GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF RESEARCH ADMINISTRATION

RESEARCH PROJECT INITIATION

Date: September 24, 1973

Project Title: "New Algebraic Methods in the Analysis and Synthesis of Complex Systems"

Project No: E-21-631

Principal Investigator Dr. Edward W. Kamen

Sponsor: U. S. Army Research Office - Durham

Agreement Period: From 9/1/73 Until 8/31/75

Type Agreement: Grant No. DA-ARO-D-31-124-73-G171

Amount:	$18,542	USARO-D	E-21-631
	2,937	GIT	E-21-321
	$21,479		

Reports Required: Semi-annual Progress Report
Final Progress Report

Sponsor Contact Person (s):

Administrative	Technical
Mr. Jack L. Harless Contracting Officer U. S. Army Research Office - Durham Box CM, Duke Station Durham, North Carolina 27706	Mr. James E. Norman Director Research - Technology Division U. S. Army Research Office - Durham Box CM, Duke Station Durham, North Carolina 27706

Assigned to: Electrical Engineering

COPIES TO:

- Principal Investigator
- School Director
- Dean of the College
- Director, Research Administration
- Director, Financial Affairs (2)
- [✓ Security-Reports-Property Office]
- Patent Coordinator
- Library
- Rich Electronic Computer Center
- Photographic Laboratory
- Project File
- Other ______

RA-3 (6-71)

Posted

GEORGIA INSTITUTE OF TECHNOLOGY

OFFICE OF RESEARCH ADMINISTRATION

RESEARCH PROJECT TERMINATION

Date: October 23, 1975

Project Title: New Algebraic Methods in the Analysis & Synthesis of Complex Systems

Project No: E-21-631

Principal Investigator: Dr. E. W. Kamen

Sponsor: U. S. Army Research Office

Effective Termination Date: 8/31/75

Clearance of Accounting Charges: all have cleared

Grant/Contract Closeout Actions Remaining:
1. Final Fiscal Report
2. Final Report of Inventions
3. Classified Material Cert.

Prop. Cert Submitted 26 Sep [illegible]

Assigned to: ELECTRICAL ENGINEERING

COPIES TO:

Principal Investigator
School Director
Dean of the College
Director of Research Administration
Associate Controller (2)
Security-Reports-Property Office ✓
Patent and Inventions Coordinator

Library, Technical Reports Section
Rich Electronic Computer Center
Photographic Laboratory
Terminated Project File No. E-21-631
Other ______

RA-4 (5/70)

PROGRESS REPORT #1

1. ARO PROPOSAL NUMBER: 11554-RT

2. PERIOD COVERED BY REPORT: September 1, 1973 through March 31, 1974

3. TITLE OF PROPOSAL: New Algebraic Methods in the Analysis and Synthesis of Complex Systems

4. CONTRACT OR GRANT NUMBER: DA-ARO-D-31-124-73-G171

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Edward W. Kamen

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

   "A New Algebraic Approach to Linear Time-Varying Systems," submitted to the Journal of Computer and System Science, March 1974.

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   Khalid Hafiz, Ph.D. Candidate

Dr. Edward W. Kamen
Georgia Institute of Technology
Atlanta, Georgia 30332

11554-RT

BRIEF OUTLINE OF RESEARCH FINDINGS

The research has resulted in a completely new algebraic approach to linear time-varying discrete-time systems, such as sampled-data systems and various types of sequential systems with switching operations. In contrast to previous work, the theory is developed in terms of a type of global-in-time representation specified by a variable time reference. The unique aspect of this setup is that it possesses previously unexplored algebraic properties which play an important role in system dynamics. These properties are a result of incorporating the time variance of systems into the algebraic structure. Using this framework, new results have been obtained on the reachability and controllability of time-varying systems, and a new approach to the important problem of realization has been developed. The algebraic setting also yields a new formulation of duality, in terms of which new results on observability have been obtained. These results can be applied to the problems of feedback control and estimation in time-varying systems. In particular, new computational procedures for the construction of controllers and state estimators are in the process of being generated by working with an algebraic framework consisting of polynomials in time. Potential applications of this work include the design of flight controllers and autopilots.

Paper: "A New Algebraic Approach to Linear Time-Varying Systems," by
Edward W. Kamen, Georgia Institute of Technology, March, 1974

ABSTRACT

A theory of linear time-varying discrete-time systems is constructed in terms of a variable time reference which yields a new type of global-in-time representation. In this approach the time-variance of systems is incorporated into an algebraic framework consisting of modules defined over noncommutative rings. In particular, input/output behavior is specified by a homomorphism between modules over a noncommutative ring of formal power series, yielding an operational calculus for computing system responses. Dynamical behavior is given in terms of a module structure defined over a skew polynomial ring. This framework is utilized to obtain general results on reachability and controllability, and is then applied to the problem of realizing time-varying discrete-time systems.

PROGRESS REPORT #2

1. ARO PROPOSAL NUMBER: 11554-RT

2. PERIOD COVERED BY REPORT: April 1, 1974 through September 30, 1974

3. TITLE OF PROPOSAL: New Algebraic Methods in the Analysis and Synthesis of Complex Systems

4. CONTRACT OR GRANT NUMBER: DA-ARO-D-31-124-73-G171

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Edward W. Kamen

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

   "Module Structure of Infinite-Dimensional Systems with Applications to Controllability", 38 pages, submitted to SIAM J. Control, September, 1974.

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

   Mr. Khalid Hafiz, Ph.D. Candidate

Dr. Edward W. Kamen
Georgia Institute of Technology
Atlanta, Georgia 30332

11554-RT

BRIEF OUTLINE OF RESEARCH FINDINGS

Research during the period April 1, 1974 through September 30, 1974 has centered on the development of algebraic techniques for the study of the following classes of systems.

1. Infinite-Dimensional Continuous-Time Systems: Here results have been obtained on causality, realization, and controllability of infinite systems. For example, the work has yielded the first algebraic results on bounded-time and minimal-time controllability of infinite systems. Via the algebra, it is now possible to compute minimal control times for many important types of infinite systems such as systems with time delays.

2. Linear Time-Varying Discrete-Time Systems: This work involves the continued development of the new algebraic framework for time-varying systems, discovered in the first work period of the grant. Preliminary results have been obtained on the solution of the linear regulator problem with constraints on the time-varying gains. The emphasis at present is on the development of computational procedures for designing controllers and state estimators in the time-varying case. Another objective here is to construct a "time-varying version" of the pole-placement theorem. This would make possible the design of feedback controllers that stabilize unstable time-varying systems.

MODULE STRUCTURE OF INFINITE-DIMENSIONAL SYSTEMS WITH APPLICATIONS TO CONTROLLABILITY*

Edward W. Kamen
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

September, 1974

ABSTRACT

A theory of infinite-dimensional time-invariant continuous-time systems is developed in terms of modules defined over a convolution ring of generalized functions. In particular, input/output operators are formulated as module homomorphisms between free modules over the convolution ring, and systems are defined in terms of a state module. Results are presented on causality and the problem of realization. The module framework is then utilized to study the reachability and controllability of states and outputs. New results are obtained on the smoothness of controls, bounded-time controls, and minimal-time controls.

---

*This research was supported by the U. S. Army Research Office, Durham, N. C., under Grant DA-ARO-D-31-124-73-G171.

PROGRESS REPORT #3

1. ARO PROPOSAL NUMBER: 11554-RT
2. PERIOD COVERED BY REPORT: October 1, 1974 through March 31, 1975
3. TITLE OF PROPOSAL: New Algebraic Methods in the Analysis and Synthesis of Complex Systems.
4. GRANT NUMBER: DA-ARO-D-31-124-73-G171
5. NAME OF INSTITUTION: Georgia Institute of Technology
6. AUTHOR OF REPORT: Edward W. Kamen
7. LIST OF MANUSCRIPTS UNDER ARO SPONSORSHIP WHICH WERE WRITTEN OR IN PREPARATION DURING THIS PERIOD:

   "New Results on Discrete-Time Time-Varying Systems," 90 pages, a Ph.D. thesis submitted by Khalid Hafiz, March 1975.

   Results in thesis are being incorporated into two separate papers which will be submitted for publication when completed.

   "Finiteness in Infinite-Dimensional Systems Applied to Regulation," an invited paper to be given at the Symposium on Algebraic System Theory in June 1975, author is E. W. Kamen

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT:

   Mr. Khalid Hafiz, Ph.D. Candidate. Mr. Hafiz' Ph.D. thesis has just been approved, so he will receive his Ph.D. in June 1975. Mr. Hafiz has received his support from this grant.

BRIEF OUTLINE OF RESEARCH FINDINGS

Research during the period October 1, 1974 through March 31, 1975 has been carried out in the following two problem areas. In both areas, major new results have been obtained.

1) The regulator or tracking problem for infinite-dimensional time-invariant systems: Here new results have been obtained on achieving exponential rates of convergence to zero of $||e(t)||$, the norm of the error $e(t)$ between a desired output (reference) and the actual output. Our approach to this important problem involves the use of the concepts of bounded-time controllability and observability in the infinite case, as developed by the author during a previous work period. Included in this work are the first general results on the computation of minimal time periods before exponential convergence occurs. This problem involves a new characterization of transient behavior in infinite-dimensional systems.

2. Continuation of a global-time algebraic theory for time-varying linear systems: We now have a new structure theory for time-varying systems based on the concept of an n-cyclic module defined over a skew polynomial ring. In terms of this framework, we have been able to formulate and solve a "time-varying version" of the important pole-placement theorem. This has resulted in many new constructive results on the design of feedback controllers. By "dualizing" our algebraic setting using a new "global-time" duality construction, we have been able to obtain several new results on the design of state estimators.

C-21-621

PROGRESS REPORT

1. ARO PROPOSAL NUMBER: 11554-RT

2. PERIOD COVERED BY REPORT: April 1, 1975 through August 31, 1975
Grant terminated on August 31, 1975; renewal began on Oct. 1, 1975

3. TITLE OF PROPOSAL: New Algebraic Methods in the Analysis and Synthesis of Complex Systems

4. CONTRACT OR GRANT NUMBER: DA-ARO-D-31-124-73-G171

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Edward W. Kamen

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

<u>ON NEXT PAGE</u>

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

Khalid Hafiz, Ph.D. Candidate

Dr. Hafiz received the Ph.D. in June 1975. His research was supported entirely by this U. S. Army Grant.

115[illegible]

EDWARD W. KAMEN
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GA 30332

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD.

Papers a.,b., and c. below were written by E. W. Kamen:

a. "Representation and Realization of Operational Differential Equations with Time-Varying Coefficients," August 1975, invited paper to appear in the Special Issue on Realization Theory, Journal of the Franklin Institute.

b. "Finiteness in Infinite-Dimensional Systems Applied to Regulation," June 1975, invited paper to appear in a volume of Lecture Notes in Mathematics, Springer-Verlag.

c. "On an Operator Theory of Linear Systems with Pure and Distributed Delays," August 1975, invited paper to appear in the Proceedings of the 1975 IEEE Conference on Decision and Control, Houston.

d. "New Results on Discrete-Time Time-Varying Linear Systems," 65 pages, Ph.D. Thesis of K. Hafiz, appeared in June, 1975.

Note: Results in d. are being incorporated into two separate papers to be submitted for publication sometime in the near future.

BRIEF OUTLINE OF RESEARCH FINDINGS

Research has been continuing on the development of an algebraic theory for various classes of linear systems. We now have an extensive global-time algebraic theory for time-varying discrete-time systems with many new results in control and estimation. For continuous-time systems, new results have been obtained on representation, controllability, and feedback control of systems given by linear functional differential equations of the retarded type. A completely new algebraic representation has been developed for operational differential equations with time-varying coefficients. In terms of this framework, the first general computable solution to the problem of realization (in the time-varying infinite case) has been obtained.

