Conventional Multipliers for Homoclinic Orbits

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Abstract In this work we introduce and describe conventional multipliers, a new characteristic of homoclinic orbits of saddle-node type periodic trajectories. We prove existence and smooth dependence of conventional multipliers on the initial point. We show that multipliers of periodic trajectories arising from the homoclinic ones as a result of the saddle-node bifurcation are close to the conventional multipliers. As an application we study behavior of a circle map inside the 'Arnold tongues'.

1 Introduction

We consider in this work some bifurcation phenomena related to so called saddle-node bifurcations. Locally, this bifurcation looks very simple: two fixed (or periodic) points of a diffeomorphism fuse, forming a saddle-node type fixed (or periodic) point, and disappear. This is a codimension one bifurcation (See for example [AAIS]). But the trivial local reorganization can be accompanied by essential complications in the global behavior of trajectories. This is so, for example, if the saddle-node point at the bifurcation moment has homoclinic orbits. In this case, nontrivial invariant sets and even strange attractors may appear as a result of the bifurcation [Af74, ACL, AAIS, LS, NPT, Vi, Yo94]. In other words, diffeomorphisms F_μ may admit chaotic behavior for positive values of a bifurcation parameter μ, even if F_μ for $\mu < 0$ are of Morse-Smale type and at the bifurcation moment F_0 satisfies only generically true assumptions.

In order to formulate results (and even assumptions) we need to introduce some characteristics of homoclinic orbits. The conventional multiplier is such a characteristic. It is closely related to the multipliers of periodic orbits. We prove that the conventional multiplier of a homoclinic orbit, say Γ, is the limit of multipliers of periodic orbits, say L_{μ_k}, as $\mu_k \to 0$, where L_{μ_k} arise from Γ as a result of the bifurcation. This result is a partial manifestation of a more general principle. That is, behavior of trajectories of the diffeomorphism F_μ, $\mu > 0$, can be treated as a realization of possibilities contained in the map F_0. The description of the mechanisms of such a realization is the main goal of studying a bifurcation problem, and the conventional multiplier is a tool for such a description.

Conventional multipliers were introduced in [ACL] for the case that the fixed point is a stable node along hyperbolic variables and with the asumption of C^3-smoothness of the diffeomorphism. In this article we assume only C^2 smoothness extend the definition of the multiplier to the general case of a saddle along the hyperbolic directions. We show that the multiplier is an invariant with respect to C^1 conjugation along homoclinic orbits. In section 3 we show that knowledge of the conventional multiplier can be used to predict the creation of periodic orbits and that the multipliers of these orbits limit to the conventional multiplier of the homoclinic orbit.

An important confirmation of the advantage of conventional multipliers is contained in section 4 of our work related to the classical problem of circle maps. Circle diffeomorphisms arise naturally as the Poincare maps of toral flows. With the rotation number, which was defined by Poincare, as the major tool, the study of circle diffeomorphisms has developed into a very deep theory. Since the groundbreaking paper of Arnold [Ar], circle maps have drawn the attention of many scientists and mathematicians. Indeed, the phenomenom of 'Arnold tongues' is now widely used as a paradigm to explain frequency locking in oscillatory systems (See for example [Ba]). Our work contributes to this theory in that the conventional multiplier can be used to predict the structure within these intervals of phase locking. In particular, Theorem 6 shows how the multiplier can be used to fully describe dynamic structure within states of principal $(1/n)$ resonances.

In the case when the homoclinic orbits of F_0 form a smooth or nonsmooth circle, conventional multipliers determine a map of an interval or circle, which contains information about the behavior of F_μ for μ small enough. Such maps in different forms were introduced in [AShi, NPT, Pr, TS, IL, Jo] and others. All of these maps were constructed as limits as $\mu \to 0^+$. It follows from this work that the main features of those maps are completely determined by conventional multipliers, which are defined intrinsically at $\mu = 0$ (see section 5).

The techniques of our work allow us to study bifurcations in both finite and infinite dimensional phase space.

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2 Saddle-Node Fixed Points

Let F_μ be a one-parameter family of C^k-diffeomorphisms, $k \geq 2$, of a manifold $\mathbf{M}$, modeled on a Banach space $\mathbf{B}$, which is C^1 in μ in the C^k topology, $\mu \in [-\mu_0, \mu_0]$. For $\mu = 0$, F_μ has a nonhyperbolic fixed point $\bar{0}$ which is a uniform saddle or node along hyperbolic directions, that is, $DF_0(\bar{0})$ has spectrum with modulus uniformly bounded away from one, with the exception of one simple eigenvalue equal to one.

With these assumptions, there is a neighborhood U of $\bar{0}$ and C^k-smooth coordinates (x, y, z) on U, $x \in \mathbf{B}_1$, $y \in \mathbf{B}_2$, and $z \in \mathbf{R}$, where $\mathbf{B} = \mathbf{B}_1 \oplus \mathbf{B}_2 \oplus \mathbf{R}$ (either $\mathbf{B}_1$, or $\mathbf{B}_2$, or both may be empty), for which F_μ has the local form Φ_μ given by:

$$\begin{array}{rclcl} \bar{x} & = & Ax + f(x,y,z,\mu) & = & Ax + \tilde{f}(x,y,z,\mu)x \\ \bar{y} & = & Cy + g(x,y,z,\mu) & = & Cy + \tilde{g}(x,y,z,\mu)y \\ \bar{z} & = & z + R(z,\mu) + h(x,y,z,\mu), & & \end{array} \tag{1}$$

where the spectrum of A and C^{-1} satisfy $|\sigma(A)| < q_0$ and $|\sigma(C^{-1})| < q_0$, for some $q_0 < 1$. Without loss of generality, we will assume hereout that

$$\begin{array}{rcl} |A| & < & q_1 \\ |C^{-1}| & < & q_1, \end{array}$$

for $q_1 < 1$. The functions f, g and h are C^k-smooth in (x, y, z), and are C^1 in μ. The functions f, g, h and their derivatives vanish for $(x, y, z, \mu) = (0, 0, 0, 0)$ and the functions $\tilde{f}$ and $\tilde{g}$ vanish for $(0, 0, 0, 0)$, and $h(0, 0, z, \mu) \equiv 0$. This form is obtained by changing variables to rectify the local center stable manifold and the local center unstable manifold. We can assume without loss of generality that R in (1) has the following form

$$R(z, \mu) = \mu + \gamma(\mu)z^2 + o(z^2), \tag{2}$$

with $\gamma(0) > 0$. We will denote by ϕ the restriction of the map to the local center manifold, so that

$$\phi(z, \mu) = z + R(z, \mu).$$

A stronger form of (1) may be obtained in the finite-dimensional case, with the restriction of C^∞-smoothness of the diffeomorphism and certain nonresonance conditions [Ta71].

For maps of the form (1), the planes $x = 0$ and $y = 0$ contain smooth invariant foliations, which we may take to have coordinates $z = constant$ (See [NPT, Af90]). Thus $h(0, y, z, \mu) = 0$ and $h(x, 0, z, \mu) = 0$. It has recently been shown that these foliations are C^k-smooth with respect to all variables [LS, Yo93]. In these coordinates it can be seen that all the points in the 'half plane', $y = 0$ and $z \leq 0$, limit to $\bar{0}$ under forward iterations. This set is known as the 'local stable set' of $\bar{0}$. The set of all points which limit to $\bar{0}$ under forward iteration, which is an extension of the local stable set, is called the stable set and is denoted by S^s. Similarly, the half plane, $x = 0$, $z \geq 0$ is known as the local unstable set. Its extension, S^u, is known as the unstable set of $\bar{0}$. For $\mu = 0$ the foliations are unique on S^s and S^u. All points of intersection of S^s and S^u are called homoclinic points, since both their forward and backward iterations under F_0 limit to $\bar{0}$. We will only be concerned with transversal intersections of these sets. (For the definition of transversal intersection in the infinite dimensional case see for example [La].)

We will repeatedly use the following result about orbits $\{z_i\}$ on the center manifold of a saddle-node point.

Lemma 1 *Let $z_0 < 0$ and $z_{n+1} = \phi(z_n, 0) \equiv z_n + R(z_n, 0)$, where R is as in (2). Then there is a constant ϵ such that $\gamma > \epsilon > 0$ and*

$$0 < -z_n < \frac{1}{(\gamma - \epsilon)n}$$

for all $n > 0$.

See section 7.1 for a proof of this lemma.

3 Conventional Multipliers of Homoclinic Points

Definition 1 *For $\mu = 0$, let $\bar{X}$ be a point of transversal intersection of S^s and S^u. This implies that $\bar{X}$ is part of an isolated, smooth, one-dimensional curve segment γ, of homoclinic points. If γ is transversal at $\bar{X}$ to the invariant foliations on both S^s and S^u, we will then call the point $\bar{X}$ a* **noncritical homoclinic point**. *If $\bar{X}$ is a point of transversal intersection, but γ is tangent at $\bar{X}$ to the foliation on S^s (S^u), then $\bar{X}$ is called a* **regular s-critical** (**regular u-critical**) *homoclinic point.*

Below we will consider only regular s-critical and noncritical homoclinic points. The case of a u-critical point can be considered by using the inverse of the original map. The case of a point which is both s-critical and u-critical belongs to the set of co-dimension 2 bifurcations and lies beyond the scope of this article.

