}_t f\|_{L^2}^2 &\leq \int_0^t |||\mathrm{Dh}_s f|||_{L^2}^2 \mathrm{d}s = -\int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|\mathrm{h}_s f\|_{L^2}\mathrm{d}s \\ &= \frac{1}{2}\left(\|f\|_{L^2}^2 - \|\mathrm{h}_t f\|_{L^2}^2\right) \leq \frac{1}{2}\|f\|_{L^2}^2. \end{aligned}$$

Finally, we use the same trick to show that $t \mapsto \|\Delta \mathrm{h}_t f\|_{L^2}$ is non-increasing, we omit the similar computation here. It follows that

$$\begin{aligned} \frac{t^2}{2}\|\Delta \mathrm{h}_t f\|_{L^2}^2 &\leq \int_0^t s\|\Delta \mathrm{h}_s f\|_{L^2}^2 \mathrm{d}s = -\int_0^t \frac{1}{2}s\frac{\mathrm{d}}{\mathrm{d}s}\|\mathrm{Dh}_s f\|_{L^2}\mathrm{d}s \\ &= \frac{1}{2}\left(\int_0^t |||\mathrm{Dh}_s f|||_{L^2}^2 \mathrm{d}s - t\|\mathrm{Dh}_t f\|_{L^2}^2\right) \leq \frac{1}{4}t\|f\|_{L^2}^2. \end{aligned}$$

In the last equality we used the estimates of $|||\mathrm{Dh}_t f|||_{L^2}$ we just derived. □

Then the 1-*Bakry-Émery* estimate proved in [58, Corollary 4.3] is stated as for any $f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$,

$$|\mathrm{Dh}_t f|(x) \leq e^{-Kt}\mathrm{h}_t|\mathrm{D}f|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathsf{X}, \tag{2.6.5}$$

which in particular implies

$$\mathrm{h}_t f \to f \qquad \text{in } H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}).$$

It is also worth pointing out that the heat flow h_t also acts on $L^p(\mathsf{X}, \mathfrak{m})$ for any $p \in [1, \infty]$ with

$$\|\mathrm{h}_t f\|_{L^p} \leq \|f\|_{L^p} \qquad \forall f \in L^p(\mathsf{X}, \mathfrak{m}).$$

We extend the heat flow theory to 1-forms. Consider the following Hodge energy defined in [54, (3.5.16)]:

$$\mathcal{E}_H(\omega) = \frac{1}{2}\int_{\mathsf{X}} |\mathrm{d}\omega|^2 + |\delta\omega|^2 \mathrm{d}\mathfrak{m}, \qquad \forall \omega \in H^{1,2}_H(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})).$$

The gradient flow associated to this energy is the heat flow $\mathrm{h}_{H,t}$ for 1-forms, it fulfills for any $t > 0$ and $\omega \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$:

$$\mathrm{h}_{H,t}\omega \in D(\Delta_{H,1}) \quad \text{and} \quad -\Delta_{H,1}\mathrm{h}_{H,t}\omega = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{h}_{H,t}\omega. \tag{2.6.6}$$

See [54, (3.6.18)], and we have the following commutative relation with the heat flow of functions:

$$\mathrm{h}_{H,t}\mathrm{d}f = \mathrm{d}\mathrm{h}_t f, \qquad \forall t > 0, \forall f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}).$$

By duality between the exterior differential d and its adjoint δ, the commutative relation also holds for δ, that is

$$\mathrm{h}_t\delta\omega = \delta\mathrm{h}_{H,t}\omega \qquad \forall t > 0, \forall \omega \in D(\delta).$$

The heat flow h_t serves as an important means of regularization, as it takes L^2 functions to $D(\Delta)$ by definition. Actually h_t improves the regularity even better, we have that

$$\mathrm{h}_t f \in \mathrm{Test}F(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \qquad \forall f \in L^2(\mathsf{X}, \mathfrak{m}) \cap L^\infty(\mathsf{X}, \mathfrak{m}), \forall t > 0. \tag{2.6.7}$$

With this regularization Honda [45, Lemma 2.2] shows that

Lemma 2.6.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $f \in D(\Delta)$. *There exists a sequence* $f \in \mathrm{Test}F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *such that* $\|f_i - f\|_{H^{1,2}} + \|\Delta f_i - \Delta_f\|_{L^2} \to 0$ *as* $i \to \infty$.

Another important use of the regularization property of heat flow is to construct good cut-off functions (see [71, Lemma 3.1]), which is widely used, for instance in the proof of

[43, Proposition 5.12].

Theorem 2.6.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $x \in \mathsf{X}$ *and* $0 < r < R < \infty$*. Then there exists* $\psi = \psi_{x,r,R} \in \mathrm{Test}F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *such that the following four properties hold:*

1. $0 \le \psi \le 1$*;*
2. $r|\nabla\psi| + r^2|\Delta\psi| \le C(K, N, R)$*;*
3. $\psi = 1$ *on* $B_r(x)$*;*
4. $\operatorname{supp}\psi \subseteq B_R(x)$.

The construction is done by regularizing a compactly supported Lipschitz function by heat flow, then compose by a C^2 cut-off function (we can certainly choose a C^∞ function as well). The estimates of $\nabla\psi$ and $\Delta\psi$ rely on the following dimensional Bakry-Ledoux gradient estimates under $\mathrm{RCD}(K, N)$ condition proved in [23, Theorem 4.2]

$$|\nabla \mathrm{h}_t f|^2 + \frac{4Kt}{N(e^{2Kt} - 1)}|\Delta \mathrm{h}_t f|^2 \le e^{-2Kt}\mathrm{h}_t(|\nabla f|^2) \qquad \mathfrak{m}\text{-a.e.}$$

Finally we point out that the heat flow $\mathrm{h}_{H,t}$ also has the similar regularization properties, we highlight that

$$\mathrm{h}_{H,t}\omega \in \mathrm{Test}T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}), \qquad \forall \omega \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})). \tag{2.6.8}$$

In particular, it implies that $\mathrm{h}_{H,t}\omega \in D(\Delta_{H,1})$ and $\Delta_{H,1}(\mathrm{h}_{H,t}\omega) \in D(\delta)$, which in turn implies that

Proposition 2.6.4. $\{\omega \in D(\Delta_{H,1}) : \omega$ *has compact support*$, \Delta_{H,1}\omega \in D(\delta)\}$ *is dense in* $L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$

Proof. Let $\omega \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$, ψ_n be good cut-off functions constructed in Theorem 2.6.3 with $\operatorname{supp}\psi_n \nearrow \mathsf{X}$, $n \in \mathbb{N}$. Since $\psi_n \in \mathrm{Test}F(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, we see that $\psi_n \mathrm{h}_{H,t}\omega \in$

$D(\Delta_{H,1})$. Note that, $\mathrm{h}_{H,t}\omega$ is a test 1-form, so $\psi_n \mathrm{h}_{H,t}\omega$ is also a test 1-form, it follows that $\Delta_{H,1}(\psi_n \mathrm{h}_{H,t}\omega) \in D(\delta)$. Now by an diagonal argument we can choose $t_n \searrow 0$ such that $\varphi_n \mathrm{h}_{H,t_n}$ converges in $L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ to ω.

□

2.6.2 Heat Kernel

After establishing the L^2 theory of the heat flow, it is natural to study the existence and properties of its integral kernel. The theory of heat kernel in a broad class of metric measure spaces (PI spaces) is well understood through Sturm's work [84, Proposition 2.3] and [85, Proposition 3.1, Corollary 3.3], some of the ideas originated from [6]. Sturm showed the existence of locally Höder continuous integral kernel of h_t if the metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is locally volume doubling, supports a local Poincaré inequality and the so-called intrinsic distance

$$\mathsf{d}_{\mathsf{Ch}}(x,y) := \inf\{|f(x)-f(y)| : f \in H^{1,2}(X) \cap C_b(X)\} \tag{2.6.9}$$

induces the same topology as d. In $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, d_{Ch} in fact coincides with the original distance d on X, as a consequence of Sobolev-to-Lipschitz property. Also, thanks to (2.4.5) and (2.4.2), Sturm's results apply, so there exists a unique (locally Hölder) continuous function $p : \mathsf{X}\times\mathsf{X}\times(0,\infty) \to (0,\infty)$, called the *heat kernel* of $(\mathsf{X},\mathsf{d},\mathfrak{m})$, such that the following holds;

$$\mathrm{h}_t f(x) = \int_{\mathsf{X}} p(x,y,t) f(y) \mathrm{d}\mathfrak{m}(y) \qquad \forall f \in L^2(\mathsf{X},\mathfrak{m}), \forall x \in \mathsf{X}. \tag{2.6.10}$$

Let us denote by $p_{y,t}(x) = p(x,y,t)$ when we consider p as a function on X for fixed $y \in \mathsf{X}$ and $t > 0$.

Let us recall the Gaussian estimates for the heat kernel p proved in [86]: For any $\epsilon \in (0,1]$ there exists a positive constant $C := C(K,N,\epsilon)$ depending only on K, N and ϵ such that for any $x, y \in \mathsf{X}$ and any $0 < t < 1$,

$$\frac{C}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}(x,y)^2}{(4-\epsilon)t} - Ct\right) \le p(x,y,t) \le \frac{C}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + Ct\right), \tag{2.6.11}$$

and for every $y \in \mathsf{X}$ and $t > 0$ we have

$$|\mathrm{d}p_{y,t}|(x) \le \frac{C}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + Ct\right) \qquad \mathfrak{m}\text{-a.e. } x \in \mathsf{X}. \tag{2.6.12}$$

Notice that (2.6.11) and Lemma 2.6.5 ensure that $p(\cdot, y, t) \in L^2(\mathsf{X}, \mathfrak{m})$ for every $y \in \mathsf{X}$, $t > 0$, therefore from (2.6.10) we deduce the Chapman-Kolmogorov equation:

$$p(x,y,t+s) = \mathrm{h}_t p(\,\cdot\,, y, s)(x) = \int_{\mathsf{X}} p(x,z,t)p(z,y,s)\,\mathrm{d}\mathfrak{m}(z) \quad \forall t,s>0, \forall x,y \in \mathsf{X}. \tag{2.6.13}$$

Also, from (2.6.2), [84, Corollary 2.7] and (2.6.11) we deduce the estimate

$$|\Delta p(\cdot, y, t)|(x) \le \frac{C}{t\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{\mathsf{d}(x,y)^2}{(4+\epsilon)t} + Ct\right) \qquad \mathfrak{m}\text{-a.e. } x \in \mathsf{X}, \tag{2.6.14}$$

for every $y \in \mathsf{X}$, $t > 0$. We are going to use these estimates only specialized to the case $\epsilon = 1$. Notice that the above discussion and estimates easily imply that

$$p_{y,t} \in \mathrm{TestF}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$$

for every $y \in \mathsf{X}$, and $t > 0$. We shall frequently use this fact. Also, for future reference we prove a technical lemma related to the Gaussian estimates.

Lemma 2.6.5. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Then for any* $t \in (0,1]$, *any* $\alpha \in \mathbb{R}$, *any* $\beta \in (0,\infty)$ *and any* $x \in \mathsf{X}$ *we have*

$$\int_{\mathsf{X}} \mathfrak{m}(B_{\sqrt{t}}(y))^{\alpha} \exp\left(-\frac{\beta \mathsf{d}^2(x,y)}{t}\right) \mathrm{d}\mathfrak{m}(y) \le C(K,N,\alpha,\beta)\mathfrak{m}(B_{\sqrt{t}}(x))^{\alpha+1}. \tag{2.6.15}$$

Proof. Considering a rescaling $\sqrt{\beta/t}\cdot\mathsf{d}$ with (2.4.2), it is enough to prove (2.6.15) assuming $\beta = t = 1$. Then by (2.4.3) and (2.4.4)

$$
\begin{aligned}
&\int_{\mathsf{X}} \mathfrak{m}(B_1(y))^{\alpha} \exp\left(-\mathsf{d}^2(x,y)\right) \mathrm{d}\mathfrak{m}(y) \\
&= \sum_{j=-\infty}^{\infty} \int_{B_{2^{j+1}}(x)\setminus B_{2^j}(x)} \mathfrak{m}(B_1(y))^{\alpha} \exp\left(-\mathsf{d}^2(x,y)\right) \mathrm{d}\mathfrak{m}(y) \\
&\le C(K,N)\mathfrak{m}(B_1(x))^{\alpha} \sum_{j=-\infty}^{\infty} \int_{B_{2^{j+1}}(x)\setminus B_{2^j}(x)} \exp\left(C(\alpha,K,N)2^{j+1} - 2^{2j}\right) \mathrm{d}\mathfrak{m}(y) \\
&= C(K,N)\mathfrak{m}(B_1(x))^{\alpha} \sum_{j=-\infty}^{\infty} \mathfrak{m}(B_{2^{j+1}}(x)\setminus B_{2^j}(x)) \exp\left(C(\alpha,K,N)2^{j+1} - 2^{2j}\right) \\
&\le C(K,N)\mathfrak{m}(B_1(x))^{\alpha} \sum_{j=-\infty}^{\infty} \mathfrak{m}(B_1(x)) \cdot \exp\left(C(K,N)2^{j}\right) \cdot \exp\left(C(\alpha,K,N)2^{j+1} - 2^{2j}\right) \\
&\le C(\alpha,K,N)\mathfrak{m}(B_1(x))^{\alpha+1}. \qquad \square
\end{aligned}
$$

Notice that (2.6.13) and the estimate (2.6.11) together with (2.6.3), (2.4.2) and Lemma 2.6.5 give

$$
\|p_{y,t}\|_{H^{1,2}} + \|\Delta p_{y,t}\|_{H^{1,2}} \le C(K,N,t)\mathfrak{m}(B_{\sqrt{t}}(y))^{-\frac{1}{2}}. \tag{2.6.16}
$$

We also notice that the identity $\partial_t p(x,y,t) = \partial_t p_{y,t}(x) = \Delta p_{y,t}(x) = \Delta_x p(x,y,t)$ valid for any $t > 0$, $y \in \mathsf{X}$ and a.e. x together with the symmetry in x, y of the heat kernel - and thus of the LHS - gives

$$
\Delta_x p(x,y,t) = \Delta_y p(x,y,t) \qquad (\mathfrak{m}\times\mathfrak{m})\text{-a.e. } (x,y),\ \forall t > 0. \tag{2.6.17}
$$

Remark 2.6.6. We point out that the continuity of the heat kernel and the estimates (2.6.11) ensure that for any $t > 0$ the map $y \mapsto p_{y,t} \in L^2(\mathsf{X},\mathfrak{m})$ is continuous. Thus by the first claim in Proposition 3.2.4 we deduce that $y \mapsto \mathrm{d}p_{y,t} \in L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ is strongly Borel. Similarly for $y \mapsto \Delta p_{y,t}$ and $y \mapsto \mathrm{d}\Delta p_{y,t}$.

Asymptotic Behavior as $t \to 0$

We present here a pointwise convergence result and a $H^{1,2}$ convergence result. The former is needed to prove the main theorem in Chapter 5 and the latter is built up on the former one and justifies the convergence of $t\mathfrak{m}(B_{\sqrt{t}}(\,\cdot\,))g_t$ to $c_n g$. See section 3.4.

Assume that $\mathrm{RCD}(K,N)$ spaces $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, p_n)$ pmGH converges to $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, p)$. We recall that in [44, Theorem 3.3 Corollary 3.6], it is proved that

Proposition 2.6.7. *Let p_n be the heat kernel of $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$, p be the heat kernel of $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. It holds that*

$$\lim_{n\to\infty} p(x_n, y_n, t_n) = p(x, y, t), \tag{2.6.18}$$

for $(x_n, y_n, t_n) \in \mathsf{X} \times \mathsf{X} \times (0,\infty) \mapsto (x, y, t) \in \mathsf{X} \times \mathsf{X} \times (0,\infty)$. In particular, for an $\mathrm{RCD}(K,N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, we have by taking tangent cones that

$$\lim_{t\to 0} \mathfrak{m}(B_{\sqrt{t}}(x))p(x, x, t) = \frac{\omega_k}{(4\pi)^{k/2}} \tag{2.6.19}$$

for any $x \in \mathcal{R}_k$.

We are particularly interested in the case where $x \in \mathcal{R}_n$, n being the essential dimension.

Then we recall that in [43, Theorem 2.19] it is proved that

Proposition 2.6.8. *$p_n(\cdot, y_n, t_n) \in H^{1,2}(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)$ $H^{1,2}$ strongly converges to $p(\cdot, y, t) \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, for $(y_n, t_n) \in \mathsf{X}_n \times (0,\infty) \mapsto (y, t) \in \mathsf{X} \times (0,\infty)$.*

The proof can be done directly by applying theorem 2.3.6 and the consequence of Gaussian estimates (2.6.16).

Heat Kernel on Compact $\mathrm{RCD}(K,N)$ Spaces

Finally, to complete the description of the heat kernel on RCD spaces, let us restrict the consideration to compact $\mathrm{RCD}(K,N)$ spaces. In this case the Laplacian has discrete spectrum, and we can express the heat kernel in another way using eigenfunctions of the Laplacian. Let us recall the spectral analysis for Laplacian on compact $\mathrm{RCD}(K,N)$ spaces. A standard argument as in [87] shows that the resolvent operator $R_\alpha := (\alpha \mathrm{Id} - \Delta)^{-1}$: $L^2(\mathsf{X},\mathfrak{m}) \to H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a well-defined, injective and bounded operator for any $\alpha > 0$, whose image $R_\alpha(L^2(\mathsf{X},\mathfrak{m}))$ is dense in $H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ and is in fact equal to $D(\Delta)$, independent of α. Then by the (metric version of) Rellich-Kondrachov theorem, all R_α's are compact and share discrete spectrum that converges to 0. This implies that $-\Delta$ has discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$. In connection with this see also Appendix B for a treatment of the sepctrum of Hodge Laplacian, in particular, a Rellich-Kondrachov theorem, i.e. the compact inclusion from Sobolev space $H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ of 1-forms, to $L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ is proved.

Let $\varphi_i \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$ be the corresponding eigenfunction of λ_i, that is φ_i satisfies $\Delta\varphi_i = -\lambda\varphi_i$. Then we have the following expansion of the heat kernel $p(x,y,t)$:

$$p(x,y,t) = \sum_{i\geq 0} e^{-\lambda_i t}\varphi(x)\varphi(y), \qquad \text{in } C(\mathsf{X}\times\mathsf{X}\times(0,\infty)). \tag{2.6.20}$$

And also

$$p(\cdot,y,t) = \sum_{i\geq 0} e^{-\lambda_i t}\varphi_i(y)\,\varphi_i \qquad \text{in } H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$$

for $y \in \mathsf{X}$ and $t > 0$.

We point out that bounds on eigenvalues and eigenfunctions can be obtained from Gaussian estimates and Bishop-Gromov inequality,

$$\|\varphi_i\|_{L^\infty} \leq C\lambda_i^{N/4}, \qquad \|\nabla\varphi_i\|_{L^\infty} \leq C\lambda_i^{(N+2)/4}, \qquad \lambda_i \geq Ci^{2/N}, \tag{2.6.21}$$

see [43, Appendix]. See also another local approach by De Girogi-Nash-Moser iteration in [88, Section 3.2], and also a heat flow approach by Jiang [89].

CHAPTER 3

SMOOTHING OF METRIC VIA HEAT FLOW

3.1 Overview

The purpose of this chapter is to prove the formula (3.3.9), originally discovered by Honda [45] for compact $\mathrm{RCD}(K,N)$ spaces, in full generality. This whole chapter is mainly taken from the joint work with Brena-Gigli-Honda [41]. In Honda's proof, the eigenfunction expansion of the heat kernel is heavily used, here we replace the manipulation of eigenfunctions by making use of suitable Gaussian decay estimates. The crucial difference between the original proof of (3.3.9) in [45] is the Lemma 3.3.4. First we introduce some version of Hille's theorem discussed in detail in [83]. It serves two purposes. On the one hand, for the later use we give a treatment of (weakly) local differential operators, on the other hand, we will frequently use integrals that take values in Banach spaces, the so-called Bochner integration, so we also present here the elements of Bochner integration tailored for our situation. We will then be able to derive enough regularity of $p(x,x,2t)$ that makes its appearance in (3.3.9), basically we will show that $\mathrm{d}\Delta p(x,x,2t)$ is well-defined as a L^2_{loc} 1-form, this is the content of Lemma 3.3.3. Then we present the crucial result, Lemma 3.3.4, which states that

$$\int_{\mathsf{X}} \Delta p_{y,t} \mathrm{d}p_{y,t}\,\mathrm{d}\mathfrak{m}(y) = \int_{\mathsf{X}} p_{y,t}\mathrm{d}\Delta p_{y,t}\,\mathrm{d}\mathfrak{m}(y).$$

That is, we can move the Laplacian, it is worth noticing that it is not a direct consequence of the fact that Δ is self-adjoint, because of lacking of regularity. We need to proceed by a heat flow regularization. Now all the ingredients are already, we can carry out a formal computation as in Theorem 3.3.5 to derive the desired formula for $\nabla^* g_t$.

Finally, we slightly strengthen the short time asymptotics proved in [43] to the non-compact setting. Mainly the strong L^2_{loc} convergence of $c_n t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t$ to g as $t \to 0$. It follows from the L^∞ bounds of g_t and the L^2_{loc} convergence that the L^p_{loc} convergence for $p \in [1, \infty)$ also holds. The key of the proof of strong convergence is the norm convergence:

$$\lim_{t \to 0^+} \int_{\mathsf{X}} \varphi |t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}} \mathrm{d}\mathfrak{m} = c_n^2 \int_{\mathsf{X}} \varphi |g|^2_{\mathrm{HS}} \mathrm{d}\mathfrak{m} = c_n^2 n \int_{\mathsf{X}} \varphi \mathrm{d}\mathfrak{m}$$

For any $\varphi \in \mathrm{Lip}_{bs}$. This reduces to the fact

$$F(z,t) := \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} |t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}} \mathrm{d}\mathfrak{m} \to c_n^2 n$$

proved in [43]. Finally we prove a technical fact, Proposition 3.4.4, which will be used in the proof of the integrate by parts formula w.r.t. to Hausdorff measure, see Theorem 4.2.1.

3.2 Local Hille's Theorem

This section is taken from [41, Section 3.1], adding some explanations why operators δ and $\Delta_{H,1}$ are local operators. First we collect some basic results about local (differentiation) operators: the main result we have in mind is the version of Hille's theorem stated in Lemma 3.2.3 below. We shall apply the notions presented here to the operators $\mathrm{d}, \Delta, \nabla^*$, but in order to highlight the similarities among the various approaches we shall give a rather abstract presentation.

Thus let us fix a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and two L^0-normed modules $\mathscr{M}, \mathscr{N}$. For $p \in [1, \infty]$ we shall denote by $L^p(\mathscr{M})$ (resp. $L^p_{loc}(\mathscr{M})$) the collection of those $v \in \mathscr{M}$ with $|v| \in L^p(\mathsf{X}, \mathfrak{m})$ (resp. $|v| \in L^p_{loc}(\mathsf{X}, \mathfrak{m})$). Similarly for $\mathscr{N}$.

Definition 3.2.1 (Weakly local operators)**.** Let $p \in [1, \infty]$ and $L : D(L) \subseteq L^p(\mathscr{M}) \to L^p(\mathscr{N})$ be a linear operator. We say that L is *weakly local* provided

$$L(v) = L(w) \qquad \mathfrak{m}\text{-a.e. on the essential interior of } \{v = w\} \text{ for any } v, w \in D(L).$$

In other words, L is weakly local provided for any $v, w \in D(L)$ and $U \subseteq \mathsf{X}$ open such that $v = w$ $\mathfrak{m}$-a.e. on U, we have $L(v) = L(w)$ $\mathfrak{m}$-a.e. on U.

Weakly local operators can naturally be extended as follows (variants of this definition are possible, but for us the following is sufficient):

Definition 3.2.2 (Extension of weakly local operators)**.** Let $p \in [1, \infty]$ and $L : D(L) \subseteq L^p(\mathscr{M}) \to L^p(\mathscr{N})$ be a weakly local operator. We then define $D_{loc}(L) \subseteq L^p_{loc}(\mathscr{M})$ as the collection of those v's such that for every $U \subseteq \mathsf{X}$ bounded and open there is $v_U \in D(L) \subseteq L^p(\mathscr{M})$ with $v_U = v$ $\mathfrak{m}$-a.e. on U.

For $v \in D_{loc}(L)$ we define $L(v) \in L^p_{loc}(\mathscr{N})$ via

$$L(v) = L(v_U) \qquad \mathfrak{m}\text{-a.e. } on\ U, \forall U \subseteq \mathsf{X} \text{ open and bounded,}$$

where v_U is as above.

It is clear from the definition that $L : D_{loc}(L) \subseteq L^p_{loc}(\mathscr{M}) \to L^p_{loc}(\mathscr{N})$ is well-posed and that the resulting operator is linear. We are interested in a version of Hille's theorem for this kind of operators and to this aim we need first to introduce the notion of integrable function with values in $L^p(\mathscr{M})$.

For the standard notion of Bochner integration of Banach valued maps we refer to [90]. Given a metric measure space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mu)$ (the topology here is not really relevant, but in our applications we shall mostly have $\mathsf{Y} = \mathsf{X}$) we shall denote by $L^1(\mathsf{Y}, \mu; L^p_{loc}(\mathscr{M}))$ the collection of (equivalence classes up to μ-a.e. equality of) maps $y \mapsto v_y \in L^p_{loc}(\mathscr{M})$ such that for any $A \subseteq \mathsf{X}$ Borel and bounded the map $y \mapsto \chi_A v_y$ is in $L^1(\mathsf{Y}, \mu; L^p(\mathscr{M}))$ (here we are endowing $L^p(\mathscr{M})$ with its natural Banach structure).

With these definitions, the following result is rather trivial (but nevertheless useful):

Lemma 3.2.3 (Local Hille's theorem - abstract version)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$*,* $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mu)$ *be metric measure spaces,* $\mathscr{M}, \mathscr{N}$ *two* L^0*-normed modules,* $p \in [1, \infty]$ *and* $L : D(L) \subseteq L^p(\mathscr{M}) \to$

$L^p(\mathscr{N})$ a weakly local and closed linear operator. Also, let $y \mapsto v_y \in L^p_{loc}(\mathscr{M})$ be in $L^1(\mathsf{Y}, \mu; L^p_{loc}(\mathscr{M}))$. Assume that

i) $v_y \in D_{loc}(L)$ *for μ-a.e. y,*

ii) *L has the following 'stability under cut-off' property: for every $V \subseteq U \subseteq \mathsf{X}$ bounded and open with $\mathsf{d}(V, \mathsf{X} \setminus U) > 0$ there is a linear map $T : L^p_{loc}(\mathscr{M}) \to L^p(\mathscr{M})$ such that:*

$$
\begin{aligned}
T(v) &= v \qquad \mathfrak{m}\text{-a.e. on } V, \\
T(v) &= T(\chi_U v) \qquad \mathfrak{m}\text{-a.e.} \\
\|T(v)\|_{L^p(\mathscr{M})} &\le C \|\chi_U v\|_{L^p(\mathscr{M})}
\end{aligned}
\tag{3.2.1}
$$

for every $v \in L^p_{loc}(\mathscr{M})$ and some $C > 0$ independent on v,

iii) *for any V, U as above and T given by item (ii) we have $T(v_y) \in D(L)$ for μ-a.e. y and the map $y \mapsto L(T(v_y))$ is in $L^1(\mathsf{Y}, \mu; L^p_{loc}(\mathscr{N}))$.*

Then $\int_{\mathsf{Y}} v_y \, \mathrm{d}\mathfrak{m}(y) \in D_{loc}(L)$, the map $y \mapsto L(v_y)$ is in $L^1(\mathsf{Y}, \mu; L^p_{loc}(\mathscr{M}))$ and

$$
L\Big(\int_{\mathsf{Y}} v_y \, \mathrm{d}\mathfrak{m}(y) \Big) = \int_{\mathsf{Y}} L(v_y) \, \mathrm{d}\mathfrak{m}(y).
$$

Proof. Fix $V \subseteq \mathsf{X}$ open bounded and then let $U \supseteq V$ open bounded be with $\mathsf{d}(V, \mathsf{X} \setminus U) > 0$. Let $T : L^p_{loc}(\mathscr{M}) \to L^p(\mathscr{M})$ be given by item (ii). By the assumption (i) we know that $y \mapsto \chi_U v_y \in L^p(\mathscr{M})$ is in $L^1(\mathsf{X}, \mathfrak{m}; L^p(\mathscr{M}))$ and the third in (3.2.1) gives that T is continuous as map from $L^p(\mathscr{M})$ to itself. It follows that $y \mapsto T(\chi_U v_y) = T(v_y)$ is in $L^1(\mathsf{Y}, \mu; L^p(\mathscr{M}))$. Then assumption (iii) and the classical theorem by Hille ensure that

$$
\int_{\mathsf{Y}} T(v_y) \, \mathrm{d}\mu(y) \in D(L) \quad \text{and} \quad L\Big(\int_{\mathsf{Y}} T(v_y) \, \mathrm{d}\mu(y) \Big) = \int_{\mathsf{Y}} L(T(v_y)) \, \mathrm{d}\mu(y). \tag{3.2.2}
$$

Now notice that the first in (3.2.1) give that $T(v_y) = v_y$ on V for every y, thus the weak locality of L also gives that $L(T(v_y)) = L(v_y)$ on V for every y. It follows that $y \mapsto \chi_V L(v_y)$ is in $L^1(\mathsf{Y},\mu;L^p(\mathscr{N}))$ and that

$$\int_{\mathsf{Y}} \chi_V v_y \,\mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V T(v_y)\,\mathrm{d}\mu(y) \quad \text{and} \quad \int_{\mathsf{Y}} \chi_V L(v_y)\,\mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V L(T(v_y))\,\mathrm{d}\mu(y).$$

Thus using again the weak locality of L it follows that

$$\begin{aligned} \chi_V L\Big(\int_{\mathsf{Y}} v_y \,\mathrm{d}\mu(y)\Big) &= \chi_V L\Big(\int_{\mathsf{Y}} T(v_y)\,\mathrm{d}\mu(y)\Big) \\ \text{(by (3.2.2))} \qquad &= \chi_V \int_{\mathsf{Y}} L(T(v_y))\,\mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V L(T(v_y))\,\mathrm{d}\mu(y) = \int_{\mathsf{Y}} \chi_V L(v_y)\,\mathrm{d}\mu(y). \end{aligned}$$

Since V was arbitrary, this is the conclusion. □

We now see how to apply this general statement to the concrete cases of

$$L = \mathrm{d}, \Delta, \nabla^*, \delta, \Delta_{H,1}.$$

The idea is to use, as map T, the multiplication with a Lipschitz cut-off function φ with support in U and identically 1 on V. For the case of the Laplacian this does not really work, as one would need to multiply by a Lipschitz function with bounded Laplacian in order to remain in the domain of the operator. The problem is that on general $\mathrm{RCD}(K,\infty)$ spaces it is not clear whether this sort of cut-off functions exist (but see [82] or [83, Lemma 6.2.15] for the case of proper RCD spaces). This issue is, however, easily dealt with by recalling that the Laplacian is the divergence of the gradient and applying the above theorem twice (this amount at localizing Δf by looking at $\mathrm{div}(\varphi\nabla(\varphi f))$).

Let us start recalling that the differential $\mathrm{d} : H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m}) \subseteq L^2(\mathsf{X},\mathfrak{m}) \to L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ is weakly local (in fact even more, as there is locality on Borel sets and not just on open ones) by [54, Theorem 2.2.3]. The same holds for the divergence operator $\mathrm{div} : D(\mathrm{div}) \subseteq L^2(T(\mathsf{X},\mathsf{d},\mathfrak{m})) \to L^2(\mathsf{X},\mathfrak{m})$. Indeed, for $v, w \in D(\mathrm{div})$ equal on some open set U, we have

$$\int_{\mathsf{X}} \varphi \operatorname{div} v \, \mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \mathrm{d}\varphi(v) \, \mathrm{d}\mathfrak{m} \overset{(*)}{=} -\int_{\mathsf{X}} \mathrm{d}\varphi(w) \, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \varphi \operatorname{div} w \, \mathrm{d}\mathfrak{m}$$

for any $\varphi \in \operatorname{Lip}(\mathsf{X}, \mathsf{d})$ with $\operatorname{supp}(\varphi) \subseteq U$, having used the locality of the differential and the assumption $v = w$ on U in the starred equality $(*)$. This is sufficient to prove the claim. Also since $\delta = -\operatorname{div}$ when acting on 1-forms, we get the locality of δ. Similarly, starting from the locality of the covariant derivative (see [54, Proposition 3.4.9]) it follows the weak locality of ∇^*. Finally, the weak locality of the Laplacian follows from that of the differential and of the divergence, and the locality of Hodge Laplacian follows from that of d and of δ.