The basic objective of this work is to develop viable procedures for the analysis and design of systems that contain time-varying parameters and/or distributed parameters (or time delays). In terms of relevance to U. S. Army problems, such systems arise when a part of the system involves one or more human operators. For example, time delays occur as a result of human reaction times or human decision times. Time-varying characteristics can also occur; for instance, as a result of fatigue of operators. Potential applications of the above-mentioned research to this class of systems include the following.

a. Techniques for realizing digital, analog, or hybrid computer simulations of systems with human operators.

b. On-line methods for monitoring operator performance (i.e. state).

c. Design of digital or analog compensators to achieve some specified degree of stability in a control process, or to achieve time-optimal control.

To appear in the Special Issue on Realization Theory,
Journal of the Franklin Institute

REPRESENTATION AND REALIZATION OF OPERATIONAL DIFFERENTIAL EQUATIONS WITH TIME-VARYING COEFFICIENTS*

E. W. Kamen

School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332
August 1975

ABSTRACT

An algebraic treatment of operational differential equations with time-varying coefficients is presented in terms of skew rings of differential polynomials defined over a Noetherian ring. Included in this framework are delay differential equations with time-varying coefficients. The operator equations are characterized by transfer matrices which are utilized to construct realizations given by first-order vector differential equations with operator coefficients. It is shown that the realization of matrix equations can be reduced to the realization of scalar equations. Finally, a simple procedure is derived for realizing scalar equations.

---

*This work was supported by the U.S. Army Research Office, Durham, N.C., under Grant DA-ARO-D-31-124-73-G171.

To appear in Lecture Notes in Mathematics, Symposium on
Algebraic System Theory, Udine, Italy 1975

FINITENESS IN INFINITE-DIMENSIONAL SYSTEMS APPLIED TO REGULATION*

Edward W. Kamen
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332/U.S.A.

ABSTRACT

After describing existing mathematical representations of linear infinite-dimensional systems, we consider the problem of regulating a system's output so that it tracks a given reference. Using a general operational framework possessing certain finiteness properties, we pursue a particular aspect of this problem by first considering the concept of bounded-time controllability. New results are then given on the construction of time-invariant input/output regulators that drive the output response (resulting from nonzero initial states) to the zero function in finite time. An example is given to illustrate that such regulators can be constructed by using ideal time delays in the feedback loop.

1. Representation of Infinite-Dimensional Systems

A good deal of work has been devoted to the development of state-space representations for linear infinite-dimensional systems. There are two basic types of mathematical models. In one setup, the system dynamics are given by an equation of evolution in an infinite-dimensional linear topological space such as a Banach or Hilbert space (e.g., see the papers by Balakrishnan [1], Aubin-Bensoussan [2], and Baras-Brockett-Fuhrmann [3]). In most of these models the state is the solution of some first-order ordinary (or operational) differential equation in a locally convex space. We shall refer to these models as abstract representations since the state is usually not defined in terms of physical attributes of the given system.

---

*This work was supported by the U. S. Army Research Office, Durham, N.C., under Grant DA-ARO-D-31-124-73-G171.

To appear in Proc. 1975 IEEE Conf. on Decision and Control

ON AN OPERATOR THEORY OF LINEAR SYSTEMS WITH PURE AND DISTRIBUTED DELAYS*

E. W. Kamen
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

Abstract

A representation theory based on convolution operations is developed for a large class of linear systems containing pure and distributed delays in state and control. In terms of this framework a necessary and sufficient condition and a sufficient condition are given for functional (null) controllability. The conditions involve the generation of modules defined over a convolution ring of functions.

1. Introduction

In many control problems the systems under consideration contain pure and distributed time delays in state and control (examples are given by MANITIUS [1]). Such systems are usually referred to as hereditary systems since the rate of change of the present state depends on past values of the state and control or input.

In this paper we consider the class of linear systems given by a first-order functional differential equation of the form

$$\dot{x}(t) = \int_{-c}^{0} A(\theta)x(t+\theta)d\theta + F_0 x(t) + \sum_{i=1}^{r} F_i x(t-a_i) \tag{1}$$
$$+ \int_{-d}^{0} B(\theta)u(t+\theta)d\theta + G_0 u(t) + \sum_{i=1}^{s} G_i u(t-b_i)$$

where c,d and the a_i, b_i are positive real numbers, the F_i (G_i) are $n \times n$ ($n \times m$) matrices over the reals R, $A(\theta)$ (resp. $B(\theta)$) is a $n \times n$ ($n \times m$) matrix of (Lebesgue) measurable and integrable functions on $[-c,0]$ ($[-d,0]$), $x(t) \in R^n$ is the "instantaneous state," and $u(t) \in R^m$ is the input.

Systems of the form (1) have been studied using mainly functional-analytical methods applied to a state space setting defined in terms of the product space $R^n \times L^p(-h,0;R^n)$ where $h = \max\{c, a_i\}$. In particular, numerous results on controllability and optimal feedback control can be found in the work of DELFOUR-MITTER [2,3], DELFOUR [4], and DELFOUR-McCALLA-MITTER [5] (see also the survey by MANITIUS [6]).

In contrast to existing methods, our approach to the study of (1) is based on an algebraic setting defined in terms of convolution operators. More precisely, in the next section it is shown that (1) can be written in the form

$$\overset{\circ}{x}(t) = (F * x)(t) + (G * u)(t) \tag{2}$$

where * denotes convolution and F and G are matrices whose elements belong to a convolution ring of functions and impulses (Dirac distributions). The convolution representation (2) is a special case of the time-domain operator framework developed by KAMEN [7].

In the latter part of the paper the representation (2) is applied to the problem of driving initial functions to the zero function in finite time (functional null controllability). New algebraic criteria for controllability are given in terms of modules defined over the convolution rings.

2. Representation by Convolution Operators

Let L^{loc} denote the space of all real-valued Lebesgue measurable functions f(t) that are locally integrable, i.e. $\int_K |f(t)|dt < \infty$ for any compact subset K of R. Let L_+^{loc} denote the subspace of L^{loc} consisting of all functions with support bounded on the left. It is easily verified that L_+^{loc} is a ring with pointwise addition and with convolution defined by

$$(g * f)(t) = \int_{-\infty}^{\infty} g(\theta)f(t-\theta)d\theta \quad .$$

---

*This work was supported by the U.S. Army Research Office, Durham, N.C., under Grant DA-ARO-D-31-124-73-G171.

MODULE STRUCTURE OF INFINITE-DIMENSIONAL SYSTEMS WITH APPLICATIONS TO CONTROLLABILITY[*]

Edward W. Kamen
School of Electrical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332

September, 1974

ABSTRACT

A theory of infinite-dimensional time-invariant continuous-time systems is developed in terms of modules defined over a convolution ring of generalized functions. In particular, input/output operators are formulated as module homomorphisms between free modules over the convolution ring, and systems are defined in terms of a state module. Results are presented on causality and the problem of realization. The module framework is then utilized to study the reachability and controllability of states and outputs. New results are obtained on the smoothness of controls, bounded-time controls, and minimal-time controls.

---

[*] This research was supported by the U. S. Army Research Office, Durham, N. C., under Grant DA-ARO-D-31-124-73-G171.

1. Introduction

The existing theory of infinite-dimensional systems is based primarily on the elements of topology and analysis (e.g. Banach spaces, Hilbert spaces, etc.). In contrast, in this paper the emphasis is on the application of modern algebra to the study of infinite-dimensional time-invariant systems. The objective here is to formulate a theory in terms of rings and modules which yield new results as a consequence of finiteness properties enjoyed by these algebraic structures.

Here the rings and modules are convolution structures that come into play as a result of the additional assumption of time invariance. In particular, as discussed in Section 2, linear time-invariant input/output (i/o) operators can be formulated as module homomorphisms between finitely-generated modules defined over a convolution ring of functions. Although the convolution structure of these i/o operators is well known, very little attention has been devoted to the relationship between the i/o module framework and the internal system structure defined in terms of the concept of state.

The first major work on the role of the convolution structure in a state space setting was Kalman's K[z]-module description of finite-dimensional discrete-time systems [1]. Kalman was also the first one to consider a module structure over a convolution ring of functions in the state space theory of continuous-time systems (see Kalman and Hautus [2]). However, the theory of [2], which centers on the problem of realization, does not apply to a very large class of infinite-dimensional systems since it is assumed that for any positive integer n, the output response resulting from the n^{th} derivative of the Dirac distribution at $\{0\}$ is infinitely differentiable on $(0,\infty)$. For example, this constraint prevents consideration of systems having time delays. The extension of Kalman's module framework to a suitably large class of infinite-dimensional systems is carried out here.

The convolution structure of the i/o description can be reflected in the internal system structure in two ways, depending on the type of internal model used. On the one hand, the dynamical equations can be given by operational-differential equations defined by convolution operators belonging to a Noetherian ring. This approach is developed in [3] and will not be considered here. In the second approach, which will be pursued here, systems are given in operational form by module homomorphisms with the state space also possessing a (topological) module structure over the convolution ring of functions.

The module structure on the state space provides a new approach to the study of dynamical properties. For instance, as revealed in Sections 5, 6, and 7, the concept of the annihilator of a module plays a central role in controllability. Using this concept, in Section 6 we obtain the surprising result that if the reachable states of a system are controllable (to the zero state), then every reachable state can be controlled within some fixed time period (bounded-time controllability). Examples of systems in which the reachable states are always controllable are given in Section 7.

In the following development, it is crucial that we work with a convolution ring of functions which contains the identity δ_o = Dirac distribution at $\{0\}$. In other words, we need to consider a convolution ring of distributions (generalized functions). Then since we want the input function space and the output function space to be modules over the convolution ring, these spaces must also be spaces of distributions. The requirement that the ring contain δ_o is mainly for algebraic reasons. For example, it is then possible to consider the operation of inversion which, as we shall see, leads to the construction of control signals.

2. Input/Output Operators

Let $\mathbb{R}$ denote the field of real numbers with the usual topology. Let $\mathcal{D}$(resp. $\mathcal{D}_-$) denote the linear space of $\mathbb{R}$-valued infinitely differentiable functions defined on $\mathbb{R}$ with compact supports (resp., with supports bounded on the right). With the Schwartz topology [4], $\mathcal{D}$ and $\mathcal{D}_-$ are Hausdorff locally convex linear topological spaces. Let $\mathcal{D}'$ (resp. $\mathcal{D}'_+$) denote the dual of $\mathcal{D}$ (resp. $\mathcal{D}_-$) with the strong topology. Then $\mathcal{D}'_+$ is the space of $\mathbb{R}$-valued distributions on $\mathbb{R}$ with support bounded on the left. The canonical injections $\mathcal{D} \to \mathcal{D}'_+ \to \mathcal{D}'$ are continuous and $\mathcal{D}$ is dense in $\mathcal{D}'_+$ (see [4]).

From the results of Schwartz [4], with the operations of addition and convolution $\mathcal{D}'_+$ is a commutative (topological) ring with no divisors of zero. Given u, v ϵ $\mathcal{D}'_+$, the convolution of u and v, denoted by $u*v$, is defined by

$$2.1 \qquad < u*v, \varphi > = <u, <v, \varphi(t+\tau) >>, \qquad \text{all } \varphi \in \mathcal{D}$$

It is easily verified that, if $u,v \neq 0$, then $\text{supp}(u*v) \subseteq (\text{supp } u) + (\text{supp } v)$ where supp denotes the support. The identity of the ring $\mathcal{D}'_+$ is the Dirac distribution δ_o. We also note that the linear structure on $\mathcal{D}'_+$ is compatible with the ring structure in that $\mathcal{D}'_+$ is a convolution algebra over $\mathbb{R}$. To simplify the notation, from here on we let V denote the ring $\mathcal{D}'_+$.

For any fixed positive integer n, let V^n denote the n-fold direct sum of V with the elements of V^n written as column vectors. Then V^n is a free n-dimensional topological module over the ring V. Given $v \in V^n$ and $\alpha \in V$, we let $\alpha * v$ denote the operation of α on v in the V-module structure of V^n.

Definition 2.1: Given fixed positive integers m and k, an input/output (i/o) operator f is a linear continuous map $f:V^m \to V^k$.