Let $\bar{X}$ be a noncritical or regular s-critical homoclinic point and let $\{\bar{X}_i\}_{i=-\infty}^{\infty}$ and $\{\gamma_i\}_{i=-\infty}^{\infty}$ be the orbits of $\bar{X}$ and γ under F_0. Fix a small neighborhood U of $\bar{0}$ and a small positive number ℓ. Let i_ℓ^- be the smallest integer such that $\bar{X}_i \in U$ for all $i \geq i_\ell^-$, and such that $\bar{X}_\ell^- \equiv \bar{X}_{i_\ell^-}$ has local z-coordinate, z_ℓ^-, greater than $-\ell$. Let i_ℓ^+ be the largest integer such that $\bar{X}_i \in U$ for all $i \leq i_\ell^+$, and such that $\bar{X}_\ell^+ \equiv \bar{X}_{i_\ell^+}$ has local z-coordinate, z_ℓ^+, less than ℓ. Let $\mathcal{F}_\ell$ be the restriction of $F_0^{i_\ell^- - i_\ell^+}$ which maps a neighborhood of $\bar{X}_\ell^+$ to a neighborhood of $\bar{X}_\ell^-$. Let π be the projection onto the z-axis and let π_ℓ^+ be defined in neighborhood of $\bar{X}_\ell^+$ as the restriction of π to $\gamma_{i_\ell^+}$. Since $\bar{X}$ is not u-critical, π_ℓ^+ is invertible. Let $\bar{\mathcal{F}}_\ell = \pi \circ \mathcal{F}_\ell \circ (\pi_\ell^+)^{-1}$, and

$$b_\ell(\bar{X}) = \bar{\mathcal{F}}_\ell'(\bar{X}_\ell^+) = \frac{dz_\ell^-}{dz_\ell^+},$$

and consider the following limit

$$b(\bar{X}) = \lim_{\ell \to 0} b_\ell(\bar{X}) = \lim_{\ell \to 0} \frac{dz_\ell^-}{dz_\ell^+}. \tag{3}$$

Theorem 1 . *The limit $b(\bar{X})$ exists, is independent of the choice of local coordinates, and is continuously differentiable with respect to $\bar{X}$ along γ. It is zero for the s-critical case and nonzero for the noncritical case.*

It follows from the definition of $b(\bar{X})$ that this number depends only on the trajectory and is independent of the choice of the point on the trajectory.

Definition 2 *The limit $b(\bar{X})$ of* (3) *is called the* **conventional multiplier** *of the homoclinic trajectory which contains the point $\bar{X}$.*

In section 7.2 we give a proof of the theorem using the following construction and lemma. Denote by ϕ the restriction of F_μ to the local center manifold, so that $\phi(z,\mu) = z + R(z,\mu)$. Choose $\alpha_0 < 0$ and let $\alpha_{i+1} = \phi(\alpha_i, 0)$ and $I_i^- = [\alpha_i, \alpha_{i+1}]$. Then given $z_0 \in I_0^-$ let $z_i = \phi^i(z_0, 0)$ and for each i let u_i be the ratio

$$u_i = \frac{z_i - \alpha_i}{\alpha_{i+1} - \alpha_i}.$$

Consider the limit

$$u(z_0) = \lim_{i \to \infty} u_i.$$

Lemma 2 *The limit $u(z_0)$ exists and is independent of the choice of local coordinates. If $u(z_0)$ and $\bar{u}(z_0)$ are defined as above for different choices α_0 and $\bar{\alpha}_0$ respectively, then there exists a constant $c(\alpha_0, \bar{\alpha}_0)$ independent of z_0 such that $u(z_0) = \bar{u}(z_0) + c(\alpha_0, \bar{\alpha}_0)$. Furthermore, the sequence $\{u_i(\cdot)\}$ converges in the C^2 topology on the interval $[\alpha_0, \alpha_1]$.*

The proof of this lemma is in section 7.3. This lemma will also be used below in the proof of Theorem 3. Note that a similar limit exists for points with positive z-coordinate as they limit to the fixed point under iteration of the inverse of ϕ. Due to Lemma 2 we can study the map ϕ in u-coordinates on the standard interval I.

The conventional multiplier $b(\bar{X})$ is a C^1 invariant in the following sense.

Proposition 1 *Let Γ and Δ be homoclinic trajectories of saddle-node points of maps F_0 and G_0 respectively. If there is a C^1 diffeomorphism $h : U \to V$, where U and V are open neighborhoods of Clos(Γ) and Clos(Δ) such that*

$$h \circ F_0|_U = G_0 \circ h|_U,$$

and h takes points on Γ to points on Δ, then $b(\Gamma) = b(\Delta)$.

The proof is in section 7.4.

4 The Multiplier and Creation of Periodic Points

In this section we discuss the most basic application of the conventional multiplier. The multiplier is shown to predict the occurrence of hyperbolic fixed points for a sequence of intervals of the bifurcation parameter greater than the bifurcation value. This is accomplished even for the case where the saddle-node point is a saddle along hyperbolic directions.

Theorem 2 *Suppose that $\bar{X}_0$ is a noncritical homoclinic point such that $b(\bar{X}_0) \neq 1$. There exist integers j_0 and m and a sequence of parameter values $\{\mu_j\}_{j=j_0}^{\infty}$, decreasing to zero, such that F_{μ_j} has a hyperbolic periodic point $\bar{X}_j^*$ of minimal period $m + j$. Furthermore, $\bar{X}_j^* \to \bar{X}_0$ as $j \to +\infty$, $DF_\mu^{j+m}|_{\bar{X}_j^*}$ has an invariant splitting*

$$TM_{\bar{X}_j^*} = E_s \oplus E_u \oplus E_c,$$

and the spectrum of this fixed point decomposes along this splitting in the following way:

$$\sigma(F_\mu^{j+m}|_{\bar{X}_j^*}) = \sigma_s \cup \sigma_u \cup (b(\bar{X}_0) + b_1),$$

where b_1 goes to 0 as j goes to $+\infty$. So, the multiplier in the direction of the subspace E^c almost coincides with $b(\bar{X})$. Also, $|\sigma_s| < a_s$ and $|\sigma_u| > a_u$ where $a_s \to 0$ as $j \to \infty$ and $a_u \to +\infty$ as $j \to \infty$.

See section 7.5 for a proof of this theorem.

Key to the proof of this theorem is the following lemma.

Lemma 3 *For given ℓ and μ, and any orbit $\{z_i\}$ along the center manifold, let i^- and i^+ be as before so that*

$$\ldots < z_{i^- -1} < -\ell < z_{i^-} < \ldots < z_{i^+} < \ell < z_{i^+ +1} < \ldots .$$

As ℓ and μ approach zero,

$$|\prod_{i=i^-}^{i^+-1} \phi'(z_i, \mu_j) - 1| = O(\ell + \sqrt{\mu}).$$

The proof is in [Af74] and some of the ideas under different assumptions were described in [AShi, AShe, IY] and others. We sketch it in section 7.6 for the sake of completeness. This key lemma will also be used in the proofs of Theorems 3, 4 and 5.

5 The Multiplier and Circle Maps

Let f_μ be a family of diffeomorphisms of a circle $\mathbf{S^1}$ with a saddle-node fixed point $\bar{0}$. Such a family can arise from a family of higher dimensional maps F_μ in the following way. Assume that F_μ has a saddle-node fixed point $\bar{0}$ as above, and that this fixed point has a connected, noncritical curve, Γ_0, of homoclinic orbits. It has been shown [Yo94] that after the saddle-node bifurcation $\mu > 0$, there is created a family of C^k-smooth invariant curves, Γ_μ, which limit to Γ_0 in the C^k topology as μ goes to 0. Thus, the restrictions of F_μ to the invariant curves are smoothly equivalent to C^k maps of the circle. In this section we study families of diffeomorphisms of the circle directly.

Generically, f_μ is given in a neighborhood U of the fixed point by a map

$$\bar{z} = \phi_\mu(z) = z + R(z, \mu), \tag{4}$$

where R has the form (2). Now we fix fundamental domains on S^1. First choose $\alpha^+ > 0$, such that $\beta^+_\mu = \alpha^+ + R(\alpha^+, \mu)$ is inside U for all μ. Denote by P_1 and P_2 the points on $\mathbf{S^1}$ with coordinates α^+ and β^+_μ respectively. Next fix m to be the smallest integer such that $f_\mu^m(P_1) \in U$ for all $0 \le \mu < \mu_0$, but $f_\mu^{m-1}(P_1) \notin U$ for some $0 \le \mu < \mu_0$. Let $Q_1 = f_0^m(P_1)$ and let α^- be the local z coordinate of Q_1. Finally, let $\beta_\mu^- = \alpha^- + R(\alpha^-, \mu)$. The fundamental domains we set to be $I_\mu^+ = [\alpha^+, \beta_\mu^+]$ and $I_\mu^- = [\alpha^-, \beta_\mu^-]$. Note that $I^+ = I_0^+ \subset I_\mu^+$ for each $0 \le \mu$, and likewise, $I^- = I_0^- \subset I_\mu^-$. Now for each μ, f_μ uniquely defines a C^k-smooth map $\kappa_{gl,\mu}$ from I_μ^+ to I_μ^-, by taking the image of $z \in I_\mu^+$ to be the first point in the forward trajectory of that point to intersect I_μ^-. It is clear that $\kappa_{gl,\mu}$ limits in the C^k sense to $\kappa_{gl} \equiv \kappa_{gl,0} \equiv f_0^m|_{I_0^+}$ as $\mu \to 0$. The definitions of the fundamental regions I^+ and I^- allow us to treat $\kappa_{gl,\mu}$ as a map from the circle $\mathbf{S^1}_+ \equiv I^+/\alpha^+ \sim \beta^+$ to the circle $\mathbf{S^1}_- \equiv I^-/\alpha^- \sim \beta^-$.

Next we will construct a family of maps from I_μ^- to I_μ^+ using formula (4). For any point $z^* \in I^+$ there exists an integer j_0 and a sequence $\{\mu_j(z^*)\}_{j=j_0}^\infty$, such that $\mu_{j_0} > \mu_{j_0+1} > \ldots > \mu_j > \ldots > 0$, $\mu_j \to 0$ as $j \to 0$ and

$$\phi_{\mu_j}^j(\beta^-) = z^*.$$

Let $\phi_{\mu_j}^{-j}(\alpha^+) = q_j^*$. We set

$$\psi_j(q) = \begin{cases} \phi_{\mu_j}^{j+1} & \text{for} \quad q \in [\alpha^-, q_j^*) \\ \phi_{\mu_j}^j & \text{for} \quad q \in [q_j^*, \beta^-] \end{cases}$$

From this definition, $\psi_j(q) \in I^+$ for $q \in I^-$. The following statement holds (see [Af74], [NPT], [AShe], [TS] and [IL]).

