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Structured Interpolation and Robust Control

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Abstract

In this paper, we extend our previous work on the spectral commutant lifting theorem [4] to the case of structured singular values which appear in certain problems in control theory. We moreover give a new characterization of the structured singular value, and formulate and prove a structured version of the matrix Nevanlinna-Pick theorem.

1 Introduction

Motivated by certain problems in robust control, this paper considers a new twist to the subject of interpolation in the disc by analytic functions in which we consider interpolation theory not with analytic matrices of bounded norm as in the classical Nevanlinna-Pick framework, but with bounded structured singular value. (See Section 2 for the precise definitions.) We will also describe a new characterization of the structured singular value from which we will derive an elementary proof to a result of Doyle [6], which relates the structured singular value to an invariant derived from certain scalings. See [6] and Section 3 below. The structured singular value was originated by Doyle [6] and Safonov [14] in connection with certain engineering control problems. Our results are based on our previous paper [4] in which we formulated and proved a spectral version of the commutant lifting theorem. Full details of this work may be found in [5].

Recall the classical Nevanlinna-Pick theory [12], [13]. We are given $2n$ points $z_1, \ldots, z_n, w_1, \ldots, w_n \in D$ (the open unit disc), and we would like to find necessary and sufficent conditions for the existence of an analytic function $f : D \to \overline{D}$ such that $f(z_j) = w_j$ for $1 \leq j \leq n$. As is well known a necessary and sufficient condition can be written for the existence of f in terms of the positivity of a certain Hermitian matrix (called the *Nevanlinna-Pick* matrix).

The standard approach to such interpolation problems is complex analytic. An important fact found by Donald Sarason [16] is that many results on interpolation by bounded analytic functions may be deduced in an operator theoretic manner, and using operator theory one can even extend interpolation theory to cover interpolation with points with infinite multiplicity. Sarason's theorem was strengthened by the Sz.-Nagy-Foias commutant lifting theorem [18], [19] which allows one to interpolate on the disc by bounded analytic *operator-valued* functions.

As alluded to above, the problem we will be considering in this paper had an applied motivation, namely from in robust system design, i.e. the design of feedback control systems in the presence of parameter uncertainty. (See [6], [8], [9], [11], [14], [15], [20].) However, we believe that the mathematical problem which arises from this also has a purely intrinsic operator-theoretic interest.

In order to describe the results of the present paper, we consider the classical Nevanlinna-Pick theory in the matrix case. As above, let $z_j \in D$ be distinct, $1 \leq j \leq n$, and let $F_1, \ldots, F_n$ be $N \times N$ matrices. Then we are interested in finding necessary and sufficient conditions for the existence of an analytic (in the disc) $N \times N$ matrix-valued function $F(z)$ with $F(z_j) = F_j$ ($1 \leq j \leq n$), and such that $\|F\| \leq 1$. The existence of F can again be reduced to the determination of the positivity of a certain Nevanlinna-Pick matrix. (This fact can be deduced, e.g., from the commutant lifting theorem [18], [19].) However, in our case we will not be interested in bounding the norm, but instead the *structured singular value* of the interpolating functions. This "structured Nevanlinna-Pick theorem" will be deduced as a consequence of a general commutant lifting result which generalizes the authors' previous spectral commutant lifting theorem [7].

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2 Classical Commutant Lifting Theorem

In what follows, $\mathcal{H}$ will denote a complex separable Hilbert space. By "operator" we shall always mean "bounded linear operator," unless explicitly stated otherwise. We let $\mathcal{L}(\mathcal{H})$ denote the set of operators on $\mathcal{H}$.

In this section, we shall give some background on the commutant lifting theorem [2], [16], [18], [19] and its connection to interpolation theory. We will follow specifically the treatment given in [4]. We have included the section to make the paper as self-contained as possible, but all the material which follows is standard. We want to write down in particular the Sarason set-up of the relationship of commutant lifting to Nevanlinna-Pick.

