Simple Newsvendor Heuristics for Two-Echelon Distribution Networks

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February 12, 2006

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We consider the problem of determining stocking levels in a multi-echelon distribution network consisting of a warehouse and *n* non-identical retail locations. Lead-times are deterministic, there are no fixed ordering costs, and unmet demand is backlogged. Both Clark and Scarf (1960) and Federgruen and Zipkin (1984b) propose heuristic solutions for such a problem based on a stochastic dynamic programming formulation. The disadvantage of their formulations lies in the very large state space needed for its solution. For a serial supply chains, Shang and Song (2003) provide single period newsvendor problems that bound the optimal stocking levels determined by the Clark and Scarf (1960) serial supply chain model. Newsvendor bounds have a number of valuable qualities; they are considerably less computationally intensive, allow for ready parametric analysis, and facilitate the development of intuition. In this paper, we extend the newsvendor bounds technique to distribution systems, thus providing a simple and surprisingly accurate heuristic. Through a simulation study, we show that our heuristic significantly outperforms other common heuristics over a wide range of parameter values. The closed form solutions provided by the newsvendor bounds also allows us to gain insights into the system behavior of a distribution network that was not previously possible through alternative solution techniques.

1. Introduction

We consider the problem of determining stocking levels in a two-echelon distribution network consisting of a warehouse and n non-identical retail locations. Optimal solutions of these systems are problematic, typically requiring stochastic dynamic programming formulations with very large state spaces. Several simpler heuristics have since been proposed, requiring a trade-off between performance and complexity.

For serial supply chains, Shang and Song (2003) provide a series of single period newsvendor problems, the solution to which bound the optimal stocking levels as determined by Clark and Scarf (1960). Newsvendor bounds have a number of valuable qualities; they are considerably less computationally intensive, allow for ready parametric analysis, and facilitate the development of intuition. In this paper, we extend the newsvendor bounds technique to distribution systems and show that it outperforms other proposed heuristics in both simplicity and performance.

This newsvendor heuristic avoids a recursive search over stocking levels, requiring only the solution of a set of simple closed form functions to set base-stock levels. To do so, we bound the costs of the distribution system by a single serial system on the low side and a set of n independent serial chains on the high side. These constructed systems also suggest bounds for the inventory base-stock levels at each installation, thus we take the average of the resulting system wide echelon base-stock levels as our heuristic for the original distribution system.

Due to the unavailability of practical analytical solution methods, we test our heuristic through an extensive and rigorous simulation experiment and compare its performance against other common heuristics. We find that our approach results in an average difference in costs from the best found solution of 0.44% for symmetric retailers and 0.87% for asymmetric retailers, and 0.60% and 0.66% for systems with 2 or 4 retailers, respectively, easily outperforming all other tested heuristics. Our closed form approach also allows us to generate insights on the effects of altering system parameters on the stocking levels and system costs. For example, we show how increasing asymmetry among retailers leads to lower stocking levels and total system costs.

The rest of the paper is organized as follows. In Section 2 we review the related literature and in Section 3 we describe the setting and foundation for our model. In Section 4 we present our heuristic and describe the simulation methodology for testing it in Section 5. In Section 6 we present our numerical results and observations based on test over a wide range of parameter values and demand distributions. In Section 7 we present our parametric analysis and conclude in Section 8. Appendix 1 provides the details from the numerical test and our proposition proofs are provided in Appendix 2.

2. Literature Review

Two main challenges exist in determining optimal supply chain strategies for distribution systems; determining the stocking policies for each installation and the allocation policy of inventory to downstream stages when demand exceeds supply at the upstream stage. Prior work on these elements of the problem is discussed in §2.1 and §2.2 below.

2.1. Allocation Policies

In their seminal analysis of serial systems, Clark and Scarf (1960) find that echelon inventory stocking policies are optimal. They suggest that arborescent systems may be approximated by a serial system under a balance allocation assumption, which is a relaxation of the traditional dynamic program formulation that states that the warehouse may reallocate downstream inventory by imposing negative inventory shipments on downstream installations (rebalance relaxation). This approach is utilized frequently in this literature (e.g. see Eppen and Schrage (1981), Federgruen and Zipkin (1984a, b), Federgruen (1993), Verrijdt and de Kok (1996), Garg and Tang (1997), van der Heijden et al. (1997)) although in practice such a policy may not always be feasible. Eppen and Schrage (1981) and Erkip et al (1990) provide simulation results suggesting that for high service level systems, such an allocation policy may sometimes be feasible. Unfortunately, the rebalance relaxation may be inappropriate when downstream installations are substantially asymmetric in inventory cost profiles and lead times (e.g. see Clark and Scarf (1960), Federgruen and Zipkin (1984a), and Axsater et al. (2002)). Additionally, the rebalance relaxation is often unrealistic in practice, as it implies the existence of costless and instantaneous transshipments.

There are also a number of proposed allocation policies that do not rely on the rebalance relaxation. Graves (1996) utilizes a virtual assignment rule, where echelon inventory is devoted to a given retailer as demand occurs. This allocation policy is essentially the opposite of the rebalance relaxation; rather than assigning inventory at the end of the supply chain, the assignment occurs before the inventory enters the system. By using a random allocation policy, Cachon (2001) develops exact results for the retailer and warehouse costs, although such a policy is somewhat crude and thus increases total costs. Myopic allocation policies, used by Federgruen and Zipkin (1984b) and Axsater et al. (2002), allocate inventory to minimize the expected costs at the retailers in the period the inventory arrives (e.g. after the warehouse to retailer shipment lead-time). Federgruen and Zipkin (1984b) show that, for identical retailers, this myopic policy is approximately optimal under general cost structures when orders may be placed every period. Jackson and Muckstadt (1989) and Jackson (1988) use a similar allocation rule, denoted the "run-out allocation rule", where the allocation is determined by solving an optimization problem over the horizon until the next arrival of inventory at the warehouse stage. Our allocation rule is most similar to that of McGavin, Schwarz, and Ward (1993), who assume identical retailers and allocate stock so as to maximize the minimum retailer inventory position. The main difference is that we minimize the maximum deviation between each installation's echelon inventory-transit position and its echelon base-stock level. This modification allows for the treatment of non-identical downstream stages.

2.2 Stocking Policies

The traditional approach used in determining stocking levels for a distribution system is to formulate the problem as a stochastic dynamic program and apply relaxations or restrictions to the system to allow for tractability. The large solution space of the optimal policy is accompanied by considerable computational burden. Hence researchers tend to approximate the system to create a policy and then compare that policy via numerical solutions or simulation to either known bounds or the "best found system" (e.g. McGavin et al, 1993).

One approach is to treat the warehouse as a cross-dock that may not hold inventory, introduced by Eppen and Schrage (1981), and extended by Erkip et al. (1990) and Garg and Tang (1997). Unlike these works, we allow the warehouse to hold inventory, thus exploiting risk pooling and holding cost savings at the warehouse.

When inventory is allowed to be held at the warehouse, Federgruen and Zipkin (1984a) show that one may approximate a two-echelon distribution system with identical retailers by relaxing a dynamic programming formulation, initially allowing rebalancing to determine shipment quantities to a collapsed retail stage. This relaxation provides a lower bound on the cost and stocking levels, and we adopt a similar approach for our lower bounds. Jackson (1988) provides an extension of the Eppen and Schrage (1981) model to allow the warehouse to hold inventory. Jackson's approximate cost function is a nested optimization problem, where internal newsvendor problems depend on the warehouse stock level. While Jacksons' approximate cost function is minimized by a search over a single variable, our heuristic does not require a recursive solution.

Axsater et al. (2002) consider a two-echelon multiple retailer distribution system with a virtual assignment heuristic that determines stocking levels. They decompose the system into multiple independent distributor-retailer systems and utilize Graves' (1996) virtual assignment rule. Although future reallocation at the warehouse stage is permitted, they argue that virtual assignment creates an upper bound on stocking levels and costs. We apply the same argument for our upper bounds.

Thus far we have discussed a number of works where the solution technique has been to relax or constrain the problem to establish tractability. Cachon (2001) considers a periodic review system with batch ordering, but shows that a random allocation policy provides exact results obtained through a recursive process. Cachon uses a bounded iterative search to determine stocking levels, and finds that other simple and commonly used heuristics fail to reliably perform well. We confirm these findings while introducing a simple closed form heuristic that does perform well. We show that as the allocation policy becomes more sophisticated, shifting from a random to a myopic allocation policy, our approach outperforms a random allocation heuristic. This suggests that, like the use of the rebalancing assumption, the use of random allocation policies improves tractability but at the cost of decreased performance.

3. Model

We consider a two-echelon supply chain with a single supplier of an abundantly available commodity. There are *n* retail sites, and installations are labeled with index $i \in (W, 1, 2, ..., n)$, where the warehouse is denoted by the index *W*. Let D_i^t denote the demand over *t* unit length periods at retailer 1_i (we omit the superscript when $t = 1$). We assume demand to be stationary and independent across retailers and time, with known but not necessarily identical distributions across retailers. In each period, the following sequence of events occurs: previously shipped replenishments arrive at each installation, demand occurs at each retailer, excess demand is fully backordered, replenishment orders are placed, costs are assessed, and replenishment orders are shipped. Inventory is reviewed every period and a centralized decision maker places replenishment orders based on knowledge of the entire supply chain's inventory positions.

We assume per unit local inventory holding costs (H_i) and backordering costs (b_i) are linear, and ordering costs throughout the system are zero, resulting in the optimality of base-stock policies at each installation (s_i). Before costs are assessed in each period, the following variables are measured:

B_i = number of backorders at installation *i*.

$I_i^{'}$ = on-hand inventory at installation *i*.

T_i = inventory in transit to stage *i*.

I_i = echelon inventory at installation *i*, $I_i = I_i^{'}$ for $i = 1, \ldots, n$, and $I_W = I_W^{'} + \sum_{j=1}^{n}\left(T_j + I_i\right)$.

IP_i = echelon inventory-transit position at installation *i*, $IP_i = I_i - B_i + T_i$.

IO_i = inventory orders outstanding for installation *i*, $IO_i = s_i - IP_i$

The total system costs in a period is the sum $h_W I_w + \sum_{j=1}^{n} \left(b_i B_i + h_i I_i \right)$. Replenishments for an installation arrive L_i periods after being shipped. While the warehouse's supplier has infinite capacity, the warehouse may not have sufficient inventory on hand to fill all retailer demands. In this case, the allocation policy is to ship all on hand inventory while minimizing IO_i. Thus, the allocation policy allocates scarce inventory to installations on the basis of their relative need.

4. Newsvendor Heuristic for Distribution Systems

In this section, we present a heuristic for determining echelon base-stock levels for a two-echelon distribution network. We construct two serial supply chain systems whose costs bound the optimal costs and echelon base-stock levels of distribution system from above and below. Our illustrative network, depicted in the center of Figure 1, faces demand processes $D_1, D_2, \ldots, D_n$ at the terminal ends of the chain segments.

To determine the upper bound, we restrict the warehouse to designate and maintain retailer specific inventories. That is, the centralized decision maker specifies which retailer each unit of inventory will eventually be shipped to as that unit of inventory is ordered from the supplier. In spirit, this is similar to the virtual assignment approach of Graves (1996), who notes that because it may be desirable to un-commit stock, this assignment rule will not perform as well as a dynamic allocation policy. The restriction decomposes the distribution network into a set of n independent serial systems, one system for each retailer, as depicted on the left of Figure 1. We refer to these serial chains as 'decomposed'.