Theorem 3 *For each point $q \in I^-$ the following limit exists:*

$$\lim_{j\to\infty} \psi_j(q) \equiv \kappa_{lc}(q, z^*) \in I^+.$$

Furthermore, $\kappa_{lc}(\alpha^-) = \kappa_{lc}(\beta^-)$, $\lim_{q\to q^{*+}} \kappa_{lc}(q) = \alpha^+$, $\lim_{q\to q^{*-}} \kappa_{lc}(q) = \beta^+$ and κ_{lc} is one-to-one.

For the sake of convenience, we provide a new proof in section 7.7 based on Lemmas 1 and 2. In this way, the map $\kappa_{lc} : I^- \to I^+$ is defined for each $z^* \in I^+$, and so we obtain a one-parameter family of maps

$$\kappa_{lc}(\cdot, z^*) : I^- \to I^+,$$

where $z^* \in I^+$. Because of the properties of κ_{lc} outlined in Theorem 3, we can treat it as a map from the circle $S_-^1 \equiv I^-/\alpha^- \sim \beta^-$ to $S_+^1 \equiv I^+/\alpha^+ \sim \beta^+$. The composition

$$\kappa_{z^*} = \kappa_{lc} \circ \kappa_{gl} : I^+ \to I^+$$

is a one-parameter family of maps of an interval that we will show to give the main information about bifurcations and the structure of limit sets of F_μ on Γ_μ. The following result is important, particularly from the standpoint of bifurcations.

Theorem 4 *If ϕ_μ is a family of C^2 diffeomorphisms then the sequence of maps ψ_j converges in the C^2 topology, when considered as maps of the circles.*

Thus the return map $\kappa_\mu \equiv \kappa_{lc,\mu} \circ \kappa_{gl,\mu}$ converges to κ_{z^*} in the C^2 topology.

The family of limit maps $\kappa_{z^*} \equiv \kappa_{lc} \circ \kappa_{gl}$ is intimately associated with the conventional multiplier, as summarized by the following. Instead of considering only κ defined for a fixed fundamental domain with end point α^+, let κ_{α^+,z^*} be the family of maps defined on $[\alpha^+, \beta^+]$ where α^+ is allowed to vary.

Theorem 5 *Given $\epsilon > 0$, there exists α^+ small enough that*

$$|\kappa'_{\alpha^+}(z) - b(z)|_1 \leq \epsilon,$$

on the interval $[\alpha^+, \beta^+]$ and $|\cdot|_1$ denotes the usual C^1 norm.

See section 7.9 for the proof.

Before we apply the conventional multiplier to 'Arnold tongues' of circle maps we first review some classical results about circle maps. (See for example [Ni]).

Definition 3 *Let $\phi : \mathbf{S^1} \to \mathbf{S^1}$ be a diffeomorphism and $\Phi : \mathbf{R} \to \mathbf{R}$ be the lift of ϕ to the real line. The rotation number, $\rho(\phi)$, is defined as*

$$\rho(\phi) = \lim_{i \to +\infty} \frac{1}{i} \Phi^i(x_0).$$

The following result is the fundamental theorem concerning rotation numbers.

1 *The rotation number $\rho(\phi)$ is well-defined, that is, the limit exists, and is independent of the initial point x_0, Furthermore,*

1. *$\rho(\phi)$ is rational if and only if ϕ has a periodic orbit.*
2. *If ϕ is C^1 and ϕ' is of bounded variation, then $\rho(\phi)$ is irrational if and only if every orbit of ϕ is dense in $\mathbf{S^1}$.*

Denjoy [De] produced an example which demonstrates the remarkable fact that if ϕ' is not of bounded variation then statement *2* does not hold. Another fundamental result is the following.

2 *If ϕ_μ is a Lipschitz family of C^2 diffeomorphisms then the rotation number $\rho(\phi_\mu)$ is a continuous function of the parameter μ.*

It is also easy to show that in our case, $\rho(\phi_\mu)$ is monotone in μ.

Now we formulate the first application of conventional multipliers to circle maps. Given any integer j, let $[\underline{\mu_j}, \overline{\mu_j}]$ be the maximal interval for which $\rho(\phi_\mu) = 1/j$. Such an interval exists by the previous remarks. Now define the function

$$\mathcal{K}(z) \equiv \alpha^+ + \int_{a^+}^{z} b(z)dz,$$

where as before we consider this as a diffeomorphism of $S^1_+ \equiv I^+/\alpha^+ \sim \beta^+$ onto itself. This is possible since $b(z)$ is periodic on S^1_+, and moreover, it follows from the definition of b that the integral of b over a fundamental domain is the length of the domain. For each μ let $z^*(\mu)$ be the first image in I^+ of α^+, and denote by κ_μ the composition $\psi_j(\mu) \circ \kappa_{gl,\mu}$. The following is an almost immediate consequence of Theorems 4 and 5.

Lemma 4 *There exists a C^2 function $G_{\alpha^+}(z,\mu)$ which is arbitrarily small in the C^2 norm for α^+ and μ sufficiently small such that*

$$\kappa_{\alpha^+,\mu}(z) = \mathcal{K}(z) + z^*(\mu) - \alpha^+ + G_{\alpha^+}(z,\mu).$$

The next theorem, which is our main result about circle maps, is a consequence of this lemma.

Theorem 6 *Suppose that $\mathcal{K}(z) - z$ has only nondegenerate critical points $\{z_i\}_{i=1}^n$ and that $\mathcal{K}(z_i) - \mathcal{K}(z_j) \neq z_i - z_j$, for any distinct z_i and z_j in this set. Then there exists an integer j_0 such that for each $j \geq j_0$, there is a monotone map $\alpha_j(\mu) : [\underline{\mu_j}, \overline{\mu_j}] \to [-\beta^+ + \alpha^+, \beta^+ - \alpha^+]$ such that the map $\kappa_\mu(z)$ is topologically conjugated to the map $\mathcal{K}(z) + \alpha_j(\mu)$.*

A critical point, z_0, i.e. $\mathcal{K}'(z_0) = 1$, is nondegenerate if $\mathcal{K}''(z_0) \neq 0$. We note that the condition of nondegeneracy implies that the critical points are finite in number and without this condition, the theorem fails. By topological conjugacy, we mean that there exists a homeomorphism $h : S^1_+ \to S^1_+$ such that

$$h \circ \kappa_\mu = ((\mathcal{K} + \alpha_j(\mu)) \circ h).$$

The meaning of Theorem 6 in terms of the original map is the following one. On the interval $[\underline{\mu_j}, \overline{\mu_j}]$ the j-periodic orbits of the diffeomorphism F_μ are in direct correspondence with the fixed points of $\mathcal{K} + \alpha_j(\mu)$. We know also from section 4 that the multipliers of the orbits are close to $\mathcal{K}'$ at the respecive fixed points. The proofs of Lemma 4 and Theorem 6 are in section 7.10.

6 Concluding Remarks

Afraimovich and Shil'nikov [Af74], Newhouse *et al.* [NPT], Pryzyticky [Pr], Turaev and Shil'nikov [TS], Il'yashenko and Li [IL], Jonker [Jo], and others define families of maps on intervals or circles which are related to κ_μ above. All of these maps correspond in a sense to homoclinic bifurcations of a saddle-node point. Their chief goal was to study the bifurcation of a critical saddle-node in the case of a single, connected curve of homoclinic points which has tangencies to the stable foliation of the stable set. Using the uniqueness of the stable foliation on S^s and the contractivity in the x directions, the construction of the previous section may be repeated, and it could be shown that the conventional multiplier is related to the derivative of these correspondence maps. If tangencies to the foliation exist, then the resulting map will not be a diffeomorphism, but an endomorphism (See [NPT, Ta89]). Study of these problems is beyond the scope of the present paper.

Conventional multipliers show us the contraction or extension rates in the central direction. Using this property, conditions of existence of Lorenz-type attractors were derived in [ACL].

Among the open questions, let us mention the following. Suppose the saddle-node fixed point is a saddle along hyperbolic directions and there exists a curve of homoclinic points which has tangencies to the foliations of both S^s and S^u. The conventional multiplier as a function of the z-coordinate will be multi-valued, but integrable as such. Given a fundamental domain on say S^u, the integral of the conventional multiplier defines a new family of multivalued correspondence maps on an interval. The question is: beyond the results implied by section 4, how can information contained in this map be used to describe the bifurcations which occur in a neighborhood of the original curve for $\mu > 0$? Some discussion of this problem is contained in [Pr].

It seems to us that conventional multipliers should be helpful in considering the case of rational rotation numbers p/q (instead of $1/n$) and we hope to study this elsewhere. [Jo]

Finally, the conventional multiplier seems to be a new tool for use in the classification of smooth equivalence of one-dimensional diffeomorphisms. Consider, for instance, the simple case in which two diffeomorphisms of circles have single saddle-node fixed points. As noted in Proposition 1, if the diffeomorphisms are smoothly conjugate, then corresponding orbits have identical multipliers. Elsewhere, we plan to study appropriate conditions under which converse statements may be made.

7 Proofs

7.1 Proof of Lemma 1

The proof of Lemma 1 is based on the following (See [ACL]).

Proposition 2 *Assume*

$$u_n > u_{n-1}^2 + \frac{1}{4}, \qquad 0 < u_1 < \frac{1}{2},$$

then $u_n \geq \frac{1}{2} - \frac{1}{n}$, *for any* $n \geq 1$.