We begin with a result due to Sz.-Nagy [17]. Let $T : \mathcal{H} \to \mathcal{H}$ be a contraction, i.e. an operator such that $\|T\| \leq 1$. Then in [17] it is proven that there exists an isometry U on a Hilbert space $\mathcal{K}$ such that

$$\mathcal{K} = \bigvee_{n=0}^{\infty} U^n \mathcal{H}$$

and $P_{\mathcal{H}} U = T P_{\mathcal{H}}$, where $P_{\mathcal{H}} : \mathcal{K} \to \mathcal{H}$ denotes orthogonal projection. U is called the *minimal isometric dilation of* T. The *commutant lifting theorem* (proved in a special case in [16], and in complete generality in [18]) may be formulated as follows:

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Theorem 1 (Commutant Lifting Theorem) *Let $\mathcal{H}$ and $\mathcal{H}'$ denote (complex separable) Hilbert spaces with $T : \mathcal{H} \to \mathcal{H}$, $T' : \mathcal{H}' \to \mathcal{H}'$ contractions. Let $A : \mathcal{H} \to \mathcal{H}'$ be an intertwining contraction for T and T', i.e. $AT = T'A$. Let $U : \mathcal{K} \to \mathcal{K}$ and $U' : \mathcal{K}' \to \mathcal{K}'$ denote the minimal isometric dilations of T and T' respectively. Then there exists a contraction $\hat{A} : \mathcal{K} \to \mathcal{K}'$ such that $U'\hat{A} = \hat{A}U$ and $P_{\mathcal{H}'}\hat{A} = AP_{\mathcal{H}}$ where $P_{\mathcal{H}'} : \mathcal{K}' \to \mathcal{H}'$ and $P_{\mathcal{H}} : \mathcal{K} \to \mathcal{H}$ denote orthogonal projections.*

Remarks 1.

(i) $\hat{A}$ is called an *intertwining dilation* of A.

(ii) For the commutant lifting theorem we do not need the minimal isometric dilation, but any isometric dilation of T satisfying $T^* = U^*|\mathcal{H}$. Moreover, in the proof we may take without loss of generality $\mathcal{H} = \mathcal{H}'$ and $T = T'$. In this case, $\hat{A}$ is called a *commuting dilation* of A.

The commutant lifting theorem allows one to derive most of the classical interpolation results involving bounded analytic functions in a unified manner even when the functions are operator-valued. In order to give the reader an idea how this is done we will sketch the classical scalar Nevanlinna-Pick problem as in [16].

Recall that the problem of Nevanlinna-Pick concerns finding necessary and sufficient conditions for the existence of an analytic $f : D \to D$, such that $f(z_j) = w_j$, $j = 1, \ldots, n$. We assume as before that the z_j are distinct. We now put the Nevanlinna-Pick problem into the commutant lifting framework.

Indeed, we let

$$m(z) = \prod_{i=1}^{n} \frac{z - z_i}{1 - \bar{z}_i z}.$$

Set $H(m) := H^2 \ominus mH^2$. (All of the Hardy spaces in this paper will be defined in the unit disc D in the standard way.) One can then show that $H(m)$ an n dimensional vector space with basis $f_1, \ldots, f_n$ where

$$f_j(z) := \left(\prod_{k \neq j} \frac{z - z_k}{1 - \bar{z}_k z} \right) \frac{1}{1 - \bar{z}_j z} \quad \text{for } j = 1, \ldots, n.$$

Now let $S : H^2 \to H^2$ denote the canonical unilateral right shift defined by multiplication by z. For $P_{H(m)} : H^2 \to H(m)$ orthogonal projection, set $S(m) := P_{H(m)} S | H(m)$ (the *compressed shift*). Again, one can compute [16] that

$$S(m) f_j = z_j f_j$$

for $1 \leq j \leq n$.

We can now define a linear operator $A : H(m) \to H(m)$ such that

$$A f_j = w_j f_j$$

for $j = 1, \ldots, n$. Clearly A commutes with T. From this it is easy to prove the Nevanlinna-Pick theorem (see [16] and [4]):

Theorem 2 (Nevanlinna-Pick) *Notation as above. Then there exists an analytic $f : D \to \overline{D}$ such that $f(z_j) = w_j$ for $j = 1, \ldots, n$ if and only if the Nevanlinna-Pick matrix*

$$\left[\frac{1 - \bar{w}_i w_j}{1 - \bar{z}_i z_j} \right]_{i,j=1,\ldots,n} \geq 0.$$

□

We will discuss the matrix structured singular value version of this result below.

3 Remarks on the Structured Singular Value

In this section, we introduce some notation and discuss some properties of the strutured singular value. We moreover state a result which we believe is new in this connection. Instead of working over diagonal sets of matrices as in [6], we work over an arbitrary finite dimensional C^*-algebra. See also Helton [11].