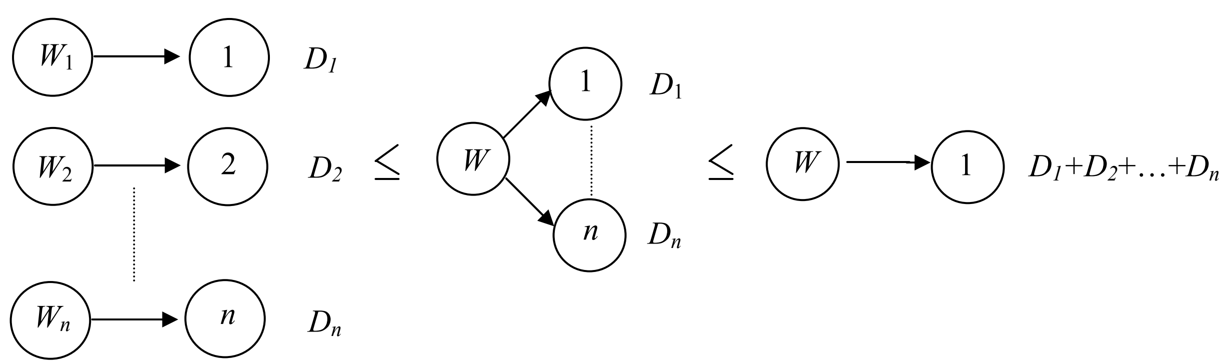

Figure 1: Constructed Serial Chains

In Figure 1, we introduce the labels W_i to denote a warehouse installation that exclusively serves retailer *i*. To describe our Newsvendor Heuristic, we need the following notation.

h_i	= echelon holding cost rate for installation i, $h_W = H_W$ and $h_i = H_i - H_W$ for $i = 1, \ldots, n$.
μ_i	= mean demand rate at installation i, $\mu_i = \mathrm{E}[D_i]$
$(x)^-$	= max{0, -x}
s_i^{x*}	= the "best found" echelon base-stock level for installation i in topology x, where $x \in (d,c,a)$
$\boldsymbol{s}^x$	= a vector of echelon base-stock levels for all installations in topology x
$C^x\left(\boldsymbol{s}^x\right)$	= the expected per period cost of the topology x under echelon base-stock vector $\boldsymbol{s}^x$, where $x \in (d,c,a)$
d, c	= superscripts denoting decomposed and collapsed systems, respectively
a	= superscript denoting the distribution system

The optimal base-stock policy of each decomposed serial chain may by obtained by solving the equations:

$$C_i(y) = \mathrm{E}\left[h_i\left(y - D_i^{L_i}\right) + \left(b_i + H_i\right)\left[y - D_i^{L_i}\right]^-\right]$$

$$s_i^{d*} = \arg\min\left\{C_i\left(y\right)\right\}$$

$$C_{W_i}^d(y) = \mathrm{E}\left[\begin{array}{l} h_{W_i}\left(y - D_i^{L_{w_i}}\right) + h_i\left(\left(s_i^{d*} - \left(y - D_i^{L_{w_i}} - s_i^{d*}\right)^-\right) - D_i^{L_i}\right) \\ +\left(b_i + H_i\right)\left(\left(s_i^{d*} - \left(y - D_i^{L_{w_i}} - s_i^{d*}\right)^-\right) - D_i^{L_i}\right)^- \end{array}\right]$$

$$s_{W_i}^* = \arg\min\left\{C_{W_i}^d\left(y\right)\right\}.$$

The optimal expected cost for each of the decomposed systems is $C_{W_i}^d\left(s_{W_i}^*\right)$, and the expected overall cost of the total system of decomposed chains is simply the sum

$$\sum_{i=1}^{n} C_{W_i}^d\left(s_{W_i}^*\right) = C^d\left(s^{d*}\right). \tag{1}$$

Because this sum is obtained by applying a constraint to the warehouse, it is an upper bound for the optimal cost of the distribution network. Additionally, removing the decomposition constraint allows for risk pooling, suggesting that if backordering costs are sufficiently high to induce installations to carry positive safety stock, the sum

$$\sum_{j=1}^{n} s_{W_j}^{d*} = s_W^{d*} \tag{2}$$

is expected to be an upper bound for the optimal echelon base stock level at the warehouse. A similar argument is made by Gallego et al. (2003).

Having constructed an upper bound for the cost of the distribution network, we next construct a single serial system that serves as a lower bound. Here, our approach is similar to Federgruen and Zipkin (1984a), who assume that instantaneous and costless transshipments within an echelon are allowable. The result of this assumption is an artificial distinction between installations in an echelon. The retailers may thus be collectively treated as a single virtual installation which fills all system demands, as shown on the right of Figure 1. We refer to this system as 'collapsed'.

As with the decomposed system, this serial system is solved by the above optimization equations. Let $\boldsymbol{s}^{c*}$ and $C^c(\boldsymbol{s}^{c*})$ represent the optimal echelon base-stock policy and expected system wide cost of the collapsed system, respectively. By introducing inventory commitment constraints on the retail stage of the collapsed system, we achieve the original distribution network. For identical retailers, $C^c(\mathbf{s}^{c*})$ is a lower bound for $C^a(\mathbf{s}^{a*})$ because the distribution network is the result of adding constraints on the collapsed network.

Additionally, the retailer echelon base-stock level under the collapsed system suggests lower bounds for the echelon base-stock levels for the distribution network. To see this, consider that by combining the retail stages from the distribution network, we gain the opportunity to exploit risk pooling. Assuming that the chain carries nonnegative safety stocks, the pooling potentially reduces inventory in this installation and also the optimal echelon base-stock level of the warehouse.

The decomposition and collapsed system results combine to give

$$C^c\left(s^{c*}\right) \le C^a\left(s^{a*}\right) \le \sum_{j=1}^{n} C^d\left(s^{d*}\right) \tag{3}$$

and suggests that

$$s_W^{c*} \le s_W^{a*} \le s_W^{d*} \ . \tag{4}$$

We use these serial systems to approximate the optimal echelon base-stock levels for the distribution network. Our approach is to utilize the Shang and Song (2003) heuristic for each of the *n*+1 constructed chains. Using an illustrative two-retailer system, for the collapsed serial chain system, the stocking level at the warehouse is

$$s_W^c = \frac{F_W^{-1}\left(\frac{b}{b + h_W + h_1}\right) + F_W^{-1}\left(\frac{b}{b + h_W}\right)}{2}. \tag{5}$$

For the decomposed serial chain system, the stocking levels at echelon *i* are, for our illustrative system,

$$s_{W_1}^d = \frac{F_{W_1}^{-1}\left(\frac{b_1}{b_1 + h_W + h_1}\right) + F_{W_1}^{-1}\left(\frac{b_1}{b_1 + h_W}\right)}{2} \quad \text{and} \quad s_{W_2}^d = \frac{F_{W_2}^{-1}\left(\frac{b_2}{b_2 + h_W + h_2}\right) + F_{W_2}^{-1}\left(\frac{b_2}{b_2 + h_W}\right)}{2}. \tag{6,7}$$

The sum of these stock levels, $s_W^d = s_{W_1}^d + s_{W_2}^d$ represents an approximation for an upper bound of the echelon base-stock policy of the distribution system.

When backorder costs or holding costs differ between retailers, we must adjust the collapsed system equation (5). To do so, we use the mean demand weighted average backorder and holding costs for the distribution stage. Thus, for a two-retailer system, the terms in equation (5) are

$$b = \frac{\mu_1 b_1 + \mu_2 b_2}{\mu_1 + \mu_2} \quad \text{and} \quad h_1 = \frac{\mu_1 h_{1,1} + \mu_2 h_{1,2}}{\mu_1 + \mu_2} \tag{8, 9}$$

In this case, our argument that the total inventory costs of the distribution system are bounded from below by that of the collapsed system may not hold. However, in our numerical experiments below, we find no instances where the collapsed system costs exceed that of the distribution system. Thus we present the results for asymmetric retailers under the numerical conjecture that the bound holds.

The Newsvendor Heuristic for the stocking level at the warehouse is a simple average of the stocking levels from the constructed systems:

$$s_W^a = \frac{s_W^c + s_W^d}{2}. \tag{10}$$

5. Simulation Methodology

A majority of previous papers on distribution system stocking policies use simulation to test the accuracy of dynamic programming relaxations because close form cost equations do not exist for most realistic allocation policies. Thus, we also use simulation to test the performance of our approach against prior work and commonly used practitioner heuristics.

Our simulation methodology is an unequal variance, two-stage screening-subset selection procedure presented in Nelson et al. (2001). We first create a set of base-stock level candidates. For distributions with finite support, these candidates are obtained by enumerating over the entire range of potential lead-time demands at each installation. For distributions with infinite support, candidates cover a range of the expected minimizing base-stock level, +/- at least 5 inventory units for each installation. For the parameter settings in these examples, this range covers approximately 50% of the cumulative distribution of the lead-time demand at each installation, centered on the cost minimizing stocking level as suggested by the Newsvendor Heuristic.

For each stocking level, we initially conduct a steady state simulation of our model and allocation policy for 50,000 periods. We batch periods into groups of 10 to reduce deviations from normality and correlations between single period costs. Based on the lead-times used in our study, we omit the first 10 periods to eliminate initialization effects. The remaining data points are used in the initial screening phase.

Potential sets of stocking levels that survive the initial screening are subjected to a second round of simulation experiments where we retain our batch mean sizes and generate a sufficient number of data points to eliminate all but one of the systems. After this experiment, the set of stocking levels that has the lowest per period cost is selected. This procedure ensures a confidence level of at least 1-α that the selected system performs within a quantity δ of the best found system cost. Hence we refer to the selected system as a δ–best system. For our purposes, we consider $\alpha = 5\%$ and $\delta = 0.2\%$ of the average per period system cost of the best system found in the first stage.

The simulation model was verified by using the same approach to simulate a serial chain, whereupon the results are identical to those found by Shang and Song (2003). In the next section, we compare the performance of our NH to other simple and widely used heuristics.

6. Problem Design and Results

6.1. Symmetric Two-Echelon Networks

Our first experimental design considers two network topologies, with either two or four symmetric retailers. We test the heuristics using a full factorial design over a range of holding cost, backorder cost, and lead-time parameters. We consider (h_W, h_i)= {(1,1), (1,2), (2,1)}, (L_W, L_i) = {(1,1), (1,2), (2,1)}, and b_i = {5,10,20}. We hold the total periodic system demand, μ, constant at 20 units per period, distributed according to a Poisson distribution. This demand is split among the retailers, resulting in $\mu_i = 10$ for the 2-retailer network and $\mu_i = 5$ for the 4-retailer network. These parameter values are similar to those used by Jackson (1988), Cachon (2001), Axsater et al. (2002) and Shang and Song (2003), and are summarized in Tables A1 and A2 in Appendix 1 for the two-retailer and four-retailer networks, respectively.

6.1.1 Random Allocation Policies

For the parameter settings in Table 1, we compare the results of the Newsvendor Heuristic (NH) to those of Cachon (2001), whose results are optimal when a random allocation policy is used. These results are presented in Table A3 in Appendix 1 and are summarized in Table 1. Based on this test, we make the following three observations.

% Error Under Random Allocation		
	Two-Retailer	Four-Retailer
Exact	0.00%	0.00%
Bounds	2.68%	3.38%

Table 1: Random Allocation Summary

Observation 1: *A small but significant error exists from using the NH under a random allocation setting. The error grows as the number of retailers increase but the heuristic reacts to parametric changes in a similar manner as the exact procedure.*

Observation 2: *The exact strategy holds more inventory at the distribution point than does the NH.*

We argue below that this is a result of poor management of inventory at the distribution point.

Observation 3: *As backorder costs increase, the total system stock held by the NH falls relative to the exact analysis.*

For backorder rates of 5, the exact analysis tends to hold less inventory than the NH. For backorder rates of 10, there is no clear trend, but for backorder rates of 20, the NH carries less total inventory than the exact analysis. We discuss this further in the next section.

6.1.2 Myopic Allocation Policies

In this section we compare the systems generated by the NH to the δ-best system found via the simulation procedure described in §5. We also investigate the performance of three other alternative heuristics. First, we use the results of Cachon's (2001) exact analysis under random allocation as a heuristic under our proposed allocation policy. Since Graves (1996) finds that holding no safety stock at the upstream stage is frequently a good (and simple) heuristic, we also consider this approach (termed the zero safety stock policy in the results below). Finally, we investigate the performance of setting a fixed service rate at the warehouse stage, as is frequently encountered in practice. We choose a 99% fill rate because in practice, managers frequently require high fill rates from the warehouse (Lee and Tang, 1997). The results of these experiments are presented in Tables A4 and A5 in Appendix 1 for the two-retailer and four-retailer networks, respectively. From these results, we make the following observations.

Observation 4: *The NH performs best of all the tested heuristics. It is followed by the Cachon exact analysis and zero safety stock heuristics, while the fixed high fill-rate heuristic performs poorly in all problems*.

A summary of these results is presented in the symmetric columns in Table 2.

% Error Under Myopic Allocation				
	2-Retailer		4-Retailer	
Heuristic	Symmetric	Asymmetric	Symmetric	Asymmetric
Newsvendor	0.40%	0.85%	0.48%	0.89%
Cachon	1.75%	NA	2.24%	NA
99% Fill Rate	22.06%	24.59%	21.21%	24.66%
Zero Safety Stock	2.21%	1.96%	2.95%	3.96%

Table 2: Myopic Allocation Summary

Observation 5: *The additional upstream inventory held by Cachon's exact analysis causes it to under perform the NH when non-random allocation is allowed*.

By allocating inventory randomly, the exact analysis increases the variance of the demand placed upon the warehouse by the terminal stages, increasing the required inventory at the warehouse. In contrast, allocating inventory myopically is more efficient. A myopic allocation reduces the penalty induced by preventing the retailer from redistributing inventory (in a random allocation), allowing inventory to be placed further downstream, as the lower inventory at the distribution point results in less frequent stock outs. This effect becomes more important as the backorder costs increase.

Observation 6: *All else held constant, increasing the number of retailers increases the total system cost. Additionally, increasing the holding costs, lead-times, or backorder costs also increases the total system cost*.

These effects are congruent with prior work and intuition. Increasing the number of retailers reduces risk-pooling savings, while increasing other parameters increases costs directly. We address these effects further in section 7.

