THEORETICAL CONSIDERATIONS OF THE CONTRIBUTION OF M-SHELL ELECTRONS TO ORBITAL ELECTRON CAPTURE

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SUMMARY

The probability for capture of electrons from the various M subshells is calculated using a V-A interaction in the Fermi theory of beta decay. The results of these calculations are applied to the problems of:

1. Calculating the nuclear mass difference between the initial and final states,
2. determining the order of forbiddenness of a unique forbidden transition, and
3. calculating nuclear matrix elements appearing in forbidden non-unique transitions.

It is shown that for small transition energies, the mass difference between the initial and final nuclear states can be, in the case of unique forbidden transitions, determined with some precision. The precision available depends on the details of the transition, but accuracies of $\pm$ 0.00005 a.m.u. are possible over a fairly wide range of conditions if an experimental determination of the M/L ratio with an accuracy of 20 per cent is available.

For unique forbidden transitions, the order of forbiddenness can be determined from a measurement of the M/L ratios provided the transition energy is sufficiently small. In fact, it is shown that even a very rough experimental estimate of the M_V/L ratio will distinguish between first and second forbidden unique transitions at small transition energies.

The calculation of the nuclear matrix elements uses the results of a theory developed by Ahrens and Feenberg. It is shown that by using this theory, the nuclear matrix elements which occur in non-unique forbidden transitions can be computed in terms of the M/L (or L/K) capture ratios. Thus an experimental determination of these ratios gives information concerning the relative magnitudes of these nuclear matrix elements. It is believed that this method for calculating nuclear matrix elements is original in this work.

The calculations of the M-shell transition probabilities proceeds along the following lines. The Fermi theory of beta decay is used to write the most general Lorentz invariant interaction hamiltonian for the transition. Experimental results are used to simplify the hamiltonian to the V-A interaction. The operators in the interaction hamiltonian are then expanded in terms of the irreducible spherical tensors by the method introduced by Rose and Osborn. After this is done, the transition probabilities are computed according to the usual perturbation theory calculation, i.e. Fermi's "Golden Rule."

These probabilities turn out to be infinite sums over one of the parameters which is introduced when the spherical tensor operators are introduced into the interaction hamiltonian. Those terms of the sum which contribute to allowed, first and second forbidden, and first, second, and third forbidden unique transitions are explicitly calculated in terms of the electron wave functions (evaluated at the nuclear radius) and the neutrino energies. These calculations are made for all electron states which make appreciable contributions to the transition.

Except for the numerical values of f and g, the radial electron wave functions, the form of the capture probability is the same for the K shell, the L_I sub-shell and the M_I sub-shell. Similarly, the form of the capture probabilities of the L_{II} (L_{III}) sub-shell are the same as for the M_{II} (M_{III}) sub-shell. The capture probabilities of the M_{IV} sub-shell can be inferred from those of the L_{III} sub-shell. Since the capture probabilities for the L sub-shells have been previously calculated by Brysk and Rose, and independently by Bouchez and Depommier, the new contribution is the calculation of the contribution of the M_V sub-shell. Nevertheless, all five M sub-shell contributions were computed in the present work. This work verifies the results of Bouchez and Depommier, but differs from the results of Brysk and Rose by the sign of some of the nuclear matrix elements, and by a small numerical factor in one of the terms.

CHAPTER I

INTRODUCTION

Historical Sketch

The theory of beta-decay was first formulated by Enrico Fermi (1) in 1934. His theory (which is still used today in slightly modified form) is based on an analogy with the theory of electromagnetic interactions in the emission of photons from excited atoms (also due to Fermi). The calculations presented here are based on the present form of this theory.

The basic process involved in beta decay is:

$$N \rightarrow P + e^- + \tilde{\nu} \; ; \tag{1}$$

the basic process involved in orbital electron capture is:

$$P + e^- \rightarrow N + \nu \tag{2}$$

which is the inverse to the process of beta-decay. For this reason, it is assumed that the interaction giving rise to orbital electron capture is identical to the interaction giving rise to beta-decay.

Definition of the Problem

The purpose of this investigation is to examine the contribution of the electrons in the M-shell to the process of orbital electron capture. Beginning with the interaction hamiltonian for a modified V-A interaction ($C_A A + C_V V$) in the usual (cartesian) relativistic form, this hamiltonian will be written in terms of irreducible spherical tensors. For calculation of the transition probabilities, only the square of the matrix element of the interaction hamiltonian is of interest. The squares and cross-product terms of the nuclear matrix elements will be written in the notation of Rose and Osborn (2). The lepton matrix elements will be written in terms of reduced matrix elements and these will, in turn, be written in explicit form.

The electron radial wave functions for the M-shell electrons were computed (by machine computations) by Brewer, Harmer, and Hay (3) from a Thomas-Fermi-Dirac potential with finite nuclear size corrections.

Experimental data on the relative probabilities of M-capture and L-capture (M/L ratios) can be applied to the problems of calculating nuclear matrix elements, determining the order of forbiddenness, and calculating nuclear mass differences. These applications are discussed in Chapter VII.

CHAPTER II

THE INTERACTION HAMILTONIAN FOR ORBITAL ELECTRON CAPTURE

General Form of the Interaction Hamiltonian

The Fermi theory of beta-decay assumes a Lorentz invariant interaction hamiltonian composed of products of the usual Dirac matrices. For spin 1/2 particles there are, in addition to the four-vector (V) coupling used originally by Fermi, four suitable choices of operators. These are designated scalar (S), tensor (T), axial vector (A), and pseudo-scalar (P) according to their transformation properties under a Lorentz transformation. These operators are given below in both their manifestly Lorentz covariant form and in the usual Dirac notation. The Dirac form can be obtained from the Lorentz covariant form by direct substitution from equation (5)*.

$$\begin{aligned} O_S &= \gamma^4 = \beta \\ O_V &= \gamma^4\gamma^\mu = i\vec{\alpha},\ 1 \\ O_T &= \gamma^4\gamma^\mu\gamma^\nu = i\beta\vec{\sigma},\ -i\beta\vec{\alpha} \\ O_A &= i\gamma^4\gamma^\mu\gamma^5 = i\gamma^5,\ \vec{\sigma} \\ O_P &= \gamma^4\gamma^5 = \beta\gamma^5 \end{aligned} \tag{3}$$

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* Specifically, for O_V: $\gamma^4\gamma^k = i\alpha_k$, $\gamma^4\gamma^4 = 1$

for O_A: $i\gamma^4\gamma^k\gamma^5 = \sigma_k$, $i\gamma^4\gamma^4\gamma^5 = i\gamma^5$

for O_T: $\gamma^4\gamma^k\gamma^4 = -i\beta\alpha_k$, $\gamma^4\gamma^i\gamma^j = -i\beta\sigma_k$ i,j,k cyclic.

The operators, γ^μ, are the usual Dirac matrices given by:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta(\mu,\nu) \qquad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4. \tag{4}$$

The $\vec{\alpha}$ and β operators are given by:

$$\alpha_k = -i\gamma^4\gamma^k, \qquad \beta = \gamma^4, \qquad \vec{\sigma} = -\gamma^5\vec{\alpha} \tag{5}$$

Thus prior to 1957 the most general form of the interaction could be written as:

$$H = \sum_x C_x \; (N|O_x|P):(\nu|O_x|e) + \text{h.c.} \tag{6}$$

where x takes on the values S, V, T, A, and P, and C_x is a constant (perhaps complex) which measures the relative strength of the various contributions. The term "h.c." means "hermitian conjugate" and must be included to make the hamiltonian hermitian.

The symbol ":" means that the two indicated integrations are not independent, but that they are performed with their integrands evaluated at the same point in space and also indicates a contraction of the tensors. Thus when the integrals are written out, the ":" becomes a Dirac delta between the nucleon coordinate, $\vec{R}$, and the lepton coordinate, $\vec{r}$, and indicates the "dot" product of the matrix elements. That is, it introduces $\delta(\vec{R}-\vec{r})$.

This hamiltonian has been constructed in analogy with the hamiltonian describing the electromagnetic radiation from excited atoms. It represents, physically, the destruction of an electron and the creation of a neutrino (the lepton term) accompanied by the destruction of a proton and the creation of a neutron (the nucleon term).

However, in 1957 the fact that parity is not conserved in weak interactions was established (4). As a result of this, the form of the interaction had to be modified to allow for the non-conservation of parity. With this modification it appears as follows:

$$H = \sum_{x}(N|O_x|P):(\nu|(C_x + C_x'\gamma^5)O_x|e) + h.c. \quad . \tag{7}$$

where the symbols have the same meaning as before except that C_x is the "parity conserving coupling constant" and C_x' is the "parity non-conserving coupling constant." Physically, this modification amounts to allowing a mixing of vector with axial vector and scalar with pseudo-scalar states since operation by γ^5 on O_V (O_S) gives $-iO_A$ ($-O_P$) etc. γ^5 operating on O_T merely interchanges the vector and axial vector nature of the elements of the tensor.

Simplification of the Interaction Hamiltonian

Certain experimental results allow a great deal of simplification of the interaction hamiltonian. Since the quantities C_x and C_x' may be complex, they represent a total of twenty independent constants. In this section it will be shown that, by using certain experimental results, the number of independent constants can be reduced to four.

The Helicity of the Neutrino

First consider the effect of the operator:

$$C_x + C_x'\gamma^5$$

on the neutrino wave function ψ_ν.

$$(C_x + C_x'\gamma^5)\psi_\nu = (C_x + C_x')\varphi_\nu^+ + (C_x - C_x')\varphi_\nu^- \tag{8}$$

where:

$$\varphi^{\pm} = \frac{1}{2}(1 \pm \gamma^5)\psi \quad . \qquad (9)$$

The wave functions $\varphi^{\pm}$ are orthogonal since:

$$(1 + \gamma^5)(1 - \gamma^5) = 1 + \gamma^5 - \gamma^5 - \gamma^5\gamma^5 = 0$$

The leptons associated with beta-decay and orbital electron capture obey the Dirac equation*:

$$W\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi \qquad (10)$$

where W is the energy of the lepton and ψ its wave function. For the electron, m=1 (in these units) while for the neutrino, m=0. Multiplying the Dirac equation by $\pm\gamma^5/2W$ and adding $(1/2)\psi$ gives:

$$\varphi^{\pm} = \left[\frac{1}{2}\left(1 \mp \frac{\vec{\sigma}\cdot\vec{p}}{W}\right) \mp \frac{m}{2W}\beta\gamma^5\right]\psi \quad . \qquad (11)$$

Since the mass of the neutrino is zero, the energy is numerically equal to the momentum. Thus letting $\hat{q}$ represent a unit vector in the direction of the neutrino's momentum, the equation becomes, for the neutrino:

$$\varphi_{\nu}^{\pm} = \frac{1}{2}(1 \mp \vec{\sigma}.\hat{q})\psi_{\nu} \quad . \qquad (12)$$

Taking $\hat{q}$ as the quantization axis, the component of $\vec{\sigma}$ along $\hat{q}$ can have only the expectation values +1 or -1. Hence $\vec{\sigma}.\hat{q}$ has the expectation value +1 for the case where the spin of the neutrino is directed parallel to $\hat{q}$, and -1 for the case where the spin of the neutrino is directed anti-parallel to $\hat{q}$. Thus the state φ^{+} corresponds to the case

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* Units such that $e = c = \hbar = m_e = 1$ are used throughout.

where the spin of the neutrino is directed anti-parallel to $\hat{q}$ whereas the state φ^- corresponds to the case where the spin of the neutrino is directed parallel to $\hat{q}$.

The helicity of a particle is defined as the scalar product between a unit vector in the direction of the particle's spin and a unit vector in the direction of the particle's momentum. Thus for the neutrino the state φ^+ corresponds to a state of negative helicity while the state φ^- corresponds to a state of positive helicity. It has been shown by several investigators (notably Goldhaber, Grodzins, and Sunyar (5)) that the neutrino emitted in orbital capture (K-capture) has a helicity of -1.00 ± 0.15. The two component neutrino theory of Lee and Yang (6), and Landau (7), requires that the helicity be either +1 or -1. Thus the helicity of the neutrino is taken to be -1. Since neutrinos characterized by φ^- are not created by orbital electron capture, transitions to this state must be suppressed. This can be accomplished (since φ^+ and φ^- are orthogonal) only by choosing $C_x = C_x'$. The result of this is that the interaction hamiltonian for orbital capture can be written:

$$H = \sum_x C_x (N|O_x|P):(\nu|(1+\gamma^5)O_x|e) + \text{h.c.} \tag{13}$$

The number of independent constants has thus been reduced to ten.

It should be noted that the condition $C_x = C_x'$ in no way affects the results of these calculations except that for arbitrary C_x and C_x' the quantity $C_x C_y$ which appears in these results would be replaced by $\frac{1}{2}(C_x C_y + C_x' C_y')$. In other words, experiments on total transition probabilities give no information on the relative amount of parity non-conservation.

Helicity of Electrons in Beta Decay

In order to reduce the number of independent constants still further, consider the interaction hamiltonian for beta decay. As stated in Chapter I the interaction for beta decay is believed to be identical to that in orbital capture, hence:

$$H_\beta = \sum_x C_x (P|O_x|N):(e|O_x(1+\gamma^5)|\nu) + h.c. \tag{14}$$

Using the commutation relations (equation 4) and the definitions of the interaction operators (equation 3) this equation can be written:

$$H_\beta = \sum_{x=S,T,P} C_x \; (P|O_x|N):(e|(1-\gamma^5)O_x|\nu) + \sum_{x=V,A} C_x \; (P|O_x|N):(e|(1+\gamma^5)O_x|\nu) + h.c.$$

In the limit of electron velocities approaching the speed of light, the last term of equation (11) vanishes (because the energy in the denominator becomes very large) while (since $|\vec{p}| \rightarrow W$ as $v \rightarrow c$) the first term becomes:

$$\frac{1}{2}\,(1 \mp \vec{\sigma}\cdot\hat{p})$$

where $\hat{p}$ is a unit vector in the direction of the electron's momentum. But:

$$\frac{1}{2}\,\psi_e^*\,(1 \pm \gamma^5) = \varphi_e^{\pm *} \,.$$

Thus in the limit of electron velocities approaching the velocity of light, φ^+ corresponds to electrons oriented with their spins directed anti-parallel to their momentum (negative helicity), whereas φ^- corresponds to electrons oriented with their spins directed parallel to their momentum (positive helicity).

Experiments by Fraunfelder (8) and others (9) have shown that the helicity, P, of electrons emitted in beta decay is given by:

$P = -v/c$ in the case of (negative) electron emission,

$P = +v/c$ in the case of positron emission.

These results have been verified (to within 15 per cent) for various types of transitions. A complete review of experiments measuring the helicity of neutrinos and beta decay electrons is given by Grodzins (10).

Since the electrons have negative helicity which approaches -1 as $v \to c$, transitions to the φ^- state must be suppressed. This can be done by choosing:

$$C_S = C_T = C_P = 0$$

If the interaction for orbital electron capture is to be identical with that for beta-decay, these constants must vanish in that case also. Thus only the vector and axial vector interactions are present in orbital electron capture.

The Modified V-A Interaction

The interaction hamiltonian can be written in the following form:

$$H = C_V (N|\gamma^4\gamma^\mu|P):(\nu|(1+\gamma^5)\gamma^4\gamma^\mu|e)$$

$$+C_A (N|i\gamma^4\gamma^\mu\gamma^5|P):(\nu|(1+\gamma^5)i\gamma^4\gamma^\mu\gamma^5|e) + \text{h.c.}$$

$$= (N|\gamma^4\gamma^\mu(C_V - C_A\gamma^5)|P):(\nu|(1+\gamma^5)\gamma^4\gamma^\mu|e) + \text{h.c.} \qquad (16)$$

The hermitian conjugate term corresponds to "positron capture" and hence gives no contribution since there are no positrons in orbital states. It will therefore be omitted in the following calculations. In the customary notation this becomes:

$$H = (N|C_V - C_A\gamma^5|P):(\nu|1+\gamma^5|e)$$

$$-(N|(C_V - C_A\gamma^5)\vec{\alpha}|P):\cdot(\nu|(1+\gamma^5)\vec{\alpha}|e) \qquad (17)$$

A series of experiments conducted at Argonne National Laboratory (11) and at the Atomic Energy of Canada laboratory at Chalk River (12) indicate that the ratio of C_A to C_V is real. These experiments measure the angular correlation between the electron and the neutrino in the decay of polarized neutrons. The results of these experiments are that the intensity of the beta radiation is given by the semi-empirical relation:

$$I \propto 1 + a\,\hat{q}\cdot\frac{\vec{v}}{c} + \hat{I}\cdot\left(A\frac{\vec{v}}{c} + B\hat{q} + D\frac{\vec{v}}{c} \times \hat{q}\right)$$

where: $\hat{q}$ is a unit vector in the direction of the neutrino momentum,

$\vec{v}$ is the electron velocity,

$\hat{I}$ is a unit vector in the direction of the neutron spin.

$A = -0.11 \pm 0.02$ $\qquad$ $D = -0.04 \pm 0.07$

$B = 0.88 \pm 0.15$ $\qquad$ $a = 0.09 \pm 0.11$

A theoretical calculation, outlined by Konipinski (9), yields the same equation with the following replacements:

$$a = \frac{|C_V|^2 - |C_A|^2}{|C_V|^2 + c|C_A|^2}$$

$$A = \frac{-2|C_A|^2 - (C_V C_A{}^* + C_V{}^* C_A)}{|C_V|^2 + 3|C_A|^2}$$

$$B = \frac{+2|C_A|^2 - (C_V C_A{}^* + C_V{}^* C_A)}{|C_V|^2 + 3|C_A|^2}$$

$$D = \frac{i(C_V C_A{}^* - C_V{}^* C_A)}{|C_V|^2 + 3|C_A|^2} \quad .$$

Writing $C_A/C_V = \rho\ e^{i\alpha}$ where ρ is real and positive gives:

$$B + A = -\frac{4\ \rho\ \cos\alpha}{1 + 3\ \rho^2} = 0.77 \pm 0.17$$

$$D = -\frac{2\ \rho\ \sin\alpha}{1 + 3\ \rho^2} = -\ 0.04 \pm 0.07 \quad .$$

The result for D puts the relative phase, α, equal to zero or π (to within about 8 degrees). The result for B + A clearly picks the case $\alpha = \pi$. Thus:

$$\frac{C_A}{C_V} = \rho\ e^{i\pi} = -\rho \quad .$$

Data based on the log ft values of O^{14}, Al^{26}, and Cl^{34} give, according to Konipinski (9):

$$|C_A|^2/|C_V|^2 = 1.47 \pm 0.06,$$

hence

$$C_A = -(1.21 \pm 0.03)\ C_V \quad .$$

Since the ratio of C_A to C_V is real, C_A and C_V can themselves be considered real since the wave functions are always determined only to within an arbitrary phase factor. Thus the number of independent constants in the interaction hamiltonian has been reduced from twenty to one. The two constants C_A and C_V will be carried through the computations even though they are not independent.

Expansion of the Interaction Hamiltonian in Terms of Irreducible Spherical Tensors

Writing out the integrations indicated in the above expressions for the interaction hamiltonian (and evaluating the lepton and nucleon parts at the same point in space as indicated by the symbol ":" in the notation of the preceding sections) gives:

$$\begin{aligned} H = &\int d^3\vec{r}\, d^3\vec{R}\; N^*(\vec{R})(C_V - C_A\gamma^5)P(\vec{R})\;\; \delta(\vec{R}-\vec{r})\, \nu^*(\vec{r})(1+\gamma^5)e(\vec{r}) \\ &- \int d^3\vec{r}\, d^3\vec{R}\; N^*(\vec{R})(C_V - C_A\gamma^5)\vec{\alpha}\;\; P(\vec{R})\cdot\overleftrightarrow{I}\cdot\delta(\vec{R}-\vec{r})\, \nu^*(\vec{r})(1+\gamma^5)\vec{\alpha}\;\, e(\vec{r}) \end{aligned} \tag{18}$$

where $\vec{R}$ and $\vec{r}$ represent the nucleon and lepton coordinates respectively, N, P, ν, and e are the wave functions of the neutron, proton, neutrino, and electron respectively. $\overleftrightarrow{I}$ is the 3 x 3 unit dyadic and "*" means hermitian conjugate.

The Dirac deltas can be expanded according to the prescription of Appendix A. When this is done, the interaction hamiltonian becomes:

$$\begin{aligned} H = &\sum_{L,M} \int d^3\vec{R}\; N^*(\vec{R})(C_V - C_A\gamma^5)P(\vec{R})Y_L^M(\Theta,\Phi)\int d^3\vec{r}\; \frac{\delta(R-r)}{r^2}\, \nu^*(\vec{r})(1+\gamma^5)e(\vec{r})Y_L^{M*}(\theta,\varphi) \\ &- \sum_{J,L,M} d^3\vec{R}\; N^*(\vec{R})(C_V - C_A\gamma^5)\vec{\alpha}P(\vec{R})\cdot\vec{\Phi}^M_{J,L}(\Theta,\Phi)\int d^3\vec{r}\; \frac{\delta(R-r)}{r^2}\;\, \nu^*(\vec{r})(1+\gamma^5)\vec{\alpha}e(\vec{r})\,\cdot \\ &\vec{\Phi}^{M*}_{J,L}(\theta,\varphi). \end{aligned} \tag{19}$$

Using for $\vec{\Phi}^{M}_{J,L}$ the quantity chosen in equation (A-4) of Appendix A, consider the quantity:

$$\vec{\sigma} \cdot \vec{\Phi}^{M}_{J,L} \quad .$$

By the definition of the solid harmonics*:

$$\vec{\sigma} \cdot \hat{e}_q = (\tfrac{4\pi}{3})^{\frac{1}{2}} \mathcal{Y}_1^{q}(\vec{\sigma}) \quad .$$

Thus one obtains:

$$\vec{\sigma} \cdot \vec{\Phi}^{M}_{J,L} = \sum_{m,q} \mathcal{Y}_L^{m}(\hat{r}) (\tfrac{4\pi}{3})^{\frac{1}{2}} \mathcal{Y}_1^{q}(\vec{\sigma}) (1Lqm|1LJM)$$

$$= (\tfrac{4\pi}{3})^{\frac{1}{2}} \sum_{\mu} (1L\ -\mu\ \mu + M|1LJM)\ \mathcal{Y}_L^{\mu+M}(\hat{r})\ \mathcal{Y}_1^{-\mu}(\vec{\sigma}) \quad .$$

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* Note that $\mathcal{Y}_1^q(\vec{A})$ is given by:

$$\mathcal{Y}_1^{0}(\vec{A}) = (\tfrac{3}{4\pi})^{\frac{1}{2}} A_z \qquad\qquad \mathcal{Y}_1^{\pm 1}(\vec{A}) = \mp(\tfrac{3}{4\pi})^{\frac{1}{2}} \tfrac{1}{\sqrt{2}} (A_x \pm iA_y) .$$

From this it is apparent that $\mathcal{Y}_1^q(\hat{r})$ is a c-number, while $\mathcal{Y}_1^q(\vec{\sigma})$ is an operator (2x2 matrix) in Pauli Space.

For $\mathcal{Y}_1^q(\vec{\sigma})$ one has

$$\mathcal{Y}_1^{0}(\vec{\sigma}) = (\tfrac{3}{4\pi})^{\frac{1}{2}} \sigma_z = (\tfrac{3}{4\pi})^{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{Y}_1^{-1}(\vec{\sigma}) = (\tfrac{3}{4\pi})^{\frac{1}{2}} \tfrac{1}{\sqrt{2}} (\sigma_x - i\sigma_y) = (\tfrac{3}{4\pi})^{\frac{1}{2}} \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

$$\mathcal{Y}_1^{1}(\vec{\sigma}) = -(\tfrac{3}{4\pi})^{\frac{1}{2}} \tfrac{1}{\sqrt{2}} (\sigma_x + i\sigma_y) = -(\tfrac{3}{4\pi})^{\frac{1}{2}} \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Defining (with Rose):

$$T_{J,L}^{\,M}(\vec{A},\vec{B}) = \sum_{\mu} (1\ L\ {-\mu}\ \mu + M|1LJM)\, \mathcal{Y}_L^{\mu+M}(\vec{A})\, \mathcal{Y}_1^{-\mu}(\vec{B}) \tag{20}$$

one obtains:

$$\vec{\sigma}\cdot\vec{\Phi}_{JL}^{M} = \left(\tfrac{4\pi}{3}\right)^{\frac{1}{2}} T_{JL}^{M}(\hat{r},\vec{\sigma}) \quad . \tag{21}$$

The interaction hamiltonian now becomes, in terms of these irreducible spherical tensors:

$$\begin{aligned} H = \sum_{L,M} \int & R^2 dR d\Omega\ N^*(R,\Omega)(C_V - C_A\gamma^5)\mathcal{Y}_L^M(\hat{R})\ P(R,\Omega) \\ & \int r^2 dr\ d\omega\ \frac{\delta(R-r)}{r^2}\ \nu^*(r,\omega)(1+\gamma^5)\mathcal{Y}_L^{M*}(r)\ e(r,\omega) \\ -\frac{4\pi}{3} \sum_{JLM} \int & R^2 dR d\Omega\ N^*(R,\Omega)(C_V\gamma^5 - C_A)T_{JL}^{\ M}(\hat{R},\vec{\sigma})P(R,\Omega) \\ & \int r^2 dr\ d\omega\ \frac{\delta(R-r)}{r^2}\ \nu^*(r,\omega)(1+\gamma^5)T_{JL}^{\ M*}(\hat{r},\vec{\sigma})\ e(r,\omega). \end{aligned}$$

Here the results of equations (4) and (5) have been used. Performing the integration over r gives:

$$\begin{aligned} H = \sum_{L,M} \int R^2\ dR & \left\{ \int d\Omega\ N^*(R,\Omega)(C_V - C_A\gamma^5)\, \mathcal{Y}_L^M\ (\hat{R})\ P(R,\Omega) \right\} \\ & \left\{ \int d\omega\ \nu^*(R,\omega)(1+\gamma^5)\mathcal{Y}_L^{M^*}(\hat{r})\ e(R,\omega) \right\} \\ -\frac{4\pi}{3} \sum_{JLM} \int R^2\ dR & \left\{ \int d\Omega\ N^*(R,\Omega)(C_V\gamma^5 - C_A)T_{JL}^{\ M}(\hat{R},\vec{\sigma})\ P(R,\Omega) \right\} \\ & \left\{ \int d\omega\ \nu^*(R,\omega)(1+\gamma^5)T_{JL}^{\ M}(\hat{r},\vec{\sigma})\ e\ (R,\omega) \right\} \quad . \end{aligned} \tag{22}$$

The integration indicated above should, strictly, be carried out over all space. Since the nuclear wave functions essentially vanish outside the nucleus, however, this integration can be restricted to cover only the nuclear volume. Since the form of the nuclear wave functions is unknown, the integration cannot be performed. This integration amounts to an averaging of the integral over the nuclear volume. Brysk and Rose have shown (13) that the results are very insensitive to the details of this averaging process; it is therefore satisfactory to evaluate the integral at the nuclear surface. When this is done, the interaction hamiltonian becomes (where R is now the nuclear radius):

$$\begin{aligned} H = R^3 \sum_{L,M} (N|(C_V - C_A\gamma^5)\mathcal{Y}_L^M(\hat{R})|P)(\nu|(1+\gamma^5)\mathcal{Y}_L^{M^*}(\hat{r})|e) \\ -\frac{4\pi}{3} R^3 \sum_{JLM} (N|(C_V\gamma^5 - C_A)T_{JL}^M(\hat{R},\vec{\sigma})|P)(\nu|(1+\gamma^5)\, T_{JL}^{M^*}(\hat{r},\vec{\sigma})|e). \end{aligned} \qquad (23)$$

In this last equation it is understood that only the integration over the solid angles remains.

