

Testing first order properties in sparse combinatorial structures

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Graph Theory at Georgia Tech

BEFORE WE START...

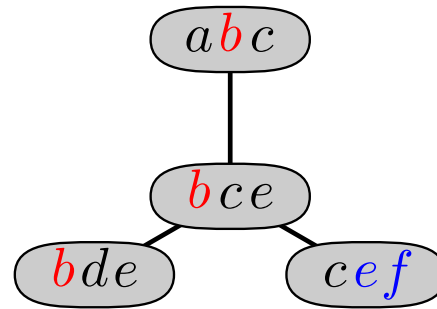
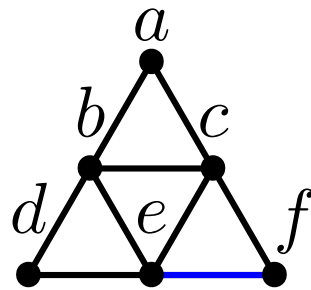
The story begins in Barcelona in the fall 2001.

OVERVIEW OF THE TALK

- Monadic second order properties
tree-width, Courcelle's theorem
- First order properties
FPT, local parameters, Frick and Grohe algorithm, other results
- Graphs with bounded expansion
structure, algorithmic aspects, short-path queries, FO prop.
- Extensions to other structures
sparse classes of relational structures, matroids
- Open problems

TREE-WIDTH, TREE-DECOMPOSITION

- decomposition tree with nodes, each has a bag of vertices
- bags containing a single vertex form a subtree
- for each edge, there is a bag containing its both end-vertices



- the width of a decomposition is its maximum bag size minus one
- the tree-width of a graph is the smallest width of its decomposition
- computable in linear time [Bodlaender, 1996]

ALGORITHMIC SUCCESS OF TREE-WIDTH

- linear-time algorithms for various problems:

existence of 3-coloring in graphs with bounded tree-width

existence of hamilton cycles in graphs with bounded tree-width

determining the choice number of graphs with bounded tree-width

database queries in instances with small hypertree-width

- What do these problems have in common?

Is there a theory unifying efficient algorithms for some problems?

MONADIC SECOND ORDER PROPERTIES

- monadic second order (MSO) formula
quantification over elements and their sets
- formula for 3-colorability

$$\exists V_1, V_2, V_3, V_1 \cup V_2 \cup V_3 = V \text{ such that } \forall uv \in E \wedge_{i=1}^3 \{u, v\} \not\subseteq V_i$$

- existence of two disjoint spanning trees
existence of hamilton cycle
- MSO_1 – only vertices and vertex sets
 MSO_2 – both vertices and edges and their sets

COURCELLE'S THEOREM

- Theorem of Courcelle (1990)
For every class of graphs with bounded tree-width,
every MSO_2 property can be decided in linear time.
- extensions to counting, etc.
- Theorem of Courcelle, Makowsky and Rotics (2000)
For every class of graphs with bounded clique-width,
every MSO_1 property can be decided in linear time.
- clique-width is another width parameter based on neighborhood complexity

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FIRST ORDER PROPERTIES

- first order (FO) formula
quantification over elements only
- containment of a subgraph/induced subgraph
dominating set has size at most 2
 $\exists u \exists v \forall w (u \sim w \vee v \sim w)$
- clearly, FO properties can be decided in polynomial time
the degree of the polynomial linear in the quantifier depth
- Can they be decided in linear/quadratic time?
fixed parameter tractability (FPT)

SOME RESULTS

- linear time algorithm for subgraph testing
for planar graphs and graphs on surfaces [Eppstein, 1994]
- FO: linear time algorithm for graphs
with bounded maximum degree [Seese, 1996]
- FO: almost-linear time algorithm for classes of graphs
with locally bounded tree-width [Frick and Grohe, 2001]
for every $\varepsilon > 0$, there exists an algorithm running in time $O(n^{1+\varepsilon})$
- FO: fixed parameter algorithms for minor-closed classes of graphs,
graphs locally excluding a minor [Dawar, Grohe, Kreutzer, 2007]