Let $\mathcal{E}$ be a finite dimensional complex Hilbert space, and $\Delta \subset \mathcal{L}(\mathcal{E})$, a C^*-algebra. For $A \in \mathcal{L}(\mathcal{E})$, $A \neq 0$, we define the *structured singular value*

$$\mu_\Delta(A) := [\inf\{\|X\| : X \in \Delta,\ -1 \in \sigma(AX)\}]^{-1}$$

Moreover, we set

$$\hat{\mu}_\Delta(A) := \inf\{\|DAD^{-1}\| : D \in \Delta'\}$$

where Δ' is the commutator of Δ. Note that for $\Delta = \mathcal{L}(\mathcal{E})$, $\mu_\Delta(A) = \|A\|$, while for $\Delta = \mathbf{C}I_{\mathcal{E}}$, $\mu_\Delta = \|A\|_{sp}$.

We now summarize some of the elementary properties of μ_Δ and $\hat{\mu}_\Delta$ based on [6]. However, we work here over an arbitrary finite dimensional C^*-algebra. Some of the properties which we mention are even valid in a more general setting. See Helton [11].

Let μ_1 denote either μ_Δ or $\hat{\mu}_\Delta$. Then μ_1 enjoys the following properties:

(i) $\mu_1(\lambda A) = |\lambda| \mu_1(A)$.

(ii) $\|A\|_{sp} \leq \mu_1(A) \leq \|A\|$.

(iii) $\mu_1(AX) \leq \|X\| \mu_1(A)$.

We also have the following useful observations [6] and [11]:

Lemma 1

$$\mu_\Delta(A) = \sup\{\|AX\|_{sp} : X \in \Delta,\ \|X\| \leq 1\}.$$

Proof. For $X \in \Delta$, $-1 \in \sigma(AX) \Rightarrow -\|X\|^{-1} \in \sigma(AX/\|X\|) \Rightarrow \|AX/\|X\|\| \geq 1/\|X\| \Rightarrow 1/\|X\| \leq \sup\{\|A\|_{sp} : X \in \Delta,\ \|X\| \leq 1\}$. But

$$\begin{aligned} \mu_\Delta(A) &= \sup\{1/\|X\| : -1 \in \sigma(AX),\ X \in \Delta\} \\ &\leq \sup\{\|AX\|_{sp} : X \in \Delta,\ \|X\| \leq 1\}. \end{aligned}$$

Conversely,

$$\begin{aligned} \|AX\|_{sp} &\leq \mu_\Delta(AX) \\ &\leq \mu_\Delta(A)\|X\| \\ &\leq \mu_\Delta(A) \end{aligned}$$

for every $X \in \Delta$, $\|X\| \leq 1$. □

Lemma 2 *μ_Δ is continuous.*

Proof. We prove the result for μ_Δ. We first show that μ_Δ is upper semicontinuous. Let $X_o \in \Delta$ be such that

$$\|X_o\| < \frac{1}{\mu_\Delta(A)}.$$

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Then $I + AX_o$ is invertible. Now let $\epsilon > 0$ be sufficiently small so that

$$\|X_o\| \leq \frac{(1-\epsilon)}{\mu_\Delta(A)}.$$

Moreover, we define

$$\nu := \max\{\|(I + AX_o)^{-1}\| : \|X_o\| \leq \frac{(1-\epsilon)}{\mu_\Delta(A)}\}.$$

For any A_1, we have

$$\begin{aligned} I + A_1X_o &= I + AX_o + (A_1 - A)X_o \\ &= (I + AX_o)^{-1}[I + (I + AX_o)^{-1}(A_1 - A)X_o] \end{aligned}$$

Thus for

$$\|A_1 - A\| < \frac{(1-\epsilon)}{\nu\mu_\Delta(A)},$$

we have $I + A_1X_o$ is invertible, and

$$\mu_\Delta(A_1) \leq \frac{\mu_\Delta(A)}{1-\epsilon}.$$

The lower semicontinuity is obvious, since from Lemma 1 we have that $\mu_\Delta(A)$ is the supremum of continuous functions. This completes the proof of the lemma. □

Lemma 3

$$\mu_\Delta(A) \leq \hat{\mu}_\Delta(A).$$

Proof. For $D \in \Delta'$, we have that

$$\mu_\Delta(A) = \mu_\Delta(DAD^{-1}) \leq \|DAD^{-1}\|.$$

□

For certain diagonal algebras of matrices, it is argued in [6], [14], and [15] that the structured singular value μ_Δ is the natural object of study in robust control. Unfortunately the structured singular value is difficult to compute, so in practice it is $\hat{\mu}_\Delta$ which is actually used for the control problems. It is therefore of interest to know when these two objects are equal.