Below for the domain $D_{loc}(\mathrm{d}) \subseteq L^2_{loc}(\mathsf{X}, \mathfrak{m})$ we shall use the more standard notation $H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. We then have the following.

Proposition 3.2.4 (Local Hille's theorem - concrete version)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space and* $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}}, \mu)$ *a metric measure space. Let* $(f_y) \in L^1(\mathsf{Y}, \mu; L^2_{loc}(\mathsf{X}, \mathfrak{m}))$ *(resp.* $(f_y) \in L^1(\mathsf{Y}, \mu; L^2_{loc}(\mathsf{X}, \mathfrak{m}))$*, resp.* $(A_y) \in L^1(\mathsf{Y}, \mu; L^2_{loc}(T^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$*) be with* $f_y \in H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *(resp.* $f_y \in D_{loc}(\Delta)$*, resp.* $A_y \in D_{loc}(\nabla^*)$*) for* μ*-a.e.* $y \in \mathsf{Y}$*.*

Then for every $U \subseteq \mathsf{X}$ *open bounded we have that* $y \mapsto \chi_U \mathrm{d} f_y$ *(resp.* $y \mapsto \chi_U \Delta f_y$*, resp.* $y \mapsto \chi_U \nabla^* A_y$*) is - the equivalence class up to* μ*-a.e. equality of - a strongly Borel function (i.e. Borel and essentially separably valued).*

Now assume also that for every $U \subseteq \mathsf{X}$ *open bounded we have* $\int_{\mathsf{Y}} \|\chi_U |\mathrm{d} f_y|\|_{L^2} \, \mathrm{d}\mu(y) < \infty$ *(resp.* $\int_{\mathsf{Y}} \|\chi_U \Delta f_y\|_{L^2} \, \mathrm{d}\mu(y) < \infty$*, resp.* $\int_{\mathsf{Y}} \|\chi_U |\nabla^* A_y|\|_{L^2} \, \mathrm{d}\mu(y) < \infty$*).*

Then $\int_{\mathsf{Y}} f_y \, \mathrm{d}\mu(y) \in H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *(resp.* $\int_{\mathsf{Y}} f_y \, \mathrm{d}\mu(y) \in D_{loc}(\Delta)$*, resp.* $\int_{\mathsf{Y}} A_y \, \mathrm{d}\mu(y) \in D_{loc}(\nabla^*)$*) with*

$$\mathrm{d} \int_{\mathsf{Y}} f_y \, \mathrm{d}\mu(y) = \int_{\mathsf{Y}} \mathrm{d} f_y \, \mathrm{d}\mu(y)$$

(resp. $\Delta \int_{\mathsf{Y}} f_y \, \mathrm{d}\mu(y) = \int_{\mathsf{Y}} \Delta f_y \, \mathrm{d}\mu(y)$*, resp.* $\nabla^* \int_{\mathsf{Y}} A_y \, \mathrm{d}\mu(y) = \int_{\mathsf{Y}} \nabla^* A_y \, \mathrm{d}\mu(y)$*).*

Proof. We start with the case of differential. We have already noticed that d is weakly local and we know from [54, Theorem 2.2.9] that it is a closed operator. Let us check that the other assumptions in Lemma 3.2.3 are satisfied. (i) holds by our assumption, thus we pass to (ii). Let U, V as in the statement and let $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be identically 1 on V and with support in U (the hypothesis $\mathsf{d}(V, \mathsf{X} \setminus U) > 0$ grants that such φ exists). We define $T(f) := \varphi f$ and notice that the properties in (3.2.1) are trivial. We pass to (iii), and notice that by the very definition of extension of d from $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ to $H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ it follows that the Leibniz rule holds even in $H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. It is then clear that we have $\varphi f \in H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with $\mathrm{d}(\varphi f) = \varphi \mathrm{d} f + f \mathrm{d}\varphi$. Thus

$$|\mathrm{d}(\varphi f)| \leq \chi_U |\mathrm{d} f| \sup |\varphi| + \chi_U C |f| \in L^2(\mathsf{X}, \mathfrak{m}), \tag{3.2.3}$$

where C denotes the Lipschitz constant of φ. Therefore φf is actually in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$.

With this said, let us verify the first claim. Fix $U \subseteq \mathsf{X}$ open bounded, let $\varphi \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$ be identically 1 on U and notice that replacing f_y with φf_y it is sufficient to prove that if $y \mapsto f_y \in L^2(\mathsf{X}, \mathfrak{m})$ is Borel and $f_y \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ for every $y \in \mathsf{Y}$, then $y \mapsto \mathrm{d} f_y \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is strongly Borel. Since $L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is separable (see [91] and [54, Proposition 2.2.5]), it is enough to check Borel regularity. Also, since $\mathrm{d} : H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \to L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is continuous, it suffices to prove that $y \mapsto f_y \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is Borel. To see this it is sufficient to show that the unit ball in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ belongs to the σ-algebra $\mathcal{A}$ generated by L^2-open sets in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. But this is obvious, because the lower semicontinuity of the $H^{1,2}$-norm w.r.t. L^2-convergence ensures that closed $H^{1,2}$-balls are also L^2-closed, and thus are in $\mathcal{A}$. Since open balls are countable unions of closed balls, the first claim follows.

For the second, we observe that what we just proved, along with our assumption that $\int_{\mathsf{Y}} \| \chi_U |\mathrm{d} f_y| \|_{L^2} \, \mathrm{d}\mu(y) < \infty$ and (3.2.3) ensure that the map $y \mapsto \mathrm{d}(\varphi f_y)$ is actually in $L^1(\mathsf{Y}, \mu; L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$, i.e. (iii) of Lemma 3.2.3 holds and the conclusion follows from such lemma.

The same line of thought gives the conclusion for ∇^*. For the Laplacian we start noticing that for V, U and φ as above we have

$$\int_{\mathsf{X}} |\varphi|^2 |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} 2f\varphi\langle \mathrm{d}f, \mathrm{d}\varphi\rangle + \varphi^2 f\Delta f \,\mathrm{d}\mathfrak{m}$$
$$\leq \int_{\mathsf{X}} \tfrac{1}{2}|\varphi|^2|\mathrm{d}f|^2 + 2|\mathrm{d}\varphi|^2|f|^2 + \tfrac{1}{2}|\varphi|^2(|f|^2 + |\Delta f|^2)\,\mathrm{d}\mathfrak{m},$$

i.e. $\frac{1}{2}\int_{\mathsf{X}} |\varphi|^2|\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m} \leq C\int_U |f|^2 + |\Delta f|^2 \,\mathrm{d}\mathfrak{m}$. This proves that if $f \in D_{loc}(\Delta) \subseteq L^2_{loc}(\mathsf{X}, \mathfrak{m})$ then $f \in H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ as well. Hence what previously proved tells that for $y \mapsto f_y \in L^2_{loc}$ Borel with $f_y \in D_{loc}(\Delta)$ for μ-a.e. y, we have that $y \mapsto \chi_U \mathrm{d}f_y \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ is strongly Borel for any $U \subseteq \mathsf{X}$ open bounded. Now we want to prove that the same assumptions ensure that $y \mapsto \chi_U \Delta f_y \in L^2(\mathsf{X}, \mathfrak{m})$ is Borel as well. Since the σ-algebra generated by the strong topology coincides with that generated by the weak topology (because the closed unit ball can be realized as countable intersection of weakly-closed halfspaces, so that closed balls are weakly Borel and thus the same holds for open balls since they are countable union of closed balls), by approximation to get the desired Borel regularity it is sufficient to prove that $y \mapsto \int_{\mathsf{X}} \psi\xi\Delta f_y \,\mathrm{d}\mathfrak{m}$ is Borel for any $\psi \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$ and ξ varying in a countable dense subset of $L^2(\mathsf{X}, \mathfrak{m})$. We pick $\xi \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and notice that

$$\int_{\mathsf{X}} \psi\xi\Delta f_y \,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \langle \nabla(\psi\xi), \nabla f_y\rangle \,\mathrm{d}\mathfrak{m} = -\int_U \langle \nabla(\psi\xi), \nabla f_y\rangle \,\mathrm{d}\mathfrak{m}$$

for any $U \subseteq \mathsf{X}$ open bounded and containing the support of ψ. By what already proved we see that the RHS is a Borel function of y, hence the desired Borel regularity follows.

With this said, the conclusion for the Laplacian follows by first applying the result to the differential and then to the divergence (the study of the divergence operator closely follows that of ∇^* that in turn, as said, is largely based on that of d). □

Remark 3.2.5. The above version of Hille's theorem is compatible with the more general one recently discussed in [92, Section 3.3]. However, being the presentation here substantially simpler we preferred giving a direct proof, rather than linking the terminology to that in [92]. ■

3.3 Smoothing metric g_t and computation of $\nabla^* g_t$

This section is taken from [41, Section 3.3]. For an n-dimensional weighted complete Riemannian manifold $(M, g, e^{-V}\mathrm{dVol}_g)$ satisfying $\mathrm{Ric}_N \geq Kg$ for some $K \in \mathbb{R}$ and some $N \in [n, \infty)$ (namely $(M, \mathsf{d}_g, e^{-V}\mathrm{Vol}_g)$ is an $\mathrm{RCD}(K, N)$ space, recall (2.1.3) and (4.1.3)), for any $t > 0$, define the map $\Phi_t : M \to L^2(M, e^{-V}\mathrm{Vol}_g)$ by

$$\Phi_t(x) := (y \mapsto p(x, y, t)) \in L^2(M, e^{-V}\mathrm{Vol}_g).$$

Then the pull-back $g_t := (\Phi_t)^* g_{L^2}$ is well-defined as a smooth tensor of type $(0, 2)$ and it satisfies

$$g_t(x) = \int_M \mathrm{d}_x p_{y,t}(x) \otimes \mathrm{d}_x p_{y,t}(x) e^{-V(y)} \mathrm{dVol}_g(y) \qquad \forall x \in M$$

where it is emphasized that the RHS of the above makes sense as Bochner integral for *any* $x \in M$ because of (2.6.12). In particular, thanks to Fubini's theorem for all smooth vector fields V_i $(i = 1, 2)$ on M with bounded supports we have

$$\int_M g_t(V_1, V_2) e^{-V} \mathrm{dVol}_g = \int_M \int_M \mathrm{d}_x p_{y,t}(V_1)(x)\, \mathrm{d}_x p_{y,t}(V_2)(x) e^{-V(x)-V(y)} \mathrm{dVol}_g(x) \mathrm{dVol}_g(y)$$

and it is easy to see that this equation also characterizes g_t.

Let us generalize this observation to an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ as follows. Start noticing that the identity $|\mathrm{d}p_{y,t} \otimes \mathrm{d}p_{y,t}|_{\mathsf{HS}} = |\mathrm{d}p_{y,t}|^2$, the bound (2.6.12) and Lemma 2.6.5 ensure that for every $t > 0$ the map $y \mapsto \mathrm{d}p_{y,t} \otimes \mathrm{d}p_{y,t}$ is in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$. Hence the following definition is well-posed:

Definition 3.3.1 (Smoothing metrics g_t)**.** Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. We define the $(0,2)$ tensor $g_t \in L^0((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ on X as

$$g_t := \int_{\mathsf{X}} \mathrm{d}_x p_{y,t} \otimes \mathrm{d}_x p_{y,t} \,\mathrm{d}\mathfrak{m}(y).$$

Notice that the basic properties of Bochner integration (Hille's Theorem) ensure that for $V_1, V_2 \in L^2(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with bounded support we have

$$\int_{\mathsf{X}} g_t(V_1,V_2)\,\mathrm{d}\mathfrak{m} = \int_{\mathsf{X}}\int_{\mathsf{X}} \mathrm{d}p_{y,t}(V_1)\,\mathrm{d}p_{y,t}(V_2)\mathrm{d}\mathfrak{m}\mathrm{d}\mathfrak{m}(y).$$

When (X,d) is compact, making use of (2.6.20), it is proved in [43, Proposition 4.7] that

$$g_t := \sum_i e^{-2\lambda_i t}\mathrm{d}\varphi_i \otimes \mathrm{d}\varphi_i, \quad \text{in } L^\infty\left((T^*)^{\otimes 2}(X,\mathsf{d},\mathfrak{m})\right). \tag{3.3.1}$$

After a normalization of g_t as follows, the smoothing metrics are uniformly bounded in L^∞:

Proposition 3.3.2. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. Then we have*

$$t\mathfrak{m}(B_{\sqrt{t}}(\,\cdot\,))g_t \le C(K,N)g \qquad \mathfrak{m}\text{-a.e., } \forall t\in(0,1] \tag{3.3.2}$$

in the sense of symmetric tensors.

In particular we have that $g_t \in L^\infty_{loc}((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and moreover that $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \in L^\infty((T^*)^{\otimes 2}(\mathsf{X},\mathsf{d},\mathfrak{m}))$.

Proof. Fix $V \in L^0(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and notice that for $\mathfrak{m}$-a.e. x we have

$$\begin{aligned} t\mathfrak{m}(B_{\sqrt{t}}(x))g_t(V,V)(x) &\le t\mathfrak{m}(B_{\sqrt{t}}(x))|V|^2(x)\int_{\mathsf{X}} |\mathrm{d}p_{y,t}|^2(x)\,\mathrm{d}\mathfrak{m}(y) \\ \text{(by (2.6.12))} \qquad &\le \frac{C|V|^2(x)}{\mathfrak{m}(B_{\sqrt{t}}(x))}\int_{\mathsf{X}} \exp\left(-\frac{\mathsf{d}(x,y)^2}{5t}+Ct\right)\mathrm{d}\mathfrak{m}(y). \end{aligned}$$

The conclusion follows from Lemma 2.6.5 (with $\alpha = 0$). ☐

We now turn to the computation of $\nabla^* g_t$. To this aim, it is convenient to introduce the following function:

$$p_t(x) := p(x,x,t) \overset{(2.6.13)}{=} \int_{\mathsf{X}} p_{y,t/2}^2(x)\,\mathrm{d}\mathfrak{m}(y).$$

Notice that thanks to the bounds (2.6.11) it is easy to see that for every $t > 0$ the map $y \mapsto p_{y,t/2}^2$ is in $L^1(\mathsf{X},\mathfrak{m};L^2(\mathsf{X},\mathfrak{m}))$. It is then clear that the identity $p_t = \int_{\mathsf{X}} p_{y,t/2}^2\,\mathrm{d}\mathfrak{m}(y)$ holds also in the sense of Bochner integrals.

Let us start collecting some estimates for this function:

Lemma 3.3.3. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Then for any* $t > 0$ *we have* $p_{2t}(x) \in D_{loc}(\Delta)$ *with*

$$\mathrm{d}p_{2t} = 2\int_{\mathsf{X}} p_{y,t/2}\mathrm{d}p_{y,t/2}\,\mathrm{d}\mathfrak{m}(y) \qquad \textit{and} \qquad |\mathrm{d}p_{2t}| \le \frac{C(K,N)}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(\cdot))} \qquad \mathfrak{m}\textit{-a.e.} \tag{3.3.3}$$

and

$$\Delta p_{2t} = 2\int_{\mathsf{X}} p_{y,t/2}\Delta p_{y,t/2} + |\mathrm{d}p_{y,t}|^2\,\mathrm{d}\mathfrak{m}(y) \qquad \textit{and} \qquad |\Delta p_{2t}| \le \frac{C(K,N)}{t\mathfrak{m}(B_{\sqrt{t}}(\cdot))} \qquad \mathfrak{m}\textit{-a.e..} \tag{3.3.4}$$

Finally, we also have $\Delta p_{2t} \in H^{1,2}_{loc}(\mathsf{X},\mathsf{d},\mathfrak{m})$ *with*

$$\mathrm{d}\Delta p_{2t} = 2\int_{\mathsf{X}} \mathrm{d}p_{y,t/2}\Delta p_{y,t/2} + p_{y,t/2}\mathrm{d}\Delta p_{y,t/2} + \mathrm{d}|\mathrm{d}p_{y,t}|^2\,\mathrm{d}\mathfrak{m}(y). \tag{3.3.5}$$

It is part of the claim the fact that the integrands in (3.3.3) *and* (3.3.5) *belong to the space* $L^1(\mathsf{X},\mathfrak{m};L^2_{loc}(T^*(\mathsf{X},\mathsf{d},\mathfrak{m})))$ *and the one in* (3.3.4) *belongs to the space* $L^1(\mathsf{X},\mathfrak{m};L^2_{loc}(\mathsf{X},\mathfrak{m}))$.

Proof. Using (2.6.11) and (2.6.12) we get

$$\begin{aligned}\int_{\mathsf{X}} |\mathrm{d}(p_{y,t/2}^2)|\,\mathrm{d}\mathfrak{m}(y) &\le 2\int_{\mathsf{X}} p_{y,t/2}|\mathrm{d}p_{y,t/2}|\,\mathrm{d}\mathfrak{m}(y)\\ &\le \frac{C}{\sqrt{t}\mathfrak{m}(B_{\sqrt{t}}(\cdot))^2}\int_{\mathsf{X}} \exp\Big(-\frac{2\mathsf{d}^2(\cdot,y)}{5t} + Ct\Big)\,\mathrm{d}\mathfrak{m}(y).\end{aligned}$$

Thus from Lemma 2.6.5 and Proposition 3.2.4 we deduce that $p_{2t} \in H^{1,2}_{loc}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and that (3.3.3) holds. Similarly, starting from

$$\int_{\mathsf{X}} |\Delta(p^2_{y,t/2})| \, \mathrm{d}\mathfrak{m}(y) \leq 2 \int_{\mathsf{X}} p_{y,t/2} |\Delta p_{y,t/2}| + |\mathrm{d}p_{y,t/2}|^2 \, \mathrm{d}\mathfrak{m}(y)$$

and using the estimates (2.6.11), (2.6.12) and (2.6.14) and then again Lemma 2.6.5 and Proposition 3.2.4, we conclude that $p_{2t}(x) \in D_{loc}(\Delta)$ and that (3.3.4) holds.

For the last claim we recall that $p_{y,t} \in \mathrm{TestF}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, thus the Leibniz rule for the Laplacian and the basic properties of test functions give $p_{y,t/2}\Delta p_{y,t/2} + |\mathrm{d}p_{y,t}|^2 \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with

$$\mathrm{d}(p_{y,t/2}\Delta p_{y,t/2} + |\mathrm{d}p_{y,t}|^2) = \mathrm{d}p_{y,t/2}\Delta p_{y,t/2} + p_{y,t/2}\mathrm{d}\Delta p_{y,t/2} + 2\mathrm{Hess}\, p_{y,t}(\nabla p_{y,t}, \,\cdot\,).$$

The fact that the first two addends in the RHS are in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$ can be proved as before. For the last one we let $A \subseteq \mathsf{X}$ be Borel and bounded and $\bar{x} \in \mathsf{X}$. Then we have $\mathsf{d}(x, y) \geq \mathsf{d}(y, \bar{x}) - R$ for any $x \in A$, $y \in \mathsf{X}$ and some $R > 0$ independent on x, y. Hence (2.6.12) implies that $\||\mathrm{d}p_{y,t}|\|_{L^\infty(A)} \leq Ce^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}}$ for some $C = C(t, K, N, A, \bar{x})$, thus

$$\begin{aligned}
\int_{\mathsf{X}} \sqrt{\int_A |\mathrm{Hess}\, p_{y,t}(\nabla p_{y,t}, \,\cdot\,)|^2 \, \mathrm{d}\mathfrak{m}(x)} \, \mathrm{d}\mathfrak{m}(y) &\leq \int_{\mathsf{X}} \||\mathrm{Hess}\, p_{y,t}|_{\mathsf{HS}}\|_{L^2} \||\mathrm{d}p_{y,t}|\|_{L^\infty(A)} \, \mathrm{d}\mathfrak{m}(y) \\
\text{(by (2.2.4))} \qquad &\leq C \int_{\mathsf{X}} (\|\Delta p_{y,t}\|_{L^2} + \||\mathrm{d}p_{y,t}|\|_{L^2}) e^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}} \, \mathrm{d}\mathfrak{m}(y) \\
\text{(by (2.6.16))} \qquad &\leq C \int_{\mathsf{X}} \mathfrak{m}(B_{\sqrt{t}}(y))^{-\frac{1}{2}} e^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}} \, \mathrm{d}\mathfrak{m}(y) < \infty,
\end{aligned}$$

where the last inequality comes from Lemma 2.6.5. The conclusion follows. □

To further analyze the link between g_t and p_t the following result will be crucial:

Lemma 3.3.4. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. Then for every $t > 0$ we have that*

$y \mapsto \Delta p_{y,t} \mathrm{d} p_{y,t}$ and $y \mapsto p_{y,t} \mathrm{d} \Delta p_{y,t}$ are both in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$ and

$$\int_{\mathsf{X}} \Delta p_{y,t} \mathrm{d} p_{y,t} \, \mathrm{d}\mathfrak{m}(y) = \int_{\mathsf{X}} p_{y,t} \mathrm{d} \Delta p_{y,t} \, \mathrm{d}\mathfrak{m}(y).$$

Proof. For the first part of the claim we argue as in the proof of Lemma 3.3.3 above: let $A \subseteq \mathsf{X}$ be Borel and bounded and $\bar{x} \in \mathsf{X}$. Then $\mathsf{d}(x, y) \geq \mathsf{d}(y, \bar{x}) - R$ for any $x \in A$, $y \in \mathsf{X}$ and some $R > 0$ independent on x, y. Hence (2.6.12) implies that

$$\| |\mathrm{d} p_{y,t}| \|_{L^\infty(A)} \leq C e^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}} \tag{3.3.6}$$

for some $C = C(t, K, N, A, \bar{x})$, and therefore

$$\int_{\mathsf{X}} \sqrt{\int_A |\Delta p_{y,t} \mathrm{d} p_{y,t}|^2 \, \mathrm{d}\mathfrak{m}(x)} \, \mathrm{d}\mathfrak{m}(y) \leq C \int_{\mathsf{X}} \|\Delta p_{y,t}\|_{L^2} e^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}} \, \mathrm{d}\mathfrak{m}(y) < \infty,$$

having used the bound (2.6.16) and Lemma 2.6.5 in the last step. This proves that $y \mapsto \Delta p_{y,t} \mathrm{d} p_{y,t}$ is in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$ and an analogous argument gives the same for $y \mapsto p_{y,t} \mathrm{d} \Delta p_{y,t}$.

Now write the Chapman-Kolmogorov equation (2.6.13) as

$$\int_{\mathsf{X}} p(y, z, s) p_{z,t} \, \mathrm{d}\mathfrak{m}(z) = p_{y,t+s}$$

and observe that the estimates (2.6.11), (2.6.12) and the same arguments just used ensure that for any $y \in \mathsf{X}$ the maps $z \mapsto p(y, z, s) p_{z,t}$ and $z \mapsto p(y, z, s) \mathrm{d} p_{z,t}$ are in $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}(\mathsf{X}, \mathfrak{m}))$ and $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$ respectively. Thus Proposition 3.2.4 gives

$$\int_{\mathsf{X}} p(y, z, s) \mathrm{d} p_{z,t} \, \mathrm{d}\mathfrak{m}(z) = \mathrm{d} p_{y,t+s}.$$

Multiplying both sides by $p_{y,t}$, integrating in y and using Fubini's theorem we obtain

$$\int_{\mathsf{X}} p_{z,t+s} \mathrm{d}p_{z,t} \, \mathrm{d}\mathfrak{m}(z) \overset{(2.6.13)}{=} \int_{\mathsf{X}} \int_{\mathsf{X}} p_{y,t} \, p(y,z,s) \mathrm{d}p_{z,t} \, \mathrm{d}\mathfrak{m}(z) \, \mathrm{d}\mathfrak{m}(y) = \int_{\mathsf{X}} p_{y,t} \mathrm{d}p_{y,t+s} \, \mathrm{d}\mathfrak{m}(y).$$

Thus to conclude it is sufficient to prove that as $s \to 0^+$ we have

$$\begin{aligned} \int_{\mathsf{X}} \frac{p_{y,t+s} - p_{y,t}}{s} \mathrm{d}p_{y,t} \, \mathrm{d}\mathfrak{m}(y) &\to \int_{\mathsf{X}} \Delta p_{y,t} \mathrm{d}p_{y,t} \, \mathrm{d}\mathfrak{m}(y), \\ \int_{\mathsf{X}} p_{y,t} \mathrm{d}\Big(\frac{p_{y,t+s} - p_{y,t}}{s} \Big) \, \mathrm{d}\mathfrak{m}(y) &\to \int_{\mathsf{X}} p_{y,t} \mathrm{d}\Delta p_{y,t} \, \mathrm{d}\mathfrak{m}(y) \end{aligned} \tag{3.3.7}$$

in $L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$. We start noticing that from (2.6.13) we have

$$\begin{aligned} F_{y,t} := \frac{p_{y,t+s} - p_{y,t}}{s} - \Delta p_{y,t} &= \int_0^1 \Delta (p_{y,t+rs} - p_{y,t}) \, \mathrm{d}r \\ &= s \int_0^1 r \Delta \Big(\int_0^1 \Delta p_{y,t+rsh} \, \mathrm{d}h \Big) \, \mathrm{d}r \\ &= s \int_0^1 r \Delta \mathrm{h}_{t/3} \Big(\int_0^1 \Delta \mathrm{h}_{t/3} p_{y,t/3+rsh} \, \mathrm{d}h \Big) \, \mathrm{d}r \end{aligned}$$

and therefore using twice (2.6.3) we obtain

$$\begin{aligned} \|F_{y,t}\|_{L^2} &\leq sC(t) \int_0^1 \int_0^1 \|p_{y,t/3+rsh}\|_{L^2} \, \mathrm{d}h \, \mathrm{d}r \\ &\overset{(2.6.4)}{\leq} sC(t) \|p_{y,t/3}\|_{L^2} \overset{(2.6.16)}{\leq} s \, C(K,N,t) \mathfrak{m}(B_{\sqrt{\frac{t}{3}}}(y))^{-\frac{1}{2}}. \end{aligned} \tag{3.3.8}$$

Thus for $A \subseteq \mathsf{X}$ Borel and bounded we have

$$\begin{aligned} &\int_{\mathsf{X}} \sqrt{\int_A |F_{y,t} \mathrm{d}p_{y,t}|^2 \, \mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m}(y) \\ &\overset{(3.3.6)}{\leq} C \int_{\mathsf{X}} \|F_{y,t}\|_{L^2} e^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}} \, \mathrm{d}\mathfrak{m}(y) \overset{(3.3.8)}{\leq} s \, C \int_{\mathsf{X}} \frac{e^{-\frac{\mathsf{d}^2(y,\bar{x})}{5t}}}{\mathfrak{m}(B_{\sqrt{\frac{t}{3}}}(y))^{\frac{1}{2}}} \, \mathrm{d}\mathfrak{m}(y) \end{aligned}$$

for some $C = C(K, N, t, A, \bar{x})$. Since the last integral is finite by Lemma 2.6.5, the LHS goes to 0 as $s \to 0^+$. This proves the first in (3.3.7). The second follows along very similar lines, we omit the details. □

We are now in a position to prove the main result of this subsection:

Theorem 3.3.5. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Then for every* $t > 0$ *we have* $g_t \in D_{loc}(\nabla^*)$ *with*

$$\nabla^* g_t = -\frac{1}{4} \mathrm{d}\Delta p_{2t}. \tag{3.3.9}$$

Proof. For any $y \in \mathsf{X}$ Proposition 2.2.9 tells $\nabla^*(\mathrm{d}p_{y,t} \otimes \mathrm{d}p_{y,t}) = -\Delta p_{y,t}\mathrm{d}p_{y,t} - \frac{1}{2}\mathrm{d}|\nabla p_{y,t}|^2$. Also, arguing as in Lemma 3.3.3 it is easy to see that $y \mapsto -\Delta p_{y,t}\mathrm{d}p_{y,t} - \frac{1}{2}\mathrm{d}|\nabla p_{y,t}|^2$ belongs to the space $L^1(\mathsf{X}, \mathfrak{m}; L^2_{loc}(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})))$. Thus taking into account Lemma 3.3.4 we obtain

$$\begin{aligned} \int_{\mathsf{X}} \nabla^*(\mathrm{d}p_{y,t} \otimes \mathrm{d}p_{y,t}) \, \mathrm{d}\mathfrak{m}(y) &= -\tfrac{1}{2} \int_{\mathsf{X}} \Delta p_{y,t}\mathrm{d}p_{y,t} + p_{y,t}\mathrm{d}\Delta p_{y,t} + \mathrm{d}|\mathrm{d}p_{y,t}|^2 \, \mathrm{d}\mathfrak{m}(y) \\ &\overset{(3.3.5)}{=} -\frac{1}{4}\mathrm{d}\Delta p_{2t}. \end{aligned}$$

The conclusion comes from the very definition of g_t and Proposition 3.2.4. □

3.4 Asymptotic behavior as $t \to 0^+$

This section is taken from [41, Section 3.4]. The goal of this subsection is to study the behaviour of g_t and $\nabla^* g_t$ as $t \to 0^+$.

We start with the following result, which generalizes to the non-compact setting the analogous statement [43, Theorem 5.10]:

Theorem 3.4.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Then* $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t \to c_n g$ *strongly in* L^p_{loc} *for any* $p \in [1, \infty)$, *where* c_n *is a positive constant depending only on* n.

Proof. Since the proof is essentially same to that in [43, Theorem 5.10] after replacing L^p by L^p_{loc} (recall that [43, Theorem 5.10] discussed only on the case when (X, d) is compact), we shall only give a sketch of the proof.

Fix $V \in L^\infty(T(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with bounded support. First let us discuss the asymptotic behaviour of the following as $t \to 0^+$ for fixed $y \in \mathsf{X}$ and $L > 0$;

$$\begin{aligned}&\int_{\mathsf{X}} t\mathfrak{m}(B_{\sqrt{t}}(x))|\mathrm{d}p_{y,t}(V)|^2(x)\mathrm{d}\mathfrak{m}(x)\\&=\int_{B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(\cdot))|\mathrm{d}p_{y,t}(V)|^2\mathrm{d}\mathfrak{m}+\int_{\mathsf{X}\setminus B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(\cdot))|\mathrm{d}p_{y,t}(V)|^2\mathrm{d}\mathfrak{m}.\end{aligned} \tag{3.4.1}$$

The key idea to control the each terms in the RHS of (3.4.1) is to apply *blow-up arguments* (i.e. we discuss the behaviour of the rescaled spaces $(\mathsf{X}, \sqrt{t}^{-1}\mathsf{d}, \mathfrak{m}(B_{\sqrt{t}}(y))^{-1}\mathfrak{m}, y)$ with respect to the pointed measured Gromov-Hausdorff convergence as $t \to 0^+$) in conjunction with the stability of the heat flow first observed in [93]. More precisely, we use the stability results proved in [60, Corollary 5.5, Theorem 5.7, Lemma 5.8], [94, Theorem 4.4], [44, Theorem 3.3] (with [43, Theorem 2.19]), [24, Theorem 6.8] with Theorem 2.4.6 and (2.6.12). Combining these, letting $t \to 0^+$ and then letting $L \to \infty$ in the RHS of (3.4.1), the following hold for $\mathfrak{m}$-a.e. $y \in \mathsf{X}$:

1. The first term of the RHS of (3.4.1) converges to $c_n|V|^2(y)$.

   More precisely, argue as [43, Proposition 5.5], we can replace $V \in L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ by some ∇f for $f \in H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$. And for $\mathfrak{m}$-a.e. $y \in \mathcal{R}_n$, on the rescaled space $(\mathsf{X}, \sqrt{t}^{-1}\mathsf{d}, \mathfrak{m}(B_{\sqrt{t}}(y))^{-1}\mathfrak{m}, y))$, there exists a subsequence t_i so that the following rescaling of f:

$$f_{\sqrt{t},y} = \frac{1}{\sqrt{t}}\left(f - \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(y))}\int_{\mathsf{X}} f\mathrm{d}\mathfrak{m}\right)$$

   converges to a harmonic and Lipschitz function $\widehat{f}$ as $t_i \to 0$ on the tangent cone $(\mathbb{R}^n, |\cdot|, \mathcal{L}^n)$. This $\widehat{f}$ must be linear. Write $\nabla \widehat{f} = \sum_{j=1}^n a_j \frac{\partial}{\partial x_j}$. Set $\mathsf{d}_t = \sqrt{t}^{-1}\mathsf{d}$, $\mathfrak{m}_t = \mathfrak{m}(B_{\sqrt{t}}(y))^{-1}\mathfrak{m}$, then let the heat kernel on $(\mathsf{X}, \mathsf{d}_t, \mathfrak{m}_t)$ be $p^t(x,y,s)$, observe the rescaling relation $p^t(x,y,s) = \mathfrak{m}(B_{\sqrt{t}}(y))p(x,y,ts)$. The blow up steps are carried out as follows(by possibly taking a subsequence of $t \to 0$):

$$
\begin{aligned}
\int_{B_{L\sqrt{t}}(y)} t\mathfrak{m}(B_{\sqrt{t}}(\cdot))|\nabla p_{y,t}(V)|^2 \mathrm{d}\mathfrak{m} &= \int_{B_L^{\mathsf{d}_t}(y)} \mathfrak{m}_t(B_1^{\mathsf{d}_t}(\cdot))|\langle \nabla_t p^t_{y,1}, \nabla_t f_{\sqrt{t},y}\rangle|^2 \mathrm{d}\mathfrak{m}_t \\
&\xrightarrow{t\to 0} \int_{B_L(0)} \widehat{\mathcal{H}}^n(B_1(\cdot))|\langle \nabla p^e(\cdot,0,1), \nabla \widehat{f}\rangle|^2 \mathrm{d}\widehat{\mathcal{H}}^n \\
&= c_n(L)|\nabla \widehat{f}|^2 = c_n \sum_{j=1}^n |a_j|^2 = c_n |\nabla f|^2
\end{aligned}
$$

where $\nabla_t = \sqrt{t}\nabla$ is the minimal relaxed slope computed under d_t, $\widehat{\mathcal{H}}^n = \frac{1}{\omega_n}\mathcal{H}^n$ is the renormalized Hausdorff measure, and p^e is the heat kernel on $\mathbb{R}^n$, the last equality holds at least for Lebesgue points of f, hence $\mathfrak{m}$-a.e.. As $L \to \infty$, $c_n(L) \to c_n$, which is a finite constant. Now substitute back V we get the convergence to $c_n|V|^2$.