LOCAL PARAMETERS

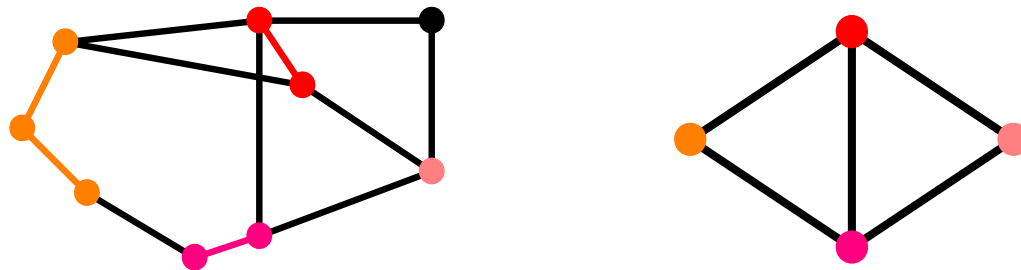
- a class \mathcal{G} of graphs has **locally bounded tree-width**, if there exists f such that the d -neighborhood of every vertex of $G \in \mathcal{G}$ has **tree-width at most $f(d)$**
In other words: the classes \mathcal{G}_d formed by d -neighborhoods of vertices of graphs in \mathcal{G} have bounded tree-width
- \mathcal{G} **locally excludes a minor**, if each \mathcal{G}_d has a forbidden minor
- introduced by Eppstein (2000), Dawar, Grohe and Kreutzer (2007)
- **planar graphs** (and more generally graphs on surfaces) have **locally bounded tree-width**

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GRAPH MINORS, SHALLOW MINORS

- a **minor** of a graph is obtained by **contracting** edges and **removing** vertices and/or edges
- alternatively, remove some vertices and edges and then contract connected subgraphs to vertices



- **d -shallow minor** if the radii of contracted subgraphs are at most d

CLASSES OF GRAPHS WITH BOUNDED EXPANSION

- introduced by Nešetřil and Ossona de Mendez in 2006
- maximum average degree

$$\text{mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$$

- let $\mathcal{G}_{\nabla d}$ be the set of all d -shallow minors of $G \in \mathcal{G}$

$$\nabla_d(\mathcal{G}) = \max_{G \in \mathcal{G}_{\nabla d}} \text{mad}(G)$$

- a class \mathcal{G} has **bounded expansion** if $\nabla_d(\mathcal{G})$ is finite for every d

EXAMPLES

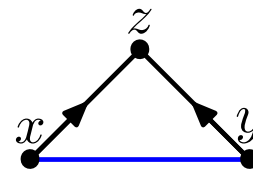
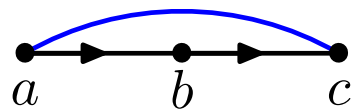
- proper minor-closed classes of graphs
every minor-closed class of graphs is degenerate
graphs on surfaces, graphs with bounded tree-width
- classes of graphs with bounded maximum degree
the maximum degree of a d -shallow minor is at most Δ^d
- proper topologically closed classes of graphs
graph classes excluding certain graphs as subdivisions

FRATERNAL ORIENTATIONS

- consider a class \mathcal{G} of graphs with bounded expansion
- every graph $G \in \mathcal{G}$ has an orientation with bounded in-degrees
- **fraternal augmentation**

transitive closure: $ab \wedge bc \Rightarrow ac$

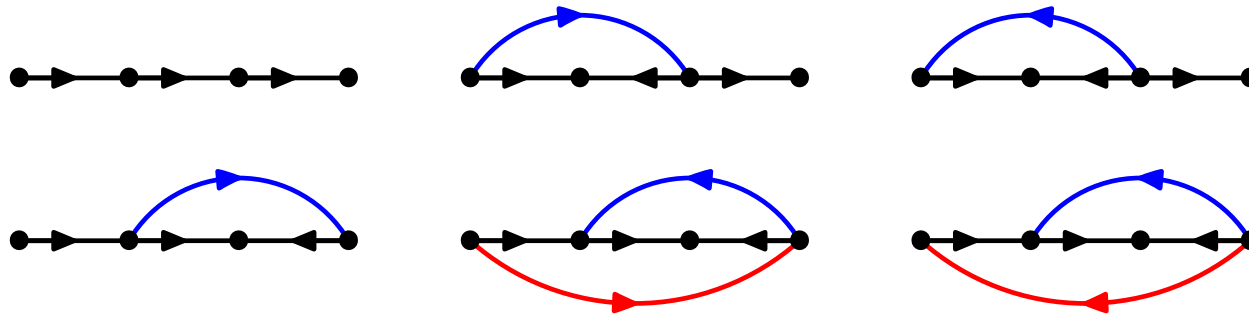
fraternal closure: $xz \wedge yz \Rightarrow xy$



- the class of augmented graphs has again bounded expansion
- the process can be iterated

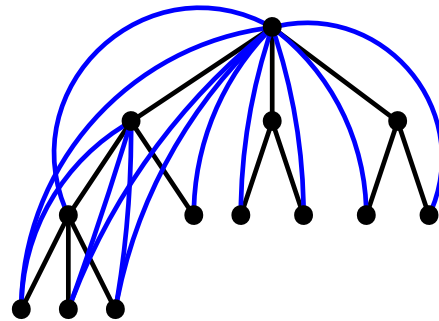
SHORT PATH QUERIES

- based on previous ideas, cf. Kowalik and Kurowski (2004)
- linear time precomputation, constant time query