Proof. We write down the recurrent formula for $\frac{1}{2} - \frac{1}{n}$.

$$\frac{1}{2} - \frac{1}{n} = (\frac{1}{2} - \frac{1}{n-1})^2 + \frac{1}{4} - \frac{1}{n(n-1)^2}.$$

So

$$\begin{aligned} u_n - (\tfrac{1}{2} - \tfrac{1}{n}) &> [u_{n-1} - (\tfrac{1}{2} - \tfrac{1}{n-1})][u_{n-1} + \tfrac{1}{2} - \tfrac{1}{n-1}] + \tfrac{1}{n(n-1)^2} \\ &> [u_{n-1} - (\tfrac{1}{2} - \tfrac{1}{n-1})][u_{n-1} + \tfrac{1}{2} - \tfrac{1}{n-1}]. \end{aligned}$$

So if $u_2 > 0$, which is always true, then by induction $u_n > \frac{1}{2} - \frac{1}{n}$, $\forall n \geq 3$.□

Suppose that $u_n > u_{n-1} + \beta u_{n-1}^2$, $u_1 < 0$, then $u_n > -\frac{1}{\beta n}$, if it is true for u_1. Now consider the asymptotic behavior of z_n^- under ϕ. We have $z_n^- = z_{n-1}^- + \gamma(z_{n-1}^-)^2 + o((z_{n-1}^-)^2)$. Since $\gamma > 0$, there exists $\gamma > \epsilon > 0$ such that $|o(z^2)| < \epsilon z^2$, so $z_n^- > z_{n-1}^- + (\gamma - \epsilon)(z_{n-1}^-)^2$. If we let $u_n = (\gamma - \epsilon) z_n + \frac{1}{2}$, then

$$u_n > u_{n-1}^2 + \frac{1}{4}.$$

Applying Proposition 2, we have proved the lemma.

7.2 Proof of Theorem 1

Denote by $\phi(z, \mu) \equiv z + R(z, \mu)$ the restriction of the map to the z-axis in a neighborhood of the fixed point. It can be easily shown for $\mu = 0$ that $\phi^{-1}(z) = z - \gamma z^2 + o(z^2)$, and we denote $R^-(z) = -\gamma z^2 + o(z^2)$. Since we are only considering the case $\mu = 0$ in this section, we will omit the parameter without ambiguity (i.e. $\phi(z) \equiv \phi(z, 0)$). Fix $\alpha_0^- < 0$ and $\alpha_0^+ < 0$ and let $\alpha_{i+1}^- = \phi(\alpha_i)$ and $\alpha_{i+1}^+ = \phi^{-1}(\alpha_i)$. For a given homoclinic orbit $\{\bar{X}_j\}_{j=-\infty}^{+\infty}$ let $\bar{X}_{j^-}$ and $\bar{X}_{j^+}^+$ be the elements of the orbit such that their z coordinates z_0^- and z_0^+, respectively satisfy

$$\alpha_0^- \leq z_0^- < \alpha_1^- \quad \text{and} \quad \alpha_1^+ < z_0^+ \leq \alpha_0^+,$$

and set $z_{i+1}^- = \phi(z_i^-)$ and $z_{i+1}^+ = \phi^{-1}(z_i^+)$. For each small $\ell > 0$ let $i^-(\ell)$ and $i^+(\ell)$ be the integers such that

$$-\ell \leq z_{i^-(\ell)}^- < \phi(-\ell) \quad \text{and} \quad \phi^{-1}(\ell) < z_{i^+(\ell)}^+ \leq \ell.$$

Simplifying notation, let $z_\ell^- \equiv z_{i^-(\ell)}^-$, $z_\ell^+ \equiv z_{i^+(\ell)}^+$, $\alpha_\ell^- \equiv \alpha_{i^-(\ell)}^-$, and $\alpha_\ell^+ \equiv \alpha_{i^+(\ell)}^+$.

Now let $u_\ell^- : [\alpha_\ell^-, \phi(\alpha_\ell^-)] \to [0,1]$ and $u_\ell^+ : [\phi^{-1}(\alpha_\ell^+), \alpha_\ell^+] \to [0,1]$ be affine maps as in Lemma 2. The induced map $\tilde{\mathcal{F}} : [0,1] \to [0,1]$ defined by

$$\tilde{\mathcal{F}}_\ell = u_\ell^- \circ \bar{\mathcal{F}}_\ell \circ (u_\ell^+)^{-1}$$

is a well-defined, smooth map. Further,

$$\tilde{\mathcal{F}}_\ell' = \frac{R^-(\alpha_\ell^+)}{R(\alpha_\ell^-)} \cdot \bar{\mathcal{F}}_\ell'.$$

It can be easily shown that

$$\frac{R^-(\alpha_\ell^+)}{R(\alpha_\ell^-)} \to 1$$

as ℓ goes to 0, and so we need only study the limit of $\tilde{\mathcal{F}}_\ell'$.

Denote by G the restriction of the map $F^{j_0^- - j_0^+}$ from a neighborhood of $\bar{X}_{j_0^+}^+$ to a neighborhood of $\bar{X}_{j_0^-}$. Let $\bar{G} = \pi \circ G \circ (\pi^+)^{-1}$, then

$$\tilde{\mathcal{F}}_\ell = \xi_\ell^- \circ \bar{G} \circ (\xi_\ell^+)^{-1},$$

where ξ_ℓ^- and ξ_ℓ^+ are defined at z_0^- and z_0^+ by

$$\xi_\ell^-(z_0^-) = u_\ell^-(z_\ell^-) \quad \text{and} \quad \xi_\ell^+(z_0^+) = u_\ell^+(z_\ell^+).$$

Now by Lemma 2, ξ_ℓ^- and ξ_ℓ^+ converge in the C^2 topology (ξ_ℓ^+ is the affine map composed with iterations of ϕ^{-1}), and so the limits $\tilde{\mathcal{F}}_\ell'$ and $\bar{\mathcal{F}}_\ell'$ exist and are continuously differentiable.

7.3 Proof of Lemma 2

First, we will simultaneously show that the limit of $u_i(z_0)$ exists as $i \to \infty$ and that the convergence is in the C^2 topology. Afterwards, we prove the other assertions about the limit. Again, we are only considering the case $\mu = 0$, so throughout this section we will omit the parameter from our notation in order to save space.

7.3.1 Existence and smoothness

In order to show that the sequence $\{u_i(\cdot)\}$ converges in the C^2 sense, consider the fact that $u_i(\alpha_0) \equiv 0$ and $u_i(\alpha_1) \equiv 1$ for all i. Because of these boundary conditions, if we can show that the second derivatives of $u_i(z)$ converge uniformly, then we will have proven C^2 convergence on $[\alpha_0, \alpha_1]$.

The first derivative of u_i is given by

$$\begin{aligned} \frac{du_i}{dz_0} &= \frac{d}{dz_0}\frac{\phi^i(z_0)-\alpha_i}{\alpha_{i+1}-\alpha_i} \\ &= \frac{1}{R(\alpha_i)}\prod_{j=0}^{i-1}\phi'(z_j). \end{aligned}$$

Let $n > m$ be large integers. Then

$$\begin{aligned} \frac{du_m}{dz_0}-\frac{du_n}{dz_0} &= \frac{1}{R(\alpha_m)}\prod_{j=0}^{m-1}\phi'(z_j) - \frac{1}{R(\alpha_n)}\prod_{j=0}^{n-1}\phi'(z_j) \\ &= A\cdot B, \end{aligned}$$

where

$$A = \frac{1}{R(\alpha_m)}\prod_{j=0}^{m-1}\phi'(z_j)$$

and

$$B = 1 - \frac{R(\alpha_m)}{R(\alpha_n)}\prod_{j=m}^{n-1}\phi'(z_j).$$

Differentiating again, we find that

$$\frac{d^2u_m}{dz_0^2}-\frac{d^2u_n}{dz_0^2} = A'\cdot B + A\cdot B'.$$

By the above remarks, we need to show that this expression is small for m sufficiently large and $n > m$, uniformly on $[\alpha_0, \alpha_1]$.

We begin by studying A. Taking the natural logarithm of the product we obtain

$$\ln\prod_{j=0}^{m-1}\phi'(z_j) = \sum_{j=0}^{m-1}\ln(1+R'(z_j)).$$

Using $z_{j+1}-z_j = R(z_j)$,

$$S_m = \sum_{j=0}^{m-1}\ln(1+R'(z_j)) = \sum_{j=0}^{m-1}\frac{\ln(1+R'(z_j))}{R(z_j)}(z_{j+1}-z_j).$$

This is a Reimann sum for the integral

$$I_m = \int_{z_0}^{z_m} \frac{\ln(1+R'(z))}{R(z)} dz. \tag{5}$$

Note that the integrand is eventually monotone increasing. If we denote by E_j the difference between the the j-th term of the sum and the integral from z_j to z_{j+1}, then by the monotonicity

$$\begin{aligned} E_j &\leq |z_{j+1} - z_j| \left| \frac{\ln(1+R'(z_{j+1}))}{R(z_{j+1})} - \frac{\ln(1+R'(z_j))}{R(z_j)} \right| \\ &\leq |z_{j+1} - z_j|^2 \max_{z_j \leq z \leq z_{j+1}} \left(\frac{\ln(1+R'(z))}{R(z)} \right)' \\ &\leq \frac{R(z_j)^2}{R(z_{j+1})^2} \max_{z_j \leq z \leq z_{j+1}} |R(z)R''(z) - \ln(1+R'(z))R'(z)(1+R'(z))| \end{aligned}$$

Now $R(z_j)^2/R(z_{j+1})^2$ approaches 1 as $z_j \to 0$ and so

$$E_j \leq k z_j^2$$

for some constant k. Using Lemma 1, $\sum_{j=j_0}^{\infty} E_j$ converges as a geometric series, and so $|I_m - S_m| = O(z_0)$, uniformly in m. The integral I_m can be represented as

$$I_m = \int_{z_0}^{z_m} \frac{1}{R} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (R')^n dz.$$