CHAPTER III

THE LEPTON MATRIX ELEMENTS

The Wigner-Eckhart Theorem and Reduced Matrix Elements

The problem of calculating the lepton matrix elements can be simplified by taking notice of the results of the Wigner-Eckhart theorem. This theorem assures that the matrix elements of operators $T^M_{JL}(\vec{A},\vec{B})$ and $\mathcal{Y}^M_J(\vec{A})$ can be written as the product of a Clebsch-Gordon coefficient times a "reduced matrix element." This reduced matrix element is independent of the magnetic quantum numbers. It will not be necessary to actually use the Wigner-Eckhart theorem since the matrix elements will be calculated explicitly. When this is done, it will be seen that the quantity

$$(f\|\Omega^*_{JL}\|i),$$

defined by the equation:

$$(f|\Omega^{M*}_{JL}|i) = (-)^M(j'J\ \mu'\ M|j'\ J\ \mu)(f\|\Omega^*_{JL}\|i), \tag{24}$$

is independent of the magnetic quantum numbers, μ, μ', and M.* The symbol Ω represents any of the operators occurring in the interaction hamiltonian, j and μ are the final state quantum numbers, and j' and μ' are the initial state quantum numbers.

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* See equations (48) and (49).

By using equation (24), the interaction hamiltonian can be written:

$$H = \sum_{JM} (-)^M (j_e J\ \mu_e\ M | j_e\ J\ j_\nu\ \mu_\nu)(\nu\|(1+\gamma^5)\mathcal{Y}_J^*(\hat{r})\|e)\ \Psi_J^M \tag{25}$$

$$- \frac{4\pi}{3} \sum_{JKM} (-)^M (j_e J\ \mu_e\ M | j_e\ J\ j_\nu\ \mu_\nu)(\nu\|(1+\gamma^5)T_{JL}^*(\hat{r},\vec{\sigma})\|e)\ \Xi_{JL}^M \quad ,$$

where:

$$\Psi_J^M = R^3 (N|(C_V - \gamma^5 C_A)\mathcal{Y}_J^M(\hat{R})|P) \quad ,$$

$$\Xi_{JL}^M = R^3 (N|(C_V\gamma^5 - C_A)\ T_{JL}^M(\hat{R},\vec{\sigma})|P) \quad . \tag{26}$$

The interaction hamiltonian then becomes:

$$H = \sum_{JM} (-)^M (j_e\ J\ \mu_e\ M | j_e\ J\ j_\nu\ \mu_\nu) \Big\{ \Psi_J^M\ (\nu\|(1+\gamma^5)\,\mathcal{Y}_J^*(\hat{r})\|e)$$

$$- \frac{4\pi}{3} \sum_L \Xi_{JL}^M (\nu\|(1+\gamma^5)T_{JL}^*(\hat{r},\vec{\sigma})\|e) \Big\} . \tag{27}$$

In the subsequent calculations the following shorthand notation will be adopted for the <u>lepton reduced</u> matrix elements:

$$(\nu\|(1+\gamma^5)T_{JL}^*(\hat{r},\vec{\sigma})\|e) \rightarrow (\nu\|\ T_{JL}^*\ \|e)$$

$$(\nu\|(1+\gamma^5)\,\mathcal{Y}_J^*\ (\hat{r})\|e) \ \rightarrow (\nu\|\mathcal{Y}_J^*\ \|e) \ . \tag{28}$$

In other words, the arguments of T and $\mathcal{Y}$ will be understood to be $(\hat{r},\vec{\sigma})$ and $(\hat{r})$ respectively; the operator $(1+\gamma^5)$ will be understood, but not written.

The probability for the capture of an electron from a given state is proportional to the average over initial magnetic sub-states and sum over final states of the square of the interaction hamiltonian. That is, it is proportional to:

$$\sum_{j_\nu \ell_\nu} \sum_{\mu_\nu \mu_e} \frac{1}{2j_e+1} |H|^2 .$$

However, the quantity of interest is the probability for the capture of any electron in the given shell. The shell is assumed to be filled, hence it contains $2j_e+1$ electrons. The probability for such an event is thus proportional to:

$$M(\kappa_e) = \sum_{j_\nu \ell_\nu} \sum_{\mu_\nu \mu_e} |H|^2 \tag{29}$$

where equation (29) is taken as the definition of $M(\kappa_e)$. The symbol κ_e represents a quantum number which specifies both j_e and ℓ_e of the state under consideration. It is defined by:

$$\kappa = \ell \qquad \text{if} \qquad j = \ell - \tfrac{1}{2}$$

$$\kappa = -\ell - 1 \qquad \text{if} \qquad j = \ell + \tfrac{1}{2}$$

Substituting equations (27) and (28) into (29) gives:

$$\begin{aligned} M(\kappa_e) = \sum_{j_\nu \ell_\nu \mu_e \mu_\nu} \sum_{J,M} \sum_{J'M'} (-)^{M+M'} & (j_e J\ \mu_e\ M | j_e J\ j_\nu \mu_\nu)(j_e J' \mu_e M' | j_e J' j_\nu \mu_\nu) \\ & \Big\{ \Psi_J^{M*}\ \Psi_J^{M'} (\nu \| \mathcal{Y}_J^* \| e)^* (\nu \| \mathcal{Y}_{J'}^* \| e) \\ & + \frac{16\pi^2}{9} \sum_{L,L'} \Xi_{JL}^{M*}\ \Xi_{J'L'}^{M'}\ (\nu \| T_{JL}^* \| e)^* (\nu \| T_{J'L'}^* \| e) \\ & - \frac{4\pi}{3} \sum_{L'} \Psi_J^{M*}\ \Xi_{J'L'}^{M'} (\nu \| \mathcal{Y}_J^* \| e)(\nu \| T_{J'L'}^* \| e) \\ & - \frac{4\pi}{3} \sum_{L} \Psi_{J'}^{M'}\ \Xi_{JL}^{M*}\ (\nu \| T_{JL}^* \| e)^* (\nu \| \mathcal{Y}_{J'}^* \| e) \Big\} . \end{aligned}$$

By anticipating the result that the reduced matrix elements are independent of μ_e and μ_ν, the sum over these quantum numbers can be made. Using equations (B-1) and (B-3) one obtains:

$$\sum_{\mu_e \mu_\nu} (j_e\ J\ \mu_e\ M | j_e\ J\ j_\nu\ \mu_\nu)(j_e\ J'\ \mu_e\ M' | j_e\ J'\ j_\nu\ \mu_\nu)$$

$$= \frac{2\ j_\nu+1}{\sqrt{(2J+1)(2J'+1)}} \sum_{\mu_e \mu_\nu} (j_\nu j_e \mu_\nu\ \ -\mu_e | j_\nu j_e JM)(j_\nu j_e \mu_\nu\ \ -\mu_e | j_\nu j_e J'M')$$

$$= \frac{2\ j_\nu+1}{2\ J+1}\ \delta(J,J')\ \delta(M,M')\ \Delta\ (j_\nu j_e J)\ .$$

where $\Delta(j_\nu j_e J) = 1$ if $|j_e - j_\nu| \leq J \leq j_e + j_\nu$

$= 0$ otherwise.

The Kronecker deltas which occur in this result allow one to perform the sum over J' and M'. Doing this (and making use of the fact that since M = M' = integer, M + M' = even integer) gives:

$$M(\kappa_e) = \sum_{J,M} \sum_{j_\nu \ell_\nu} \frac{2\ j_\nu+1}{2J+1} \Big\{ |\Psi_J^M|^2 |(\nu\|\mathcal{Y}_J^*\|e)|^2 + \frac{16\pi^2}{9} \sum_{L,L'} \Xi_{JL}^{M*}\, \Xi_{JL'}^{M}$$

$$(\nu\|T_{JL}^*\|e)^*(\nu\|T_{JL'}^*\|e) - \frac{4\pi}{3} \sum_L \Psi_J^{M*}\Xi_{JL}^M (\nu\|\mathcal{Y}_J^*\|e)^* \tag{30}$$

$$(\nu\|T_{JL}^*\|e) - \frac{4\pi}{3} \sum_L \Psi_J^M\, \Xi_{JL}^{M*}\ (\nu\|T_{JL}^*\|e)^*(\nu\|\mathcal{Y}_J^*\|e) \Big\}\ .$$

Explicit Form for the Lepton Matrix Elements

The purpose of this chapter is to compute the reduced lepton matrix elements which occur in equation (30). In order to do this, it is expedient to make use of the following relation:

$$(\nu|\Omega^*|e) = (e|\Omega|\nu)^* \tag{31}$$

i.e., the hermiticity of the interaction hamiltonian. Writing out the "reduced matrix element" corresponding to the last matrix element in the above equation gives:

$$(e|\Omega|\nu)^* = (j_\nu J\ \mu_\nu\ M|j_\nu J\ j_e\ \mu_e)(e||\Omega||\nu)^* \tag{32}$$

where $(e||\Omega||\nu)^*$ is (as will be seen later) independent of μ_e, μ_ν, and M. According to equation (B-1) this can be written:

$$(e|\Omega|\nu)^* = (-)^{j_e-j_\nu+M}\left[\frac{2\ j_e+1}{2\ j_\nu+1}\right]^{\frac{1}{2}}(j_e J\ \mu_e\ \ -M|j_e J\ j_\nu\mu_\nu)(e||\Omega||\nu)^* \quad . \tag{33}$$

By comparing these results with equation (24), one obtains:

$$(\nu||\Omega^*||e) = (-)^{j_e-j_\nu}\left[\frac{2\ j_e+1}{2\ j_\nu+1}\right]^{\frac{1}{2}}(e||\Omega||\nu)^* \quad . \tag{34}$$

Hence the lepton matrix elements which are of interest are those of the form:

$$(e|\Omega|\nu)^* \quad .$$

In order to write these matrix elements in explicit form, it will be assumed that the electrons move in a spherically symmetric potential. The wave function can then be separated into products of a radial wave function and an angular wave function. The wave function for the electron is written as follows:

$$|e) = \begin{pmatrix} i\ f_{\kappa_e}(r) & \chi^{\mu_e}_{-\kappa_e}(\omega) \\ g_{\kappa_e}(r) & \chi^{\mu_e}_{\kappa_e}(\omega) \end{pmatrix} \tag{35}$$

where f and g are the radial wave functions and χ is the angular wave function. The "i" is included in order to follow the usual convention. That f and g are real has been shown by Rose (22)*.

The symbol κ has been defined previously.

Similarly, the neutrino wave functions are given by:

$$|\nu) = \begin{pmatrix} i\varphi_{\kappa_\nu}(r)\ \chi^{\mu_\nu}_{-\kappa_\nu}(\omega) \\ \gamma_{\kappa_\nu}(r)\ \chi^{\mu_\nu}_{\kappa_\nu}(\omega) \end{pmatrix}. \tag{36}$$

Since the neutrino is uncharged the wave function is that of a free particle. The radial parts of this wave function are therefore spherical Bessel functions. This can be seen by substituting equation (36) into the Dirac equation with no potential field and observing that the resulting radial equations are spherical Bessel equations.

The angular wave functions, χ^{μ}_{κ}, are given by:

$$\chi^{\mu}_{\kappa} = \sum_{\tau} (\ell\ \tfrac{1}{2}\ \mu-\tau\ \tau\,|\,\ell\tfrac{1}{2}\ j\ \mu)\ \chi^{\tau}_{\frac{1}{2}}\quad Y^{\mu-\tau}_{\ell} \tag{37}$$

where:

Y^{m}_{ℓ} denotes the ordinary spherical harmonic,

$\chi^{\tau}_{\frac{1}{2}}$ denotes a spinor which takes on the following two values:

---

*More specifically, the ratio of f to g is real. The absolute phase of the wave function is, as always, arbitrary.

$$\chi^{\frac{1}{2}}_{\frac{1}{2}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \chi^{-\frac{1}{2}}_{\frac{1}{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

The functions φ_{κ_ν} and γ_{κ_ν} are given by:

$$\varphi_{\kappa_\nu} = \frac{\kappa_\nu}{|\kappa_\nu|} \; q \; j_{\ell(-\kappa_\nu)}(qr) \qquad \gamma_{\kappa_\nu} = q \; j_{\ell(\kappa_\nu)}(qr) \tag{38}$$

where q is the relativistic energy of the neutrino. $j_{\ell(\pm\kappa)}(qr)$ is the usual spherical Bessel function given by:

$$j_n(z) = \sqrt{\frac{\pi}{2z}} \; J_{n+\frac{1}{2}}(z). \tag{39}$$

The normalization is such that there is one neutrino in a sphere of unit radius, i.e.

$$\int_0^1 (\varphi^2 + \gamma^2) \; r^2 \, dr = 1.$$

The lepton matrix elements of interest can now be written in explicit form. In addition to the equations and definitions given above it will be necessary to recall that, in the representation adopted, γ^5 can be expressed as:

$$\gamma^5 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} .$$

The matrix elements become:

$$(e|\mathcal{Y}_J^M(\hat{r})|\nu) = \int d\omega\, (-if_{\kappa_e}\, \chi_{-\kappa_e}^{\mu_e *} \quad g_{\kappa_e}\, \chi_{\kappa_e}^{\mu_e *})\; Y_J^M \begin{pmatrix} i\varphi_{\kappa_\nu}\, \chi_{-\kappa_\nu}^{\mu_\nu} \\ \gamma_{\kappa_\nu}\, \chi_{\kappa_\nu}^{\mu_\nu} \end{pmatrix}$$

$$= \int d\omega\, Y_J^M \left\{ f_{\kappa_e}\, \varphi_{\kappa_\nu}\, \chi_{-\kappa_e}^{\mu_e *}\, \chi_{-\kappa_\nu}^{\mu_\nu} + g_{\kappa_e}\, \gamma_{\kappa_\nu}\, \chi_{\kappa_e}^{\mu_e *}\, \chi_{\kappa_\nu}^{\mu_\nu} \right\} \quad (40)$$

$$(e|\gamma^5 \mathcal{Y}_J^M(\hat{r})|\nu) = \int d\omega\, (-if_{\kappa_e}\, \chi_{-\kappa_e}^{\mu_e *} \quad g_{\kappa_e}\, \chi_{\kappa_e}^{\mu_e *})\; Y_J^M$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i\varphi_{\kappa_\nu}\, \chi_{-\kappa_\nu}^{\mu_\nu} \\ \gamma_{\kappa_\nu}\, \chi_{\kappa_\nu}^{\mu_\nu} \end{pmatrix} \quad (41)$$

$$= +i \int d\omega\, Y_J^M \left\{ f_{\kappa_e} \gamma_{\kappa_\nu}\, \chi_{-\kappa_e}^{\mu_e *}\, \chi_{\kappa_\nu}^{\mu_\nu} - g_{\kappa_e}\, \varphi_{\kappa_\nu}\, \chi_{\kappa_e}^{\mu_e *}\, \chi_{-\kappa_\nu}^{\mu_\nu} \right\}$$

$$(e|T_{JL}^M(\hat{r},\vec{\sigma})|\nu) = \int d\omega\, (-if_{\kappa_e}\, \chi_{-\kappa_e}^{\mu_e *} \quad g_{\kappa_e}\, \chi_{\kappa_e}^{\mu_e *})\; T_{JL}^M(\hat{r},\vec{\sigma}) \begin{pmatrix} i\varphi_{\kappa_\nu}\, \chi_{-\kappa_\nu}^{\mu_\nu} \\ \gamma_{\kappa_\nu}\, \chi_{\kappa_\nu}^{\mu_\nu} \end{pmatrix}$$

$$= \int d\omega \left\{ f_{\kappa_e} \varphi_{\kappa_\nu}\, \chi_{-\kappa_e}^{\mu_e *}\, T_{JL}^M\, \chi_{-\kappa_\nu}^{\mu_\nu} + g_{\kappa_e} \gamma_{\kappa_\nu}\, \chi_{\kappa_e}^{\mu_e *}\, T_{JL}^M\, \chi_{\kappa_\nu}^{\mu_\nu} \right\} \quad (42)$$

$$(e|\gamma^5 T_{JL}^M(\hat{r},\vec{\sigma})|\nu) = \int d\omega\, (-if_{\kappa_e}\, \chi_{-\kappa_e}^{\mu_e *} \quad g_{\kappa_e}\, \chi_{\kappa_e}^{\mu_e *})\; T_{JL}^M(\hat{r},\vec{\sigma})$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i\varphi_{\kappa_\nu}\, \chi_{-\kappa_\nu}^{\mu_\nu} \\ \gamma_{\kappa_\nu}\, \chi_{\kappa_\nu}^{\mu_\nu} \end{pmatrix}$$

$$= i\int d\omega \left\{ f_{\kappa_e} \gamma_{\kappa_\nu} \chi_{-\kappa_e}^{\mu_e *} T_{JL}^{M} \chi_{\kappa_\nu}^{\mu_\nu} - g_{\kappa_e} \varphi_{\kappa_\nu} \chi_{\kappa_e}^{\mu_e *} T_{JL}^{M} \chi_{-\kappa_\nu}^{\mu_\nu} \right\} \quad (43)$$

Recall that $T(\hat{r},\vec{\sigma})$ is an operator in Pauli space, whereas γ^5 is an operator in Dirac space.

Calculation of the Reduced Lepton Matrix Elements

Each term of the matrix elements given by equations (40) and (41) contains the factor:

$$\int d\omega \, Y_J^M \chi_\kappa^{\mu *} \chi_{\kappa'}^{\mu'}$$

In Appendix C it is shown that:

$$\int d\omega \, Y_J^M \chi_\kappa^{\mu *} \chi_{\kappa'}^{\mu'} = (4\pi)^{-\frac{1}{2}}(-)^{\ell' + j - \frac{1}{2}} \left[(2J+1)(2\ell+1)(2j'+1)\right]^{\frac{1}{2}} (\ell J 0 0|\ell J \ell' 0) \, W(\ell J \ell' j'; \tfrac{1}{2} J)(j' J \mu' M|j' J j \mu). \quad (44)$$

Using this result, and combining the matrix elements of equations (40) and (41) gives:

$$\begin{aligned}
(e|(1+\gamma^5)\mathcal{Y}_J^M(\hat{r})|\nu)^* &= (4\pi)^{-\frac{1}{2}}(-)^{j_e-\frac{1}{2}} \left[(2J+1)(2j_\nu+1)\right]^{\frac{1}{2}}(j_\nu J \mu_\nu M|j_\nu J j_e \mu_e) \\
&\Big\{\Big[\varphi_{\kappa_\nu} f_{\kappa_e} (-)^{\bar{\ell}_\nu}(2\bar{\ell}_e+1)^{\frac{1}{2}}(\bar{\ell}_e J\, 00|\bar{\ell}_e J \bar{\ell}_\nu 0)\, W(\bar{\ell}_e\, j_e\, \bar{\ell}_\nu\, j_\nu; \tfrac{1}{2} J) \\
&+\gamma_{\kappa_\nu} g_{\kappa_e} (-)^{\ell_\nu}(2\ell_e+1)^{\frac{1}{2}}(\ell_e J\, 00|\ell_e J \ell_\nu\, 0)\, W(\ell_e j_e \ell_\nu j_\nu; \tfrac{1}{2} J)\Big] \quad (45)\\
&+i\Big[\varphi_{\kappa_\nu} g_{\kappa_e} (-)^{\bar{\ell}_\nu}(2\bar{\ell}_e+1)^{\frac{1}{2}}(\ell_e J\, 00|\ell_e J \bar{\ell}_\nu 0)\, W(\ell_e j_e \bar{\ell}_\nu j_\nu; \tfrac{1}{2} J) \\
&-\gamma_{\kappa_\nu} f_{\kappa_e} (-)^{\ell_\nu}(2\bar{\ell}_e+1)^{\frac{1}{2}}(\bar{\ell}_e J\, 00|\bar{\ell}_e J\, {}_\nu 0)\, W(\bar{\ell}_e\, j_e \ell_\nu\, j_\nu; \tfrac{1}{2} J)\Big]\Big\}
\end{aligned}$$

where the symbol ℓ_e means the value of ℓ associated with κ_e and the symbol $\bar{\ell}_e$ means the value of ℓ associated with $-\kappa_e$.