In [6], Doyle has shown that in fact $\mu_\Delta = \hat{\mu}_\Delta$ when the relevant diagonal algebra has three or fewer blocks. In [5], we give a very different proof of this fact based on the following result which we believe has independent interest. More precisely set $\mathcal{H} := \mathcal{L}(\mathcal{E})$, and define $M_A \in \mathcal{L}(\mathcal{E})$ by $M_A := AX$. Notice that $\mathcal{E}$ is a Hilbert space with respect to

$$\langle T_1, T_2 \rangle := Tr(T_2^*T_1),$$

where Tr denotes the trace. Define

$$\tilde{\mu}_\Delta(A) := \mu_{\tilde{\Delta}}(M_A)$$

where

$$\tilde{\Delta} := \{M_X : X \in \Delta'\}'.$$

We now have (see [5] for the proof):

Theorem 3

$$\hat{\mu}_\Delta(A) := \tilde{\mu}_\Delta(A).$$

Remarks 2. 1. Note that the theorem implies that $\hat{\mu}_\Delta$ can be regarded as a structured singular value on a bigger space. We note that the theorem is true for an arbitrary finite dimensional C^*-algebra Δ.
2. From the theorem and Lemma 2, we can immediately infer that $\hat{\mu}_\Delta$ is continuous.

It is now eleemntary to prove the aforementioned result of Doyle (see [5] for the details):

Corollary 1 (Doyle) *For Δ block diagonal with three or fewer blocks,*

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

4 Structured Singular Values and Dilations

In this section, we will give a structured analogue of the commutant lifting theorem [18]. This will be applied to the structured version of classical matricial Nevanlinna-Pick interpolation below. The techniques we employ are very closely related to those in [4]. Again see [5] for all the proofs.

Set

$$T := S(m) \otimes I_\mathcal{E}$$

where m is a finite Blaschke product, $S(m)$ is the compressed shift, and $\mathcal{E}$ is a finite dimensional complex Hilbert space. Fix $\Delta \subset \mathcal{L}(\mathcal{E})$, a C^*-algebra. Define

$$I_{H^2} \otimes \Delta := \{I_{H^2} \otimes X : X \in \Delta\}$$

$$I_{H^2} \otimes \Delta' := \{I_{H^2} \otimes X : X \in \Delta'\}.$$

Notice that $\mathcal{H} := H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$ reduces both $I_{H^2} \otimes \Delta$ and $I_{H^2} \otimes \Delta'$. Now define for $A \in \{T\}'$ (the commutant of T),

$$\rho_T^\Delta(A) := \inf\{\|XAX^{-1}\| : X \text{ invertible, } X \in \{T\}',\ X \in (I_{H^2} \otimes \Delta|\mathcal{E})'\}.$$

Let U be the isometric dilation of T on $H^2(\mathcal{E})$ (so that U is defined by multiplication by z), and let $P : H^2(\mathcal{E}) \to \mathcal{H}$ denote orthogonal projection. For $B \in \{U\}'$, define

$$\rho_U^\Delta(B) := \inf\{\|YBY^{-1}\| : Y \text{ invertible, } Y \in \{U\}',\ Y \in (I_{H^2} \otimes \Delta)'\}.$$

Lemma 4 *Let $D(\zeta) \in \Delta'$ for $\zeta \in \partial D$ almost everywhere. Suppose that there exists $\Theta(\zeta)$ outer such that*

$$\Theta(\zeta)^*\Theta(\zeta) = D(\zeta)^*D(\zeta),\ \ \zeta \in \partial D \text{ almost everywhere.} \qquad (1)$$

Then there exists an (analytic) outer function $\Theta_1 : D \to \Delta'$ such that

$$\Theta_1(\zeta)^*\Theta_1(\zeta) = D(\zeta)^*D(\zeta),\ \ \zeta \in \partial D \text{ almost everywhere.}$$

We now have:

Proposition 1 *Let $A(z) : \bar{D} \to \mathcal{L}(\mathcal{E})$ be continuous, and analytic in D. Suppose that*

$$\hat{\mu}_\Delta(A(z)) < 1\ \ \forall z \in \bar{D}.$$

Then there exists an analytic map $\Theta(z) : D \to \Delta'$ with $\Theta(z)^{-1}$ analytic, such that

$$\|\Theta(z)A(z)\Theta(z)^{-1}\| < 1\ \ \forall z \in D.$$

We can now state the following result:

Corollary 2 *Let $B \in \{U\}'$, and suppose B is continuous on $\bar{D}$. Then*

$$\rho_U^\Delta(B) = \sup_{z \in D} \hat{\mu}_\Delta(B(z)).$$

Proof. Immediate from Proposition 1. □

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5 Structured Commutant Lifting Theorem

In this section, we will state our main result namely a structured generalization of the spectral commutant lifting theorem [4]. This will be applied to the a structured version of classical matricial Nevanlinna-Pick interpolation below. All the proofs may be found in [5].