6.2. Asymmetric Two-Echelon Networks

We now consider networks where the terminal stages are asymmetric or non-identical. We consider a full factorial design over $h_i = \{1,2\}$ and $b_i = \{5,10,20\}$ for both two and four-retailer chains, while holding $L_2 = L_1 = 1$ and the system demand as described in section 6.1. The parameters for each problem investigated are presented in Tables A1 and A2. We compare the performance of the NH, Zero Safety Stock, and 99% Fill Rate heuristics to that of the δ-best system. The results are presented in Tables A6 and A7 in Appendix 1 and are summarized in the asymmetric columns in Table 2.

Observation 7: *Observations 4 and 6 hold in the asymmetric case. Additionally, asymmetric networks introduce slightly more error in the NH performance*.

This increase is present in the other tested heuristics as well, and may be due in part to a larger number of candidate policies. The NH returns an average error of 0.87%, while the holding costs between retail locations vary by 100% and the backorder costs between locations vary by 400%. We believe this range covers most realistic distribution systems.

6.3 Heuristic Robustness Tests

Having established that the NH performs well over a broad range of cost parameters, we next examine its robustness. Our primary goal in this section is to determine where the NH breaks down, thus the range for the tested parameter values may exceed those ever found in practice. We begin by examining the performance of the NH across other demand distributions than Poisson. We investigate three other demand distributions: discrete uniform (5,15), negative binomial (with $\mu = 10$ and $\sigma^2 = 16.54$), and a constructed bimodal distribution whose pmf is depicted in Figure 2. The variance of the negative binomial distribution was selected to match that of the constructed distribution, while the mean demand of each distribution matches those of the Poisson distribution from the previous tests.

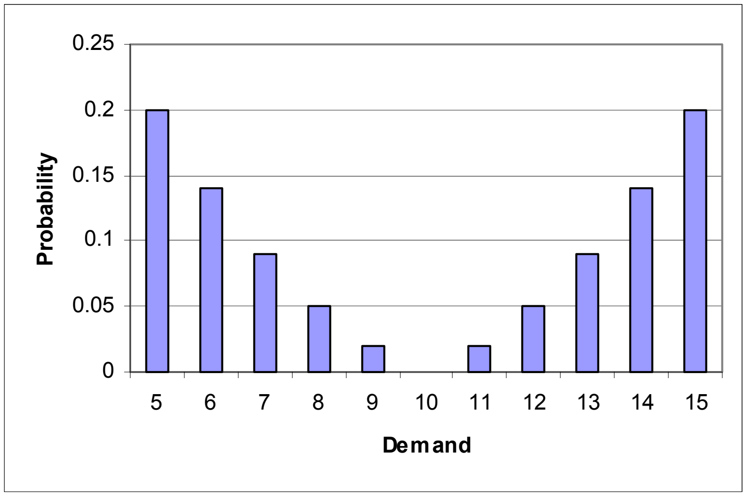

Figure 2: PMF of the Constructed Bimodal Distribution

We test the NH on these distributions over a broader range of parameters. We consider a full factorial design of $n = \{2,10\}$, $b_1 = \{1,10,50\}$, $(h_W, h_1) = \{(1,1), (10,1), (1,10)\}$, and $(L_W, L_1) = \{(1,1), (1,3), (3,1)\}$, and continue to use the simulation methodology presented in §5. A summary of the results of these tests is presented in Table 3 (full results are available in Tables A8-A10), where we report the number of test cases where the cost exceeded the δ-best policy by the range given in the left most column.

	Number of Retailers					
	2			10		
Range of Error	Discrete Uniform	Negative Binomial	Constructed Bimodal	Discrete Uniform	Negative Binomial	Constructed Bimodal
0%	13	12	11	1	2	0
<1%	7	10	12	11	11	11
1% to 5%	5	5	4	4	4	6
>5%	2	0	0	11	10	10

Table 3: Robustness Tests Across Demand Distributions and Number of Retailers

Observation 8: *The accuracy of the NH does not significantly depend on the type of demand distribution.*

The error the NH incurs is approximately the same across each demand distribution investigated. Additionally, the errors generated by the two retailer tests are approximately equal to those under the Poisson tests in §6.1.2.

Observation 9: *Increasing the number of retailers significantly decreases the performance of the NH.*

As the number of retailers increase from two to ten, the performance of the NH drops precipitously. This trend is common across demand distributions but is not common across cost parameters. A further investigation into the trend cited in Observation 9 reveals that the NH's performance is heavily dependent on the relative holding cost patterns, as depicted in Table 4. This leads to our next observation.

	Holding Cost (h_W, h_1)		
Range of Error	(1,1)	(10,1)	(1,10)
0%	2	1	0
<1%	12	21	0
1% to 5%	7	5	2
>5%	6	0	25

Table 4: Performance of the NH with 10 Retailers

Observation 10: *The NH fails when there are many retailers and the holding cost increases dramatically between the warehouse and the retail stages.*

Interestingly, the warehouse echelon base-stock level under these conditions is approximately equal to that of the δ-best system. The majority of the error arises instead from the allocation of inventory within the system. In these cases, the NH carries too little inventory at the retail stages, overcompensating for the exceptionally high holding costs. Thus the NH is useful in setting the total inventory stock, but should not be used to determine the base-stock levels at the retailers.

We believe these scenarios, where holding costs at the retailers exceed that at the warehouse by orders of magnitude are rare in practice. For instance, in the electronics industry in the United States, warehouse and retail space rents are approximately $4/ft^2 and $7.1/ft^2, respectively (www.bizstats.com). On average, the firms generate $355/ft^2 in sales on 13.8 inventory turns a year, generating $58/ft^2 in gross profit. If the cost of capital is 10%, the capital cost component of local holding costs at the retail and warehouse locations are $9.25/ft^2 and $6.15/ft^2 for the average US electronics seller respectively. Taking backordering costs as solely the forfeited margin of lost sales, and scaling such that $h_1 = 1$, these correspond to cost parameter settings of $h_1 = 1$, $h_2 = 0.5$, and $b = 9.52$. These values easily fall within the parameter value range where the NH performs well.

We next consider the effects of asymmetry in the retail parameter values. There were no significant differences in the accuracy of our results between the four different demand distributions so we utilize the discrete uniform distribution in the following tests. Because the discrete uniform distribution has finite support, we are able to fully enumerate over all possible lead-time demands. We consider two sets of problems, each with two retailers, where the demand or lead-time to one retailer varies, respectively. The first set of examples is a full factorial design over the parameter values $(b_1, b_2) = \{(1,1), (1,50), (50,1)\}$, $h_2 = 1$, $L_W = L_1 = L_2 = 1$, $(h_1, h_2) = \{(1,1), (1,10), (10,1)\}$, $D_1 \sim U(5,15)$, and $D_2 \sim \{U(15,25), U(35,45), U(55,65), U(75,85), U(95,105)\}$. This test captures asymmetries in retail demand; the results are summarized in Table 5 (full data are presented in Table A11).

	Demand Ratio (μ_2/μ_1)				
Range of Error	2	4	6	8	10
0%	1	0	1	0	1
<1%	2	3	5	5	4
1% to 5%	6	6	2	3	3
>5%	0	0	1	1	1

Table 5: NH Performance over Demand and Cost Asymmetry

The second set of test problems is a full factorial design over the parameter values $(b_1, b_2) = \{(1,1), (1,50), (50,1)\}$, $h_W = 1$, $L_W = L_1 = 1$, $L_2 = \{2,3,4,5\}$, $(h_1, h_2) = \{(1,1), (1,10), (10,1)\}$, $D_1 \sim U(5,15)$, and $D_2 \sim U(5,15)$. This test captures asymmetries in the lead-times between the warehouse and the two retailers. The results of this test are summarized in Table 6 (full data are presented in Table A12). Our final observation summarizes our asymmetric results.

	Lead-time Ratio (L_2/L_1)			
Range of Error	2	3	4	5
0%	0	0	0	3
<1%	1	3	4	1
1% to 5%	8	5	4	4
>5%	0	0	1	1

Table 6: NH Performance over Lead-time and Cost Asymmetry

Observation 11: *The NH is robust to asymmetry in both the lead-times and demand rates.*

Tables 5 and 6 show the NH is robust across widely varying cost, lead-time, and demand rates. Although the NH typically fails to identify the δ-best policy, it performs within 5% of the δ-best policy costs in 76 of the 81 test cases. That this performance occurs under a wide disparity in retailer parameter values gives further support to the robustness of the NH.

7. Cost Functions and Analysis of Parameter Value Effects

As noted by Shang and Song (2003), the simple newsvendor bounds presented above enable the analysis of the effects of the system parameters much more readily than previous solution methods. Although these bounds and cost functions are general, assuming normally distributed demand allows us to obtain some analytical results. Hence, for Propositions 1 and 2 below, we assume demand at each retailer is normally distributed.

Recall that our method of bounding the distribution system is through the construction of a set of serial systems. The analysis of the resulting cost functions has a number of parallels to the serial supply chain system studied by Shang and Song (2003). Under symmetric profiles, increasing either the backordering cost or the lead-time increases both total system costs and echelon stocking levels. Increasing the warehouse echelon holding cost rate increases system costs and stocking levels at the retailers, while increasing the echelon holding cost rate at the retailers while decreasing the echelon base stock levels at the warehouse. Thus the parametric results for symmetric distribution systems are identical to those of a serial chain. The analysis becomes slightly more complex when considering asymmetric problems. Here, a change in a given retailer's parameter value does not affect the base-stock levels of the other retailers. Proposition 1 describes the impact on the retailer whose parameter value is modified along with the impact on the warehouse.

Proposition 1. *For $i = 1, 2, \ldots n$, and $i \neq j$*

(a) *as b_j increases, $C^a(s^{a^*})$ increases, s_W and s_j increase, while s_i remains unchanged.*

(b) *as h_j increases, $C^a(s^{a^*})$ is non-decreasing, s_W and s_j decrease, and s_i remains unchanged.*

(c) *as L_j increases, $C^a(s^{a^*})$ increases, s_W and s_j increase, while s_i remains unchanged.*

Thus, increasing the backorder cost or lead-time at one retailer increases the total system costs and increases the echelon stocking levels of both the warehouse and that retailer. However, the stocking levels of the other retailers are independent of the effects of the change in the parameter value. An increase in the echelon holding cost at the retailer decreases the echelon stocking levels, while the total system costs are non-decreasing (and likely increasing).

Standard risk pooling arguments yield the intuition that, keeping the system demand constant, increasing the number of retailers increases both system stocking levels and total system costs. We formalize this intuition in Proposition 2.

Proposition 2. *For $i = 1, 2, \ldots n$, and $j = 1, 2, \ldots n, n+1$, and assuming safety stocks are positive, $C^a(s^{a^*})$ and s_W are non-decreasing, and $s_j < s_i$ where $i \neq j$.*

Proposition 2 states that while increasing the number of retailers in a distribution network while keeping the total system demand constant reduces the inventory held at each retailer, it also likely increases the total amount of system stock and total system costs. These effects arise due to the limited ability of the centralized decision maker to exploit risk-pooling opportunities.

Finally, we examine the effects of increasing asymmetry in the retailer parameter values. We begin by addressing asymmetry in backorder costs. Consider an initially symmetric system, and increase b_1 while decreasing b_2 by Δ, such that $b_1 = b_i\,(1+\Delta)$ and $b_2 = b_i\,(1-\Delta)$. Because there is no closed form for the inverse of the normal cdf, we condition Propositions 3 and 4 assuming uniform lead-time demand distributions. Our numerical tests verify the results hold for normal distributions as well, although we note that certain pathological distributions exist for which the results will not hold.

Proposition 3. *For $i = 3, \ldots, n$, $b_1 = b_i(1+\Delta)$ and $b_2 = b_i(1-\Delta)$, s_W and $C^a(s^{a^*})$ are non-increasing with Δ.*

Proposition 3 states that increasing asymmetry in backordering costs does not increase and often decreases stocking levels and system costs, as the decomposed echelon stock levels and system costs are non-increasing while the collapsed values remain unchanged. This seemingly counter intuitive result arises due to the tendency of the system to behave as a serial chain as asymmetry increases. Taken to an extreme, the retailer with the high backordering cost captures the majority of the inventory related costs. Thus, this high-backorder cost retailer dominates the system, which begins to resemble a serial chain consisting solely of the high-backorder cost retailer. Recall that we expect the collapsed serial chain to serve as an approximate lower bound for the distribution system. In effect, we find that symmetric sub-chains may be thought of as 'worst case' scenarios for total system costs. Proposition 3 is illustrated by a small set of numerical test problems, as presented in Figure 3. We compare the difference in $C^a(s^a)$ for two-retailer networks with $b_i = 10$, $\Delta = \{0,0.5\}$, normally distributed demands with $\mu = 20$ and $\sigma^2 = 20$, and three holding cost cases where $(h_1,h_2) = \{(1,1), (1,2), (2,1)\}$.

