

Unavoidable minors in large 2- and 3-connected matroids

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Theorem

For every positive integer k , there is an integer N such that every 2-connected loopless graph G with at least N edges has a cycle or bond with at least k edges.

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G has, as a minor, a k -edge cycle or k parallel edges.

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For every integer $k \geq 3$, every sufficiently large 3-connected simple graph has, as a minor, a k -spoke wheel \mathcal{W}_k or $K_{3,k}$.

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- (OOT) Unavoidable topological minors for large 3-connected graphs.

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- (Kawarabayashi and Maharry 2012) Unavoidable minors for large almost 5-connected non-planar graphs.

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Example (A matroid from a graph G : a **graphic matroid**)

A **matroid** M is a pair (E, \mathcal{C}) consisting of:

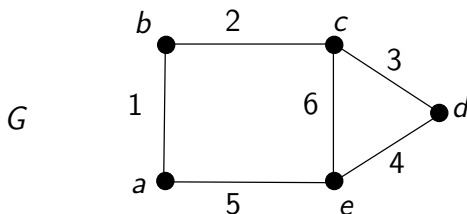
- a finite set E (take the **edge-set of G**)
- a collection \mathcal{C} of subsets of E called **circuits** (take the edge sets of all the cycles)

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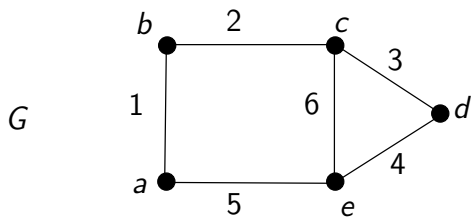
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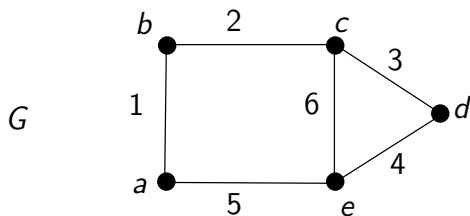
$$E = \{1, 2, 3, 4, 5, 6\} \text{ and } \mathcal{C} = \{\{1, 2, 5, 6\}, \{3, 4, 6\}, \{1, 2, 3, 4, 5\}\}$$

Matroids from matrices

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Matroids from matrices



The vertex-edge incidence matrix of G (over the 2-element field):

$$\begin{array}{c} \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ a & \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right) \\ b \\ c \\ d \\ e \end{array} \end{array}$$

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Every graphic matroid is binary.

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Problem (Thomas 1991)

Does every sufficiently large 2-connected matroid have a big circuit or a big cocircuit?

Unavoidable minors in 2-connected matroids

Theorem (Lovász, Schrijver, and Seymour 1991)

Let M be a 2-connected matroid whose largest circuit and largest cocircuit have c and c^ elements, respectively. Then*

$$|E(M)| < 2^{c+c^*}.$$

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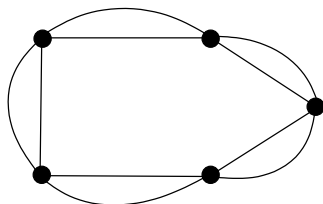
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Here $|E(M)| = \frac{1}{2}cc^*$.

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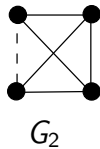
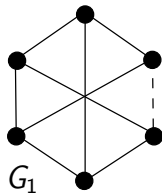
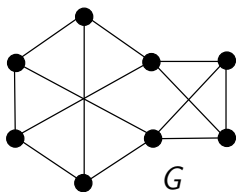
Pou-Lin Wu (2000) determined all graphic matroids attaining the bound.

Problem (Royle)

Find all of the non-graphic matroids attaining the bound.

3-connectivity in matroids

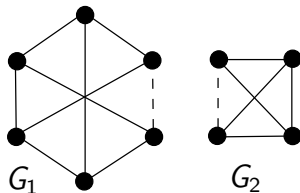
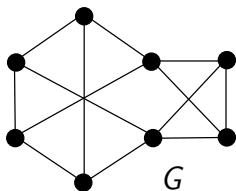
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G is 2-connected but not 3-connected.

G is the 2-sum of G_1 and G_2 .

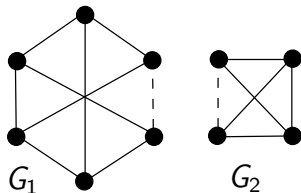
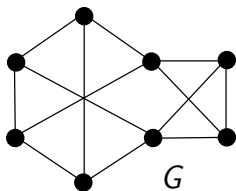
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G is the **2-sum** of G_1 and G_2 .

All cycles in G are easily specified in terms of cycles of G_1 and cycles of G_2 .