Similarly, one observes that the matrix elements given by equations (42) and (43) all contain factors of the form:

$$\int d\omega\, \chi_{\kappa}^{\mu *}\, T_{JL}^{M}\, \chi_{\kappa'}^{\mu'}$$

In Appendix C it is shown that:

$$\int d\omega\, \chi_{\kappa}^{\mu *}\, T_{JL}^{M}(\hat{r},\vec{\sigma})\chi_{\kappa'}^{\mu'} = \frac{3\sqrt{2}}{4\pi}\left[(2L+1)(2j'+1)(2J+1)(2\ell+1)\right]^{\frac{1}{2}} (-)^{j'-j} \tag{46}$$

$$(\ell L\; 00|\ell L\; \ell' 0)\; X(\tfrac{1}{2}1\tfrac{1}{2};\; j\; J\; j';\ell\; L\; \ell')(j'\; J\; \mu'\; M|\; j'\; J\; j\; \mu).$$

Using this result, and combining the matrix elements of equations (42) and (43) gives:

$$(e|(1+\gamma^5)T_{JL}^{M}(\hat{r},\vec{\sigma})|\nu)^* = \frac{3\sqrt{2}}{4\pi}\left[(2L+1)(2J+1)(2j_\nu+1)\right]^{\frac{1}{2}}(-)^{j_\nu-j_e}(j_\nu J\; \mu_\nu M|j_\nu J\; j_e\mu_e)$$

$$\Big\{\Big[\varphi_{\kappa_\nu} f_{\kappa_e}\; (2\,\bar{\ell}_e+1)^{\frac{1}{2}}\; (\bar{\ell}_e L\; 00|\; \bar{\ell}_e L\; \bar{\ell}_\nu 0)\; X(\tfrac{1}{2}1\tfrac{1}{2};\; j_e\; J\; j_\nu;\; \bar{\ell}_e\; L\; \bar{\ell}_\nu)$$

$$+\gamma_{\kappa_\nu} g_{\kappa_e}(2\ell_e+1)^{\frac{1}{2}}(\ell_e\; L\; 00|\ell_e L\; \ell_\nu 0)\; X(\tfrac{1}{2}1\tfrac{1}{2};\; j_e\; J\; j_\nu;\ell_e\; L\; \ell_\nu)\Big]$$

$$+i\Big[\varphi_{\kappa_\nu} g_{\kappa_e}\; (2\ell_e+1)^{\frac{1}{2}}(\ell_e L\; 00|\ell_e\; L\; \bar{\ell}_\nu 0)\; X(\tfrac{1}{2}1\tfrac{1}{2};\; j_e J\; j_\nu;\; \ell_e L\; \bar{\ell}_\nu)$$

$$-\gamma_{\kappa_\nu} f_{\kappa_e}\; (2\,\bar{\ell}_e+1)^{\frac{1}{2}}(\bar{\ell}_e\; L\; 00|\bar{\ell}_e\; L\; \ell_\nu 0)\; X(\tfrac{1}{2}1\tfrac{1}{2};\; j_e\; J\; j_\nu;\; \bar{\ell}_e\; L\; \ell_\nu)\Big]\Big\}$$

In order to facilitate the calculation of the reduced lepton matrix elements, the following symmetric coefficient is introduced:

$$Z(a\; b\; c) = (-)^c\; \left[(2a+1)(2b+1)\right]^{\frac{1}{2}}\; (a\; b\; 00\; |\; a\; b\; c\; 0) \tag{47}$$

The symmetry of this coefficient is apparent when it is expressed in terms of the symmetrized V-coefficients. Applying equation (B-5) gives:

$$Z(a\ b\ c) = \left[(2a+1)(2b+1)(2c+1)\right]^{\frac{1}{2}} V(a\ b\ c;\ 0\ 0\ 0)$$

In terms of this notation the reduced lepton matrix elements of equation (30) become (using equation (34) and (32)):

$$(\nu\|\mathcal{Y}^{*}_{J}\|e) = (4\pi)^{-\frac{1}{2}} (-)^{j_\nu - \frac{1}{2}} (2j_e+1)^{\frac{1}{2}}$$

$$\Big\{\Big[\varphi_{\kappa_\nu} f_{\kappa_e}\ Z(\bar{\ell}_e\ J\ \bar{\ell}_\nu)\ W(\bar{\ell}_e\ j_e\ \bar{\ell}_\nu\ j_\nu;\ \tfrac{1}{2}\ J)$$

$$+\gamma_{\kappa_\nu} g_{\kappa_e}\ Z(\ell_e\ J\ \ell_\nu)\ W(\ell_e\ j_e\ \ell_\nu\ j_\nu;\ \tfrac{1}{2}\ J)\Big]$$

$$+i\ \Big[\varphi_{\kappa_\nu} g_{\kappa_e}\ Z(\ell_e\ J\ \bar{\ell}_\nu)\ W(\ell_e\ j_e\ \bar{\ell}_\nu\ j_\nu;\ \tfrac{1}{2}\ J)$$

$$-\gamma_{\kappa_\nu} f_{\kappa_e}\ Z(\bar{\ell}_e\ J\ \bar{\ell}_\nu)\ W(\bar{\ell}_e\ j_e \ell_\nu\ j_\nu;\ \tfrac{1}{2}\ J)\Big]\Big\}$$

$$(\nu\|T^{*}_{JL}\|e) = \frac{3\sqrt{2}}{4\pi}\left[(2J+1)(2\ j_e+1)\right]^{\frac{1}{2}}$$

$$\Big\{\Big[\varphi_{\kappa_\nu} f_{\kappa_e} (-)^{\bar{\ell}_\nu}\ Z(\bar{\ell}_e\ L\ \bar{\ell}_\nu)\quad X(\tfrac{1}{2} 1 \tfrac{1}{2};\ j_e\ J\ j_\nu;\ \bar{\ell}_e\ L\ \bar{\ell}_\nu)$$

$$+\gamma_{\kappa_\nu} g_{\kappa_e} (-)^{\ell_\nu}\ Z(\ell_e\ L \ell_\nu)\ X(\tfrac{1}{2} 1 \tfrac{1}{2};\ j_e\ J\ j_\nu; \ell_e\ L\ \ell_\nu)\Big]$$

$$+i\Big[\varphi_{\kappa_\nu} g_{\kappa_e} (-)^{\bar{\ell}_\nu}\ Z(\ell_e\ L\ \bar{\ell}_\nu)\ X(\tfrac{1}{2} 1 \tfrac{1}{2};\ j_e\ J\ j_\nu; \ell_e\ L\ \bar{\ell}_\nu)$$

$$-\gamma_{\kappa_\nu} f_{\kappa_e} (-)^{\ell_\nu}\ Z(\bar{\ell}_e\ L\ \ell_\nu)\ X(\tfrac{1}{2} 1 \tfrac{1}{2};\ j_e\ J\ j_\nu;\ \bar{\ell}_e\ L\ \ell_\nu)\Big]\Big\}$$

For the actual calculation, these formulae were rewritten in the following form:

$$(\nu\|\mathcal{Y}_J^*\|e) = (4\pi)^{-\frac{1}{2}}\, 2(j_e+1)^{\frac{1}{2}}\, (-)^{\kappa_\nu-1} \tag{48}$$

$$\Big\{\Big[\varphi_{\kappa_\nu}\, f_{\kappa_e}\, Z(\bar{\ell}_\nu J \bar{\ell}_e)\, W(\bar{\ell}_\nu j_\nu \bar{\ell}_e j_e;\, \tfrac{1}{2}\, J)$$

$$+\gamma_{\kappa_\nu}\, g_{\kappa_e}\, Z(\ell_\nu J \ell_e)\, W(\ell_\nu j_\nu \ell_e j_e;\, \tfrac{1}{2}\, J)\Big]$$

$$+i\,\Big[\varphi_{\kappa_\nu}\, g_{\kappa_e}\, Z(\bar{\ell}_\nu J \ell_e)\, W(\bar{\ell}_\nu j_\nu \ell_e j_e;\, \tfrac{1}{2}\, J)$$

$$-\gamma_{\kappa_\nu}\, f_{\kappa_e}\, Z(\ell_\nu J\, \bar{\ell}_e)\, W(\ell_\nu j_\nu \bar{\ell}_e j_e;\, \tfrac{1}{2}\, J)\Big]\Big\}$$

$$(\nu\|T_{JL}^*\|e) = \frac{3\sqrt{2}}{4\pi}\, \Big[(2J+1)(2j_e+1)\Big]^{\frac{1}{2}}\, (-)^{\kappa_\nu + \kappa_e + J + L + 1 + \ell_e} \tag{49}$$

$$\Big\{\Big[-\varphi_{\kappa_\nu}\, f_{\kappa_e}\, Z(\bar{\ell}_\nu L\, \bar{\ell}_e)\, X(\tfrac{1}{2} 1 \tfrac{1}{2};\, j_\nu J\, j_e;\, \bar{\ell}_\nu L\, \bar{\ell}_e)$$

$$+\gamma_{\kappa_\nu}\, g_{\kappa_e}\, Z(\ell_\nu L \ell_e)\, X(\tfrac{1}{2} 1 \tfrac{1}{2};\, j_\nu J\, j_e;\, \ell_\nu L\, \ell_e)$$

$$+i\Big[\varphi_{\kappa_\nu}\, g_{\kappa_e}\, Z(\bar{\ell}_\nu L \ell_e)\, X(\tfrac{1}{2} 1 \tfrac{1}{2};\, j_\nu J\, j_e;\, \bar{\ell}_\nu L\, \ell_e)$$

$$+\gamma_{\kappa_\nu}\, f_{\kappa_e}\, Z(\ell_\nu L\, \bar{\ell}_e)\, X(\tfrac{1}{2} 1 \tfrac{1}{2};\, j_\nu\, J\, j_e;\, \ell_\nu\, L\, \bar{\ell}_e)\Big]\Big\}$$

The last result has been obtained by interchanging the elements of X in accordance with the symmetry properties given in Appendix B.

CHAPTER IV

THE NUCLEAR MATRIX ELEMENTS

Nuclear Matrix Element Combinations

The nuclear matrix elements which appear in the interaction hamiltonian are (see equation 23):

$$
\begin{array}{ll}
N|C_V \mathcal{Y}_J^M(\hat{R})|P) & (N|C_A\ T_{JL}^M(\hat{R},\vec{\sigma})|P) \\
(N|C_A \mathcal{Y}_J^M(\hat{R})\gamma^5|P) & (N|C_V\ T_{JL}^M(\hat{R},-\vec{\alpha})|P)
\end{array} \qquad (50)
$$

where in the last result, use has been made of the form of T_{JL}^M and of the fact that $\gamma^5\vec{\sigma} = -\vec{\alpha}$.

Rose and Osborn (2) have introduced symbols for the combinations of nuclear matrix elements which appear in the square of the interaction hamiltonian. This notation will be used here; it is as follows*:

$$
\begin{aligned}
&\sum_M (N|T_{JL}^M(\vec{R},\vec{A})|P)(N|T_{JL'}^M(\vec{R},\vec{B})|P)^* = I_J(L,L';\ \vec{A},\vec{B})\ R^{-6} \\
&\sum_M (N|\mathcal{Y}_J^M(\vec{R})|P)(N|T_{JL}^M(\vec{R},\vec{A})|P)^* = J_J(L,\vec{A})\ R^{-6} \\
&\sum_M |(N|\mathcal{Y}_J^M(\vec{R})|P)|^2 = K_J\ R^{-6} \\
&\sum_M |(N|\mathcal{Y}_J^M(\vec{R})\ \gamma^5|P)|^2 = \mathcal{K}_J\ R^{-6} \\
&\sum_M (N|\mathcal{Y}_J^M(\vec{R})\ \gamma^5|P)(N|T_{JL}^M(\vec{R},\vec{A})|P)^* = \mathcal{J}_J(L,\vec{A})\ R^{-6}\ .
\end{aligned}
$$

---

*The symbol $\mathcal{J}_J$ is not introduced by Rose. Also, in Rose's paper the non-relativistic replacements: $\gamma^5 \to \vec{\sigma}\cdot\vec{p}/M$ and $-\vec{\alpha} \to \vec{p}/M$ have been made.

Notice that the arguments of the spherical tensor operators in these matrix element combinations differ from those of the spherical tensor operators used previously in that the position vector $\vec{R}$ rather than the unit vector $\hat{R}$ appears. From the definitions of the tensor operators involved, it is apparent that these equations can be written in terms of the unit vector arguments as follows:

$$\sum_M (N|T^M_{JL}(\hat{R},\vec{A})|P)(N|T^M_{JL'}(\hat{R},\vec{B})|P)^* = R^{-(L+L')}\ I_J(L\ L';\vec{A}\ \vec{B})\ R^{-6}$$

$$\sum_M (N|\mathcal{Y}^M_J(\hat{R})|P)(N|T^M_{JL}(\hat{R},\vec{A})|P)^* = R^{-(J+L)}\ J_J(L,\vec{A})\ R^{-6}$$

$$\sum_M |(N|\mathcal{Y}^M_J(\hat{R})|P)|^2 = R^{-2J}\ K_J\ R^{-6} \qquad (51)$$

$$\sum_M |(N|\mathcal{Y}^M_J(\hat{R})\gamma^5|P)|^2 = R^{-2J}\ \mathcal{K}_J\ R^{-6}$$

$$\sum_M (N|\mathcal{Y}^M_J(\hat{R})\gamma^5|P)(N|T^M_{JL}(\hat{R},\vec{A})|P)^* = R^{-(J+L)}\ \mathcal{J}_J(L,\vec{A})\ R^{-6}$$

In addition (following Rose) the following "short hand" notation is adopted:

$$I_J(L\ L;\ \vec{A}\ \vec{A}) = I_J(L,\vec{A})\ .$$

By means of these definitions, the quantities:

$$\sum_M |\Psi^M_J|^2 \qquad \sum_M \Xi^{M*}_{JL}\ \Xi^M_{JL'} \qquad \sum_M \Psi^{M*}_J\ \Xi^M_{JL}$$

which appear in equation (30) can be expressed in a somewhat more compact form. These quantities can be evaluated directly from their definition, equation (26). For the first of these, one obtains:

$$\sum_M |\Psi_J^M|^2 = R^6 \sum_M (N|(C_V-\gamma^5 C_A)\mathcal{Y}_J^M(\hat{R})|P)^*(N|(C_V-\gamma^5 C_A)\mathcal{Y}_J^M(\hat{R})|P)$$

$$= R^6 \sum_M \left\{(N|C_V \mathcal{Y}_J^M(\hat{R})|P)^* - (N|\gamma^5 C_A \mathcal{Y}_J^M(\hat{R})|P)^*\right\}$$

$$\left\{(N|C_V \mathcal{Y}_J^M(\hat{R})|P) - (N|\gamma^5 C_A \mathcal{Y}_J^M(\hat{R})|P)\right\}$$

$$= R^6 \sum_M \left\{|C_V|^2 |(N|\mathcal{Y}_J^M(\hat{R})|P)|^2 + |C_A|^2 |(N|\gamma^5 \mathcal{Y}_J^M(\hat{R})|P)|^2\right\}.$$

In the last result, the cross product term vanishes because the operators in the two factors are of opposite parity*. In terms of the quantities defined above this becomes:

$$\sum_M |\Psi_J^M|^2 = |C_V|^2 \, R^{-2J} K_J + |C_A|^2 \, R^{-2J} \, \mathcal{K}_J \quad . \tag{52}$$

In the same manner, one obtains for the second of these terms:

$$\sum_M \Xi_{JL}^{M*} \, \Xi_{JL'}^M = R^6 \cdot \sum_M (N|(\gamma^5 C_V - C_A) T_{JL}^M(\hat{R},\vec{\sigma})|P)^* (N|(\gamma^5 C_V - C_A) T_{JL'}^M(\hat{R},\vec{\sigma})|P)$$

$$= R^6 \sum_M \left\{C_V^*(N|\gamma^5 T_{JL}^M(\hat{R},\vec{\sigma})|P)^* - C_A^*(N|T_{JL}^M(\hat{R},\vec{\sigma})|P)^*\right\}$$

$$\left\{C_V(N|\gamma^5 T_{JL'}^M(\hat{R},\vec{\sigma})|P) - C_A(N|T_{JL'}^M(\hat{R},\vec{\sigma})|P)\right\}$$

---

*If one makes the very plausible assumption that the hamiltonian describing the nucleus is invariant under reflection of the spatial axes, then the wave function describing the nucleus is an eigenfunction of the parity operator. If this is the case, the nuclear states (initial and final) have definite parity (though it may be different for the two states). If the initial and final states have the same parity, the factor which contains the operator of odd parity vanishes; if the initial and final states have opposite parity, then the factor which contains the operator of even parity vanishes. In either event one of the factors vanishes and their product is zero.

$$= R^6 \sum_M \Big\{ |C_V|^2 (N|T^M_{JL}(\hat{R},\vec{\alpha})|P)^* (N|T^M_{JL'}(\hat{R},\vec{\alpha})|P)$$

$$+|C_A|^2 (N|T^M_{JL}(\hat{R},\vec{\sigma})|P)^* (N|T^M_{JL'}(\hat{R},\vec{\sigma})|P)$$

$$+C_V^* C_A (N|T^M_{JL}(\hat{R},\vec{\alpha})|P)^* (N|T^M_{JL'}(\hat{R},\vec{\sigma})|P) \tag{53}$$

$$+C_V C_A^* (N|T^M_{JL}(\hat{R},\vec{\sigma})|P)^* (N|T^M_{JL'}(\hat{R},\vec{\alpha})|P) \Big\}$$

$$= R^{-(L+L')} \Big\{ |C_V|^2 \, I_J \, (L' \, L; \, \vec{\alpha} \, \vec{\alpha}) + |C_A|^2 \, I_J (L'L; \, \vec{\sigma}\vec{\sigma})$$

$$+C_V^* C_A \, I_J \, (L'L; \, \vec{\sigma}\vec{\alpha}) + C_A^* C_V \, I_J \, (L'L; \, \vec{\alpha},\vec{\sigma}) \Big\}$$

Finally for the last of these terms, one obtains:

$$\sum_M \Psi^{M*}_J \, \Xi^M_{JL} = R^6 \sum_M (N|(C_V - \gamma^5 C_A) \mathcal{Y}^M_J(\hat{R})|P)^* (N|(\gamma^5 C_V - C_A) T^M_{JL}(\hat{R},\vec{\sigma})|P)$$

$$= R^6 \sum_M \Big\{ |C_V|^2 (N|T^M_{JL}(\hat{R},-\vec{\alpha})|P)(N|\mathcal{Y}^M_J(\hat{R})|P)^*$$

$$+ |C_A|^2 (N|T^M_{JL}(\hat{R},\vec{\sigma})|P)(N|\mathcal{Y}^M_J(\hat{R}) \, \gamma^5|P)^*$$

$$- C_V^* C_A \, (N|T^M_{JL}(\hat{R},\vec{\sigma})|P)(N|\mathcal{Y}^M_J(\hat{R})|P)^* \tag{54}$$

$$- C_V C_A^* \, (N|T^M_{JL}(\hat{R},-\vec{\alpha})|P)(N|\mathcal{Y}^M_J(\hat{R}) \, \gamma^5|P)^* \Big\}$$

$$= R^{-(J+L)} \Big\{ |C_V|^2 \, J_J^*(L,-\vec{\alpha}) + |C_A|^2 \, \mathcal{J}_J^*(L,\vec{\sigma})$$

$$- C_V^* C_A \, J_J^*(L,\vec{\sigma}) - C_V C_A^* \, \mathcal{J}_J^*(L,-\vec{\alpha}) \Big\} \quad .$$

$M(\kappa_e)$

Using the results of the preceding section, one can write the expression for $M(\kappa_e)$ as follows:

$$M(\kappa_e) = \sum_J \sum_{j_\nu \ell_\nu} \frac{2j_\nu + 1}{2J + 1} \Big\{ R^{-2J} \left[|C_V|^2 K_J + |C_A|^2 \mathcal{K}_J\right] |(\nu\|\mathcal{Y}_J^*\|e)|^2$$

$$+ \frac{16\pi^2}{9} \sum_{L,L'} R^{-(L+L')} \Big[|C_V|^2 I_J(L'L;\vec{\alpha}\vec{\alpha}) + |C_A|^2 I_J(L'L;\vec{\sigma}\vec{\sigma})$$

$$+ C_V^* C_A I_J(L'L;\vec{\sigma}\vec{\alpha}) + C_V C_A^* I_J(L'L;\vec{\alpha}\vec{\sigma})\Big] (\nu\|T^*_{JL}\|e)^*(\nu\|T^*_{JL'}\|e)$$

$$- \frac{4\pi}{3} \sum_L R^{-(J+L)} \Big[|C_V|^2 J_J^*(L,-\vec{\alpha}) + |C_A|^2 \mathcal{J}_J^*(L,\vec{\sigma}) - C_V^* C_A J_J^*(L,\vec{\sigma})$$

$$+ C_V C_A^* \mathcal{J}_J^*(L,\vec{\alpha})\Big] (\nu\|\mathcal{Y}_J^*\|e)^*(\nu\|T^*_{JL}\|e) \tag{55}$$

$$- \frac{4\pi}{3} \sum_L R^{-(J+L)} \Big[|C_V|^2 J_J(L,-\vec{\alpha}) + |C_A|^2 \mathcal{J}_J(L,\vec{\sigma}) - C_V C_A^* J_J(L,\vec{\sigma})$$

$$+ C_V^* C_A \mathcal{J}_J(L,\vec{\alpha})\Big] (\nu\|\mathcal{Y}_J^*\|e) (\nu\|T^*_{JL}\|e)^* \Big\}$$

Time Reversal Invariance

It has been shown by Longmire and Messiah (14) that if time reversal invariance is to be preserved, the ratios of the matrix elements (50) must be either real or pure imaginary. From the properties of the time reversal operator, Rose and Osborn (2) have shown that if time reversal invariance is preserved the products of the matrix elements of the following operators are real:

$$i\,\mathcal{Y}_J^M(\hat{R}), \quad i\,T_{JL}^M(\hat{R},\vec{\sigma}), \quad T_{JL}^M(\hat{R},\vec{\alpha}), \quad \mathcal{Y}_J^M(\hat{R})\,\gamma^5 \quad .$$

Experiments conducted at Argonne National Laboratories by Burgy, et. al. (11) indicate that time reversal invariance is preserved. Thus the matrix element combinations:

$$J_J(L,\vec{\sigma}) \qquad \text{and} \qquad \mathcal{J}_J(L,\vec{\alpha})$$

are real, while the matrix element combinations:

$$J_J(L,\vec{\alpha}), \qquad \mathcal{J}_J(L,\vec{\sigma}) \qquad \text{and} \qquad I_J(L'L;\vec{\sigma}\vec{\alpha})$$

are pure imaginary. Using this result along with the reality of C_V and C_A, one obtains from equation (55):

$$\begin{aligned}
M(\kappa_e) = \sum_J \sum_{j_\nu,\ell_\nu} \frac{2j_\nu+1}{2J+1} \Big\{ & R^{-2J}\left[C_V^2\,K_J + C_A^2\,\mathcal{K}_J\right] |(\nu\|\mathcal{Y}_J^*\|e)|^2 \\
& + \frac{16\,\pi^2}{9} \sum_{L,L'} R^{-(L+L')}\left[C_V^2\,I_J(L'L;\vec{\alpha}\vec{\alpha}) + C_A^2\,I_J(L'L;\vec{\sigma}\vec{\sigma})\right] \\
& \qquad (\nu\|T_{JL}^*\|e)^*(\nu\|T_{JL'}^*\|e) \\
& + \frac{16\,\pi^2}{9} \sum_{L,L'} C_V C_A\,I_J(L'L;\vec{\sigma}\vec{\alpha})\left[(\nu\|T_{JL}^*\|e)^*(\nu\|T_{JL'}^*\|e) - c.c.\right] \\
& + \frac{4\pi}{3}\sum_L R^{-(J+L)}\left[C_V^2\,J_J(L,\vec{\alpha}) - C_A^2\,\mathcal{J}_J(L,\vec{\sigma})\right]\left[(\nu\|\mathcal{Y}_J^*\|e)(\nu\|T_{JL}^*\|e)^* - c.c.\right] \\
& + \frac{4\pi}{3}\sum_L R^{-(J+L)} C_V C_A\left[J_J(L,\vec{\sigma}) - \mathcal{J}_J(L,\vec{\alpha})\right]\left[(\nu\|\mathcal{Y}_J^*\|e)(\nu\|T_{JL}^*\|e)^* + c.c.\right]\Big\} \quad .
\end{aligned}$$

CHAPTER V

SELECTION RULES AND FORBIDDENNESS OF THE IRREDUCIBLE SPHERICAL TENSOR OPERATORS

Parity of the Irreducible Spherical Tensor Operators

In order to discuss the parity of the irreducible spherical tensor operators, one must make the following definitions. Two states are said to have the same parity if the "large" components of the wave functions describing these states have the same parity. An operator, Ω, is said to have even parity if $\psi_2^*\Omega\psi_1$ has the same parity as $\psi_2^*\psi_1$. The operator, Ω, is said to have odd parity if $\psi_2{}^*\Omega\psi_1$ has opposite parity from $\psi_2{}^*\psi_1$. Recall also that the large and small components of the wave functions, ψ, have opposite parity.

Consider first the operator $\mathcal{Y}_J^M(\vec{R}) = R^J\ Y_J^M(\theta,\varphi)$. Since this operator connects the large component of ψ_2 with the large component of ψ_1, and the small component of ψ_2 with the small component of ψ_1, its parity is simply that of the spherical harmonic $Y_J^M(\theta,\varphi)$, namely $(-)^J$.

Similarly, the operator $T_{JL}^M(\vec{R},\vec{\sigma})$ connects the large component of ψ_2 with the large component of ψ_1 etc. From the definition of T_{JL}^M one sees that its parity must be that of $\mathcal{Y}_L^{\mu+M}(\vec{R})$, namely $(-)^L$.

Now consider the effect of the operator γ^5. This operator mixes the large and small components of ψ_2 and ψ_1. In other words, if one sets:

$$\psi_2 = \begin{bmatrix} L_2 \\ \\ S_2 \end{bmatrix} \qquad\qquad \psi_1 = \begin{bmatrix} L_1 \\ \\ S_1 \end{bmatrix}$$

where L and S represent the large and small components respectively, then $\psi_2{}^*\psi_1 = L_2{}^*L_1 + S_2{}^*S_1$ but $\psi_2{}^*\gamma^5\,\psi_1 = -L_2{}^*S_1 - S_2{}^*L_1$. Since L_1 and S_1 have opposite parity (as do L_2 and S_2) the operator γ^5 has odd parity. Hence $\mathcal{Y}_J^M(\vec{R})\,\gamma^5$ and $T_{JL}^M(\vec{R},\vec{\alpha}) = -T_{JL}^M(\vec{R},\vec{\sigma})\,\gamma^5$ must have parities $(-)^{J+1}$ and $(-)^{L+1}$ respectively.

Order of Forbiddenness of the Irreducible Spherical Tensor Operators

The order of forbiddenness of the irreducible spherical tensor operators depends on two properties of these operators. The first of these is the R dependence of the operator. As can be seen from the definitions of these operators, $T_{JL}^M(\vec{R},\vec{B})$ is proportional to R^L while $\mathcal{Y}_J^M(\vec{R})$ is proportional to R^J, where R is the nuclear radius (in Compton wavelengths of the electron for the natural units). Since R is less than 0.02 for atomic mass number, A, less than 260, each power of R decreases the magnitude of the matrix element by at least a factor of 50.

The second property of the operator which affects its order of forbiddenness is whether or not the operator mixes large and small components of the wave functions of the initial and final states. As was seen in the preceding section, the operators $\mathcal{Y}_J^M(\vec{R})\,\gamma^5$ and $T_{JL}^M(\vec{R},\vec{\alpha})$ do mix these components. Since the nucleons have velocities small compared with the velocity of light, the small component of the wave function is indeed small compared with the large component. For this reason matrix elements which couple $L_2{*}L_1 + S_2{*}S_1$ are large compared with those which couple $L_2{*}S_1 + S_2{*}L_1$. This mixing of large and small components has about the same effect on the order of magnitude of the matrix element as does increasing the power of R by one.* These properties are summarized in Table I below.

Table I. Parity and Forbiddenness of the Operators

Operator	Parity	Order of Forbiddenness
$\mathcal{Y}_J^M(\vec{R})$	$(-)^J$	J
$T_{JL}^M(\vec{R},\vec{\sigma})$	$(-)^L$	L
$T_{JL}^M(\vec{R},\vec{\alpha})$	$(-)^{L+1}$	L+1
$\mathcal{Y}_J^M(\vec{R})\ \gamma^5$	$(-)^{J+1}$	J+1

Approximations in $M(\kappa_e)$

From the definition of T_{JL}^M one sees that due to the triangular properties of the Clebsch-Gordon coefficients, $T_{JL}^M = 0$ unless $J = L$, $L \pm 1$. For a given argument, terms with $L = J + 1$ have the same parity as terms with $L = J - 1$, but their forbiddenness is two orders higher.

* An alternative approach to this second property which illustrates this fact is the following. It has been shown that in the non-relativistic limit the operator $\gamma^5 \to \frac{\vec{\sigma}\cdot\vec{p}}{M}$, and $\vec{\alpha} \to -\vec{p}/M$ where $\vec{p}$ and M are the momentum and mass of the nucleon respectively. Making the further replacement $\vec{p} \to -i\vec{\nabla}$, in effect brings down one additional power of R into the matrix element. In this sense, the two properties described above are similar.