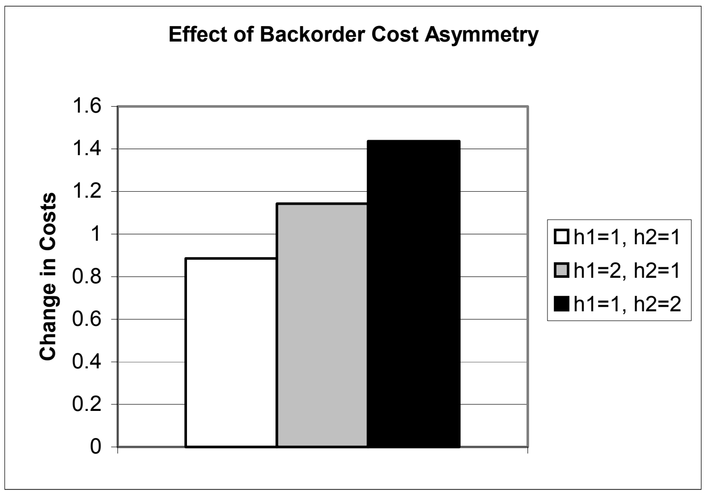

Figure 3: Effect of Backorder Cost Asymmetry

In a similar manner, we examine the effect of asymmetry in the holding costs in Proposition 4. To do so, we modify the retailer's echelon holding cost such that $h_1 = h_i(1+\Delta)$ and $h_2 = h_i(1-\Delta)$.

Proposition 4. *For $i = 1, ..., n$, $h_1 = h_i(1+\Delta)$ and $h_2 = h_i(1-\Delta)$, s_2 increases while s_1 decreases with increasing Δ. Also, s_W is non-increasing and $C^a(s^{a*})$ is non-decreasing in Δ.*

Proposition 4 shows that the system echelon stocking level is non-increasing but total system inventory costs are non-decreasing in asymmetry of the retailer holding costs. This is illustrated by a set of numerical test problems, as presented in Figure 4. We compare the difference in $C^a(s^a)$ for two-retailer networks with $h_i = 1$, $\Delta = \{0,0.5\}$ demand distributed normally with $\mu = 20$ and $\sigma^2 = 20$ and three backorder cases where $b = \{5,10,15\}$.

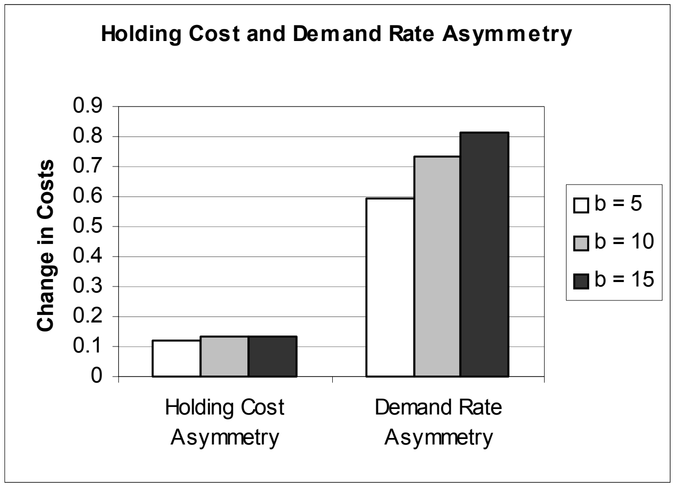

Figure 4: Effects of Holding Cost and Demand Rate Asymmetry

Finally, in Proposition 5 we present the effect of demand asymmetry on the stocking levels and supply chain costs. The critical fractile computations of our newsvendor approach are independent of the demand distribution, hence we return to considering normal distributions for this result.

Proposition 5. *For* $i = 3, \ldots, n$, $\mu_1 = (1+\Delta)\mu_i$, *and* $\mu_2 = (1-\Delta)\mu_i$, *both* s_W *and* $C^a(s^{a^*})$ *are non-increasing with* Δ.

Proposition 5 states that increasing asymmetry in demand rates does not increase and often decreases both echelon stocking levels and system costs. By a similar argument as for Proposition 3, the results of Proposition 5 arise due to the tendency of the resulting network to more closely resemble a serial chain. Although the increase in asymmetry decreases the risk pooling savings at the warehouse, it also introduces a virtual pooling effect in the retail stages of the network. A numerical depiction of Proposition 5 is illustrated in Figure 4 where we compare the difference in $C^a(s^a)$ for two-retailer networks with $\mu_i = 10$ and $\Delta = \{0, 0.5\}$.

8. Concluding Remarks

In this paper, we present a simple heuristic for two-echelon distribution system with n non-identical retailers. The Newsvendor Heuristic requires only the computation of $4(n+1)$ newsvendor problems, but performs well over a wide range of parameters, resulting in an average cost that is 0.44% and 0.87% greater than the cost of the best found stocking policies for symmetric and asymmetric cost parameters, respectively, outperforming all other commonly used heuristics. The heuristic is robust over multiple demand distributions and widely varying cost parameters. For asymmetric (non identical) systems, the heuristic is shown to perform well with backordering costs ranging from 50% to 2,500% of the local holding costs, and mean demand and lead-times varying by up to 500%. Although the heuristic does break down when holding costs increase by 1,000% between the warehouse and retailers, even here the NH provides useful insights, correctly predicting the amount of total system stock. The simplicity of our heuristic also facilitates insights on parametric analysis that are difficult or impossible to obtain based on the competing heuristics. For example, we show that the supply chain's inventory and costs increase in the number of retailers, but decrease as backordering costs and demands at the retailers become asymmetric. Our results simplify the teaching of supply chain distribution system concepts in the classroom and provide practical insights for managers.

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Appendix 1: Numeric Experimental Data and Results

Two-Echelon Two-Retailer Problem Parameter Settings							
Problem	h_W	h_1	H_2	b_1	b_2	L_W	L_1
1	1	1	1	5	5	1	1
2	1	1	1	10	10	1	1
3	1	1	1	20	20	1	1
4	1	2	2	5	5	1	1
5	1	2	2	10	10	1	1
6	1	2	2	20	20	1	1
7	2	1	1	5	5	1	1
8	2	1	1	10	10	1	1
9	2	1	1	20	20	1	1
10	1	1	1	5	5	1	2
11	1	1	1	10	10	1	2
12	1	1	1	20	20	1	2
13	1	2	2	5	5	1	2
14	1	2	2	10	10	1	2
15	1	2	2	20	20	1	2
16	2	1	1	5	5	1	2
17	2	1	1	10	10	1	2
18	2	1	1	20	20	1	2
19	1	1	1	5	5	2	1
20	1	1	1	10	10	2	1
21	1	1	1	20	20	2	1
22	1	2	2	5	5	2	1
23	1	2	2	10	10	2	1
24	1	2	2	20	20	2	1
25	2	1	1	5	5	2	1
26	2	1	1	10	10	2	1
27	2	1	1	20	20	2	1
28	1	1	1	5	10	1	1
29	1	1	1	5	20	1	1
30	1	1	1	10	20	1	1
31	1	1	2	5	5	1	1
32	1	1	2	10	10	1	1
33	1	1	2	20	20	1	1
34	1	1	2	5	10	1	1
35	1	1	2	5	20	1	1
36	1	1	2	10	20	1	1
37	1	2	1	5	10	1	1
38	1	2	1	5	20	1	1
39	1	2	1	10	20	1	1
40	2	1	1	5	10	1	1
41	2	1	1	5	20	1	1

42	2	1	1	10	20	1	1
43	2	1	2	5	5	1	1
44	2	1	2	10	10	1	1
45	2	1	2	20	20	1	1
46	2	1	2	5	10	1	1
47	2	1	2	5	20	1	1
48	2	1	2	10	20	1	1

Table A1: Two Retailer Parameter Settings

Two-Echelon, Four-Retailer Network Problem Parameter Settings											
Problem	h_2	$h_{1,1}$	$h_{1,2}$	$h_{1,3}$	$h_{1,4}$	b_1	b_2	b_3	b_4	L_2	L_1
49	1	1	1	1	1	5	5	5	5	1	1
50	1	1	1	1	1	10	10	10	10	1	1
51	1	1	1	1	1	20	20	20	20	1	1
52	1	2	2	2	2	5	5	5	5	1	1
53	1	2	2	2	2	10	10	10	10	1	1
54	1	2	2	2	2	20	20	20	20	1	1
55	2	1	1	1	1	5	5	5	5	1	1
56	2	1	1	1	1	10	10	10	10	1	1
57	2	1	1	1	1	20	20	20	20	1	1
58	1	1	1	1	1	5	5	5	5	1	2
59	1	1	1	1	1	10	10	10	10	1	2
60	1	1	1	1	1	20	20	20	20	1	2
61	1	2	2	2	2	5	5	5	5	1	2
62	1	2	2	2	2	10	10	10	10	1	2
63	1	2	2	2	2	20	20	20	20	1	2
64	2	1	1	1	1	5	5	5	5	1	2
65	2	1	1	1	1	10	10	10	10	1	2
66	2	1	1	1	1	20	20	20	20	1	2
67	1	1	1	1	1	5	5	5	5	2	1
68	1	1	1	1	1	10	10	10	10	2	1
69	1	1	1	1	1	20	20	20	20	2	1
70	1	2	2	2	2	5	5	5	5	2	1
71	1	2	2	2	2	10	10	10	10	2	1
72	1	2	2	2	2	20	20	20	20	2	1
73	2	1	1	1	1	5	5	5	5	2	1
74	2	1	1	1	1	10	10	10	10	2	1
75	2	1	1	1	1	20	20	20	20	2	1
76	1	1	1	1	1	5	5	10	10	1	1
77	1	1	1	1	1	5	5	20	20	1	1
78	1	1	1	1	1	10	10	20	20	1	1
79	1	1	2	1	2	5	5	5	5	1	1

80	1	1	2	1	2	10	10	10	10	1	1
81	1	1	2	1	2	20	20	20	20	1	1
82	1	1	2	1	2	5	5	10	10	1	1
83	1	1	2	1	2	5	5	20	20	1	1
84	1	1	2	1	2	10	10	20	20	1	1
85	2	1	1	1	1	5	5	10	10	1	1
86	2	1	1	1	1	5	5	20	20	1	1
87	2	1	1	1	1	10	10	20	20	1	1
88	2	1	2	1	2	5	5	5	5	1	1
89	2	1	2	1	2	10	10	10	10	1	1
90	2	1	2	1	2	20	20	20	20	1	1
91	2	1	2	1	2	5	5	10	10	1	1
92	2	1	2	1	2	5	5	20	20	1	1
93	2	1	2	1	2	10	10	20	20	1	1

Table A2: Four Retailer Parameter Settings

Random Allocation Policy Results							
Problem	Exact Results			Bounds Heuristic Results			
	s_2	s_1	Cost	s_2	s_1	Cost	% Error
1	41	22	48.15	39	23	48.40	0.53
2	43	24	55.29	39	25	56.44	2.08
3	44	26	62.28	40	26	64.57	3.68
4	42	21	54.22	41	22	54.76	0.99
5	43	23	64.32	42	23	64.47	0.23
6	44	25	74.58	43	24	75.52	1.26
7	38	22	77.35	35	24	79.12	2.29
8	41	23	88.56	36	25	90.67	2.38
9	42	25	99.71	38	26	102.27	2.56
10	40	33	72.42	37	35	73.34	1.27
11	42	35	80.75	38	36	81.90	1.42
12	43	37	88.89	38	38	91.70	3.16
13	41	31	79.95	40	33	80.87	1.15
14	42	34	91.80	40	35	92.72	1.01
15	44	36	103.67	41	36	104.78	1.07
16	38	32	122.56	34	35	124.40	1.50
17	40	34	135.45	34	37	137.96	1.85
18	42	36	148.42	35	38	152.71	2.89
19	62	22	49.91	61	23	49.91	0.01
20	64	24	57.38	60	25	58.33	1.65
21	66	26	64.69	62	26	66.15	2.25
22	62	21	56.03	62	22	56.48	0.80
23	64	23	66.50	64	23	66.50	0.00
24	66	25	77.04	65	24	77.84	1.04
25	58	22	79.85	56	24	81.08	1.54

26	60	24	91.86	58	25	92.62	0.82
27	63	25	103.76	59	26	105.61	1.79
49	41	11	57.35	39	12	58.08	1.27
50	43	12	67.60	39	13	68.46	1.27
51	44	14	77.35	39	14	80.19	3.67
52	42	10	65.74	42	11	66.90	1.76
53	43	12	80.28	42	12	80.31	0.04
54	45	13	94.32	43	13	94.68	0.38
55	38	11	89.05	32	13	93.33	4.81
56	40	12	104.10	36	13	106.35	2.16
57	42	13	119.62	36	14	124.19	3.82
58	39	17	83.73	37	18	84.94	1.44
59	42	18	95.32	33	20	99.02	3.89
60	43	20	106.86	34	21	112.41	5.20
61	40	16	94.19	39	17	96.40	2.35
62	43	17	110.60	40	18	111.80	1.09
63	44	19	127.73	41	19	128.53	0.63
64	37	16	136.55	29	19	140.77	3.09
65	40	17	154.55	30	20	158.54	2.58
66	41	19	172.19	30	21	179.29	4.12
67	62	11	58.94	61	12	59.52	0.99
68	63	13	69.47	61	13	69.64	0.24
69	65	14	79.40	62	14	80.25	1.07
70	63	10	67.45	63	11	68.44	1.48
71	64	12	82.15	64	12	82.15	0.00
72	66	13	96.55	65	13	96.69	0.14
73	57	11	91.15	53	13	94.71	3.90
74	61	12	106.99	58	13	108.17	1.10
75	63	13	123.20	58	14	125.13	1.56