2. The second term of the RHS of (3.4.1) converges to 0.

   More precisely, this term is controlled by using (2.6.12).

Thus as $t \to 0^+$ we obtain

$$
\int_{\mathsf{X}} t\mathfrak{m}(B_{\sqrt{t}}(x))|\mathrm{d}p_{y,t}(V)|^2(x)\mathrm{d}\mathfrak{m}(x) \to c_n|V|^2(y) \qquad \mathfrak{m}\text{-a.e. } y \in \mathsf{X}.
$$

Thus combining this with (3.3.2) and the dominated convergence theorem we get

$$
\int_{\mathsf{X}} t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t(V,V)\mathrm{d}\mathfrak{m} \to c_n \int_{\mathsf{X}} |V|^2 \mathrm{d}\mathfrak{m}
$$

which proves that $t\mathfrak{m}(B_{\sqrt{t}}(\,\cdot\,))g_t$ L^p-weakly converge to $c_n g$ on any bounded subset A of X because g_t is symmetric and V is arbitrary.

In order to get the L^p_{loc}-strong convergence it suffices to check

$$
\lim_{t\to 0^+} \int_{\mathsf{X}} \varphi |t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}} \mathrm{d}\mathfrak{m} = c_n^2 \int_{\mathsf{X}} \varphi |g|^2_{\mathrm{HS}} \mathrm{d}\mathfrak{m} = c_n^2 n \int_{\mathsf{X}} \varphi \mathrm{d}\mathfrak{m} \tag{3.4.2}
$$

for every $\varphi \in \mathrm{Lip}_{bs}(\mathsf{X},\mathsf{d})$, because this implies the L^2_{loc}-strong convergence and the improvement to the L^p_{loc}-strong one comes from (3.3.2). Let us check (3.4.2) as follows.

For any $z \in \mathcal{R}_n$, applying blow-up arguments as explained above again allows us to deduce

$$F(z,t) := \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} |t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}} \mathrm{d}\mathfrak{m} \to c_n^2 n$$

and thus (recalling (3.3.2) to use the dominate convergence theorem) for $\varphi \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$ we have

$$\lim_{t\to 0^+} \int_{\mathsf{X}} \varphi(z) F(z,t) \,\mathrm{d}\mathfrak{m}(z) = c_n^2 n \int_{\mathsf{X}} \varphi \,\mathrm{d}\mathfrak{m}. \tag{3.4.3}$$

On the other hand, we have

$$\int_{\mathsf{X}} \varphi(z) F(z,t) \,\mathrm{d}\mathfrak{m}(z) = \int_{\mathsf{X}} |t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}}(x) \underbrace{\int_{B_{\sqrt{t}}(x)} \frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))} \mathrm{d}\mathfrak{m}(z)}_{=:G(x,t)} \,\mathrm{d}\mathfrak{m}(x). \tag{3.4.4}$$

Now notice that $\sup_{t,x} G(x,t) < \infty$ (because of (2.4.4)) and $\lim_{t\to 0^+} G(x,t) = \varphi(x)$ for $\mathfrak{m}$-a.e. x (because of the convergence of the blow-ups to the Euclidean space). It follows (again using (3.3.2) to use the dominate convergence theorem) that

$$\lim_{t\to 0^+} \int_{\mathsf{X}} |t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t|^2_{\mathrm{HS}}(x) \left| \int_{B_{\sqrt{t}}(x)} \frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))} \mathrm{d}\mathfrak{m}(z) - \varphi(x) \right| \mathrm{d}\mathfrak{m}(x) = 0$$

which together with (3.4.3) and (3.4.4) gives (3.4.2) and the conclusion. □

Remark 3.4.2. In Theorem 3.4.1, the conclusion can not be improved to the case when $p = \infty$ in general. For example, the $\mathrm{RCD}(0,1)$ space $([0,\pi], \mathsf{d}_{\mathbb{R}}, \mathcal{H}^1)$ satisfies

$$\liminf_{t\to 0^+} \left\| t\mathcal{H}^1(B_{\sqrt{t}}(\cdot))g_t - c_1 g_{\mathbb{R}} \right\|_{L^\infty} > 0.$$

See [43, Remark 5.11] for details. It is worth pointing out that the validity of

$$\|t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t - c_n g\|_{L^\infty_{loc}} \to 0 \tag{3.4.5}$$

is closely related to the nonexistence of singular points (actually the singular points are $\{0, \pi\}$ in this example). See also [95, Theorem 1.1].

In connection with this pointing out, if $(M, g, e^{-V}\mathrm{dVol}_g)$ is any weighted complete n-dimensional Riemannian manifold with $\mathrm{Ric}_N \geq Kg$ for some $K \in \mathbb{R}$ and some $N \in [n, \infty)$, applying a construction of the heat kernel by *parametrix*, we can actually prove that (3.4.5) holds. More precisely we have as $t \to 0^+$.

$$\begin{aligned} &4(8\pi)^{n/2}t^{(n+2)/2}g_t \\ &= e^V g - e^V \left(\frac{2}{3}\left(\mathrm{Ric}_g - \frac{1}{2}\mathrm{Scal}_g g\right) - \mathrm{d}V \otimes \mathrm{d}V - \Delta^g V g + \frac{|\nabla^g V|^2}{2} g\right) t + O(t^2), \end{aligned}$$

which is uniform on any bounded set. We will derive this expansion in next Chapter. See also [42, Theorem 5]. ■

Corollary 3.4.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Let* A *be a bounded Borel subset of* X *with*

$$\inf_{r \in (0,1), x \in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0. \tag{3.4.6}$$

Then $\mathcal{H}^n \llcorner A$ *is a Radon measure absolutely continuous w.r.t.* $\mathfrak{m}$ *and*

$$\chi_A \omega_n t^{(n+2)/2} g_t \to \chi_A c_n \frac{\mathrm{d}\mathcal{H}^n \llcorner A}{\mathrm{d}\mathfrak{m}} g \qquad \textit{in } L^p((T^*)^{\otimes 2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})),\ \forall p \in [1, \infty).$$

Proof. The first part of the claim follows from Lemma 2.4.7 and (3.4.6). Then Theorem 2.4.6 ensures that as $r \to 0^+$

$$\frac{\omega r^n}{\mathfrak{m}(B_r(x))} \to \frac{\mathrm{d}\mathcal{H}^n \llcorner A}{\mathrm{d}\mathfrak{m}} \qquad \mathfrak{m}\text{-a.e. } x \in A.$$

Thus (3.4.6), the dominated convergence theorem and Theorem 3.4.1 give the conclusion. □

We now turn to the asymptotic of Δp_{2t}:

Proposition 3.4.4. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space,* $K \in \mathbb{R}$*,* $N \in [1, \infty)$*. Then as* $t \to 0^+$ *we have*

$$t\mathfrak{m}(B_{\sqrt{t}}(\cdot))\Delta p_{2t}(\cdot) \to 0 \qquad \textit{in } L^p_{loc}(\mathsf{X}, \mathfrak{m}),\ \forall p \in [1, \infty).$$

Proof. The proof is based on blow-up arguments which is similar to that of Theorem 3.4.1. Therefore we give only a sketch of the proof (see also [43]).

Let us first prove that for any $z \in \mathcal{R}_n$, as $t \to 0^+$,

$$\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} t\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\mathrm{d}\mathfrak{m}(x) \to 0. \tag{3.4.7}$$

In order to prove this, consider the pointed measured Gromov-Hausdorff convergent sequence of the rescaled space:

$$(\mathsf{X}^{t,z}, \mathsf{d}^{t,z}, \mathfrak{m}^{t,z}, z) := \left(\mathsf{X}, \frac{1}{\sqrt{t}}\mathsf{d}, \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(z))}\mathfrak{m}, z\right) \overset{\mathrm{pmGH}}{\to} \left(\mathbb{R}^n, \mathsf{d}_{\mathbb{R}^n}, \frac{1}{\omega_n}\mathcal{H}^n, 0_n\right) \tag{3.4.8}$$

and denote by $p^{t,z}$, $\Delta^{t,z}$, the heat kernel, the Laplacian of $(\mathsf{X}^{t,z}, \mathsf{d}^{t,z}, \mathfrak{m}^{t,z})$, respectively, namely $p^{t,z}(x, y, s) = \mathfrak{m}(B_{\sqrt{t}}(z))p(x, y, ts)$, $\Delta^{t,z} f = t\Delta f$. Thus the LHS of (3.4.7) is equal to

$$\int_{B_1^{\mathsf{d}^{t,z}}(z)} \mathfrak{m}^{t,z}(B_1^{\mathsf{d}^{t,z}}(x))|\Delta^{t,z} p_2^{t,z}(x)|\mathrm{d}\mathfrak{m}^{t,z}(x). \tag{3.4.9}$$

Applying the stability results already used in the proof of Theorem 3.4.1 shows that $\Delta^{t,z}p_2^{t,z}$ L^2_{loc}-strongly converge to $\Delta^{g_{\mathbb{R}^n}}(\omega_n\tilde{p}_2)$ with respect to (3.4.8), where $\tilde{p}(x)$ denotes the heat kernel of the n-dimensional Euclidean space evaluated at (x, x). Since $\Delta^{g_{\mathbb{R}^n}}(\omega_n\tilde{p}_2) = 0$ because $\tilde{p}_2$ is constant, (3.4.9) converges to

$$\int_{B_1(0_n)} |\Delta^{g_{\mathbb{R}^n}}(\omega_n\tilde{p}_2)|\mathrm{d}\left(\frac{1}{\omega_n}\mathcal{H}^n\right) = 0$$

as $t \to 0^+$, which proves (3.4.7).

Fix a $\varphi \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$. Applying (3.4.7) with (3.3.4), the dominated convergence theorem yields

$$\int_{\mathsf{X}} \frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} t\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\mathrm{d}\mathfrak{m}(x)\mathrm{d}\mathfrak{m}(z) \to 0.$$

On the other hand (3.3.4) and dominated convergence (recall (2.4.4)) imply

$$\left|\int_{\mathsf{X}} \frac{\varphi(z)}{\mathfrak{m}(B_{\sqrt{t}}(z))} \int_{B_{\sqrt{t}}(z)} t\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\mathrm{d}\mathfrak{m}(x)\mathrm{d}\mathfrak{m}(z) - \int_{\mathsf{X}} \varphi(z)t\mathfrak{m}(B_{\sqrt{t}}(z))|\Delta p_{2t}(z)|\mathrm{d}\mathfrak{m}(z)\right|$$
$$\leq C(K,N)\int_{\mathsf{X}} \left|\varphi(z) - \int_{B_{\sqrt{t}}(z)} \frac{\varphi(x)}{\mathfrak{m}(B_{\sqrt{t}}(x))}\mathrm{d}\mathfrak{m}(x)\right|\mathrm{d}\mathfrak{m}(z) \to 0.$$

Thus

$$\int_{\mathsf{X}} \varphi(x)t\mathfrak{m}(B_{\sqrt{t}}(x))|\Delta p_{2t}(x)|\mathrm{d}\mathfrak{m}(x) \to 0. \tag{3.4.10}$$

The desired L^p_{loc}-strong convergence comes from (3.4.10) and (3.3.4). □

Corollary 3.4.5. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. Also, let* A *be a bounded Borel subset of* X *and* $n \in \mathbb{N}$ *be such that*

$$\inf_{r\in(0,1),x\in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0.$$

Then as $t \to 0^+$

$$t^{(n+2)/2}\Delta p_{2t} \to 0 \qquad \text{in } L^2(A, \mathfrak{m}).$$

Proof. Direct consequence of Proposition 3.4.4. □

Remark 3.4.6. Although the above convergence results are stated for the strong convergence for the sake of generality, the weak convergence is enough to justify the main results in Chapter 4 and 5. ■

3.5 Further Studies

The local Dirichlet heat kernel theory in an $\mathrm{RCD}(K,N)$ space is discussed in [88]. Given any bounded open set $U \subseteq \mathsf{X}$, there is a heat flow h_t^U and heat kernel $p^U(x,y,t) \in H_0^{1,2}(U,\mathsf{d},\mathfrak{m})$ called Dirichlet heat kernel associated to the local Dirichlet form

$$\mathcal{E}(f) = \int_{\mathsf{X}} |\nabla f|^2 \mathrm{d}\mathfrak{m} \qquad \forall f \in H_0^{1,2}(U,\mathsf{d},\mathfrak{m})$$

It is natural to ask if this method of smoothing of metric by pulling-back using heat kernel admits a local counterpart. That is, define a family of metrics

$$g_t^U = \int_X \mathrm{d}p^U(x,y,t) \otimes \mathrm{d}p^U(x,y,t)\mathrm{d}\mathfrak{m}(y),$$

does it holds that $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t^U \to c_n g$ in $L^2((T^*)^{\otimes 2}(U,\mathsf{d},\mathfrak{m}))$?

What becomes different from the global case is that the gradient estimates of p^U. In [85, Theorem 4.8] Sturm proved the following Gaussian lower bound in very general setting, now we state it in the framework of $\mathrm{RCD}(K,N)$ space:

Theorem 3.5.1. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space,* $U \subseteq \mathsf{X}$ *be an open subset. There exists constant* $C(K,N)$ *so that*

$$p^U(x,y,t) \geq \frac{1}{C}\mathfrak{m}^{-1}(B_{\sqrt{s}}) \exp\left\{-C\frac{\mathsf{d}^2(x,y)}{s}\right\} \exp\left\{-\frac{C}{R^2}t\right\} \tag{3.5.1}$$

for all points (x,y) *that are joined by a curve* $\gamma : [0,1] \to U$ *of length* $\mathsf{d}(x,y)$*, where* $s = \inf\{t, R^2\}$ *and* $R = \inf_{s\in[0,1]} \mathsf{d}(\gamma(s), \mathsf{X}\setminus U)$.

Examining the proof carefully, it is notable that the γ can be any curve completely in U so that its length $\ell(\gamma)$ is comparable to $\mathsf{d}(x,y)$, that is, there exist $c, C > 0$ so that $c\mathsf{d}(x,y) < \ell(\gamma) < C\mathsf{d}(x,y)$, which is sufficient to apply the parabolic Harnack inequality. We can specialize this general result to $U = B_R$ and apply the gradient estimate in [96] to get the a gradient estimate for Dirichlet heat kernel. Recall [96, Theorem 1.1]:

Theorem 3.5.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathrm{RCD}(-K, N)$ *space for* $K \geq 0$ *and* $N \in (1, \infty)$ *and let* $T \in (0, \infty)$*,* B_R *be a geodesic ball of radius* R*. If* $u(x,t) \in W^{1,2}_{loc}(B_{R,T})$ *is a local weak solution of heat equation in* $B_{R,T} := B_R \times (0,T)$*. Suppose that* $0 < m \leq u \leq M$ *on* $B_{R,T}$*, then the following estimate holds:*

$$|\nabla_x f(x,t)| \leq C_N \left(\frac{\log(M/m)}{R^2} + \frac{1}{T} + K \right) \cdot \log \frac{M}{u(x,t)}$$

for a.e. $(x,t) \in B_{R/2} \times (T/2, T)$*, where* $f = \log u$ *and* C_N *is a constant depending on* N*.*

Theorem 3.5.3 (gradient estimate)**.** *Let* $p^R(x,y,t) := p^R_{y,t}(x)$ *be the Dirichlet heat kernel on* $B_R(q) \subseteq \mathsf{X}$*, then there exists a constant* $C(K,N)$ *such that for fixed* $y \in B_R(q)$*, almost every* $x \in B_{R/8}(q)$ *and almost every* $t < T_y$*, such that* $B_{\sqrt{T_y}} \subseteq B_R(q)$ *and* $T_y \leq \inf\{(R/6)^2, \mathsf{d}(y, \mathsf{X} \setminus B_R(q)))\}$*, the following estimate holds*

$$|\nabla \log p^R_{y,t}(x)|^2 \leq C \left(\frac{t+1}{R^2} + \frac{t}{R^2 \mathsf{d}^2(y, \partial B_R(q))} + \frac{2}{t} + |K| \right) \left(1 + \frac{\mathsf{d}^2(x,y)}{t} + t \right).$$

Proof. Fix $y_0 \in B_R(q)$ and $T > 0$ small so that $T < T_y$. Set $u(x,t) = p^R(x, y_0, t)$. Then u is a local solution to heat equation on $B_{R/4}(q)$. Let $M = \sup_{B_{R/4}(q) \times (T/2,T)} u$, $m = \inf_{B_{R/4}(q) \times (T/2,T)} u$. We first do some computations. Thanks to Sturm's heat kernel lower bound, the fact that $p^R \leq p$ (from weak maximum principle) and the Gaussian estimates of p, i.e., (2.6.11), we have

$$\begin{aligned} \frac{M}{m} &\leq C_1 C_3 \frac{\mathfrak{m}(B_{\sqrt{T}}(y_0))}{\mathfrak{m}(B_{\sqrt{T/2}}(y_0))} e^{C_4 \frac{25R^2}{16t} + C_2 t + \frac{C_5 t}{\mathsf{d}^2(y, \partial B_R(q))}} \\ &\leq C e^{\sqrt{(N-1)|K|T} + C_4 \frac{25R^2}{16t} + C_2 t + \frac{C_5 t}{\mathsf{d}^2(y, \partial B_R(q))}}. \end{aligned}$$

Taking logarithm,

$$\log\frac{M}{m} \le C + \sqrt{(N-1)|K|T} + C_4\frac{25R^2}{8T} + C_2T + \frac{C_5T}{\mathsf{d}^2(y,\partial B_R(q))}$$
$$\le C\left(1 + T + \frac{T}{\mathsf{d}^2(y,\partial B_R(q))} + \frac{R^2}{T}\right).$$

The same computation gives

$$\log\frac{M}{u(x,t)} \le C\left(1 + T + \frac{\mathsf{d}(x,y_0)^2}{T}\right).$$

Now it follows from [96, Theorem 1.1] that for $\mathfrak{m}\times\mathcal{L}^1$-a.e. $(x,t)\in B_{R/8}\times(T/2,T)$ it holds that

$$|\nabla_x \log p^R(x,y_0,t)|^2 \le C\left(\frac{1+t}{R^2} + \frac{t}{R^2\mathsf{d}^2(y_0,\mathsf{X}\setminus B_R(q))} + \frac{2}{t} + |K|\right)\left(1 + t + \frac{\mathsf{d}(x,y_0)^2}{t}\right),$$

as desired. □

Further manipulation gives a Gaussian estimate of $|\nabla p^R|$. Note that $\frac{t}{\mathsf{d}(y,X\setminus B_R)}$ blows up when y approaches the boundary of B_R. This makes the integral

$$\int_X t\mathfrak{m}(B_{\sqrt{t}}(\cdot))\mathrm{d}p^U(x,y,t)\otimes \mathrm{d}p^U(x,y,t)\mathrm{d}\mathfrak{m}(y)$$

hard to be uniformly estimated when $t\to 0$. Even in Euclidean space, the presence of such a term $\frac{1}{\mathsf{d}(y,X\setminus B_R(0))}$ in the gradient estimates of local heat kernel of the ball $B_R(0)$ is inevitable. So the argument in [43] cannot be applied directly. Since it is relatively hard to compute the explicit form of a local heat kernel, there is also no counterexample of the convergence $t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t^U \to c_n g$ is known.

Another question closely related to g_t^U is the convergence to the global metric g_t, when enlarging the open set U. More precisely, denote by g_t^R the corresponding pull-back metric on the ball B_R, is there an L^2 convergence $g_t^R \to g_t$ as $R\to\infty$?

CHAPTER 4

WEAKLY NON-COLLAPSED SPACES ARE STRONGLY NON-COLLPASED

4.1 Overview

This Chapter is a review of the joint work with Brena-Gigli-Honda [41]. The notion $\mathrm{wncRCD}(K, N)$ spaces is introduced in Definition 2.5.6. We provide a motivation for it here. It starts with finding criteria for non-collapsed spaces. The analysis carried out by Cheeger-Colding and the analogy with the study of the Bakry-Émery N-Ricci curvature tensor (see (4.1.3) and the subsequent discussion) suggest that in fact $\mathrm{ncRCD}(K, N)$ spaces should be identifiable among $\mathrm{RCD}(K, N)$ ones by properties seemingly weaker than the one $\mathfrak{m} = \mathcal{H}^N$. To be more precise we need to introduce the N-dimensional (Bishop-Gromov) density $\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}] : \mathsf{X} \to [0, \infty]$ as

$$\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}](x) := \lim_{r \to 0^+} \frac{\mathfrak{m}(B_r(x))}{\omega_N r^N} \qquad \forall x \in \mathsf{X},$$

notice that the existence of the limit is a consequence of the Bishop-Gromov inequality. It is worth pointing out that standard results about differentiation of measures ensure that if $\mathcal{H}^N$ is a Radon measure on X, then

$$\limsup_{r \to 0^+} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} \leq 1 \qquad \text{for } \mathcal{H}^N\text{-a.e. } x \in \mathsf{X}.$$

In particular, if X is a $\mathrm{ncRCD}(K, N)$ space we have

$$\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}](x) \leq 1 \qquad \forall x \in \mathsf{X}. \tag{4.1.1}$$

Then the following conjecture is raised in [78]:

Conjecture 4.1.1 (De Philippis-Gigli)**.** *If*

$$\mathfrak{m}\left(\{x \in \mathsf{X} : \theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}](x) < \infty\}\right) > 0, \tag{4.1.2}$$

then $\mathfrak{m} = c\mathcal{H}^N$ *for some* $c \in (0, \infty)$*. In particular* $(\mathsf{X}, \mathsf{d}, c^{-1}\mathfrak{m})$ *is a* $\mathrm{ncRCD}(K, N)$ *space.*

Let us make few comments about the statement of the conjecture and its validity.

First of all, we remark that condition (4.1.2) cannot be replaced by the weaker one

$$\{x \in \mathsf{X} : \theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}](x) < \infty\} \neq \emptyset$$

because for instance the metric measure space $([0, \pi], \mathsf{d}_{\mathbb{R}}, \sin^{N-1} t\mathrm{d}t)$ is an $\mathrm{RCD}(N-1, N)$ space, the density θ_N is finite on $\{0, \pi\}$ which is null with respect to the reference measure $\sin^{N-1} t\mathrm{d}t$, and for any $N > 1$, $\sin^{N-1} t\mathrm{d}t$ does not coincides with $c\mathcal{H}^N$ for any $c \in (0, \infty)$.

Moreover let us point out that the Hausdorff dimension of any $\mathrm{CD}(K, N)$ space X is at most N [18]. In this sense, the assumption in Conjecture 4.1.1 amounts at asking for the existence of a 'big' portion of the space with maximal dimension (notice for instance that if $\mathfrak{m} \ll \mathcal{H}^\alpha$ for some $\alpha < N$, then $\theta_N = +\infty$ $\mathfrak{m}$-a.e.). Such 'maximality' of N in the conjecture plays an important role. To see why, consider an n-dimensional weighted Riemannian manifold $(M, g, e^{-V}\mathrm{dVol}_g)$, and recall that for $N \geq 1$ the definition of Bakry-Émery N-Ricci curvature tensor (2.1.3) and the fact that

$$(M, \mathsf{d}_g, e^{-V}\mathrm{Vol}_g) \text{ is an } \mathrm{RCD}(K, N) \text{ space if and only if } \mathrm{Ric}_N \geq Kg. \tag{4.1.3}$$

On the other hand, it is clear that $(e^{-V}\mathrm{Vol}_g)(B_r(x)) \sim r^n$ for every $x \in M$ as $r \to 0^+$, thus assumption (4.1.2) holds if and only if $n = N$, and this information together with $\mathrm{Ric}_N \geq Kg$ forces V to be constant by the very definition of Ric_N.

It is now time to point out that thanks to the main result of [68] - and the aforementioned structural properties - we now know that any $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ admits an *essen-*

tial dimension $n \in \mathbb{N} \cap [1, N]$ by Theorem 2.4.4, meaning in particular that $\mathfrak{m} \ll \mathcal{H}^n \ll \mathfrak{m}$ on the Borel set $\mathcal{R}_n^*$ (recall (2.4.7)), where $\mathfrak{m}(\mathsf{X} \setminus \mathcal{R}_n^*) = 0$. We thus see from general results about differentiation of measures that

$$\text{if (4.1.2) holds, then we have } \theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}] < \infty \quad \mathfrak{m}\text{-a.e.}. \tag{4.1.4}$$

$\mathrm{RCD}(K, N)$ spaces for which $\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}]$ is finite $\mathfrak{m}$-a.e. have been called *weakly non-collapsed* RCD *spaces* in [78], while spaces such that $\theta_N[\mathsf{X}, \mathsf{d}, \mathfrak{m}]$ is finite for *every* point have been called 'non-collapsed' in [77]. It is then clear from (4.1.1) that

$$\text{non-collapsed} \Longrightarrow \text{non-collapsed in the sense of [77]} \Longrightarrow \text{weakly non-collapsed}$$

and from (4.1.4) that proving Conjecture 4.1.1 is equivalent to proving that these three 'non-collapsing conditions' are equivalent (up to multiplying the reference measure by a scalar).

It is known that the conjecture holds true in the following three cases:

1. (X, d) has an upper bound on sectional curvature in a synthetic sense, namely, it is a CAT(κ) space for some $\kappa > 0$: [97]

2. (X, d) is isometric to a smooth Riemannian manifold, possibly with boundary: [98].

3. (X, d) is compact: [45].

The main result of this section is the resolution of Conjecture 4.1.1 in full generality:

Theorem 4.1.2. *Conjecture 4.1.1 holds true.*

We emphasize that our proof also yields the following result, which is of independent interest and will play a prominent role in the proof of Theorem 4.1.2.

Theorem 4.1.3. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space of essential dimension* n*, and let* U *be a connected open subset of* X *with*

$$\inf_{r\in(0,1),x\in A}\frac{\mathfrak{m}(B_r(x))}{r^n}>0 \tag{4.1.5}$$

for any compact subset $A\subseteq U$*. Then the following two conditions are equivalent:*

1. *for every* $f\in D(\Delta)$*,*

$$\Delta f=\mathrm{tr}(\mathrm{Hess}f)\quad \mathfrak{m}\text{-a.e. on } U;$$

2. *for some* $c\in(0,\infty)$*,*

$$\mathfrak{m}\llcorner U=c\mathcal{H}^n\llcorner U.$$

Notice that this has nothing to do with non-collapsing properties and, in particular, it can very well be that the assumption (4.1.5) holds for $U=\mathsf{X}$. Moreover items 1 and 2 may hold only on some $U\subsetneq\mathsf{X}$: just consider the case of a weighted Riemannian manifold as before with V constant on U but non-constant outside U.

Remark 4.1.4. As a consequence of our main result, we obtain that if the Hausdorff dimension of an $\mathrm{RCD}(K,N)$ space is N, then also its topological dimension is N (we refer to [99] for the relevant definitions). Indeed, under this assumption Theorem 2.5.5 and our main result imply that the space is, up to a scalar multiple of the reference measure, a $\mathrm{ncRCD}(K,N)$ space. Then from the Reifenberg flatness around a regular point (see [12] and then [78], [53]) we see that any regular point has a neighbourhood which is homeomorphic to $\mathbb{R}^N$. This proves that the topological dimension is at least N and since in general this is at most the Hausdorff one (see e.g. [100, Theorem 8.14]), our claim is proved.

Finally the main result easily implies the follwowing characterization of non-collapsed spaces:

Theorem 4.1.5 (Characterization of non-collapsed $\mathrm{RCD}(K,N)$ space)**.** *Let* $(\mathsf{X},\mathsf{d},\mathcal{H}^n)$ *be an* $\mathrm{RCD}(K,N)$ *space. Then the following two conditions are equivalent:*

1. $(\mathsf{X}, \mathsf{d}, \mathcal{H}^n)$ *is a non-collapsed* $\mathrm{RCD}(K, n)$ *space.*

2. *For any compact subset* $A \subseteq \mathsf{X}$*, we have*

$$\inf_{x \in A, r \in (0,1)} \frac{\mathcal{H}^n(B_r(x))}{r^n} > 0. \tag{4.1.6}$$

4.1.1 Strategy of proof

The basic strategy we adopt in proving the conjecture is the one introduced by Honda in [45] to handle the compact case. Still, moving from compact to non-compact creates additional technical complications that must be handled: one of the things is to replace the expansion of the heat kernel via eigenfunctions - used in [45] - with suitable decay estimates based on Gaussian bounds. We have done this in Chapter 3. Also, in the course of the proof we obtain (by making explicit some ideas that were implicitly used in [45]) interesting intermediate results that are new even in the smooth context, see in particular formula (4.1.14). Finally, on general RCD spaces X of essential dimension n and $U \subseteq \mathsf{X}$ open we establish relevant links between the properties

- $\mathrm{tr}(\mathrm{Hess} f) = \Delta f$ on $U \subseteq \mathsf{X}$ for every f sufficiently regular,

- $\mathfrak{m} = c\mathcal{H}^n$ on $U \subseteq \mathsf{X}$ for some $c > 0$,

see Theorem 4.1.3 below for the precise statement.

With this said, let us describe the main idea by having a look at the case of a weighted n-dimensional Riemannian manifold $(M, g, e^{-V}\mathrm{dVol}_g)$. Let us consider the reference measure $\mathfrak{m} := e^{-V}\mathrm{Vol}_g$ and the Hausdorff measure $\mathcal{H}^n = \mathrm{Vol}_g$. Assume that $\mathrm{Ric}_N \geq Kg$ for some $K \in \mathbb{R}$ and some $N \in [n, \infty)$ (namely $(M, \mathsf{d}_g, \mathfrak{m})$ is an $\mathrm{RCD}(K, N)$ space, recall (2.1.3) and (4.1.3)). Now notice that the following integration by parts formulas hold: for every $f, \varphi \in C_c^\infty(M)$ we have

$$-\int_M \langle \mathrm{d}f, \mathrm{d}\varphi\rangle \,\mathrm{d}\mathfrak{m} = \int_M \varphi\, \Delta f \,\mathrm{d}\mathfrak{m}, \tag{4.1.7a}$$

$$-\int_M \langle \mathrm{d}f, \mathrm{d}\varphi\rangle \,\mathrm{d}\mathcal{H}^n = \int_M \varphi\, \mathrm{tr}(\mathrm{Hess} f) \,\mathrm{d}\mathcal{H}^n. \tag{4.1.7b}$$

From these identities it is easy to conclude that

$$\mathfrak{m} = c\mathcal{H}^n \qquad \Leftrightarrow \qquad \mathrm{tr}(\mathrm{Hess} f) = \Delta f \quad \forall f \in C_c^\infty(M). \tag{4.1.8}$$

Thus recalling (4.1.4) we see that the desired result will follow if we can show that

$\theta_N[M, \mathsf{d}_g, \mathfrak{m}] < \infty$ a.e. implies that $\mathrm{tr}(\mathrm{Hess} f) = \Delta f$ for any smooth function f.