As usual, an i/o operator $f:V^m \to V^k$ characterizes the correspondence between input functions (in V^m) and the resulting output functions (in V^k) for some m-input terminal k-output terminal linear continuous-time system. There are two main reasons for taking $V = \mathcal{D}'_+$ as the space of "admissible signals" appearing at the input and output terminals. First, V is a convolution ring containing δ_o which, as mentioned in the introduction, is necessary for the algebraic constructions that follow. Second, the class of systems describable by an i/o operator $f:V^m \to V^k$ is extremely large, including, for example, distributed-parameter devices such as LC and RC transmission lines.

Unfortunately, the topology on V is not normable, and in some applications it may be highly desirable to work with a convolution ring (with δ_o) having a nice topological structure, such as a Banach convolution algebra (Bensoussan and Kamen [5]). However, most of the results that follow can be carried over to these other rings.

In this paper, we restrict attention to i/o operators f having the property that $f(\delta_\tau * v) = \delta_\tau * f(v)$, all $\tau \in \mathbb{R}$, $v \in V^m$; that is, f commutes with the shift operator δ_τ. Such i/o operators are said to be time invariant or constant.

Letting $V^{k \times m}$ denote the V-module of $k \times m$ matrices over V, we have the following result on the representation of i/o operators.

Theorem 1.1: For each time-invariant i/o operator $f:V^m \to V^k$, there exists a unique $W \in V^{k\times m}$ such that $f(v) = W*v$ for all $v \in V^m$. Conversely, given $W \in V^{k\times m}$, the operator $V^m \to V^k : v \mapsto W*v$ is a time-invariant i/o operator.

Proof: Follows from the Schwartz kernel theorem [4] using the fact that the canonical injections $\mathcal{D} \to V \to \mathcal{D}'$ are continuous and $\mathcal{D}$ is dense in V.

Corollary 1.1: With respect to the topological V-module structure on V^m and V^k, every time-invariant i/o operator $f:V^m \to V^k$ is a (topological) V-module homomorphism.

Corollary 1.2: For fixed positive integers m and k, the V-module consisting of all time-invariant i/o operators $f:V^m \to V^k$ is isomorphic to $V^{k\times m}$.

The matrix W, whose existence is asserted in Theorem 1.1, is usually referred to as the impulse response matrix. A major point here is that the existence of W is directly connected to the fact that the i/o operator is a V-module homomorphism. The basic idea of this work is to exploit the module structure. But before we begin to do this, we need to consider the notion of causality in the space V.

Definition 2.2: An i/o operator $f:V \to V : v \mapsto w*v$, $w \in V$, is causal if whenever $u|_{(-\infty,\tau)} = v|_{(-\infty,\tau)}$, $u,v \in V$, $\tau \in \mathbb{R}$, then $f(u)|_{(-\infty,\tau)} = f(v)|_{(-\infty,\tau)}$, where $|_{(-\infty,\tau)}$ denotes restriction to the open interval $(-\infty,\tau)$ in the sense of distributions.

Proposition 2.1: Given $f:V \to V : v \mapsto w*v$, the following are equivalent:

i. f is causal

ii. if $\operatorname{supp} v \subseteq [\tau,\infty)$, $v \in V$, $\tau \in \mathbb{R}$, then $\operatorname{supp} f(v) \subseteq [\tau,\infty)$

iii. $\operatorname{supp} w \subseteq [0,\infty)$

Proof: i. $\Rightarrow$ ii.: Let $v \in V$ with supp $v \subseteq [\tau,\infty)$. Then since $v|_{(-\infty,\tau)} = 0$ and f is causal, $f(v)|_{(-\infty,\tau)} = f(0)|_{(-\infty,\tau)} = 0$. Thus supp $f(v) \subseteq [\tau,\infty)$.

ii $\Rightarrow$ iii.: Since supp $\delta_o = \{0\}$, by ii. supp $f(\delta_o) \subseteq [0,\infty)$. But $f(\delta_o) = w*\delta_o = w$.

iii. $\Rightarrow$ i.: Suppose that $u|_{(-\infty,\tau)} = v|_{(-\infty,\tau)}$, then supp $(u-v) \subseteq [\tau,\infty)$ and since supp $w \subseteq [0,\infty)$, supp $[w*(u-v)] \subseteq [0,\infty) + [\tau,\infty) = [\tau,\infty)$. Therefore, supp $f(u-v) \subseteq [\tau,\infty)$ which implies that $f(u)|_{(-\infty,\tau)} = f(v)|_{(-\infty,\tau)}$ since f is additive.

Even though supp $w \subseteq [0,\infty)$ for a causal operator, in general it is not possible to construct the impulse response w from the restriction $w|_{(o,\infty)}$. This situation can occur when w is not regular on any neighborhood of the origin. (A distribution $v \in V$ is regular on an open set U if $v|_U$ can be generated in the usual manner from a locally integrable function on U.) For example, if $w = \delta_o + e^{-t}H(t)$, $H(t)$ = Heaviside function, then $w|_{(o,\infty)} = e^{-t}|_{(o,\infty)}$ which does not contain any knowledge of the singular component δ_o.

Many system problems, such as the problem of realization, involve the restriction $w|_{(o,\infty)}$ of the impulse response w, assuming that w can be determined uniquely from $w|_{(o,\infty)}$. A sufficient condition for the determination of w from $w|_{(o,\infty)}$ is given in the following

Proposition 2.2: Let $w \in V$ with supp $w \subseteq [0,\infty)$. If there exists an open neighborhood U of the origin such that $w|_U$ is a regular distribution, then w can be completely and uniquely determined from $w|_{(o,\infty)}$.

Proof: Suppose that w satisfies the hypothesis of the proposition. Let a be a positive number belonging to U and write $w_+ = w|_{(o,\infty)}$. Since w_+ is regular on (0,a), from w_+ we can construct the following regular distribution on $(-\infty,a)$:

$$w_a(t) = \begin{cases} w_+(t), \ 0 < t < a \\ 0, \ t \le 0 \end{cases}$$

Then since $w_+ = w_a$ on $(0,a)$, by the theorem on "piecing together distributions" (Zemanian [6, page 34]), from w_+ and w_a it is possible to construct one and only one distribution θ on $\mathbb{R}$ such that $\theta|_{(-\infty,a)} = w_a$ and $\theta|_{(o,\infty)} = w_+$. Further, θ is clearly independent of the value chosen for a. Now by construction, $\theta = w$ on $(-\infty,0) \bigcup (0,\infty)$. Hence $\theta = w$ on $\mathbb{R}$ since the Lebesgue measure of $\{0\}$ is zero and both w and θ are regular on the open neighborhood U.

Definition 2.3: A causal i/o operator f with impulse response w is said to be strictly causal if $w|_U$ is regular for some open neighborhood U of $\{0\}$.

The term strictly causal is taken from the work of Saeks [7]. Although Saek's formulation of causality is developed in terms of an abstract Hilbert space rather than a space of distributions, his definition of strictly causal is similar to that given here.

In many cases the impulse response w is an ordinary function (i.e., a regular distribution) with supp $w \subseteq [0,\infty)$, and thus the i/o operator is strictly causal as defined above. On the other hand, there exist important examples of systems whose impulse responses are not regular and yet the corresponding i/o operators are strictly causal. These systems are necessarily infinite dimensional; that is, the Laplace transform of the impulse response is not rational. A simple example is the ideal delay line with impulse response δ_τ, $\tau > 0$.

An interesting class of causal operators which are not strictly causal is the class of operators having supp $w = \{0\}$. By a well-known theorem of distribution theory (Zemanian [6, page 98]) supp $w = \{0\}$ if and only if w is a finite $\mathbb{R}$-linear combination of δ_o and its derivatives. If we let $\delta_o^{(n)}$ denote the n^{th} derivative of δ_o, then since $\delta_o^{(n)} * v = v^{(n)} = n^{th}$ derivative of $v \in V$, for an i/o operator f with supp $w = \{0\}$ the response $f(v)$ is a finite linear combination of the input v and its derivatives.

Most causal operators of interest can be decomposed uniquely into the sum of a strictly causal operator and an operator with impulse response concentrated at the origin:

Proposition 2.3: Given a causal operator $f: V \to V: v \mapsto w*v$, if there exists an $a > 0$ such that $w|_{(o,a)}$ is regular, then f can be decomposed uniquely into the sum $f = f_{sc} + f_o$ where f_{sc} is strictly causal and $f_o: v \mapsto w_o * v$ with $w_o = 0$ or supp $w_o = \{0\}$.

Proof: Let w satisfy the hypothesis. As in the proof of Proposition 2.2, from $w|_{(o,\infty)}$ we can construct a distribution θ on $\mathbb{R}$ such that $\theta = w$ on $\mathbb{R} - \{0\}$ and the operator $f_{sc}: v \mapsto \theta * v$ is strictly causal. Now define $w_o = \theta - w$, then $w_o = 0$ or supp $w_o = \{0\}$ and $f = f_{sc} + f_o$ where $f_o: v \mapsto w_o * v$. Uniqueness: Suppose that $f = \bar{f}_{sc} + \bar{f}_o$ where $\bar{f}_{sc}$ is strictly causal and $\bar{f}_o(v) = \bar{w}_o * v$, $\bar{w}_o \neq w_o$, $\bar{w}_o = 0$ or supp $\bar{w}_o = \{0\}$. Then since $f_{sc} + f_o = \bar{f}_{sc} + \bar{f}_o$, $f_o - \bar{f}_o = \bar{f}_{sc} - f_{sc}$. But this is impossible since the operator $\bar{f}_{sc} - f_{sc}$ is strictly causal and supp$(w_o - \bar{w}_o) = \{0\}$.

The above results are easily extendable to the multi-input multi-output case. In particular, the i/o operator $f: V^m \to V^k: v \mapsto W*v$, $W = (w_{ij}) \in V^{k\times m}$, is strictly causal if for each i,j there exists an open neighborhood U_{ij} of $\{0\}$ such that $w_{ij}|_{U_{ij}}$ is regular. In the remainder of this paper, we limit our study to strictly causal i/o operators.

3. State in a Module Framework

In this section we formulate a definition of systems which reflects the convolution module structure of the i/o representation. In order to express the concept of state in terms of the convolution structure, we need to define another type of i/o operator which is a module homomorphism between modules defined over a proper subring of $V = \mathcal{D}'_+$.

Let Ω denote the subring of V consisting of all distributions having compact support contained in $(-\infty,0]$. With the induced topology, Ω is a topological subring of V, and the m-fold direct sum Ω^m is a free m-dimensional topological module over the ring Ω. (Throughout this paper it is understood that the topology of all modules considered is Hausdorff and locally convex.)

Let Γ denote the set $\{v|_{(o,\infty)} : v \in V\}$. With the induced operations, Γ is a linear subspace of $\mathcal{D}'(0,\infty)$, the space of all distributions defined on $(0,\infty)$. Further, it follows from the discussion given by Treves [8, page 246] that Γ is a proper subspace of $\mathcal{D}'(0,\infty)$. We give Γ the strongest topology such that the map

$$\rho : V \to \Gamma : v \mapsto v|_{(o,\infty)} \tag{3.1}$$

is continuous. Note that ρ is also an open mapping since a set U_1 is a neighborhood of zero in Γ if and only if there is a neighborhood U_2 of zero in V such that $\rho(U_2) = U_1$.

<u>Proposition 3.1</u>: Γ is a topological module with multiplication

$$\Omega\times\Gamma \to \Gamma : (\omega,\gamma) \mapsto \omega\gamma \overset{\Delta}{=} (\omega * \bar{\gamma})|_{(o,\infty)} \tag{3.2}$$

where $\bar{\gamma} \in V$ is any extension of γ to V (i.e. $\bar{\gamma}|_{(o,\infty)} = \gamma$).