Theorem 4 (Structured Commutant Lifting Theorem) *Notation as above. Then for* $A \in \{T\}'$,

$$\rho_T^{\Delta}(A) = \inf\{\rho_U^{\Delta}(B) : B \textit{ is a commuting dilation of } A\}.$$

Remark 3. From the Structured Commutant Lifting Theorem and Corollary 2 we have that

$$\rho_T^{\Delta}(A) = \inf_B \sup_{z \in D} \hat{\mu}_{\Delta}(B(z))$$

where $B \in \{U\}'$ is a commuting dilation of A.

For T as above with minimal isometric dilation U a shift, we will say that a commuting dilation B of $A \in \{T\}'$ is *rational* if it is defined by multiplication by a rational (matrix-valued) function.

Lemma 5 *Notation as above. Then for every invertible*

$$X \in \{T\}' \cap (I_{H^2} \otimes \Delta|\mathcal{H})',$$

there exists an invertible commuting dilation $\hat{X}$ *of* X, $\hat{X} \in \{U\}' \cap (I_{H^2} \otimes \Delta)'$ *such that* $\hat{X}$ *is rational.*

We now have the following corollary to Theorem 4:

Corollary 3 *Notation as above. Then*

$$\rho_T^{\Delta}(A) = \inf\{\rho_U^{\Delta}(B) : B \textit{ is a commuting dilation of } A,\ B(z) \textit{ is rational}\} \tag{2}$$

6 Structured Nevanlinna-Pick Theory

In this section, we apply the above theory to a structured version of the Nevanlinna-Pick interpolation problem. In order to do this, we first put the Nevanlinna-Pick theory into the commutant lifting framework [16], [19], [7]. First let us recall the general problem of Nevanlinna-Pick in the matrix case. See also the closely related discussion above in the scalar case.

Let $\mathcal{E}$ be a finite dimensional Hilbert space, let $z_1, \ldots, z_n \in D$ be mutually distinct, and let $F_1, \ldots, F_n \in \mathcal{L}(\mathcal{E})$. Then we want necessary and sufficient conditions for the existence of an analytic function $F : D \to \mathcal{L}(\mathcal{E})$ with $\|F\|_\infty \leq 1$ such that

$$F(z_j) = F_j \tag{3}$$

for $j = 1, \ldots, n$. Define

$$m(z) := \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z}_j z},$$

$$f_j := \left(\prod_{k \neq j} \frac{z - z_k}{1 - \overline{z}_k z} \right) \frac{1}{1 - \overline{z}_j z},$$

and

$$\mathcal{H} := H^2(\mathcal{E}) \ominus m H^2(\mathcal{E}).$$

We set $T := S(m) \otimes I_{\mathcal{E}}$, and note that

$$\mathcal{H} = f_1 \otimes \mathcal{E} + f_2 \otimes \mathcal{E} + \cdots + f_n \otimes \mathcal{E}.$$

This sum is direct but not orthogonal.

For the given interpolation data above, we define $A : \mathcal{H} \to \mathcal{H}$ by linearity and by

$$A(f_j \otimes \xi) = f_j \otimes F_j \xi$$

for all $\xi \in \mathcal{E}$, $j = 1, 2, \ldots, n$. Note that

$$T(f_j \otimes \xi) = z_j f_j \otimes \xi$$

for all $\xi \in \mathcal{E}$, $j = 1, 2, \ldots, n$. Thus

$$A \in \{T\}'.$$

It is easy to show that F satisfies the interpolation conditions

$$F(z_j) = F_j, \quad \forall j = 1, \ldots, n \tag{4}$$

if and only if

$$P_{\mathcal{H}} M_F = A P_{\mathcal{H}}$$

Thus, the commutant lifting theorem implies that there exists an F satisfying (4) and $\|F\|_\infty \leq 1$, if and only if $\|A\| \leq 1$. We will now show that the structured Nevanlinna-Pick problem can be given a similar solution, based on the structured commutant lifting theorem.