To see this recall that, as already noticed, having $\theta_N[M, \mathsf{d}_g, \mathfrak{m}](x) < \infty$ for some point $x \in M$ implies that M is N-dimensional (and thus in particular that N is an integer), then recall (4.1.3) and the definition (2.1.3) of the N-Ricci curvature tensor.

This establishes the claim in the smooth setting. In the general case we follow the same general ideas, but we have to deal with severe technical complications. Start observing that the analogue of (4.1.7a) holds in general RCD spaces by the very definition of Δ (see [22]) and that, less trivially, the analogue of (4.1.8) is also in place on $\mathrm{RCD}(K, N)$ spaces of essential dimension equal to N (from the results in [81], using the fact that local dimension is equal to the essential dimension, see also [54]). Thus to conclude along the lines above it is sufficient to prove that (4.1.7b) holds on $\mathrm{RCD}(K, N)$ spaces of essential dimension n. We do not have exactly such result, but have instead the following result which is anyway sufficient to conclude:

Theorem 4.1.6. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n. *Let* $U \subseteq$

X *be bounded open and assume that*

$$\inf_{r\in(0,1),x\in U}\frac{\mathfrak{m}(B_r(x))}{r^n}>0. \tag{4.1.9}$$

Then for every $\varphi\in \mathrm{Lip}(\mathsf{X},\mathsf{d})$*,* $f\in D(\Delta)$ *with* $\mathrm{supp}(\varphi),\mathrm{supp}(f)\subseteq U$ *formula* (4.1.7b) *holds.*

See Theorem 4.2.1 for a slightly sharper statement. Notice also that by the Bishop-Gromov inequality, assumption (4.1.9) holds trivially with $n=N$ for any bounded subset U of a weakly non-collapsed RCD space. Also, the statement above is interesting regardless of the application we just described, and valid also in possibly 'collapsed' RCD spaces.

Thus everything boils down to the proof of such result. The basic idea for the proof is to perform a smoothing of the metric tensor via heat flow. Let us describe the procedure, introduced in [42], in the smooth setting. Consider a compact smooth Riemannian manifold (M,g,dVol_g) and, for every $t>0$, let $\Phi_t:M\to L^2(M,\mathrm{Vol}_g)$ be defined as

$$\Phi_t(x):=(y\mapsto p(x,y,t)),$$

where p is the heat kernel. We can use this map to pull-back the flat metric g_{L^2} of $L^2(M,\mathrm{Vol}_g)$ and obtain the metric tensor $g_t:=\Phi_t^*g_{L^2}$ that is explicitly given by

$$g_t=\int_M \mathrm{d}p(\cdot,y,t)\otimes\mathrm{d}p(\cdot,y,t)\mathrm{dVol}_g(y)\in C^\infty((T^*)^{\otimes 2}M). \tag{4.1.10}$$

The intriguing fact we discussed in Chapter 3 is that, after appropriate rescaling, the tensors g_t converge to the original one g. More precisely, we have

$$\|4(8\pi)^{n/2}t^{(n+2)/2}g_t-g\|_{L^\infty}\to 0 \qquad \text{as } t\to 0^+, \tag{4.1.11}$$

where n denotes the dimension of M. In fact in [42] more is proved, as it is provided the first order Taylor expansion of $t^{(n+2)/2}g_t$, we will discuss the first order expansion, i.e. the second term in the expansion in Chapter 5. A way to read this convergence is via the stability of the heat flow under measured-Gromov-Hausdorff convergence of spaces with Ricci curvature uniformly bounded from below; this observation is more recent than [42], as it has been made by Gigli in [93], still, this is the argument used in the RCD setting so let us present this viewpoint. It is clear that for $M = \mathbb{R}^n$ the tensor g_t is just a rescaling of the Euclidean tensor. On the other hand, denoting by M^λ the manifold M equipped with the rescaled metric tensor λg, and by p^λ the associated heat kernel, it is also clear that $p(x, y, t) = p^\lambda(x, y, \lambda^{-1}t)$. Thus the asymptotics of $p(x, y, t)$ as $t \to 0^+$ corresponds to that of $p^\lambda(x, y, 1)$ as $\lambda \to \infty$ and, as said, these kernels converge to the Euclidean ones where the evolution of the metric tensors g_t is trivial.

Coming back to the RCD setting, we recall that the heat kernel is well-defined in this context [101], and a differential calculus is available in this framework [54]. Thus the same definition as in (4.1.10) can be given and one can wonder whether the same convergence result as in (4.1.11) holds. Interestingly, in this case one has

$$\|t\mathfrak{m}(B_{\sqrt{t}}(\cdot))g_t - c_n g\|_{L^p_{loc}} \to 0 \qquad \text{as } t \to 0^+,\ \forall p \in [1, \infty) \tag{4.1.12}$$

for some constant c_n depending only on the essential dimension of X (this has been proved in [43] for compact $\mathrm{RCD}(K, N)$ spaces, and is generalized to the non-compact setting in Chapter 3). Notice that the loss from convergence in L^p to convergence in L^p_{loc} is unavoidable, but unharmful for our purposes. It is important to remark that the factor in front of g_t is now not constant anymore: this has to do with Gaussian gradient estimates for the heat kernel. Now let $U \subseteq \mathsf{X}$ be open bounded and assume that $\mathcal{H}^n$ is a Radon measure on U (this is always the case if (4.1.9) holds). In this case by standard results about differentiations of measures we have

$$\lim_{t\to 0^+}\frac{t\mathfrak{m}(B_{\sqrt{t}}(\cdot))}{t^{\frac{n+2}{2}}}=c'_n\frac{\mathrm{d}\mathfrak{m}}{\mathrm{d}\mathcal{H}^n}\qquad \mathfrak{m}\text{-a.e. } on\ U.$$

Thus if (4.1.9) holds, from (4.1.12) we deduce that

$$\|t^{\frac{n+2}{2}}g_t-c''_n\tfrac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}g\|_{L^p(U)}\to 0\qquad \text{as } t\to 0^+. \tag{4.1.13}$$

We couple this information with the following explicit computation of the adjoint ∇^* of the covariant derivative of g_t:

$$\nabla^* g_t(x)=-\frac{1}{4}\mathrm{d}_x\Delta_x p(x,x,2t). \tag{4.1.14}$$

This formula was obtained in [45] in the compact setting by expanding the heat kernel via eigenfunctions of the Laplacian. As pointed out in Chapter 3 this approach does not work in our current framework and we have proceeded via a more direct approach based on 'local' Bochner integration.

We are almost done: by explicit computations based on Gaussian estimates one can see that

$$t^{\frac{n+2}{2}}\mathrm{d}_x\Delta_x p(x,x,2t)\rightharpoonup 0\qquad \text{as } t\to 0^+$$

where $\rightharpoonup$ means weak L^p convergence, thus combining this information with (4.1.14), (4.1.13) we conclude that

$$\nabla^*(\tfrac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}g)=0\qquad \text{in } U.$$

This latter equation is a restatement of (4.1.7b) for f,φ with support in U, i.e. this argument gives Theorem 4.1.6, as desired.

4.2 Proof of the conjecture 4.1.1

This section is taken from [41, Section 4]. From both the technical and conceptual points of view, the following is the crucial result in this section. Its proof is basically a combination of the convergence results established in Corollaries 3.4.3, 3.4.5 together with formula (3.3.9):

Theorem 4.2.1 (Integration-by-parts formula)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space of essential dimension* n*. Let also* $U \subseteq \mathsf{X}$ *be open and assume that*

$$\inf_{r\in(0,1),x\in A} \frac{\mathfrak{m}(B_r(x))}{r^n} > 0$$

for every compact subset A *of* U*. Then for any* $\varphi \in \mathrm{Lip}_{bs}(\mathsf{X}, \mathsf{d})$ *with* $\mathrm{supp}(\varphi) \subseteq U$ *and* $f \in D(\Delta)$*, it holds that*

$$\int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f\rangle \mathrm{d}\mathcal{H}^n = -\int_{\mathsf{X}} \varphi \, \mathrm{tr}(\mathrm{Hess} f) \mathrm{d}\mathcal{H}^n.$$

Proof. The assumptions on φ, f ensure that $\varphi \mathrm{d}f$ is in the domain of the covariant derivative with $\nabla(\varphi \mathrm{d}f) = \mathrm{d}\varphi \otimes \mathrm{d}f + \varphi \mathrm{Hess} f$ (see [54, Theorem 3.4.2, Proposition 3.4.5]), with identifications under the Riesz isomorphisms. Thus (3.3.9) gives

$$\begin{aligned} \int_{\mathsf{X}} \langle t^{(n+2)/2} g_t, \nabla(\varphi \mathrm{d}f)\rangle_{\mathsf{HS}} \, \mathrm{d}\mathfrak{m} &= -\frac{1}{4}\int_{\mathsf{X}} \langle \nabla \Delta(t^{(n+2)/2} p_{2t}), \varphi \nabla f\rangle \mathrm{d}\mathfrak{m} \\ &= \frac{1}{4}\int_{\mathsf{X}} \Delta(t^{(n+2)/2} p_{2t}) \mathrm{div}(\varphi \nabla f) \mathrm{d}\mathfrak{m}. \end{aligned} \tag{4.2.1}$$

Let us take the limit $t \to 0^+$ in (4.2.1). The RHS converge to 0 because of Corollary 3.4.5 applied with $A := \mathrm{supp}(\varphi)$. On the other hand by Corollary 3.4.3 applied with $A := \mathrm{supp}(\varphi)$, the LHS of (4.2.1) converges to, up to multiplying by a constant,

$$\int_{\mathsf{X}} \langle g, \nabla(\varphi \mathrm{d}f)\rangle_{\mathsf{HS}} \, \mathrm{d}\mathcal{H}^n = \int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f\rangle \mathrm{d}\mathcal{H}^n + \int_{\mathsf{X}} \varphi \, \mathrm{tr}(\mathrm{Hess} f) \mathrm{d}\mathcal{H}^n.$$

This completes the proof. □

To deduce from the above the equivalence of the 'weak' and 'strong' non-collapsed conditions we shall use the following simple result:

Lemma 4.2.2. *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathrm{RCD}(K,N)$ *space. Also, let* $U\subseteq\mathsf{X}$ *be an open connected set and let* $\xi\in L^\infty_{loc}(U,\mathfrak{m})$. *Assume that for every* $\psi\in\mathrm{Lip}_{bs}(\mathsf{X},\mathsf{d})$ *with support in* U *and* $f\in D(\Delta)$ *it holds*

$$\int_{\mathsf{X}}\xi\langle\nabla\psi,\nabla f\rangle\,\mathrm{d}\mathfrak{m}=-\int_{\mathsf{X}}\xi\psi\Delta f\mathrm{d}\mathfrak{m}. \tag{4.2.2}$$

Then ξ *is constant on* U.

Proof. It suffices to check that ξ is locally constant on U because U is connected. Let $z\in\mathsf{X}$ and $r\in(0,\frac{1}{6})$ with $B_{3r}(z)\subseteq U$ and let $\psi\in\mathrm{Lip}(\mathsf{X},\mathsf{d})$ be identically 1 on $B_{2r}(z)$ and with support in $B_{3r}(z)$. Also, set $\xi_t:=\mathrm{h}_t\left(\chi_{B_{2r}(z)}\xi\right)\in D(\Delta)$, namely $\xi_t(y)=\int_{B_{2r}(z)}p(x,y,t)\xi(x)\mathrm{d}\mathfrak{m}(x)$ for $\mathfrak{m}$-a.e. $y\in\mathsf{X}$ and notice that Hille's theorem (see also Proposition 3.2.4) gives

$$\Delta\xi_t(y)=\int_{B_{2r}(z)}\Delta_y p(x,y,t)\xi(x)\mathrm{d}\mathfrak{m}(x)\overset{(2.6.17)}{=}\int_{B_{2r}(z)}\xi\Delta p_{y,t}\,\mathrm{d}\mathfrak{m}\,.$$

This identity and the assumption (4.2.2) (with $f=p_{y,t}$) give

$$\Delta\xi_t(y)=\int_{\mathsf{X}}(\chi_{B_{2r}(z)}-\psi)\xi\Delta p_{y,t}\,\mathrm{d}\mathfrak{m}-\int_{\mathsf{X}}\xi\langle\nabla\psi,\nabla p_{y,t}\rangle\,\mathrm{d}\mathfrak{m}$$

for $\mathfrak{m}$-a.e. $y\in\mathsf{X}$. Therefore the assumption $\xi\in L^\infty_{loc}(U,\mathfrak{m})$ tells that for $y\in B_r(z)$ we have

$$\begin{aligned}|\Delta\xi_t|(y)&\le C\int_{B_{3r}(z)\setminus B_{2r}(z)}|\Delta p_{y,t}|\,\mathrm{d}\mathfrak{m}+C\int_{B_{3r}(z)\setminus B_{2r}(z)}|\nabla p_{y,t}|\mathrm{d}\mathfrak{m}\\ \text{(by (2.6.12), (2.6.14))}\qquad&\le C\left(t^{-1}+t^{-1/2}\right)\exp\left(-\frac{r^2}{5t}\right)\int_{B_{3r}(z)}\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(x))}\,\mathrm{d}\mathfrak{m}(x),\end{aligned}$$

where C is a positive constant which is independent with t and y. Now notice that (2.4.2) and the assumption $r \in (0, \frac{1}{6})$ ensure that $\frac{1}{\mathfrak{m}(B_{\sqrt{t}}(x))} \leq \frac{C(K,N)}{\mathfrak{m}(B_1(z))} t^{-\frac{N}{2}}$ for every $t \in (0,1)$ and $x \in B_{3r}(z)$. It then follows that $\Delta \xi_t$ uniformly converge to 0 on $B_r(z)$.

Let now $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be with support in $B_r(z)$ and notice that

$$\begin{aligned}\int_{\mathsf{X}} |\mathrm{d}(\varphi \xi_t)|^2 \,\mathrm{d}\mathfrak{m} &= \int_{\mathsf{X}} |\xi_t|^2 |\mathrm{d}\varphi|^2 + 2\xi_t \varphi \langle \mathrm{d}\xi_t, \mathrm{d}\varphi \rangle + |\varphi|^2 |\mathrm{d}\xi_t|^2 \,\mathrm{d}\mathfrak{m} \\ &= \int_{\mathsf{X}} |\xi_t|^2 |\mathrm{d}\varphi|^2 - |\varphi|^2 \xi_t \Delta \xi_t \,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By what we proved we see that the RHS is bounded as $t \to 0^+$, hence the lower semicontinuity of the Cheeger energy ensures that $\varphi\xi \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. Now choose $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ identically 1 on $B_{r/2}(z)$ and with support in $B_r(z)$ and let $\eta \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ be arbitrary with support in $B_{r/2}(z)$. Since $\mathrm{supp}(\eta) \subseteq \{\varphi = 1\}$, from (4.2.2) it follows that

$$\int_{\mathsf{X}} \varphi \xi \langle \nabla \eta, \nabla f \rangle \,\mathrm{d}\mathfrak{m} = -\int_{\mathsf{X}} \eta \xi \varphi \Delta f \mathrm{d}\mathfrak{m} \tag{4.2.3}$$

for any $f \in D(\Delta)$. Moreover, by what we just proved the following computations are justified:

$$-\int_{\mathsf{X}} \varphi \xi \eta \Delta f \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \langle \nabla(\varphi \xi \eta), \nabla f \rangle \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \varphi \xi \langle \nabla \eta, \nabla f \rangle + \eta \langle \nabla(\varphi \xi), \nabla f \rangle \mathrm{d}\mathfrak{m}.$$

This and (4.2.3) imply that $\int_{\mathsf{X}} \eta \langle \nabla \xi, \nabla f \rangle \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \eta \langle \nabla(\varphi\xi), \nabla f \rangle \mathrm{d}\mathfrak{m} = 0$. The arbitrariness of η then gives $\langle \nabla(\varphi\xi), \nabla f \rangle = 0$ $\mathfrak{m}$-a.e. on $B_{r/2}(z)$. Then the density of $D(\Delta)$ in $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ gives $\nabla(\varphi\xi) = 0$ $\mathfrak{m}$-a.e. on $B_{r/2}(z)$. In turn this implies (e.g. from the Sobolev to Lipschitz property) that $\varphi\xi$, and thus ξ, has a representative which is constant in $B_{r/2}(z)$, which is sufficient to conclude. □

We have now all the ingredients to prove the main equivalence.

Proof of Theorem 4.1.3. Under (4.1.5), we can apply Theorem 4.2.1 and deduce the integration-by-parts formula:

$$\int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f \rangle \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m} = - \int_{\mathsf{X}} \varphi \, \mathrm{tr}(\mathrm{Hess} f) \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} \, \mathrm{d}\mathfrak{m} \, ,$$

valid for any $\varphi \in \mathrm{Lip}(\mathsf{X}, \mathsf{d})$ with support in U and any $f \in D(\Delta)$. Now notice that (4.1.5) together with Theorem 2.4.6 imply that $\frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} \in L^\infty_{loc}(U, \mathfrak{m})$. Hence if item 1 holds, we can apply Lemma 4.2.2 with $\xi = \chi_U \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}$ to deduce that item 2 holds as well.

Conversely, if item 2 holds, for all φ and f as above, we have

$$- \int_{\mathsf{X}} \varphi \Delta f \, \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \langle \mathrm{d}\varphi, \mathrm{d}f \rangle \, \mathrm{d}\mathfrak{m} = - \int_{\mathsf{X}} \varphi \, \mathrm{tr}(\mathrm{Hess} f) \, \mathrm{d}\mathfrak{m} \, ,$$

having used item 2 and the integration-by-parts formula in the last step. By the arbitrariness of φ, this proves item 1. □

Proof of Theorem 4.1.2. From the Bishop-Gromov inequality (2.4.2) it easily follows that for any bounded set A of X we have

$$\inf_{r \in (0,1), x \in A} \frac{\mathfrak{m}(B_r(x))}{r^N} > 0. \tag{4.2.4}$$

On the other hand, Theorem 2.5.5 gives that the essential dimension of X is N, thus Theorem 2.5.7 with (2.1.2) shows

$$\Delta f = \mathrm{tr}(\mathrm{Hess} f) \qquad \forall f \in D(\Delta). \tag{4.2.5}$$

Then the conclusion follows from (4.2.4), (4.2.5) and Theorem 4.1.3. □

4.3 Applications

This section is taken from [41, Section 1.3]. The following applications seem to be already known to experts if Theorem 4.1.2 is established (for instance [53] and [69]). However for readers' convenience let us give them precisely. Roughly speaking, they are based on a fact that the space of weakly non-collapsed spaces is open in the space of $\mathrm{RCD}(K, N)$ spaces because of the lower semicontinuity of the essential dimensions with respect to pointed measured Gromov-Hausdorff convergence proved in [77] (Theorem 2.4.9).

It is known that pointed Gromov-Hausdorff (pGH) and pointed measured Gromov-Hausdorff (pmGH) convergences are metrizable (see for instance in [24]). Thus 'ϵ-pGH close' and 'ϵ-pmGH close' make sense as appeared in the following theorem. Note that as the sequential compactness of $\mathrm{RCD}(K, N)$ spaces is known (Theorem 2.3.3), any such metric determines the same compact topology.

The first application is stated as follows.

Theorem 4.3.1. *For any $K \in \mathbb{R}$, any $N \in \mathbb{N}$, any $\delta \in (0, \infty)$ and any $v \in (0, \infty)$, there exists $\epsilon := \epsilon(K, N, \delta, v) \in (0, 1)$ such that if a pointed $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$ is so that $(\mathsf{X}, \mathsf{d}, x)$ is ϵ-pGH close to $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, y)$ for some non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathcal{H}^N)$ with*

$$\mathcal{H}^N(B_1(y)) \geq v, \tag{4.3.1}$$

then we have $\mathfrak{m} = c\mathcal{H}^N$ for some $c \in (0, \infty)$, and moreover $\left|\mathcal{H}^N(B_1(x)) - \mathcal{H}^N(B_1(y))\right| < \delta$.

Next application shows that the non-collapsed condition can be recognized from the point of an infinitesimal view.

Theorem 4.3.2. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. If the essential dimension of some tangent cone $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_Y, y)$ at some point $x \in \mathsf{X}$ is equal to N, then $\mathfrak{m} = c\mathcal{H}^N$ for some $c \in (0, \infty)$.*

Note that the converse implication also holds in Theorem 4.3.2, namely if $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)$ is a non-collapsed $\mathrm{RCD}(K, N)$ space, then any tangent cone at any point is also a pointed non-collapsed $\mathrm{RCD}(0, N)$ space (see Theorem 2.5.3).

The following final application shows that the non-collapsed condition can be also recognized from the asymptotical point of view. Note that the LHS of (4.3.3) exists by the Bishop-Gromov inequality, and does not depend on the choice of $x \in \mathsf{X}$.

Theorem 4.3.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(0, N)$ *space and assume that*

$$\sup_{x \in \mathsf{X}} \mathfrak{m}(B_1(x)) < \infty \tag{4.3.2}$$

and that for some (hence all) $x \in \mathsf{X}$

$$\lim_{r \to \infty} \frac{\mathfrak{m}(B_r(x))}{r^N} > 0. \tag{4.3.3}$$

Then $\mathfrak{m} = c\mathcal{H}^N$ *for some* $c \in (0, \infty)$.

Notice that the assumption (4.3.2) is essential, as this simple example shows: just consider the $\mathrm{RCD}(0, N)$ space $([0, \infty), \mathsf{d}_{\mathbb{R}}, x^{N-1}\mathcal{H}^1)$, which satisfies (4.3.3) but is clearly not non-collapsed. Conversely, any non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)$ satisfies (4.3.2), as a consequence of the Bishop-Gromov inequality and (4.1.1).

We now proceed to the proof of the above results.

Proof of Theorem 4.3.1. From the continuity of $\mathcal{H}^N$ in the compact (as a consequence of Theorem 2.3.3) space of unit balls in $\mathrm{RCD}(K, N)$ spaces stated in Theorem 2.5.4, we see that picking ϵ sufficiently small, the conclusion $\left|\mathcal{H}^N(B_1(x)) - \mathcal{H}^N(B_1(y))\right| < \delta$ holds true. Thus we concentrate on the first part of the claim.

The proof is done by contradiction. If not, there exist a sequence $\epsilon_i \to 0^+$, a sequence of pointed $\mathrm{RCD}(K, N)$ spaces $(\mathsf{X}_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ and a sequence of non-collapsed $\mathrm{RCD}(K, N)$ spaces $(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, \mathcal{H}^N, y_i)$ with $\mathcal{H}^N(B_1(y_i)) \geq v$ such that $(\mathsf{X}_i, \mathsf{d}_i, x_i)$ ϵ_i-pGH close to $(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, y_i)$ and so that $\mathfrak{m}_i$ is not proportional to $\mathcal{H}^N$.

Thanks to Theorem 2.3.3, after passing to a non-relabelled subsequence, there exists a pointed $\mathrm{RCD}(K, N)$ space $(\mathsf{Z}, \mathsf{d}_\mathsf{Z}, \mathfrak{m}_\mathsf{Z}, z)$ such that

$$\left(\mathsf{X}_i, \mathsf{d}_i, \frac{1}{\mathfrak{m}_i(B_1(x_i))}\mathfrak{m}_i, x_i\right) \stackrel{\mathrm{pmGH}}{\to} (\mathsf{Z}, \mathsf{d}_\mathsf{Z}, \mathfrak{m}_\mathsf{Z}, z)$$

and

$$(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, y_i) \stackrel{\mathrm{pGH}}{\to} (\mathsf{Z}, \mathsf{d}_\mathsf{Z}, z).$$

Thanks to Theorem 2.5.3 with (4.3.1), we have

$$(\mathsf{Y}_i, \mathsf{d}_{\mathsf{Y}_i}, \mathcal{H}^N, y_i) \stackrel{\mathrm{pmGH}}{\to} (\mathsf{Z}, \mathsf{d}_\mathsf{Z}, \mathcal{H}^N, z),$$

with $\mathcal{H}^N(B_1(z)) \geq v$. Recalling Theorem 2.5.5, we see that $(\mathsf{Z}, \mathsf{d}_\mathsf{Z}, \mathfrak{m}_\mathsf{Z})$ is weakly non-collapsed, in particular, has essential dimension N. Then the lower semicontinuity statement given by Theorem 2.4.9 gives

$$N \geq \liminf_{i \to \infty} \mathrm{essdim}(\mathsf{X}_i) \geq \mathrm{essdim}(\mathsf{Z}) = N.$$

It follows that $\mathrm{essdim}(\mathsf{X}_i) = N$ for any sufficiently large i. Thus from the characterization of weakly non-collapsed spaces in Theorem 2.5.5 and our main result Theorem 4.1.2 it follows that $\mathfrak{m}_i = c_i\mathcal{H}^N$ for every i sufficiently large. This provides the desired contradiction. □

Proof of Theorem 4.3.2. Let us take $r_i \to 0^+$ with (2.4.6), according to the assumption $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y}, y) \in \mathrm{Tan}(\mathsf{X}, \mathsf{d}, \mathfrak{m}, x)$. As the essential dimension does not change under rescaling as in the LHS of (2.4.6), we see, by Theorem 2.4.9 and the assumption $\mathrm{essdim}(\mathsf{Y}) = N$, that $\mathrm{essdim}(\mathsf{X}) = N$. Thus we conclude by our main result Theorem 4.1.2, taking into account also Theorem 2.5.5. □

Proof of Theorem 4.3.3. Let us take $r_i \to \infty$ and a sequence of rescaled spaces as in the LHS of (2.4.6); by Theorem 2.3.3 (here we use the fact that the space is an $\mathrm{RCD}(K, N)$

space with $K = 0$) we can extract a non relabelled subsequence of $\{r_i\}_i$ such that such rescaled spaces converge to the $\mathrm{RCD}(0, N)$ space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y}, y)$ in the pmGH topology. Therefore, if $z \in \mathsf{Y}$, we take a sequence $\{y_i\}_i \subseteq \mathsf{X}$ that converges to z under this pmGH convergence,

$$\begin{aligned}\frac{\mathfrak{m}_\mathsf{Y}(B_r(z))}{r^N} &= \lim_i \frac{\mathfrak{m}(B_{rr_i}(y_i))}{r^N \mathfrak{m}(B_{r_i}(x))} = \lim_i \frac{\mathfrak{m}(B_{rr_i}(y_i))}{(rr_i)^N} \frac{r_i^N}{\mathfrak{m}(B_{r_i}(x))} \\ &\leq \limsup_i \mathfrak{m}(B_1(y_i)) \frac{r_i^N}{\mathfrak{m}(B_{r_i}(x))} \leq C\end{aligned}$$

where C is independent of r. Here we have used the Bishop-Gromov inequality (2.4.2) for the first inequality and our assumptions for the last inequality. Therefore, using Theorem 2.5.5 we see that $\mathrm{essdim}(\mathsf{Y}) = N$, so that we can conclude as in the proof of Theorem 4.3.2. □

4.4 Further studies

For any connect open set U, the equivalence between

- $\mathrm{trHess} f = \Delta f$, $\mathfrak{m}$-a.e. in U
- $m \llcorner U = \mathcal{H}^n \llcorner U$

should hold without assuming the volume ratio bound (4.1.5), since the bound is only for technical use when we take the limit $t^{(n+2)/2} g_t \to c_n g$. In particular, from the equivalence established in Theorem 2.5.7, it is expected that the local Bochner inequality (2.5.2) should implies (4.1.5), hence $m \llcorner U = \mathcal{H}^n \llcorner U$. But there is no clear argument to prove (4.1.5) under (2.5.2).

Another question raised by Honda in [102] is also closely related to non-collapsed spaces. We recall the notion of non-collapsing convergence.

We start by the following example. For every $\varepsilon > 0$ the metric horn constructed by Cheeger and Colding is $(\mathbb{S}^4 \times \mathbb{R}_{\geq 0}, \mathsf{d}_{g_\varepsilon}, \nu_\varepsilon, p)$ where $g_\varepsilon = \mathrm{d}t^2 + \left(\frac{t^{1+\varepsilon}}{2}\right)^2 \mathrm{d}g_{\mathbb{S}^4}$, p is the tip, and $g_{\mathbb{S}^4}$ is the round metric on $\mathbb{S}^4$. This is a (collapsed) Ricci limit space coming from the sequence of manifolds that are topologically $\mathbb{R}^8$ with non-negative Ricci curvature. The question is, can it be an (intrinsically) non-collapsed $\mathrm{RCD}(0,5)$ or $\mathrm{RCD}(K,5)$ space for any $K \in \mathbb{R}$. The answer is no. Theorem 4.1.5 asserts that for $(\mathbb{S}^4 \times \mathbb{R}_{\geq 0}, \mathsf{d}_{g_\varepsilon}, \mathcal{H}^5)$ to be non-collapsed $\mathrm{RCD}(K,5)$, $\mathcal{H}^5$ must satisfy that for any compact subset A,

$$\inf_{x \in A, r \in (0,1)} \frac{\mathcal{H}^5(B_r(x))}{r^5} > 0.$$

However, at the tip p, from the expression of metric g_ε we see that for any $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{\mathcal{H}^5(B_r(p))}{r^5} = 0.$$

This shows that the metric horn is "intrinsically" collapsed. On the other hand, there are trivial examples such as $\mathbb{S}^1(1) \times \mathbb{S}^1(r) \xrightarrow{r \to 0} \mathbb{S}^1(1)$, where the limit space $\mathbb{S}^1(1)$ coming from a collapsing sequence, but it is intrinsically non-collapsed.

In RCD context, we also have the notion of collapsing and non-collapsing convergence, it is exactly Theorem 2.5.3([78, Theorem 1.2]), we now restate it to fit into our current situation.

Theorem 4.4.1. *Let* $(\mathsf{X}_i, \mathsf{d}_i, \mathcal{H}^N, x_i)$ *be a sequence of pointed* $\mathrm{ncRCD}(K,N)$ *spaces. Assume that* $(\mathsf{X}_i, \mathsf{d}_i, x_i)$ *pGH converges to* $(\mathsf{X}, \mathsf{d}, x)$*, then exactly one of the following happens:*

- *Non-collapsing convergence:* $\limsup_{i \to \infty} \mathcal{H}^N(B_1(x_i)) > 0$. *In this case the* $\limsup$ *is actually a limit and the convergence is in pmGH topology to* $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N, x)$*, moreover, it is a pointed* $\mathrm{ncRCD}(K,N)$ *space.*
- *Collapsing convergence:* $\lim_{i \to \infty} \mathcal{H}^N(B_1(x_i)) = 0$. *In this case there is a dimension gap* $\dim_{\mathcal{H}}(\mathsf{X}) \leq N-1$.

where $\dim_{\mathcal{H}}$ *is the Hausdorff dimension.*

Compare to Definition 2.5.1. In the first case of the theorem we can call the sequence non-collapsing and in the second case collapsing. How can one identify the intrinsically non-collapsed ones among all the Ricci limit spaces, or $\mathrm{RCD}(K, N)$ spaces coming from a collapsing sequence? Honda made the following conjecture ([102, Conjecture 4.2]),

Conjecture 4.4.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space. If* $\mathfrak{m} = b\mathcal{H}^k$ *for some* $b > 0$ *and* $k \in \mathbb{N}$*, then* $(\mathsf{X}, \mathsf{d}, \mathcal{H}^k)$ *is an* $\mathrm{RCD}(K, k)$ *space.*

Clearly, according to the structure results in Section 2.4, k must be the essential dimension of X.