Proof: Multiplication (3.2) is independent of the extension considered: Let $\gamma \in \Gamma$, then by definition of Γ, γ has at least one extension $\bar{\gamma} \in V$. Suppose that $\bar{\gamma}$ and $\bar{\gamma}'$ are two extensions of γ and let $\omega \in \Omega$. Then

$$3.3 \qquad < \bar{\gamma},\varphi > = < \bar{\gamma}',\varphi >, \text{ all } \varphi \in \mathcal{D}:\text{supp } \varphi \subset (0,\infty)$$

Now since supp $\omega \subset (-\infty,0]$, for every $\tau \leq 0$,

$$< \omega,\varphi(t+\tau) > = 0, \text{ all } \varphi:\text{supp } \varphi \subset (0,\infty)$$

Thus, viewed as a function of τ, $< \omega,\varphi(t+\tau) >$ is an element of $\mathcal{D}$ with support contained in $(0,\infty)$. Then from (3.3), we have

$$< \bar{\gamma},< \omega,\varphi(t+\tau) >> = < \bar{\gamma}',< \omega,\varphi(t+\tau) >>, \text{ all } \varphi:\text{supp } \varphi \subset (0,\infty)$$

From the definition of convolution (2.1), we get

$$< \omega*\bar{\gamma},\varphi > = < \omega*\bar{\gamma}',\varphi >, \text{ all } \varphi:\text{supp } \varphi \subset (0,\infty)$$

Thus $(\omega*\bar{\gamma})|_{(o,\infty)} = (\omega*\bar{\gamma}')|_{(o,\infty)}$, showing that multiplication (3.2) is properly defined. The proof that Γ with (3.2) is a topological module follows from the fact that ρ, given by (3.1), is open and continuous. The straightforward details are omitted.

Corollary 3.1: The k-fold direct sum Γ^k is a (nonfinite) topological module over the ring Ω.

Let $I:\Omega^m \rightarrow V^m$ denote the inclusion map, and define the map $P:V^k \rightarrow \Gamma^k:(v_1,\ldots,v_k)^{TR} \mapsto (\rho(v_1),\ldots,\rho(v_k))^{TR}$, where TR denotes the transpose.

Theorem 3.1: Given a strictly causal i/o operator $f:V^m \rightarrow V^k:v \mapsto W*v$, let f^* denote the composition PfI. Then

i. f^* is a (topological) Ω-module homomorphism.

ii. f^* is completely and uniquely determined by $W|_{(o,\infty)}$ and vice versa.

iii. f can be completely and uniquely constructed from f^*.

Proof: i. By definition of Ω^m, the inclusion map $I:\Omega^m \to V^m$ is an Ω-module homomorphism with V^m viewed as an Ω-module. It is also clear that $P:V^k \to \Gamma^k$ is an Ω-module homomorphism with V^k viewed as an Ω-module. Hence the composition $PfI = f^*:\Omega^m \to \Gamma^k$ is an Ω-module homomorphism.

ii. Let $\omega \in \Omega^m$, then $\omega = \sum\limits_i \omega_i * e_i$ where $e_i = (0\ 0 \ldots \underset{\uparrow\!\_\_\_\_\ i^{th}\text{ place}}{\delta_o}\ 0 \ldots 0)^{TR}$.

Since f^* is an Ω-module homomorphism, $f^*(\omega) = \sum\limits_i \omega_i f^*(e_i)$, and by definition of f^*, $f^*(e_i) = (W*e_i)|_{(o,\infty)} = \gamma_i|_{(o,\infty)}$ where γ_i is the i^{th} column of W. Hence $W|_{(o,\infty)}$ determines f^* uniquely and conversely.

iii. Follows from ii. and Proposition 2.2.

The operator $f^* = PfI$ characterizes the input/output behavior relative to the time reference $t = 0$. As will be done shortly, we can define the state space to be some space through which f^* is factored. The module structure comes into play by requiring that the factorization consist of Ω-module homomorphisms.

Let ℓ denote the map

3.4 $$\ell : V \to \mathbb{R} : v \mapsto \ell(v) = \begin{cases} \inf\limits_t \{t \in \text{supp } v\}, & v \neq 0 \\ 0, & v = 0 \end{cases}$$

Since $\text{supp}(u*v) \subseteq (\text{supp } u) + (\text{supp } v)$, $u,v \in V$, $u,v \neq 0$,

3.5 $$\ell(u*v) \geq \ell(u) + \ell(v), \text{ all } u,v \neq 0.$$

Note that $\ell(v) \leq 0$ for any $v \in \Omega$. Finally, ℓ can be extended to V^m by defining

3.6 $$\ell:(v_1,\ldots,v_m)^{TR} \mapsto \min_i\{\ell(v_i)\}$$

Definition 3.1: An m-input k-output strictly causal linear time-invariant system Σ is a sextuple $\Sigma = (\Omega^m, X, \Gamma^k, \mu, \eta, \psi)$ where

i. X is a topological Ω-module with multiplication denoted by $\pi\cdot x$, $\pi \in \Omega$, $x \in X$.

ii. $\mu:\Omega^m \to X$ and $\eta:X \to \Gamma^k$ are (topological) Ω-module homomorphisms with the composition $\eta\mu$ equal to PfI for some strictly causal time-invariant i/o operator f.

iii. ψ is a map defined by

$$\psi:\Omega^m \times X \times \mathbb{R}^- \to X:(\omega,x,a) \mapsto \delta_{a+\ell(\omega)}\cdot x + \mu(\omega)$$

where $\mathbb{R}^- = \{a \in \mathbb{R}: a < 0\}$.

In this definition, X is the module of states, $\mu(\omega)$ is the state at time $t = 0$ due to input $\omega \in \Omega^m$, and $\eta(x)$ is the output response on $(0,\infty)$ resulting from state x at $t = 0$. The map ψ is a state transition operator: $\psi(\omega,x,a)$ is the state at $t = 0$ due to input ω and initial state x at time $t = a+\ell(\omega)$ prior to the application of ω. The parameter a in the definition of ψ cannot be zero in general because the input ω may contain Dirac distributions at $\{\ell(\omega)\}$. Since the input (function) module Ω^m and the output (function) module Γ^k are fixed, we shall usually write $\Sigma = (X,\mu,\eta,\psi)$.

Note that since the composition $\eta\mu$ equals PfI for some strictly causal operator f, by Theorem 3.1 knowledge of $\eta\mu$ is equivalent to knowledge of f. Therefore f can be (and will be) taken as <u>the</u> i/o operator of the system $\Sigma = (X,\mu,\eta,\psi)$.

Although the definition of a system Σ is specified with respect to the time reference $t = 0$, this does not result in any special restrictions, other than those already given, since Σ is time invariant. The time invariance of Σ is a consequence of the fact that μ and η are Ω-module homomorphisms.

The requirement that the state set X admit a module structure over the convolution ring Ω is actually a very natural condition since we are considering systems whose input/output behavior is given by an Ω-module homomorphism. Furthermore, as shown in the next section every strictly causal i/o operator can be realized by a system having an Ω-module structure.

One final point here is that since $\mathbb{R}$ can be viewed as a subring of Ω under the embedding $\mathbb{R} \to \Omega : a \mapsto a\delta_0$, X is also a linear space over $\mathbb{R}$. Thus the module structure on X "contains" the usual linear space structure. A system Σ is infinite-dimensional in the usual sense if X is infinite dimensional as a linear space over $\mathbb{R}$.

4. Realization of Input/Output Operators

Following the standard definitions, we say that a system $\Sigma = (X, \mu, \eta, \psi)$ is completely reachable (resp. completely observable) if μ is surjective (resp. η is injective). In the first part of this section it is proved that every strictly causal i/o operator can be realized by a system that is completely reachable and observable. Then we consider realizations given by differential equations in the sense of distributions.

<u>Definition 4.1</u>: A realization of a strictly causal i/o operator $f: V^m \to V^k$ is a system $\Sigma = (X, \mu, \eta, \psi)$ with $\eta\mu = PfI$. A realization is said to be canonical if it is completely reachable and observable.

Theorem 4.1: Every strictly causal time-invariant i/o operator f has a canonical realization.

Proof: Given f, let $f^* = PfI$. Since Γ^k is a Hausdorff space, $\{0\}$ is a closed set in Γ^k, and by the continuity of f^*, $\operatorname{Ker} f^* \triangleq \{\omega \in \Omega^m : f^*(\omega) = 0\}$ is a closed set in Ω^m. Hence the quotient space $\Omega^m/\operatorname{Ker} f^* \triangleq \{[\omega] \triangleq \omega + \operatorname{Ker} f^* : \omega \in \Omega^m\}$ with the quotient topology is a Hausdorff locally convex linear topological space. Further, it is easily checked that $\Omega^m/\operatorname{Ker} f^*$ is a topological Ω-module with multiplication $\pi \cdot [\omega] \triangleq [\pi * \omega]$, $\pi \in \Omega$, $\omega \in \Omega^m$. Now take $X_f \triangleq \Omega^m/\operatorname{Ker} f^*$ to be the state module, and define the following Ω-module homomorphisms

$$\mu_f : \Omega^m \to X_f : \omega \mapsto [\omega]$$
$$\eta_f : X_f \to \Gamma^k : [\omega] \mapsto f^*(\omega)$$

Clearly, μ_f is surjective and η_f is injective. Given $\omega \in \Omega^m$, $\mu_f(\omega) = [\omega]$ is defined to be the state at time $t = 0$ due to input ω, and for every $\tau \leq 0$, $\mu_f(\delta_\tau * \omega) = \delta_\tau \cdot [\omega]$ is the state at time $t = 0$ due to state $[\omega]$ at time τ. Therefore, if the input $\omega \in \Omega^m$ is applied with initial state $x = [\beta]$ at time $a + \ell(\omega)$, $a < 0$, the state $\psi_f(\omega, x, a)$ at $t = 0$ is given by $\psi_f(\omega, x, a) = \delta_{a+\ell(\omega)} \cdot x + \mu_f(\omega)$. Finally, since $f^* = \eta_f \mu_f$, $(X_f, \mu_f, \eta_f, \psi_f)$ is a canonical realization of f.

Regarding the uniqueness of canonical realizations, we have the following

Proposition 4.1: If (X, μ, η, ψ) and $(\hat{X}, \hat{\mu}, \hat{\eta}, \hat{\psi})$ are two canonical realizations of an i/o operator f, then with respect to the algebraic structure there exists a unique ξ-module homomorphism $\xi : X \to \hat{X}$ with $\xi\mu = \hat{\mu}$ and $\hat{\eta}\xi = \eta$.

Proof: Follows from a standard isomorphism theorem.

Corollary 4.1: If the composition PfI is an open mapping, ξ is a topological Ω-module isomorphism; that is, ξ is also a homeomorphism.

Proof: Suppose that $f^* = PfI$ is open and let U be an open set in $\hat{X}$. Then $(f^*\hat{\mu}^{-1})(U)$ is open in Γ^k since $\hat{\mu}$ is continuous. Since η is injective and $\hat{\mu}$ is surjective and $f^* = \hat{\eta}\hat{\mu}$, $(\eta^{-1}f^*\hat{\mu}^{-1})(U) = \xi^{-1}(U)$ which is open in X because η is continuous. Hence ξ is continuous. A similar proof shows that ξ^{-1} is continuous.

In many applications it is desirable to have a realization given by dynamical differential equations. For example, with such a realization it would be possible to apply the theory of differential equations to the study of optimal control. As we now show, i/o operators can be realized by differential equations in the sense of distributions.

Given the i/o operator $f:V^m \to V^k$, let $\Sigma = (X,\mu,\eta,\psi)$ denote the canonical realization of f constructed in the proof of Theorem 4.1. Following Kalman and Hautus [2], define the truncation operator $\mathscr{S}:\mathscr{D}^m \to \Omega^m:\alpha \mapsto \mathscr{S}\alpha$ where $(\mathscr{S}\alpha)(t) = 0, t > 0$, and $(\mathscr{S}\alpha)(t) = \alpha(t)$, $t \le 0$.
For every $\omega \in \Omega^m$, define

$$x_\omega : \mathscr{D} \to X : \varphi \mapsto [\mathscr{S}(\check{\varphi} * \omega)],\ \check{\varphi}(t) = \varphi(-t)$$

Note that since $\pi * \varphi \in \mathscr{D}$, all $\pi \in \Omega$, $\varphi \in \mathscr{D}$ (see [6]), x_ω is properly defined. As proved in [2], x_ω is an X-valued distribution. The interpretation of x_ω is that it is the generalized state trajectory resulting from the application of the input ω.

