ON THE SECOND EIGENVALUE OF THE LAPLACE OPERATOR PENALIZED BY CURVATURE

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ABSTRACT. Consider the operator $-\nabla^2 - q(\kappa)$, where $-\nabla^2$ is the (positive) Laplace-Beltrami operator on a closed manifold of the topological type of the two-sphere S^2 and q is a symmetric non-negative quadratic form in the principal curvatures. Generalizing a well-known theorem of J. Hersch for the Laplace-Beltrami operator alone, it is shown in this note that the second eigenvalue λ_1 is uniquely maximized, among manifolds of fixed area, by the true sphere.

Dimensionally, the Laplace operator $-\nabla^2$ is comparable to the square of curvature, both having dimensions $(\text{length})^{-2}$. Thus one might expect to encounter partial differential operators of the form $-\nabla^2 - q$ in applications, where q is a quadratic expression in the principal curvatures.

This was recently the case when N. Alikakos and G. Fusco performed a stability analysis of the interfacial surface separating two phases in one of the simpler reaction-diffusion models, the Allen-Cahn system. It was already realized in the first article about this model [5] that it exhibits interfaces moving according mean-curvature, as a consequence of which the model has attractive geometric features; for current state of mathematical knowledge of this see [9]. While simplified in comparison to most realistic reaction-diffusion systems, Allen-Cahn is a reasonable model at least for bistable alloys of iron and aluminum. Picture a bubble of material of phase I in a background of phase II. It undergoes slow motions and deformations, and if it is not at an external boundary, the surface Ω smooths out and eventually becomes round. According to Alikakos and Fusco, instabilities of the surface are associated with negative eigenvalues of an operator emerging from linearization at Ω of the form

$$-\nabla^2 - \sum_{j=1}^{2} \kappa_j^2, \tag{1}$$

where κ_j are the principal curvatures at any given point of Ω, and ∇^2 is the Laplace-Beltrami operator on Ω. This can be thought of as a geometric Schrödinger operator with a negative potential determined by curvature. It is evident that (1) is a highly symmetric, reasonable object, and that the second eigenvalue is special because it equals 0 when Ω attains its target shape of a sphere. (While the analysis by Alikakos and Fusco is not accessible in print at the time of this writing, the lower dimensional analogy is worked out in the recent thesis of V. Stephanopoulos [12, see Proposition 5.4 and Theorem 7.1; see also 2]. Related work and an entry point

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for the literature on Allen-Cahn and similar reaction-diffusion models can be found in [1-3, 9].)

The conjecture was that for any other shape of Ω the second eigenvalue of (1) is strictly negative, and this is a special case of the theorem proved below. The conjecture calls to mind the theorem of J. Hersch [8] for the Laplace operator without the curvature penalty, whereby the unique such Ω maximizing the second eigenvalue is the sphere. For the Laplace operator plain and simple, the lowest eigenvalue is trivial, so the second eigenvalue is often referred to as the "first." Unambiguously, the eigenvalues will be written here $\lambda_0 < \lambda_1 \leq \ldots$, and the one at issue is λ_1. Actually, Hersch's problem was a bit more general, since he was concerned with

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}, \tag{2}$$

and he also allowed arbitrary weights on Ω. The variational principle he used as a lower bound for (2) is not available for (1), because the latter is not positive or even bounded from below *a priori*. Hersch's technique is particular to two dimensions, since it relies on the ability to map a generic Ω conformally to S^2, among other special facts, and in more than two dimensions the conformal equivalence class is not large enough to do this. Moreover, the natural extension of Hersch's theorem to higher dimensions has been shown to be false by H. Urakawa [14]. As for Hersch, the operative definition of the topological type of the sphere used here will be conformal equivalence, so the theorem of this article is likewise restricted to two dimensions. More specifically, the following lemma from [8] will be needed:

Lemma. *(J. Hersch). Let Ω be a two-dimensional, closed, smooth Riemannian manifold of the topological type of the sphere, and specify a bounded, positive, measurable function ρ on Ω. Then there exists a conformal transformation $\Phi : \Omega \to S^2 \subset R^3$, embedded in the standard way as the unit sphere, such that*

$$\int_{S^2} \mathrm{x} \rho(\Phi^{-1}(\mathrm{x})) J d\hat{S} = \mathbf{0}. \tag{3}$$

Here $\mathrm{x} = (x, y, z)$ is the position vector in R^3, J is the Jacobian factor of the transformation, and $d\hat{S}$ is the standard area element on S^2. Thinking of ρ as a mass distribution, the statement means that the center of mass is mapped to the origin in R^3.