CHAPTER 5

APPROXIMATE EINSTEIN TENSOR AND NON-COLLAPSED SPACES

5.1 Overview

This chapter is mainly a review of the joint work with Honda [40]. A partial new proof to the main theorem is added. To motivate our study, we consider Einstein tensor in the classical setting. For a closed Riemannian manifold (M^n, g), the *Einstein tensor* G_g is defined by

$$G_g := \mathrm{Ric}_g - \frac{1}{2}\mathrm{Scal}_g g, \tag{5.1.1}$$

where Ric_g and Scal_g denote the Ricci and the scalar curvature, respectively. It is well-known that G_g is divergence free:

$$\nabla^* G_g = 0 \tag{5.1.2}$$

which is a direct consequence of the second Bianchi identity.

The main purpose of this chapter is to establish (5.1.2) for *non-collapsed* $\mathrm{RCD}(K,N)$ *spaces.* More precisely, for a compact $\mathrm{RCD}(K,N)$ space $(X, \mathsf{d}, \mathfrak{m})$, (5.1.2) holds in some sense as explained below if and only if $(X, \mathsf{d}, c\mathfrak{m})$ is non-collapsed for some positive constant c. It is worth pointing out that our argument allows us to provide a new proof of (5.1.2) even for a closed Riemannian manifold (M^n, g) without using the Bianchi identity.

This characterization of non-collapsed spaces is discovered when studying the second term in the short time expansion of $t^{(n+2)/2}g_t$. Since we do not a-priori know the existence of the second term, we instead study the family of tensors indexed by t:

$$\frac{c_n t^{(n+2)/2} g_t - \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} g}{t} \tag{5.1.3}$$

called the *approximate Einstein tensor* of $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, the desired second term of the short time expansion of $t^{(n+2)/2}g_t$ would be the limit of (5.1.3) if any existence of limit is guaranteed. In order to make sense of (5.1.2) in the metric measure setting, we propose a weaker divergence free property, called asymptotically divergence free, refer to Definition 5.3.5 for the precise definition. In connection with the smooth case, it is natural ask when (5.1.3) is weakly asymptotically divergence free, that,

$$\lim_{t\to 0^+}\int_{\mathsf{X}}\left\langle \frac{c_n t^{(n+2)/2}g_t - \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}g}{t}, \nabla\omega\right\rangle \mathrm{d}\mathfrak{m} = 0. \tag{5.1.4}$$

holds for a regular and large enough of 1-forms ω.

Before stating the main result of this chapter, recall that $D(\Delta_{H,1})$ and $D(\delta)$ denote the domain of the Hodge Laplacian $\Delta_{H,1} = \delta\mathrm{d} + \mathrm{d}\delta$ on 1-forms defined in [54] and the domain of the adjoint operator $\delta = \mathrm{d}^*$ of the exterior derivative d on 1-forms, respectively.

Theorem 5.1.1 ("Weakly asymptotically divergence free" characterizes the non-collapsed condition)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(K, N)$ *space of essential dimension* n*. The following two conditions are equivalent:*

1.

$$\inf_{x\in A}\frac{\mathfrak{m}(B_r(x))}{r^n} > 0 \tag{5.1.5}$$

for any compact subset $A \subseteq \mathsf{X}$*, and (5.1.4) for any* $\omega \in D(\Delta_{H,1})$ *compactly supported, with* $\Delta_{H,1}\omega \in D(\delta)$*.*

2.

$$\mathfrak{m} = c\mathcal{H}^n, \tag{5.1.6}$$

for some constant $c > 0$*.*

Since the space $\{\omega \in D(\Delta_{H,1}) : \omega$ has compact support, $\Delta_{H,1}\omega \in D(\delta)\}$ is dense in the space of L^2-1-forms, (5.1.4) can be interpreted as that the approximate Einstein tensor (5.1.3) is actually weakly asymptotically divergence free. See also Appendix B (Corollary

B.0.4). Let us remark that (5.1.6) implies that $(X, \mathsf{d}, \mathcal{H}^n)$ is a non-collapsed $\mathrm{RCD}(K, n)$ space.

The following is a direct consequence of Theorem 5.1.1 in the smooth context, which is also new (recall $e^{-f}\mathrm{vol}_g(A) = \int_A e^{-f}\mathrm{dvol}_g$):

Corollary 5.1.2. *Let $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ be a closed weighted Riemannian manifold. Then there exists a $G_{g,f} \in C^\infty((T^*)^{\otimes 2}M^n)$ called the weighted Einstein tensor such that the following expansion holds,*

$$c_n t^{(n+2)2} g_t = e^f g - \frac{2t}{3} G_{g,f} + O(t^2) \quad (t \to 0^+). \tag{5.1.7}$$

Moreover, f is constant if and only if $G_{g,f}$ is divergence free with respect to $\mathrm{vol}_{g,f}$, that is,

$$\int_{M^n} \langle G_{g,f}, \nabla\omega \rangle \mathrm{dvol}_{g,f} = 0 \tag{5.1.8}$$

holds for any $\omega \in C^\infty(T^*M^n)$.

It is worth noticing that although the left hand side of (5.1.4) converges as $t \to 0^+$, the approximate Einstein tensor itself (5.1.3) may not L^2-converge to a limit tensor in general. This is because lack of L^2 bounds, see section 5.3.4 for the explicit construction of a non-collapsed $\mathrm{RCD}(K, 3)$ space with $K > 1$ such that the L^2 norm of (5.1.3) tends to $+\infty$ as $t \to 0^+$. On the other hand, assuming the uniform L^2 bound, we can prove that all limit tensors are actually divergence free as follows, which is an easy consequence of Theorem 5.1.1.

Corollary 5.1.3. *Let $(X, \mathsf{d}, \mathcal{H}^n)$ be a compact non-collapsed $\mathrm{RCD}(K, n)$ space. If*

$$\sup_{0<t<1} \left\| \frac{c_n t^{(n+2)/2} g_t - g}{t} \right\|_{L^2} < \infty \tag{5.1.9}$$

holds, then any $G \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathcal{H}^n))$ that is a L^2-weak limit of some subsequence of

$$\frac{c_n t^{(n+2)/2} g_t - g}{t} \tag{5.1.10}$$

as $t \to 0^+$ satisfies $G \in D(\nabla^*)$ with $\nabla^* G = 0$, where $D(\nabla^*)$ denotes the domain of the divergence operator ∇^*.

Applying Corollary 5.1.3 to a closed Riemannian manifold $(M^n, \mathsf{d}^g, \mathrm{vol}^g)$ gives a new proof of (5.1.2) without using the Bianchi identity.

5.1.1 Strategy of proof

We will first provide a direct proof of this corollary with an explicit formula for $G_{g,f}$ in the next section, see Proposition 5.2.7 for the main statement. The key of the proof in the smooth context is to construct the local parametrix for the weighted Laplacian and find the short time expansion for the weighted heat kernel. To this end the factor $A(x,y) := \frac{f(x)+f(y)}{2}$ is added to the equation (5.2.12), which becomes

$$(\Delta_{f,x} - \partial_t) S_k = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathsf{d}_g(x,y)^2}{4t}\right) \cdot t^k \cdot \Delta_{f,x}\left(u_k e^A\right)$$

Where u_j are some undetermined smooth functions. The computation to find the short time expansion of weighted heat kernel is essentially the same as that of the Minakshisundaram-Pleijel expansion formula for (unweighted) heat kernel, which is in many literature, to name some, see for example [103] and [104]. The point is to compare the coefficients for each power of t to derive a recurrence formula for the undetermined u_j as in Lemma 5.2.2. We emphasize that in this computation the completeness is not needed, given that the nature of this expansion is local.

Once we have the short time expansion for the weighted heat kernel (5.2.20), we can follow the computation of [42] closely to find out the second term in the short time expansion $t^{(n+2/2)}g_t$ for closed weighted manifolds, which we call the weighted Einstein tensor

$$G_{g,f} =:= e^f G_g - \frac{3e^f}{2}\left(\mathrm{d}f\otimes\mathrm{d}f + \Delta f g - \frac{|\nabla f|^2}{2}g\right)$$

The computation is done in Theorem 5.2.4. The key point is to compute the symmetric second derivative, i.e. $\mathrm{d}y\,\mathrm{d}x$ of weighted heat kernel p_f. This allows us to check directly that the weighted divergence $\nabla^* G_{g,f}$ vanishes if and only if f is a constant, we see that the weighted divergence $\nabla_f^* G_{g,f} = 0$ is equivalent to $\Delta f \mathrm{d}f + \mathrm{d}\Delta f = 0$. We can multiple by an integrating factor e^f to see that $\mathrm{d}\big(e^f\Delta f\big) = e^f(\Delta f\mathrm{d}f + \mathrm{d}\Delta f) = 0$ which means $e^f\Delta f$ is constant, but this is impossible unless the the constant is 0, which in turn implies that f is harmonic (w.r.t. the Laplace-Beltrami operator, i.e. the unweighted one) and a harmonic function is constant on any closed manifold, we concludes the proof.

For the proof in RCD context, we make use of the $\nabla^* g_t$. i.e. , (3.3.9), Although all the essential facts about g_t are discussed in great details in both Chapter 3 and Chapter 4, to keep the presentation of this chapter independent, we recall again that (3.3.9) can be restated for $\omega\in H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ as

$$\int_{\mathsf{X}}\langle g_t,\nabla\omega\rangle\mathrm{d}\mathfrak{m} = -\frac{1}{4}\int_{\mathsf{X}}\langle\omega,\mathrm{d}_x\Delta_x p(x,x,2t)\rangle\mathrm{d}\mathfrak{m}.$$

Also, under 5.1.5, we can restate Theorem 4.2.1 as (recall in the overview of Chapter 4 that formally $\nabla^*(\frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}g) = 0$)

$$\int_{\mathsf{X}}\mathrm{tr}\nabla\omega\mathrm{d}\mathcal{H}^n = 0,$$

for $\omega\in H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ with compact support. The difficulty for the current situation is that, we are now dealing with $t^{n/2}g_t$, and we do not have enough knowledge of the asymptotic behavior of $t^{n/2}\mathrm{d}\Delta p(x,x,2t)$ as $t\to 0$, but we understand that of $t^{n/2}p(x,x,2t)$. So we apply all the differential operators to compactly supported 1-form ω, after imposing suitable regularity on ω. Now we see that

$$\langle t^{n/2} g_t, \nabla\omega\rangle = t^{n/2} p(x,x,t)\delta(\Delta_{H,1}\omega)$$

for regular enough ω. Recall the short time behavior of the heat kernel (2), after taking $t \to 0$ we get that

$$\int_{\mathsf{X}} \delta(\Delta_{H,1}\omega) \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} \mathrm{d}\mathfrak{m} = 0,$$

which is enough to conclude that $\frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}$ is a constant.

5.2 Smooth context

This section is taken from [40, Section 3]. Throughout the section we fix a smooth weighted (not necessarily complete) Riemannian manifold without boundary $(M^n, g, \mathrm{vol}_{g,f})$, where $f \in C^\infty(M^n)$, and for any Borel subset A of M^n,

$$\mathrm{vol}_{g,f} := \int_A e^{-f} \mathrm{dvol}_g. \tag{5.2.1}$$

Recall that $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ is an $\mathrm{RCD}(K,N)$ space if and only if $n \le N$, the Bakry-Émery N-Ricci tensor satisfies:

$$\mathrm{Ric}_N \ge Kg, \tag{5.2.2}$$

and (M^n, d_g) is a complete metric space. If $n = N$ holds, then (5.2.2) is understood as that f is constant and that $\mathrm{Ric}_g \ge Kg$ holds. In particular if M^n is closed, then for any $N > n$ there exists $K \in \mathbb{R}$ such that $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ is an $\mathrm{RCD}(K,N)$ space whose essential dimension is trivially equal to n.

We first discuss the *Dirichlet Laplacian* on $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ without assuming completeness of (M^n, d_g). To be precise, let us clarify the meaning of the Dirichlet heat kernel p_f of $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ for the reader's convenience.

Let $H_0^{1,2}(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ denote the completion of $C_c^\infty(M^n)$ with respect to the $H^{1,2}$-

norm and let $h_{f,t}$ denotes the associated semigroup, so-called the *heat flow* associated with the Dirichlet weighted Laplacian Δ_f:

$$\Delta_f\varphi := \mathrm{tr}(\mathrm{Hess}_\varphi) - g(\nabla f, \nabla\varphi), \tag{5.2.3}$$

that is, for any $\varphi \in L^2(M^n, \mathrm{vol}_{g,f})$, $h_{f,t}\varphi \in C^\infty(M^n) \cap H_0^{1,2}(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} h_{f,t}\varphi = \Delta_f^g h_{f,t}\varphi \quad \text{in } L^2(M^n, \mathrm{vol}_{g,f}), \tag{5.2.4}$$

and that $h_{f,t}\varphi \to \varphi$ in $L^2(M^n, \mathrm{vol}_{g,f})$ as $t \to 0^+$. The existence of such a semigroup is known given that Δ_f is self-adjoint, see [105, Theorem. 4.9]. Then the Riesz representation theorem yields that for any $t \in (0, \infty)$ and any $x \in M^n$, there exists a unique $p_{t,x} \in L^2(M^n, \mathrm{vol}_{g,f})$ such that

$$h_{f,t}\varphi(x) = \int_{M^n} p_{t,x}(y)\varphi(y)\mathrm{dvol}_{g,f} \tag{5.2.5}$$

holds for any $\varphi \in L^2(M^n, \mathrm{vol}_{g,f})$. Then the *heat kernel* $p_f(x, y, t)$ is defined by

$$p_f(x, y, t) := \int_{M^n} p_{t/2,x}(z)p_{t/2,y}(z)e^{-f(z)}\mathrm{dvol}_g(z) \tag{5.2.6}$$

which is smooth on $M^n \times M^n \times (0, \infty)$, see [105, Definition 7.12].

From now on, let $(r, \xi^1, \xi^2, \ldots, \xi^n) := (r, \xi)$ be the normal coordinates around $x \in M^n$, and $g(r, \xi)$ be the Riemannian metric at the point (r, ξ) in the normal coordinates. We introduce the following elementary lemma which will play a role later.

Lemma 5.2.1. *For any $x \in M^n$ we have the following asymptotic expansion as $r \to 0^+$*

$$\mathrm{vol}_{g,f}(B_r(x)) = \omega_n r^n e^{-f(x)} \left(1 - \frac{\mathrm{Scal}_g + 3\Delta f - 3|\nabla f|^2}{6(n+2)} r^2 + O(r^3)\right). \tag{5.2.7}$$

Moreover, the asymptotic behavior (5.2.7) is uniform for any compact subset $K \subseteq M^n$ in *the sense that*

$$\sup_{x\in K, r<1} r^{-3-n}\left|\mathrm{vol}_{g,f}(B_r(x)) - \omega_n r^n e^{-f(x)}\left(1-\tfrac{\mathrm{Scal}_g+3\Delta f-3|\nabla f|^2}{6(n+2)}r^2\right)\right| < \infty. \quad (5.2.8)$$

Proof. Recall that for any unit vector $v \in T_xM$ and any geodesic γ emanating from x with $\dot\gamma(0) = v$, it follows from Taylor expansion at $x = \gamma(0)$ that

$$\sqrt{\det g(\gamma(t))} = 1 - \frac{\mathrm{Ric}_g(v,v)}{6}t^2 + O(t^3), \quad (5.2.9)$$

$$e^{-f(\gamma(t))+f} = 1 - \langle \nabla f, v\rangle t + \frac{1}{2}\left(|\langle \nabla f, v\rangle|^2 - \mathrm{Hess}_f(v,v)\right)t^2 + O(t^3). \quad (5.2.10)$$

Thus we have

$$\begin{aligned}\mathrm{vol}^g_f(B_r(x)) = &\int_0^r\int_{S^{n-1}}\left(1-\frac{(\mathrm{Ric}_g)_{ij}}{6}\xi^i\xi^j t^2 + O(t^3)\right)\cdot \\ &\left[1-\nabla f_i\xi^i t + \frac{1}{2}\left((\mathrm{d}f\otimes\mathrm{d}f - \mathrm{Hess}_f)_{ij}\right)\xi^i\xi^j t^2 + O(t^3)\right]e^{-f}t^{n-1}\mathrm{d}\xi\mathrm{d}t \\ &= \omega_n r^n e^{-f(x)}\left(1-\frac{\mathrm{Scal}_g+3\Delta f-3|\nabla f|^2}{6(n+2)}r^2 + O(r^3)\right)\end{aligned}$$

as desired, where Hess_f, $\mathrm{d}f\otimes\mathrm{d}f$, ∇f and Ric_g are all evaluated at x. By expanding the left hand side of (5.2.9) and (5.2.10) to the t^3 or higher order terms, we can infer that the coefficients involve the derivatives of the Riemannian curvature tensor, and the derivatives of f, respectively. Since they are all smooth objects, they are uniformly bounded on any compact set K, the uniform bound (5.2.8) then follows. □

5.2.1 The weighted heat kernel expansion

For each $y \in M^n$, choose $\epsilon_y = \mathrm{inj}(y)/2$, where $\mathrm{inj}(y)$ denotes the injective radius at y, and consider

$$V = \{(x,y) \in M^n \times M^n : \mathsf{d}_g(x,y) < \epsilon_y\}.$$

Fix $k \in \mathbb{Z}_{>0}$, we seek for $u_j \in C^\infty(V)$, $j = 1, 2, \dots, k$ such that

$$(\Delta_{f,x} - \partial_t) S_k = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathsf{d}_g(x,y)^2}{4t}\right) \cdot t^k \cdot \Delta_{f,x}\left(u_k e^A\right), \quad \forall (x,y) \in V \tag{5.2.11}$$

holds, where $A = A(x,y) = \frac{f(x)+f(y)}{2}$ and

$$S_k(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathsf{d}_g(x,y)^2}{4t} + A(x,y)\right) \cdot \sum_{j=0}^{k} t^j u_j(x,y). \tag{5.2.12}$$

We claim that u_j satisfies the following recurrence formula.

Lemma 5.2.2. *We have*

$$\begin{aligned} u_0(x,y) =& D^{-\frac{1}{2}}(y) \\ u_j(x,y) =& \mathsf{d}_g(x,y)^{-j} D^{-1/2}(y) \left[\int_0^{\mathsf{d}_g(x,y)} D^{1/2}(\gamma(s)) \Delta_{\gamma(s)} u_{j-1}(x,\gamma(s)) s^{j-1} \mathrm{d}s \right. \\ &\left. + \int_0^{\mathsf{d}_g(x,y)} D^{1/2}(\gamma(s)) \left(\frac{1}{2}\Delta f(\gamma(s)) - \frac{1}{4}|\nabla f(\gamma(s))|^2\right) u_{j-1}(x,\gamma(s)) s^{j-1} \mathrm{d}s \right] \end{aligned} \tag{5.2.13}$$

where $j \geq 1$ and γ is the unit speed minimal geodesic from x to y, and $D(y) = \frac{\sqrt{\det g(r,\xi)}}{\mathsf{d}_g(x,y)^{n-1}}$ which is the volume density at y in normal coordinates (r,ξ) around x.

Proof. From (5.2.11) with (5.2.3), we obtain that (5.2.11) is equivalent to

$$\begin{aligned} 0 &= \mathsf{d}_g(x,y)\partial_r u_0 + \frac{\mathsf{d}_g(x,y)}{2}\frac{\partial_r D}{D} u_0 \\ 0 &= \mathsf{d}_g(x,y)\partial_r u_j + \left(j + \frac{\mathsf{d}_g(x,y)}{2}\frac{\partial_r D}{D}\right) u_j - \Delta u_{j-1} - \left(\frac{1}{2}\Delta f - \frac{1}{4}|\nabla f|^2\right) u_{j-1} \end{aligned} \tag{5.2.14}$$

where $j \geq 1$ and $r = \mathsf{d}_g(x,y)$ and ∂_r is the radial derivative from x, we give a sketch of this computation. Solve the first equation of (5.2.14), to get $u_0(x,y) = C(\xi) D^{-\frac{1}{2}}(y)$, note that $u_0(x,x) = 1$, so $C(\xi) = 1$, then we get the first equality of (5.2.13). To yield the second equation of (5.2.13), we first solve the corresponding homogeneous equation of the second equation of (5.2.14), which is

$$\mathsf{d}_g(x,y)\partial_r u_j + \left(j + \frac{\mathsf{d}_g(x,y)}{2}\frac{\partial_r D}{D}\right) u_j = 0, \tag{5.2.15}$$

then we use the method of variation of parameters to finish the computation. □

Now we extend S_k to whole $M^n \times M^n$ by multiplying a cut-off function $\varphi(x,y) \in C^\infty(M^n \times M^n)$ so that for each $y \in M^n$, $\varphi(x,y) = 0$ on $X \setminus B_{\epsilon_y}(y)$, $\varphi(x,y) = 1$ on $B_{\epsilon_y/2}(y)$ and $0 \le \varphi(x,y) \le 1$. Let

$$H_k(x,y,t) := \varphi(x,y) S_k(x,y,t) \in C^\infty\left(M^n \times M^n \times (0,\infty)\right). \tag{5.2.16}$$

The following properties are known for H_k:

1. $(\partial_t - \Delta_f^g) H_k \in C^\ell(M^n \times M^n \times [0,\infty))$ for any integer $\ell < k - \frac{n}{2}$;

2. For every $x \in M^n$, $H_k(x,y,t) \to \delta_y(x)$ for all $y \in M^n$.

See [103, p. 152 Lemma 1], and [104, Lemmma 3.18]. Note that in both references compactness and completeness of M^n are assumed, but it is irrelevant here since the computation is local. In particular it implies that H_k is a parametrix of p_f when $k > \frac{n}{2} + 2$.

We are now in position to establish the following asymptotic expansion of p_f. It is worth pointing out that (5.2.21) is established in [106] with a slightly different normalization of the heat kernel.

For the proof, we introduce the (weighted) convolution $F * H$ for $F, H \in C^0(M^n \times M^n \times (0,\infty))$:

$$F * H(x,y,t) = \int_0^t \int_M F(x,z,s) H(z,y,t-s) e^{-f} \mathrm{dvol}_g(z) ds,$$

and denote $H^{*j} = H * H * \cdots * H$ for j-fold convolution. Let

$$F_k = \sum_{j\geq 0}(-1)^{j+1}((\partial_t - \Delta_f)H_k)^{*j}. \tag{5.2.17}$$

It is also proved that $F_k \in C^\ell(M^n \times M^n \times [0,\infty))$ for any integer $\ell < k - \frac{n}{2}$, see for instance [103, p. 154]. It follows from a direct computation and induction that for any $t_0 > 0$ and any compact subset $K \subseteq M^n$,

$$\left\|(\partial_t - \Delta_f^g)H_k(\cdot,\cdot,t)\right\|_{L^\infty(K\times K)} < C(K)t^{k-n/2}, \quad \forall t \in [0,t_0]. \tag{5.2.18}$$

$$\|F_k(\cdot,\cdot,t)\|_{L^\infty(K\times K)} < C(K)t^{k-n/2}, \quad \forall t \in [0,t_0]. \tag{5.2.19}$$

Theorem 5.2.3. *For any $y \in M^n$ there exists $\epsilon > 0$ such that for any $x \in B_{\epsilon/2}(y)$, the heat kernel $p_f(x,y,t)$ has the following asymptotic expansion:*

$$p_f(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\mathsf{d}_g(x,y)^2}{4t} + A(x,y)\right)\left(\sum_{j=0}^{k} t^j u_j(x,y) + O(t^{k+1})\right) \tag{5.2.20}$$

as $t \to 0^+$. Moreover if $x = y$, then the expansion is uniform in the sense of Lemma 5.2.1, that is there exists $t_0 > 0$ such that (5.2.23) holds. In particular, we have

$$u_1(x,x) = \frac{\mathrm{Scal}_g(x)}{6} - \frac{1}{2}\Delta f(x) + \frac{1}{4}|\nabla f(x)|^2. \tag{5.2.21}$$

Proof. It is shown above that S_k hence H_k has this expansion. From the fact that for every $k > \frac{n}{2}+2$, H_k is a parametrix, (5.2.18) and (5.2.19), we infer that for every $k > \frac{n}{2}+2$, $p_f = H_k - H_k * F_k \in C^{k-\frac{n}{2}}(M^n \times M^n \times (0,\infty))$ (see [104, Thm 3.22]), hence $p_f \in C^\infty(M^n \times M^n \times (0,\infty))$. For x, y such that $\mathsf{d}_g(x,y) \leq \epsilon_y/2$, from (5.2.16) and (5.2.19) it holds that for every $k > \frac{n}{2}+2$ and every compact set K,

$$\|H_k * F_k\|_{L^\infty(K\times K)} < Ct^{k+1-\frac{n}{2}} \qquad \forall t \in [0,t_0], \tag{5.2.22}$$

where t_0 is in (5.2.19). Apply the inequality (5.2.22) to yield that $p_f(x,y,t) = S_k(x,y,t) + O(t^{k+1-n/2})$, so p_f has the same expansion as S_k up to order k.

When $x = y$ we have that for each integer $k \geq 1$, and compact subset $K \subseteq M^n$, we have

$$\sup_{x\in K, t<t_0} t^{\frac{n}{2}-k}\left|p_f(x,x,t) - \frac{1}{(4\pi t)^{n/2}}e^{f(x)}\sum_{j=0}^{k-1} t^j u_j(x,x)\right| < \infty. \tag{5.2.23}$$

For the computation of u_1, recall in (5.2.13), we found that $u_0(x,y) = D^{-1/2}(y)$. Let γ be as in Lemma 5.2.2, with (5.2.9) we have

$$u_0(x,y) = 1 + \frac{1}{12}\mathrm{Ric}(\dot\gamma(0),\dot\gamma(0))\mathsf{d}_g(x,y)^2 + O(\mathsf{d}_g(x,y)^3), \tag{5.2.24}$$

in particular $u_0(x,x) = 1$. Then it follows that $\Delta u_0(x,x) = \mathrm{Scal}_g(x)/6$. Finally letting $y \to x$ in the second equation of (5.2.13) for $j = 1$ leads to

$$u_1(x,x) = \Delta u_0(x,x) + \frac{1}{2}\Delta f(x) - \frac{1}{4}|\nabla f(x)|^2 = \frac{\mathrm{Scal}_g(x)}{6} + \frac{1}{2}\Delta f(x) - \frac{1}{4}|\nabla f(x)|^2.$$

□

5.2.2 Divergence free property of the weighted Einstein tensor on a closed manifold

From now on we make a further assumption that M^n is a closed manifold. Let us consider the heat kernel embedding:

$$\Phi_{f,t} : M^n \hookrightarrow L^2(M^n, \mathrm{vol}_{g,f}) \tag{5.2.25}$$

defined by

$$x \mapsto (y \mapsto p_f(x,y,t)). \tag{5.2.26}$$

Put $g_{f,t} := (\Phi_{f,t})^* g_{L^2}$.

To study the second term of $g_{f,t}$ along the same way as in [42], it is necessary to generalize the heat kernel expansion in [42, p.380] to weighted Riemannian manifolds. We claim:

Theorem 5.2.4 (Weighted version of Bérard-Besson-Gallot theorem)**.** *We have the following asymptotic formula as* $t \to 0^+$

$$c_n t^{(n+2)/2} g_{f,t} = e^f g - e^f \left(\frac{2}{3} G_g - \mathrm{d}f \otimes \mathrm{d}f - \Delta f g + \frac{|\nabla f|^2}{2} g \right) t + O(t^2), \quad (5.2.27)$$

where the convergence is uniform, that is,

$$\sup_{x\in M^n, t<1} \left| t^{-2} \left(c_n t^{(n+2)/2} g_{f,t} - \left(e^f g - e^f \left(\tfrac{2}{3} G_g - \mathrm{d}f \otimes \mathrm{d}f - \Delta f g + \tfrac{|\nabla f|^2}{2} g \right) t \right) \right) \right| (x) < \infty. \quad (5.2.28)$$

In particular, we have the uniform convergence:

$$\left\| \frac{c_n t^{(n+2)/2} g_{f,t} - e^f g}{t} - e^f \left(-\frac{2}{3} G_g + \mathrm{d}f \otimes \mathrm{d}f + \Delta f g - \frac{|\nabla f|^2}{2} g \right) \right\|_{L^\infty} \to 0. \quad (5.2.29)$$

Proof. By (3.3.1), which remains valid on weighted manifolds because of the characterization (5.2.2) for being an $\mathrm{RCD}(K, N)$ space, and the fact that the set of eigenfunctions $\{\varphi_i\}_{i\geq 0}$ forms an orthonormal basis of $L^2(M^n, \mathrm{vol}_{g,f})$, we see that for every $x \in M^n$ and $v \in T_x M^n$,

$$g_{f,t}(v, v) = \sum_i e^{-2\lambda_i t} |\mathrm{d}_x \varphi_i(v)|^2 = (\partial_y \partial_x p_f)_{(x,x,2t)}(v, v) =: (\mathrm{d}_S p_f)_{(x,x,2t)}(v, v) \quad (5.2.30)$$

where we used a fact that the expansion (2.6.20) is satisfied in $C^\infty(M^n)$ (see [105, Thm.10.3]), and we followed the notation in [42], denoting $\mathrm{d}_S := \partial_y \partial_x$ for the mixed second derivative. Let us compute $(\mathrm{d}_S p_f)(x, x, 2t)$. Put $U = \sum_{j\geq 0} t^j u_j(x, y)$, from the regularity and uniform estimates (5.2.19), (5.2.18) of H_k and F_k respectively, we see that the differentiation of U can be carried out term by term. Then for the last term in (5.2.20), we see that

$$(8\pi t)^{n/2}(\mathrm{d}_S p_f)(x,y,2t) = \left(-\tfrac{\mathrm{d}_S r_x^2}{8t}e^A U - \tfrac{\partial_x r_x^2}{8t}\partial_y(e^A U) + \mathrm{d}_S(e^A U)\right)e^{-r_x^2/(8t)} - \tfrac{\partial_y r_x^2}{8t}\partial_x p_f$$

where $r_x := \mathsf{d}_g(x,\cdot)$. Since at (x,x), $\partial_x r_x^2 = \partial_y r_x^2 = 0$ and $\mathrm{d}_S r_x^2 = -2g$ hold in normal coordinates, we have

$$(8\pi t)^{n/2}(\mathrm{d}_S p_f)_{(x,x,2t)} = -\frac{e^{f(x)}U(x,x,2t)}{8t}(\mathrm{d}_S)_{(x,x)}r_x^2 + \mathrm{d}_S(e^A U)_{(x,x,2t)}$$

Thanks to (5.2.24) we have $(\partial_x u_0)_{(x,x)} = (\partial_y u_0)_{(x,x)} = 0$ and $(\mathrm{d}_S u_0)_{(x,x)} = -\frac{1}{6}\mathrm{Ric}_g(x)$, which imply

$$(\partial_x U)_{(x,x)} = (\partial_x u_0)_{(x,x)} + O(t) = O(t).$$

Similarly $(\partial_y U)_{(x,x)} = O(t)$, and

$$(\mathrm{d}_S U)_{(x,x)} = (\mathrm{d}_S u_0)_{(x,x)} + O(t) = -\frac{1}{6}\mathrm{Ric}_g(x) + O(t).$$

It follows that

$$\begin{aligned}\mathrm{d}_S(e^A U)_{(x,x,2t)} &= \left(U\mathrm{d}_S e^A + \partial_x e^A \partial_y U + \partial_y e^A \partial_x U + e^A \mathrm{d}_S U\right)_{(x,x,2t)} \\ &= \left(U\mathrm{d}_S e^A + e^A\mathrm{d}_S U + O(t)\right)_{(x,x,2t)} \\ &= \frac{1}{4}e^{f(x)}\mathrm{d}f\otimes\mathrm{d}f - \frac{1}{6}e^{f(x)}\mathrm{Ric}_g + O(t).\end{aligned}$$

This allows us to show that (recall $\mathrm{d}_S r_x^2 = -2g$)

$$\begin{aligned}&(8\pi t)^{n/2}(\mathrm{d}_S p_f)_{(x,x,2t)} \\ &= \frac{1}{4t}e^{f(x)}\left(u_0(x,x) + 2tu_1(x,x) + O(t^2)\right)g + \frac{1}{4}e^{f(x)}\mathrm{d}f\otimes\mathrm{d}f - \frac{1}{6}e^{f(x)}\mathrm{Ric}_g + O(t)\end{aligned}$$

Recall that we have (5.2.21), we finally deduce that

$$\begin{aligned}4t(8\pi t)^{n/2}(\mathrm{d}_S p_f)_{(x,x,2t)} &= e^{f(x)}\left[1+2t\left(\frac{\mathrm{Scal}_g}{6}+\frac{\Delta f}{2}-\frac{|\nabla f|^2}{4}\right)\right]g \\ &\quad + \frac{1}{2}e^{f(x)}\mathrm{d}f\otimes\mathrm{d}f\cdot 2t - \frac{1}{3}e^{f(x)}\mathrm{Ric}_g\cdot 2t + O(t^2) \\ &= e^f g - e^f\left(\frac{2}{3}G_g - \mathrm{d}f\otimes\mathrm{d}f - \Delta f g + \frac{|\nabla f|^2}{2}g\right)t + O(t^2)\end{aligned}$$

as claimed. □

Based on Theorem 5.2.4, let us give the following definitions in order to prove Corollary 5.1.2.