Table A3: Random Allocation Results

Myopic Allocation Policy Results for 2-Echelon, 2-Retailer Symmetric Networks															
Problem	*g*-Optimal System			Bounds Heuristic			Cachon Heurisitc			99% Fill Rate Heuristic			Zero Safety Stock Heuristic		
	s_2	s_1	Cost	s_2	s_1	%Error	s_2	s_1	%Error	s_2	s_1	%Error	s_2	s_1	%Error
1	17	14	38.53	19	13	0.00	21	12	1.21	31	13	23.44	20	13	0.50
2	20	14	43.47	19	15	0.54	23	14	2.49	31	15	23.12	20	15	1.32
3	18	16	47.82	20	16	1.63	24	16	6.33	31	16	20.14	20	16	1.63
4	19	12	42.88	21	12	0.96	22	11	1.48	31	12	20.23	20	12	0.31
5	21	13	49.99	22	13	0.62	23	13	1.42	31	13	14.93	20	13	0.45
6	22	14	57.40	23	14	0.28	24	15	3.35	31	14	10.95	20	14	1.69
7	14	14	63.66	15	14	0.18	18	12	0.65	31	14	39.90	20	14	7.61
8	16	15	71.22	16	15	0.12	21	13	1.73	31	15	33.03	20	15	4.66
9	19	15	78.75	18	16	0.64	22	15	2.90	31	16	27.71	20	16	2.73
10	14	26	64.11	17	25	0.29	20	23	0.79	31	25	18.09	20	25	2.49
11	18	26	70.70	18	26	0.00	22	25	1.19	31	26	13.63	20	26	0.60
12	18	28	76.18	18	28	1.27	23	27	2.79	31	28	14.15	20	28	1.83
13	17	24	70.60	20	23	0.46	21	21	1.06	31	23	13.77	20	23	0.46
14	17	26	80.13	20	25	0.24	22	24	0.30	31	25	11.97	20	25	0.24
15	19	27	88.09	21	26	1.63	24	26	3.22	31	26	10.35	20	26	1.59
16	11	26	110.56	14	25	0.22	18	22	0.93	31	25	24.27	20	25	5.62
17	9	29	120.20	14	27	0.28	20	24	1.23	31	27	22.93	20	27	5.50
18	15	28	130.40	15	28	0.00	22	26	1.82	31	28	18.22	20	28	2.71
19	40	13	40.64	41	13	0.27	42	12	1.32	55	13	26.97	40	13	0.23
20	40	15	46.05	40	15	0.00	44	14	1.09	55	15	24.97	40	15	0.34
21	41	16	51.35	42	16	-0.08	46	16	3.63	55	16	19.73	40	16	0.84
22	40	12	45.20	42	12	0.35	42	11	1.13	55	12	22.99	40	12	0.23
23	43	13	52.72	44	13	0.38	44	13	0.38	55	13	16.65	40	13	2.09
24	45	14	60.46	45	14	0.00	46	15	2.24	55	14	12.03	40	14	6.02
25	35	14	66.91	36	14	0.32	38	12	0.99	55	14	45.13	40	14	5.07
26	37	15	75.32	38	15	0.22	40	14	0.34	55	15	36.48	40	15	2.19
27	39	16	83.70	39	16	0.00	43	15	1.38	55	16	29.78	40	16	0.69

Table A4: Myopic Allocation Policy Results for Two-Retailer Symmetric Networks

Myopic Allocation Policy Results for 2-Echelon, 4-Retailers Symmetric Networks															
Problem	*g*-Optimal System			Bounds Heuristic			Cachon Heuristic			99% Fill Rate Heuristic			Zero Safety Stock Heuristic		
	s_2	s_1	Cost	s_2	s_1	%Error	s_2	s_1	% Error	s_2	s_1	%Error	s_2	s_1	%Error
49	17	7	44.4	19	7	0.53	21	6	2.6	31	7	22	20	7	1.336
50	18	8	51.3	19	8	0.21	23	7	3.06	31	8	18.7	20	8	0.814
51	18	9	58.3	19	9	0.1	24	9	5.02	31	9	16.4	20	9	0.622
52	19	6	50.5	22	6	1.55	22	5	4.79	31	6	16.6	20	6	0.241
53	20	7	60.4	22	7	0.87	23	7	1.71	31	7	13.2	20	7	0
54	20	8	70.7	23	8	1	25	8	2.93	31	8	10.8	20	8	0
55	15	7	70.8	12	8	0.4	18	6	1.14	31	8	44	20	8	13.87
56	15	8	81	16	8	0.16	20	7	1.45	31	8	28.8	20	8	4.331
57	16	9	91.6	16	9	0	22	8	2.49	31	9	25.5	20	9	3.707
58	15	13	72.7	17	13	0.26	19	12	0.59	31	13	15.7	20	13	2.136

59	17	14	81.7	13	15	0.01	22	13	1.71	31	15	17.5	20	15	4.398
60	14	16	90.8	14	16	0	23	15	2.14	31	16	14.3	20	16	2.836
61	16	12	81.1	19	12	0.85	20	11	0.7	31	12	13.5	20	12	1.505
62	18	13	94.4	20	13	0.3	23	12	1.78	31	13	10.1	20	13	0.297
63	19	14	108	21	14	0.1	24	14	1.57	31	14	7.73	20	14	0
64	0	16	121	9	14	0.08	17	11	1.29	31	14	28.7	20	14	10.96
65	10	15	135	10	15	0	20	12	2.2	31	15	24.3	20	15	8.429
66	3	18	149	10	16	0.05	21	14	1.32	31	16	20.5	20	16	6.224
67	38	7	46.2	41	7	1.17	42	6	3.08	55	7	26	40	7	0.64
68	39	8	53.5	41	8	0.42	43	8	2.05	55	8	21.5	40	8	0.107
69	40	9	60.7	42	9	0.54	45	9	3.35	55	9	18.3	40	9	0
70	40	6	52.5	43	6	1.15	43	5	5.07	55	6	19.6	40	6	0
71	41	7	62.8	44	7	1	44	7	1	55	7	15.4	40	7	0.546
72	42	8	73.3	45	8	0.9	46	8	1.68	55	8	12.3	40	8	1.428
73	31	8	73.5	33	8	0.64	37	6	1.97	55	8	49.5	40	8	11.03
74	36	8	84.5	38	8	0.6	41	7	1.7	55	8	33.1	40	8	2.483
75	37	9	95.6	38	9	0.14	43	8	2.16	55	9	28.6	40	9	1.783

Table A5: Myopic Allocation Policy Results for Four-Retailer Symmetric Networks

Myopic Allocation Policy Results for Two-Echelon, Two-Retailer Asymmetric Networks																
Problem	g-Optimal System				Bound Heuristic				99% Fill Rate Heuristic				Zero Safety Stock Heuristic			
	s_2	$s_{1,1}$	$s_{1,2}$	Cost	s_2	$s_{1,1}$	$s_{1,2}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	% Error
28	19	13	14	40.97	20	13	15	1.31	31	13	15	23.33	20	13	15	1.31
29	19	13	16	43.42	21	13	16	1.26	31	13	16	20.85	20	13	16	0.45
30	19	15	16	45.93	20	15	16	0.92	31	15	16	20.75	20	15	16	0.64
31	19	13	12	40.66	20	13	12	0.39	31	13	12	21.84	20	13	12	0.37
32	20	14	13	46.73	21	15	13	0.91	31	15	13	18.77	20	15	13	0.47
33	22	15	14	52.88	22	16	14	0.72	31	16	14	14.33	20	16	14	0.79
34	20	13	13	44.29	21	13	13	0.46	31	13	13	18.55	20	13	13	0.00
35	19	13	15	47.97	22	13	14	1.02	31	13	14	16.19	20	13	14	0.50
36	21	14	14	50.41	21	15	14	0.65	31	15	14	16.34	20	15	14	1.15
37	20	12	14	43.18	20	12	15	1.19	31	12	15	21.61	20	12	15	0.80
38	19	12	16	45.65	21	12	16	0.97	31	12	16	19.51	20	12	16	0.53
39	21	13	15	49.21	21	13	16	0.61	31	13	16	16.60	20	13	16	0.69
40	15	14	15	67.39	16	14	15	0.27	31	14	15	36.33	20	14	15	5.87
41	17	13	16	71.27	18	14	16	1.51	31	14	16	32.96	20	14	16	4.92
42	16	15	16	74.98	17	15	16	0.35	31	15	16	30.21	20	15	16	3.26
43	17	13	12	65.47	17	14	12	0.59	31	14	12	34.76	20	14	12	4.62
44	18	14	13	74.04	18	15	13	0.41	31	15	13	28.18	20	15	13	2.17
45	18	16	15	82.93	19	16	15	0.79	31	16	15	25.21	20	16	15	1.79
46	16	13	14	70.35	18	14	13	1.15	31	14	13	30.79	20	14	13	3.33
47	17	13	15	75.33	18	14	15	1.33	31	14	15	30.06	20	14	15	3.47
48	18	14	15	79.09	18	15	15	0.52	31	15	15	27.57	20	15	15	2.49

Table A6: Myopic Allocation Policy Results for Two-Retailer Asymmetric Networks

Myopic Allocation Policy Results for Two-Echelon, Four-Retailer Asymmetric Networks																								
Problem	*g*-Optimal System						Bounds Heuristic						99% Fill Rate Heuristic						Zero Safety Stock Heuristic					
	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	Cost	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	% Error	s_2	$s_{1,1}$	$s_{1,2}$	$s_{1,3}$	$s_{1,4}$	% Error
76	18	7	7	8	8	47.8	19	7	7	8	8	0.5	31	7	7	8	8	20.4	20	7	7	8	8	1.2
77	18	7	7	9	9	51.2	20	7	7	9	9	1.0	31	7	7	9	9	19.1	20	7	7	9	9	1.0
78	18	8	8	9	9	54.7	19	8	8	9	9	0.3	31	8	8	9	9	17.6	20	8	8	9	9	0.8
79	19	7	6	7	6	47.4	21	7	6	7	6	1.3	31	7	6	7	6	19.1	20	7	6	7	6	0.5
80	19	8	7	8	7	55.8	20	8	7	8	7	0.4	31	8	7	8	7	15.9	20	8	7	8	7	0.4
81	19	9	8	9	8	64.5	21	9	8	9	8	0.7	31	9	8	9	8	13.3	20	9	8	9	8	0.3
82	18	7	6	8	7	51.6	21	7	6	8	7	1.2	31	7	6	8	7	17.4	20	7	6	8	7	0.5
83	19	7	6	9	8	55.8	21	7	6	9	8	1.2	31	7	6	9	8	16.1	20	7	6	9	8	0.2
84	19	8	7	9	8	60.1	21	8	7	9	8	1.0	31	8	7	9	8	14.6	20	8	7	9	8	0.4
85	15	7	7	8	8	75.9	14	8	8	8	8	0.4	31	8	8	8	8	36.0	20	8	8	8	8	8.5
86	15	7	7	9	9	81.0	16	8	8	9	9	2.0	31	8	8	9	9	33.9	20	8	8	9	9	8.2
87	15	8	8	9	9	86.2	16	8	8	9	9	0.3	31	8	8	9	9	27.4	20	8	8	9	9	4.2
88	15	7	7	7	6	73.2	14	8	7	8	7	0.9	31	8	7	8	7	40.2	20	8	7	8	7	11.2
89	15	8	8	8	7	84.9	16	8	8	8	8	1.0	31	8	8	8	8	30.3	20	8	8	8	8	6.2
90	16	9	9	9	8	97.1	16	9	9	9	9	0.9	31	9	9	9	9	26.8	20	9	9	9	9	5.5
91	16	7	6	8	7	79.0	15	8	7	8	8	1.2	31	8	7	8	8	35.0	20	8	7	8	8	8.5
92	16	7	6	9	8	85.1	15	8	7	9	9	0.9	31	8	7	9	9	32.6	20	8	7	9	9	7.8
93	17	8	7	9	8	91.0	16	8	8	9	9	0.9	31	8	8	9	9	28.4	20	8	8	9	9	5.8