The first term is

$$\int_{z_0}^{z_m} \frac{R'}{R} dz = \ln \frac{R(z_m)}{R(z_0)},$$

and the n-th term is

$$\int_{z_0}^{z_m} r_n(z) dz,$$

where

$$r_n(z) = \frac{(-1)^{n+1}}{n} \cdot \frac{(R'(z))^n}{R(z)}, \quad n \geq 2.$$

To estimate this integral, first find the maximal value of $|r_n(z)|$ on the interval $[z_0, z_m]$. Each term takes on the maiximal value at the end point z_0 and in fact

$$|r_n(z)| = \frac{1}{n} \frac{|(R')^{n-2}| \cdot (R')^2}{R} \leq \frac{1}{n} A_1 (b \cdot z_0)^{n-2},$$

where we may choose b and A so that $|R'| \leq |2\gamma z + o(z)| \leq b \cdot z_0$ and

$$\frac{(R')^2(z_0, \mu)}{R(z_0, \mu)} = \frac{4\gamma^2 z_0^2 + o(z_0)}{\gamma z_0^2 + \mu + o(z_0)} \leq A.$$

Now

$$\begin{aligned} I_i &\leq \ln \frac{R(z_i)}{R(z_0)} + \int_{z_0}^{z_i} \sum_{n=2}^{\infty} A \cdot (b \cdot z_0)^{n-2} dz \\ &\leq \ln \frac{R(z_i)}{R(z_0)} + A \frac{2z_0}{1 - bz_0}; \end{aligned}$$

and similarly

$$I_i \geq \ln \frac{R(z_i)}{R(z_0)} - A \frac{2z_0}{1 - bz_0}.$$

Therefore,

$$A = \frac{1}{R(\alpha_m)} \frac{R(z_m)}{R(z_0)} (1 + O(z_0)).$$

Now

$$z_i - R(z_i) < \alpha_i < z_i,$$

so that

$$\frac{R(z_i)}{\alpha_i} = \frac{z_i^2 + o(z_i^2)}{z_i^2 + o(z_i^2)}.$$

This ratio goes to 1 as i goes to $+\infty$ and so the factor A is uniformly bounded.

Next consider A'. We have

$$A' = \frac{1}{R(\alpha_m)} \frac{d}{dz_0} \prod_{j=0}^{m-1} (\phi'(z_j)).$$

Now

$$\begin{aligned} \frac{d}{dz_0} \prod_{j=0}^{m-1} (1 + R'(z_j)) &= \phi''(z_{m-1})(\phi'(z_{m-2}) \cdots \phi'(z_0))^2 \\ &\quad + \phi'(z_{m-1})\phi''(z_{m-2})(\phi'(z_{m-3}) \cdots \phi'(z_0))^2 \\ &\quad + \cdots \\ &\quad + \phi'(z_{m-1}) \cdots \phi'(z_2)\phi''(z_1)\phi'(z_0)^2 \\ &\quad + \phi'(z_{m-1}) \cdots \phi'(z_1)\phi''(z_0). \end{aligned}$$

We may uniformly bound $\phi''(z_j)$ by some constant K and so

$$A' = \frac{K}{R(\alpha_m)} \phi'(z_{m-1}) \cdots \phi'(z_0) \cdot S_m,$$

where

$$\begin{aligned} S_m &= \frac{\phi'(z_{m-1})}{\phi'(z_{m-2})} \phi'(z_{m-3}) \cdots \phi'(z_0) + \frac{\phi'(z_{m-2})}{\phi'(z_{m-3})} \phi'(z_{m-4}) \cdots \phi'(z_0) \\ &\quad + \cdots + \frac{\phi'(z_2)}{\phi'(z_1)} \phi'(z_0) + \frac{1}{\phi'(z_0)}. \end{aligned}$$

As j becomes large $\phi'(z_j)/\phi'(z_{j+1}$ approaches 1. We have already shown that

$$\phi'(z_{j-3})\cdots\phi'(z_0) \leq k\frac{R(z_{j-3})}{R(z_0)}$$

as j approaches ∞, so the j term in the sum S_m is of the order z_j^2. By Lemma 1 the sum is bounded.

Next consider the factor B. The logarithm of the product

$$\ln \prod_{j=m}^{n-1}(1+R'(z_j))$$

is approximated to order $O(z_m)$ by the integral

$$\int_{z_m}^{z_n}\frac{\ln(1+R'(z))}{R(z)}dz.$$

The leading term of this integral is $\ln(R(z_n)/R(z_m))$ and it can be shown that subsequent terms are of order $O(z_m)$. By preceeding remarks, B is small for m large.

Finally, differentiating B we obtain

$$B' = -\frac{R(\alpha_m)}{R(\alpha_n)}\frac{d}{dz_0}\prod_{j=m}^{n-1}(\phi'(z_j)).$$

But,

$$\frac{d}{dz_0}\prod_{j=m}^{n-1}(\phi'(z_j)) = \phi'(z_{n-1})\cdots\phi'(z_0)\cdot S_{n,m},$$

where

$$\begin{aligned} S_{n,m} &= \frac{\phi'(z_{n-1})}{\phi'(z_{n-2})}\phi'(z_{n-3})\cdots\phi'(z_m) + \frac{\phi'(z_{n-2})}{\phi'(z_{n-3})}\phi'(z_{n-4})\cdots\phi'(z_m) \\ &\quad + \cdots + \frac{\phi'(z_{m+2})}{\phi'(z_{m+1})}\phi'(z_m) + \frac{1}{\phi'(z_m)}]. \end{aligned}$$

As before, the sum $S_{n,m}$ is bounded,

$$\frac{K}{R(\alpha_n)}\phi'(z_{n-1})\cdots\phi'(z_0)$$

is bounded, and $R(\alpha_m) \to 0$ as m becomes large. These estimates are uniform on $[\alpha_0, \alpha_1]$ and therefore we have shown C^2 convergence.

7.3.2 Translation invariance

Let $\bar{u}(z_0)$ defined with respect to $\bar{\alpha}_0$, $\bar{\alpha}_0 \neq \alpha_0$. That is, let

$$\bar{u}_i = \frac{z_i - \bar{\alpha}_i}{\bar{\alpha}_{i+1} - \bar{\alpha}_i} = \frac{z_i - \alpha_i}{\bar{\alpha}_{i+1} - \bar{\alpha}_i} - \frac{\bar{\alpha}_i - \alpha_i}{\bar{\alpha}_{i+1} - \bar{\alpha}_i}.$$

By the previous section,

$$\lim_{i\to+\infty}\frac{\bar\alpha_i-\alpha_i}{\bar\alpha_{i+1}-\bar\alpha_i}=c(\bar\alpha_0,\alpha_0),$$

independent of z_0. Next we will prove that

$$\lim_{i\to+\infty}\frac{z_i-\alpha_i}{\bar\alpha_{i+1}-\bar\alpha_i}=u(z_0).$$

Indeed,

$$\frac{z_i-\alpha_i}{\bar\alpha_{i+1}-\bar\alpha_i}-\frac{z_i-\alpha_i}{\alpha_{i+1}-\alpha_i}=(z_i-\alpha_i)\cdot\frac{\alpha_{i+1}-\alpha_i-\bar\alpha_{i+1}+\bar\alpha_i}{(\bar\alpha_{i+1}-\bar\alpha_i)(\alpha_{i+1}-\alpha_i)}.$$

By (2), $\alpha_{i+1}-\alpha_i=R(\alpha_i)$ so that

$$\alpha_{i+1}-\alpha_i-\bar\alpha_{i+1}+\bar\alpha_i=\gamma(\bar\alpha_i+\alpha_i)(\bar\alpha_i-\alpha_i)+o(\alpha^2)-o(\bar\alpha^2).$$

Therefore,

$$\frac{z_i-\alpha_i}{\bar\alpha_{i+1}-\bar\alpha_i}-\frac{z_i-\alpha_i}{\alpha_{i+1}-\alpha_i}=\gamma(\bar\alpha_i+\alpha_i)\cdot u_i\cdot\frac{\bar\alpha_i-\alpha_i}{\bar\alpha_{i+1}-\bar\alpha_i}+o(\alpha)-o(\bar\alpha).$$

The latter two terms of this product approach a limit by previous observations (in fact we need only boundedness) and $|\bar\alpha_i+\alpha_i|$ goes to zero as $i\to+\infty$, therefore the limits of the two sequences are identical.

7.3.3 Coordinate invariance

Next, suppose that a different coordinate chart in a neighborhood of 0 is given by coordinate $\bar z$, and h is the differentiable coordinate transformation. Then

$$\begin{aligned}|u_i-\bar u_i| &= \left|\frac{z_i-\alpha_i}{\alpha_{i+1}-\alpha_i}-\frac{h(z_i)-h(\alpha_i)}{h(\alpha_{i+1})-h(\alpha_i)}\right| \\ &= \left|\frac{z_i-\alpha_i}{\alpha_{i+1}-\alpha_i}-\frac{(z_i-\alpha_i)\cdot h'(0)+o(z_i)-o(\alpha_i)}{(\alpha_{i+1}-\alpha_i)\cdot h'(0)+o(z_i)-o(\alpha_i)}\right| \\ &= o(z_i),\end{aligned}$$

which goes to zero as i goes to $+\infty$.