More precisely, define

$$\mathcal{I} := \{F : D \to \mathcal{L}(\mathcal{E}) : F \text{ is rational, bounded in } D,\ F(z_j) = F_j\}.$$

We are interested in finding

$$\mu(\mathcal{I}) := \inf\{\sup_{z \in D} \mu_{\Delta}(F(z)) : F \in \mathcal{I}\}$$

$$\hat{\mu}(\mathcal{I}) := \inf\{\sup_{z \in D} \hat{\mu}_{\Delta}(F(z)) : F \in \mathcal{I}\}.$$

We denote the operator A associated with the matrices $F_1, \ldots, F_n$ by $A(F_1, \ldots, F_n)$. Set

$$\rho_T^{\Delta}(A) = \inf\{\|A(D_1 F_1 D_1^{-1}, \ldots, D_n F_n D_n^{-1})\| \ :\ D_j \in \Delta',\ 1 \leq j \leq n\}$$

(where the D_j are invertible) and

$$\rho_{T,\Delta}(A) = \sup\{\rho_T(A(F_1 X_1, \ldots, F_n X_n)) : \|A(X_1, \ldots X_n)\| \leq 1,\ X_1, \ldots, X_n \in \mathcal{L}$$

where for an operator $M \in \{T\}'$ we let

$$\rho_T(M) = \inf\{\|X M X^{-1}\| : X \text{ is invertible and } X \in \{T\}' \}.$$

(In [7], $\rho_T(M)$ is called the *generalized spectral radius.*) It is easy to show that

$$\rho_T^{\Delta}(A) \geq \rho_{T,\Delta}(A).$$

Indeed, just note that

$$\begin{aligned} \rho_T(A(F_1 X_1, \ldots, F_n X_n)) &= \inf\{\|A(Y_1 F_1 X_1 Y_1^{-1}, \ldots, Y_n F_n X_n Y_n^{-1})\|\} \\ &\leq \|A(D_1 F_1 D_1^{-1} X_1, \ldots, D_n F_n D_n^{-1} X_n)\| \\ &\leq \|A(D_1 F_1 D_1^{-1}, \ldots, D_n F_n D_n^{-1})\| \\ &\quad \|A(X_1, \ldots, X_n)\| \\ &\leq \|A(D_1 F_1 D_1^{-1}, \ldots, D_n F_n D_n^{-1})\| \end{aligned}$$

We can now state:

Theorem 5 (Structured Nevanlinna-Pick) *Notation as above. Then*

$$\hat{\mu}(\mathcal{I}) = \rho_T^{\Delta}(A).$$

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Proof. Follows immediately from the Structured Commutant Lifting Theorem, Remark 3, and the above set-up for the Nevanlinna-Pick theorem. □

Theorem 6 *Notation as above. Then*

$$\mu(\mathcal{I}) \geq \rho_{T,\Delta}(A).$$

Proof. Let $F \in \mathcal{I}$ satisfy

$$\mu_\Delta(F(z)) \leq \mu(\mathcal{I}) + \epsilon, \quad \forall z \in D.$$

By our above discussion (see the proof of the Structured Commutant Lifting Theorem 4), if

$$\|A(X_1, \ldots, X_n)\| \leq 1, \;\; X_1, X_2, \ldots, X_n \in \Delta,$$

there exists $X(z) : D \to \Delta$ with $\|X\| \leq 1$, $X(z_j) = X_j$, $1 \leq j \leq n$. Then

$$\|F(z)X(z)\|_{sp} \leq \mu_\Delta(F(z)) \leq \mu(\mathcal{I}) + \epsilon.$$

By the spectral Nevanlinna-Pick theorem [4],

$$\rho_T(A(F(z_1)X(z_1), \ldots, F(z_n)X(z_n)) \leq \mu(\mathcal{I}) + \epsilon,$$

that is

$$\rho_T(A(F_1X_1, \ldots, F_nX_n)) \leq \mu(\mathcal{I}) + \epsilon, \;\; \forall \epsilon > 0.$$

Thus

$$\rho_T(A(F_1X_1, \ldots, F_nX_n)) \leq \mu(\mathcal{I})$$

for all $\|A(X_1, \ldots, X_n)\| \leq 1$, $X_1, \ldots, X_n \in \Delta$. □

Remark 4. We do not know at this point if we have equality in Theorem 7. This is still an interesting research problem.

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