Table A7: Myopic Allocation Policy Results for Four-Retailer Asymmetric Networks

Problem	h_W	h_i	b	L_W	L_i	n	s^*_W	s^*_i	s^a_W	s^a_i	Cost	% Error
bp1	1	1	1	1	1	2	10	13	11	14	30.1	0.26
bp2	1	1	10	1	1	2	23	15	23	15	41.6	0.00
bp3	1	1	50	1	1	2	28	15	28	15	44.3	0.00
bp4	1	10	1	1	1	2	18	6	23	5	35.4	4.34
bp5	1	10	10	1	1	2	21	12	23	12	81.2	0.20
bp6	1	10	50	1	1	2	27	15	25	15	91.5	3.07
bp7	10	1	1	1	1	2	10	8	0	13	216.7	0.02
bp8	10	1	10	1	1	2	10	15	10	15	285.4	0.00
bp9	10	1	50	1	1	2	20	15	21	15	351.2	0.16
bp10	1	1	1	3	1	2	50	14	50	14	32.4	0.00
bp11	1	1	10	3	1	2	66	15	66	15	47.4	0.00
bp12	1	1	50	3	1	2	77	15	77	15	54.4	0.00
bp13	1	10	1	3	1	2	58	6	61	5	38.0	1.68
bp14	1	10	10	3	1	2	64	14	65	12	87.2	4.49
bp15	1	10	50	3	1	2	77	15	72	15	99.5	0.61

bp16	10	1	1	3	1	2	32	15	32	15	222.6	0.00
bp17	10	1	10	3	1	2	50	15	50	15	306.8	0.00
bp18	10	1	50	3	1	2	63	15	64	15	397.9	0.18
bp19	1	1	1	1	3	2	10	34	10	34	74.9	0.00
bp20	1	1	10	1	3	2	14	41	14	41	97.1	0.00
bp21	1	1	50	1	3	2	20	44	19	44	108.4	0.45
bp22	1	10	1	1	3	2	12	25	23	24	86.7	3.57
bp23	1	10	10	1	3	2	26	31	27	31	158.0	0.27
bp24	1	10	50	1	3	2	26	36	30	36	227.1	0.35
bp25	10	1	1	1	3	2	10	25	0	31	627.0	0.09
bp26	10	1	10	1	3	2	10	34	0	40	726.8	0.02
bp27	10	1	50	1	3	2	6	44	6	44	836.0	0.00
bp28	1	1	1	1	1	10	68	12	49	14	140.9	0.01
bp29	1	1	10	1	1	10	102	15	96	15	171.8	2.18
bp30	1	1	50	1	1	10	118	15	113	15	180.3	1.25
bp31	1	10	1	1	1	10	55	11	118	5	169.0	14.58
bp32	1	10	10	1	1	10	101	15	103	12	276.9	33.26
bp33	1	10	50	1	1	10	114	15	100	15	260.6	17.23
bp34	10	1	1	1	1	10	68	8	8	13	1065.6	0.14
bp35	10	1	10	1	1	10	62	13	45	15	1396.8	0.04
bp36	10	1	50	1	1	10	97	15	89	15	1605.2	1.51
bp37	1	1	1	3	1	10	237	15	245	14	142.9	0.16
bp38	1	1	10	3	1	10	309	15	305	15	182.0	0.78
bp39	1	1	50	3	1	10	330	15	350	15	206.2	3.76
bp40	1	10	1	3	1	10	239	13	311	5	170.2	12.35
bp41	1	10	10	3	1	10	302	15	307	12	292.3	31.97
bp42	1	10	50	3	1	10	330	15	324	15	245.2	1.59
bp43	10	1	1	3	1	10	205	13	176	15	1079.1	0.31
bp44	10	1	10	3	1	10	251	15	245	15	1409.1	0.06
bp45	10	1	50	3	1	10	299	15	298	15	1675.0	0.07
bp46	1	1	1	1	3	10	68	33	45	34	362.4	0.04
bp47	1	1	10	1	3	10	60	43	45	41	462.9	5.13
bp48	1	1	50	1	3	10	91	45	60	44	547.6	17.51
bp49	1	10	1	1	3	10	65	29	121	24	403.5	6.87
bp50	1	10	10	1	3	10	58	40	117	31	633.9	23.50
bp51	1	10	50	1	3	10	86	44	114	36	845.1	44.80
bp52	10	1	1	1	3	10	50	24	6	32	3138.1	0.26
bp53	10	1	10	1	3	10	68	33	5	39	3600.7	0.01
bp54	10	1	50	1	3	10	66	40	8	44	4142.8	1.17

Table A8: Constructed Bimodal Distribution Robustness Tests

Problem	h_W	h_i	b	L_W	L_i	n	S^*_W	s^*_i	s^a_W	s^a_i	Cost	% Error
up1	1	1	1	1	1	2	11	13	15	12	29.1	1.48
up2	1	1	10	1	1	2	20	15	20	15	37.6	0.00
up3	1	1	50	1	1	2	25	15	25	15	41.6	0.00
up4	1	10	1	1	1	2	15	8	23	6	33.4	5.54
up5	1	10	10	1	1	2	25	10	26	10	70.2	0.71
up6	1	10	50	1	1	2	24	14	24	14	85.3	0.00
up7	10	1	1	1	1	2	11	9	0	15	213.3	0.01
up8	10	1	10	1	1	2	10	15	10	15	265.1	0.00
up9	10	1	50	1	1	2	18	15	18	15	318.9	0.00
up10	1	1	1	3	1	2	47	15	54	12	29.6	0.11
up11	1	1	10	3	1	2	63	15	63	15	42.0	0.00
up12	1	1	50	3	1	2	69	15	71	15	48.3	-0.03
up13	1	10	1	3	1	2	56	8	61	6	35.0	2.79
up14	1	10	10	3	1	2	66	12	68	10	74.8	6.89
up15	1	10	50	3	1	2	69	15	68	14	92.7	1.40
up16	10	1	1	3	1	2	35	15	35	15	217.8	0.00
up17	10	1	10	3	1	2	50	15	50	15	282.4	0.00
up18	10	1	50	3	1	2	59	15	61	15	354.6	0.13
up19	1	1	1	1	3	2	13	32	14	32	71.5	0.10
up20	1	1	10	1	3	2	17	38	17	38	88.6	0.00
up21	1	1	50	1	3	2	18	42	19	41	98.9	0.52
up22	1	10	1	1	3	2	14	27	23	25	81.1	3.62
up23	1	10	10	1	3	2	26	30	28	30	139.0	1.14
up24	1	10	50	1	3	2	25	36	24	36	179.7	-0.02
up25	10	1	1	1	3	2	13	25	9	36	621.4	0.05
up26	10	1	10	1	3	2	13	33	2	39	697.3	-0.07
up27	10	1	50	1	3	2	9	41	9	41	782.8	0.00
up28	1	1	1	1	1	10	76	12	69	12	130.3	0.50
up29	1	1	10	1	1	10	93	15	81	15	167.8	5.74
up30	1	1	50	1	1	10	108	15	101	15	175.5	3.86
up31	1	10	1	1	1	10	70	11	117	6	160.0	17.20
up32	1	10	10	1	1	10	85	15	114	10	279.6	45.12
up33	1	10	50	1	1	10	106	15	97	14	278.3	31.53
up34	10	1	1	1	1	10	62	9	6	15	1054.7	0.00
up35	10	1	10	1	1	10	57	14	45	15	1288.5	0.10
up36	10	1	50	1	1	10	85	15	75	15	1526.3	2.47
up37	1	1	1	3	1	10	269	13	266	12	132.9	0.97
up38	1	1	10	3	1	10	296	15	294	15	166.9	0.28
up39	1	1	50	3	1	10	318	15	324	15	183.1	0.66
up40	1	10	1	3	1	10	247	13	308	6	157.4	12.85
up41	1	10	10	3	1	10	287	15	323	10	292.3	44.88
up42	1	10	50	3	1	10	312	15	316	14	260.8	15.36
up43	10	1	1	3	1	10	220	13	187	15	1066.6	0.47
up44	10	1	10	3	1	10	269	13	245	15	1306.8	0.10
up45	10	1	50	3	1	10	288	15	285	15	1542.8	0.10

up46	1	1	1	1	3	10	68	33	66	32	348.1	0.23
up47	1	1	10	1	3	10	50	42	64	38	426.3	4.64
up48	1	1	50	1	3	10	66	44	64	41	501.8	14.14
up49	1	10	1	1	3	10	50	32	118	25	383.6	6.67
up50	1	10	10	1	3	10	60	39	123	30	589.4	27.03
up51	1	10	50	1	3	10	53	44	96	36	665.2	25.47
up52	10	1	1	1	3	10	58	26	7	33	3102.5	0.12
up53	10	1	10	1	3	10	68	33	5	39	3458.4	0.03
up54	10	1	50	1	3	10	56	40	25	41	3914.1	1.67

Table A9: Discrete Uniform Distribution Robustness Results

Problem	h_W	h_i	b	L_W	L_i	n	s^*_W	s^*_i	s^a_W	s^a_i	Cost	% Error
nb1	1	1	1	1	1	2	11	12	15	11	29.0	0.28
nb2	1	1	10	1	1	2	19	16	19	16	45.2	0.00
nb3	1	1	50	1	1	2	20	20	22	19	57.5	0.93
nb4	1	10	1	1	1	2	13	8	21	6	34.5	3.01
nb5	1	10	10	1	1	2	24	10	25	10	78.1	0.16
nb6	1	10	50	1	1	2	27	14	27	14	123.9	0.00
nb7	10	1	1	1	1	2	9	9	1	13	212.2	-0.06
nb8	10	1	10	1	1	2	8	15	5	17	276.1	0.06
nb9	10	1	50	1	1	2	9	20	9	20	357.7	0.00
nb10	1	1	1	3	1	2	34	21	53	11	31.7	0.08
nb11	1	1	10	3	1	2	63	16	63	16	51.2	0.00
nb12	1	1	50	3	1	2	59	22	68	19	65.1	0.48
nb13	1	10	1	3	1	2	53	8	57	6	37.1	1.13
nb14	1	10	10	3	1	2	65	12	67	10	85.1	2.90
nb15	1	10	50	3	1	2	72	16	72	14	132.9	2.70
nb16	10	1	1	3	1	2	19	22	29	16	217.7	-0.06
nb17	10	1	10	3	1	2	44	17	44	17	299.8	0.00
nb18	10	1	50	3	1	2	46	23	52	20	402.0	0.01
nb19	1	1	1	1	3	2	8	32	11	32	73.1	-0.11
nb20	1	1	10	1	3	2	8	43	17	39	97.6	0.23
nb21	1	1	50	1	3	2	15	46	16	45	115.4	0.59
nb22	1	10	1	1	3	2	11	25	23	23	84.2	3.69
nb23	1	10	10	1	3	2	28	29	29	29	159.3	0.44
nb24	1	10	50	1	3	2	28	36	28	36	229.7	0.00
nb25	10	1	1	1	3	2	0	13	3	37	613.7	-0.07
nb26	10	1	10	1	3	2	10	33	4	41	710.7	-0.08
nb27	10	1	50	1	3	2	11	41	0	46	831.2	-0.01
nb28	1	1	1	1	1	10	34	15	79	11	133.1	-0.15
nb29	1	1	10	1	1	10	67	18	86	16	189.3	0.48
nb30	1	1	50	1	1	10	68	21	73	19	256.0	9.92
nb31	1	10	1	1	1	10	39	13	108	6	147.2	4.18

nb32	1	10	10	1	1	10	74	16	123	10	272.6	18.40
nb33	1	10	50	1	1	10	86	18	110	14	367.1	18.50
nb34	10	1	1	1	1	10	33	11	8	13	1048.7	0.05
nb35	10	1	10	1	1	10	42	15	25	17	1319.0	0.05
nb36	10	1	50	1	1	10	35	20	39	20	1668.6	0.05
nb37	1	1	1	3	1	10	198	18	275	11	137.2	0.29
nb38	1	1	10	3	1	10	223	22	295	16	198.6	0.98
nb39	1	1	50	3	1	10	263	22	310	19	245.3	2.22
nb40	1	10	1	3	1	10	160	21	301	6	153.4	5.89
nb41	1	10	10	3	1	10	248	18	327	10	319.3	31.44
nb42	1	10	50	3	1	10	259	21	334	14	427.9	32.53
nb43	10	1	1	3	1	10	141	19	166	16	1057.3	0.09
nb44	10	1	10	3	1	10	205	18	225	17	1351.8	0.14
nb45	10	1	50	3	1	10	214	22	248	20	1739.1	1.03
nb46	1	1	1	1	3	10	73	30	65	32	353.1	-0.20
nb47	1	1	10	1	3	10	50	42	65	39	455.1	2.35
nb48	1	1	50	1	3	10	69	46	50	45	554.3	9.24
nb49	1	10	1	1	3	10	46	30	131	23	407.8	10.58
nb50	1	10	10	1	3	10	67	38	137	29	682.1	31.99
nb51	1	10	50	1	3	10	75	44	114	36	832.5	32.33
nb52	10	1	1	1	3	10	76	21	6	32	3072.6	0.24
nb53	10	1	10	1	3	10	33	35	5	39	3514.6	0.04
nb54	10	1	50	1	3	10	67	39	8	44	4089.0	0.36