7.4 Proof of Proposition 1

First note that if $h\circ F_0=G_0\circ h$ on a neighborhood of Γ, then $h\circ F_0^i=G_0^i\circ h$ on the neighborhood. For each $\ell>0$ let z_ℓ^+ be as in section 3. Denote by b^{F_0} the conventional multiplier of the homoclinic orbit of F_0, and by b^{G_0} the multiplier for the corresponding orbit of G_0. Then

$$\begin{aligned} b_\ell^{F_0}(\bar{X}) &= (\pi \circ F_0^m \circ \pi^{-1})'(z_\ell^+) \\ & (\pi \circ h^{-1} \circ G_0^m \circ h \circ \pi^{-1})'(z_\ell^+). \end{aligned}$$

But as $\ell \to 0$, $\bar{X}_\ell^+$ and $\bar{X}_\ell^- = F_0^m(\bar{X}_\ell^+)$ both approach $\bar{0}$. Since h is C^1, we have

$$|Dh(\bar{X}_\ell^+) - Dh(\bar{X}_\ell^-)| \to 0,$$

and so $|b^{F_0}(\bar{X})_\ell - b^{G_0}(h(\bar{X}))_\ell| \to 0$ as $\ell \to 0$.

7.5 Proof of Theorem 2

Let γ be a small segment of the intersection of S^s and S^u containing $\bar{X}_0$ and fix a small neighborhood U of $\bar{0}$. Let $\gamma^+ \equiv \gamma_{i^+}$ be such that $\gamma_i \in U$ for all $i \leq i^+$. Denote by α^+ and β^+ the minimum and maximum z-coordinates on γ^+. In fact we can take the (x,y) coordinates of γ^+ to be $(0,0)$. Let R^+ be a rectangular region on S^u given by $\{(y,z) : |y| \leq y^+, \alpha^+ \leq z \leq \beta^+\}$. Next define the following function space.

$$H_\epsilon = \{\zeta \in C^{k-1,1}(R^+, \mathbf{B}_1) : \|\zeta\|_{k-1,1} \leq \epsilon\}.$$

Hereout, $C^{k-1,1}$ indicates the Banach spaces of C^{k-1}-smooth functions for which the $(k-1)$-th degree derivative is Lipschitz. Now consider the first iterate of R^+ which reenters U, and denote this object by $D^-$and the number of iterations needed by m. By the transversality of S^u to S^s, if y^+ is sufficiently small then D^- will be the graph of a C^k-smooth function, ζ_0, over its projection onto the yz-coordinate plane, $\tilde{D}^-$. Similarly, if y^+ and ϵ are sufficiently small and $\eta(y,z,\mu) \in H_\epsilon$, then the image of the graph of η under F_μ^m will be a C^k-smooth surface which is close to the graph of ζ_0. It will also be the graph of a C^k-smooth function, ζ, over its projection onto the yz-coordinate plane. Note that ζ and ζ_0 will be close in the C^k sense on the intersection of their domians.

The next proposition is the crucial part of the proof of Theorem 2.

Proposition 3 *Given ϵ there exists an integer j_0 and a sequence of nonempty intervals*

$$\{(\overline{\mu_j}, \underline{\mu_j})\}_{j=j_0}^{\infty}$$

such that if $\mu \in (\overline{\mu_j}, \underline{\mu_j})$, then F_μ^{j+m} induces a map $\mathcal{T}_\mu$ from H_ϵ to itself.

Proof. If $\eta \in H_\epsilon$ then it is clear from (1) that points in the image $\overline{\eta} = \mathcal{T}_\mu(\eta)$ must satisfy

$$\begin{aligned} \overline{\eta} &= (A^j + a(\eta, y, z, \mu)) D_x F^m_{\mu,1}(\eta, y, z, \mu)\eta + \bar{F}_1(\eta, y, z, \mu) \\ \bar{y} &= (C^j + c(\eta, y, z, \mu)) D_y F^m_{\mu,2}(\eta, y, z, \mu) y + \bar{F}_2(\eta, y, z, \mu) \\ \bar{z} &= D_z \Phi^j(\eta, y, z, \mu) D_z F^m_{\mu,3}(\eta, y, z, \mu) z + \bar{F}_3(\eta, y, z, \mu), \end{aligned} \tag{6}$$

where a and c are small for μ and y^+ small and where $\bar{F}_i$ are the nonlinear components of F^m_μ with respect to the x, y and z coordinates. Given (x, y, z) in the graph of η, let $(x_1, y_1, z_1) = F^m_\mu(x, y, z)$ and let $\{(x_i, y_i, z_i)\}$ be the sequence of points defined by $(x_i, y_i, z_i) = \Phi_\mu(x_{i-1}, y_{i-1}, z_{i-1})$. Denote by ϕ_μ the one dimensional restriction of Φ to the center manifold. Provided that $|y_i|$ remains less than y^+, then

$$D_z \Phi^j(x_1, y_1, z_1, \mu) = \prod_{i=1}^{j} \phi'(z_i, \mu_j) + O(\mu, y_+).$$

This follows directly from [Yo94]. For any $\ell > 0$, we may write

$$\prod_{i=1}^{j} \phi'(z_i, \mu_j) = \prod_{i=1}^{i^-} \phi'(z_i, \mu_j) \cdot \prod_{i=i^-+1}^{i^+-1} \phi'(z_i, \mu_j) \cdot \prod_{i=i^+}^{j} \phi'(z_i, \mu_j) \tag{7}$$

where $\ldots < z_{i^-} < -\ell < z_{i^-+1} < \ldots < z_{i^+-1} < \ell < z_{i^+} < \ldots$. Given any $\ell > 0$ and $\epsilon > 0$, there exists j large enough (μ_j small enough) that the following holds

$$\left| \prod_{i=1}^{i^-} \phi'(z_i, \mu_j) \cdot \prod_{i=i^+}^{j} \phi'(z_i, \mu_j) \cdot D_y F^m_{\mu,3}(\eta, y, z, \mu) - b(z) \right| < \frac{\epsilon}{2}.$$

We use here that $z_i \approx \phi^{i-j}(z, \mu_j)$ for $i^+ \leq i \leq j$. This statement is then clear since it involves only a fixed finite number of iterations.

Next we consider the remaining factor of (7). By Lemma 3, as ℓ and μ approach zero,

$$\left| \prod_{i=i^-+1}^{i^+-1} \phi'(z_i, \mu_j) - 1 \right| = O(\ell + \sqrt{\mu}).$$

Thus from (7) have

$$\bar{z} = (b(z) + b_1(z, \mu)) z + \bar{F}_3(\eta, y, z, \mu), \tag{8}$$

where b_1 is small for μ and y^+ small.

Using the ideas of the Lambda Lemma [Pa], we can show (as in [Yo94]) that the image of the graph of ζ as it passes through U will become approximately a rectangular shape. Because the multiplier $b(\bar{X}_0)$ was assumed to be greater than one, the length in the z direction of this object will be greater than that of the original graph of η. In addition, the width in the y direction of the image will also be greater than that of the original for j sufficiently large. Thus, since the projection onto the yz-plane of this image is larger than the original, this projection must contain R^+ for a whole open interval $(\overline{\mu_j}, \underline{\mu_j})$ of parameter values.□

It follows from [Yo94] that $\mathcal{T}_\mu$ is a contraction on H_ϵ and that the unique fixed point, ζ^*_μ, is in fact C^k-smooth function in y and z. The graph of this function is thus a smooth surface Σ_μ, which is invariant under F^{j+m}_μ.

Now from the form of the map (6) and (8), the surface Σ_μ must contain a unique hyperbolic fixed point. The spectrum of this fixed point obviously decomposes in the way described by the theorem and thus $DF^{j+m}_\mu|_{\bar{p}}$ has the prescribed invariant splitting. This spectrum can be represented in the following form:

$$\sigma(F^{j+m}_\mu|_{\bar{X}^*_j}) = \sigma((A^j + a)D_x F^m_\mu) \cup (b(\bar{X}_0) + b_1) \cup \sigma((C^j + c)D_y F^m_\mu).$$

Finally, by a diagonal process we may take the length of γ to approach zero and simultaneously choose one $\bar{X}_j$ from each collection $\bar{X}_\mu$, $\mu \in (\overline{\mu_j}, \underline{\mu_j})$, thereby obtaining the sequence $\{\bar{X}_j\}$.

7.6 Proof of Lemma 3

In order to show that the value of the product $\prod_{i=i^-+1}^{i^+-1} \phi'(z_i, \mu_j)$ is close to unity, we begin by taking the natural logarithm of the product to obtain

$$S_{j,\ell} \equiv \ln \prod_{i=i^-+1}^{i^+-1} \phi'(z_i, \mu_j) = \sum_{i=i^-+1}^{i^+-1} \ln(1 + R'(z_i, \mu_j)).$$

Using $z_{i+1} - z_i = R(z_i, \mu_j)$, we have

$$S_{j,\ell} = \sum_{i=i^-+1}^{i^+-1} \frac{\ln(1 + R'(z_i, \mu_j))}{R(z_i, \mu_j)} (z_{i+1} - z_i).$$

For ℓ and μ_j small, $S_{j,\ell}$ is a Riemann sum for the integral

$$I_{j,\ell} = \int_{-\ell}^{\ell} \frac{\ln(1 + R'(z, \mu_j))}{R(z, \mu_j)} dz. \tag{9}$$

Using the fact that the integrand has a minimum at $z = \sqrt{\mu/\gamma} + O(\mu)$, we may repeat the argument of section 7.4 and find that the integral approximates the sum with accuracy of order $O(\ell + \sqrt{\mu})$.