Recall for later purposes that the restrictions of the Cartesian coordinate functions (x, y, z) to the unit sphere are the spherical harmonics, which are the eigenfunctions of $-\nabla^2$ associated with its (multiple) second eigenvalue [10]. Following Hersch's notation, let X, Y, Z denote the functions on Ω obtained by mapping (x, y, z) back to Ω with Φ^{-1}. Thus

$$X^2 + Y^2 + Z^2 = 1, \tag{4}$$

and (3) implies that X, Y, and Z are orthogonal to ρ on the Hilbert space $L^2(\Omega)$ (endowed with the measure dS arising from the metric on Ω).

Theorem. *Let $\kappa_{1,2}(p)$ denote the principal curvatures at the point $p \in \Omega$, a closed manifold of the topological type of S^2. Let $q(\xi, \eta)$ be a nonnegative quadratic polynomial in ξ, η, such that $q(\eta, \xi) = q(\xi, \eta)$, and let $\lambda_0 < \lambda_1 \leq \dots$ denote the eigenvalues of*

$$-\nabla^2 - q(\kappa_1, \kappa_2)$$

as an operator on $L^2(\Omega)$. Then

$$\lambda_1 \leq \frac{4\pi(2 - q(1,1) + q(0,0))}{|\Omega|} - q(0,0). \tag{5}$$

Equality is attained if and only if $\Omega = rS^2$ (sphere of radius r).

A plausibility argument for the theorem is to recall Hersch's theorem and observe that the quadratic form q is minimized on average by the sphere. The obstacle to a rigorous argument along these lines is that since λ_1 is not the lowest eigenvalue, it is characterized by a min-max principle, and one must cope with the orthogonalization to the ground-state eigenfunction u_0 for λ_0. For a general Ω this u_0 is not accessible.

Proof. Without loss of generality, subtract a constant from q so that $q(0,0) = 0$. Inequality (5) then becomes

$$\lambda_1 \leq \frac{4\pi(2 - q(1,1))}{|\Omega|} \tag{6}$$

Note that equality is attained if $\Omega = rS^2$. Inequality (6) will follow if a function ζ can be exhibited, which is orthogonal to u_0, and for which the Rayleigh quotient for this operator,

$$R(\zeta) := \frac{\int_\Omega |\nabla\zeta|^2 dS - \int_\Omega q(\kappa_1, \kappa_2)|\zeta|^2 dS}{\int_\Omega |\zeta|^2 dS} \tag{7}$$

is less than the stated bound. As constructed, the functions X, Y, and Z are suitable candidates for the Rayleigh quotient for λ_1 if ρ is identified with u_0, which is always positive [7, Theorem 4.2.1; 11].

Next, recall that the Dirichlet integral in the numerator of (7) is conformally invariant, i.e.,

$$\int_\Omega |\nabla X|^2 dS = \int_{S^2} |\nabla x|^2 d\hat{S} = \frac{8\pi}{3} \tag{8}$$

(because x is an eigenfunction for $-\nabla^2$ on S^2 with eigenvalue 2). With (8) and setting $\zeta = X$ in (7), for example:

$$R(X) = \frac{\frac{8\pi}{3} - \int_\Omega q(\kappa_1, \kappa_2) X^2 dS}{\int_\Omega X^2 dS}$$

and similarly for $R(y)$ and $R(z)$.

Now, it is an elementary fact that if c_j are positive numbers and for each j,

$$a \leq \frac{b_j}{c_j}$$

then

$$a \leq \frac{\sum_j b_j}{\sum_j c_j}$$

(multiply by c_j and sum). Thus, with (4),

$$\min(R_X, R_Y, R_Z) \leq \frac{8\pi - \int_\Omega q dS}{\int_\Omega 1 dS}.$$

A simple calculation (writing $q(\xi, \eta) = a\xi\eta + b(\xi^2 + \eta^2)$ and completing the square) shows that

$$q(\xi, \eta) \geq q(1,1)\xi\eta, \tag{9}$$

with equality only for $\xi = \eta$. Hence:

$$\min(R_X, R_Y, R_Z) \leq \frac{8\pi - \int_\Omega q(1,1)\kappa_1, \kappa_2 dS}{|\Omega|} = \frac{4\pi(2 - q(1,1))}{|\Omega|}$$

by the Gauß-Bonnet theorem, establishing (6). Because equality in (9) requires $\xi = \eta$, equality in (6) requires $\kappa_1 = \kappa_2$ a.e., and by a theorem of Liebmann [13, p. 122] this characterizes Ω as the sphere rS^2. QED

As remarked above, a simple extension to more than two dimensions is unlikely, but related theorems probably apply to a) surfaces of higher genus as in [15], b) surfaces with boundaries which are predetermined, and c) one dimension. These cases all occur physically and are under investigation [4]. An interface may meet the predetermined edge of a piece of alloy, producing case b) for example; while the one-dimensional situation of a curve describes an interface in a thin sheet of metal. Curiously, even the one-dimensional analogue is more complicated than this two-dimensional theorem, due to the lack of conformal invariance of the Dirichlet integral.

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