As a result of this, one would expect the L = J - 1 terms to be large compared with the L = J + 1 terms in the sum, and the latter will be neglected in the ensuing calculations. The case J = 0 is an exception to this since the L = J - 1 term does not exist.

As a result of this approximation, equation (56) becomes:

$$
\begin{aligned}
M(\kappa_e) = \sum_{\kappa_\nu} 2|\kappa_\nu| \Big\{ & \left[C_V^2 K_0 + C_A^2 \mathcal{K}_0\right] |(\nu\|\mathcal{Y}_0^*\|e)|^2 \\
& + \frac{16\pi^2}{9} R^{-2} \left[C_V^2 I_0(1,\vec{a}) + C_A^2 I_0(1,\vec{\sigma})\right] |(\nu\|T_{01}^*\|e)|^2 \\
& + \frac{4\pi}{3} R^{-1} \left[C_V^2 J_0(1,\vec{a}) - C_A^2 \mathcal{J}_0(1,\vec{\sigma})\right] \left[(\nu\|\mathcal{Y}_0^*\|e)(\nu\|T_{01}^*\|e)^* - c.c.\right]\Big\} \\
+ \sum_{J=1}^{\infty} \sum_{\kappa_\nu} \frac{2|\kappa_\nu|}{2J+1} \Big\{ & R^{-2J} \left[C_V^2 K_J + C_A^2 \mathcal{K}_J\right] |(\nu\|\mathcal{Y}_J^*\|e)|^2 \\
& + \frac{16\pi^2}{9} \sum_L R^{-2L} \left[C_V^2 I_J(L,\vec{a}) + C_A^2 I_J(L,\vec{\sigma})\right] |(\nu\|T_{JL}^*\|e)|^2 \\
& + \frac{16\pi^2}{9} R^{1-2J} C_V C_A I_J(J,J-1;\vec{\sigma},\vec{a}) \left[(\nu\|T_{J,J-1}^*\|e)^*(\nu\|T_{JJ}\|e) - c.c.\right] \\
& + \frac{4\pi}{3} R^{1-2J} \left[C_V^2 J_J(J-1,\vec{a}) - C_A^2 \mathcal{J}_J(J-1,\vec{\sigma})\right] \\
& \qquad \left[(\nu\|\mathcal{Y}_J^*\|e)(\nu\|T_{J,J-1}^*\|e)^* - c.c.\right] \\
& + \frac{4\pi}{3} R^{-2J} C_V C_A \left[J_J(J,\vec{\sigma}) - \mathcal{J}_J(J,\vec{a})\right] \left[(\nu\|\mathcal{Y}_J^*\|e)(\nu\|T_{JJ}^*\|e)^* + c.c.\right]\Big\}.
\end{aligned}
\tag{57}
$$

<u>Final Form for $M(\kappa_e)$</u>

Equation (57) still contains terms which can be neglected compared with other terms in the sum. Consider first the case J = 0. The terms $I_0(1,\vec{a})$ and $J_0(1,\vec{a})$ both have even parity as does K_0, but the order of forbiddenness of $I_0(1,\vec{a})$ and of $J_0(1,\vec{a})$ is two, while K_0 is "allowed", i.e. its order of forbiddenness is zero. Thus one would expect the magnitude of K_0 to be of the order of R^{-4} times as great as that of $I_0(1,\vec{\alpha})$ or $J_0(1,\vec{\alpha})$. Consequently, these latter terms will be neglected in the ensuing calculations.

Consider next the case $J \neq 0$. The terms $\mathcal{K}_J$, $I_J(J,\vec{\alpha})$, $\mathcal{J}_J(J,\vec{\alpha})$ and $\mathcal{J}_J(J-1,\vec{\sigma})$ have parity $(-)^{J+1}$ as does $I_J(J-1,\vec{\sigma})$. The order of forbiddenness of $I_J(J-1,\vec{\sigma})$ is J-1, while the order of forbiddenness of $\mathcal{K}_J$, $I_J(J,\vec{\alpha})$ and $\mathcal{J}_J(J,\vec{\alpha})$ is J+1 and that of $\mathcal{J}_J(J-1,\vec{\sigma})$ is J. Thus these latter four terms will be neglected compared with $I_J(J-1,\vec{\sigma})$.

When these approximations are made, the equation for $M(\kappa_e)$ becomes:

$$
\begin{aligned}
M(\kappa_e) = \sum_{\kappa_\nu} 2|\kappa_\nu| \Big\{ & \left[C_V^2 K_0 + C_A^2 \mathcal{K}_0\right] |(\nu\|\mathcal{Y}_0^*\|e)|^2 \\
& + \frac{16\pi^2}{9} R^{-2} C_A^2 I_0(1,\vec{\sigma}) |(\nu\|T_{01}^*\|e)|^2 \\
& - \frac{4\pi}{3} R^{-1} C_A^2 \mathcal{J}_0(1,\vec{\sigma}) \left[(\nu\|\mathcal{Y}_0^*\|e)\ (\nu\|T_{01}^*\|e)^* - c.c.\right]\Big\} \\
& + \sum_{J=1}^{\infty} \sum_{\kappa_\nu} \frac{2|\kappa_\nu|}{2J+1} \Big\{ R^{-2J} C_V^2 K_J\ |(\nu\|\mathcal{Y}_J^*\|e)|^2 \\
& + \frac{16\pi^2}{9} R^{-2J} C_A^2 I_J(J,\vec{\sigma})\ |(\nu\|T_{JJ}^*\|e)|^2 \qquad (58) \\
& + \frac{16\pi^2}{9} R^{2-2J} \left[C_V^2 I_J(J-1,\vec{\alpha}) + C_A^2 I_J(J-1,\vec{\sigma})\right] |(\nu\|T_{J,J-1}^*\|e)|^2 \\
& + \frac{16\pi^2}{9} R^{1-2J} C_V C_A I_J(J,J-1;\vec{\sigma},\vec{\alpha}) \left[(\nu\|T_{J,J-1}^*\|e)^*(\nu\|T_{JJ}^*\|e) - c.c.\right] \\
& + \frac{4\pi}{3} R^{1-2J} C_V^2 J_J(J-1,\vec{\alpha}) \left[(\nu\|\mathcal{Y}_J^*\|e)(\nu\|T_{J,J-1}^*\|e)^* - c.c.\right] \\
& - \frac{4\pi}{3} R^{-2J} C_V C_A J_J(J,\vec{\sigma}) \left[(\nu\|\mathcal{Y}_J^*\|e)(\nu\|T_{JJ}^*\|e)^* + c.c.\right]\Big\} .
\end{aligned}
$$

This is the desired form for calculation of the transition probabilities.

CHAPTER VI

CALCULATION OF TRANSITION PROBABILITIES

Expansion of the Neutrino Wave Functions

In order to evaluate the lepton matrix elements which appear in equation (58) it is expedient to expand the neutrino wave functions (which are spherical Bessel functions) in a power series. From the power series expansion for the (cylindrical) Bessel functions and the definition of the spherical Bessel functions (equation 39), one obtains:

$$j_n(z) = z^n \sum_{k=0} \frac{(-)^k \; z^{2k}}{2^k \; k! \; (2n+2k+1)!!} \tag{59}$$

Since these wave functions are to be evaluated only over the nuclear dimensions, it is sufficient to keep only the first term in the series.* Thus one obtains from equations (38) and (59) the following form for the

---

*The sum for $j_n(z)$ can be written:

$$j_n(z) = \frac{z^n}{(2n+1)!!}\left[1 - \frac{z^2}{2(2n+3)} + \ldots\right]$$

Thus even for $n = 0$, the second term is smaller than the first by the factor $(1/6)z^2$ which corresponds to $(1/6)(qR)^2$ in the wave function. For $q < 2$ the second term would make a contribution comparable to the contribution from transitions of one higher order of forbiddenness. Since these contributions have been neglected already, this approximation is consistent with the others.

neutrino wave functions:

$$\varphi_{\kappa_\nu} = \frac{\kappa_\nu}{|\kappa_\nu|} \frac{q\ (qR)^{\bar{\ell}_\nu}}{(2\bar{\ell}_\nu+1)!!}$$

$$\gamma_{\kappa_\nu} = \frac{q\ (qR)^{\ell_\nu}}{(2\ell_\nu+1)!!} \quad . \qquad (60)$$

By using these equations for φ_{κ_ν} and γ_{κ_ν}, one can evaluate the reduced lepton matrix elements which occur in (58) directly from equations (48) and (49). The Z coefficients can be evaluated directly from their definitions and tables of Clebsch-Gordon coefficients such as those found in Condon and Shortly (15). The values of the Z coefficients which occur in the calculations are listed in Appendix B. The Racah coefficients, W, are tabulated by Biedenharn (16). The explicit forms of the Fano coefficients, X, are given by Rose (2).

The values of the reduced lepton matrix elements which occur in the interaction hamiltonian are listed in Appendix D.

Results for $M_J(\kappa_e)$ for κ_e = -1, -2, -3; J = 0, 1, 2, 3

In this section are recorded the results of performing the calculations indicated in equation (58). The first four terms in the sum are evaluated (J = 0, 1, 2, 3) for each of the sub-shells of negative κ_e which give an appreciable contribution. Corresponding equations for positive κ_e can be obtained from these by making the substitutions: $f \rightarrow g$ and $g \rightarrow -f$.

In the following equations, the J'th term in the sum over J of equation (58) is represented by $M_J(\kappa_e)$.

Transitions from the M_I shell ($\kappa_e = -1$)

$$M_0(-1) = \frac{1}{\pi}\left[C_V^2 K_0 + C_A^2 \mathcal{K}_0\right] q^2g^2 + \frac{4}{3} q^2R^{-2} C_A^2 I_0(1,\vec{\sigma})\left[f^2 + \frac{2}{3} qRfg + \frac{1}{9} q^2R^2g^2\right] - i2\sqrt{\frac{4}{3\pi}}\, q^2R^{-1}C_A^2 \mathcal{J}_0(1,\vec{\sigma})\, g\left[f + \frac{1}{3} qRg\right]$$

$$M_1(-1) = \frac{1}{3\pi} q^2R^{-2}C_V^2 K_1\left[f^2 + \frac{2}{3} qRfg + \frac{1}{3} q^2R^2g^2\right] + \frac{4}{3} q^2\left[C_V^2 I_1(0,\vec{a}) + C_A^2 I_1(0,\vec{\sigma})\right] g^2 + \frac{8}{9} q^2R^{-2}C_A^2 I_1(1,\vec{\sigma})\left[f^2 - \frac{2}{3} qRfg + \frac{1}{6} q^2R^2g^2\right]$$
$$+ i\frac{16}{3\sqrt{6}} q^2R^{-1} C_VC_A I_1(1,0;\vec{\sigma},\vec{a})\, g\left[f - \frac{1}{3} qrg\right]$$
$$+ i\,\frac{4}{3\sqrt{\pi}} q^2R^{-1} C_V^2 J_1(0,\vec{a})\, g\left[f + \frac{1}{3} qRg\right] + \frac{8}{3\sqrt{6\pi}} q^2R^{-2} C_VC_A J_1(1,\vec{\sigma}) f^2.$$

$$M_2(-1) = \frac{2}{45\pi} q^4R^{-2} C_V^2 K_2\left[f^2 + \frac{2}{5} qRfg + \frac{1}{10} q^2R^2g^2\right] + \frac{4}{27} q^4 \left[C_V^2 I_2(1,\vec{a}) + C_A^2 I_2(1,\vec{\sigma})\right] g^2 + \frac{4}{45} q^4R^{-2} C_A^2 I_2(2,\vec{\sigma})$$
$$\left[f^2 - \frac{2}{5} qRfg + \frac{1}{15} q^2R^2g^2\right] + i\frac{8}{9\sqrt{15}} q^4R^{-1} C_VC_A I_2(2,1;\vec{\sigma},\vec{a})g\left[f-\frac{1}{5}qRg\right]$$
$$+ i\frac{8}{9\sqrt{30\pi}} q^4R^{-1} C_V^2 J_2(1,\vec{a})\, g\left[f + \frac{1}{5} qRg\right] + \frac{4}{45}\sqrt{\frac{2}{\pi}}\, q^4R^{-2}C_VC_A J_2(2,\vec{\sigma})f^2.$$

$$M_3(-1) = \frac{1}{525\pi} q^6R^{-2}C_V^2 K_3\left[f^2 + \frac{2}{7} qRfg + \frac{1}{21}q^2R^2g^2\right] + \frac{4}{675} q^6\left[C_V^2 I_3(2,\vec{a}) + C_A^2 I_3(2,\vec{\sigma})\right] g^2 + \frac{16}{4725} q^6R^{-2} C_A^2 I_3(3,\vec{\sigma})\left[f^2 - \frac{2}{7} qRfg + \frac{1}{28} q^2R^2g^2\right]$$
$$+ i\frac{16}{675\sqrt{7}} q^6R^{-1} C_VC_A I_3(3,2;\vec{\sigma},\vec{a})\, g\left[f - \frac{1}{7} qRg\right] + i\frac{4}{225\sqrt{7\pi}} q^6R^{-1} C_V^2 J_3(2,\vec{a})\, g\left[f + \frac{1}{7} qRg\right] + \frac{8}{1575\sqrt{\pi}} q^6R^{-2} C_VC_A J_3(3,\vec{\sigma}) f^2$$

$M_4(-1) = \frac{4}{33075} q^8 C_A^2 I_4(3,\vec{\sigma}) g^2$ + terms of higher orders of forbiddenness.

Transitions from the M_{III} shell ($\kappa_e = -2$)

$$M_2(-2) = \frac{2}{5\pi} q^2 R^{-4} C_V^2 K_2 \left[f^2 + \frac{2}{3} qRfg + \frac{2}{9} q^2R^2g^2\right] + \frac{4}{3} q^2 R^{-2}$$

$$\left[C_V^2 I_2(1,\vec{\alpha}) + C_A^2 I_2(1,\vec{\sigma})\right] g^2 + \frac{4}{5} q^2 R^{-4} C_A^2 I_2(2,\vec{\sigma}) \left[f^2 - \frac{2}{3} qRfg\right.$$

$$\left. + \frac{1}{9} q^2R^2g^2\right] + i\frac{8}{\sqrt{15}} q^2 R^{-3} C_V C_A I_2(2,1;\vec{\sigma},\vec{\alpha}) \; g\left[f - \frac{1}{3} qRg\right]$$

$$+ i\frac{8}{\sqrt{30\pi}} q^2 R^{-3} C_V^2 J_2(1,\vec{\alpha}) \; g\left[f + \frac{1}{3} qRg\right] + \frac{8}{5\sqrt{2\pi}} q^2 R^{-4} C_V C_A$$

$$J_1(2,\vec{\sigma}) \left[f^2 - \frac{1}{9} q^2R^2g^2\right]$$

$$M_3(-2) = \frac{2}{35\pi} q^4 R^{-4} C_V^2 K_3\left[f^2 + \frac{2}{5} qRfg + \frac{1}{15} q^2R^2g^2\right] + \frac{8}{45} q^4 R^{-2}$$

$$\left[C_V^2 I_3(2,\vec{\alpha}) + C_A^2 I_3(2,\vec{\sigma})\right] g^2 + \frac{32}{315} q^4 R^{-4} C_A^2 I_3(3,\vec{\sigma})$$

$$\left[f^2 - \frac{2}{5} qRfg + \frac{1}{24} q^2R^2g^2\right] + i \frac{32}{45\sqrt{7}} q^4 R^{-3} C_V C_A I_3(3,2;\vec{\sigma},\vec{\alpha})$$

$$g\left[f - \frac{1}{5} qRg\right] + i \frac{8}{15\sqrt{7\pi}} q^4 R^{-3} C_V^2 J_3(2,\vec{\alpha}) \; g\left[f + \frac{1}{5} qRg\right]$$

$$+ \frac{16}{105\sqrt{\pi}} q^4 R^{-4} C_V C_A J_3(3,\vec{\sigma})\left[f^2 - \frac{1}{30} q^2R^2g^2\right]$$

$M_4(-2) = \frac{4}{525} q^6 R^{-2} C_A^2 I_4(3,\vec{\sigma}) \; g^2$ + terms of higher orders of forbiddenness.

Transitions from the M_V shell ($\kappa_e = -3$)

$$M_3(-3) = \frac{3}{7\pi} q^2R^{-6} C_V^2 K_3\left[f^2 + \frac{2}{3} qRfg + \frac{1}{5} q^2R^2g^2\right] + \frac{4}{3} q^2R^{-4}$$

$$\left[C_V^2 I_3(2,\vec{\alpha}) + C_A^2 I_3(2,\vec{\sigma})\right] g^2 + \frac{16}{21} q^2R^{-6} C_A^2 I_3(3,\vec{\sigma})$$

$$\left[f^2 - \frac{2}{3} qRfg + \frac{7}{60} q^2R^2g^2\right] + i \frac{16}{3\sqrt{7}} q^2R^{-5} C_VC_A I_3(3,2;\vec{\sigma}\vec{\alpha})$$

$$g\left[f - \frac{1}{3} qRg\right] + i\frac{4}{\sqrt{7\pi}} q^2R^{-5} C_V^2 J_3(2,\vec{\alpha}) g\left[f + \frac{1}{3} qRg\right]$$

$$+ \frac{8}{7\sqrt{\pi}} q^2R^{-6} C_VC_A J_3(3,\vec{\sigma})\left[f^2 - \frac{2}{15} q^2R^2g^2\right]$$

$$M_4(-3) = \frac{4}{21} q^4 R^{-4} C_A^2 I_4(3,\vec{\sigma}) g^2 + \text{terms of higher forbiddenness.}$$

Transition Probabilities

In order to use the results stated above and/or to compare them with experimental data, it is expedient to regroup the terms. The transition probability for any transition is given by the sum of all terms of the type given above. For any given transition, however, the initial and final states of the nucleus require that the nuclear matrix elements of some of the terms vanish. Of the remaining terms, some will be small enough to be neglected compared with others. In order to examine this situation, one regroups the above terms into combinations of like parity and like order of forbiddenness.* The parity and order of forbiddenness of the nuclear matrix element combinations can be

*By the term "parity of a matrix element" (or of a matrix element combination) is meant the following. A matrix element is said to have even parity if it connects states of like parity, it is said to have odd parity if it connects states of opposite parity. In other words, matrix elements of odd operators have odd parity, etc.

determined from the parity and order of forbiddenness of the irreducible spherical tensor operators (given in Table 1). For the nuclear matrix element combinations: K_J, $I_J(j-1,\vec{\alpha})$, $I_J(J,\vec{\sigma})$, $I_J(J,J-1;\vec{\sigma},\vec{\alpha})$, $J_J(J-1,\vec{\alpha})$, and $J_J(J,\vec{\sigma})$, the parity is given by $(-)^J$ and the order of forbiddenness is J. For $I_J(J-1,\vec{\sigma})$ the parity is $(-)^{J-1}$ and the order of forbiddenness is J-1. The remaining two nuclear matrix element combinations which occur, $\mathcal{K}_0$ and $\mathcal{J}_0(1,\vec{\sigma})$, have odd parity and are first-forbidden.

The regrouped transition probabilities, $S_n^{(J)}$, are given below. It is clear that $M(\kappa_e)$ which is given by:

$$M(\kappa_e) = \sum_J M_J(\kappa_e)$$

is equally well given by:

$$M(\kappa_e) = \sum_J \left[S_J^{(J)} + S_{J-1}^{(J)} + S_{J+1}^{(J)}\right] ,$$

(where $S_{J+1}^{(J)}$ is significant only for J = 0).

<u>Allowed (no parity change)</u>

$J = 0$
K, L_I, M_I: $\quad S_0^{(0)} = \frac{1}{\pi} q^2 C_V{}^2 K_0 g^2$

$J = 1$
K, L_I, M_I: $\quad S_0^{(1)} = \frac{4}{3} q^2 C_A{}^2 I_1(0,\vec{\sigma}) g^2$

<u>First Forbidden (yes parity change)</u>

$J = 0$
K, L_I, M_I:

$$S_1^{(0)} = \frac{1}{\pi} q^2 C_A{}^2 \left\{ \mathcal{K}_0 g^2 + \frac{4\pi}{3} R^{-2} I_0(1,\vec{\sigma}) \left[f^2 + \frac{2}{3} qRfg + \frac{1}{9} q^2R^2g^2\right] - i2\sqrt{\frac{4\pi}{3}} R^{-1} \mathcal{J}_0(1,\vec{\sigma}) g\left[f + \frac{1}{3} qRg\right]\right\}$$

J = 1

K, L_I, M_I: $$S_1^{(1)} = \frac{1}{3\pi} q^2 C_V^2 \Big\{ R^{-2} K_1 \left[f^2 + \frac{2}{3} qRfg + \frac{1}{3} q^2R^2g^2\right]$$

$$+ 4\pi I_1(0,\vec{\alpha}) g^2 + i4\sqrt{\pi} R^{-1} J_1(0,\vec{\alpha}) g\left[f + \frac{1}{3} qRg\right]\Big\}$$

$$+ \frac{8}{9} q^2R^{-2} C_A^2 I_1(1,\vec{\sigma}) \left[f^2 - \frac{2}{3} qRfg + \frac{1}{6} q^2R^2g^2\right]$$

$$+ \frac{8}{3\sqrt{6\pi}} q^2 C_V C_A \Big\{R^{-2} J_1(1,\vec{\sigma}) f^2 + i2\sqrt{\pi} R^{-1} I_1(1,0;\vec{\sigma},\vec{\alpha}) g\left[f - \frac{1}{3} qRg\right]\Big\}$$

First-Forbidden Unique (yes parity change) J = 2

K, L_I, M_I: $$S_1^{(2)} = \frac{4}{27} q^4 C_A^2 I_2(1,\vec{\sigma}) g^2$$

L_{III}, M_{III}: $$S_1^{(2)} = \frac{4}{3} q^2R^{-2} C_A^2 I_2(1,\vec{\sigma}) g^2$$

M_V: $$S_1^{(2)} = \frac{4}{15} q^4 C_A^2 I_2(1,\vec{\sigma}) g^2$$

Second-Forbidden (no parity change) J = 2

K, L_I, M_I: $$S_2^{(2)} = \frac{2}{45\pi} q^4 C_V^2 \Big\{ R^{-2} K_2 \left[f^2 + \frac{2}{5} qRfg + \frac{1}{10} q^2R^2g^2\right]$$

$$+ \frac{10\pi}{3} I_2(1,\vec{\alpha}) g^2 + i2\sqrt{\frac{10\pi}{3}} R^{-1} J_2(1,\vec{\alpha}) g\left[f + \frac{1}{5} qRg\right]\Big\}$$

$$+ \frac{4}{45} q^4R^{-2} C_A^2 I_2(2,\vec{\sigma}) \left[f^2 - \frac{2}{5} qRfg + \frac{1}{15} q^2R^2g^2\right]$$

$$+ \frac{8}{45} \frac{1}{\sqrt{2\pi}} q^4 C_V C_A \Big\{ R^{-2} J_2(2,\vec{\sigma}) f^2 + i\sqrt{\frac{10\pi}{3}} R^{-1} I_2(2,1;\vec{\sigma},\vec{\alpha}) g\left[f - \frac{1}{5} qRg\right]\Big\}$$

L_{III}, M_{III}: $$S_2^{(2)} = \frac{2}{5\pi} q^2 R^{-2} C_V^2 \Big\{ R^{-2} K_2 \left[f^2 + \frac{2}{3} qRfg + \frac{2}{9} q^2R^2g^2\right]$$

$$+ \frac{10\pi}{3} I_2(1,\vec{\alpha}) g^2 + i2\sqrt{\frac{10\pi}{3}} R^{-1} J_2(1,\vec{\alpha}) g\left[f + \frac{1}{3} qRg\right]\Big\}$$

$$+ \frac{4}{5} q^2R^{-4} C_A^2 I_2(2,\vec{\sigma}) \left[f^2 - \frac{2}{3} qRfg + \frac{1}{9} q^2R^2g^2\right] + \frac{8}{5} \frac{1}{\sqrt{2\pi}} q^2R^{-2} C_V C_A$$

$$\Big\{R^{-2} J_2(2,\vec{\sigma}) \left[f^2 - \frac{1}{9} q^2R^2g^2\right] + i\sqrt{\frac{10\pi}{3}} R^{-1} I_2(2,1;\vec{\sigma},\vec{\alpha}) g\left[f - \frac{1}{3} qRg\right]\Big\}$$

Second-Forbidden Unique (no parity change) J = 3

K, L_I, M_I: $$S_2^{(3)} = \frac{4}{675} q^6 |C_A|^2 I_3(2,\vec{\sigma}) g^2$$

L_{III}, M_{III}: $$S_2^{(3)} = \frac{8}{45} q^4 R^{-2} |C_A|^2 I_3(2,\vec{\sigma}) g^2$$

M_V: $$S_2^{(3)} = \frac{4}{3} q^2 R^{-4} |C_A|^2 I_3(2,\vec{\sigma}) g^2$$

Third Forbidden (yes parity change) J = 3

K, L_I, M_I: $$S_3^{(3)} = \frac{1}{525\pi} q^6 |C_V|^2 \Big\{ R^{-2} K_3\Big[f^2 + \frac{2}{7} qRfg + \frac{1}{21} q^2R^2g^2\Big] + \frac{28\pi}{9} I_3(2,\vec{\alpha}) g^2 + i2\sqrt{\frac{28\pi}{9}} R^{-1} J_3(2,\vec{\alpha}) g\Big[f + \frac{1}{7} qRg\Big]\Big\} + \frac{16}{4725} q^6 R^{-2} |C_A|^2 I_3(3,\vec{\sigma})\Big[f^2 - \frac{2}{7} qRfg + \frac{1}{28} q^2R^2g^2\Big] + \frac{8}{1575\sqrt{\pi}} q^6 C_V^* C_A\Big\{R^{-2} J_3(3,\vec{\sigma}) f^2 + i\sqrt{\frac{28\pi}{9}} R^{-1} I_3(3,2;\vec{\sigma},\vec{\alpha}) g \Big[f - \frac{1}{7} qRg\Big]\Big\}$$