The integral $I_{j,\ell}$ can be represented as

$$I_{j,\ell} = \int_{-\ell}^{\ell} \frac{1}{R} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (R')^n dz.$$

The first term is

$$\int_{-\ell}^{\ell} \frac{R'}{R} dz = \ln \frac{R(\ell,\mu)}{R(-\ell,\mu)} = O(\ell).$$

The n-th term is

$$\int_{-\ell}^{\ell} r_n(z,\mu) dz,$$

where

$$r_n(z,\mu) = \frac{(-1)^{n+1}}{n} \cdot \frac{(R'(z,\mu))^n}{R(z,\mu)}, \quad n \geq 2.$$

To estimate this integral, first find the maximal value of $|r_n(z,\mu)|$ on the interval $[-\ell,\ell]$. On the boundaries $z = \pm\ell$

$$|r_n(\pm\ell,\mu)| = \frac{1}{n} \frac{|(R')^{n-2}| \cdot (R')^2}{R} \leq \frac{1}{n} A_1 (b \cdot \ell)^{n-2},$$

where A_1 and b are some constants. In fact, $|R'| \leq |2\gamma z + o(z)| \leq b \cdot \ell$ and

$$\frac{(R')^2(\pm\ell,\mu)}{R(\pm\ell,\mu)} = \frac{4\gamma^2\ell^2 + o(\ell)}{\gamma\ell^2 + \mu + o(\ell)} \leq A_1.$$

On the interior of $[-\ell,\ell]$ if

$$\frac{dr_n}{dz} = \frac{(-1)^{n+1}}{n} \cdot [\frac{n(R')^{n-1} \cdot R'' \cdot R - (R')^{n+1}}{R^2}] = 0$$

then

$$(R')^n = n(R')^{n-2} \cdot R'' \cdot R.$$

Thus,

$$\max |r_n(z,\mu)| = \frac{1}{n} \cdot [n(R')^{n-1} \cdot R''] \leq A_2 \cdot (b \cdot \ell)^{n-2},$$

where $A_2 = \max_{-\ell<z<\ell} R''$. Therefore in both cases

$$\max |r_n(z,\mu)| \leq A_0 \cdot (b \cdot \ell)^{n-2}.$$

Now

$$\begin{aligned} I_{j,\ell} &\leq \ln \frac{R(\ell,\mu)}{R(-\ell,\mu)} + \int_{-\ell}^{\ell} \sum_{n=2}^{\infty} A_0 \cdot (b \cdot \ell)^{n-2} dz \\ &= \ln \frac{R(\ell,\mu)}{R(-\ell,\mu)} + A_0 \frac{2\ell}{1 - b\ell} \end{aligned}$$

and similarly

$$I_{j,\ell} \geq \ln \frac{R(\ell,\mu)}{R(-\ell,\mu)} - A_0 \frac{2\ell}{1 - b\ell}.$$

The ratio $R(\ell,\mu)/R(-\ell,\mu)$ approaches 1 as $\ell \to 0$. Therefore, $I_{j,\ell} = O(\ell)$ uniformly in μ and so $\prod_{i=i^-+1}^{i^+-1} \phi'(z_i,\mu_j) = 1 + O(\ell, \sqrt{\mu})$.

7.7 Proof of Theorem 3

Fix z^* and let $\bar{z}^*_\mu = \phi_\mu(z^*)$. Instead of considering the maps ψ_j from I^- to I^+, consider the sequence of maps $\phi^j_\mu(\cdot)$ from I^- to $[z^*, \bar{z}^*]$ (so that $q^*_j = \beta^-$). It is clear that if one of these sequences has a pointwise limit then the other does as well.

Fix a small number $\ell > 0$. Let $i^-(\ell)$ be the smallest integer such that $\alpha^-_\ell = \phi_0^{i^-(\ell)}(\alpha^-) \geq -\ell$ and $i^+(\ell)$ the largest integer such that $\alpha^+_\ell = \phi_0^{-i^+(\ell)}(\alpha^+) \leq \ell$.

Now $\phi^j_{\mu_j} = \phi^{i^+}_{\mu_j} \circ \phi^{j-i^--i^+}_\mu \circ \phi^{i^-}_{\mu_j}$. Let $u^-_\ell : [\alpha^-_\ell, \beta^-_\ell] \to [0,1]$ and $u^+_\ell : [\alpha^+_\ell, \beta^+_\ell] \to [0,1]$ be affine maps as in Lemma 1. Observe that the induced map $\tilde{\phi}_{\ell,\mu_j} : [0,1] \to [0,1]$ given by

$$\tilde{\phi}_{\ell,\mu_j} \equiv (u^+_\ell)^{-1} \circ \phi^{j-i^--i^+}_\mu \circ u^-_\ell$$

is a well-defined, smooth map. Note that $\tilde{\phi}_{\mu_j}(0) = 0$ and $\tilde{\phi}_{\mu_j}(1) = 1$ and

$$\tilde{\phi}'_{\ell,\mu_j} = \frac{R^-(\alpha^+_\ell)}{R(\alpha^-_\ell)} (\phi^{j-i^--i^+}_\mu)'.$$

As in section 7.1 the ratio $R^-(\alpha^+_\ell)/R(\alpha^-_\ell) \to 1$ as ℓ and μ_j go to zero. Also, by Lemma 3, $(\phi^{j-i^--i^+}_\mu)' \to 1$ as ℓ and μ_j go to zero. Therefore

$$\tilde{\phi}_{\ell,\mu_j} \to id_{[0,1]}$$

in the C^1 topology as $\ell, \mu_j \to 0$.

Next consider the maps $\xi^-_\ell : [\alpha^-_\ell, \beta^-_\ell] \to [0,1]$ and $\xi^+_\ell : [\alpha^+_\ell, \beta^+_\ell] \to [0,1]$ given by

$$\xi^-_\ell(z^-_0) = u^-_\ell(z^-_\ell) \quad \text{and} \quad \xi^+_\ell(z^+_0) = u^+_\ell(z^+_\ell)$$

By Lemma 2, these maps converge in the C^2 sense as ℓ and μ_j go to zero. But, observe that

$$\xi^-_\ell = u^-_\ell \circ \phi^{i^-}_{\mu_j} \quad \text{and} \quad \xi^+_\ell = u^+_\ell \circ \phi^{-i^+}_{\mu_j},$$

and so

$$\phi^j_{\mu_j} = (\xi^+_\ell)^{-1} \circ \tilde{\phi}_{\ell,\mu_j} \circ u^-_\ell.$$

Each map in this composition converges as $\ell, \mu_j \to 0$ and so by a diagonal process the limit of the composition exists in the C^1 topology as μ_j goes to zero..

Now observe from the definition of ψ_j that $\psi_j(\alpha^-) = \psi_j(\beta^-)$ for all j. Thus $\kappa_{lc}(\alpha^-) = \kappa_{lc}(\beta^-)$.

Finally observe that

$$\lim_{z \to q^{*+}_j} \psi_j(z) = \alpha^+ \quad and \quad \lim_{z \to q^{*-}_j} \psi_j(z) = \beta^+,$$

for all j. By the uniformity of the convergence these limits are preserved for κ_{lc}. Therefore, κ_{lc} has the correct limit as $z \to q^*_j$.

7.8 Proof of Theorem 4

We will show that ψ_k is a Cauchy sequence in the C^2 topology. We have already noted in the previous section that convergence of the functions is in the C^1 topology.

Consider the second derivative of the iterated map ϕ^n_μ;

$$\begin{aligned}(\phi^n(z_0))'' &= \phi''(z_{n-1})(\phi'(z_{n-2})\cdots\phi'(z_0))^2 \\ &\quad +\phi'(z_{n-1})\phi''(z_{n-2})(\phi'(z_{n-3})\cdots\phi'(z_0))^2 \\ &\quad +\cdots \\ &\quad +\phi'(z_{n-1})\cdots\phi'(z_2)\phi''(z_1)(\phi'(z_0))^2 \\ &\quad +\phi'(z_{n-1})\cdots\phi'(z_1)\phi''(z_0).\end{aligned}$$

Let i^- and i^+ be as above and consider the following. Given $\epsilon > 0$, let ℓ and μ be sufficiently small that the quantity in the Lemma 2 is less than $\epsilon/3$. The second derivative of ϕ^n may be written as the sum of three parts

$$(\phi^n(z_0))'' = \Sigma^+ + \Sigma^0 + \Sigma^-,$$

where

$$\begin{aligned}\Sigma^+ &= \phi''(z_{n-1})(\phi'(z_{n-2})\cdots\phi'(z_{i^+})\cdots\phi'(z_{i^-})\cdots\phi'(z_0))^2 \\ &\quad +\cdots \\ &\quad +\phi'(z_{n-1})\cdots\phi'(z_{i^++2})\phi''(z_{i^++1})(\phi'(z_{i^+})\cdots\phi'(z_{i^-})\cdots\phi'(z_0))^2,\end{aligned}$$

$$\begin{aligned}\Sigma^- &= \phi'(z_{n-1})\cdots\phi'(z_{i^+})\cdots\phi'(z_{i^-})\phi''(z_{i^--1})(\phi'(z_{i^--2})\cdots\phi'(z_0))^2 \\ &\quad +\cdots \\ &\quad \phi'(z_{n-1})\cdots\phi'(z_{i^+})\cdots\phi'(z_{i^-})\cdots\phi'(z_1)\phi''(z_0),\end{aligned}$$

and

$$\begin{aligned}\Sigma^0 &= \phi'(z_{n-1})\cdots\phi'(z_{i^++1})\phi''(z_{i^+})(\phi'(z_{i^+-1})\cdots\phi'(z_{i^-+1})\cdots\phi'(z_0))^2 \\ &\quad +\phi'(z_{n-1})\cdots\phi'(z_{i^+})\phi''(z_{i^+-1})(\phi'(z_{i^+-2})\cdots\phi'(z_{i^-+2})\cdots\phi'(z_0))^2 \\ &\quad +\cdots \\ &\quad +\phi'(z_{n-1})\cdots\phi'(z_{i^+-1})\cdots\phi'(z_{i^-+1})\phi''(z_{i^-})(\phi'(z_{i^--1})\cdots\phi'(z_0))^2.\end{aligned}$$

First consider Σ^+ and Σ^-. Let m and n be large enough so that μ_m and μ_n are small. It follows from Lemma 3 that the terms

$$\phi'(z_{i^+})\cdots\phi'(z_{i^-})$$

are approximately 1, regardless of m and n. Furthermore, with μ_m and μ_n small enough $z_0, z_1, \ldots, z_{i^-}$ for μ_m are arbitrarily close to the respective values of the same iterations for μ_n. Also since, $z_m = z_n$ for μ_m and μ_n respectively, the values $z_{m-1}, \ldots, z_{i^+}$ for μ_m will be arbitrarily close to $z_{n-1}, \ldots, z_{i^+}$ for μ_n. With these observations it can be seen that Σ^+ and Σ^- for m is arbitrarily close to the corresponding quantities for n, provided only that m and n are sufficiently large.