Now define

$$F: X \to X : [\omega] \mapsto [\delta_o^{(1)} * \omega]$$

$$G: \mathbb{R}^m \to X : (a_1, \ldots, a_m)^{TR} \mapsto [(a_1\delta_o, \ldots, a_m\delta_o)^{TR}]$$

Then for all $\varphi \in \mathcal{D}$, x_ω satisfies the differential equation

4.1 $$\frac{dx_\omega(\varphi)}{dt} = Fx_\omega(\varphi) + G\omega(\varphi)$$

The proof follows from [2].

Hence we have an internal differential equation describing the realization. However we do not have an output equation as constructed in [2] because here the output response on $(0,\infty)$ may not be an ordinary function. Nevertheless, in most cases it is possible to formulate an output equation as follows.

Let $\overline{X} = \{[\sigma] : \sigma \in \mathcal{S}(\mathcal{D}^m)\}$ which is a linear subspace of X viewed as a linear space over $\mathbb{R}$. Suppose that for each $\sigma \in \mathcal{S}(\mathcal{D}^m)$, $f(\sigma)$ is continuous on some neighborhood of zero. Then since $f(\beta) = f(\sigma)$ on $(0,\infty)$ for every $\beta \in [\sigma]$, we can define the operator

$$H: \overline{X} \to \mathbb{R}^k : [\sigma] \mapsto f(\sigma)(0) = \lim_{t \downarrow o} f(\sigma)(t)$$

Let $\omega \in \Omega^m$, then for every $\varphi \in \mathcal{D}$ we have that

$< f(\omega), \varphi > = (\check{\varphi} * f(\omega))(0)$

$\Rightarrow$ $< f(\omega), \varphi > = (f(\check{\varphi} * \omega))(0)$ since f is a V-module homomorphism

$\Rightarrow$ $< f(\omega), \varphi > = (f(\mathcal{S}(\check{\varphi} * \omega)))(0)$ since f is strictly causal

$\Rightarrow$ $< f(\omega), \varphi > = H[\mathcal{S}(\check{\varphi} * \omega)]$ by definition of H.

Thus

4.2 $$< f(\omega), \varphi > = Hx_\omega(\varphi), \quad \omega \in \Omega^m, \quad \varphi \in \mathcal{D}$$

Hence we have proved

Theorem 4.2: Given the i/o operator $f:V^m \to V^k : v \mapsto W*v$, if for each $\sigma \in \mathcal{S}(\mathcal{D}^m)$, $f(\sigma)$ is continuous on some neighborhood of zero, then f has a canonical realization which can be described by dynamical differential equations (given by 4.1-2).

Instead of working with dynamical differential equations, in the remainder of this paper we consider only the operational form of a system Σ as given in Definition 3.1. The objective is to study dynamical properties by using the module structure on $\Sigma = (X,\mu,\eta,\psi)$.

5. Controllability in a Module Framework

In terms of the $\mathbb{R}$-linear structure, few algebraic results exist on the controllability of infinite systems simply because the state space is infinite dimensional as a linear space. However, as a consequence of finiteness properties of the module structure, it is possible to study control from an algebraic standpoint. We shall do this here, setting up the theory in terms of a general framework that includes state and output function controllability. In the following development the topological structure is not considered.

Let M be an Ω-module, and let $\lambda:\Omega^m \to M$ be an Ω-module homomorphism.

Definition 5.1: An element $x \in M$ is reachable if there exists an $\omega \in \Omega^m$ such that $\lambda(\omega) = x$. An element $x \in M$ is controllable if there exist $\tau < 0$ and $\omega \in \Omega^m$ with $\ell(\omega) > \tau$, such that $\delta_\tau \cdot x + \lambda(\omega) = 0$. The element ω is called a control for x and $-\tau$ is a control time.

Given a system $\Sigma = (X,\mu,\eta,\psi)$, state controllability and a type of output function controllability are particular cases of the above definition:

1. State Controllability: Take $M = X$ and $\lambda = \mu$. Then $x \in X$ is reachable if there exists an input in Ω^m which sets up (from the zero state) the state x at time $t = 0$, and x is controllable if there exists an input in Ω^m which drives the system to the zero state at $t = 0$ starting from state x at some time τ prior to the application of the input.

2. Output Function Controllability: Take $M = \Gamma^k$ and $\lambda = PfI = f^*$ where f is the i/o operator of the system Σ. Then an output function $\gamma \in \Gamma^k$ is reachable if there exists an input in Ω^m which produces this response with zero initial state prior to the application of the input. An element $\gamma \in \Gamma^k$ is controllable if there exist $\tau < 0$, $\omega \in \Omega^m$, $\ell(\omega) > \tau$, such that $\delta_\tau\gamma + f^*(\omega) = 0$ which implies that

$$\gamma|_{(-\tau,\infty)} + f(\delta_{-\tau}*\omega)|_{(-\tau,\infty)} = 0 \tag{5.1}$$

Therefore, viewing γ as an output response on $(0,\infty)$ due to an input and/or initial state occurring in the time interval $(-\infty,0]$, by (5.1) we have that the input $\delta_{-\tau}*\omega$ (applied during the interval $(0,-\tau]$) drives the output response to zero on $(-\tau,\infty)$.

The objective here is to study controllability in terms of the general framework given in Definition 5.1. All of the following results specialize to state and output function controllability by setting $\lambda = \mu$ or PfI as done above. We begin with the following basic definitions from module theory.

Given an Ω-module M, $x \in M$ is said to be a free element if $\pi x = 0$ for some $\pi \in \Omega$, then $\pi = 0$. If there exists a nonzero $\pi \in \Omega$ such that $\pi x = 0$, x is called a torsion (or nonfree) element. Since Ω is an integral domain (i.e., Ω is a commutative ring with no divisors of zero), the set $T(M)$ of torsion elements of M is a submodule of M.

Let S be a subset of M. The annihilator of S, denoted by $\mathrm{Ann}(S)$, is the set of elements $\pi \in \Omega$ such that $\pi x = 0$ for all $x \in S$. For any subset $S \subset M$, $\mathrm{Ann}(S)$ is an ideal of the ring Ω. If $S = \{x\}$, we write $\mathrm{Ann}(S) = \mathrm{Ann}(x)$.

Given an Ω-module homomorphism $\lambda : \Omega^m \to M$, let M_r denote the submodule of M consisting of all reachable elements; that is, $M_r = \lambda(\Omega^m)$. Since Ω^m is a finitely-generated Ω-module, M_r is also finitely generated, in particular, $M_r = \sum_{i=1}^{m} \Omega q_i$ where $q_i = \lambda(e_i)$, $e_i = (0\ 0 \ldots \underset{\uparrow\, i^{th} \text{ place}}{\delta_o}\ 0 \ldots 0)^{TR}$.

It is easily verified that $\mathrm{Ann}(M_r) = \bigcap_i \mathrm{Ann}(\Omega q_i) = \bigcap_i \mathrm{Ann}(q_i)$.

Using this fact, we can prove

<u>Proposition 5.1</u>: Suppose that $M_r \neq \{0\}$, then the following are equivalent

i. $\mathrm{Ann}(M_r) \neq \{0\}$

ii. $M_r \subset T(M)$

iii. Each nontrivial submodule Ωq_i contains a nonzero torsion element.

<u>Proof</u>: Obviously, i. $\Rightarrow$ ii. and ii. $\Rightarrow$ iii.

iii. $\Rightarrow$ i.: Suppose that for each $q_i \neq 0$ there exists $0 \neq x_i \in \Omega q_i$ and $0 \neq \pi_i \in \Omega$ such that $\pi_i x_i = 0$. Then since $x_i = \omega_i q_i$ for some $\omega_i \in \Omega$, $\omega_i \neq 0$, we have that $\pi_i x_i = (\pi_i * \omega_i) q_i = 0$. Hence $0 \neq \pi_i * \omega_i \in \mathrm{Ann}(q_i)$ and the product $\prod_i (\pi_i * \omega_i) \neq 0$ annihilates each q_i, $i=1,2,\ldots,m$. Thus $\prod_i (\pi_i * \omega_i) \in \bigcap_i \mathrm{Ann}(q_i) = \mathrm{Ann}(M_r)$.

The following result shows that if $M_r \neq \{0\}$ and any one of the equivalent statements of Proposition 5.1 is not true, then for at least one i such that $q_i \neq 0$, every nonzero state in Ωq_i is uncontrollable.

Proposition 5.2: If $M_r \neq \{0\}$ and for each i such that $q_i \neq 0$, the submodule Ωq_i contains a nonzero controllable element, then $\mathrm{Ann}(M_r) \neq \{0\}$.

Proof: Suppose that the hypothesis is satisfied. Then for each i such that $q_i \neq 0$, there exist $0 \neq x_i = \omega_i q_i$, $\tau_i < 0$, and $u_i \in \Omega^m$ with $\ell(u_i) > \tau_i$, such that

$$\delta_{\tau_i}(\omega_i q_i) + \lambda(u_i) = (\delta_{\tau_i} * \omega_i) q_i + \lambda(u_i) = 0$$

Writing $u_i = \sum_j u_{ij} e_j$, $u_{ij} \in \Omega$ with $\ell(u_{ij}) > \tau_i$, we have

$$(\delta_{\tau_i} * \omega_i) q_i + \sum_j u_{ij} q_j = 0$$

Hence

$$(\delta_{\tau_i} * \omega_i + u_{ii}) q_i + \sum_{j \neq i} u_{ij} q_j = 0 \tag*{5.2}$$

Now for each i such that $q_i = 0$, we have

$$\delta_o q_i = 0 \tag*{5.2}$$

Let C denote the m×m matrix consisting of the coefficients of the q_i in equations (5.2-3) such that the diagonal elements of C are $\delta_{\tau_i} * \omega_i + u_{ii}$ or δ_o. Then (Lang [9, page 335]) the determinant of C, denoted by det C, annihilates each q_i, and thus $\det C \in \mathrm{Ann}(M_r)$. It must be shown that $\det C \neq 0$:

By construction, det C is of the form

$$\det C = \Big(\prod_i (\delta_{\tau_i} * \omega_i)\Big) + \pi$$

where $\ell(\pi) > \sum_i \tau_i$ since $\ell(u_{ij}) > \tau_i$.

Hence the support of π does not intersect the support of the product $\prod_i (\delta_{\tau_i} * \omega_i)$ and since $\prod_i (\delta_{\tau_i} * \omega_i) \neq 0$, det $C \neq 0$.

Corollary 5.1: If M_r contains a free element, then for at least one i, $q_i \neq 0$ and every nonzero element of Ωq_i is uncontrollable.

Corollary 5.2: If $m = 1$ and $T(M_r) = \{0\}$, no nonzero element of M is both reachable and controllable.

Proposition 5.3: Suppose that for some fixed i, q_i is free and $q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m$ have torsion, then M_r can be written as an internal direct sum

$$M_r = M_1 \oplus \Omega q_i \quad \text{where} \quad M_1 = \sum_{\substack{j=1 \\ j \neq i}}^{m} \Omega q_i$$

and every element $x = x_1 + x_i$ is uncontrollable, where $x_1 \in M$, $x_i \in \Omega q_i$, $x_i \neq 0$.

Proof: Clearly, $M = M_1 + \Omega q_i$. Suppose that there exists an $x \neq 0$ with $x \in M_1 \cap \Omega q_i$. Then since $\mathrm{Ann}(M_1)$ is nontrivial by Proposition 5.1, there exists a $\pi \in \Omega$, $\pi \neq 0$, such that $\pi x = 0$. Now $x = \omega q_i$ for some $\omega \in \Omega$, $\omega \neq 0$, and thus $(\pi * \omega) q_i = 0$, $\pi * \omega_i \neq 0$, which is a contradiction if q_i is free.