Table A10: Negative Binomial Distribution Robustness Results

Problem	H_1	h_2	B_1	b_2	μ_2	s^*_W	s^*_1	s^*_2	s^a_W	s^a_1	s^a_2	Cost	Error
ad1	1	1	1	1	20	22	13	22	22	12	22	38.7	0.08
ad2	1	1	50	1	20	35	15	18	32	15	22	51.8	3.82
ad3	1	1	1	50	20	33	9	25	33	12	25	50.5	3.25
ad4	1	1	1	1	40	42	13	42	42	12	42	58.7	0.05
ad5	1	1	50	1	40	55	15	38	51	15	42	73.3	4.80
ad6	1	1	1	50	40	53	9	45	54	12	45	70.8	2.76
ad7	1	1	1	1	60	62	13	62	62	12	62	78.7	0.04
ad8	1	1	50	1	60	75	15	58	70	15	62	94.8	5.42
ad9	1	1	1	50	60	74	8	65	74	12	65	90.8	1.93
ad10	1	1	1	1	80	80	14	83	82	12	82	98.7	0.04
ad11	1	1	50	1	80	95	15	78	89	15	82	117.1	6.59
ad12	1	1	1	50	80	94	8	85	94	12	85	110.8	1.58
ad13	1	1	1	1	100	100	14	103	102	12	102	118.7	-0.03
ad14	1	1	50	1	100	114	15	97	109	15	102	137.1	5.50
ad15	1	1	1	50	100	114	8	105	114	12	105	130.8	1.33
ad16	10	1	1	1	20	27	7	20	26	6	22	41.7	1.87
ad17	1	10	1	1	20	24	10	17	25	12	16	41.3	0.22

ad18	10	1	1	1	40	47	7	39	46	6	42	61.7	1.14
ad19	1	10	1	1	40	48	10	36	45	12	36	61.3	0.76
ad20	10	1	1	1	60	67	7	60	67	6	62	81.7	0.98
ad21	1	10	1	1	60	66	11	57	65	12	56	81.3	0.59
ad22	10	1	1	1	80	85	8	81	87	6	82	101.7	0.64
ad23	1	10	1	1	80	88	10	76	85	12	76	101.3	0.46
ad24	10	1	1	1	100	107	7	100	107	6	102	121.7	0.66
ad25	1	10	1	1	100	106	11	97	105	12	96	121.3	0.39
ad26	10	1	50	1	20	35	14	18	31	14	22	91.5	2.54
ad27	1	10	50	1	20	35	15	16	35	15	16	51.4	0.00
ad28	10	1	50	1	40	55	14	38	50	14	42	112.5	3.07
ad29	1	10	50	1	40	54	15	36	53	15	36	70.8	0.30
ad30	10	1	50	1	60	72	15	57	70	14	62	132.5	0.66
ad31	1	10	50	1	60	73	15	56	73	15	56	91.5	0.00
ad32	10	1	50	1	80	95	14	78	89	14	82	154.5	3.56
ad33	1	10	50	1	80	95	15	75	92	15	76	112.1	0.71
ad34	10	1	50	1	100	115	14	98	109	14	102	174.6	3.19
ad35	1	10	50	1	100	115	15	95	112	15	96	132.1	0.60
ad36	10	1	1	50	20	34	6	25	37	6	25	52.3	3.53
ad37	1	10	1	50	20	33	10	24	31	12	24	89.4	1.60
ad38	10	1	1	50	40	54	6	45	58	6	45	73.1	3.76
ad39	1	10	1	50	40	54	9	44	52	12	44	109.2	1.02
ad40	10	1	1	50	60	74	6	65	78	6	65	93.2	2.97
ad41	1	10	1	50	60	73	10	64	72	12	64	129.2	0.95
ad42	10	1	1	50	80	94	6	85	98	6	85	113.2	2.43
ad43	1	10	1	50	80	94	9	84	92	12	84	149.2	0.75
ad44	10	1	1	50	100	114	6	105	118	6	105	133.2	2.06
ad45	1	10	1	50	100	113	10	104	112	12	104	169.2	0.73

Table A11: Asymmetric Demand Robustness Results

Problem	h_W	h_1	h_2	b_1	b_2	L_2	s^*_W	s^*_1	s^*_2	s^a_W	s^a_1	s^a_2	Cost	% Error
al1	1	1	1	1	1	2	11	13	21	15	12	22	40.4	1.19
al2	1	1	1	50	1	2	24	15	17	21	15	22	54.4	4.70
al3	1	1	1	1	50	2	18	10	29	20	12	29	55.7	1.68
al4	1	1	1	1	1	3	11	13	30	14	12	32	51.2	0.33
al5	1	1	1	50	1	3	24	15	26	21	15	32	65.5	5.27
al6	1	1	1	1	50	3	14	12	42	19	12	41	70.0	0.30
al7	1	1	1	1	1	4	14	11	38	13	12	43	62.3	0.56
al8	1	1	1	50	1	4	23	15	35	21	15	43	77.4	5.18
al9	1	1	1	1	50	4	17	10	52	18	12	53	83.4	0.02
al10	1	10	1	1	1	2	16	7	19	18	6	22	43.2	2.49
al11	1	1	10	1	1	2	17	10	14	19	12	15	44.6	2.83
al12	1	10	1	1	1	3	16	7	28	18	6	32	54.4	2.01

al13	1	1	10	1	1	3	12	13	24	18	12	25	56.5	2.11
al14	1	10	1	1	1	4	16	7	37	17	6	43	65.9	2.43
al15	1	1	10	1	1	4	12	13	33	19	12	34	68.7	2.97
al16	1	10	1	50	1	2	24	14	16	21	14	22	92.5	2.31
al17	1	1	10	50	1	2	23	15	13	26	15	15	55.1	1.49
al18	1	10	1	50	1	3	23	14	26	21	14	32	103.6	2.40
al19	1	1	10	50	1	3	23	15	21	26	15	25	68.1	2.54
al20	1	10	1	50	1	4	24	14	35	20	14	43	116.6	3.83
al21	1	1	10	50	1	4	23	15	31	27	15	34	80.4	3.46
al22	1	10	1	1	50	2	21	6	28	24	6	29	58.1	2.52
al23	1	1	10	1	50	2	23	8	24	22	12	25	111.7	0.51
al24	1	10	1	1	50	3	19	6	40	23	6	41	72.4	1.33
al25	1	1	10	1	50	3	21	9	35	22	12	36	137.7	0.56
al26	1	10	1	1	50	4	16	6	52	22	6	53	86.9	0.57
al27	1	1	10	1	50	4	19	10	46	23	12	46	162.2	0.10
al28	1	1	1	1	1	5	10	13	49	13	12	53	73.2	0.51
al29	1	1	1	50	1	5	23	15	45	20	15	53	89.7	5.30
al30	1	1	1	1	50	5	15	11	64	17	12	64	96.7	-0.24
al31	1	10	1	1	1	5	16	7	46	17	6	53	76.7	1.95
al32	1	1	10	1	1	5	13	11	41	20	12	43	80.7	1.84
al33		10	1	50	1	5	23	14	44	20	14	53	127.5	3.49
al34	1	1	10	50	1	5	23	15	39	27	15	43	91.6	2.63
al35	1	10	1	1	50	5	21	6	64	21	6	64	98.7	0.00
al36	1	1	10	1	50	5	21	9	56	21	12	57	184.0	-0.16

Table A12: Asymmetric Lead-time Robustness Results

Appendix 2: Proofs of Propositions 1-5

Define the critical fractiles

$$\Theta_i = \frac{b_i + h_W}{b_i + h_W + h_i} \quad \Theta_{W,i}^{l,d} = \frac{b_i}{b_i + h_W + h_i} \quad \Theta_{W,i}^{u,d} = \frac{b_i}{b_i + h_W} \quad \Theta_W^{l,c} = \frac{b}{b + h_W + h_i} \quad \Theta_W^{u,c} = \frac{b}{b + h_W}$$

Let

$$z_i = \Phi^{-1}\left(\Theta_i\right),\; z_{W,i}^{l,d} = \Phi^{-1}\left(\Theta_{W,i}^{l,d}\right),\; z_{W,i}^{u,d} = \Phi^{-1}\left(\Theta_{W,i}^{u,d}\right),\; z_W^{l,c} = \Phi^{-1}\left(\Theta_W^{l,c}\right) \text{and } z_W^{u,c} = \Phi^{-1}\left(\Theta_W^{u,c}\right).$$

Let $\phi(\cdot)$ and $\Phi(\cdot)$ represent the standard normal pdf and cdf, respectively. Following the approach in Zipkin (2000) (see also Shang and Song, 2003),

$$s_i = \mu_i \tilde{L}_i + z_i \sqrt{\sigma_i^2 \tilde{L}_i} \tag{A1}$$

$$s_{W,i}^{l,d} = \mu_i \tilde{L}_W + z_{W,i}^{l,d} \sqrt{\sigma_i^2 \tilde{L}_{W,i}} \tag{A2}$$

$$s_{W,i}^{u,d} = \mu_i \tilde{L}_W + z_{W,i}^{u,d} \sqrt{\sigma_i^2 \tilde{L}_{W,i}} \tag{A3}$$

$$s_W^{l,c} = \mu \tilde{L}_W + z_W^{l,c} \sqrt{\sigma^2 \tilde{L}_W} \tag{A4}$$

$$s_W^{u,c} = \mu \tilde{L}_W + z_W^{u,c} \sqrt{\sigma^2 \tilde{L}_W} \tag{A5}$$

$$s_W^a = \mu \tilde{L}_W + \frac{(z_W^{u,c} + z_W^{l,c})}{4} \sqrt{\sigma^2 \tilde{L}_W} + \sum_{i=1}^{n} \frac{(z_{W,i}^{u,d} + z_{W,i}^{l,d})}{4} \sqrt{\sigma_i^2 \tilde{L}_W} \tag{A6}$$

$$C_i\left(s_i\right) = \left(b_i + h_i + h_W\right) \phi\left(z_i\right) \sqrt{\sigma_i^2 \tilde{L}_i} \tag{A7}$$

$$C_{W,i}^{l,d}\left(s_{W,i}^{u,d}\right) = \left(b_i + h_W\right) \phi\left(z_{W,i}^{u,d}\right) \sqrt{\sigma_i^2 \tilde{L}_{W,i}} + h_W \mu_i \tilde{L}_{W,i} \tag{A8}$$

$$C_W^{l,d} = \sum_{i=1}^{n} C_{W,i}^{l,d} \tag{A9}$$

$$C_{W,i}^{u,d}\left(s_{W,i}^{l,d}\right) = \left(b_i + h_W + h_i\right) \phi\left(z_{W,i}^{l,d}\right) \sqrt{\sigma_i^2 \tilde{L}_{W,i}} + h_W \mu_i \tilde{L}_{W,i} \tag{A10}$$

$$C_W^{u,d} = \sum_{i=1}^{n} C_{W,i}^{u,d} \tag{A11}$$

$$C_W^{l,c}\left(s_W^{u,c}\right) = \left(b + h_W\right) \phi\left(z_W^{u,c}\right) \sqrt{\sigma^2 \tilde{L}_W} + h_W \mu \tilde{L}_W \tag{A12}$$

$$C_W^{u,c}\left(s_W^{l,c}\right) = \left(b + h_W + h_i\right) \phi\left(z_W^{l,c}\right) \sqrt{\sigma^2 \tilde{L}_W} + h_W \mu \tilde{L}_W \tag{A13}$$

$$C_W^{l,d} \le C_W^d \le C_W^{u,d} \tag{A14}$$

$$C_W^{l,c} \le C_W^c \le C_W^{u,c} \tag{A15}$$

and from (3),

$C^c \le C^a\left(s_W^a\right) \le C^d$ (A16)

Proof of Proposition 1.

Proposition 1 follows by inspection of equations A1 through A16.

(a) As b_i increases,
   a. Θ_i, $\Theta_{W,i}^{l,d}$, $\Theta_{W,i}^{u,d}$, $\Theta_W^{l,c}$, and $\Theta_W^{u,c}$ increase, increasing A1 to A13.
   b. Θ_j remains unchanged where $j \neq i$

(b) As h_i increases,
   a. Θ_i, $\Theta_{W,i}^{l,d}$, and $\Theta_W^{l,c}$, decrease, decreasing A1, A2, A4, and A6.
   b. Examination of equations A7, A8, A10, A12, and A13 shows that for a fixed y, $C_i(y)$ increases with h_i due to the increase in the first coefficient in A7, A10, and A13. Meanwhile, A8 and A12 are independent of changes in h_i. Hence the collapsed system remains unchanged while the decomposed systems increase in costs.
   c. Θ_j and hence A1 remains unchanged where $j \neq i$

(c) As L_i increases,
   a. equations A1 through A13 increase
   b. equations A1 and A7 remain unchanged where $j \neq i$

Proof of Proposition 2.