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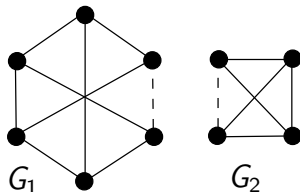
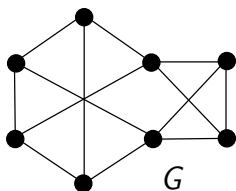


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- does not break up as a 2-sum.

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The matroid of a graph G is 3-connected \iff G is 3-connected and simple.

Matroid duals and minors

Graph G gives a **cycle matroid** $M(G)$: circuits are **cycles**.

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$$M^*(G) = M(G^*).$$

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Matroid minors

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M/x , the **contraction** of x :

Circuits are the minimal non-empty sets of the form $C - \{x\}$ where C is a circuit of M .

Unavoidable minors for large 3-connected binary matroids

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Theorem (Ding, Oporowski, O, Vertigan, 1996)

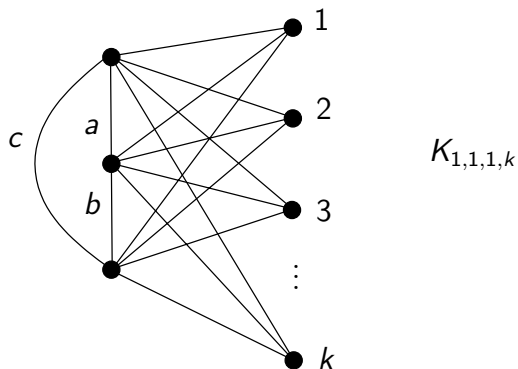
For all $k \geq 3$, every sufficiently large 3-connected binary matroid has, as a minor, $M(\mathcal{W}_k)$, $M(K_{3,k})$, $M^(K_{3,k})$, or $M[I_k|Z_k]$.*

Capturing elements in unavoidable minors

Try to put two edges in a highly structured large 3-connected graph.

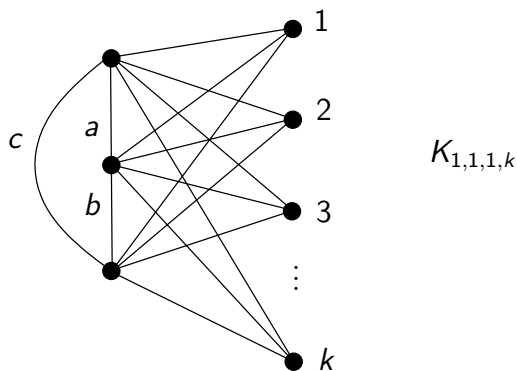
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We cannot capture a and b in a big $K_{3,n}$ -minor or a big \mathcal{W}_n -minor.

Capturing elements in unavoidable minors



Theorem (Deb Chun, O, 2012)

Let $k \geq 3$ and e and f be edges of a 3-connected simple graph G . When G is sufficiently large, it has a minor using $\{e, f\}$ and isomorphic to $K_{1,1,1,k}$ or \mathcal{W}_k .

Capturing elements in unavoidable minors

Theorem (Deb Chun, O, 2012)

Let $k \geq 3$ and $\{e, f, g\}$ be a 3-cycle in 3-connected simple graph G . When G is sufficiently large, it has a minor *using* $\{e, f, g\}$ and isomorphic to $K_{1,1,1,k}$ or \mathcal{W}_k .

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The last two results are special cases of results for binary matroids.

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What about $M[I_k|Z_k]$?

Z_k has all ones with zeros on the diagonal.

Z'_k is Z_k with a one rather than zero in the bottom right entry.

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Theorem (Deb Chun, O, 2012)

*Let $k \geq 3$ and e and f be elements of a 3-connected binary matroid M . When M is sufficiently large, it has a minor **using $\{e, f\}$** and isomorphic to $M(K_{1,1,1,k})$, $M^*(K_{1,1,1,k})$, $M(\mathcal{W}_k)$, or $M[I_k|Z'_k]$.*

What about non-binary matroids?

Having a $(0, 1)$ -matrix representation makes binary matroids easier to handle than general matroids.

What about non-binary matroids?

Ding, Oporowski, O, and Vertigan (1997) found the unavoidable minors for arbitrary large 3-connected matroids.

Problem

Does the result on capturing two elements extend to all matroids?

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OPEN

Can capture one element (D. Chun, O, Whittle, 2012).

Capturing one element: general case

Theorem (Deb Chun, O, Whittle 2012)

Let $k \geq 3$ and e be an element of a 3-connected matroid M .

*When M is sufficiently large, M or its dual has a minor **using e** and isomorphic to*

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- k points on a line
- a rank- k whirl — (a relative of the wheel)
- $M(K_{3,k})$ extended by an element freely placed on the guts line.

Many problems remain

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Problem

Find the unavoidable minors for large internally 4-connected binary matroids.