Suppose that $x = x_1 + x_i$ is controllable where $x_1 \in M_1$ and $x_i = \pi q_i \neq 0$. Then there exist $\tau < 0$, $\omega \in \Omega^m$, $\ell(\omega) > \tau$, such that $\delta_\tau (x_1 + \pi q_i) + \lambda(\omega) = 0$. Writing $\omega = \sum_j \omega_j e_j$ and since $x_1 = \sum_{\substack{j \\ j \neq i}} \alpha_j q_j$, $\alpha_j \in \Omega$, we obtain

$$\sum_{\substack{j \\ j \neq i}} (\delta_\tau * \alpha_j + \omega_j) q_j + (\delta_\tau * \pi + \omega_i) q_i = 0$$

Multiplying both sides of this equation by some $\beta \in \mathrm{Ann}(M_1)$, $\beta \neq 0$, gives $\beta * (\delta_T * \pi + \omega_i) q_i = 0$, which is a contradiction since q_i is free.

As seen from the following results, the condition $\mathrm{Ann}(M_r) \neq \{0\}$ is also related to the controllability of elements that are not necessarily reachable.

Proposition 5.4: If $\mathrm{Ann}(M_r)$ is nontrivial, every controllable element of M is contained in T(M), the torsion submodule of M.

Proof: If $x \in M$ is controllable, there exist $\tau < 0$, $\omega \in \Omega^m$, $\ell(\omega) > \tau$, such that $\delta_\tau x = -\lambda(\omega)$. Since $\lambda(\omega) \in M_r$, if $\mathrm{Ann}(M_r) \neq \{0\}$, there exists $\pi \in \Omega$, $\pi \neq 0$, such that $(\pi * \delta_\tau) x = -\pi\lambda(\omega) = 0$. Hence x is a torsion element.

Corollary 5.3: If $\mathrm{Ann}(M_r) \neq \{0\}$, every free element of M is not controllable and not reachable.

Proposition 5.5: Suppose that for each $i \in \{1,2,\ldots,m\}$ there exists a nonzero torsion element of M which is controllable with control $\omega_i e_i \neq 0$, $\omega_i \in \Omega$, then $\mathrm{Ann}(M_r)$ is nontrivial.

Proof: Let $x_1, x_2, \ldots, x_m$ be nonzero torsion elements of M such that for each i, there exist $\tau_i < 0$, $\omega_i e_i \neq 0$, $\ell(\omega_i) > \tau_i$, with $\delta_{\tau_i} x_i + \lambda(\omega_i e_i) = 0$. Then if $\alpha_i x_i = 0$, $\alpha_i \neq 0$, $(\alpha_i * \delta_{\tau_i}) x_i = -\alpha_i \lambda(\omega_i e_i) = -(\alpha_i * \omega_i) q_i = 0$, and thus $\alpha_i * \omega_i \in \mathrm{Ann}(q_i)$. Therefore the product $\prod_i (\alpha_i * \omega_i)$ is a nonzero element of $\mathrm{Ann}(M_r)$.

It follows from Corollary 5.1 that for every element of M_r to be controllable, it is necessary that $\mathrm{Ann}(M_r)$ be nontrivial. Whether or not the reachable states are controllable is an important question since for any $x_1, x_2 \in M_r$, there exists a control $\omega \in \Omega^m$ which sets up x_2 from x_1 (i.e.,

$\delta_\tau x_1 + \lambda(\omega) = x_2$ for some $\tau < \ell(\omega)$) if and only if every element of M_r is controllable. It is interesting to note that when M_r is finite dimensional as a linear space over $\mathbb{R}$, every $x \in M_r$ is controllable. The easy proof is omitted.

In terms of the module structure, we now develop a necessary and sufficient condition for controllability of M_r. We begin with the following ring-theoretic result.

Lemma 5.1: Let A be an ideal of the ring Ω and suppose that there exist $\tau < 0$ and $\alpha \in \Omega$, $\ell(\alpha) > \tau$, such that $\delta_\tau + \alpha \in A$. Given $\omega \in \Omega$, let $s \geq 0$ be an integer such that $(s+1)(\tau - \ell(\alpha)) < \ell(\omega)$. Then $\delta_{(s+1)\tau} * \omega + \pi \in A$, where $\pi = (-1)^s \alpha^{s+1} * \omega$, $\ell(\pi) > (s+1)\tau$, α^{s+1} = (s+1)-fold convolution of α.

Proof: Given $\tau < 0$, $\alpha \in \Omega$, $\ell(\alpha) > \tau$, such that $\delta_\tau + \alpha \in A$, by induction it is easily verified that for any integer $s \geq 0$

$$\delta_{(s+1)\tau} + (-1)^s \alpha^{s+1} = (\delta_\tau + \alpha) * \left(\sum_{i=0}^{s} (-1)^i \delta_{(s-i)\tau} * \alpha^i \right)$$

Then since A is an ideal of Ω and $\delta_\tau + \alpha \in A$, $(\delta_{(s+1)\tau} + (-1)^s \alpha^{s+1}) * \omega \in A$ for any $\omega \in \Omega$. Now given a fixed $\omega \in \Omega$, we pick an integer $s \geq 0$ such that $(s+1)(\tau - \ell(\alpha)) < \ell(\omega)$. Such an integer can always be found since $\tau - \ell(\alpha) \leq 0$.

Then $\quad (s+1)\tau < (s+1)\ell(\alpha) + \ell(\omega)$

$\Rightarrow (s+1)\tau < \ell(\alpha^{s+1}) + \ell(\omega)$ using (3.5)

$\Rightarrow (s+1)\tau < \ell(\alpha^{s+1} * \omega)$ again using (3.5)

$\Rightarrow (s+1)\tau < \ell(\pi)$ by definition of π.

Theorem 5.1: Every $x \in M_r$ is controllable if and only if there exists $\delta_\tau + \alpha \in \mathrm{Ann}(M_r)$ with $\ell(\alpha) > \tau$.

Proof: Recall that $M_r = \sum_{i=1}^{m} \Omega q_i$. If every $x \in M_r$ is controllable, each q_i is controllable, and thus for each i, there exist $\tau_i < 0$, $u_i \in \Omega^m$, $\ell(u_i) > \tau_i$, with $\delta_{\tau_i} q_i + \lambda(u_i) = 0$. Using the construction given in the proof of Proposition 5.2, we have that $\mathrm{Ann}(M_r)$ contains an element of the form $\delta_\tau + \pi$, $\ell(\pi) > \tau \overset{\Delta}{=} \sum_i \tau_i$. Conversely, suppose that there exists $\delta_\tau + \alpha \in \mathrm{Ann}(M_r)$, $\ell(\alpha) > \tau$, and let $x \in M_r$. Then $x = \lambda\left(\sum_i \omega_i e_i\right)$, $\omega_i \in \Omega$. Let $s \geq 0$ be an integer such that $(s+1)(\tau - \ell(\alpha)) < \ell(\omega_i)$, $i=1,2,\ldots,m$. Then by Lemma 5.1, for each i, $\delta_{(s+1)\tau} * \omega_i + \pi_i \in \mathrm{Ann}(M_r)$, $\ell(\pi_i) > (s+1)\tau$, $\pi_i = (-1)^s \alpha^{s+1} * \omega_i$.

Hence $\qquad (\delta_{(s+1)\tau} * \omega_i + \pi_i) q_i = 0, \quad i=1,2,\ldots,m.$

$$\Rightarrow \quad \sum_i \left(\delta_{(s+1)\tau} * \omega_i + \pi_i\right) q_i = 0$$

$$\Rightarrow \quad \delta_{(s+1)\tau} x + \sum_i \pi_i q_i = 0$$

Since $\ell(\pi_i) > (s+1)\tau$, all i, the element $\sum_i \pi_i e_i \in \Omega^m$ is a control for x.

Corollary 5.4: Every $x \in M_r$ is controllable if and only if each generator q_i is controllable.

Examples for which the condition in Theorem 5.1 is satisfied will be given in Section 7.

6. Bounded and Minimal Time Controllability

Given an Ω-module homomorphism $\lambda : \Omega^m \to M$, the submodule $M_r = \lambda(\Omega^m)$ is said to be reachable (resp. controllable) in bounded time N if for each $x \in M_r$, there exists an $\omega \in \Omega^m$ with $\ell(\omega) > -N$, such that $x = \lambda(\omega)$ (resp.

$\delta_{-N}x + \lambda(\omega) = 0$). In the first part of this section, we prove that if every element of M_r is controllable, then M_r is reachable and controllable in bounded time. Then we consider the determination of the smallest time period during which all the elements of M_r can be controlled. In the last part of the section results are given on the smoothness of the controls constructed here.

Let A be an ideal of the ring Ω, and let Ω/A denote the residue class ring of Ω by A. The elements of Ω/A will be denoted by $[\omega] = \omega + A$, $\omega \in \Omega$. Recall that Ω is a subring of $V = \mathcal{D}'_+$, the ring of distributions with support bounded on the left. We also note that for any $\gamma \in V$, $\varphi \in \mathcal{D}$, the multiplication $\gamma\varphi$ of γ by φ is defined by $< \gamma\varphi, \chi > = < \gamma, \varphi\chi >$ where $\varphi\chi$ is the usual pointwise multiplication of functions.

Lemma 6.1: Let A be an ideal of Ω and suppose that there exists a $\beta \in A$ having an inverse $\beta^{-1} \in V$. Let $\tau < \ell(\beta)$, then for each $[\omega] \in \Omega/A$ there exists an $\alpha \in [\omega]$ such that $\ell(\alpha) > \tau$.

Proof: Assume that there exists $\beta \in A$ with $\beta^{-1} \in V$, and fix $\tau < \ell(\beta)$. Given $[\omega] \in \Omega/A$, if $\ell(\omega) > \tau$ there is nothing to prove. Therefore assume that $\ell(\omega) \le \tau$. Now $\beta*(\beta^{-1}*\omega) = \omega$, and thus $\ell(\omega) \ge \ell(\beta) + \ell(\beta^{-1}*\omega)$. Since $\ell(\omega) \le \tau$ and $\tau < \ell(\beta)$,

$$\ell(\beta^{-1}*\omega) \le \tau - \ell(\beta) < 0.$$

Choose $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $-\infty < a_2 < a_1 < \ell(\beta^{-1}*\omega)$ and $\tau - \ell(\beta) < b_1 < b_2 < 0$. By a well-known result of distribution theory (see [6, page 31]), there exists a $\varphi \in \mathcal{D}$ such that $\varphi(t) = 1$ on $[a_1, b_1]$, $\varphi(t) = 0$ on $\mathbb{R} - [a_2, b_2]$, and $0 \le \varphi(t) \le 1$, all $t \in \mathbb{R}$.
Then $\text{supp}[(\beta^{-1}*\omega)\varphi] \subseteq [\ell(\beta^{-1}*\omega), b_2]$, which implies that $(\beta^{-1}*\omega)\varphi \in \Omega$.

Now define $\alpha = -\beta*[(\beta^{-1}*\omega)\varphi - \beta^{-1}*\omega]$. Then $\alpha = -\beta*[(\beta^{-1}*\omega)\varphi] + \omega \in [\omega]$. It is claimed that $\ell(\alpha) > \tau$:
By construction, $(\beta^{-1}*\omega)\varphi = \beta^{-1}*\omega$ on $(-\infty, b_1)$, and thus $\mathrm{supp}[-(\beta^{-1}*\omega)\varphi + \beta^{-1}*\omega] \subset [b_1, \infty)$. Then by definition of α, $\mathrm{supp}\ \alpha \subseteq [b_1+\ell(\beta), b_2]$. Therefore, $\ell(\alpha) \geq b_1+\ell(\beta)$, but by definition of b_1, $\tau < b_1 + \ell(\beta)$, and hence $\ell(\alpha) > \tau$.

Using Lemma 6.1, we obtain the following sufficient condition for M_r to be reachable and controllable in bounded time.