L_{III}, M_{III}: $$S_3^{(3)} = \frac{2}{35\pi} q^4 R^{-2} |C_V|^2 \Big\{R^{-2} K_3\Big[f^2 + \frac{2}{5} qRfg + \frac{1}{15} q^2R^2g^2\Big] + \frac{28\pi}{9} I_3(2,\vec{\alpha}) g^2 + i2\sqrt{\frac{28\pi}{9}} R^{-1} J_3(2,\vec{\alpha}) g \Big[f + \frac{1}{5} qRg\Big]\Big\} + \frac{32}{315} q^4 R^{-4} |C_A|^2 I_3(3,\vec{\sigma})\Big[f^2 - \frac{2}{5} qRfg + \frac{1}{24} q^2R^2g^2\Big] + \frac{16}{105\sqrt{\pi}} q^4 R^{-2} C_V^* C_A\Big\{R^{-2} J_3(3,\vec{\sigma})\Big[f^2 - \frac{1}{30} q^2R^2g^2\Big] + i\sqrt{\frac{28\pi}{9}} R^{-1} I_3(3,2;\vec{\sigma},\vec{\alpha}) g\Big[f - \frac{1}{5} qRg\Big]\Big\}$$

M_V: $$S_3^{(3)} = \frac{3}{7\pi} q^2R^{-4}|C_V|^2 \left\{ R^{-2}K_3 \left[f^2 + \frac{2}{3} qRfg + \frac{1}{5} q^2R^2g^2\right] + \frac{28\pi}{9} I_3(2,\vec{\alpha})g^2 + i2\sqrt{\frac{28\pi}{9}} R^{-1}J_3(2,\vec{\alpha})g\left[f + \frac{1}{3} qRg\right]\right\} + \frac{16}{21} q^2R^{-6}|C_A|^2 I_3(3,\vec{\sigma})\left[f^2 - \frac{2}{3} qRfg + \frac{7}{60} q^2R^2g^2\right]$$

$$+ \frac{8}{7\sqrt{\pi}} q^2R^{-4} C_V{}^*C_A\left\{J_3(3,\vec{\sigma})\left[f^2 - \frac{2}{15}q^2R^2g^2\right] + i\sqrt{\frac{28\pi}{9}} R^{-1}I_3(3,2;\vec{\sigma}\vec{\alpha})\ g\left[f - \frac{1}{3} qRg\right]\right\}$$

Third Forbidden Unique (yes parity change) J = 4

K, L_I, M_I: $$S_3^{(4)} = \frac{4}{33075}\ q^8\ C_A{}^2\ I_4(3,\vec{\sigma})\ g^2$$

L_{III}, M_{III}: $$S_3^{(4)} = \frac{4}{525}\ q^6R^{-2}\ C_A{}^2\ I_4(3,\vec{\sigma})\ g^2$$

M_V: $$S_3^{(4)} = \frac{4}{21}\ q^4R^{-4}\ C_A{}^2\ I_4(3,\vec{\sigma})\ g^2$$

Total Transition Probability

Even though the transition probability is given by the sum of all of the above terms, at most three terms are significant. For transitions in which the initial and final nuclear states have the same parity, all of the nuclear matrix element combinations with odd parity vanish. The transition probability in this case is given by:

$$S_0^{(0)} + S_0^{(1)} + S_2^{(2)} + S_2^{(3)} + \ldots \quad .$$

For transitions in which the initial and final nuclear states have opposite parity, the even parity nuclear matrix element combinations vanish. The transition probability is then given by:

$$S_1^{(0)} + S_1^{(1)} + S_1^{(2)} + S_3^{(3)} + S_3^{(4)} + \ldots \quad .$$

A given transition is characterized by a definite change in the angular momentum of the nucleus, δI. The nuclear matrix elements with $J < |\delta I|$ vanish. Hence for a given transition, the sums stated above in effect begin with the $J = |\delta I|$ term. There are two possible situations which arise. Either the term with $J = |\delta I| + 1$ is two orders of forbiddenness higher than the term with $J = |\delta I|$ or it is of the same order of forbiddenness.* In the first case, the higher order term may be neglected; only one term in the sum contributes significantly to the transition. In fact, inspection of the equations for $S_n^{(j)}$ reveals that in this case only one nuclear matrix element combination, namely $I_J(J-1,\vec{\sigma})$, contributes significantly to the transition. In the other case it is not immediately clear whether the $J = |\delta I| + 1$ term is of importance compared with the $J = |\delta I|$ term. Brysk and Rose (13) indicate that this term can be neglected, but experimental work by Harmer and Perlman (17) indicates that the $S_1^{(2)}$ term does contribute, at least in some cases, to $|\delta I| = 1$ beta decay transitions. In either event, terms with $J > |\delta I| + 1$ are of higher order of forbiddenness and therefore are negligible.

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* The case $|\delta I| = 0$, yes parity change, is an exception to this. In this case, the three terms:

$$S_1^{(0)} + S_1^{(1)} + S_1^{(2)}$$

may all contribute.

CHAPTER VII

M/L RATIOS AND THEIR APPLICATIONS

General Discussion of the M/L Ratios

A general discussion of the M/L ratios for the forbidden non-unique transitions is difficult due to the multiplicity of the (unknown) nuclear matrix elements. This is not the case for the unique transitions since in these transitions the nuclear matrix elements cancel out in the ratio of M/L.

Figures 1 and 2 display the M/L ratios for the first forbidden unique and second forbidden unique transitions respectively for the four values of Z, Z = 60, 70, 80, and 90. These ratios are obtained by taking the sum of the ratios of the various M sub-shell probabilities to the total L shell probability.

The L-shell electron wave functions used in these calculations are obtained from the graphs in Brysk and Rose (13). The method by which they were obtained is described in some detail in reference (13). The M-shell electron wave functions used in these calculations are those obtained by Brewer, Harmer and Hay (3). The method by which these were calculated is described in reference (3). The significant difference in the two methods is that the L-shell wave functions were obtained by taking Coulomb wave functions and applying corrections for finite nuclear size and for screening, whereas the M-shell wave functions were computed directly from a Thomas-Fermi-Dirac potential which included these effects.

From Figure 1 it is observed that for first forbidden unique transitions, the total M/L ratio approaches a constant value for large values of the transition energy E_{EC}. This value is on the order of 22 per cent for all values of atomic number. As the transition energy decreases, the M/L ratios increase and of course tend toward infinity as the transition energy approaches the L-shell binding energy (the threshold for L capture). From Figure 2 it is observed that for second forbidden unique transitions, the M/L ratios have quite a different energy dependence. For large values of the transition energy the ratios approach a constant value of about the same magnitude as in the first forbidden unique case. However, in the second forbidden unique case, the ratios do not increase as rapidly at first (going from higher to lower E_{EC}) but finally become much larger at low transition energies than in the first forbidden unique case.

Figure 3 shows the individual contributions to the M/L ratios from the various M sub-shells for Z = 90. The shapes of the curves are the same for other values of Z but the abscissa is shifted slightly. Since the binding energies of the M_I and M_{II} sub-shells are very nearly the same, and since the form of the transition probabilities from these two sub-shells is the same (as pointed out previously), their contributions to the M/L ratio have the same form (as a function of E_{EC}) and are plotted together as M_I+M_{II}/L.* The M_{II} contribution is only about one

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*The symbol M_I+M_{II}/L would be more properly written as $(M_I+M_{II})/L$, but the accepted convention is to omit the parentheses.

per cent as large as the M_I contribution, thus the M_I/L ratio can be obtained by multiplying the M_I+M_{II}/L ratio by 0.99, while the M_{II}/L ratio can be obtained by multiplying the M_I+M_{II}/L ratio by 0.01. Similarly, the M_{III} and M_{IV} contributions are plotted together as $M_{III}+M_{IV}/L$. Once again, the M_{IV} contribution is about one per cent of the M_{III} contribution.

From these figures it is apparent how the two cases, first forbidden unique and second forbidden unique, differ from one another. Consider first the M_I+M_{II}/L ratios for the two cases. At high transition energies this ratio is about the same for the two cases. However, as the transition energy decreases, the M_I+M_{II}/L ratio begins to fall off sooner in the second forbidden case. The ratio $M_{III}+M_{IV}/L$ becomes significant sooner (at higher transition energies) in the first forbidden unique case than in the second forbidden unique case. In fact, the $M_{III}+M_{IV}/L$ ratio is larger for first forbidden unique transitions than it is for second forbidden unique transitions at all values of the transition energy until very near the L-capture threshold. Thus as far as the first four M sub-shells are concerned, the M/L ratios for first forbidden unique transitions would, at any given value of the transition energy (except very near L capture threshold) exceed the M/L ratios for second forbidden unique transitions.

The contribution from the M_V sub-shell, which is very small for large values of the transition energy in both cases, becomes the predominant term at low energies in the second forbidden unique case while remaining small for the first forbidden unique case. This contribution is so important that at small values of the transition energy, the M/L ratio for second forbidden unique transitions far exceeds that for first forbidden unique transitions.

Equations for the M/L Ratios

The equations from which the M/L ratios in this chapter were computed are given below for reference.

$$M_I + M_{II}/L = \frac{q_{M_I}^{p+2}\ g_{M_I}^2 + q_{M_{II}}^{p+2}\ f_{M_{II}}^2}{q_{L_I}^p\ g_{L_I}^2 + q_{L_{II}}^p\ f_{L_{II}}^2 + A\ q_{L_{III}}^{p-2}\ g_{L_{III}}^2} \quad (61)$$

$$M_{III} + M_{IV}/L = \frac{A(\ q_{M_{III}}^p\ g_{M_{III}}^2 + q_{M_{IV}}^p\ f_{M_{IV}}^2\ )}{q_{L_I}^p\ g_{L_I}^2 + q_{L_{II}}^p\ f_{L_{II}}^2 + A\ q_{L_{III}}^{p-2}\ g_{L_{III}}^2} \quad (62)$$

$$M_V/L = \frac{B(\ q_{M_V}^2\ g_{M_V}^2\ )}{q_{L_I}^p\ g_{L_I}^2 + q_{L_{II}}^p\ f_{L_{II}}^2 + A\ q_{L_{III}}^{p-2}\ g_{L_{III}}^2}\ . \quad (63)$$

In these equations, one sets:

For First Forbidden Unique:

$A = 9\ R^{-2}$, $B = 0$,* $p = 2$

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*For first forbidden unique, $M_V/L = \frac{9}{5} \left[\frac{g_V}{g_I}\right]^2 M_I/L < 5 \times 10^{-12}\ M_I/L$ for all Z.

For Second Forbidden Unique:

$A = 30\ R^{-2},\ B = 225\ R^{-4},\ p = 4$

These equations can also be made to apply to second forbidden non-unique equations in the event $|qRg| \ll |f|$ and $S_2^{(3)}$ does not contribute. Under these approximations, one needs only make the replacements:

$$g \rightarrow C_V\ (K_2)^{\frac{1}{2}}\ \theta_2\ \mp\ (2\pi)^{\frac{1}{2}}\ C_A\ (I_2)^{\frac{1}{2}}\ f$$

$$f \rightarrow C_V\ (K_2)^{\frac{1}{2}}\ \theta_2'\ \pm\ (2\pi)^{\frac{1}{2}}\ C_A\ (I_2)^{\frac{1}{2}}\ g$$

where:

$$\theta_2' = g - (10\pi/3)^{\frac{1}{2}}\ \Gamma_2\ f, \quad A = 9\ R^{-2},\ B = 0,\ p = 2,$$

and θ_2, Γ_2, and I_2 are defined in equations (70), (69), and (68) respectively.

The numerical results of calculating these ratios for first and second forbidden unique transitions for Z = 60, 70, 80, and 90 are given in Appendix E.

Calculation of Nuclear Mass Differences

It is apparent from the graphs (Figures 1 through 3) that the M/L ratios for the unique transitions depend on the transition energies. This dependence is quite pronounced whenever the M/L ratio exceeds about 0.3. Hence a good experimental determination of the M/L ratio would, in these cases, give an accurate determination of the mass difference between the parent and daughter nuclei.

Consider two examples. First suppose some nucleus is found to undergo a first forbidden unique transition with an M/L ratio of

0.40 ± 0.08 (an accuracy of 20 per cent). This value is near the flat portion of the curve, nevertheless from these data, the transition energy can be determined to be (choosing $Z = 90$ as an example) $370\,^{+200}_{-80}$ KeV. The mass difference between the parent and daughter nuclei is thus $0.00038\,^{+0.00021}_{-0.00008}$ a.m.u.

As a second example consider a more favorable portion of the graphs. Suppose some nucleus is found to undergo a second forbidden unique transition with $M/L = 10 \pm 2$. Once again taking $Z = 90$ one obtains $E_{EC} = 32.5\,^{+1.5}_{-1.0}$ KeV. This gives a mass difference between parent and daughter nuclei of $0.0000348\,^{+0.0000016}_{-0.0000011}$ a.m.u.

Determination of the Order of Forbiddenness

From the graphs of the M/L ratios, it is apparent that one can obtain some information concerning the order of forbiddenness of the unique transitions. For small transition energies the second forbidden unique transitions exhibit a much larger M/L ratio than do the first forbidden unique transitions (at the same transition energy). This is due entirely to the M_V sub-shell whose contribution is enhanced at small energies in the second forbidden unique case because the L_{III} contribution (in the denominator of the M/L ratio) is suppressed, while at small energies in the first forbidden unique case, the M_V contribution is itself suppressed. In both cases, the reason that one term is suppressed is the fact that the conservation of angular momentum and the requirements on the parity of the operators involved forces the neutrino associated with the transition to carry off at least one unit of orbital angular momentum. This causes its

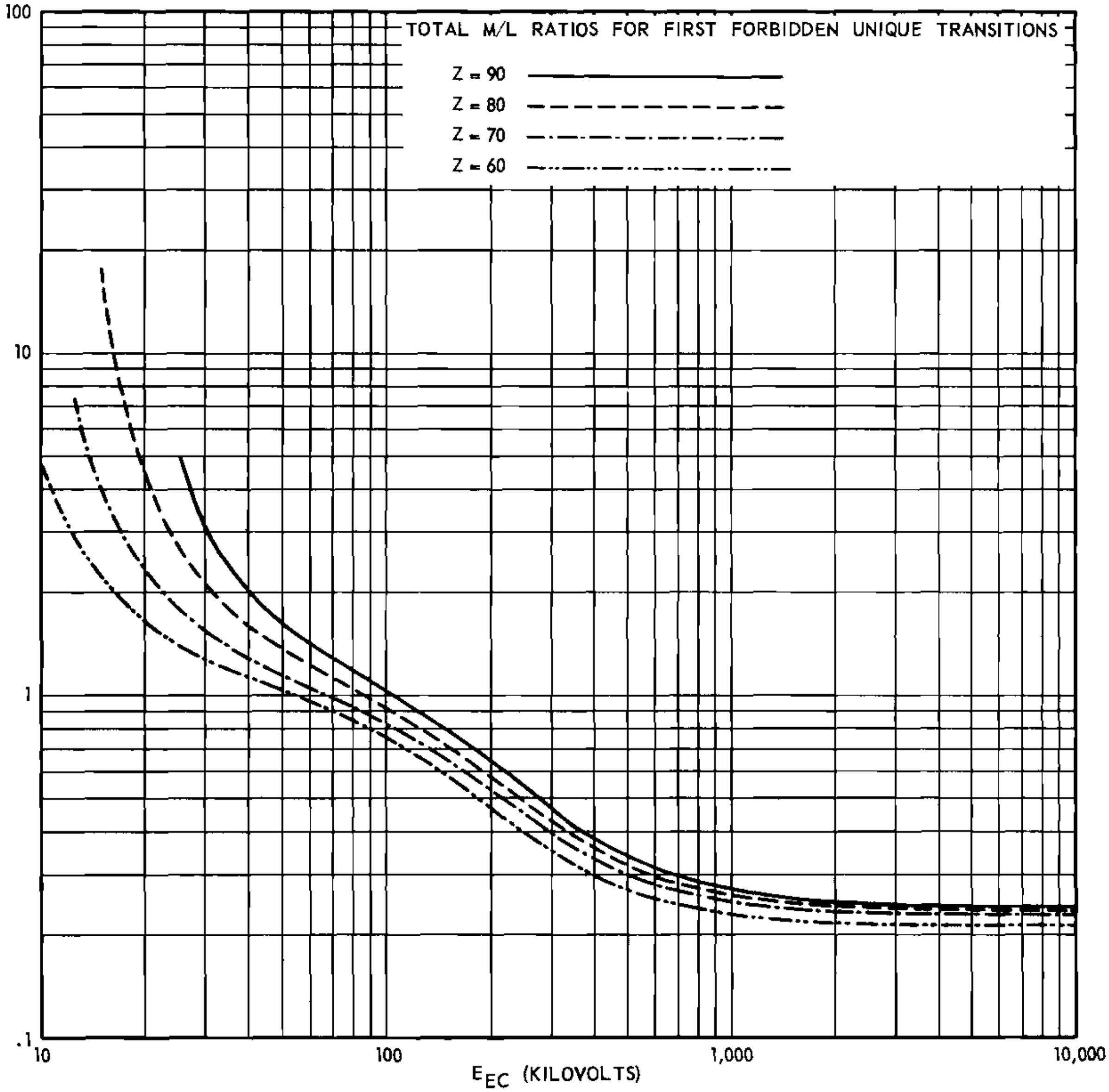

Figure 1. Ratio of the Probability for M-Shell Electron Capture to that for L-Shell Electron Capture for First Forbidden Unique Transitions.

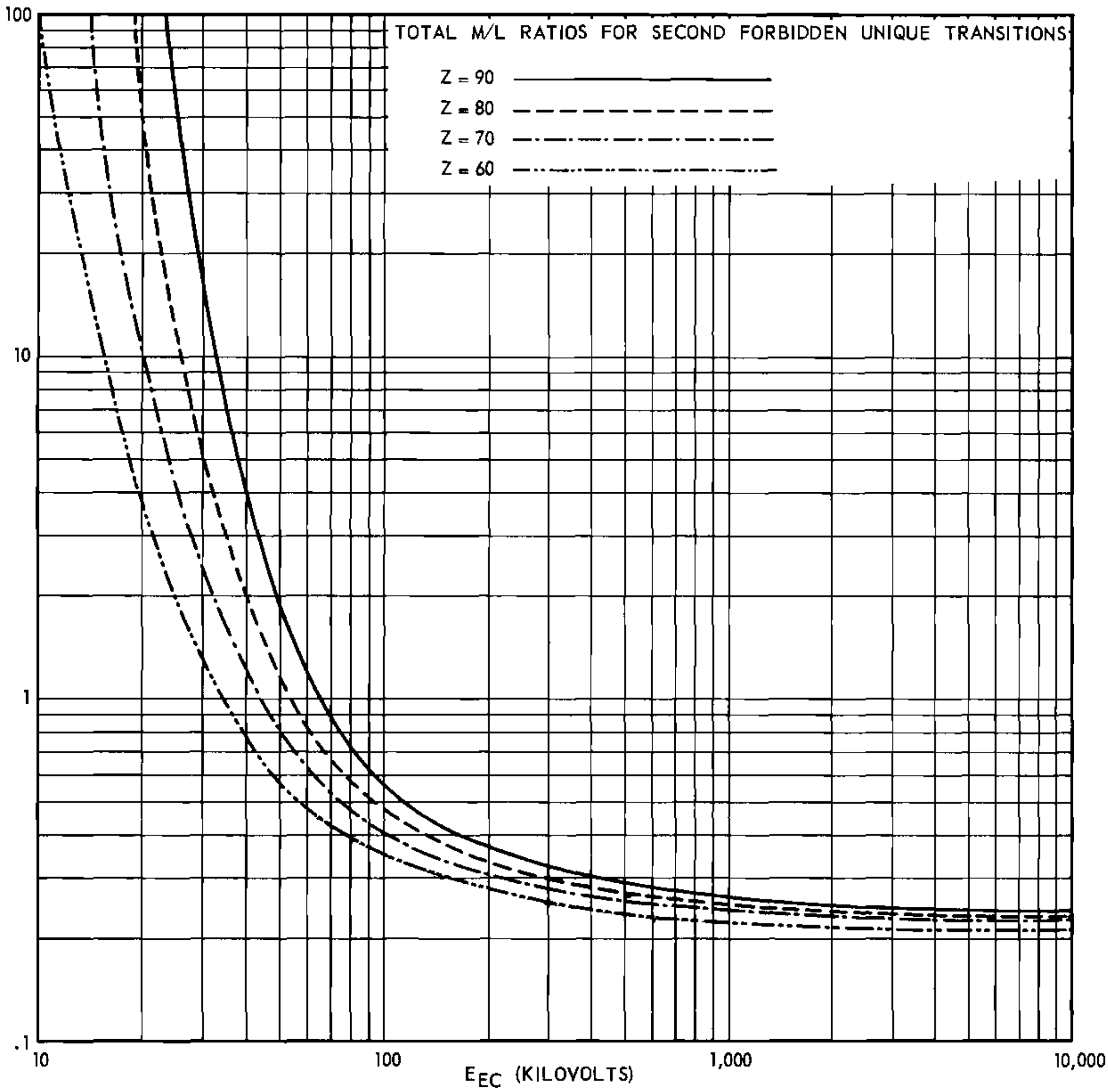

Figure 2. Ratio of the Probability for M-Shell Electron Capture to that for L-Shell Electron Capture for Second Forbidden Unique Transitions.

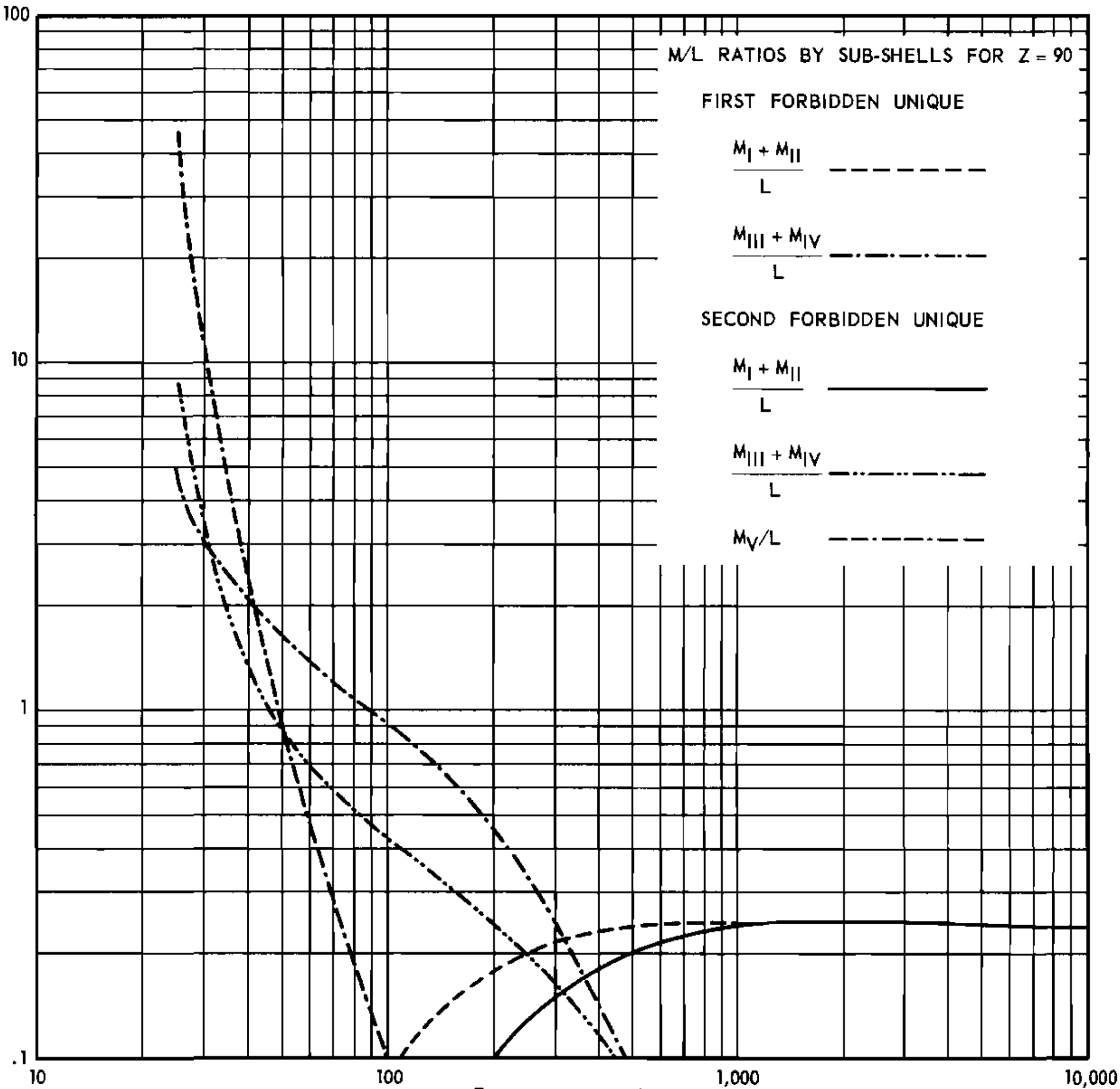

Figure 3. Ratio of the Probability for Electron Capture from Individual M-Subshells to the Total Probability for L-Shell Electron Capture for First and Second Forbidden Unique Transitions (Z = 90).

wave function to vary for small R as $(1/3)q^2R$. If the neutrino had no orbital angular momentum this factor would be q. Thus this extra power of qR causes the wave function to become small at small values of q.