Definition 5.2.5 (Weighted Einstein tensor)**.** Define $G_{g,f}$ by

$$G_{g,f} := e^f G_g - \frac{3e^f}{2}\left(\mathrm{d}f\otimes\mathrm{d}f + \Delta f g - \frac{|\nabla f|^2}{2}g\right). \tag{5.2.31}$$

Definition 5.2.6 (Weighted adjoint operator ∇_f^*)**.** For any $T \in C^\infty((T^*)^{\otimes 2}M^n)$, define $\nabla_f^* T$ by

$$\nabla_f^* T := \nabla^* T + T(\nabla f, \cdot), \tag{5.2.32}$$

where ∇^* is the adjoint operator of the covariant derivative ∇ of (M^n, g), in fact, ∇^* coincides with minus divergence. Moreover, we say that T is *divergence free on* $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ if $\nabla_f^* T = 0$ holds.

Note that $\nabla_f^* T$ is characterized by the equation

$$\int_{M^n} e^{-f}\langle\nabla_f^* T, \omega\rangle \mathrm{dvol}_g = \int_{M^n} e^{-f}\langle T, \nabla\omega\rangle \mathrm{dvol}_g, \quad \forall \omega \in C^\infty(T^*M^n), \tag{5.2.33}$$

that is, ∇_f^* is the adjoint operator of the covariant derivative with respect to $e^{-f}\mathrm{dvol}_g$. Although the next proposition is a direct consequence of Theorem 5.2.4 with more general results (Theorem 5.1.1 and Proposition 5.3.6), we give a direct proof.

Proposition 5.2.7. *It holds that the weighted Einstein tensor $G_{g,f}$ is divergence free on $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ if and only if f is constant.*

Proof. It is enough to check the "only if" part because the other implication reduces to (5.1.2). Assume that $\nabla_f^* G_{g,f} \equiv 0$ holds. Then it is easy to see

$$\nabla^* \left(\mathrm{d}f \otimes \mathrm{d}f + \Delta f g - \frac{|\nabla f|^2}{2} g \right) \equiv 0 \tag{5.2.34}$$

because of (5.1.2). Thus we have

$$\Delta f \mathrm{d}f + \mathrm{d}\Delta f \equiv 0 \tag{5.2.35}$$

see also (2.2.5). Let us consider an open subset U of M^n:

$$U := \{x \in M^n; \Delta f(x) \neq 0\}. \tag{5.2.36}$$

It is enough to prove $U = \emptyset$ because then f is harmonic on (M^n, g), thus f is constant. Assume $U \neq \emptyset$ and take $x \in U$. Define a function $F(z) := e^{f(z)} \Delta f(z)$. Note that F is locally constant on U because

$$\mathrm{d}F(z) = e^{f(z)} \Delta f(z) \mathrm{d}f(z) + e^{f(z)} \mathrm{d}\Delta f(z) = -e^{f(z)} \mathrm{d}\Delta f(z) + e^{f(z)} \mathrm{d}\Delta f(z) = 0, \tag{5.2.37}$$

where we used (5.2.35) in the second equality. Let

$$X := \{z \in M^n; F(z) = F(x)\} \subseteq U. \tag{5.2.38}$$

Since F is continuous on M^n, X is closed in M^n. On the other hand, since F is locally constant on U, we see that X is an open subset of M^n. Thus $X = M^n$. In particular

$$0 = \int_{M^n} \Delta f \mathrm{dvol}_g = F(x) \int_M e^{-f} \mathrm{dvol}_g \neq 0 \tag{5.2.39}$$

which is a contradiction. Thus we have $U = \emptyset$. □

This completes the treatment of the equivalence between being non-collapsed and the divergence free property of weighted Einstein tensor in the smooth context. We now turn to the case of $\mathrm{RCD}(K,N)$ spaces.

5.3 RCD context

5.3.1 Variant of the formula of $\nabla^* g_t$

We rewrite (3.3.9) as

Proposition 5.3.1. *For any $\omega \in H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and any $t \in (0,\infty)$ we have*

$$\int_{\mathsf{X}} \langle g_t, \nabla\omega\rangle \mathrm{d}\mathfrak{m} = -\frac{1}{4}\int_{\mathsf{X}} \langle \omega, \mathrm{d}_x\Delta_x p(x,x,2t)\rangle \mathrm{d}\mathfrak{m}. \tag{5.3.1}$$

Next, we impose better regularity on ω and move the differential operators d,Δ on $p_{2t}(x) := p(x,x,2t)$ in (5.3.1) to ω. Let us first prove a technical lemma.

Lemma 5.3.2. *We have $\mathrm{d}p_{2t} \in D_{loc}(\Delta_{H,1})$ and that $\Delta_{H,1}\mathrm{d}p_{2t} = \mathrm{d}\Delta p_{2t}$*

Proof. Lemma 3.3.3 in particular yields that $p_{2t}(x) \in D_{loc}(\Delta)$ and $\delta \mathrm{d}p_{2t}(x) = -\Delta p_{2t}(x) \in H^{1,2}_{loc}(\mathsf{X},\mathsf{d},\mathfrak{m})$, which in turn implies that $\mathrm{d}p_{2t}(x) \in D_{loc}(\Delta_{H,1})$, now take a good cut off function φ that is 1 on B_r and 0 outside B_R for some $R > r > 0$. [54, Proposition 3.6.1] shows that

$$\Delta_{H,1}\varphi \mathrm{d}p_{2t} = \Delta\varphi \mathrm{d}p_{2t} + 2\mathrm{Hess}_{p_{2t}}(\nabla\varphi,\cdot) + \varphi \mathrm{d}\Delta p_{2t}. \tag{5.3.2}$$

The locality of Δ and ∇(essentially D) yields that $\nabla\varphi$ and $\Delta\varphi$ are $\mathfrak{m}$-a.e. 0 in B_r. So $\Delta_{H,1}\mathrm{d}p_{2t} = \mathrm{d}\Delta p_{2t}$ holds on B_r, by the arbitrariness of r we complete the proof. □

Now if $\omega \in D(\Delta_{H,1})$ compactly supported, with $\Delta_{H,1}\omega \in D(\delta)$, denoting $p_{2t}(x) := p(x,x,2t)$, we are able to show:

Theorem 5.3.3. *For any $t \in (0,\infty)$ and any $\omega \in D(\Delta_{H,1})$ compactly supported with $\Delta_{H,1}\omega \in D(\delta)$ we have*

$$\int_{\mathsf{X}} \langle g_t, \nabla\omega\rangle \mathrm{d}\mathfrak{m} = \frac{1}{4}\int_{\mathsf{X}} \delta(\Delta_{H,1}\omega)p_{2t}(x)\mathrm{d}\mathfrak{m}. \tag{5.3.3}$$

Proof. Proposition 5.3.1 and Lemma 5.3.2 yield that

$$\begin{aligned}\int_X \langle g_t, \nabla\omega\rangle \mathrm{d}\mathfrak{m} &= -\frac{1}{4}\int_X \langle \omega, \mathrm{d}_x\Delta_x p_{2t}(x)\rangle \mathrm{d}\mathfrak{m} = \frac{1}{4}\int_X \langle \omega, \Delta_{H,1}(\mathrm{d}_x p_{2t}(x))\rangle \mathrm{d}\mathfrak{m}\\ &= \frac{1}{4}\int_X \langle \Delta_{H,1}\omega, \mathrm{d}_x p_{2t}(x)\rangle \mathrm{d}\mathfrak{m} = \frac{1}{4}\int_X \delta(\Delta_{H,1}\omega)p_{2t}(x)\mathrm{d}\mathfrak{m}.\end{aligned}$$

□

5.3.2 Proof of Theorem 5.1.1

Before we start, we point out that under (5.1.5), we can rewrite Theorem 4.2.1 as

$$\int_{\mathsf{X}} \mathrm{tr}\nabla\omega \mathrm{d}\mathcal{H}^n = 0, \quad \text{or} \quad \int_{\mathsf{X}} \left\langle \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} g, \nabla\omega \right\rangle \mathrm{d}\mathfrak{m} = 0 \tag{5.3.4}$$

for any $\omega \in H^{1,2}_C(\mathsf{X},\mathsf{d},\mathfrak{m})$, because of the density of test 1-forms in $H^{1,2}_C(\mathsf{X},\mathsf{d},\mathfrak{m})$, see also the proof of Theorem B.0.2. This is in particular satisfied if $(\mathsf{X},\mathsf{d},\mathcal{H}^n)$ is $\mathrm{ncRCD}(K,n)$. So the proof of both directions in Theorem 5.1.1 reduce to the computation of the integral of $\langle t^{n/2}g_t, \nabla\omega\rangle$.

In what follows, when we use the notation ω, we always assume that $\omega \in D(\Delta_{H,1})$, ω has compact support, $\Delta_{H,1}\omega \in D(\delta)$. We start with a technical lemma.

Lemma 5.3.4. *Let $\xi \in L^\infty_{loc}(\mathsf{X},\mathfrak{m})$, φ be a good cut-off function that is supported in $B_R(z)$ for some $R > 0$ and $z \in \mathsf{X}$. It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathsf{X}} \delta(\varphi \mathrm{h}_{H,t}\omega)\xi \mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \delta(\varphi\Delta_{H,1}\mathrm{h}_{H,t}\omega)\xi \mathrm{d}\mathfrak{m}, \qquad \forall t \in (0,\infty). \tag{5.3.5}$$

Proof. It suffices to show for any $t > 0$ the convergence

$$\lim_{s\to 0}\int_{\mathsf{X}} \delta\left(\varphi\frac{\mathrm{h}_{H,t+s}-\mathrm{h}_{H,t}}{s}\omega\right)\xi\mathrm{d}\mathfrak{m} = \int_{\mathsf{X}} \delta(\varphi\Delta_{H,1}\mathrm{h}_{H,t}\omega)\xi\mathrm{d}\mathfrak{m}. \tag{5.3.6}$$

We denote the heat kernel by $p_{y,t}(x) := p(x,y,t)$, and let $F_{y,t} = \frac{p_{y,t+s}-p_{y,t}}{s} - \Delta p_{y,t}$. From the proof of the Lemma 3.3.4 we see that $\|F_{y,t}\|_{(L^2;\mathrm{d}\mathfrak{m}(x))} \leq sC(K,N,t)\mathfrak{m}(B_{\sqrt{t}}(y))^{-\frac{1}{2}}$.

First, we observe that

$$\begin{aligned}\delta\left(\varphi\frac{\mathrm{h}_{H,t+s}-\mathrm{h}_{H,t}}{s}\omega\right) &= \left\langle \mathrm{d}\varphi, \frac{\mathrm{h}_{H,t+s}-\mathrm{h}_{H,t}}{s}\omega\right\rangle + \varphi\delta\left(\frac{\mathrm{h}_{t+s}-\mathrm{h}_t}{s}\omega\right) \\ &= \left\langle \mathrm{d}\left(\frac{\mathrm{h}_{t+s}-\mathrm{h}_t}{s}\varphi\right), \omega\right\rangle + \varphi\left(\frac{\mathrm{h}_{t+s}-\mathrm{h}_t}{s}\delta\omega\right).\end{aligned} \tag{5.3.7}$$

In the same way we have $\delta(\varphi\Delta_{H,1}\mathrm{h}_{H,t}\omega) = \langle \mathrm{d}(\Delta\mathrm{h}_t\varphi), \omega\rangle + \varphi(\Delta\mathrm{h}_t\delta\omega)$.

Next, we split the LHS of (5.3.6) as in (5.3.7) and prove the convergence separately. For the second term, we use Cauchy-Schwarz inequality:

$$\begin{aligned}&\int_{\mathsf{X}}\int_{\mathsf{X}} \varphi\left(\frac{p_{y,t+s}(x)-p_{y,t}(x)}{s} - \Delta p_{y,t}(x)\right)\delta\omega(y)\,\xi(x)\mathrm{d}\mathfrak{m}(x)\,\mathrm{d}\mathfrak{m}(y) \\ &\leq \|\chi_{B_R(z)}\xi^2\|_{L^\infty}^{\frac{1}{2}}\int_{\mathsf{X}} \|F_{y,t}\|_{(L^2;\mathrm{d}\mathfrak{m}(x))}\delta\omega(y)\mathrm{d}\mathfrak{m}(y) \\ &\leq Cs\|\delta\omega\|_{L^2}\int_{B_R(z)} \frac{1}{\mathfrak{m}(B_{\sqrt{t}}(y))}\mathrm{d}\mathfrak{m}(y) \leq C(K,N,R,t)s.\end{aligned} \tag{5.3.8}$$

In the last inequality we used (5.1.5) for $t < 1$ and Bishop-Gromov inequality for $t \geq 1$.

For the first term, we use the flow a-priori estimates:

$$\begin{aligned}&\int_{\mathsf{X}} \left\langle \mathrm{d}\left(\frac{\mathrm{h}_{t+s}-\mathrm{h}_t}{s} - \Delta\mathrm{h}_t\right)\varphi, \omega\right\rangle \xi\mathrm{d}\mathfrak{m}(x) \\ &\leq \frac{1}{t}\|\chi_{B_R(z)}\xi\|_{L^\infty}\|\delta\omega\|_{L^2}\left\|\left(\frac{\mathrm{h}_{t/2+s}-\mathrm{h}_{t/2}}{s} - \Delta\mathrm{h}_{t/2}\right)\varphi\right\|_{L^2} \\ &\leq C(K,N,R,t)\int_{\mathsf{X}} \|F_{y,t/2}\|_{(L^2;\mathrm{d}\mathfrak{m}(x))}\mathrm{d}\mathfrak{m}(y) \leq C(K,N,R,t)s.\end{aligned} \tag{5.3.9}$$

□

Let us finally embark on the proof the Theorem 5.1.1, we will use the compactness assumption minimally, presenting most of the proof in a general setting and we will give 2 proofs of $(1) \Rightarrow (2)$.

proof of Theorem 5.1.1. We first prove the implication (2) ⇒ (1). In this case we have $\mathfrak{m} = \mathcal{H}^n$. It is clear from Bishop-Gromov inequality that (5.1.5) holds. We first notice that from (2.6.19) we have for every $x \in \mathcal{R}_n$:

$$\lim_{t\to 0} t^{n/2} p_{2t}(x) = \frac{1}{\cdot 2^{n/2}} \frac{(2t)^{n/2}}{\mathcal{H}^n(B_{\sqrt{2t}}(x))} \mathcal{H}^n(B_{\sqrt{2t}}(x)) p_{2t}(x) = \frac{1}{(8\pi)^{/2}}, \tag{5.3.10}$$

then also recall (5.3.4), it follows that

$$\int_{\mathsf{X}} \left\langle \frac{c_n t^{(n+2)/2} g_t - g}{t}, \nabla\omega \right\rangle \mathrm{d}\mathcal{H}^n \overset{(5.3.3)}{=} -\frac{c_n}{4} \int_{\mathsf{X}} t^{n/2} p_{2t}\, \delta(\Delta_{H,1}\omega) \mathrm{d}\mathcal{H}^n$$
$$\xrightarrow{t\to 0} -\frac{c_n}{4(8\pi)^{n/2}} \int_{\mathsf{X}} \delta(\Delta_{H,1}\omega) \mathrm{d}\mathcal{H}^n = 0.$$

The convergence is justified since that

$$\sup_{t>0, x\in \operatorname{supp}\omega} t^{n/2} p_{2t}(x) = \sup_{t>0, x\in \operatorname{supp}\omega} \frac{(2t)^{n/2}}{2^{n/2}\mathfrak{m}(B_{\sqrt{2t}}(x))} \mathfrak{m}(B_{\sqrt{2t}}(x)) p_{2t}(x) < \infty,$$

which follows from Gaussian estimates (2.6.11) and Bishop-Gromov inequality.

We then deal with the implication (1) ⇒ (2). We have that

$$0 = \lim_{t\to} \int_{\mathsf{X}} \left\langle \frac{c_n t^{(n+2)/2} g_t - \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} g}{t}, \nabla\omega \right\rangle \mathrm{d}\mathfrak{m} \overset{(5.3.3)}{=} -\lim_{t\to 0} \frac{c_n}{4} \int_{\mathsf{X}} t^{n/2} p_{2t}\, \delta(\Delta_{H,1}\omega) \mathrm{d}\mathfrak{m}$$

Notice that it can be deduced from (5.1.5) and Gaussian estimates (2.6.11) that, again

$$\sup_{t>0, x\in \operatorname{supp}\omega} t^{n/2} p_{2t}(x) < \infty,$$

then (2.6.19) implies that for $x \in \mathcal{R}_n$

$$\lim_{t\to 0} t^{n/2} p_{2t}(x) = \frac{1}{\cdot 2^{n/2}} \frac{(2t)^{n/2}}{\mathfrak{m}^n(B_{\sqrt{2t}}(x))} \mathfrak{m}^n(B_{\sqrt{2t}}(x)) p_{2t}(x) = \frac{\omega_n}{(8\pi)^{/2}} \frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}$$

Denote $\frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}} := \theta$, the same argument shows

$$0 = -\frac{c_n}{4} \int_{\mathsf{X}} \lim_{t\to 0} t^{n/2} p_{2t}\, \delta(\Delta_{H,1}\omega) = \int_{\mathsf{X}} \delta(\Delta_{H,1}\omega)\theta \mathrm{d}\mathfrak{m}. \tag{5.3.11}$$

Then, if X is compact, we see that for any $t > 0$, $\mathrm{h}_t\omega$ has compact support and $\mathrm{h}_{H,t}\omega \in D(\Delta_{H,1})$ and $\Delta_{H,1}\mathrm{h}_{H,t}\omega \in D(\delta)$, we have that

$$0 = \int_{\mathsf{X}} \delta(\Delta_{H,1}(\mathrm{h}_{H,t}\omega))\theta\, \mathrm{d}\mathfrak{m}. \tag{5.3.12}$$

Lemma 5.3.4 yields that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathsf{X}} \delta(\mathrm{h}_{H,t}\omega)\theta\, \mathrm{d}\mathfrak{m}. \tag{5.3.13}$$

Integrate the above equation w.r.t to t to find that there exists constant C_ω so that

$$C_\omega = \int_{\mathsf{X}} \delta(\mathrm{h}_{H,t}\omega)\theta\, \mathrm{d}\mathfrak{m}, \qquad \forall t \in (0, \infty). \tag{5.3.14}$$

Meanwhile, by Cauchy-Schwarz inequality and the heat flow a-priori estimates, we have

$$|C_\omega| \le \frac{C}{\sqrt{t}} \|\chi_{\mathrm{supp}\,\omega}\theta\|_{L^2} \|\omega\|_{L^2} \xrightarrow{t\to+\infty} 0, \tag{5.3.15}$$

which forces $C_\omega = 0$. Now by letting $t \to 0$ in (5.3.14) with dominated convergence theorem, we see that

$$0 = \int_{\mathsf{X}} (\delta\omega)\theta\, \mathrm{d}\mathfrak{m}. \tag{5.3.16}$$

Now following Lemma 4.2.2(with $\omega = \varphi \mathrm{d}f$ for some good cut-off φ and test function f), we get $\theta = c$ for some $c > 0$. This completes the proof.

Alternatively, to proof the implication (1) $\Rightarrow$ (2), after getting to (5.3.11) we can take $\omega = \mathrm{d}f_i$, where f_i is the eigenfunction of eigenvalue $\lambda_i > 0$, for all positive integer i. We derive that,

$$\lambda_i^2 \int_{\mathsf{X}} f_i \theta \mathrm{d}\mathfrak{m} = 0 \tag{5.3.17}$$

Since f_i along with $f_0 = \frac{1}{\mathfrak{m}(\mathsf{X})}$ consists of an orthonormal basis of $L^2(\mathsf{X}, \mathfrak{m})$, we conclude that $\theta = c$ for some $c > 0$, which also completes the proof. □

5.3.3 Weakly asymptotically divergence free

This section is taken from [40, Section 4.5]. In order to prove Corollary 5.1.3 let us recall the

Definition 5.3.5 (Weakly asymptotically divergence free)**.** Let $\{T_t\}_{t\in(0,1)}$ be a family of L^2-tensor fields of type $(0, 2)$ on X. We say that it is *weakly asymptotically divergence free as* $t \to 0^+$ if there exists a dense subset V of $H^{1,2}_C(T^*(X, \mathsf{d}, \mathfrak{m}))$ such that for any $\omega \in V$ we have

$$\int_X \langle T_t, \nabla\omega \rangle \mathrm{d}\mathfrak{m} \to 0 \tag{5.3.18}$$

as $t \to 0^+$.

Note that Theorem 5.1.1 implies that a family of L^∞-tensors (5.1.3) is weakly asymptotically divergence free as $t \to 0^+$ if an $\mathrm{RCD}(K, n)$ space $(X, \mathsf{d}, \mathfrak{m})$ satisfies $\dim_{\mathsf{d},\mathfrak{m}}(X) = n$ because the space

$$\{\omega \in D(\Delta_{H,1}); \omega \text{ has compact support, } \Delta_{H,1}\omega \in D(\delta)\} \tag{5.3.19}$$

is dense in $H^{1,2}_C(T^*(X, \mathsf{d}, \mathfrak{m}))$, see for instance Proposition 2.6.4. Corollary 5.1.3 is a direct consequence of Theorem 5.1.1 with the following proposition.

Proposition 5.3.6. *Let $\{T_t\}_{t\in(0,1)}$ be a family of L^2-tensor fields of type $(0,2)$ on X with*

$$\limsup_{t\to 0^+} \|T_t\|_{L^2} < \infty \tag{5.3.20}$$

Then the following two conditions are equivalent:

1. *$\{T_t\}_{t\in(0,1)}$ is weakly asymptotically divergence free as $t \to 0^+$.*
2. *If $G \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ is the L^2-weak limit of T_{t_i} for some convergent sequence $t_i \to 0^+$, then $G \in D(\nabla^*)$ with $\nabla^* G = 0$.*

Proof. Let us first prove the implication from (1) to (2). Assume that $\{T_t\}_{t\in(0,1)}$ is weakly asymptotically divergence free as $t \to 0^+$. Let V be as in Definition 5.3.5 and let t_i, G be as in the assumption of (2). By definition we have

$$\int_X \langle G, \nabla\omega\rangle \mathrm{d}\mathfrak{m} = \lim_{i\to\infty} \int_X \langle T_{t_i}, \nabla\omega\rangle \mathrm{d}\mathfrak{m} = 0 \tag{5.3.21}$$

holds for any $\omega \in V$. Since V is dense in $H^{1,2}_C(T^*(X, \mathsf{d}, \mathfrak{m}))$, we have

$$\int_X \langle G, \nabla\omega\rangle \mathrm{d}\mathfrak{m} = 0, \quad \forall \omega \in H^{1,2}_C(T^*(X, \mathsf{d}, \mathfrak{m})) \tag{5.3.22}$$

which shows $G \in D(\nabla^*)$ with $\nabla^* G = 0$.

Next let us prove the remaining implication. Assume that (2) holds. Let us fix $\omega \in H^{1,2}_C(T^*(X, \mathsf{d}, \mathfrak{m}))$. If (5.3.18) is not satisfied for this ω, then combining this fact with the L^2-weak compactness, it follows that there exist a convergent sequence $t_i \to 0^+$ and $G \in L^2((T^*)^{\otimes 2}(X, \mathsf{d}, \mathfrak{m}))$ such that $T_{t_i} \to G$ in the L^2-weak topology and

$$\int_X \langle G, \nabla\omega\rangle \mathrm{d}\mathfrak{m} = \lim_{i\to\infty} \int_X \langle T_{t_i}, \nabla\omega\rangle \mathrm{d}\mathfrak{m} \neq 0 \tag{5.3.23}$$

are satisfied, which contradicts the assumption (2). □

5.3.4 The L^p divergence of the approximate Einstein tensor for $p > 1$

This section is taken from [40, Section 5]. In this section, we explain why it is necessary to state the main theorem (Theorem 5.1.1) using the weakly asymptotically divergence free property by giving an example. In fact, we cannot hope that (5.1.3) has a limit in a reasonable sense, let alone in $D(\nabla^*)$, more precisely, the L^p convergence of (5.1.3) can fail for any $p > 1$. To show this we will construct a compact non-collapsed $\mathrm{RCD}(0, n)$ space such that

$$\left\| \frac{c_n t^{(n+2)/2} g_t - g}{t} \right\|_{L^p} \xrightarrow{t \to 0^+} +\infty \tag{5.3.24}$$

We first point out that the computation in Section 5.2 can be generalized to a *smooth* open subset U in a compact $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$, where $(U, \mathsf{d}, \mathfrak{m}\llcorner_U)$ is said to be locally isometric to a weighted (not necessary complete) Riemannian manifold $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$ if there exists a homeomorhism $\Phi : U \to M^n$ such that $\Phi_*(\mathfrak{m}\llcorner_U) = \mathrm{vol}_{g,f}$ and that Φ is a local isometry as metric spaces.

Proposition 5.3.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a compact $\mathrm{RCD}(K, N)$ space. If there exists an open subset $U \subseteq X$ such that $(U, \mathsf{d}, \mathfrak{m}\llcorner_U)$ is locally isometric to an n-dimensional weighted (not necessary complete) Riemannian manifold $(M^n, \mathsf{d}_g, \mathrm{vol}_{g,f})$, then Theorem 5.2.4 holds on U in the sense that*

$$\frac{c_n t^{n+2/2} g_t - e^{f(x)} g}{t} \to -\frac{2}{3} G_{g,f} \tag{5.3.25}$$

holds uniformly on any compact subset of U.

Proof. Fix $y \in U$ and take a sufficiently small $\epsilon > 0$ such that $B_\epsilon(y) \in U$ and that $\partial B_\epsilon(y)$ is smooth. With no loss of generality we can assume $\mathfrak{m}(B_\epsilon(y)) = 1$. Let $p_{f,\epsilon}$ be the Dirichlet heat kernel on $B_\epsilon(y)$. Thanks to the smoothness of $\partial B_\epsilon(y)$, we know that $p_{f,\epsilon}$ has the continuous extension, denoted $p_{f,\epsilon}$ again, to $\overline{B}_\epsilon(y) \times \overline{B}_\epsilon(y) \times (0, \infty)$ such that $p_{f,\epsilon}(x, z, t) = 0$ whenever $x \in \partial B_\epsilon(y)$ which is justified by regularity results for parabolic equations on Euclidean balls. The key point in the proof of (5.3.25) is to show that the global heat kernel p on X and $p_{f,\epsilon}$ are exponentially close on $B_\epsilon(y)$, that is, for sufficiently small t,

$$\sup_{x\in B_\epsilon(y)} |p(x,y,t) - p_{f,\epsilon}(x,y,t)| < C(K,N)e^{-\epsilon^2/6t}, \tag{5.3.26}$$

where $C(K,N)$ denotes a positive constant with dependence on K and N. Then since the restriction of p to $B_\epsilon(y)\times B_\epsilon(y)\times(0,\infty)$ is smooth (see for instance the proof of [105, Thm.7.20]), (5.3.26) implies the power series expansion in t for p and $p_{f,\epsilon}$ are the same. In particular p has the same expansion as in (5.2.20) on $B_\epsilon(y)$. Then the desired convergence (5.3.25) comes from the same proof of Theorem 5.2.4.

To prove (5.3.26), applying the Gaussian estimates (2.6.11) when $\epsilon = 1$, together with the maximum principle yields for small $t > 0$

$$\begin{aligned}\sup_{x\in B_\epsilon(y)} |p(x,y,t) - p_{f,\epsilon}(x,y,t)| &\le \sup_{\partial B_\epsilon(y)\times(0,t]} (p(x,y,s) - p_{f,\epsilon}(x,y,s))\\ &\le C_1e^{C_2t}\sup_{s\in(0,t]}\frac{e^{-\epsilon^2/5s}}{\mathfrak{m}(B_{\sqrt{s}}(y))}\\ &\le C_1Ce^{C_2t}\sup_{s\in(0,t]}\frac{e^{-\epsilon^2/5s}}{s^{n/2}}\\ &\le C_1Ce^{C_2t}\frac{e^{-\epsilon^2/5t}}{t^{n/2}} \le C_1Ce^{C_2t}e^{-\epsilon^2/6t},\end{aligned} \tag{5.3.27}$$

where we used the Bishop-Gromov inequality for $\mathfrak{m}$ in the second inequality, and a fact that the function $\frac{e^{-\epsilon^2/5s}}{s^{n/2}}$ is monotone increasing for $s\in(0,t]$ when t is small enough. □

Example 5.3.8. Given $p>1$, let $\alpha = 1-\frac{1}{p}\in(0,1)$ and Z be the metric completion of $(0,1)\times\mathbb{S}^1$ with the warped product metric $g_Z = dr^2 + (r-r^{1+\alpha})^2d\theta^2$. This metric is $C^{1,\alpha}$ at the origin 0 and smooth elsewhere. It follows from direct computation that

$$\begin{aligned}&\mathrm{Ric}_{g_Z} = \alpha(1+\alpha)r^{\alpha-1}g_Z \ge 0,\\ &\|\mathrm{Scal}_{g_Z}\|_{L^p}^p = [2\pi\alpha(1+\alpha)]^p\int_0^1 r^{(\alpha-1)p}dr = +\infty.\end{aligned} \tag{5.3.28}$$

Let $f(r) = r - r^{1+\alpha}$, then since $f(0) = f(1) = 0$ the metric completion Z is a closed $C^{1,\alpha}$

manifold. Moreover it is esay to verify that this f satisfies the condition in [107, Theorem 6.2], which implies that (Z, d_{g_Z}) is an Alexandrov space with non-negative curvature. In particular it follows from [48, Main thm] that $(Z, \mathsf{d}_{g_Z}, \mathcal{H}^2)$ is a non-collapsed $\mathrm{RCD}(0,2)$ space.

For $n \geq 3$ take $X := Z \times \mathbb{T}^{n-2}$ with product metric $g_X = g_Z + g_{\mathbb{T}^{n-2}}$, where $(\mathbb{T}^{n-2}, g_{\mathbb{T}^{n-2}})$ is the $(n-2)$ dimensional flat torus. Then $(X, \mathsf{d}_{g_X}, \mathcal{H}^n)$ is a non-collapsed $\mathrm{RCD}(0,n)$ space. Let $X_{\mathrm{sing}} := \{0\} \times \mathbb{T}^{n-2}$ and $X_{\mathrm{reg}} := X \setminus X_{\mathrm{sing}}$, we have the Einstein tensor on X_{reg}:

$$G^{g_X}_{\mathrm{reg}} = \mathrm{Ric}_{g_X} - \frac{1}{2}\mathrm{Scal}_{g_X} g_X = \mathrm{Ric}_{g_Z} - \frac{1}{2}\mathrm{Scal}_{g_Z}(g_Z + g_{\mathbb{T}^{n-2}}) = -\frac{1}{2}\mathrm{Scal}_{g_Z} g_{\mathbb{T}^{n-2}}. \tag{5.3.29}$$

We used the fact that in dimension 2 the Einstein tensor vanishes in the last equality.