Theorem 6.1: Given $M_r = \lambda(\Omega^m)$, if $\mathrm{Ann}(M_r)$ contains an element β having an inverse $\beta^{-1} \in V$, for any $a > 0$ and $x_1, x_2 \in M_r$, there exists a control $\omega \in \Omega^m$, with $\ell(\omega) > \tau \triangleq \ell(\beta) - a$, such that $\delta_\tau x_1 + \lambda(\omega) = x_2$.

Proof: Let β satisfy the hypothesis, fix $a > 0$, and set $\tau = \ell(\beta) - a$. Then given $x_1 = \sum_i \omega_i q_i$ and $x_2 = \sum_i \pi_i q_i$, by Lemma 6.1 (taking $A = \mathrm{Ann}(M_r)$), for each i there exists an $\alpha_i \in (\delta_\tau * \omega_i - \pi_i) + \mathrm{Ann}(M_r)$, with $\ell(\alpha_i) > \tau$. Hence $\delta_\tau x_1 - x_2 = \sum_i \alpha_i q_i$, which proves that x_2 can be set up from x_1 by control $-\sum \alpha_i q_i$.

Corollary 6.1: If there exists a $\beta \in \mathrm{Ann}(M_r)$ with $\beta^{-1} \in V$, then M_r is reachable and controllable in bounded time $-\ell(\beta) + a$ where a is an arbitrarily small positive number.

Referring back to Theorem 5.1, we had that every element of M_r is controllable if and only if there exists $\delta_\tau + \alpha \in \mathrm{Ann}(M_r)$ with $\ell(\alpha) > \tau$. As we shall see, this condition implies that $\mathrm{Ann}(M_r)$ contains a β with $\beta^{-1} \in V$, giving the following surprising result.

Theorem 6.2: M_r is reachable and controllable in bounded time if and only if every element of M_r is controllable.

The proof of Theorem 6.2 follows from

Lemma 6.2: Any element of the form $\delta_\tau+\alpha \in \Omega$, $\ell(\alpha) > \tau$, has an inverse in V.

Proof: Given $\delta_\tau+\alpha \in \Omega$, $\ell(\alpha) > \tau$, consider $\delta_{-\tau}(\delta_\tau+\alpha) = \delta_o+\delta_{-\tau}\alpha$, which is an element of V. It will be shown that $\delta_o+\delta_{-\tau}\alpha$ has a (unique) inverse in V. Viewing $(\delta_o+\delta_{-\tau}\alpha)^{-1}$ as an element in the quotient field of V, we can expand by long division giving

$$(\delta_o+\delta_{-\tau}\alpha)^{-1} = \sum_{n=o}^{\infty} (-\delta_{-\tau}\alpha)^n \tag{6.1}$$

Let $\{\gamma_i\}$ denote the sequence of partial sums obtained from the sum (6.1). Now since $\ell(\delta_{-\tau}\alpha) = a$, some $a > 0$, $\ell((\delta_{-\tau}\alpha)^n) \geq na$. Then given $\varphi \in \mathcal{D}$, since φ has compact support there exist an integer i_o and a constant K such that $\gamma_i(\varphi) = K$, all $i \geq i_o$. Hence $\{\gamma_i(\varphi)\}$ converges in $\mathbb{R}$, proving that $\{\gamma_i\}$ converges in V. Therefore $(\delta_o+\delta_{-\tau}\alpha)^{-1}$ is a distribution with support contained in $[0,\infty)$, and since $(\delta_\tau+\alpha)^{-1} = \delta_{-\tau}(\delta_o+\delta_{-\tau}\alpha)^{-1}$, $\delta_\tau+\alpha$ has an inverse in V.

If M_r is controllable in bounded time, the question then arises as to what is the smallest time interval during which all the elements of M_r can be controlled. This minimal control time, denoted by N_{min}, is defined to be the infimum over all N such that M_r is controllable in time N. We have the following results on the magnitude of N_{min}.

Let Ker λ denote the submodule $\{\omega \in \Omega^m : \lambda(\omega) = 0\} \subset \Omega^m$, and define

$$S_1 = \{\omega = (\omega_1,\ldots,\omega_m)^{TR} \in \text{Ker } \lambda : \omega_i^{-1} \in V,\ i=1,2,\ldots,m\}$$

$$S_2 = \{\omega = (\omega_1,\ldots,\omega_m)^{TR} \in S_1 : \omega_i \in \text{Ann}(q_i),\ i=1,2,\ldots,m\}$$

In terms of S_1 and S_2, we have the following bounds on $N_{\min}$.

Theorem 6.3: If M_r is controllable in bounded time, then

$$\inf_{\omega \in S_1} \{-\ell(\omega)\} \leqq N_{\min} \leqq \inf_{\omega \in S_2} \{-\ell(\omega)\}$$

Proof: If M_r is controllable in time N, for each i=1,2,...,m, there exists $u_i \in \Omega^m$, $\ell(u_i) > -N$, such that $\delta_{-N}q_i + \lambda(u_i) = 0$. Thus $\delta_{-N}e_i + u_i \in \text{Ker } \lambda$, all i, which implies that $\sum_i (\delta_{-N}e_i + u_i) \in \text{Ker } \lambda$. Since $\ell(u_i) > -N$, all i, $\sum_i (\delta_{-N}e_i + u_i) = \sum_i (\delta_{-N} + \pi_i)e_i$ for some $\pi_i \in \Omega$ with $\ell(\pi_i) > -N$, all i. By Lemma 6.2, each $\delta_{-N} + \pi_i$ has an inverse in V, and thus $\sum_i (\delta_{-N}e_i + u_i) \in S_1$. Therefore, $N_{\min} \geqq \inf_{\omega \in S_1} \{-\ell(\omega)\}$.

Now S_2 is not empty since $\text{Ann}(M_r) \neq \{0\}$. Let $\omega \in S_2$, then it follows from Lemma 6.1 that M_r is controllable in bounded time $-\ell(\omega)+a$, any $a > 0$. Hence $N_{\min} \leqq \inf_{\omega \in S_2} \{-\ell(\omega)\}$.

When m = 1, $\text{Ker } \lambda = \text{Ann}(M_r)$ and $S_1 = S_2$ so we have

Corollary 6.2: If m = 1 and each $x \in M_r$ is controllable, M_r is controllable in minimal time $N_{\min} = \inf\{-\ell(\pi) : \pi \in \text{Ann}(M_r),\ \pi^{-1} \in V\}$.

In the next section, we shall use this result to compute minimal control times for delay-differential systems.

Given $\beta \in \text{Ann}(M_r)$ with $\beta^{-1} \in V$, by Theorem 6.1 every $x \in M_r$ can be controlled in bounded time τ for any $\tau < \ell(\beta)$. In particular, if $x = \sum \omega_i q_i$, by the construction given in the proof of Lemma 6.1 a control $u \in \Omega^m$ for x is

6.2 $\quad u = \sum_i u_i e_i$ where $u_i = -\beta * [(\beta^{-1} * \delta_\tau * \omega_i)\varphi_i] + \delta_\tau * \omega_i,\ \tau < \ell(\beta)$.

However the control u may be so "rough" that it is not possible to generate an actual signal which is a good approximation to u. For example, this is the case if u contains derivatives of the Dirac distribution. Therefore some indication of the smoothness of the control (6.2) is very desirable. We now consider this by using the concept of the order of a distribution.

Let $\gamma \in \mathcal{D}'$ and let U be an open set contained in $\mathbb{R}$. The order of γ on U, denoted by ord $\gamma|_U$, is the smallest integer r such that $\gamma = h_\gamma^{(r)}$ on U where $h_\gamma^{(r)}$ is the r^{th} derivative of some continuous function h_γ on U. If no such positive integer r exists, γ is said to be of infinite order on U. If γ is infinitely differentiable on U, we write ord $\gamma|_U = -\infty$. The order of any distribution on a bounded set U is finite or $-\infty$ and so is the order of any distribution on $\mathbb{R}$ having compact support (see [6, page 95]). It is easily verified that for any u, v $\in \mathcal{D}'$ having order $< +\infty$, $\text{ord}(u*v) \le (\text{ord } u) + (\text{ord } v)$.

Now given $x = \sum_i \omega_i q_i$, consider the control (6.2). We have the following upper bound on the order of the components of u:

<u>Theorem 6.4</u>: $\text{ord}(u_i) \le \text{ord}\,\beta + \text{ord}(\omega_i) + \text{ord}\,\beta^{-1}|_{(o,-\tau-\ell(\omega_i))}$

<u>Proof</u>: Given $u_i = -\beta*[(\beta^{-1}*\delta_\tau*\omega_i)\varphi_i] + \delta_\tau*\omega_i$, since supp $u_i \subset (\tau,0)$ and $\text{supp}(\delta_\tau*\omega_i) \subseteq [\tau+\ell(\omega_i),\tau]$, $u_i = -\beta*[(\beta^{-1}*\delta_\tau*\omega_i)\varphi_i]$ on $(\tau,0)$, $u_i = 0$, otherwise. Thus

$$\text{ord}(u_i) = \text{ord}\,\beta*[(\beta^{-1}*\delta_\tau*\omega_i)\varphi_i]|_{(\tau,o)}$$

$$\text{ord}(u_i) \le \text{ord}\,\beta + \text{ord}[(\beta^{-1}*\delta_\tau*\omega_i)\varphi_i]|_{(\tau,o)}$$

$$\text{ord}(u_i) \le \text{ord}\,\beta + \text{ord}(\beta^{-1}*\delta_\tau*\omega_i)|_{(\tau,o)} \qquad \text{since } \varphi \in \mathcal{D}$$

$$\text{ord}(u_i) \le \text{ord}\,\beta + \text{ord}(\beta^{-1}*\omega_i)|_{(o,-\tau)}$$

$$\text{ord}(u_i) \le \text{ord}\,\beta + \text{ord}\,\omega_i + \text{ord}\,\beta^{-1}|_{(o,-\tau-\ell(\omega_i))} \qquad \text{since supp } \omega_i \subseteq [\ell(\omega_i),0]$$

Corollary 6.3: If β^{-1} is infinitely differentiable on $(0,\infty)$, every element of M_r has a control whose components are infinitely differentiable.

Proof: In this case, ord $\beta^{-1}|_{(o,\infty)} = -\infty$, so that by the theorem, ord$(u_i) = -\infty$, implying that u_i is infinitely differentiable.

As will be seen in the next section, there exist controls that are infinitely differentiable when M_r is finite dimensional as a linear space over $\mathbb{R}$.

7. Role of the Impulse Response Matrix in Controllability

The results of the preceding two sections reveal that the annihilating ideal $\mathrm{Ann}(M_r)$ plays a crucial role in the controllability of $M_r = \lambda(\Omega^m)$. Given a system $\Sigma = (X,\mu,\eta,\psi)$, for the special cases $\lambda = \mu$ and $\lambda = \eta\mu = f^*$ we now investigate the properties of $\mathrm{Ann}(M_r)$ by relating it to the impulse response matrix W of the system Σ. Here we obtain particular results on output function and state controllability, expressed in terms of the properties of W.

For the system $\Sigma = (X,\mu,\eta,\psi)$, let $X_r = \mu(\Omega^m)$ and $(\Gamma^k)_r = f^*(\Omega^m)$ denote the finitely-generated submodules of reachable states and reachable outputs, respectively. Letting $\{e_1,\dots,e_m\}$ denote the standard basis of Ω^m as before, we have that X_r is generated by $g_i \triangleq \mu(e_i)$, $i=1,2,\dots,m$, and $(\Gamma^k)_r$ is generated by $h_i \triangleq f^*(e_i)$, $i=1,2,\dots,m$.

Since f^* is equal to the composition $\eta\mu$, $(\Gamma^k)_r = \eta(X_r)$, and thus $\mathrm{Ann}(X_r) \subseteq \mathrm{Ann}(\Gamma^k)_r$. However in general $\mathrm{Ann}(X_r) \neq \mathrm{Ann}(\Gamma^k)_r$. A necessary and sufficient condition for equality is given in the following

Proposition 7.1: $\mathrm{Ann}(X_r) = \mathrm{Ann}(\Gamma^k)_r$ if and only if the restriction of η to the submodule Ωg_i is injective for each i such that $g_i \neq 0$.