Thus an experiment which would determine the M/L ratio would be of use in determining the order of forbiddenness of a unique transition and would therefore help in assigning the values of spin and parity to the initial and final states of the nucleus. Below 400 KeV the M/L ratios for first and second forbidden unique transitions differ by more than 25 per cent except in a relatively small region where the M/L ratio for second forbidden unique transitions crosses the M/L ratio for first forbidden unique transitions. In fact, around 100 KeV the ratios differ by more than a factor of two.

An experiment which would be especially useful in determining the order of forbiddenness for transitions at low transition energies (below about 100 KeV) would be one which could determine the M_V/L ratio. As noted earlier, for first forbidden unique transitions the M_V/L ratio goes as:

$$M_V/L = (9/5)\ (g_{M_V}/g_{M_I})^2 < 5 \times 10^{-12}\ M_I/L \quad .$$

Below 100 KeV the M_V/L ratio is thus always less than 4×10^{-13} (for $Z > 60$) for first forbidden unique transitions, while for second forbidden unique transitions it is, under these conditions, always greater than 0.029.

Calculation of Nuclear Matrix Elements and the Theory of Ahrens and Feenberg

In this section the theory developed thus far for the transition probabilities in terms of the nuclear matrix elements will be coupled with the theory of Ahrens and Feenberg (19) and (20) in order to calculate the relative magnitudes of the nuclear matrix elements which enter the forbidden non-unique transitions. A calculation will be made, as an example, using L/K ratios since no M/L ratios are known with enough precision at the present time. As will be pointed out later, M/L ratios would be much better for such a calculation.

Following the suggestion of Ahrens and Feenberg (19), one can write:

$$\int T_{J,J-1}(\vec{R},\vec{\alpha}) = -i\, \Gamma_J\, R^{-1} \int \mathcal{Y}_J(\vec{R}) \ .$$

This can be compared with the notation of Ahrens as follows. From (2) one obtains the following relationship:*

$$J_1(0,\vec{\alpha}) = -\frac{3}{(4\pi)^{3/2}}\, R \int \frac{\vec{r}}{R} \cdot \int \vec{\alpha}^* \qquad .$$

From the definition of $J_J(0,\vec{\alpha})$ this becomes:

$$\int \mathcal{Y}_1(\vec{R}) \int T_{10}(\vec{R},\vec{\alpha})^* = \frac{3}{(4\pi)^{3/2}}\, R \int \frac{\vec{r}}{R} \cdot \frac{-i}{2}\, \Lambda\alpha Z \int \frac{\vec{r}}{R}^*$$

$$= -i\, \frac{3}{2}\, \frac{1}{(4\pi)^{3/2}}\, R \left| \int \frac{\vec{r}}{R} \right|^2 \Lambda\alpha Z$$

---

*Actually (2) has a different sign for $J_1(0,\alpha) = -J_1(0,p/M)$. This sign appears to be in error.

where, following Ahrens,

$$\tfrac{1}{2}\Lambda \alpha Z \int \frac{\vec{r}}{R} = -i \int \vec{\alpha} \quad .$$

From (2) one also obtains:

$$K_1 = \frac{3}{4\pi} R^2 \left| \int \frac{\vec{r}}{R} \right|^2 .$$

Using these relationships and the definition of K_1, one obtains:

$$T_{10}(\vec{R},\vec{\alpha}) = -i \frac{\Lambda \alpha Z}{4(\pi)^{\frac{1}{2}}} R^{-1} \int \mathcal{Y}_1(\vec{R}) .$$

From this relationship, one obtains for the nuclear matrix element combinations which appear in the theory:

$$I_1(0,\vec{\alpha}) = \frac{\Lambda^2 \alpha^2 Z^2}{16\pi} R^{-2} K_1 = \Gamma_1^2 R^{-2} K_1$$

$$J_1(0,\vec{\alpha}) = +i \frac{\Lambda \alpha Z}{4(\pi)^{\frac{1}{2}}} R^{-1} K_1 = -i \Gamma_1 R^{-1} K_1$$

$$I_1(1,0;\vec{\sigma},\vec{\alpha}) = +i \frac{\Lambda \alpha Z}{4(\pi)^{\frac{1}{2}}} R^{-1} J_1^*(1,\vec{\sigma}) = -i \Gamma_1 R^{-1} J_1(1,\vec{\sigma}) .$$

Thus in general:

$$\Gamma_1 = - \frac{\Lambda \alpha Z}{4(\pi)^{\frac{1}{2}}} \quad . \tag{64}$$

The following general notation is adopted:

$$I_J(J-1,\vec{\alpha}) = \Gamma_J^2 R^{-2} K_J \tag{65}$$

$$J_J(J-1,\vec{\alpha}) = -i \Gamma_J R^{-1} K_J \tag{66}$$

$$I_J(J,J-1;\vec{\sigma},\vec{\alpha}) = -i \Gamma_J R^{-1} J_J(J,\vec{\sigma}) \quad . \tag{67}$$

Thus it is possible to write all of the nuclear matrix elements in terms of the three:*

$$K_J,\ I_J(J,\vec{\sigma}),\ J_J(J,\vec{\sigma}).$$

In the following these will be designated simply as:

$$K_J, \qquad I_J, \qquad J_J. \tag{68}$$

In terms of these nuclear matrix elements, the transition probabilities become:

$$S_1^{(1)} = \frac{1}{3\pi} q^2R^{-2}|C_V|^2K_1\left[f^2 + \frac{2}{3} qRfg + \frac{1}{3} q^2R^2g^2 + 4\pi\Gamma_1^2g^2 + 2\sqrt{4\pi}\,\Gamma_1 g(f + \frac{1}{3} qRg)\right]$$

$$+ \frac{8}{9} q^2R^{-2}|C_A|^2I_1\left[f^2 - \frac{2}{3} qRfg + \frac{1}{6} q^2R^2g^2\right]$$

$$+ \frac{2}{3}\sqrt{\frac{8}{3\pi}}\ q^2R^{-2}\ C_V^*C_A\ J_1\left[f^2 + \sqrt{4\pi}\,\Gamma_1\ g(f - \frac{1}{3} qRg)\right]$$

$$= \frac{1}{3\pi} q^2R^{-2}\left\{|C_V|^2K_1\ \left[(f + \sqrt{4\pi}\,\Gamma_1 g)^2 + \frac{2}{3} qRg(f + \sqrt{4\pi}\,\Gamma_1 g) + \frac{1}{3} q^2R^2g^2\right]\right.$$

$$+ 2\sqrt{\frac{8\pi}{3}}\,C_V^*C_AJ_1\left[f(f + \sqrt{4\pi}\,\Gamma_1\ g) - \frac{1}{3}\sqrt{4\pi}\,\Gamma_1\ qRg^2\right]$$

$$\left. + \frac{8\pi}{3}\ |C_A|^2\ I_1\left[\ f^2 - \frac{2}{3} qRfg + \frac{1}{6} q^2R^2g^2\right]\right\}$$

Setting $\theta_1 = f + \sqrt{4\pi}\ \Gamma_1\ g$ this becomes:

$$S_1^{(1)} = \frac{1}{3\pi} q^2R^{-2}\left\{|C_V|^2K_1\left[\theta_1^2 + \frac{2}{3} qRg\theta_1 + \frac{1}{3} q^2R^2g^2\right] + 2\sqrt{\frac{8\pi}{3}}\,C_V^*\ C_A\ J_1\right.$$

$$\left.\left[(f - \frac{1}{3} qRg)\theta_1 + \frac{1}{3} qRgf\right] + \frac{8\pi}{3}|C_A|^2I_1\left[f^2 - \frac{2}{3} qRfg + \frac{1}{6} q^2R^2g^2\right]\right\}$$

---

*The nuclear matrix element $I_J(J-1,\vec{\sigma})$ will of course still occur in the allowed and unique transitions, but this should cause no confusion.

Similarly, setting $\theta_2 = f + \sqrt{\frac{10\pi}{3}}\,\Gamma_2 g$ we have for K, L_I, M_I:

$$S_2^{(2)} = \frac{2}{45\pi} q^4 R^{-2}\Big\{|C_V|^2 K_2\Big[\theta_2^2 + \frac{2}{5} qRg\theta_2 + \frac{1}{10} q^2R^2g^2\Big] + 2\sqrt{2\pi}\, C_V^* C_A J_2 \Big[\theta_2(f - \frac{1}{5} qRg) + \frac{1}{5} qRfg\Big] + 2\pi|C_A|^2 I_2\Big[f^2 - \frac{2}{5} qRfg + \frac{1}{15} q^2R^2g^2\Big]\Big\}$$

For L_{III} and M_{III}:

$$S_2^{(2)} = \frac{2}{5\pi} q^2 R^{-4}\Big\{|C_V|^2 K_2\Big[\theta_2^2 + \frac{2}{3} qRg\theta_2 + \frac{2}{9} q^2R^2g^2\Big] + 2\sqrt{2\pi}\, C_V^* C_A J_2 \Big[(f - \frac{1}{3} qRg)\theta_2 + \frac{1}{3} qRfg\Big] + 2\pi|C_A|^2 I_2\Big[f^2 - \frac{2}{3} qRfg + \frac{1}{9} q^2R^2g^2\Big]\Big\}$$

The form of these last three equations indicates that the S's are almost perfect squares when one recalls that:

$$J_J^2 = K_J I_J \quad , \quad J_J \text{ real}$$

Making use of this fact, one can write these equations in a form which will facilitate the computation of the nuclear matrix elements from the capture ratios.

For K, L_I, M_I:

$$S_1^{(1)} = \frac{1}{3\pi} q^2 R^{-2}\Big\{|C_V|^2 K_1\Big[\theta_1 + \frac{1}{3} qRg\Big]^2 + 2\sqrt{\frac{8\pi}{3}}\, C_V^* C_A J_1 \Big[(f - \frac{1}{3} qRg)\theta_1 + \frac{1}{3} qRgf\Big] + \frac{8\pi}{3}|C_A|^2 I_1\Big[f - \frac{1}{3} qRg\Big]^2 + |C_V|^2 K_1 \frac{2}{9} q^2R^2g^2 + \frac{8\pi}{3}|C_A|^2 I_1 \frac{1}{18} q^2R^2g^2\Big\}$$

$$= \frac{1}{3\pi} q^2 R^{-2}\Big\{\Big[C_V\sqrt{K_1}\,(\theta_1 + \frac{1}{3} qRg) \pm \sqrt{\frac{8\pi}{3}}\, C_A\sqrt{I_1}\,(f - \frac{1}{3} qRg)\Big]^2 + \frac{2}{9} q^2R^2g^2\Big[C_V^2 K_1 \pm 2\sqrt{\frac{8\pi}{3}}\, C_V^* C_A\sqrt{I_1K_1}\;\frac{1}{2} + \frac{1}{4}\cdot\frac{8\pi}{3} C_A^2 I_1\Big]\Big\}$$

$$= \frac{1}{3\pi} q^2R^{-2}\left\{\left[C_V\sqrt{K_1}\,(\theta_1 + \tfrac{1}{3} qRg) \pm\sqrt{\tfrac{8\pi}{3}}\; C_A\sqrt{I_1}\,(f - \tfrac{1}{3} qRg)\right]^2 + \frac{2}{9} q^2R^2g^2 \left[C_V\sqrt{K_1} \pm \tfrac{1}{2}\sqrt{\tfrac{8\pi}{3}}\, C_A\sqrt{I_1}\right]^2\right\}$$

For K, L_I, M_I:

$$S_2^{(2)} = \frac{2}{45\pi} q^4R^{-2}\left\{C_V^2 K_2 \left[\theta_2 + \tfrac{1}{5} qRg\right]^2 + 2\sqrt{2\pi}\, C_VC_AJ_2 \left[(f - \tfrac{1}{5} qRg)\theta_2 + \tfrac{1}{5} qRfg\right] + 2\pi\, C_A^2\, I_2\left[f - \tfrac{1}{5} qRg\right]^2 + \frac{3}{50} q^2R^2g^2\left[C_V^2 K_2 + \tfrac{4}{9}(2\pi)\, C_A^2\, I_2\right]\right\}$$

$$= \frac{2}{45\pi} q^4R^{-2}\left\{\left[C_V\sqrt{K_2}\,(\theta_2 + \tfrac{1}{5} qRg) \pm \sqrt{2\pi}\, C_A\sqrt{I_2}\,(f - \tfrac{1}{5} qRg)\right]^2 + \frac{3}{50} q^2R^2g^2 \left[C_V\sqrt{K_2} \pm \tfrac{2}{3}\sqrt{2\pi}\, C_A\sqrt{I_2}\right]^2\right\}$$

For L_{III}, M_{III}:

$$S_2^{(2)} = \frac{2}{5\pi} q^2R^{-4}\left\{C_V^2K_2 \left[\theta_2 + \tfrac{1}{3} qRg\right]^2 + 2\sqrt{2\pi}\, C_VC_AJ_2 \left[(f - \tfrac{1}{3} qRg)\theta_2 + \tfrac{1}{3} qRfg\right] + 2\pi\, C_A^2\, I_2 \left[f - \tfrac{1}{3} qRg\right]^2 + \frac{1}{9} q^2R^2g^2\, C_V^2K_2\right\}$$

$$= \frac{2}{5\pi} q^2R^{-4}\left\{\left[C_V\sqrt{K_2}\,(\theta_2 + \tfrac{1}{3} qRg) \pm \sqrt{2\pi}\, C_A\sqrt{I_2}\,(f - \tfrac{1}{3} qRg)\right]^2 + \frac{1}{9} q^2R^2g^2\, C_V\sqrt{K_2}\left[C_V\sqrt{K_2} \pm 2\sqrt{2\pi}\, C_A\sqrt{I_2}\right]\right\}$$

In the above equations use has been made of the fact that C_V and C_A are real. The upper sign is chosen if J_J is positive, the lower sign if it is negative.

In many cases $|qRg| \ll |f|$. This is true if $q \lessapprox 200$ KeV. (For q=250 KeV. $|qRg|$ amounts to about 10 per cent of $|f|$ or less). When this approximation can be made, the above equations simplify to:

For K, L_I, M_I:

$$S_1^{(1)} = \frac{1}{3\pi} q^2 R^{-2} \left[C_V(K_1)^{\frac{1}{2}} \theta_1 \pm \left(\frac{8\pi}{3}\right)^{\frac{1}{2}} C_A (I_1)^{\frac{1}{2}} f\right]^2$$

For K, L_I, M_I:

$$S_2^{(2)} = \frac{2}{45\pi} q^4 R^{-2} \left[C_V(K_2)^{\frac{1}{2}} \theta_2 \pm (2\pi)^{\frac{1}{2}} C_A(I_2)^{\frac{1}{2}} f\right]^2$$

For L_{III}, M_{III}:

$$S_2^{(2)} = \frac{2}{5\pi} q^2 R^{-4} \left[C_V(K_2)^{\frac{1}{2}} \theta_2 \pm (2\pi)^{\frac{1}{2}} C_A (I_2)^{\frac{1}{2}} f\right]^2$$

As an example, the nuclear matrix elements of ${}_{76}Os^{185}$ are calculated below.* This isotope has been carefully studied by Johns et al (21). They find the L/K ratio to be 1.04 ± 0.04, and the energy for the transition to be 112 ± 7 KeV. The L and K shell wave functions are obtained from the graphs in (22). Λ is evaluated from the result given in (19).

$f(L_I) = -.16$ $\quad$ $g(L_I) = .69$ $\quad$ $E_{EC} = .219 \pm .014$

$f(K) = -.45$ $\quad$ $g(k) = 1.75$ $\quad$ $q_K = 38 \pm 7$ KeV

$q_{L_I} = 99 \pm 7$ KeV

$$\theta_1 = f + (4\pi)^{\frac{1}{2}}\Gamma_1 g = f - \frac{\Lambda\, \alpha Z}{2}\, g = f - .276\, \Lambda\, g$$

$$\Lambda = 1 + (\Delta M + 1.5)\, \frac{A^{1/3}}{Z} = 1 + (1.72 \pm .01)\, \frac{5.7}{76} = 1.13$$

$$\theta_1 = f - .309\, g$$

$\theta_1(K) = -1.0$ $\quad$ $\theta_1(L_I) = -.37$

---

*In these calculations, it has been assumed that $S_1^{(2)}$ is negligible compared with $S_1^{(1)}$.

$$\frac{L}{K} = \left(\frac{q_L}{q_K}\right)^2 \left(\frac{C_V \sqrt{K_1}\ \theta_1(L) \pm \sqrt{\frac{8\pi}{3}}\, C_A \sqrt{I_1}\ f(L)}{C_V \sqrt{K_1}\ \theta_1(K) \pm \sqrt{\frac{8\pi}{3}}\, C_A \sqrt{I_1}\ f(K)}\right)^2$$

$$(1.04 \pm .04)^{\frac{1}{2}} \left(\frac{38 \pm 7}{99 \pm 7}\right) = \frac{C_V \sqrt{K_1}\ \theta_1(L) \pm \sqrt{\frac{8\pi}{3}}\ C_A \sqrt{I_1}\ f(L)}{C_V \sqrt{K_1}\ \theta_1(K) \pm \sqrt{\frac{8\pi}{3}}\ C_A \sqrt{I_1}\ f(K)}$$

Set

$$\rho = \sqrt{\frac{L}{K}}\ \frac{q_K}{q_L} = (1.02 \pm .02) \left(\frac{38 \pm 7}{99 \pm 7}\right) = .39 \pm .05$$

$$C_V \sqrt{K_1} \left[\rho\theta_1(K) - \theta_1(L)\right] \pm \sqrt{\frac{8\pi}{3}}\ C_A \sqrt{I_1} \left[\rho f(K) - f(L)\right] = 0$$

$$K_1 = \left(\frac{C_A}{C_V}\right)^2 \frac{8\pi}{3} I_1 \left(\frac{\rho\ f(K) - f(L)}{\rho\ \theta_1(K) - \theta_1(L)}\right)^2$$

$$= 11.7\ I_1 \left(\frac{(.39 \pm .05)(-.45) - .16}{(.39 \pm .05)(-1) - .37}\right)^2 = 11.7\ I_1 \left(\frac{.34 \pm .03}{.76 \pm .03}\right)^2$$

$$= (2.3 \pm .2)\ I_1$$

The error quoted above does not take into account errors of any kind in the lepton wave functions. Assuming an error in the electron wave function of the order of 10 per cent, the result would become

$$K_1 = (2.3 \pm .6)\ I_1$$

In order to investigate the case of Second-Forbidden transitions, one must use the extension of Ahrens' theory given in (20). Here Ahrens gives the relationship:

$$\int A_{ij} = i\ \frac{\alpha Z \Lambda}{2}\ R^{-1} \int R_{ij}$$

From the table in (2), once again with the change in sign for $J_2(1,\vec{a})$, one obtains:

$$J_2(1,\vec{a}) = - \frac{(135)^{\frac{1}{2}}}{(8\pi)^{3/2}} \int R_{ij} \int A^*_{ij} = +i \frac{(135)^{\frac{1}{2}}}{(8\pi)^{3/2}} \frac{\alpha Z\Lambda}{2} R^{-1} \left| \int R_{ij} \right|^2$$

$$K_2 = \frac{15}{8\pi} \left| \int R_{ij} \right|^2 \qquad I_2(1,\vec{a}) = \frac{-9}{64\pi^2} \left| \int A_{ij} \right|^2$$

$$\int \mathcal{Y}_2(\vec{R}) \int T_{21}(\vec{R},\vec{a})^* = + i \frac{(135)^{\frac{1}{2}}}{30\sqrt{8\pi}} \alpha Z\Lambda R^{-1} \int \mathcal{Y}_2(\vec{R}) \int \mathcal{Y}_2(\vec{R})^*$$

$$T_{21}(\vec{R},\vec{a}) = -i \frac{\Lambda\alpha Z}{4\sqrt{\frac{10\pi}{3}}} R^{-1} \int \mathcal{Y}_2(\vec{R})$$

$$I_2(1,\vec{a}) = - \frac{\Lambda^2\alpha^2 Z^2}{\frac{160\pi}{3}} R^{-2} K_2$$

$$J_2(1,\vec{a}) = + i \frac{\Lambda\alpha Z}{4\sqrt{\frac{10\pi}{3}}} R^{-1} K_2$$

$$I_2(2,1;\vec{\sigma},\vec{a}) = + i \frac{\Lambda\alpha Z}{4\sqrt{\frac{10\pi}{3}}} R^{-1} J_2(2,\vec{\sigma})$$

$$\Gamma_2 = - \frac{\Lambda\alpha Z}{4\sqrt{\frac{10\pi}{3}}} \tag{69}$$

$$\theta_2 = f - \frac{\Lambda\alpha Z}{4} g \tag{70}$$

The M/L ratio has one definite advantage over the K/L ratio for applying the calculations outlined above to second forbidden transitions. Namely, since the K electrons are often bound with energies which are on the order of the transition energy, the value of q for the K-capture neutrino is quite uncertain due to the uncertainty in E_{EC}.* For example, the transition energy of $_{93}Np^{235}$ is unusually well known. Its value is $E_{EC} = 123 \pm 1$ KeV. Nevertheless the value of the energy of the K-capture neutrino is 4.4 ± 1 KeV (after subtracting the well known binding energy). If the M/L ratio were known for this decay, however, the q which would be of significance, i.e. whose value would limit the accuracy of the calculation, would be that due to L-capture. This value is 101 ± 1. Thus the error in q_L is 1 per cent whereas the error in q_K is 23 per cent. Thus calculations of the nuclear matrix elements of the type described above for the second forbidden transitions are limited only by the accuracy of the M/L ratios and the accuracy of the electron wave functions at the nuclear radius provided a reasonably good value of E_{EC} is available.

If both the L/K and the M/L ratios were known for a given transition, one would be provided with two equations in two unknowns. He would then be able to calculate the absolute value of the nuclear matrix elements (provided of course that the numerical factors did not render these equations linearly dependent).

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*Recall that the neutrino energy in orbital capture is given by:

$$q = E_{EC} - BE$$

where BE is the binding energy of the captured electron.

There are at present only seven second-forbidden (not unique) transitions (which the author has found in the literature) which are fairly well established. These have been established by the spins of the initial and final nuclear states (which are questionable in some cases) and the log ft values. They are:

$$_{17}Cl^{36},\ _{57}La^{137},\ _{28}Ni^{59},\ _{42}Mo^{93},\ _{41}Nb^{91},\ _{43}Tc^{97},\ _{25}Mn^{53}.$$

There are many isotopes which are known to decay by orbital capture but for which either the initial or final state angular momentum is unknown; some of these are probably second-forbidden transitions also.

In order to calculate the nuclear matrix elements one needs to know the decay energy (E_{EC}) and either the L/K ratio or the M/L ratio, preferably the latter as explained above. The following table shows the presently known experimental results:

Table 2. Transition Energy and L/K Ratio for Certain Second Forbidden Transitions

Isotope	L/K	E_{EC}
$_{17}Cl^{36}$		380 KeV
$_{25}Mn^{53}$		598 KeV
$_{28}Ni^{59}$		1070 KeV
$_{41}Nb^{91}$		1200 KeV
$_{42}Mo^{93}$		
$_{43}Tc^{97}$	$.21^{+.14}_{-.10}$	
$_{57}La^{137}$		

No M/L ratios are available for these decays. The angular momentum of the daughter nucleus is known by actual measurement (except in the case of Cl^{36}), but that of the parent nucleus is inferred from other data (except for Mn^{53}).

Thus it seems that more experimental data are necessary for the determination of the nuclear matrix elements in second forbidden decay by these methods. In order to make these calculations both the M/L ratio and the transition energy must be known fairly accurately. In fact, for any useful calculations along these lines, these two quantities must be known within about 5 or at worst 10 per cent.

APPENDIX A

EXPANSION OF THE DIRAC DELTAS

In this section two identities will be established between the Dirac deltas and the spherical harmonics. The first of these is:

$$\delta(\vec{R} - \vec{r}) = \frac{\delta(R - r)}{r^2} \sum_{L,M} Y_L^M(\Theta, \Phi)\, Y_L^{M*}(\theta, \varphi). \tag{A-1}$$

Let $f(\vec{r}) = f(r, \theta, \varphi)$ be an arbitrary function. Since $Y_L^M(\theta, \varphi)$ form a complete set, one can write:

$$f(\vec{r}) = \sum_{\ell, m} f_{\ell m}(r)\, Y_\ell^m(\theta, \varphi) . \tag{A-2}$$

Thus one can write:

$$\int d^3\vec{r}\, f(\vec{r}) \left[\frac{\delta(R - r)}{r^2} \sum_{L,M} Y_L^M(\Theta, \Phi)\, Y_L^{M*}(\theta, \varphi) \right]$$

$$= \int_0^{2\pi}\!\!\int_0^{\pi}\!\!\int_0^{\infty} \sum_{\ell, m} f_{\ell, m}(r)\, Y_\ell^m(\theta, \varphi)\, \frac{\delta(R - r)}{r^2} \sum_{L,M} Y_L^M(\Theta, \Phi)\, Y_L^{M*}(\theta, \varphi)$$

$$r^2\, dr\, \sin\theta\, d\theta\, d\varphi$$

Changing the order of summation and integration, performing the integration over r, and using the orthonormality properties of the spherical harmonics, one obtains:

$$\sum_{\ell, m} \sum_{L,M} \int_0^{2\pi}\!\!\int_0^{\pi} f_{\ell, m}(R)\, Y_\ell^m(\theta, \varphi)\, Y_L^{M*}(\theta, \varphi)\, \sin\theta\, d\theta\, d\varphi\, Y_L^M(\Theta, \Phi)$$

$$= \sum_{L,M} f_{L,M}(R)\, Y_L^M(\Theta, \Phi) = f(\vec{R})$$

Thus the right hand member of (A-1) satisfies the operational definition of the Dirac delta; hence (A-1) is proved.