Let us show the L^p divergence of (5.1.3) as $t \to 0^+$ in this example. Proposition 5.3.7 yields

$$\int_X \left\langle \frac{c_n t^{(n+2)/2} g_t - g}{t}, T \right\rangle \mathrm{d}\mathcal{H}^n \to -\frac{2}{3} \int_X \left\langle G^{g_X}_{X_{\mathrm{reg}}}, T \right\rangle \mathrm{d}\mathcal{H}^n \tag{5.3.30}$$

for any tensor T of type $(0,2)$ with compact support in X_{reg}. In particular for any T with $\|T\|_{L^q} \leq 1$, where q is the conjugate index such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|G^{g_X}_{X_{\mathrm{reg}}}\|^2_{L^p(X_{\mathrm{reg}})} = \left| \int_{X_{\mathrm{reg}}} \left\langle G^{g_X}_{X_{\mathrm{reg}}}, T \right\rangle \mathrm{d}\mathcal{H}^n \right| \leq \frac{3}{2} \liminf_{t \to 0^+} \left\| \frac{c_n t^{(n+2)/2} g_t - g}{t} \right\|_{L^p} \tag{5.3.31}$$

Taking the supremum over T in (5.3.31), we have

$$\|G^{g_X}_{X_{\mathrm{reg}}}\|_{L^p} \leq \frac{3}{2} \liminf_{t \to 0^+} \left\| \frac{c_n t^{(n+2)/2} g_t - g}{t} \right\|_{L^p}. \tag{5.3.32}$$

Since the left hand side of (5.3.32) is $+\infty$ because of (5.3.28) and (5.3.29)

the divergence of the right hand side of (5.3.32) follows.

5.4 Further Studies

The first question to be asked is that if Theorem 5.1.1 holds without assuming compactness of the space. The proof already shows the implication (2) $\Rightarrow$ (1) does not need compactness. In the implication (1) $\Rightarrow$ (2), we can get along the same line

$$\int_{\mathsf{X}} \delta(\Delta_{H,1}\omega)\frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}\mathrm{d}\mathfrak{m} = 0$$

for all $\omega \in D(\Delta_{H,1})$ compactly supported with $\Delta_{H,1}\omega \in D(\delta)$. However, without compactness of (X, d), we do not have eigenfunctions at disposal, and the heat flow regularization $\mathrm{h}_t\omega$ loses compact support for $t > 0$, it then becomes a problem to control $\frac{\mathrm{d}\mathcal{H}^n}{\mathrm{d}\mathfrak{m}}$. We expect the answer to be negative. To get a feel of this, fix a weighted manifold $(M, g, \mathrm{vol}_{g,V})$, where $\mathrm{vol}_{g,V} = e^{-V}\,\mathrm{vol}_g$, one can follow the same argument to get that

$$\int_M \delta(\Delta_H\omega)e^V \mathrm{d}\,\mathrm{vol}_{g,V} = 0.$$

Now take the weighted Hodge decomposition, so that $\omega = \omega_{\mathrm{harm}} + \mathrm{d}f + \delta\eta$, for some L^2 function f, and some L^2 2-form η with bounded support. Then this integral reduces to for every compactly supported L^2 function f,

$$0 = \int_M \Delta_V\Delta_V f e^V \mathrm{d}\,\mathrm{vol}_{g,V} = \int_M \Delta_V f \Delta_V e^V \mathrm{d}\,\mathrm{vol}_{g,V}$$

For this to hold, since $\Delta_V e^V = e^V \Delta V$, it suffices to have that $\Delta V = 0$, i.e., V is a (unweighted) harmonic function. Meanwhile V is supposed to satisfy the Bakry-Émery Ricci curvature lower bound. To this end, it is enough to know that $\mathrm{d}V$ and Hess_V are bounded, such an example is expected to be constructed on manifolds of negative curvature.

Another question worth looking at is the L^1 unboundedness of the quotient (5.1.3). It is expected the convergence of quotient in the form of (5.1.4) is optimal, that is, there is in general no L^p convergence of (5.1.3) for any $p \in [1, \infty]$. We have constructed an example to show that L^p-weak convergence of (5.1.3) can fail for $p \in (1, \infty)$, but we are unable to construct an example of an compact $\mathrm{RCD}(K, N)$ space so that the L^1 norm of (5.1.3) diverges. Such an example is expected to be found in the class of stratified spaces with a low regularity ($C^{1,\alpha}$ for $\alpha \in (0, 1)$) metric.

Yet another intriguing question is the uniqueness of the limit:

$$G := \lim_{t \to 0} \frac{t^{(n+2)/2} g_t - g}{t},$$

given the L^2 bound

$$\sup_{t>0} \left\| \frac{t^{(n+2)/2} g_t - g}{t} \right\|_{L^2} < \infty,$$

on a non-collapsed space $(\mathsf{X}, \mathsf{d}, \mathcal{H}^n)$. An argument proposed by Honda is that one can use G and $\mathrm{tr}G$ to construct the scalar curvature $\mathrm{Scal} := \frac{6}{n-2}\mathrm{tr}G$ and then produce a Ricci tensor $\mathrm{Ric} := -3G + \frac{1}{2}\mathrm{Scal}$ according to the expansion formula (5.2.27). One is then expected to check the uniqueness by proving the following Weitzenböck formula:

$$\frac{1}{2}\Delta|\omega| = |\nabla\omega|^2 - \langle \Delta_{H,1}\omega, \omega\rangle + \langle \mathrm{Ric}, \omega \otimes \omega\rangle.$$

In connection with it, looking again at (5.2.27), the quotient (5.1.3) is also expected to define scalar curvature lower bound given the measure-valued Ricci tensor in [54, Theorem 3.6.7].

CHAPTER 6

CONVEX PROPERTIES OF $\mathrm{RCD}(K,N)$ SPACES

6.1 Overview

This chapter is inspired by the communication with Prof. Vitali Kapovitch and his student Qin Deng. They suggested the combination of Hölder continuity of tangent cone and the one dimensional localization technique. In this chapter, by the word geodesic we always intend a minimizing geodesic. The goal of this Chapter is to present a light improvement of the almost convexity derived by Deng [33, Theorem 6.5] for $\mathrm{RCD}(K,N)$ spaces:

Theorem 6.1.1. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space with $\mathrm{essdim}=n$. For $\mathfrak{m}\times\mathfrak{m}$-a.e. every $(x,y)\in\mathcal{R}_n\times\mathcal{R}_n$, there exists a geodesic joining x,y, and entirely contained in $\mathcal{R}_n$.*

See also the identical statement for Ricci limit spaces in [35]. In both cases the proof relies on the Hölder continuity of tangent cones along the interior of any geodesic, this is a deep result and only an easy consequence of it is needed for our purposes, which is stated as follows:

Proposition 6.1.2. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space. For each integer $1\le k\le N$, and for every geodesic γ in X, $\gamma\cap\mathcal{R}_k$ is closed relative to the interior of γ. If in addition $\gamma\cap\mathcal{R}_k$ is dense in the interior of γ, then it is all of the interior.*

This is because at every regular point the tangent cone is unique. We can strengthen Theorem 6.1.1 to the following:

Theorem 6.1.3. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space with $\mathrm{essdim}=n$. For every $x\in\mathcal{R}_n$, there exists a subset $R_x\subseteq\mathcal{R}_n$ so that $\mathfrak{m}(\mathsf{X}\setminus R_x)=0$ and for any $y\in R_x$ there is a minimizing geodesic joining x,y and entirely contained in $\mathcal{R}_n$.*

We need the technique of localization via transport rays of some 1 Lipschitz function, developed by Cavalletti and Mondino in non-smooth setting, which generalizes Klartag's needle decomposition on Riemannian manifolds. The key fact is that there exists a strongly consistent disintegration of $\mathfrak{m}$ into measures $\mathfrak{m}_\alpha$ concentrated on geodesics, which are also transport rays, so that their densities w.r.t. $\mathcal{H}^1$ are all $\mathrm{CD}(K,N)$ density (recall Definition 2.1.3). An observation is that the defining inequality of $\mathrm{CD}(K,N)$ density ensures that such a density is $\mathcal{H}^1$-a.e. positive, so it holds that $\mathfrak{m}_\alpha \ll \mathcal{H}^1 \ll \mathfrak{m}_\alpha$, hence the equivalence $\mathfrak{m}_\alpha(\gamma \setminus (\gamma \cap \mathcal{R}_n)) = 0 \Leftrightarrow \mathcal{H}^1(\gamma \setminus (\gamma \cap \mathcal{R}_n)) = 0$, for any geodesic γ. The latter in particular implies that $\gamma \cap \mathcal{R}_n$ is dense in γ, then Proposition 6.1.2 yields that the interior of γ is all in $\mathcal{R}_n$.

6.2 Almost convexity of $\mathcal{R}_n$ and interior of $\mathrm{RCD}(K,N)$ spaces

We minimally collect the elements of the localization technique introduced in [108] and [109], we remark that this technique is available for a much general class of metric measure spaces, the so called essentially non-branching $\mathrm{MCP}(K,N)$(MCP stands for measure contraction property) spaces, which contains essentially non-branching $CD(K,N)$ spaces, hence $\mathrm{RCD}(K,N)$ spaces.

Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space, u be a 1-Lipschitz function. Define the transport set induced by u as:

$$\Gamma(u) := \{(x,y) \in \mathsf{X} \times \mathsf{X} : u(x) - u(y) = \mathsf{d}(x,y)\},$$

and its transpose as $\Gamma^{-1}(u) := \{(x,y) \in \mathsf{X} \times \mathsf{X} : (y,x) \in \Gamma(u)\}$. The union $R_u := \Gamma^{-1}(u) \cup \Gamma(u)$ defines a relation on X. By excluding negligible isolated and branching points, one can find a transport set $\mathcal{T}_u$ such that $\mathfrak{m}(\mathsf{X} \setminus \mathcal{T}_u) = 0$ and R_u restricted to $\mathcal{T}_u$ is an equivalence relation. So there is a partition of $\mathcal{T}_u := \cup_{\alpha \in Q} \mathsf{X}_\alpha$, where Q is a set of indices, denote by $\mathfrak{Q} : \mathcal{T}_u \to Q$ the quotient map. In [108, Proposition 5.2], it is shown that there exists a measurable selection $s : \mathcal{T}_u \to \mathcal{T}_u$ such that if xR_uy then $s(x) = s(y)$, so we can identify Q as $s(\mathcal{T}_u) \subseteq \mathsf{X}$. Equip Q with the σ-algebra induced by $\mathfrak{Q}$ and the measure $\mathfrak{q} := \mathfrak{Q}_\sharp(\mathfrak{m} \llcorner \mathcal{T}_u)$, we can hence view $\mathfrak{q}$ as a Borel measure on X. Furthermore, each X_α is shown ([109, Lemma 3.1]) be to isometric to an interval I_α, the distance preserving map $\gamma_\alpha : I_\alpha \to \mathsf{X}_\alpha$ extend to an geodesic still denoted by $\gamma_\alpha : \bar{I}_\alpha \to \mathsf{X}$. Putting several results together, we have ([53, Theorem A.5]):

Theorem 6.2.1. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space. u be a 1-Lipschitz function. Then $\mathfrak{m}$ admits a disintegration:*

$$\mathfrak{m} = \int_Q \mathfrak{m}_\alpha \mathfrak{q}(\mathrm{d}\alpha),$$

where $\mathfrak{m}_\alpha$ is a non-negative Radon measure on X, such that

1. *For any $\mathfrak{m}$-measurable set B, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable.*

2. *for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathsf{X}_\alpha = \mathfrak{Q}^{-1}(\alpha)$. This property is called strong consistency of the disintegration.*

3. *for any $\mathfrak{m}$-measurable set B and $\mathfrak{q}$-measurable set C, it holds*

$$\mathfrak{m}(B \cap \mathfrak{Q}^{-1}(C)) = \int_C \mathfrak{m}_\alpha(B)\mathfrak{q}(\mathrm{d}\alpha).$$

4. *for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha = h_\alpha \mathcal{H}^1 \llcorner \mathsf{X}_\alpha \ll \mathcal{H}^1 \llcorner \mathsf{X}_\alpha$, where h_α is a $\mathrm{CD}(K, N)$ density, and $(\overline{\mathsf{X}}_\alpha, \mathsf{d}, \mathfrak{m}_\alpha)$ is an $\mathrm{RCD}(K, N)$ space.*

We are now ready to prove Theorem 6.1.3.

Proof of Theorem 6.1.3. Take $x \in \mathcal{R}_n$, disintegrate $\mathfrak{m}$ w.r.t $\mathsf{d}_x := \mathsf{d}(x, \cdot)$. Item 3 in Theorem 6.2.1 yields that

$$0 = \mathfrak{m}(\mathsf{X} \setminus \mathcal{R}_n) = \int_Q \mathfrak{m}_\alpha(\mathsf{X} \setminus \mathcal{R}_n)\mathfrak{q}(\mathrm{d}\alpha). \tag{6.2.1}$$

Then for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha(\mathsf{X} \setminus \mathcal{R}_n) = 0$, we set $\widetilde{Q} := \{\alpha \in Q : \mathfrak{m}_\alpha(\mathsf{X} \setminus \mathcal{R}_n) = 0\}$, then $R_x := (\cup_{\alpha \in \widetilde{Q}} \mathsf{X}_\alpha) \cap \mathcal{R}_n$ is the desired set. Indeed, for any $y \in R_x$, there is a geodesic (segment) γ contained in X_α joining x, y, for some $\alpha \in \widetilde{Q}$, with $\mathfrak{m}_\alpha(\gamma \setminus \mathcal{R}_n) = 0$. As pointed out in the overview, h_α is positive on X_α, so we get that $\mathcal{H}^1 \llcorner \mathsf{X}_\alpha(\gamma \setminus \mathcal{R}_n) = 0$, which in turn implies that regular points of essential dimension is dense in the interior of γ. Now apply Proposition 6.1.2, we see that the interior of γ is entirely in $\mathcal{R}_n$ and the end points are also in $\mathcal{R}_n$. □

Theorem 6.1.3 implies also the almost convexity of the interior of an $\mathrm{ncRCD}(K, N)$ space with boundary. To make the statement precise, let us present here some facts about the boundary of RCD spaces. To this end, we first introduce the singular set. The singular set $\mathcal{S}$ of $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, is the complement of regular sets, $\mathcal{S} := \mathsf{X} \setminus \cup_k \mathcal{R}_k$. Since the regular set of essential dimension already has full $\mathfrak{m}$ measure, we see that $\mathfrak{m}(\mathcal{S}) = 0$, beyond that we have relatively little information of $\mathcal{S}$ compare to $\mathcal{R}$. Without the space being non-collapsed, $\mathcal{S}$ can be wild. Pan-Wei in [110] constructed the following example: For any real number $\beta > 0$, there exists a Ricci limit space, that is $\mathrm{RCD}(0, N(\beta))$ for some $N(\beta) \geq 2$, homeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$, with regular set (topologically) $\mathbb{R} \times \mathbb{R}_+$ having Hausdorff dimension 2 and singular set (topologically) $\mathbb{R} \times \{0\}$ with Hausdorff dimension $1 + \beta$.

When restricted to $\mathrm{ncRCD}(K, N)$ spaces, more is known about $\mathcal{S}$. Thanks to the volume cone to metric cone property established in [79], in a $\mathrm{ncRCD}(K, N)$ space, $\mathcal{S}$ is stratified into

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_{N-1},$$

where for $0 \leq k \leq N-1$, $k \in \mathsf{Z}$, $\mathcal{S}_k = \{x \in \mathcal{S} :$ no tangent cone at x is isometric to $\mathbb{R}^{k+1} \times C(\mathsf{Z})$ for any metric space $\mathsf{Z}\}$, where $C(\mathsf{Z})$ is the metric cone over a metric space Z. It is proved in [78, Theorem 1.8] that $\dim_{\mathcal{H}}(\mathcal{S}_k) \leq k$. Much finer structure results are known for singular sets of Ricci limit spaces, see [111]. Based on the stratification result, De

Philippis-Gigli proposed the following definition of boundary of a $\mathrm{ncRCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$:

$$\partial\mathsf{X} := \overline{\mathcal{S}_{n-1}\setminus\mathcal{S}_{n-2}}. \tag{6.2.2}$$

On the other hand, Kapovitch-Mondino ([53]) proposed another recursive definition of boundary analogous to that of Alexandrov spaces:

$$\mathcal{F}\mathsf{X} := \{x\in\mathsf{X} : \exists\mathsf{Y}\in\mathrm{Tan}(\mathsf{X},\mathsf{d},\mathfrak{m},x), \mathsf{Y}=C(\mathsf{Z}), \mathcal{F}\mathsf{Z}\neq\emptyset\}. \tag{6.2.3}$$

In this definition Z must be a non-collapsed $\mathrm{RCD}(N-2,N-1)$ space with suitable metric and measure ([53, Lemma 4.1]), so one can recursively reduce the consideration to the case $N=1$, in which case the classification of $\mathrm{RCD}(K,1)$ is completed in [112].

The relation between 2 definitions of boundary (6.2.2) and (6.2.3) is studied in [52, Section 6], it is known that $\mathcal{F}\mathsf{X}\subseteq\partial\mathsf{X}$ and $\mathcal{F}\mathsf{X}\neq\emptyset\Leftrightarrow\partial\mathsf{X}\neq\emptyset$. It is conjectured that $\mathcal{F}\mathsf{X}=\partial\mathsf{X}$, we will discuss a related consequence in the sequel.

We adapt De Philippis-Gigli's definition of boundary, and call $\mathrm{Int}(\mathsf{X}) := \mathsf{X}\setminus\partial\mathsf{X}$ the interior of X. Now we can state the precise corollary of Theorem 6.1.3 as follows

Corollary 6.2.2. *Let* $(\mathsf{X},\mathsf{d},\mathcal{H}^N)$ *be an* $\mathrm{ncRCD}(K,N)$ *space. For every* $x\in\mathrm{Int}(\mathsf{X})$*, there exists a subset* $R_x\subseteq\mathrm{Int}(\mathsf{X})$ *so that* $\mathfrak{m}(\mathsf{X}\setminus R_x)=0$ *and for any* $y\in\mathrm{Int}(\mathsf{X})$ *there is a minimizing geodesic joining* x,y *and entirely contained in* $\mathrm{Int}(\mathsf{X})$.

To show this, we just replace $\mathcal{R}_n$ by $\mathrm{Int}(\mathsf{X})$ in the proof of Theorem 6.1.3, since $\mathcal{H}^N(\partial\mathsf{X})=0$.

6.3 Further studies

The interior of a $\mathrm{ncRCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is expected to be strongly convex. That is,

Conjecture 6.3.1. *For every* $x,y\in\mathrm{Int}(\mathsf{X})$*, every geodesic joining* x,y *is entirely contained in* $\mathrm{Int}(\mathsf{X})$.

It is pointed out by Kapovitch that this conjecture can follow from the equivalence of two definitions of boundary, (6.2.2) and (6.2.3). The expected argument goes as follows:

Given the equivalence, $x, y \in \mathrm{Int}(\mathsf{X})$, and a geodesic $\gamma_{x,y}$ connecting x, y, the intersection $\gamma_{x,y} \cap \partial\mathsf{X}$ is closed and is exactly the set of points at which there exists a tangent cone with boundary in $\gamma_{x,y}$, so $\gamma_{x,y} \cap \mathrm{Int}(\mathsf{X})$ is the set of points in $\gamma_{x,y}$ at which no tangent cone has boundary. [52, Theorem 1.6] asserts that, the set of all points at which no tangent cone has boundary is closed, in particular $\gamma_{x,y} \cap \mathrm{Int}(\mathsf{X})$ is closed, since $\gamma_{x,y}$ is connected, either $\gamma_{x,y} \cap \partial\mathsf{X}$ or $\gamma_{x,y} \cap \mathrm{Int}(\mathsf{X})$ is empty, given endpoints in $\gamma_{x,y} \cap \mathrm{Int}(\mathsf{X})$, we see that $\gamma_{x,y} \cap \partial\mathsf{X} = \emptyset$.

A closely related question is the strong convexity of $\mathcal{R} := \cup_k \mathcal{R}_k$. More precisely, given any $x, y \in \mathcal{R}$ is there a geodesic joining x, y completely in $\mathcal{R}$?

Another question related the convexity is the inverse of [21, Theorem 6.18], the theorem goes as follows:

Theorem 6.3.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, \infty)$ *space,* $\mathsf{Y} \subseteq \mathsf{X}$ *be a closed convex subset such that* $\mathfrak{m}(\mathsf{Y}) > 0$ *and* $\mathfrak{m}(\partial\mathsf{Y}) = 0$*, where* $\partial\mathsf{Y}$ *is the topological boundary. Then* $(\mathsf{Y}, \mathsf{d}, \mathfrak{m}\llcorner \mathsf{Y})$ *is also* $\mathrm{RCD}(K, N)$.

It is interesting to know the converse, that is: Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space, $\mathsf{Y} \subseteq \mathsf{X}$ be an open connected subset such that $\mathfrak{m}(\mathsf{Y}) > 0$, $\mathfrak{m}(\partial\mathsf{Y}) = 0$ and $(\mathsf{Y}, \mathsf{d}_{\mathsf{Y}}, \mathfrak{m} \llcorner Y)$ is also $\mathrm{RCD}(K, N)$, where d_{Y} is the length metric, then it is true that Y is geodesically convex, in the sense that every geodesic in distance d_{Y} is also a geodesic in distance d?

Appendices

APPENDIX A

ESSENTIAL DIMENSION IS A METRIC CONCEPT

We explain why essential dimension is a metric concept. First we give the precise statement of this fact.

Theorem A.0.1. *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}_1)$ *and* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}_2)$ *be* $\mathrm{RCD}(K, N)$ *spaces. Then their essential dimension is the same and equal to the maximal* n *such that* $(\mathbb{R}^n, |\cdot|, \mathcal{L}^n)$ *is a tangent cone of* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}_1)$*, hence also* $(\mathsf{X}, \mathsf{d}, \mathfrak{m}_2)$.

Remark A.0.2. it is proved in [46, Theoerm 1.2] that if there exists integer n such that for an $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, $\mathfrak{m}(\mathsf{X} \setminus \mathcal{R}_n) = 0$, then for any integer $k \in (n, N]$, $\mathcal{R}_k = \emptyset$, this already shows that the essential dimension is the maximal number of $\mathbb{R}$-factors can be split off in any tangent cone. It also worth mentioning that this proof relies on a fact that for any integer k, if there exists $x \in \mathcal{R}_k$, then there is a set of positive measure in the neighborhood of x such that the tangent cone at every point in this set splits at least k $\mathbb{R}$-factors, see [46, Propsition 3.6].

We start the proof.

Proof. Let $n_i := \mathrm{essdim}(\mathsf{X}, \mathsf{d}, \mathfrak{m}_i)$ and $\mathcal{R}^i_{n_i}$ be the regular set for $\mathfrak{m}_i$, $i = 1, 2$. By symmetry it suffices to prove that $n_1 \leq n_2$. Fix $x \in \mathcal{R}^1_{n_1}$. Then $\mathrm{Tan}(\mathsf{X}, \mathsf{d}, \mathfrak{m}_1, x) = \{(\mathbb{R}^n, |\cdot|, \mathcal{L}^{n_1}, 0)\}$, in particular $(\mathsf{X}, r^{-1}\mathsf{d}, x)$ pGH converges to $(\mathbb{R}^{n_1}, |\cdot|, 0)$ as $r \to 0$. It follows that $\mathrm{Tan}(\mathsf{X}, \mathsf{d}, \mathfrak{m}_2, x) = \{(\mathbb{R}^{n_1}, |\cdot|, \nu, 0) : \nu \text{ some Radon measure on } \mathbb{R}^n\}$. Note that by the stability of $\mathrm{RCD}(K, N)$ condition, for each ν, $(\mathbb{R}^{n_1}, |\cdot|, \nu, 0)$ is an $\mathrm{RCD}(0, N)$ space that contains n_1 lines. Applying the splitting theorem [27] recursively, we get that $\nu = \mathcal{L}^{n_1}$. $\square$

APPENDIX B

SPECTRUM OF HODGE LAPLACIAN ON COMPACT $\mathrm{RCD}(K,N)$ SPACES

This appendix is taken from [40, Appendix]. We provide a Rellich type compactness theorem for 1-forms on compact $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, stated as follows

Theorem B.0.1 (Rellich compactness)**.** *Let* $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *be a compact* $\mathrm{RCD}(K,N)$ *space. Then the canonical inclusion map:*

$$H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m})) \hookrightarrow L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m})) \tag{B.0.1}$$

is a compact operator.

This theorem is independently obtained in [113] as an application of the heat flow. Our proof is based on δ-splitting maps which is different from that of [113].

This theorem in particular proves that the space (5.3.19) is dense in $H^{12}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$;

$$\overline{\{\omega \in D(\Delta_{H,1}); \Delta_{H,1}\omega \in D(\delta)\}} = H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m})) \tag{B.0.2}$$

Let us mention that $h_{H,t}\omega$ is in (5.3.19) for any $\omega \in L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ and any $t>0$, which gives another proof of (B.0.2) without the compactness of (X,d), where $h_{H,t}$ is the heat flow acting on $L^2(T^*(X,\mathsf{d},\mathfrak{m}))$ associated to the energy

$$\omega \mapsto \frac{1}{2}\int_{\mathsf{X}} (|\mathrm{d}\omega|^2 + |\delta\omega|^2)\mathrm{d}\mathfrak{m}, \tag{B.0.3}$$

as discussed around 2.6.6, see also [54, (3.6.18)].

For the proof, we need several analytic notions, including the local Sobolev spaces $H^{1,p}(U,\mathsf{d},\mathfrak{m})$, the domain of local Laplacian $D(\Delta,U)(\subseteq H^{1,2}(U,\mathsf{d},\mathfrak{m}))$ with the Laplacian

$\Delta_U = \Delta$ for any open subset U of X and so on. We refer [94, 60, 114] for the detail. Let us recall that for $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ we have:

1. (Good cut-off function, Theorem 2.6.3) for any $x \in X$ and all $0 < r < R < \infty$, there exists $\varphi \in D(\Delta) \cap \mathrm{Lip}_b(X, \mathsf{d})$ such that $0 \le \varphi \le 1$ holds, that $\varphi \equiv 1$ holds on $B_r(x)$, that $\mathrm{supp}\,\varphi \subseteq B_R(x)$ holds, and that $|\nabla \varphi| + |\Delta \varphi| \le C(K, N, r, R)$ holds for $\mathfrak{m}$-a.e. $x \in X$;

2. (Hessian estimates for harmonic functions) For any harmonic function f on $B_R(x) \subseteq \mathsf{X}$ with $|\nabla f| \le L$, that is, $f \in D(\Delta, B_R(x))$ with $\Delta f \equiv 0$, and for any $r < R$, we have
$$\int_{B_r(x)} |\mathrm{Hess}_f|^2 \mathrm{d}\mathfrak{m} \le C(K, N, r, R, L). \tag{B.0.4}$$

Note that the Hessian of a harmonic function f as above is well-defined as a measurable tensor over $B_R(x)$ because of the locality of the Hessian proved in [54, Prop.3.3.24], see also [70, (1.1)]. The proof of (B.0.4) is easily done by applying (2.1.2) with the good cut-off function.

Finally let us recall a useful notation from the convergence theory;

$$\Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_l; c_1, c_2, \ldots, c_m) \tag{B.0.5}$$

denotes a function $\Psi : (\mathbb{R}_{>0})^l \times \mathbb{R}^m \to (0, \infty)$ satisfying

$$\lim_{(\epsilon_1, \ldots, \epsilon_k) \to 0} \Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_l; c_1, c_2, \ldots, c_m) = 0, \quad \forall c_i. \tag{B.0.6}$$

Proof. Proof of Theorem B.0.1. With no loss of generality we can assume that $\mathfrak{m}(\mathsf{X}) = 1$ and $N > 1$. Let ω_i be a bounded sequence in $H^{1,2}_C(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$. By the L^2-weak compactness with no loss of generality we can assume that ω_i L^2-weakly converge to some $\omega \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$. Our goal is to prove that this is an L^2-strong convergence.

Let us remark that thanks to [54, Prop.3.4.6] (recall that for any $\omega \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$, $\omega \in W^{1,2}_C(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ holds if and only if $\omega^\sharp \in W^{1,2}_C(T(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$ holds), we have $|\omega_i|^2 \in H^{1,1}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with $|\nabla|\omega_i|^2| \le 2|\nabla\omega_i||\omega_i|$ for $\mathfrak{m}$-a.e. $x \in X$. In particular the Sobolev embedding theorem proved in [66, Thm.5.1] yields

$$\sup_i \||\omega_i|^2\|_{L^{p_N}} < \infty, \tag{B.0.7}$$

where $p_N := N/(N-1)$ because a Poincaré inequality 2.4.5 is satisfied, and the Bishop-Gromov inequality implies the inequality $\mathfrak{m}(B_s(y)) \ge C(s/r)^N \mathfrak{m}(B_r(x))$ for all $x \in \mathsf{X}$, $y \in B_r(x)$ and $s \in (0, r]$.

Fix $\epsilon > 0$ and put $n := \mathrm{essdim}(\mathsf{X})$. For any $x \in \mathcal{R}_n$ there exists $r_x > 0$ such that for any $r \in (0, r_x)$ there exists a harmonic map $\Phi_{r,x} = (\varphi_{r,x,1}, \varphi_{r,x,2}, \ldots, \varphi_{r,x,n}) : B_{2r}(x) \to \mathbb{R}^n$ (that is, each $\varphi_{r,x,i}$ is a harmonic function on $B_{2r}(x)$) such that $|\nabla\varphi_{r,x,i}| \le C(K, N)$ holds for any i, that

$$\frac{1}{\mathfrak{m}(B_{2r}(x))}\int_{B_{2r}(x)} |\langle \nabla\varphi_{r,x,i}, \nabla\varphi_{r,x,j}\rangle - \delta_{ij}| \,\mathrm{d}\mathfrak{m} + \frac{r^2}{\mathfrak{m}(B_{2r}(x))}\int_{B_{2r}(x)} |\mathrm{Hess}_{\varphi_{r,x,i}}|^2 \mathrm{d}\mathfrak{m} \le \epsilon \tag{B.0.8}$$

holds for all i, j (see [70, Prop.1.4]). Note that the L^2-weak convergence of ω_i to ω yields that $\langle \mathrm{d}\varphi_{r,x,j}, \omega_i\rangle$ L^2-weakly converge to $\langle \mathrm{d}\varphi_{r,x,j}, \omega\rangle$ on $B_{2r}(x)$ for any j.

On the other hand applying [54, Prop.3.4.6] (with a good cut-off function as above) again yields $\langle \mathrm{d}\varphi_{r,x,j}, \omega_i\rangle \in H^{1,1}(B_r(x), \mathsf{d}, \mathfrak{m})$ with

$$|\nabla\langle \mathrm{d}\varphi_{r,x,j}, \omega\rangle| \le |\mathrm{Hess}_{\varphi_{r,x,j}}||\omega_i| + |\nabla\varphi_{r,x,j}||\nabla\omega_i|, \quad \text{for } \mathfrak{m} - \text{a.e. } x \in B_r(x). \tag{B.0.9}$$

To show this, take $\varphi \in D(\Delta) \cap \mathrm{Lip}_b(\mathsf{X}, \mathsf{d})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ holds on $B_r(x)$, and that $\mathrm{supp}\,\varphi \subseteq B_{2r}(x)$, $|\nabla\varphi| + |\Delta f| \le C(K, N, r)$ holds for $\mathfrak{m}$-a.e. $x \in \mathsf{X}$. Then since $\varphi\varphi_{r,x,j} \in D(\Delta) \cap \mathrm{Lip}_b(X, \mathsf{d})$, applying [54, Prop.3.4.6] yields that $\langle \mathrm{d}(\varphi\varphi_{r,x,j}), \omega_i\rangle \in$

$H^{1,1}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ with

$$|\nabla\langle \mathrm{d}(\varphi\varphi_{r,x,j}), \omega\rangle| \le |\mathrm{Hess}_{\varphi\varphi_{r,x,j}}||\omega_i| + |\nabla\varphi_{r,x,j}||\nabla(\varphi\omega_i)|, \quad \text{for } \mathfrak{m}-\text{a.e. } x \in \mathsf{X}.$$

Restricting this observation to $B_r(x)$ with the locality properties of the gradient (for instance [54, Thm.2.2.6]) and of the Hessian [54, Prop.3.3.24] proves the desired statement.

In particular (B.0.4) shows

$$\sup_i \|\langle \mathrm{d}\varphi_{r,x,j}, \omega_i\rangle\|_{H^{1,1}(B_r(x),\mathsf{d},\mathfrak{m})} < \infty. \tag{B.0.10}$$

Therefore applying the Rellich compactness theorem for $H^{1,1}$-functions proved in [66, Thm.8.1] shows that $\langle \mathrm{d}\varphi_{r,x,j}, \omega_i\rangle$ L^p-strongly converge to $\langle \mathrm{d}\varphi_{r,x,j}, \omega\rangle$ on $B_r(x)$ for all $p \in [1, p_N)$. By (B.0.7) we see that $\langle \mathrm{d}\varphi_{r,x,j}, \omega_i\rangle$ L^2-strongly converge to $\langle \mathrm{d}\varphi_{r,x,j}, \omega\rangle$ on $B_r(x)$ for any j.