TREE-DEPTH

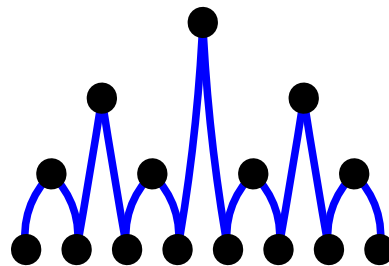
- transitive closure \overline{T} of a tree T
- the tree-depth $\text{td}(G)$ of G is the smallest depth of a tree T such that G is a subgraph of \overline{T}



- observe that $\text{tw}(G) \leq \text{td}(G) - 1$
- the ranking number of a graph
the minimal number of colors such that every path joining two vertices of color i contains a vertex of color bigger than i

TREE-WIDTH VS. TREE-DEPTH

- tree-depth is not bounded by a function of tree-width
 $\text{td}(P_n) = \lceil \log_2(n + 1) \rceil$
- bounded by a function of the tree-width and the order
 $\text{td}(G) \leq (\text{tw}(G) + 1) \log_2 n$
- related to the length of the longest path
 $\text{td}(G) \leq \ell(G) \leq 2^{\text{td}(G)}$
- other characterizations of tree-depth by separations



BOUNDED TREE-DEPTH COLORING

- since $\nabla_0(G) = \text{mad}(G)$ for $G \in \mathcal{G}$ is bounded, the chromatic number of all graphs in \mathcal{G} is bounded
- the chromatic number remains bounded for augmentations!
- bounded tree-depth coloring [Nešetřil and Ossona de Mendez, 2006]
For every class \mathcal{G} of graphs with bounded expansion and every k , there exists K such that every $G \in \mathcal{G}$ has a vertex coloring with K colors such that every union of $i \leq k$ color classes induces a subgraph of tree-depth at most i .
- generalization of tree-width coloring for minor-closed classes of graphs of DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan (2004)

NEW ALGORITHMS

- Theorem [Dvořák, K., Thomas, FOCS'10]
Every FO property can be decided for graphs with **bounded expansion** in a **linear** time.
- Theorem [Dvořák, K., Thomas, FOCS'10]
Every FO property can be decided for graphs with **locally bounded expansion** in **almost-linear** time.
- **main ingredients**: low tree-depth colorings, locality of FO formulas
- independently proven by Dawar and Kreutzer

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EXTENSION TO RELATIONAL STRUCTURES

- for a relational structure (X, \mathcal{R}) , the Gaifman graph is a graph with vertex set X where $x, y \in X$ are adjacent if there are in a common relation in \mathcal{R}
- class of relational structure has bounded expansion / locally bounded expansion if the class of their Gaifman graphs has bounded expansion / locally bounded expansion
- our algorithms work for classes of relational structures with (locally) bounded expansion

MATROIDS

- a **matroid** is a pair (A, \mathcal{A}) , $\mathcal{A} \subseteq 2^A$
 - A —the set of **elements** of the matroid
 - \mathcal{A} —a hereditary set of **independent sets**

Augmentation axiom: For every $X, Y \in \mathcal{A}$, $|X| < |Y|$,
there exists $y \in Y$ such that $X \cup \{y\} \in \mathcal{A}$.
- **edges of a graph**
 - independent sets formed by acyclic subgraphs
- **vectors of a vector space**
 - independent sets formed by linearly independent vectors
- a matroid is **representable over a field F**
 - if there exist vectors in F^d that correspond to its elements
- **regular matroids** are representable over all fields F

EXTENSIONS TO MATROIDS

- testing MSO properties for matroids represented over a **fixed finite field** with **bounded branch-width** can be done in cubic time [Hliněný, 2002]
- matroids with locally bounded branch-width defined through containment in a short cycle
- testing FO properties is fixed parameter tractable for **regular matroids** with **locally bounded branch-width** [Gavenčiak, K., Oum, 2011]

SOME DETAILS OF THE PROOF

- Gaifman's theorem on the locality of FO sentences
- covering Gaifman graph with sets with small branch-width and applying the algorithm for MSO properties
- *difficult*: constructing Gaifman graph
- *regular matroids*: joins of graphic, cographic matroids and R_{10}
- *color coding* of Alon et al. and *powercuts* of Kawarabayashi et al.

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OPEN PROBLEMS

- testing FO in classes of nowhere-dense graphs
a class \mathcal{G} is nowhere-dense if mad of d -shallow minors $\nabla_d(\mathcal{G})$ is $O(n^\varepsilon)$ for every d and every $\varepsilon > 0$
Is testing FO properties in nowhere-dense classes FPT?
- structural characterization of graphs with bounded expansion
proper minor-closed classes of graphs [Robertson, Seymour]
graphs excluding a subdivision [Grohe, Marx]

Happy birthday, Robin!