In a similar manner, it will now be shown that:

$$\overleftrightarrow{I}\,\delta(\vec{R}-\vec{r}) = \frac{\delta(R-r)}{r^2} \sum_{JLM} \vec{\Phi}^{M}_{JL}(\Theta,\emptyset)\, \vec{\Phi}^{M*}_{JL}(\theta,\varphi) \qquad \text{(A-3)}$$

where $\vec{\Phi}^{M}_{JL}$ constitute any complete, orthonormal set of vector functions.* As before, let $\vec{f}(\vec{r})$ be an arbitrary function. Since $\vec{\Phi}^{M}_{JL}$ form a complete set, one can write:

$$\vec{f}(\vec{r}) = \sum_{j\ell m} f_{j\ell m}(r)\, \vec{\Phi}^{m}_{j}(\theta,\varphi) \quad .$$

---

*The direct (or dyadic) product of two vectors, written

$$\vec{A}\,\vec{B}$$

with neither a dot nor a cross between them, is represented by the tensor:

$$\vec{A}\,\vec{B} = \begin{pmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{pmatrix} \quad .$$

A third vector, $\vec{V}$, dotted into $\vec{A}\,\vec{B}$ is given by:

$$\vec{V} \cdot \vec{A}\,\vec{B} = (\vec{V} \cdot \vec{A})\,\vec{B} \; .$$

Similarly:

$$\vec{A}\,\vec{B} \cdot \vec{V} = \vec{A}\,(\vec{B} \cdot \vec{V}) \; .$$

The unit dyadic, $\overleftrightarrow{I}$, is represented by the tensor:

$$\overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

Thus one can write:

$$\int d^3\vec{r}\ \vec{f}(\vec{r}) \cdot \left[\frac{\delta(R-r)}{r^2} \sum_{JLM} \vec{\Phi}^{M}_{JL}(\Theta,\emptyset)\ \vec{\Phi}^{M*}_{JL}(\theta,\varphi) \right]$$

$$= \int d^3\vec{r} \sum_{j\ell m} f_{j\ell m}(r) \vec{\Phi}^{m}_{j\ell}(\theta,\varphi) \cdot \frac{\delta(R-r)}{r^2} \sum_{JLM} \vec{\Phi}^{M*}_{JL}(\theta,\varphi)\ \vec{\Phi}^{M}_{JL}(\Theta,\emptyset)$$

$$= \sum_{JLM} \sum_{j\ell m} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f_{j\ell m}(r)\ \delta(R-r)\ dr \vec{\Phi}^{m}_{j\ell}(\theta,\varphi)\cdot\vec{\Phi}^{M*}_{JL}(\theta,\varphi)\ \sin\theta$$

$$d\theta\ d\varphi\ \vec{\Phi}^{M}_{JL}(\Theta,\emptyset)$$

$$= \sum_{JLM} f_{JLM}(R) \vec{\Phi}^{M}_{JL}(\Theta,\emptyset) = \vec{f}(\vec{R}) \quad .$$

It will prove expedient to choose the following representation for $\vec{\Phi}^{M}_{JL}$.

$$\vec{\Phi}^{M}_{JL}(\theta,\varphi) = \sum_{m,q} \mathcal{Y}^{m}_{L}(\hat{r})\ \hat{e}_q\ (1\ L\ q\ m|1\ L\ J\ M) \tag{A-4}$$

where: $\mathcal{Y}^{m}_{L}(\vec{B}) = B^L\ Y^{m}_{L}(\hat{B})$, and $\hat{e}_1 = -\frac{1}{\sqrt{2}}(\hat{e}_x + i\hat{e}_y)$, $\hat{e}_o = \hat{e}_z$,

$\hat{e}_{-1} = \frac{1}{\sqrt{2}}(\hat{e}_x - i\hat{e}_y)$, $(1\ L\ q\ m|1\ L\ J\ M)$ is a Clebsch Gordon coefficient.

That this choice satisfies the requirements on $\vec{\Phi}^{M}_{JL}$ can be seen as follows.

$$\int_0^{\pi}\int_0^{2\pi} \vec{\Phi}^{M*}_{JL}(\theta,\varphi)\cdot \vec{\Phi}^{M'}_{J'L'}(\theta,\varphi)\ \mathrm{Sin}\theta\ d\theta\ d\varphi = \sum_{m,q} \sum_{m',q'} \int_0^{\pi}\int_0^{2\pi} Y^{m*}_{L}(\theta,\varphi)\ Y^{m'}_{L'}(\theta,\varphi)$$

$$\hat{e}^*_q \cdot \hat{e}_{q'}\ (1\ L\ q\ m|1\ L\ J\ M)(1\ L'\ q'\ m'|1\ L'\ J'\ M')$$

$$= \sum_{m,m'} \sum_{q,q'} \delta(m,m')\ \delta(L,L')\ \hat{e}^*_q \cdot \hat{e}_{q'} (1\ L\ q\ m|1\ L\ J\ M)(1\ L\ q'm'|1\ L\ J'M')$$

$$= \sum_{m,q} \delta(L,L')(1\ L\ q\ m|1\ L\ J\ M)(1\ L\ q\ m|1\ L\ J'M')$$

Since: $\hat{e}^*_q \cdot \hat{e}_q = 1$; $\hat{e}^*_{\pm 1} \cdot \hat{e}_o = 0$; $\hat{e}^*_1 \cdot \hat{e}_{-1} = 0$.

Using the orthogonality properties of Clebsch Gordon coefficients (which are discussed in Appendix B) this becomes:

$$\int_0^{\pi} \int_0^{2\pi} \vec{\Phi}^{M*}_{JL}(\theta,\varphi) \cdot \vec{\Phi}^{M'}_{J'L'}(\theta,\varphi) \sin\theta \, d\varphi = \delta(L,L') \, \delta(M,M') \, \delta(J,J') \, \Delta(1,L,J)$$

where $\Delta(1,L,J) = 1$ if $|1-L| \leq J \leq |1+L|$

$= 0$ otherwise.

APPENDIX B

SOME PROPERTIES OF VECTOR COUPLING COEFFICIENTS

Clebsch-Gordon Coefficients

The following properties of Clebsch-Gordon coefficients are proved in Condon and Shortly (15) and/or by Racah (18). Only the results will be stated here.

The Clebsch-Gordon coefficients are defined by the relation:

$$\psi(\gamma,\ J_1\ j_2\ j\ m) = \sum_{m_1} \sum_{m_2} (j_1\ j_2\ m_1\ m_2 | j_1\ j_2\ j\ m)\ \varphi(\gamma,\ j_1\ j_2\ m_1\ m_2)$$

where ψ describes a certain state of a system of two particles in terms of the quantum numbers: γ, the energy of the system; j_1 and j_2, the angular momenta of the individual particles; j the total angular momentum of the system; and m, the projection of j along z; and φ describes the same state of the system in terms of the quantum numbers: γ; j_1; j_2; and m_1 and m_2, the projections of j_1 and j_2 respectively along z. Explicit forms for these coefficients are quite lengthy. They are unnecessary for the present calculations, but can be found in (18).

The Clebsch-Gordon coefficients satisfy the following symmetry properties:

$$(j_1\ j_2\ m_1\ m_2|j_1\ j_2\ j\ m) = (-)^{j_1+j_2-j}\ (j_2\ j_1\ m_2\ m_1|j_2\ j_1\ j\ m)$$

$$= (-)^{j_2+m_2} \left[\frac{2j+1}{2j_1+1}\right]^{\frac{1}{2}} (j_2\ j\ -m_2\ m|j_2\ j\ j_1\ m_1)$$

$$= (-)^{j_1-m_1} \left[\frac{2j+1}{2j_2+1}\right]^{\frac{1}{2}} (j\ j_1\ m\ -m_1|j\ j_1\ j_2\ m_2) \qquad \text{(B-1)}$$

$$= (-)^{j_1+j_2-j}\ (j_1\ j_2\ -m_1\ -m_2|\ j_1\ j_2\ j\ -m)$$

The Clebsch-Gordon coefficients satisfy the following orthogonality properties.

$$\sum_{j,m} (j_1\ j_2\ m_1\ m_2|j_1\ j_2\ j\ m)(j_1\ j_2\ m_1'\ m_2'|j_1\ j_2\ j\ m) = \delta(m_1, m_1')\ \delta(m_2, m_2') \quad \text{(B-2)}$$

$$\sum_{m_1 m_2} (j_1\ j_2\ m_1\ m_2|j_1\ j_2\ j\ m)(j_1\ j_2\ m_1\ m_2|j_1\ j_2\ j'm') = \delta(j, j')\ \delta(m, m')$$

$$\Delta(j_1, j_2, j) \qquad \text{(B-3)}$$

where $\Delta(j_1, j_2, j) = 1$ if $|j_1-j_2| \leq j \leq j_1 + j_2$

$= 0$ otherwise.

In the special case where all of the magnetic quantum numbers vanish, the Clebsch-Gordon coefficients reduce to the following explicit form:

$$(j_1\ j_2\ 00|j_1\ j_2\ j\ 0) = 0 \qquad \text{for } j_1 + j_2 + j \text{ odd} \quad \text{(B-4)}$$

$$= (-)^{j+g}\ (2j + 1)^{\frac{1}{2}} \left[\frac{(j_1+j_2-j)!(j_1+j-j_2)!(j_2+j-j_1)!}{(j_1+j_2+j + 1)!}\right]^{\frac{1}{2}}$$

$$\frac{g\ !}{(g - j_1)!(g - j_2)!(g - j)!} \qquad \text{for } j_1 + j_2 + j \text{ even.}$$

where: $g = \frac{1}{2}(j_1 + j_2 + j)$.

Symmetrized V Coefficients

For some applications it is expedient to write the Clebsch-Gordon coefficients in terms of a different set of coefficients defined as follows:

$$V(j_1\ j_2\ j;\ m_1\ m_2\ -m) = (-)^{j+m}\,(2j+1)^{-\frac{1}{2}}\,(j_1\ j_2\ m_1\ m_2|j_1\ j_2\ j\ m) \qquad \text{(B-5)}$$

These V coefficients satisfy the following symmetry relations:

$$V(a\ b\ c;\ \alpha\ \beta\ \gamma) = (-)^{a+b-c}\,V(b\ a\ c;\ \beta\ \alpha\ \gamma) = (-)^{a+b+c}\,V(a\ c\ b;\ \alpha\ \gamma\ \beta)$$

$$= (-)^{a-b+c}\,V(c\ b\ a;\ \gamma\ \beta\ \alpha) = (-)^{2b}V(c\ a\ b;\ \gamma\ \alpha\ \beta) = (-)^{2c}\,V(b\ c\ a;\ \beta\ \gamma\ \alpha)$$

$$= (-)^{2\gamma}\,V(b\ a\ c;\ -\beta\ -\alpha\ -\gamma) = (-)^{a+b+c}\,V(a\ b\ c;\ -\alpha\ -\beta\ -\gamma). \qquad \text{(B-6)}$$

The V coefficients satisfy the following orthogonality relations:

$$\sum_{c,\gamma} (2c+1)\,V(a\ b\ c;\ \alpha\ \beta\ \gamma)\,V(a\ b\ c;\ \alpha'\beta'\gamma) = \delta(\alpha,\alpha')\,\delta(\beta,\beta') \qquad \text{(B-7)}$$

$$\sum_{\alpha\beta} V(a\ b\ c;\ \alpha\ \beta\ \gamma)\,V(a\ b\ c';\ \alpha\ \beta\gamma') = \frac{\delta(c,c')\,\delta(\gamma,\gamma')}{2c+1}\,\Delta(a\ b\ c). \qquad \text{(B-8)}$$

In the event that all of the magnetic quantum numbers are zero, these reduce to the following explicit forms:

$$V(a\ b\ c;\ 0\ 0\ 0) = 0 \qquad \text{if } a + b + c \text{ odd}$$

$$= (-)^g \left[\frac{(a+b-c)!(a+c-b)!(b+c-a)!}{(a+b+c+1)!}\right]^{\frac{1}{2}} \tag{B-9}$$

$$\frac{g!}{(g-a)!(g-b)!(g-c)!} \qquad \text{if } a + b + c \text{ even}$$

where $g = \frac{1}{2}(a + b + c)$.

Racah Coefficients

The Racah coefficient W(abcd;ef) is defined explicitly in reference (18). The following properties of the Racah coefficient (which are derived in (18)) are of use in the present calculations.

$$(-)^{e+\alpha+\beta}\ V(abe;\alpha,\ \beta,\ -\alpha-\beta)\ V(edc;\ \alpha+\beta,\ \delta,\ -\alpha-\beta-\delta)$$
$$= \sum_f (2f+1)\ (-)^{f+\delta+\beta}\ V(bdf;\ \beta,\ \delta,\ -\beta-\delta)\ V(afc;\ \alpha,\ \beta+\delta,\ -\alpha-\beta-\delta) \tag{B-10}$$

$$W(abcd;ef)$$
$$\sum_\beta (-)^{e+f+\gamma+\beta}\ V(abe;\ a,\ \beta,\ -\alpha-\beta)\ V(edc;\ a+\beta,\ \gamma-\alpha-\beta,\ -\gamma)$$
$$V(bdf;\ \beta,\ \gamma-\alpha-\beta,\ -\gamma+\alpha) = V(afc;\ \alpha,\ \gamma-\alpha,\ -\gamma)\ W(abcd;ef) \tag{B-11}$$

The elements of the Racah coefficients may be either integral or half-integral, but the following triads must have integral sums and be triangular:

abe cde acf bdf .

The following symmetry properties apply:

$$W(abcd;ef) = W(badc;ef) = W(acbd;fe) = W(cdab;ef)$$
$$= (-)^{e+f-a-d}\ W(abcf;ad) = (-)^{e+f-b-c}\ W(aefd;bc)\ . \tag{B-12}$$

The X-Coefficient of Fano

Yet another coupling coefficient of use is the Fano X-coefficient defined by the relation:

$$X(abc;def;ghi) = (-)^{E} \sum_{s} (2s+1)\ W(bdcg;sa)\ W(dbfh;se)\ W(gchf;si) \qquad \text{(B-13)}$$

where $E = a+b+c+d+e+f+g+h+i$.

This coefficient has the following properties. If it is written in the following form:

$$X \begin{pmatrix} abc \\ def \\ ghi \end{pmatrix}$$

the interchange of any two rows or any two columns multiplies X by $(-)^{E}$. Interchanging rows and columns (transpose) leaves X unaffected. The elements of any row or any column must satisfy the triangular inequality. Explicit forms for all cases of interest in these calculations are given in reference (2).

Products of Spherical Harmonics

The product of two spherical harmonics can be expressed in terms of a sum of spherical harmonics by the relationship:

$$Y^{m}_{\ell}(\theta,\varphi)\ Y^{m'}_{\ell'}(\theta,\varphi) = \sum_{L,M} \left[\frac{(2\ell+1)(2\ell'+1)}{4\pi\ (2L+1)}\right]^{\frac{1}{2}}$$

$$(\ell\ \ell'\ m\ m'|\ell\ \ell'\ L\ M)(\ell\ \ell'\ 0\ 0|\ell\ \ell'\ L\ 0)\ Y^{M}_{L}(\theta,\varphi). \qquad \text{(B-14)}$$

From this and the orthonormality properties of the spherical harmonics, the following result can be obtained:

$$\int d\omega\, Y_{\ell}^{m}(\omega)\, Y_{\ell'}^{m'}(\omega)\, Y_{\ell''}^{m''}(\omega) = (-)^{\ell - m - m''} \left[\frac{2\ell+1}{4\pi}\right]^{\frac{1}{2}} \tag{B-15}$$

$$(\ell'' \ell\, -m''\, -m \mid \ell'' \ell\, \ell'\, m')(\ell\, \ell'\, 0\, 0 \mid \ell\, \ell'\, \ell''\, 0).$$

The Z Coefficient

The following values of the Z coefficient defined in equation (47) are used in the calculations.

$Z(000) = 1$ $\qquad$ $Z(112) = (6)^{\frac{1}{2}}$

$Z(011) = -(3)^{\frac{1}{2}}$ $\qquad$ $Z(123) = -3$

$Z(022) = (5)^{\frac{1}{2}}$ $\qquad$ $Z(222) = -5(2/7)^{\frac{1}{2}}$

$Z(033) = -(7)^{\frac{1}{2}}$ $\qquad$ $Z(233) = 2(7/3)^{\frac{1}{2}}$.

APPENDIX C

DETAILS OF THE CALCULATIONS OF THE REDUCED LEPTON MATRIX ELEMENTS

Evaluation of $\int d\omega\, Y_J^M\, \chi_\kappa^{\mu *}\, \chi_{\kappa'}^{\mu'}$

From the definition of χ_κ^μ (equation (37)) this becomes:

$$\int d\omega\, Y_J^M\, \chi_\kappa^{\mu *}\, \chi_{\kappa'}^{\mu'} = \int d\omega\, Y_J^M \sum_{\tau,\tau'} (\ell \tfrac{1}{2}\ \mu-\tau\ \tau | \ell \tfrac{1}{2}\ j\ \mu)$$

$$(\ell' \tfrac{1}{2}\ \mu'-\tau'\ \tau' | \ell' \tfrac{1}{2}\ j'\ \mu')\, \chi_{\frac{1}{2}}^{\tau *}\, \chi_{\frac{1}{2}}^{\tau'}\, Y_\ell^{\mu-\tau *}\, Y_{\ell'}^{\mu'-\tau'} \quad .$$

From the definition of $\chi_{\frac{1}{2}}^{\tau}$ one sees immediately that:

$$\chi_{\frac{1}{2}}^{\tau *}\, \chi_{\frac{1}{2}}^{\tau'} = \delta(\tau,\tau') \quad .$$

From the property of spherical harmonics with the phase chosen, $Y_\ell^{m*} = (-)^m\, Y_\ell^{-m}$, one obtains:

$$\int d\omega\, Y_J^M\, \chi_\kappa^{\mu *} \chi_{\kappa'}^{\mu'} = \sum_\tau (-)^{\mu-\tau}\, (\ell\ \tfrac{1}{2}\ \mu-\tau\ \tau | \ell \tfrac{1}{2}\ j\ \mu)(\ell' \tfrac{1}{2}\ \mu'-\tau\ \tau | \ell' \tfrac{1}{2}\ j'\ \mu')$$

$$\int d\omega\, Y_J^M\, Y_\ell^{\tau-\mu}\, Y_{\ell'}^{\mu'-\tau} \quad .$$

Applying equation (B-15) from appendix B this becomes:

$$\int d\omega\, Y_J^M\, \chi_\kappa^{\mu *} \chi_{\kappa'}^{\mu'} = (-)^{J-M} \left[\frac{2J+1}{4\pi}\right]^{\frac{1}{2}} (J\ \ell'\ 00 |\ J\ \ell' \ell 0) \sum_\tau (\ell J\ \mu-\tau\ -M | \ell J\ \ell' \mu'-\tau)$$

$$(\ell \tfrac{1}{2}\ \mu-\tau\ \tau | \ell \tfrac{1}{2}\ j\ \mu)(\ell'\ \tfrac{1}{2}\ \mu'-\tau\ \tau | \ell' \tfrac{1}{2}\ j'\ \mu') \quad .$$

In order to facilitate the calculation of the sum over τ, it is expedient to rewrite this sum in terms of Racah's V-coefficients. These are defined by equation (B-5) of appendix B. The sum then becomes:

$$(2j+1)^{\frac{1}{2}}(2j'+1)^{\frac{1}{2}}(2\,\ell'+1)^{\frac{1}{2}}\,(-)^{j'-\mu'}\sum_{\tau}(-)^{\tau+\ell+j'+\mu'-\mu}\,V(j\tfrac{1}{2}\ell;-\mu\;\tau\;\mu-\tau)$$

$$V(\ell\,\ell' J;\;\tau-\mu\;\mu'-\tau\;\mu-\mu')\;V(\tfrac{1}{2}\ell' j';\;\tau\;\mu'-\tau\;-\mu').$$

Applying a "Racah Recoupling" according to the prescription of equation (B-11) of appendix B one obtains:

$$(2j+1)^{\frac{1}{2}}(2j'+1)^{\frac{1}{2}}(2\ell'+1)^{\frac{1}{2}}\,(-)^{j'-\mu'}\;V(j\;j'\;J;\;-\mu\;\mu'\;\mu-\mu')\;W(j\;\tfrac{1}{2}\;J\,\ell';\ell\;j')$$

Thus one finally obtains:

$$\int d\omega\, Y_J^M\, \chi_\kappa^{\mu *}\chi_{\kappa'}^{\mu'} = (4\pi)^{-\frac{1}{2}}\,(-)^{\ell'+j-\frac{1}{2}}\left[(2J+1)(2\ell+1)(2j'+1)\right]^{\frac{1}{2}}\,(\ell J\;00|\ell J\;\ell'\;0)$$

$$(j'J\;\mu'\;M|j'J\;j\;\mu)\;W(\ell j\,\ell' j';\tfrac{1}{2}\;J)$$

which is the desired result.

<u>Evaluation of $\int d\omega\, \chi_\mu^{\tau *}\, T_{JL}^{-M}\,(\hat{r},\vec{\sigma})\,\chi_{\kappa'}^{\mu'}$</u>

Proceeding in the same manner as before:

$$\int d\omega\, \chi_\mu^{\tau *}\, T_{JL}^{-M}\,\chi_{\kappa'}^{\mu'} = \sum_{\tau,m'\tau'}(\ell\tfrac{1}{2}\;\mu-\tau\;\tau|\ell\tfrac{1}{2}\;j\;\mu)(\ell'\tfrac{1}{2}\;\mu'-\tau'\;\tau'|\ell'\tfrac{1}{2}\;j'\;\mu')$$

$$(1L\;-m'\;m'-M|1\;L\;J\;-M)\,\chi_{\frac{1}{2}}^{\tau *}\,\mathcal{Y}_1^{-m'}(\vec{\sigma})\,\chi_{\frac{1}{2}}^{\tau'}\int d\omega\; Y_\ell^{\mu-\tau *}\;Y_L^{m'-M}\;Y_{\ell'}^{\mu'-\tau'}$$

$$= (-)^{\ell+\ell'} \left[\frac{2L+1}{4\pi}\right]^{\frac{1}{2}} (\ell L\ 00|\ell L \ell'\ 0) \sum_{\tau, m' \tau'} (L \ell'\ M-m'\ \tau'-\mu | L \ell' \ell\ \tau-\mu)$$

$$(\ell \tfrac{1}{2}\ \mu-\tau\ \tau | \ell \tfrac{1}{2}\ j\ \mu)(\ell' \tfrac{1}{2}\ \mu'-\tau'\ \tau' | \ell' \tfrac{1}{2}\ j' \mu')(1L\ -m'\ m'-M | 1LJ\ -M)$$

$$\chi_{\frac{1}{2}}^{\tau *} \mathcal{Y}_1^{-m'}(\vec{\sigma})\, \chi_{\frac{1}{2}}^{\tau'} .$$

From the definition of $\chi_{\frac{1}{2}}^{\tau}$ and of $\mathcal{Y}_1^{-m'}(\vec{\sigma})$ one finds by inspection that:

$$\chi_{\frac{1}{2}}^{\tau *} \mathcal{Y}_1^{-m'}(\vec{\sigma}) \chi_{\frac{1}{2}}^{\tau'} = \delta(\tau, \tau'-m') \frac{3}{\sqrt{4\pi}} (-)^{m'} (\tfrac{1}{2}\ 1\ \tau'-m'\ m' | \tfrac{1}{2} 1 \tfrac{1}{2}\ \tau') .$$

Substituting these results into the preceding equation and performing the summation over τ one obtains:

$$\int d\omega\, \chi_{\kappa}^{\mu *}\ T_{JL}^{-M}\ \chi_{\kappa'}^{\mu'} = \frac{3}{4\pi}(2L+1)^{\frac{1}{2}} (-)^{\ell+\ell'} (\ell L\ 00 | \ell L \ell' 0) \sum_{m'} (-)^{m'}$$

$$(1L\ -m'\ m'-M | 1LJ\ -M) \sum_{\tau'} (\ell \tfrac{1}{2}\ \mu-\tau'+m'\ \tau'-m' | \ell \tfrac{1}{2}\ j\mu)$$

$$(\ell' \tfrac{1}{2}\ \mu'-\tau'\ \tau' | \ell' \tfrac{1}{2}\ j'\ \mu')(\tfrac{1}{2} 1 \tau'-m'\ m' | \tfrac{1}{2} 1 \tfrac{1}{2}\ \tau')(L\ \ell'\ m'-M\ \mu'-\tau' | L \ell' \ell\ \mu-\tau'+m') .$$

From the last Clebsch-Gordon coefficient is obtained the relationship: $\mu = \mu' - M$. Eliminating μ by this equation, and writing the summation over τ' in terms of the symmetric V coefficients to expedite the calculation one obtains:

$$\left[2(2j+1)(2j'+1)(2\ell+1)\right]^{\frac{1}{2}} (-)^{\ell' - m' + \mu' + \frac{1}{2}} \sum_{\tau'} V(\ell L \ell';\ \mu'-\tau'+m'-M\ M-m'\ \tau'-\mu')$$

$$V(\ell' j' \tfrac{1}{2}; \mu'-\tau'\ -\mu'\ \tau')\ V(\ell j\ \tfrac{1}{2}; \mu'-\tau'+m'-M\ M-\mu'\ \tau'-m)\ V(\tfrac{1}{2} 1 \tfrac{1}{2}; m'-\tau'\ -m'\ -\tau') .$$

Performing a Racah Recoupling according to equation (B-10) of appendix B this becomes:

$$\left[2(2j+1)(2j'+1)(2\ell+1)\right]^{\frac{1}{2}} \sum_{f,f'} (2f+1)(2f'+1)\; W(\ell\tfrac{1}{2}\; L\; j';f\ell')\; W(\ell\tfrac{1}{2}\; j\; 1;f'\tfrac{1}{2})$$

$$V(L\; j'f;M-m'\; -\mu'\; m'-M+\mu')\; V(j\; 1\; f;\; M-\mu'\; -m'\; m'-M+\mu')$$

$$\sum_{\tau',x} V(\ell\tfrac{1}{2}\; f;\; x\; \tau'\; M-m'-\mu')\; V(\ell\tfrac{1}{2}\; f';\; x\; \tau'\; M-m'-\mu')\; (-)^{1}\; .$$

Using the orthogonality properties of the V-coefficients, one can make the sum over τ' and x. Performing this sum and substituting this result back into the original expression gives:

$$\int d\omega\, \chi_{\kappa}^{\mu *}\, T_{JL}^{-M}\, \chi_{\kappa'}^{\mu'} = \frac{3}{4\pi}\left[2(2L+1)(2j'+1)(2\ell+1)(2j+1)\right]^{\frac{1}{2}}(-)^{\ell+\ell'+1+J-M}\delta(M,\mu'-\mu)$$

$$(\ell L\; 00|\ell L \ell' 0)\sum_{s}\Delta(\ell\tfrac{1}{2}s)(2s+1)\; W(\ell\tfrac{1}{2}\; j1;s\tfrac{1}{2})\; W(\ell\tfrac{1}{2}Lj';s\ell')(2J+1)^{\frac{1}{2}}\sum_{m'}(-)^{m'}$$

$$V(1LJ;-m'\; m'-M\; M)\; V(Lj's;M-m'\; -\mu'\; m'-M+\mu')\; V(j1s;\; M-\mu'\; -m'\; m'-M+\mu').$$

In order to make the sum over m' it will be necessary once again to perform a Racah Recoupling. When this is done the sum over m' becomes:

$$(-)^{L+j+\mu'}\; V(j'Jj;\mu'\; -M\; M-\mu')\; W(1j\; L\; j';sJ)\; .$$

Substituting this result into the preceding equation gives:

$$\int d\omega\, \chi_{\kappa}^{\mu *}\, T_{JL}^{-M}\, \chi_{\kappa'}^{\mu'} = \frac{3\sqrt{2}}{4\pi}\left[(2L+1)(2j'+1)(2J+1)(2\ell+1)\right]^{\frac{1}{2}}\; (-)^{J+1}\; \delta(M,\mu'-\mu)$$

$$(\ell L\; 00|\ell L \ell' 0)(j'J\; \mu'\; -M|j'J\; j\; \mu'-M)\; \sum_{s}\Delta(\ell\tfrac{1}{2}\; s)(2s+1)$$

$$W(1j\; L\; j';\; s\; J)\; W(j\; 1\,\ell\tfrac{1}{2};\; s\; \tfrac{1}{2})\; W(j'L\,\tfrac{1}{2}\,\ell;\; s\; \ell')$$

$$= \frac{3\sqrt{2}}{4\pi} \left[(2L+1)(2j'+1)(2J+1)(2\ell+1)\right]^{\frac{1}{2}} (-)^{j'-j} (\ell L\ 00|\ell\ L\ \ell' 0)$$

$$X(\tfrac{1}{2} 1 \tfrac{1}{2};\ j\ J\ j';\ \ell\ L\ \ell')(j'\ J\ \mu'\ -M;\ j'\ J\ j\ \mu'-M).$$

This is the desired result.