Next we estimate Σ^0 for ℓ small and n large. Note that for $-\ell < z < \ell$ we have

$$\phi''(z) = 2\gamma + o(\ell).$$

Thus we may factor this out of each of the terms of Σ^0. As well, we may factor out other repeated terms to obtain

$$\begin{aligned}\Sigma^0 \;=\; & (2\gamma + o(\ell)) \cdot (\phi'(z_{n-1}) \cdots \phi'(z_{i^++1})) \cdot [(\phi'(z_{i^--1}) \cdots \phi'(z_0))^2 \\ & \cdot(\phi'(z_{i^+-1}) \cdots \phi'(z_{i^-+1}))^2 (\phi'(z_{i^-}))^2 \\ & + \phi'(z_{i^+})(\phi'(z_{i^+-2}) \cdots \phi'(z_{i^-+2}))^2 (\phi'(z_{i^-+1})\phi'(z_{i^-}))^2 \\ & + \cdots].\end{aligned}$$

Applying Lemma 3 to appropriate products of the last expression, we find that Σ^0 may be approximated as

$$\begin{aligned}\Sigma^0 \;=\; & (4\gamma + o(\ell)) \cdot (\phi'(z_{n-1}) \cdots \phi'(z_{i^++1})) \cdot (\phi'(z_{i^--1}) \cdots \phi'(z_0))^2 \\ & \cdot[\phi'(z_{i^-}) + \phi'(z_{i^-+1})\phi'(z_{i^-}) \\ & + \ldots + \phi'(z_{i^0}) \cdots \phi'(z_{i^-})],\end{aligned}$$

where i^0 is the integer closest to $(i^+ - i^-)/2$. For ℓ and μ small z_{i^0} is approximately zero. Also note that for ℓ sufficiently small, the factor

$$(\phi'(z_{n-1}) \cdots \phi'(z_{i^++1})) \cdot (\phi'(z_{i^--1}) \cdots \phi'(z_0))^2$$

can be made arbitrarily small. Therefore we need only show that the sequences

$$\sigma^0 = (\phi'(z_{i^-}) + \phi'(z_{i^-+1})\phi'(z_{i^-}) + \ldots + \phi'(z_{i^0}) \cdots \phi'(z_{i^-}))$$

are uniformly bounded to imply that Σ^0 is arbitarily small for n large. If we take the logarithm of the j-th term of σ^0, and use $z_{i+1} - z_i = R(z_i, \mu)$, we obtain

$$s_j = \ln \prod_{i=1}^{j} \phi'(z_{i^-+i}) = \sum_{i=1}^{j} \frac{\ln(\phi'(z_{i^-+i}))}{R(z_{i^-+i})}(z_{i^-+i+1} - z_{i^-+i}).$$

For ℓ and μ_j small, this is a Reimann sum for the integral

$$I_{j,\ell} = \int_{z_{i^-}}^{z_{i^-+j}} \frac{\ln(1 + R'(z_i, \mu_j))}{R(z_i, \mu_j)} dz. \tag{10}$$

Using the fact that the integrand has a minimum at $z = \sqrt{\mu/\gamma} + O(\mu)$, we may repeat the arguments of previous sections and so the integral approximates the sum with accuracy of order $O(\ell + \sqrt{\mu})$. Expanding the integrand in power series and integrating we find

$$I_{j,\ell} = \ln \frac{\mu/\gamma + z_{i^-+j}^2}{\mu/\gamma + z_{i^-}^2} + O(\ell).$$

For given μ and ℓ the number of iterations from $-\ell$ to 0 is approximately $n = \ell/\mu$. Thus the sum σ^0 is given by

$$\sigma^0 = \sum_{i=1}^{n} \frac{\frac{\ell}{\gamma n} + z_{i^-+i}^2}{\frac{\ell}{\gamma n} + z_{i^-}^2} \leq \frac{1}{\frac{\ell}{\gamma n} + z_{i^-}^2}(\frac{\ell}{\gamma} + \sum_{i=1}^{n} z_{i^-+i}^2) + O(\ell, \sqrt{\mu}).$$

By Lemma 1, the sum is bounded and the proof of the lemma is complete.

7.9 Proof of Theorem 5

We will use Lemmas 2 and 3 to prove Theorem 5. If we let $\alpha^- = -\ell$ and $\beta^+ = \ell$ for ℓ sufficiently small, then κ_{gl} locally is exactly $\pi \circ \mathcal{F}_\ell \circ (\pi_\ell^+)^{-1}$. That is, $\kappa_{gl}'(z) = b_\ell(z)$ for each z. By Lemma 3 the derivative of the local map $\kappa_{lc,\mu}$ is approximately 1, and so $\kappa'(z) \equiv (\kappa_{lc} \circ \kappa_{gl})'(z) \approx b(z)$.

Also $\kappa_{gl}''(z) = b_\ell'(z)$. Since $\kappa'' = \kappa_{lc}'' \cdot \kappa_{gl}' + \kappa_{lc}' \cdot \kappa_{gl}''$ we need only to show that the second derivative of $\kappa_{lc,\mu}$ is close to 0 for ℓ and μ sufficiently small.

Let κ_{lc} the local map for some fixed $\alpha^- < 0$ and $\alpha^+ > 0$. For $\ell > 0$ small, let i^- be the first integer such that $\alpha_\ell^- \equiv \phi(\alpha^-, 0) > -\ell$ and let i^+ be the first integer such that $\beta_\ell^+ \equiv \phi^{-i^+}(\beta^+, 0) < \ell$. Define κ_ℓ to be the local map from $[\alpha_\ell^-, \beta_\ell^-]$ to $[\alpha_\ell^+, \beta_\ell^+]$. It can be seen that $\kappa_{lc} = \phi^{i^+} \circ \kappa_\ell \circ \phi^{i^-}$. Let $\xi_\ell^- : [\alpha^-, \beta^-] \to I = [0, 1]$ be the ratio

$$\xi^-(z) = \frac{\phi^{i^-}(z, 0) - \phi^{i^-}(\alpha^-, 0)}{\phi^{i^-}(\beta^-, 0) - \phi^{i^-}(\alpha^-, 0)}$$

and let $\psi_\ell^- : [\alpha^+, \beta^+] \to I = [0, 1]$ be the map

$$\xi^+(z) = \frac{\phi^{-i^+}(z, 0) - \phi^{-i^+}(\alpha^+, 0)}{\phi^{-i^+}(\beta^+, 0) - \phi^{-i^+}(\alpha^+, 0)}.$$

Define $\tilde{\kappa}_\ell$ to be the map induced on I by κ_ℓ, given by

$$\tilde{\kappa}_\ell = \xi_\ell^+ \circ \kappa_{lc} \circ (\xi_\ell^-)^{-1}.$$

By Lemma 2, both ξ_ℓ^- and ξ_ℓ^+ converge in the C^2 sense as ℓ goes to zero. This implies that κ_ℓ also approaches a limit in the C^2 sense as ℓ goes to zero, since

$$\tilde{\kappa}_\ell' = \frac{R(\alpha_\ell^-, 0)}{R(\alpha_\ell^+, 0)} \kappa_\ell'.$$

Now by lemma 3 the limit function $\tilde{\kappa}'_\ell$ must be the identity function on the unit interval. The second derivative of κ_ℓ must also be small by the previous equation.

7.10 Proofs of Lemma 4 and Theorem 6

We begin by noting that the choice of fundamental domains is immaterial. This is because κ_1, defined on a fundamental domain I_1, and κ_2, defined on a second domain, I_2, are C^k-smoothly conjugated by iterations of the original map ϕ_0. Therefore, given $\epsilon > 0$ choose α^+ sufficiently small that from Theorem 5 we have for κ defined on $[\alpha^+, \beta^+]$

$$|\kappa' - b|_1 \leq \frac{\epsilon}{2}.$$

Theorem 4 then allows us to choose μ_0 such that

$$|\kappa'_\mu - b|_1 \leq \epsilon.$$

Thus we may write

$$\kappa'_\mu(z) = b(z) + g(z, \mu),$$

where $|g|_1 \leq \epsilon$. Now integrating both sides of the last equality from α^+ to z we find that

$$\kappa_\mu(z) - \kappa_\mu(\alpha^+) = \mathcal{K}(z) - \alpha^+ + G(z, \mu),$$

where $|G|_2 \leq \epsilon \cdot (\beta^+ - \alpha^-)$, or since $\kappa_\mu(\alpha^+) = z^*$ by assumption

$$\kappa_\mu(z) = \mathcal{K}(z) + z^*(\mu) - \alpha^+ + G(z, \mu).$$

Thus the family $\kappa_\mu(z)$ is just a C^2-small perturbation of translations of $\mathcal{K}$ as asserted by the lemma. Noting that $z^*(\mu)$ is strictly monotone increasing in μ, it is clear that $\kappa_\mu(z)$ undergoes exactly the same fixed point structure as the family $\mathcal{K}(z) + \alpha$. If we set $\alpha(\mu)$ to be the bifurcation values for $\mathcal{K}(z) + \alpha$ for the corresponding values of μ, and require that $\alpha(\mu)$ be monotone between these critical values, then we have the map α in the theorem. Finally, we note that two circle diffeomorphisms with fixed points are topologically conjugate if and only if each of the maps have the same number of fixed points with the same stability types and are arranged in the same order.

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