Let

$$A(r,x) := \left\{ y \in B_r(x); |\langle \nabla\varphi_{r,x,i}, \nabla\varphi_{r,x,j}\rangle(y) - \delta_{ij}| \le \epsilon^{1/2}, \forall i, \forall j \right\}. \tag{B.0.11}$$

Then the Markov inequality with (B.0.8) shows

$$\frac{\mathfrak{m}(B_r(x) \setminus A(r,x))}{\mathfrak{m}(B_r(x))} \le \epsilon^{1/2}. \tag{B.0.12}$$

Note that for any $\eta \in L^2(T^*(\mathsf{X}, \mathsf{d}, \mathfrak{m}))$

$$\left| |\eta|^2(y) - \sum_j \langle \mathrm{d}\varphi_{r,x,j}, \eta\rangle^2(y) \right| \le \Psi(\epsilon; n)\, |\eta|^2, \quad \text{for a.e. } y \in A(r,x). \tag{B.0.13}$$

See also [43, (5.36) and (5.37)]. Applying the Vitali covering theorem to a family $\mathcal{F} := \{\overline{B}_r(x)\}_{x \in \mathcal{R}_n, r < r_x}$ yields that there exists a pairwise disjoint subfamily $\{\overline{B}_{r_j}(x_j)\}_{j \in \mathbb{N}}$ of $\mathcal{F}$

such that

$$\mathcal{R}_n \setminus \bigsqcup_{j=1}^{k} \overline{B}_{r_j}(x_j) \subseteq \bigcup_{j\ge k+1} \overline{B}_{5r_j}(x_j), \quad \forall k \tag{B.0.14}$$

holds. Take k_0 with $\sum_{j\ge k_0+1} \mathfrak{m}(B_{r_j}(x_j)) < \epsilon$. Then by (B.0.12) we have

$$\begin{aligned}
\mathfrak{m}\left(\mathsf{X} \setminus \bigsqcup_{j=1}^{k_0} A(r_j,x_j)\right) &\le \mathfrak{m}\left(\mathsf{X} \setminus \bigsqcup_{j=1}^{k_0} B_{r_j}(x_j)\right) + \sum_{j=1}^{k_0} \mathfrak{m}(B_{r_j}(x_j) \setminus A(r_j,x_j)) \\
&\le \sum_{j\ge k_0+1} \mathfrak{m}(B_{5r_j}(x_j)) + \epsilon^{1/2} \sum_{j=1}^{k_0} \mathfrak{m}(B_{r_j}(x_j)) \\
&\le C(K,N) \sum_{j\ge k_0+1} \mathfrak{m}(B_{r_j}(x_j)) + \epsilon^{1/2} \\
&\le \Psi(\epsilon; K, N).
\end{aligned} \tag{B.0.15}$$

Thus for any sufficiently large i we have

$$\begin{aligned}
&\int_{\mathsf{X}} |\omega_i|^2 \mathrm{d}\mathfrak{m} \\
&= \sum_{j=1}^{k_0} \int_{A(r_j,x_j)} |\omega_i|^2 \mathrm{d}\mathfrak{m} + \int_{X \setminus \bigsqcup_{j=1}^{k_0} A(r_j,x_j)} |\omega_i|^2 \mathrm{d}\mathfrak{m} \\
&\le \sum_{j=1}^{k_0} \sum_{l=1}^{n} \int_{A(r_j,x_j)} (\langle \mathrm{d}\varphi_{r_j,x_j,l}, \omega_i\rangle^2 + \Psi(\epsilon;n)|\omega_i|^2) \mathrm{d}\mathfrak{m} + \mathfrak{m}^{\frac{1}{q_N}} \left(X \setminus \bigsqcup_{j=1}^{k_0} A(r_j,x_j)\right) \||\omega_i|^2\|_{L^{p_N}} \\
&\le \sum_{j=1}^{k_0} \sum_{l=1}^{n} \int_{A(r_j,x_j)} \langle \mathrm{d}\varphi_{r_j,x_j,l}, \omega\rangle^2 \mathrm{d}\mathfrak{m} + \Psi(\epsilon;n) \sup_m \|\omega_m\|_{L^2}^2 + \Psi(\epsilon;K,N) \sup_m \||\omega_m|^2\|_{L^{p_N}} \\
&\le \sum_{j=1}^{k_0} \sum_{l=1}^{n} \int_{A(r_j,x_j)} (1+\Psi(\epsilon;n))|\omega|^2 \mathrm{d}\mathfrak{m} + \Psi(\epsilon;K,N)(\sup_m \|\omega_m\|_{L^2}^2 + \sup_m \||\omega_m|^2\|_{L^{p_N}}) \\
&\le \int_X |\omega|^2 \mathrm{d}\mathfrak{m} + \Psi(\epsilon;K,N)(\sup_m \|\omega_m\|_{L^2}^2 + \sup_m \||\omega_m|^2\|_{L^{p_N}}),
\end{aligned} \tag{B.0.16}$$

where q_N is the conjugate exponent of p_N. Since ϵ is arbitrary, (B.0.16) shows that

$$\limsup_{i\to\infty} \int_X |\omega_i|^2 \mathrm{d}\mathfrak{m} \le \int_X |\omega|^2 \mathrm{d}\mathfrak{m} \tag{B.0.17}$$

which completes the proof of the L^2-strong convergence of ω_i to ω. $\square$

The following fact is a refinement of $\mathrm{trHess} = \Delta$.

Theorem B.0.2. *Let* $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)$ *be a non-collapsed* $\mathrm{RCD}(K, N)$ *space. Then we have* $H^{1,2}_C(T^*(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)) \subseteq D(\delta)$ *with*

$$\delta\omega = -\mathrm{tr}\nabla\omega, \quad \forall \omega \in H^{1,2}_C(T^*(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)). \tag{B.0.18}$$

Proof. we see that for $f_i \in \mathrm{Test}F(X, \mathsf{d}, \mathcal{H}^N)$,

$$\begin{aligned}\delta(f_1 \mathrm{d}f_2) &= -\langle \mathrm{d}f_1, \mathrm{d}f_2\rangle - f_1 \Delta f_2 \\ &= -\langle \mathrm{d}f_1, \mathrm{d}f_2\rangle - f_1 \mathrm{tr}(\mathrm{Hess}_{f_2}) \\ &= -\langle g, \mathrm{d}f_1 \otimes \mathrm{d}f_2\rangle - \langle g, f_1 \mathrm{Hess}_{f_2}\rangle = -\langle g, \nabla(f_1 \mathrm{d}f_2)\rangle = -\mathrm{tr}\nabla(f_1 \mathrm{d}f_2)\end{aligned}$$

holds, which shows that (B.0.18) holds for all $\omega \in \mathrm{Test}T^*(X, \mathsf{d}, \mathcal{H}^N)$. Thus we have the conclusion because by definition $\mathrm{Test}T^*(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)$ is dense in $H^{1,2}_C(T^*(X, \mathsf{d}, \mathcal{H}^N))$. $\square$

It directly follows from Theorem B.0.2 that for a non-collapsed $\mathrm{RCD}(K, N)$ space $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)$ and any $f \in D(\Delta)$, we have $fg \in D(\nabla^*)$ with

$$\nabla^*(fg) = -\mathrm{d}f \tag{B.0.19}$$

because for any $\omega \in H^{1,2}_C(T^*(\mathsf{X}, \mathsf{d}, \mathcal{H}^N))$,

$$\int_{\mathsf{X}} \langle \omega, \nabla^*(fg)\rangle \mathrm{d}\mathcal{H}^N = \int_{\mathsf{X}} \langle \nabla\omega, fg\rangle \mathrm{d}\mathcal{H}^N = \int_{\mathsf{X}} f\delta\omega \mathrm{d}\mathcal{H}^N = \int_{\mathsf{X}} \langle \mathrm{d}f, \omega\rangle \mathrm{d}\mathcal{H}^N. \tag{B.0.20}$$

The following is also a direct consequence of (2.2.14), (2.2.16) and Theorem B.0.2:

Corollary B.0.3. *Let* $(\mathsf{X}, \mathsf{d}, \mathcal{H}^N)$ *be a non-collapsed* $\mathrm{RCD}(K, N)$ *space. Then we have*

$H^{1,2}_H(T^*(\mathsf{X},\mathsf{d},\mathcal{H}^N)) = H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathcal{H}^N))$ *with*

$$\frac{1}{2}\|\omega\|_{H^{1,2}_H} \le \|\omega\|_{H^{1,2}_C} \le (1+K^-)\|\omega\|_{H^{1,2}_H}, \quad \forall \omega \in H^{1,2}_H(T^*(X,\mathsf{d},\mathcal{H}^N)), \tag{B.0.21}$$

where $K^- = \max\{0, -K\}$.

Then the following corollary is a direct consequence of Corollary B.0.3 and Theorem B.0.1 (see for instance the appendix of [115]).

Corollary B.0.4. *The spectrum of the Hodge Laplacian* $\Delta_{H,1}$ *acting on* 1*-forms is discrete and unbounded. If we denote the spectrum by*

$$0 \le \lambda_{(H,1),1} \le \lambda_{(H,1),2} \le \lambda_{(H,1),3} \le \cdots \le \lambda_{(H,1),k} \le \cdots \to \infty \tag{B.0.22}$$

*counted with multiplicities, then corresponding eigen-*1*-forms* $\omega_1, \omega_2, \ldots$ *with* $\|\omega_k\|_{L^2} = 1$ *give an orthogonal basis of* $L^2(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$.

Remark B.0.5. Under the same notation as in Corollary B.0.4, it is easy to see that for any $\omega \in H^{1,2}_H(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$,

$$\omega = \sum_i \left(\int_X \langle \omega, \omega_i\rangle \mathrm{d}\mathfrak{m}\right)\omega_i \tag{B.0.23}$$

in $H^{1,2}_H(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$. In particular (B.0.23) also holds in $H^{1,2}_C(T^*(\mathsf{X},\mathsf{d},\mathfrak{m}))$ because of (2.2.14).

Remark B.0.6. As an immediate consequence of Theorem B.0.1, we are able to prove a similar spectral decomposition result as in Corollary B.0.4 for the *connection Laplacian* $\Delta_{C,1}$ acting on 1-forms. Moreover the technique provided in the proof of Theorem B.0.1 allows us to prove similar decomposition results for the connection Laplacian acting on differential forms and tensor fields of any type. Compare with [116, 115].

REFERENCES

[1] X. Dai and G. Wei, “Comparison geometry for ricci curvature,” Preprint, 2012.

[2] S.-Y. Cheng, “Eigenvalue comparison theorems and its geometric applications,” *Mathematische Zeitschrift*, vol. 143, no. 3, pp. 289–297, 1975.

[3] M. OBATA, “Certain conditions for a riemannian manifold to be isometric with a sphere,” *Journal of the Mathematical Society of Japan*, vol. 14, no. 3, pp. 333–340, 1962.

[4] J. Cheeger and D. Gromoll, “The splitting theorem for manifolds of nonnegative Ricci curvature,” *J. Differential Geometry*, vol. 6, pp. 119–128, 1971/72.

[5] P. Li and S. T. Yau, “On the parabolic kernel of the Schrödinger operator,” *Acta Mathematica*, vol. 156, no. none, pp. 153–201, 1986.

[6] L. Saloff-Coste, “Uniformly elliptic operators on Riemannian manifolds,” *Journal of Differential Geometry*, vol. 36, no. 2, pp. 417–450, 1992.

[7] J. Lohkamp, “Metrics of negative ricci curvature,” *Annals of Mathematics*, vol. 140, no. 3, pp. 655–683, 1994.

[8] M. T. Anderson, “Short geodesics and gravitational instantons,” *J. Differential Geom.*, vol. 31, no. 1, pp. 265–275, 1990.

[9] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, vol. 33, pp. xiv+415, ISBN: 0-8218-2129-6.

[10] M. Gromov, “Sign and geometric meaning of curvature,” *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 61, no. 1, pp. 9–123, 1991.

[11] J. Cheeger and T. H. Colding, “Lower bounds on Ricci curvature and the almost rigidity of warped products,” *Ann. of Math. (2)*, vol. 144, no. 1, pp. 189–237, 1996.

[12] ——, “On the structure of spaces with Ricci curvature bounded below. I,” *J. Differential Geom.*, vol. 46, no. 3, pp. 406–480, 1997.

[13] ——, “On the structure of spaces with Ricci curvature bounded below. II,” *J. Differential Geom.*, vol. 54, no. 1, pp. 13–35, 2000.

[14] ——, “On the structure of spaces with Ricci curvature bounded below. III,” *J. Differential Geom.*, vol. 54, no. 1, pp. 37–74, 2000.

[15] C. Villani, *Optimal transport. Old and new*, ser. Grundlehren der Mathematischen Wissenschaften. Berlin: Springer-Verlag, 2009, vol. 338, pp. xxii+973, ISBN: 978-3-540-71049-3.

[16] R. J. McCann, "A convexity principle for interacting gases," *Advances in Mathematics*, vol. 128, no. 1, pp. 153–179, 1997.

[17] M.-K. von Renesse and K.-T. Sturm, "Transport inequalities, gradient estimates, entropy and ricci curvature," *Communications on Pure and Applied Mathematics*, vol. 58, no. 7, pp. 923–940, 2005. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.20060.

[18] K.-T. Sturm, "On the geometry of metric measure spaces. II," *Acta Math.*, vol. 196, no. 1, pp. 133–177, 2006.

[19] S.-i. Ohta, "Finsler interpolation inequalities," *Calc. Var. Partial Differential Equations*, vol. 36, no. 2, pp. 211–249, 2009.

[20] ——, "Splitting theorems for Finsler manifolds of nonnegative Ricci curvature," Preprint, arXiv:1203.0079, 2012.

[21] L. Ambrosio, N. Gigli, and G. Savaré, "Metric measure spaces with Riemannian Ricci curvature bounded from below," *Duke Mathematical Journal*, vol. 163, no. 7, pp. 1405–1490, 2014.

[22] N. Gigli, "On the differential structure of metric measure spaces and applications," *Mem. Amer. Math. Soc.*, vol. 236, no. 1113, pp. vi+91, 2015.

[23] M. Erbar, K. Kuwada, and K.-T. Sturm, "On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces," *Invent. Math.*, vol. 201, no. 3, pp. 1–79, 2014.

[24] N. Gigli, A. Mondino, and G. Savaré, "Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows," *Proc. Lond. Math. Soc. (3)*, vol. 111, no. 5, pp. 1071–1129, 2015.

[25] C. Ketterer, "Cones over metric measure spaces and the maximal diameter theorem," *J. Math. Pures Appl. (9)*, vol. 103, no. 5, pp. 1228–1275, 2015.

[26] ——, "Obata's Rigidity Theorem for Metric Measure Spaces," *Analysis and Geometry in Metric Spaces*, vol. 3, no. 1, Oct. 2015.

[27] N. Gigli, "The splitting theorem in non-smooth context," Preprint, arXiv:1302.5555, 2013.

[28] K.-T. Sturm, “On the geometry of metric measure spaces. I,” *Acta Math.*, vol. 196, no. 1, pp. 65–131, 2006.

[29] T. Rajala, “Failure of the local-to-global property for $CD(K, N)$ spaces,” Preprint, arXiv:1207.6754, 2013.

[30] K. Bacher and K.-T. Sturm, “Localization and tensorization properties of the curvature-dimension condition for metric measure spaces,” *J. Funct. Anal.*, vol. 259, no. 1, pp. 28–56, 2010.

[31] T. Rajala and K.-T. Sturm, “Non-branching geodesics and optimal maps in strong $CD(K, \infty)$-spaces,” *Calc. Var. Partial Differential Equations*, vol. 50, no. 3-4, pp. 831–846, 2012.

[32] F. Cavalletti and E. Milman, “The globalization theorem for the curvature dimension condition,” Preprint, arXiv:1612.07623, 2016.

[33] Q. Deng, “Hölder continuity of tangent cones in RCD(K,N) spaces and applications to non-branching,” 2020. arXiv: 2009.07956 `[math.DG]`.

[34] J. Cheeger and A. Naber, “Regularity of einstein manifolds and the codimension 4 conjecture,” *Annals of Mathematics*, vol. 182, no. 3, pp. 1093–1165, 2015.

[35] T. H. Colding and A. Naber, “Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications,” *Ann. of Math. (2)*, vol. 176, no. 2, pp. 1173–1229, 2012.

[36] A. Naber, “Characterizations of bounded ricci curvature on smooth and nonsmooth spaces,” Preprint arXiv:1306.6512, 2013.

[37] K.-T. Sturm, “Remarks about synthetic upper Ricci bounds for metric measure spaces,” *Tohoku Mathematical Journal*, vol. 73, no. 4, pp. 539–564, 2021.

[38] Y. Guo, “The measure preserving isometry groups of metric measure spaces,” *SIGMA*, vol. 16, no. 114, 2020.

[39] V. Kapovitch and C. Ketterer, “CD meets CAT,” *J. Reine Angew. Math.*, vol. 766, pp. 1–44, 2020.

[40] S. Honda and X. Zhu, “A characterization of non-collapsed $RCD(K, N)$ spaces via Einstein tensors,” 2020. arXiv: 2010.02530 `[math.DG]`.

[41] C. Brena, N. Gigli, S. Honda, and X. Zhu, “Weakly non-collapsed RCD spaces are strongly non-collapsed,” preprint arXiv:2110.02420, 2021.

[42] P. Bérard, G. Besson, and S. Gallot, “Embedding Riemannian manifolds by their heat kernel,” *Geom. Funct. Anal.*, vol. 4, no. 4, pp. 373–398, 1994.

[43] L. Ambrosio, S. Honda, J. W. Portegies, and D. Tewodrose, “Embedding of $\mathrm{RCD}^*(K,N)$ spaces in L^2 via eigenfunctions,” *J. Funct. Anal.*, vol. 280, no. 10, Paper No. 108968, 72, 2021.

[44] L. Ambrosio, S. Honda, and D. Tewodrose, “Short-time behavior of the heat kernel and Weyl’s law on $\mathrm{RCD}^*(K,N)$ spaces,” *Ann. Global Anal. Geom.*, vol. 53, no. 1, pp. 97–119, 2018.

[45] S. Honda, “New differential operator and noncollapsed RCD spaces,” *Geom. Topol.*, vol. 24, no. 4, pp. 2127–2148, 2020.

[46] Y. Kitabeppu, “A sufficient condition to a regular set being of positive measure on RCD spaces,” *Potential Anal.*, vol. 51, no. 2, pp. 179–196, 2019.

[47] J. Cheeger, “Differentiability of Lipschitz functions on metric measure spaces,” *Geom. Funct. Anal.*, vol. 9, no. 3, pp. 428–517, 1999.

[48] A. Petrunin, “Alexandrov meets Lott-Villani-Sturm,” *Münster J. Math.*, vol. 4, pp. 53–64, 2011.

[49] D. Bakry and M. Émery, “Diffusions hypercontractives,” in *Séminaire de probabilités, XIX, 1983/84*, ser. Lecture Notes in Math. Vol. 1123, Berlin: Springer, 1985, pp. 177–206.

[50] L. Ambrosio, N. Gigli, and G. Savaré, “Bakry-émery curvature-dimension condition and Riemannian Ricci curvature bounds,” *The Annals of Probability*, vol. 43, no. 1, pp. 339–404, 2015.

[51] J. Bertrand, C. Ketterer, I. Mondello, and T. Richard, “Stratified spaces and synthetic Ricci curvature bounds,” *Annales de l’Institut Fourier*, vol. 71, no. 1, pp. 123–173, 2021.

[52] E. Bruè, A. Naber, and D. Semola, “Boundary regularity and stability for spaces with Ricci bounded below,” Preprint, arXiv:2011.08383, 2020.

[53] V. Kapovitch and A. Mondino, “On the topology and the boundary of N-dimensional $\mathrm{RCD}(\mathrm{K},\mathrm{N})$ spaces,” *Geom. Topol.*, vol. 25, no. 1, pp. 445–495, 2021.

[54] N. Gigli, “Nonsmooth differential geometry - an approach tailored for spaces with Ricci curvature bounded from below,” *Mem. Amer. Math. Soc.*, vol. 251, no. 1196, pp. v+161, 2014.

[55] N. Weaver, "Lipschitz algebras and derivations. II. Exterior differentiation," *J. Funct. Anal.*, vol. 178, no. 1, pp. 64–112, 2000.

[56] N. Gigli, "Lecture notes on differential calculus on RCD spaces," *Publ. Res. Inst. Math. Sci.*, vol. 54, no. 4, pp. 855–918, 2018.

[57] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Second, ser. Lectures in Mathematics ETH Zürich. Basel: Birkhäuser Verlag, 2008, pp. x+334, ISBN: 978-3-7643-8721-1.

[58] G. Savaré, "Self-improvement of the Bakry-émery condition and Wasserstein contraction of the heat flow in $\mathrm{RCD}(K,\infty)$ metric measure spaces," *Discrete Contin. Dyn. Syst.*, vol. 34, no. 4, pp. 1641–1661, 2014.

[59] S. Honda, "Isometric immersions of RCD spaces," To appear in Comment. Math. Helv., arXiv: 2005.01222, 2020. arXiv: 2005.01222 [math.DG].

[60] L. Ambrosio and S. Honda, "Local spectral convergence in $\mathrm{RCD}^*(K,N)$ spaces," *Nonlinear Anal.*, vol. 177, no. part A, pp. 1–23, 2018.

[61] M.-C. Lee, A. Naber, and R. Neumayer, "d_p Convergence and ϵ-regularity theorems for entropy and scalar curvature lower bounds," Preprint arXiv:2010.15663, 2021.

[62] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, English, ser. Modern Birkhäuser Classics. Boston, MA: Birkhäuser Boston Inc., 2007, pp. xx+585, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates, ISBN: 978-0-8176-4582-3; 0-8176-4582-9.

[63] P. Petersen, *Riemannian geometry*, Third, ser. Graduate Texts in Mathematics. Springer, Cham, 2016, vol. 171, pp. xviii+499, ISBN: 978-3-319-26652-7; 978-3-319-26654-1.

[64] J. Lott and C. Villani, "Ricci curvature for metric-measure spaces via optimal transport," *Ann. of Math. (2)*, vol. 169, no. 3, pp. 903–991, 2009.

[65] T. Rajala, "Local Poincaré inequalities from stable curvature conditions on metric spaces," *Calc. Var. Partial Differential Equations*, vol. 44, no. 3-4, pp. 477–494, 2012.

[66] P. Hajłasz and P. Koskela, "Sobolev met Poincaré," *Mem. Amer. Math. Soc.*, vol. 145, May 2000.

[67] T. H. Colding and A. Naber, “Characterization of tangent cones of noncollapsed limits with lower ricci bounds and applications,” *Geometric and Functional Analysis*, vol. 23, no. 1, pp. 134–148, 2013.

[68] E. Brué and D. Semola, “Constancy of the dimension for $\mathrm{RCD}(K, N)$ spaces via regularity of Lagrangian flows,” *Comm. Pure Appl. Math.*, vol. 73, no. 6, pp. 1141–1204, 2020.

[69] S. Honda and I. Mondello, “Sphere theorems for RCD and stratified spaces,” *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, vol. 22, no. 2, pp. 903–923, 2021.

[70] E. Bruè, E. Pasqualetto, and D. Semola, “Rectifiability of $\mathrm{RCD}(\mathrm{K}, \mathrm{N})$ spaces via δ-splitting maps,” *Ann. Fenn. Math.*, vol. 46, no. 1, pp. 465–482, 2021.

[71] A. Mondino and A. Naber, “Structure theory of metric measure spaces with lower Ricci curvature bounds,” *Journal of the European Mathematical Society*, vol. 21, no. 6, pp. 1809–1854, Mar. 2019.

[72] M. Kell and A. Mondino, “On the volume measure of non-smooth spaces with Ricci curvature bounded below,” *Annali della Scuola normale superiore di Pisa, Classe di scienze*, vol. XVII, pp. 593–610, Apr. 2018.

[73] N. Gigli and E. Pasqualetto, “Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces,” To appear in Comm. Anal. Geom., arXiv: 1611.09645, 2016.

[74] G. De Philippis, A. Marchese, and F. Rindler, “On a conjecture of Cheeger,” in *Measure theory in non-smooth spaces*, ser. Partial Differ. Equ. Meas. Theory, De Gruyter Open, Warsaw, 2017, pp. 145–155.

[75] N. Gigli and E. Pasqualetto, “Behaviour of the reference measure on RCD spaces under charts,” Accepted at Comm. Anal. Geom., arXiv: 1607.05188, 2016.

[76] L. Ambrosio and P. Tilli, *Topics on analysis in metric spaces*, ser. Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2004, vol. 25, pp. viii+133, ISBN: 0-19-852938-4.

[77] Y. Kitabeppu, “A Bishop-type inequality on metric measure spaces with Ricci curvature bounded below,” *Proc. Amer. Math. Soc.*, vol. 145, no. 7, pp. 3137–3151, 2017.

[78] G. De Philippis and N. Gigli, “Non-collapsed spaces with Ricci curvature bounded from below,” *J. Éc. polytech. Math.*, vol. 5, pp. 613–650, 2018.

[79] ——, “From volume cone to metric cone in the nonsmooth setting,” *Geom. Funct. Anal.*, vol. 26, no. 6, pp. 1526–1587, 2016.

[80] T. H. Colding, “Ricci curvature and volume convergence,” *Ann. of Math. (2)*, vol. 145, no. 3, pp. 477–501, 1997.

[81] B.-X. Han, “Ricci tensor on $\mathrm{RCD}^*(K,N)$ spaces,” *J. Geom. Anal.*, vol. 28, no. 2, pp. 1295–1314, 2018.

[82] L. Ambrosio, A. Mondino, and G. Savaré, “On the Bakry-émery condition, the gradient estimates and the Local-to-Global property of $RCD^*(K,N)$ metric measure spaces,” *The Journal of Geometric Analysis*, vol. 26, no. 1, pp. 1–33, 2014.

[83] N. Gigli and E. Pasqualetto, *Lectures on Nonsmooth Differential Geometry*. Springer International Publishing, 2020.

[84] K.-T. Sturm, “Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations,” *Osaka J. Math.*, vol. 32, no. 2, pp. 275–312, 1995.

[85] ——, “Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality,” *J. Math. Pures Appl. (9)*, vol. 75, no. 3, pp. 273–297, 1996.

[86] R. Jiang, H. Li, and H. Zhang, “Heat Kernel Bounds on Metric Measure Spaces and Some Applications,” *Potential Analysis*, vol. 44, 2014.

[87] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. De Gruyter, 2011, ISBN: 9783110889741.

[88] H. Zhang and X.-P. Zhu, “Weyl’s law on $\mathrm{RCD}^*(K,N)$ metric measure spaces,” *Communications in Analysis and Geometry*, vol. 27, no. 8, pp. 1869–1914, Jan. 2019.

[89] R. Jiang, “Cheeger–harmonic functions in metric measure spaces revisited,” *Journal of Functional Analysis*, vol. 266, pp. 1373–1394, 2014.

[90] J. Diestel and J. J. Uhl Jr., *Vector measures*. American Mathematical Society, Providence, R.I., 1977, pp. xiii+322, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[91] L. Ambrosio, M. Colombo, and S. Di Marino, “Sobolev spaces in metric measure spaces: Reflexivity and lower semicontinuity of slope,” in *Variational methods for evolving objects*, ser. Adv. Stud. Pure Math. Vol. 67, Math. Soc. Japan, [Tokyo], 2015, pp. 1–58.

[92] E. Caputo, N. Gigli, and E. Pasqualetto, “Parallel transport on non-collapsed $\mathrm{RCD}(K, N)$ spaces,” Preprint, arXiv:2108.07531.

[93] N. Gigli, “On the heat flow on metric measure spaces: Existence, uniqueness and stability,” *Calc. Var. PDE*, vol. 39, no. 1-2, pp. 101–120, 2010.

[94] L. Ambrosio and S. Honda, “New stability results for sequences of metric measure spaces with uniform Ricci bounds from below,” *Measure Theory in Non-Smooth Spaces*, pp. 1–51, 2017.

[95] S. Honda and Y. Sire, “Sobolev mappings between RCD spaces and applications to harmonic maps: a heat kernel approach,” Preprint, arXiv: 2105.08578, 2021.

[96] J.-C. Huang and H.-C. Zhang, “Sharp gradient estimate for heat kernels on $RCD^*(K, N)$ metric measure spaces,” Jan. 2017.

[97] V. Kapovitch and C. Ketterer, “Weakly noncollapsed RCD spaces with upper curvature bounds,” *Anal. Geom. Metr. Spaces*, vol. 7, no. 1, pp. 197–211, 2019.

[98] B.-X. Han, “Measure rigidity of synthetic lower Ricci curvature bound on Riemannian manifolds,” *Adv. Math.*, vol. 373, pp. 107327, 31, 2020.

[99] J.-i. Nagata, *Modern dimension theory*, Revised, ser. Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin, 1983, vol. 2, pp. ix+284, ISBN: 3-88538-002-1.

[100] J. Heinonen, *Lectures on Analysis on Metric Spaces*, ser. Universitext. New York: Springer-Verlag, 2001, pp. x+140, ISBN: 0-387-95104-0.

[101] L. Ambrosio, N. Gigli, and G. Savaré, “Metric measure spaces with Riemannian Ricci curvature bounded from below,” *Duke Mathematical Journal*, vol. 163, no. 7, pp. 1405–1490, May 2014.

[102] S. Honda, “Collapsed Ricci limit spaces as non-collapsed RCD spaces,” *SIGMA*, vol. 16, no. 021, 2020.

[103] I. Chavel, B. Randol, and J. Dodziuk, *Eigenvalues in Riemannian Geometry*, ser. Pure and Applied Mathematics. Elsevier Science, 1984, ISBN: 9780080874340.

[104] S. Rosenberg, *The Laplacian on a Riemannian Manifold: An Introduction to Analysis on Manifolds*, ser. London Mathematical Society Student Texts. Cambridge University Press, 1997.

[105] A. Grigor’yan, *Heat Kernel and Analysis on Manifolds*, ser. AMS/IP studies in advanced mathematics. American Mathematical Society, 2012, ISBN: 9780821893937.

[106] J. H. P. McKean and I. M. Singer, "Curvature and the eigenvalues of the Laplacian," *Journal of Differential Geometry*, vol. 1, no. 1-2, pp. 43–69, 1967.

[107] S. B. Alexander and R. L. Bishop, "Curvature bounds for warped products of metric spaces," *Geometric & Functional Analysis GAFA*, vol. 14, no. 6, pp. 1143–1181, 2004.

[108] F. Cavalletti, "Monge problem in metric measure spaces with riemannian curvature-dimension condition," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 99, pp. 136–151, 2014.

[109] F. Cavalletti and A. Mondino, "Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower ricci curvature bounds," *Inventiones mathematicae*, vol. 208, no. 3, pp. 803–849, 2017.

[110] J. Pan and G. Wei, "Examples of ricci limit spaces with non-integer hausdorff dimension," Preprint arXiv: 2106.03967, 2021.

[111] J. Cheeger, W. Jiang, and A. Naber, "Rectifiability of singular sets of noncollapsed limit spaces with Ricci curvature bounded below," *Annals of Mathematics*, vol. 193, no. 2, pp. 407–538, 2021.

[112] Y. Kitabeppu and S. Lakzian, "Characterization of low dimensional rcd*(k, n) spaces," *Analysis and Geometry in Metric Spaces*, vol. 4, 2015.

[113] M. Braun, "Heat flow on 1-forms under lower ricci bounds. functional inequalities, spectral theory, and heat kernel," arXiv:2010.01849, 2020.

[114] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson, *Sobolev spaces on metric measure spaces*, ser. New Mathematical Monographs. Cambridge University Press, Cambridge, 2015, vol. 27, pp. xii+434, An approach based on upper gradients, ISBN: 978-1-107-09234-1.

[115] S. Honda, "Spectral convergence under bounded ricci curvature," *Journal of Functional Analysis*, vol. 273, no. 5, pp. 1577–1662, 2017.

[116] ——, "Elliptic PDEs on compact Ricci limit spaces and applications," Preprint, arXiv:1410.3296.