APPENDIX D

REDUCED LEPTON MATRIX ELEMENTS EXPRESSED IN ALGEBRAIC FORM

Appendix D lists the results of evaluating equations (48) and (49) for the various reduced lepton matrix elements which enter the allowed, first forbidden, first forbidden unique, second forbidden, second forbidden unique, and third forbidden unique transitions. In these results terms which are small compared with other terms listed have been neglected. The matrix elements which are significant are (in every case) the ones connecting the initial electron state with the neutrino states having the two lowest possible angular momenta consistent with the order of the spherical tensor operator, that is the states with $j_\nu = |J - j_e|$ and $j_\nu = |J - j_e| + 1$. Matrix elements for which the inequality:

$$J + j_e \geq j_\nu \geq |J - j_e|$$

is not satisfied vanish identically. Matrix elements which satisfy this inequality but which are not listed are small compared with the matrix elements listed (by at least a factor of qR).

Reduced Lepton Matrix Elements For J = 0 $\kappa_e = -1$

$$(-1\|(1+\gamma^5)\, \mathcal{Y}_0^*(\hat{r})\|-1) = \frac{1}{2\sqrt{\pi}}\, qg$$

$$(1\|(1+\gamma^5)\, \mathcal{Y}_0^*(\hat{r})\|-1) = +\frac{i}{2\sqrt{\pi}}\, qg$$

$$(-1\|(1+\gamma^5)\, T_{01}^*(\hat{r},\vec{\sigma})\|-1) = \frac{-i\sqrt{3}}{4\pi}\, q\left[f+\frac{1}{3}\, qRg\right]$$

$$(1\|(1+\gamma^5)\, T_{01}^*(\hat{r},\vec{\sigma})\|-1) = \frac{\sqrt{3}}{4\pi}\, q\left[f+\frac{1}{3}\, qRg\right]$$

Reduced Lepton Matrix Elements For J = 1 $\kappa_e = -1$

$$(-1\|(1+\gamma^5)\, \mathcal{Y}_1^*(\hat{r})\|-1) = +\frac{i}{2\sqrt{\pi}}\, q\left[f+\frac{1}{3}\, qRg\right]$$

$$(1\|(1+\gamma^5)\, \mathcal{Y}_1^*(\hat{r})\|-1) = -\frac{1}{2\sqrt{\pi}}\, q\left[f+\frac{1}{3}\, qRg\right]$$

$$(-2\|(1+\gamma^5)\, \mathcal{Y}_1^*(\hat{r})\|-1) = \frac{1}{2\sqrt{\pi}}\, q\, \frac{1}{3}\, qRg$$

$$(2\|(1+\gamma^5)\, \mathcal{Y}_1^*(\hat{r})\|-1) = +\frac{i}{2\sqrt{\pi}}\, q\, \frac{1}{3}\, qRg$$

$$(-1\|(1+\gamma^5)\, T_{10}^*(\hat{r},\vec{\sigma})\|-1) = \frac{3}{4\pi}\, qg$$

$$(1\|(1+\gamma^5)\, T_{10}^*(\hat{r},\vec{\sigma})\|-1) = +i\,\frac{3}{4\pi}\, qg$$

$$(-1\|(1+\gamma^5)\, T_{11}^*(\hat{r},\vec{\sigma})\|-1) = +i\,\frac{3}{2\pi\sqrt{6}}\, q\left[f-\frac{1}{3}\, qRg\right]$$

$$(1\|(1+\gamma^5)\, T_{11}^*(\hat{r},\vec{\sigma})\|-1) = -\frac{3}{2\pi\sqrt{6}}\, q\left[f-\frac{1}{3}\, qRg\right]$$

$$(-2\|(1+\gamma^5)\, T_{11}^*(\hat{r},\vec{\sigma})\|-1) = \frac{1}{4\pi\sqrt{6}}\, q^2Rg$$

$$(2\|(1+\gamma^5)\, T_{11}^*(\hat{r},\vec{\sigma})\|-1) = +i\,\frac{1}{4\pi\sqrt{6}}\, q^2Rg$$

Reduced Lepton Matrix Elements For $J = 2 \quad \kappa_e = -1$

$$(-2\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-1) = +\frac{i}{6\sqrt{\pi}}\, q^2R\left[f + \frac{1}{5}\, qRg\right]$$

$$(2\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-1) = -\frac{1}{6\sqrt{\pi}}\, q^2R\left[f + \frac{1}{5}\, qRg\right]$$

$$(-3\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-1) = \frac{1}{6\sqrt{\pi}}\, q^2R\cdot\frac{1}{5}\cdot qRg$$

$$(3\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-1) = +\frac{i}{6\sqrt{\pi}}\, q^2R\cdot\frac{1}{5}\cdot qRg$$

$$(-2\|(1+\gamma^5)\; T^*_{21}(\hat{r},\vec{\sigma})\|-1) = \frac{1}{4\pi}\sqrt{\frac{5}{6}}\; q^2Rg$$

$$(2\|(1+\gamma^5)\; T^*_{21}(\hat{r},\vec{\sigma})\|-1) = \frac{+i}{4\pi}\sqrt{\frac{5}{6}}\; q^2Rg$$

$$(-2\|(1+\gamma^5)\; T^*_{22}(\hat{r},\vec{\sigma})\|-1) = +i\,\frac{\sqrt{2}}{8\pi}\, q^2R\left[f - \frac{1}{5}\, qRg\right]$$

$$(2\|(1+\gamma^5)\; T^*_{22}(\hat{r},\vec{\sigma})\|-1) = -\frac{\sqrt{2}}{8\pi}\, q^2R\left[f - \frac{1}{5}\, qRg\right]$$

$$(-3\|(1+\gamma^5)\; T^*_{22}(\hat{r},\vec{\sigma})\|-1) = \frac{\sqrt{2}}{60\pi}\, q^2R\cdot qRg$$

$$(3\|(1+\gamma^5)\; T^*_{22}(\hat{r},\vec{\sigma})\|-1) = +i\,\frac{\sqrt{2}}{60\pi}\, q^2R\cdot qRg$$

Reduced Lepton Matrix Elements For $J = 2 \quad \kappa_e = -2$

$$(-1\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-2) = \frac{-i}{\sqrt{2\pi}}\, q\left[f + \frac{1}{3}\, qRg\right]$$

$$(1\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-2) = \frac{1}{\sqrt{2\pi}}\, q\left[f + \frac{1}{3}\, qRg\right]$$

$$(-2\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-2) = -\frac{1}{6\sqrt{\pi}}\, q^2Rg$$

$$(2\|(1+\gamma^5)\mathcal{Y}^*_2(\hat{r})\|-2) = \frac{-i}{6\sqrt{\pi}}\, q^2Rg$$

$$(-1\|(1+\gamma^5)\ T^*_{21}(\hat{r},\vec{\sigma})\|-2) = -\frac{\sqrt{15}}{4\pi}\ qg$$

$$(1\|(1+\gamma^5)\ T^*_{21}(\hat{r},\vec{\sigma})\|-2) = -i\ \frac{\sqrt{15}}{4\pi}\ qg$$

$$(-1\|(1+\gamma^5)\ T^*_{22}(\hat{r},\vec{\sigma})\|-2) = -i\ \frac{3}{4\pi}\ q\left[f-\frac{1}{3}\ qRg\right]$$

$$(1\|(1+\gamma^5)\ T^*_{22}(\hat{r},\vec{\sigma})\|-2) = \frac{3}{4\pi}\ q\left[f-\frac{1}{3}\ qRg\right]$$

$$(-2\|(1+\gamma^5)\ T^*_{22}(\hat{r},\vec{\sigma})\|-2) = 0$$

$$(2\|(1+\gamma^5)\ T^*_{22}(\hat{r},\vec{\sigma})\|-2) = 0$$

<u>Reduced Lepton Matrix Elements For $J = 3\quad \kappa_e = -1$</u>

$$(-3\|(1+\gamma^5)\mathcal{Y}^*_3(\hat{r})\|-1) = \frac{i}{30\sqrt{\pi}}\ q^3R^2\left[f+\frac{1}{7}\ qRg\right]$$

$$(3\|(1+\gamma^5)\mathcal{Y}^*_3(\hat{r})\|-1) = \frac{-1}{30\sqrt{\pi}}\ q^3R^2\left[f+\frac{1}{7}\ qRg\right]$$

$$(-4\|(1+\gamma^5)\mathcal{Y}^*_3(\hat{r})\|-1) = \frac{1}{30\sqrt{\pi}}\ q^3R^2\cdot\frac{1}{7}\ qRg$$

$$(4\|(1+\gamma^5)\mathcal{Y}^*_3(\hat{r})\|-1) = \frac{i}{30\sqrt{\pi}}\ q^3R^2\cdot\frac{1}{7}\ qRg$$

$$(-3\|(1+\gamma^5)\ T^*_{32}(\hat{r},\vec{\sigma})\|-1) = \frac{\sqrt{7}}{60\pi}\ q^3R^2g$$

$$(3\|(1+\gamma^5)\ T^*_{32}(\hat{r},\vec{\sigma})\|-1) = i\ \frac{\sqrt{7}}{60\pi}\ q^3R^2g$$

$$(-3\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-1) = \frac{i}{30\pi}\ q^3R^2\left[f-\frac{1}{7}\ qRg\right]$$

$$(3\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-1) = -\frac{1}{30\pi}\ q^3R^2\left[f-\frac{1}{7}\ qRg\right]$$

$$(-4\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-1) = \frac{1}{40\pi}\ q^3R^2\ \frac{1}{7}\ qRg$$

$$(4\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-1) = \frac{i}{40\pi}\ q^3R^2\ \frac{1}{7}\ qRg$$

Reduced Lepton Matrix Elements For J = 3 $\kappa_e = -2$

$$(-2\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-2) = \frac{-i}{2\sqrt{5\pi}}\ q^2R\ \left[f+\frac{1}{5}\ qRg\right]$$

$$(2\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-2) = \frac{1}{2\sqrt{5\pi}}\ q^2R\ \left[f+\frac{1}{5}\ qRg\right]$$

$$(-3\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-2) = -\frac{1}{3\sqrt{5\pi}}\ q^2R\ (\frac{1}{5}\ qRg)$$

$$(3\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-2) = \frac{-i}{3\sqrt{5\pi}}\ q^2R\ (\frac{1}{5}\ qRg)$$

$$(-2\|(1+\gamma^5)\ T^*_{32}(\hat{r},\vec{\sigma})\|-2) = -\frac{1}{4\pi}\sqrt{\frac{7}{5}}\ q^2Rg$$

$$(2\|(1+\gamma^5)\ T^*_{32}(\hat{r},\vec{\sigma})\|-2) = \frac{-i}{4\pi}\sqrt{\frac{7}{5}}\ q^2Rg$$

$$(-2\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-2) = \frac{-i}{2\pi\sqrt{5}}\ q^2R\ \left[f-\frac{1}{5}\ qRg\right]$$

$$(2\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-2) = \frac{1}{2\pi\sqrt{5}}\ q^2R\ \left[f-\frac{1}{5}\ qRg\right]$$

$$(-3\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-2) = -\frac{1}{12\pi\sqrt{5}}\ q^2R\ \cdot\ \frac{1}{5}\ qRg$$

$$(3\|(1+\gamma^5)\ T^*_{33}(\hat{r},\vec{\sigma})\|-2) = \frac{-i}{12\pi\sqrt{5}}\ q^2R\ \cdot\ \frac{1}{5}\ qRg$$

Reduced Lepton Matrix Elements For J = 3 $\kappa_e = -3$

$$(-1\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-3) = +\frac{i}{2}\sqrt{\frac{3}{\pi}}\ q\left[f+\frac{1}{3}\ qRg\right]$$

$$(1\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-3) = -\frac{1}{2}\sqrt{\frac{3}{\pi}}\ q\ \left[f+\frac{1}{3}\ qRg\right]$$

$$(-2\|(1+\gamma^5)\ \mathcal{Y}^*_3(\hat{r})\|-3) = \sqrt{\frac{3}{10\pi}}\ q\ \cdot\ \frac{1}{3}\ qRg$$

$$(2\|(1+\gamma^5)\mathcal{Y}_3^*(\hat{r})\|-3) = +i\sqrt{\frac{3}{10\pi}}\, q \cdot \frac{1}{3}\, qRg$$

$$(-1\|(1+\gamma^5)\ T_{32}^*(\hat{r},\vec{\sigma})\|-3) = \frac{\sqrt{21}}{4\pi}\, qg$$

$$(1\|(1+\gamma^5)\ T_{32}^*(\hat{r},\vec{\sigma})\|-3) = +i\frac{\sqrt{21}}{4\pi}\, qg$$

$$(-1\|(1+\gamma^5)\ T_{33}^*(\hat{r},\vec{\sigma})\|-3) = +i\ \frac{\sqrt{3}}{2\pi}\, q\left[f - \frac{1}{3}\, qRg\right]$$

$$(1\|(1+\gamma^5)\ T_{33}^*(\hat{r},\vec{\sigma})\|-3) = -\ \frac{\sqrt{3}}{2\pi}\, q\ \left[f - \frac{1}{3}\, qRg\right]$$

$$(-2\|(1+\gamma^5)\ T_{33}^*(\hat{r},\vec{\sigma})\|-3) = -\ \frac{1}{4\pi}\sqrt{\frac{3}{10}}\, q \cdot \frac{1}{3}\, qRg$$

$$(2\|(1+\gamma^5)\ T_{33}^*(\hat{r},\vec{\sigma})\|-3) = \frac{-i}{4\pi}\sqrt{\frac{3}{10}}\, q \cdot \frac{1}{3}\, qRg$$

Reduced Lepton Matrix Elements for J = 4, $\kappa_e = -1$

$$(-4\|T_{43}^*\|-1) = \frac{\sqrt{3}}{280\pi}\, q^4\ R^3\ g$$

$$(+4\|T_{43}^*\|-1) = i\ \frac{\sqrt{3}}{280\pi}\, q^4\ R^3\ g$$

Reduced Lepton Matrix Elements for J = 4, $\kappa_e = -2$

$$(-3\|T_{43}^*\|-2) = -\ \frac{3}{20\pi\sqrt{7}}\ q^3\ R^2\ g$$

$$(+3\|T_{43}^*\|-2) = -i\ \frac{3}{20\pi\sqrt{7}}\ q^3\ R^2\ g$$

Reduced Lepton Matrix Elements for J = 4, $\kappa_e = -3$

$$(-2\|T_{43}^*\|-3) = \frac{3}{4\pi}\sqrt{\frac{3}{14}}\ q^2\ R\ g$$

$$(+2\|T_{43}^*\|-3) = i\ \frac{3}{4\pi}\sqrt{\frac{3}{14}}\, q^2\ R\ g$$

APPENDIX E

NUMERICAL VALUES OF THE M/L CAPTURE RATIOS

The following tables give the values of the contributions of the various M sub-shells to the M/L capture ratios. These values were computed on the Georgia Institute of Technology's Burroughs 220 computer from the equations (61), (62), and (63).

Table 3. First Forbidden Unique M/L Ratios for Z=60

E_{EC}(KeV)	M_I+M_{II}/L	M_{III}+M_{IV}/L	M_V/L	M/L
10000.0	0.211	0.000	0.000	0.211
5000.0	0.211	0.001	0.000	0.212
3000.0	0.212	0.002	0.000	0.213
2000.0	0.212	0.004	0.000	0.216
1000.0	0.211	0.016	0.000	0.227
700.0	0.210	0.032	0.000	0.241
500.0	0.205	0.061	0.000	0.266
300.0	0.188	0.156	0.000	0.345
200.0	0.160	0.301	0.000	0.461
100.0	0.087	0.665	0.000	0.752
80.0	0.065	0.782	0.000	0.847
60.0	0.042	0.920	0.000	0.962
40.0	0.022	1.106	0.000	1.128
30.0	0.013	1.264	0.000	1.278
25.0	0.010	1.397	0.000	1.407
20.0	0.007	1.627	0.000	1.634
15.0	0.005	2.160	0.000	2.165
12.5	0.004	2.821	0.000	2.825
10.0	0.004	4.683	0.000	4.687

Table 4. Second Forbidden Unique M/L Ratios for Z=60

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.211	0.000	0.000	0.211
5000.0	0.211	0.001	0.000	0.212
3000.0	0.211	0.002	0.000	0.213
2000.0	0.211	0.004	0.000	0.215
1000.0	0.205	0.015	0.000	0.221
700.0	0.197	0.030	0.000	0.227
500.0	0.181	0.054	0.000	0.235
300.0	0.140	0.116	0.001	0.257
200.0	0.096	0.182	0.005	0.282
100.0	0.037	0.287	0.029	0.354
80.0	0.026	0.319	0.052	0.396
60.0	0.016	0.363	0.106	0.486
40.0	0.009	0.453	0.305	0.767
30.0	0.006	0.563	0.693	1.261
25.0	0.005	0.674	1.222	1.901
20.0	0.004	0.902	2.645	3.551
15.0	0.004	1.574	8.698	10.275
12.5	0.004	2.678	22.35	25.03
10.0	0.006	7.369	103.4	110.8

Table 5. First Forbidden Unique M/L Ratios for Z=70

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.227	0.000	0.000	0.227
5000.0	0.227	0.001	0.000	0.228
3000.0	0.228	0.002	0.000	0.230
2000.0	0.229	0.005	0.000	0.234
1000.0	0.228	0.020	0.000	0.248
700.0	0.226	0.040	0.000	0.266
500.0	0.221	0.077	0.000	0.298
300.0	0.199	0.194	0.000	0.393
200.0	0.164	0.363	0.000	0.527
100.0	0.082	0.750	0.000	0.833
80.0	0.060	0.871	0.000	0.931
60.0	0.039	1.020	0.000	1.059
40.0	0.020	1.259	0.000	1.280
30.0	0.013	1.516	0.000	1.529
25.0	0.010	1.769	0.000	1.779
20.0	0.008	2.290	0.000	2.298
15.0	0.007	3.966	0.000	3.972
12.5	0.008	7.432	0.000	7.440

Table 6. Second Forbidden Unique M/L Ratios for Z=70

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.227	0.000	0.000	0.227
5000.0	0.228	0.001	0.000	0.228
3000.0	0.228	0.002	0.000	0.230
2000.0	0.227	0.005	0.000	0.232
1000.0	0.221	0.019	0.000	0.240
700.0	0.209	0.037	0.000	0.247
500.0	0.190	0.066	0.000	0.256
300.0	0.140	0.137	0.002	0.279
200.0	0.092	0.206	0.007	0.306
100.0	0.035	0.319	0.047	0.400
80.0	0.025	0.358	0.083	0.466
60.0	0.016	0.423	0.178	0.617
40.0	0.009	0.579	0.574	1.161
30.0	0.007	0.810	1.492	2.309
25.0	0.006	1.090	2.999	4.095
20.0	0.006	1.811	8.236	10.05
15.0	0.009	5.408	48.11	53.53
12.5	0.019	18.98	263.3	282.3

Table 7. First Forbidden Unique M/L Ratios for Z=80

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.232	0.000	0.000	0.232
5000.0	0.233	0.001	0.000	0.234
3000.0	0.234	0.002	0.000	0.236
2000.0	0.235	0.005	0.000	0.240
1000.0	0.236	0.022	0.000	0.258
700.0	0.234	0.045	0.000	0.279
500.0	0.229	0.086	0.000	0.315
300.0	0.206	0.219	0.000	0.425
200.0	0.169	0.408	0.000	0.576
100.0	0.083	0.837	0.000	0.920
80.0	0.061	0.977	0.000	1.037
60.0	0.039	1.166	0.000	1.205
40.0	0.021	1.536	0.000	1.557
30.0	0.015	2.037	0.000	2.051
25.0	0.012	2.642	0.000	2.655
20.0	0.012	4.314	0.000	4.325
15.0	0.022	17.58	0.000	17.60

Table 8. Second Forbidden Unique M/L Ratios for Z=80

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.232	0.000	0.000	0.233
5000.0	0.233	0.001	0.000	0.234
3000.0	0.234	0.002	0.000	0.237
2000.0	0.234	0.005	0.000	0.240
1000.0	0.228	0.021	0.000	0.249
700.0	0.217	0.042	0.000	0.258
500.0	0.196	0.074	0.001	0.270
300.0	0.143	0.152	0.003	0.298
200.0	0.094	0.228	0.011	0.332
100.0	0.036	0.365	0.070	0.471
80.0	0.026	0.422	0.129	0.578
60.0	0.017	0.528	0.296	0.842
40.0	0.011	0.840	1.129	1.980
30.0	0.010	1.441	3.670	5.120
25.0	0.011	2.407	9.308	11.72
20.0	0.016	6.389	41.90	48.30
15.0	0.119	106.0	1426.	1533.

Table 9. First Forbidden Unique M/L Ratios for Z=90

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.241	0.000	0.000	0.241
5000.0	0.242	0.001	0.000	0.243
3000.0	0.244	0.003	0.000	0.246
2000.0	0.245	0.006	0.000	0.251
1000.0	0.248	0.024	0.000	0.272
700.0	0.247	0.049	0.000	0.296
500.0	0.243	0.094	0.000	0.337
300.0	0.220	0.241	0.000	0.461
200.0	0.178	0.451	0.000	0.629
100.0	0.086	0.927	0.000	1.013
80.0	0.063	1.092	0.000	1.154
60.0	0.041	1.342	0.000	1.383
40.0	0.023	1.958	0.000	1.982
30.0	0.018	3.082	0.000	3.101
25.0	0.019	4.994	0.000	5.013

Table 10. Second Forbidden Unique M/L Ratios for Z=90

E_{EC}(KeV)	M_I+M_{II}/L	$M_{III}+M_{IV}/L$	M_V/L	M/L
10000.0	0.241	0.000	0.000	0.242
5000.0	0.243	0.001	0.000	0.244
3000.0	0.244	0.003	0.000	0.247
2000.0	0.245	0.006	0.000	0.251
1000.0	0.240	0.023	0.000	0.263
700.0	0.229	0.045	0.000	0.274
500.0	0.207	0.081	0.001	0.288
300.0	0.150	0.166	0.004	0.320
200.0	0.098	0.250	0.014	0.362
100.0	0.038	0.420	0.101	0.560
80.0	0.028	0.505	0.195	0.729
60.0	0.020	0.685	0.490	1.195
40.0	0.015	1.367	2.396	3.778
30.0	0.018	3.337	11.38	14.73
25.0	0.029	8.737	46.23	54.99

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*Abbreviations here follow the form of Science Abstracts, Section A, Physics Abstracts.

VITA

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