The proof of this result is straightforward, and therefore will not be given.

Recalling that the system Σ is completely observable if η is injective, we have

Corollary 7.1: If Σ is completely observable, $\mathrm{Ann}(X_r) = \mathrm{Ann}(\Gamma^k)_r$.

Now let $W = (w_{ij})$ denote the impulse response matrix of the system Σ. For each $i=1,2,\ldots,k$ and $j=1,2,\ldots,m$, define $A_{ij} = \{\pi \in \Omega : w_{ij} * \pi \in \Omega\}$. Each A_{ij} is an ideal of the ring Ω. In terms of the A_{ij}, the following result establishes a direct relationship between the impulse response matrix and the annihilating ideal of $(\Gamma^k)_r$.

Proposition 7.2: $\mathrm{Ann}(\Gamma^k)_r = \bigcap_{i,j} A_{ij}$.

Proof: Let $\pi \in \mathrm{Ann}(\Gamma^k)_r$, then $\pi h_j = 0$, $j=1,2,\ldots,m$, $h_j = f^*(e_j)$. Hence $(w_{ij} * \pi)\big|_{(o,\infty)} = 0$, $i=1,2,\ldots,k$, $j=1,2,\ldots,m$, which implies that $w_{ij} * \pi \in \Omega$, all i,j. Thus $\pi \in \bigcap_{i,j} A_{ij}$. Conversely, let $\pi \in \bigcap_{i,j} A_{ij}$, then $(w_{ij} * \pi)\big|_{(o,\infty)} = 0$, all i,j. Thus $f^*(\pi e_j) = 0$, $j=1,2,\ldots,m \Rightarrow \pi h_j = 0$, all j $\Rightarrow \pi \in \mathrm{Ann}(\Gamma^k)_r$.

From the results of Section 6, the reachable states and outputs are controllable if and only if each ideal, $\mathrm{Ann}(X_r)$ and $\mathrm{Ann}(\Gamma^k)_r$, contains an element that is invertible in $V = \mathcal{D}'_+$. Since $\mathrm{Ann}(X_r) \subseteq \mathrm{Ann}(\Gamma^k)_r = \bigcap_{i,j} A_{ij}$, controllability is therefore connected to the existence of invertible elements in $\bigcap_{i,j} A_{ij}$, which we now consider. The approach given below is developed in terms of fields and rings of fractions.

Since V (resp. Ω) is an integral domain, the smallest field in which V (Ω) can be embedded is its quotient field, denoted by $Q(V)$ $(Q(\Omega))$. Let $\mathcal{J}:V \to Q(V):v \mapsto \frac{v}{\delta_o}$ denote the embedding of V in $Q(V)$. Note that since $\Omega \subset V$, $Q(\Omega)$ is a subfield of $Q(V)$.

Proposition 7.3: Given a system Σ with impulse response matrix W, $\bigcap_{i,j} A_{ij} \neq \{0\}$ if and only if $\mathcal{J}(w_{ij}) \in Q(\Omega)$, all i,j.

Proof: Clear.

From Corollary 5.1, we have

Corollary 7.2: If any one of the elements of W cannot be embedded in $Q(\Omega)$, there exist at least one $g_i \neq 0$ and $h_j \neq 0$ such that every nonzero state in Ωg_i is uncontrollable and every nonzero output in Ωh_j is uncontrollable.

As we now show, a condition for controllability is that the elements of W belong to a ring of fractions of Ω. Let $D = \{\pi \in \Omega:\pi^{-1} \in V\}$, which is clearly a multiplicative subset of the ring Ω. Let $D^{-1}\Omega$ denote the ring of fractions of Ω defined by D. Note that $D^{-1}\Omega$ can be viewed as a subring of V under the embedding $D^{-1}\Omega \to V:\frac{\omega}{\pi} \mapsto \pi^{-1}*\omega$. Then combining Theorems 6.1-2 and Propositions 7.1-2, we have

Theorem 7.1: Given a system $\Sigma = (X,\mu,\eta,\psi)$ with impulse response matrix W, the following are equivalent:

i. $w_{ij} \in D^{-1}\Omega$, all i,j
ii. Every reachable output is controllable
iii. The submodule of reachable outputs is reachable and controllable in bounded time.

Furthermore, for the reachable states to be controllable (or controllable in bounded time) it is necessary that one of these conditions be true.

Theorem 7.2: If the restriction of η on each nontrivial submodule Ωg_i is injective or if Σ is completely observable, the following are equivalent:

i. $w_{ij} \in D^{-1}\Omega$, all i,j
ii. Every reachable state is controllable
iii. The submodule of reachable states is reachable and controllable in bounded time.

Some important consequences of Theorem 7.2 are

Corollary 7.3: If $\Sigma = (X,\mu,\eta,\psi)$ is completely reachable and observable, then X is completely controllable (or controllable in bounded time) if and only if $w_{ij} \in D^{-1}\Omega$, all i,j.

Corollary 7.4: A strictly causal i/o operator $f:V^m \to V^k : v \mapsto W*v$ has a canonical realization (X,μ,η,ψ) with X completely controllable if and only if W is over $D^{-1}\Omega$.

These results show that controllability properties of the systems considered here are nice if the impulse response matrix is over $D^{-1}\Omega$. There exist systems for which this is not the case. For example, consider a single-input single-output system with impulse response $w(t) = e^{-t^2}H(t)$, where $H(t)$ is the Heaviside function. Because $e^{-(t-\tau)^2}$ contains the factor $e^{2\tau t}$, it follows that there does not exist a $\beta \in \Omega$ with $\beta^{-1} \in V$, such that $\beta * w \in \Omega$. The details are rather involved and will not be given.

Examples of classes of systems having impulse response matrix defined over $D^{-1}\Omega$ can be generated in the following manner. Let K be a multiplicative subset of Ω with $K \subseteq D$. Let $\mathcal{K}$ denote the class of all strictly causal systems (Definition 3.1) whose impulse response matrix is over $K^{-1}\Omega \subseteq D^{-1}\Omega$.

Example 1: Let $\mathbb{R}[p] = \left\{ \sum_{i=o}^{n} a_i p^i : a_i \in \mathbb{R},\ n \geq 0 \right\}$ where p^i denotes the i^{th} derivative of δ_o. Clearly, $\mathbb{R}[p]$ is a subring of Ω. Further, it is well known (see [6]) that every nonzero element of $\mathbb{R}[p]$ has an inverse in V which is infinitely differentiable on $(0,\infty)$. Thus we can take $K = \mathbb{R}[p] - \{0\}$. The resulting class $\mathcal{K}$ of systems includes all finite-dimensional systems. Let Σ be a system in this class. Since for any $\beta \in K$, $\ell(\beta) = 0$ and β^{-1} is infinitely differentiable on $(0,\infty)$, by Theorem 6.1 and Corollary 6.3 every reachable state and every reachable output of Σ can be controlled in an arbitrarily small time interval by a control whose components are infinitely differentiable.

Example 2: Let $\mathbb{R}[d_1,\ldots,d_r,p]$ denote the smallest subring of Ω containing $d_i = \delta_{a_i}$, $a_i < 0$, $p = \delta_o^{(1)}$, and $b\delta_o$, all $b \in \mathbb{R}$. Any $\beta \in \mathbb{R}[d_1,\ldots,d_r,p]$ can be written as a finite sum

$$\beta = \sum_{j_1,\ldots,j_{r+1}} c_{j_1,\ldots,j_{r+1}} d_1^{j_1} * \cdots * d_r^{j_r} * p^{j_{r+1}}$$

where the j_i are non-negative integers, $c_{j_1,\ldots,j_{r+1}} \in \mathbb{R}$, and $d_i^j = j^{th}$-fold convolution of d_i.

From the results of Kamen [3], every nonzero element of $\mathbb{R}[d_1,\ldots,d_r,p]$ has an inverse in V, so that we can take $K = \mathbb{R}[d_1,\ldots,d_r,p] - \{0\}$. In this case the class $\mathcal{K}$ consists of systems having time delays equal to multiples of $-a_i$, $i=1,2,\ldots,r$ ($d_i = \delta_{a_i}$). In particular, $\mathcal{K}$ contains a subclass of delay-differential systems, i.e. systems whose inputs and outputs are related by delay-differential equations. By Theorems 7.1-2 we have the interesting result that for systems with time delays (as defined here), the submodules of reachable states and outputs are reachable and controllable in bounded time.

For the case $K = \mathbb{R}[d_1,\ldots,d_r,p] - \{0\}$, in many instances we can readily compute the minimal control time $N_{\min}$: Let $\Sigma = (X,\mu,\eta,\psi)$ be a single-input single-output completely reachable and observable system belonging to the class $\mathcal{K}$. Then the impulse response w is given by $w = \beta^{-1} * \pi$, some $\beta \in K$, $\pi \in \Omega$, and $\mathrm{Ann}(X) = \{\alpha \in \Omega : w*\alpha \in \Omega\}$. Suppose that $\beta\Omega + \pi\Omega = \Omega$, then we claim that $\mathrm{Ann}(X) = \beta\Omega$, that is, $\mathrm{Ann}(X)$ is a principal ideal:

Let $\alpha \in \mathrm{Ann}(X)$, then $\beta^{-1} * \pi * \alpha = \sigma$, some $\sigma \in \Omega, \Rightarrow \pi*\alpha = \beta*\sigma$

$\Rightarrow \beta | \pi*\alpha$ (i.e. β divides $\pi*\alpha$ in Ω)

Now $\beta*\gamma + \pi*\xi = \delta_o$ some $\gamma, \xi \in \Omega$, and since $\beta | \pi*\alpha*\xi$ and $\beta | \beta*\gamma*\alpha$, $\beta | (\beta*\gamma + \pi*\xi)\alpha \Rightarrow \beta | \alpha$. Hence $\mathrm{Ann}(X) \subset \beta\Omega$, while it is clear that $\beta\Omega \subset \mathrm{Ann}(X)$. Now by definition of K, $\ell(u*v) \leq \ell(u)$ for any $u,v \in K$. Thus $\ell(\alpha) \leq \ell(\beta)$ for any $\alpha \in \beta\Omega$, and by Corollary 6.2, $N_{\min} = -\ell(\beta)$. For example, let $K = \mathbb{R}[\delta_{-1},p]$ and suppose that the impulse response of the system is

$$w(t) = \sum_{n=o}^{\infty} \frac{(n-t)^n}{n!} e^{-(t-n)} H(t-n)$$

Using the operational calculus given in [3], we have $(\delta_{-1}*p+\delta_{-1}+\delta_o)*w = \delta_{-1}$, so that we can take $\beta = \delta_{-1}*p+\delta_{-1}+\delta_o$ and $\pi = \delta_{-1}$. Since

$\beta - (p+\delta_o)*\pi = \delta_o$, $\beta\Omega + \pi\Omega = \Omega$, and thus $N_{min} = -\ell(\beta) = 1$. In words, every state of Σ can be controlled to zero within a minimal time period of one second.

REFERENCES

1. R. Kalman, P. Falb, and M. Arbib, Topics in Mathematical System Theory, McGraw-Hill, 1969.

2. R. Kalman and M. Hautus, Realization of Continuous-Time Linear Dynamical Systems, Proc. Conf. Diff. Equ., NRL Math. Research Center, June 1971.

3. E. Kamen, On an Algebraic Theory of Systems Defined by Convolution Operators, to appear in Math. Systems Theory, Vol. 9, No. 1.

4. L. Schwartz, Theorie des Distributions, Hermann, Paris, 1966.

5. A. Bensoussan and E. Kamen, Continuous-Time Systems Defined over a Banach Convolution Algebra, IRIA Report INF 72 023, France, 1972.

6. A. Zemanian, Distribution Theory and Transform Analysis, McGraw-Hill, 1965.

7. R. Saeks, Causality in Hilbert Space, SIAM Review, Vol. 12, pp. 357-383, 1970.

8. F. Treves, Topological Vector Spaces, Distributions, and Kernels, Academic Press, 1967.

9. S. Lang, Algebra, Addison-Wesley, 1967.