In this proposition, we hold the total system demand constant. Assuming demand is distributed normally with mean μ and variance σ^2, splitting among n identical terminal locations gives $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$, or $\mu_i = \frac{\mu}{n}$ and $\sigma_i^2 = \frac{\sigma^2}{n}$ while splitting the demand process across n+1 identical terminal locations gives $\mu_j = \frac{\mu}{n+1}$ and $\sigma_j^2 = \frac{\sigma^2}{n+1}$.

(a) We consider three cases, the retail stages, and the collapsed and decomposed serial systems.
   a. For the retail stages, $\mu_j < \mu_i$ and $\sigma_j^2 < \sigma_i^2$, hence $s_i > s_j$ from equations A1.
   b. For the collapsed serial system, $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ remain unchanged. Hence Equations A4 and A5 remain unchanged.

c. For the decomposed systems, consider $\sum_{i=1}^{n} s_{W,i}^{l,d} - \sum_{j=1}^{n+1} s_{W,j}^{l,d}$. From equations A2,

$$= \sum_{i=1}^{n}\left(\mu_i \tilde{L}_i + z_i^{l,d}\sqrt{\sigma_i^2 \tilde{L}_i}\right) - \sum_{j=1}^{n+1}\left(\mu_j \tilde{L}_i + z_i^{l,d}\sqrt{\sigma_j^2 \tilde{L}_i}\right)$$

$$= \sum_{i=1}^{n}\left(\mu_i \tilde{L}_i + z_i^{l,d}\sqrt{\sigma_i^2 \tilde{L}_i}\right) - \sum_{j=1}^{n+1}\left(\frac{n\mu_i}{n+1}\tilde{L}_i + z_i^{l,d}\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i}\right)$$

$$= \sum_{i=1}^{n}\left(z_i^{l,d}\sqrt{\sigma_i^2 \tilde{L}_i}\right) - \sum_{j=1}^{n+1}\left(z_i^{l,d}\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i}\right)$$

$$= z_i^{l,d}\sqrt{\tilde{L}_i}\left(n\sqrt{\sigma_\alpha^2} - (n+1)\sqrt{\frac{n\sigma_i^2}{n+1}}\right)$$

$$= z_i^{l,d}\sqrt{\tilde{L}_i}\left(\sqrt{n^2\sigma_i^2} - \sqrt{(n+1)n\sigma_i^2}\right)$$

$$= -z_i^{l,d}\sigma_i\sqrt{n\tilde{L}_i}$$

$$< 0$$

The above holds for $\sum_{i=1}^{n} s_{W,i}^{u,d} - \sum_{j=1}^{n+1} s_{W,j}^{u,d}$ as well.

Hence the echelon base stock level of the warehouse is non-decreasing in the number of retailers. To see the effects on the system costs, consider

d. For the retail installations, let K be a positive constant equal to $(b_i + h_i + h_W)\phi(z_i)$. Then

$$\sum_{i=1}^{n} C_i - \sum_{j=1}^{n+1} C_j$$

$$= \sum_{i=1}^{n}\left(K\sqrt{\sigma_i^2 \tilde{L}_i}\right) - \sum_{j=1}^{n+1}\left(K\sqrt{\sigma_j^2 \tilde{L}_i}\right)$$

$$= \sum_{i=1}^{n}\left(K\sqrt{\sigma_i^2 \tilde{L}_i}\right) - \sum_{j=1}^{n+1}\left(K\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i}\right)$$

$$= n\left(K\sqrt{\sigma_i^2 \tilde{L}_i}\right) - (n+1)\left(K\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i}\right)$$

$$= K\left(n\sqrt{\sigma_i^2 \tilde{L}_i} - \sqrt{n(n+1)\sigma_i^2 \tilde{L}_i}\right)$$

$$= -K\sqrt{n\sigma_\alpha^2 \tilde{L}_i}$$

$$< 0$$

e. For the collapsed system, $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ remain unchanged. Hence Equations A11 and A12 remain unchanged.

f. For the decomposed systems, let K be a positive constant equal to $\left(b + h_W\right)\phi\left(z_{W,i}^{u,d}\right)$. Then

$$
\begin{aligned}
&\sum_{i=1}^{n} C_{W,i}^{l,d} - \sum_{j=1}^{n+1} C_{W,j}^{l,d} \\
&= \sum_{i=1}^{n}\left(K\sqrt{\sigma_i^2 \tilde{L}_i} + h_W \mu_i \tilde{L}_i\right) - \sum_{j=1}^{n+1}\left(K\sqrt{\sigma_j^2 \tilde{L}_i} + h_W \mu_j \tilde{L}_i\right) \\
&= \sum_{i=1}^{n}\left(K\sqrt{\sigma_i^2 \tilde{L}_i} + h_W \mu_i \tilde{L}_i\right) - \sum_{j=1}^{n+1}\left(K\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i} + \frac{n h_W \mu_i}{n+1}\tilde{L}_i\right) \\
&= \sum_{i=1}^{n}\left(K\sqrt{\sigma_i^2 \tilde{L}_i}\right) - \sum_{j=1}^{n+1}\left(K\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i}\right) \\
&= K\left(n\left(\sqrt{\sigma_i^2 \tilde{L}_i}\right) - (n+1)\left(\sqrt{\frac{n\sigma_i^2}{n+1}\tilde{L}_i}\right)\right) \\
&= -K\sqrt{n\sigma_i^2 \tilde{L}_i} \\
&< 0
\end{aligned}
$$

The proof for $C_{W,i}^{u,d}$ follows exactly as above if we instead let $K =$ $\left(b + h_W + h_i\right)\phi\left(z_{W,i}^{l,c}\right)$.

Hence the costs of both the retailer and warehouse echelons are non-decreasing in the number of retailers.

For Propositions 3 and 4, we assume the lead-time demand at retailer i is uniformly distributed. Specifically, we will consider Uniform(0,1) distributions. Let f(·) and F(·) represent the Uniform(0,1) pdf and cdf, respectively. The base stock levels become $F^{-1}(\Theta) = \Theta$.

Following the standard approach (e.g. see pp 205-209 in Zipkin (2000); proofs of these derivations are available from the authors upon request)

$$C_i\left(s_i\right) = \frac{1}{2}\left(1-\Theta_i\right)\left(b_i + h_W\right) \tag{A17}$$

$$C_{W,i}^{l,d}\left(s_{W,i}^{u,d}\right) = \frac{1}{2}\left(1-\Theta_{W,i}^{u,d}\right)b_i + h_W \mu_i \tilde{L}_i \tag{A18}$$

$$C_{W,i}^{u,d}\left(s_{W,i}^{l,d}\right)=\frac{1}{2}\left(1-\Theta_{W,i}^{l,d}\right)b_i+h_W\mu_i\tilde{L}_i \tag{A19}$$

$$C_W^{l,c}\left(s_W^{u,c}\right)=\frac{1}{2}\left(1-\Theta_W^{u,c}\right)b+h_W\mu\tilde{L}_i \tag{A20}$$

$$C_W^{u,c}\left(s_W^{l,c}\right)=\frac{1}{2}\left(1-\Theta_W^{l,c}\right)b+h_W\mu\tilde{L}_i \tag{A21}$$

Proof of Proposition 3.

Here we investigate the effects of backorder asymmetry. We consider two cases, the collapsed and decomposed systems. Let $b_1 = b_i(1+\Delta)$ and $b_2 = b_i(1-\Delta)$ where $i \neq 1,2$.

(a) For the collapsed systems, $b=\frac{1}{\mu}\sum_{i=1}^{n}\mu_i b_i$ remains unchanged. Hence the critical fractiles $\Theta_i^{u,c}$ and $\Theta_i^{l,c}$ remain unchanged and hence base stocking level equations A4 and A5, and equations A20 and A21 remain unchanged.

(b) For the decomposed systems, let $B = h_i + h_W$, and note that $B > 0$. Consider

$$\begin{aligned}
&2s_{W,i}^{l,d}-s_{W,1}^{l,d}-s_{W,2}^{l,d}\\
&=2F^{-1}\left(\Theta_{i,\alpha}^{d}\right)-F^{-1}\left(\Theta_{i,1}^{d}\right)-F^{-1}\left(\Theta_{i,2}^{d}\right)\\
&=2F^{-1}\left(\frac{b_i}{b_i+B}\right)-F^{-1}\left(\frac{b_1}{b_1+B}\right)-F^{-1}\left(\frac{b_2}{b_2+B}\right)\\
&=2F^{-1}\left(\frac{b}{b+B}\right)-F^{-1}\left(\frac{b+\Delta}{b+B+\Delta}\right)-F^{-1}\left(\frac{b-\Delta}{b+B-\Delta}\right)\\
&=2\frac{b}{b+B}-\frac{b+\Delta}{b+B+\Delta}-\frac{b-\Delta}{b+B-\Delta}\\
&=\frac{2(b)(b+B+\Delta)(b+B-\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)}-\frac{(b+B)(b+\Delta)(b+B-\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)}\\
&\qquad-\frac{(b+B)(b+B+\Delta)(b-\Delta)}{(b+B)(b+B+\Delta)(b+B-\Delta)}
\end{aligned}$$

After expansion and intermediate collection of like terms,

$$\begin{aligned}
&=\frac{2\left(b^3+2b^2B+bB^2-b\Delta^2\right)}{(b+B)(b+B+\Delta)(b+B-\Delta)}-\frac{b^3+2b^2B+bB\Delta+bB^2+B^2\Delta-b\Delta^2-B\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)}\\
&-\frac{b^3+2b^2B+bB^2-b^2\Delta-bB\Delta-B^2\Delta-B\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)}
\end{aligned}$$

Which, further simplified is

$$\frac{2B\Delta^2}{(b+B)(b+B+\Delta)(b+B-\Delta)}$$

> 0, and increasing in Δ.

The above also holds for $2s_{W,i}^{u,d} - s_{W,1}^{u,d} - s_{W,2}^{u,d}$ if we define B $= h_W$.

Thus increasing asymmetry in backordering costs decreases equations A2 and A3. To see the results for the effects on system costs, let $A = h_W + h_i$. First consider

$$2C_{W,i}^{u,d}\left(s_{W,i}^{l,d}\right) - C_{W,1}^{u,d}\left(s_{W,2}^{l,d}\right) - C_{W,1}^{u,d}\left(s_{W,2}^{l,d}\right)$$

$$= \left(1-\Theta_{W,i}^{l,d}\right)(b_i) - \frac{1}{2}\left(1-\Theta_{W,1}^{l,d}\right)(b_1) - \frac{1}{2}\left(1-\Theta_{W,2}^{l,d}\right)(b_2) + 2h_W \mu_i \tilde{L}_i - h_W \mu_1 \tilde{L}_i - h_W \mu_2 \tilde{L}_i$$

$$= \left(1-\Theta_{W,i}^{l,d}\right)b_i - \frac{1}{2}\left(1-\Theta_{W,1}^{l,d}\right)(b_i+\Delta) - \frac{1}{2}\left(1-\Theta_{W,2}^{l,d}\right)(b_i-\Delta)$$

$$= \left(1-\frac{b_i}{b_i+h_W+h_i}\right)(b_i) - \frac{1}{2}\left(1-\frac{b_i+\Delta}{b_i+\Delta+h_W+h_i}\right)(b_i+\Delta) - \frac{1}{2}\left(1-\frac{b_i-\Delta}{b_i-\Delta+h_W+h_i}\right)(b_i-\Delta)$$

$$= \left(1-\frac{b_i}{b_i+A}\right)(b_i) - \frac{1}{2}\left(1-\frac{b_i+\Delta}{b_i+\Delta+A}\right)(b_i+\Delta) - \frac{1}{2}\left(1-\frac{b_i-\Delta}{b_i-\Delta+A}\right)(b_i-\Delta)$$

$$= b - \frac{b^2}{b+A} - \frac{(b+\Delta)}{2} + \frac{(b+\Delta)^2}{2(b+\Delta+A)} - \frac{(b-\Delta)}{2} + \frac{(b-\Delta)^2}{2(b-\Delta+A)}$$

$$= -\frac{b^2}{b+A} + \frac{(b+\Delta)^2}{2(b+\Delta+A)} + \frac{(b-\Delta)^2}{2(b-\Delta+A)}$$

$$= \frac{1}{2}\frac{2b^2(b+\Delta+A)(b-\Delta+A) - (b+A)(b+\Delta)^2(b-\Delta+A) - (b+A)(b-\Delta)^2(b+\Delta+A)}{(b+A)(b+\Delta+A)(b-\Delta+A)}$$

And after expansion and collection of terms,

$$= \frac{\Delta^2 A^2}{(b+A)(b+\Delta+A)(b-\Delta+A)}$$

> 0, and increases with Δ

Note that the above analysis also holds for $2C_{i,\alpha}^{l,d}\left(s_{i,\alpha}^{u,d}\right)-C_{i,1}^{l,d}\left(s_{i,2}^{u,d}\right)-C_{i,1}^{l,d}\left(s_{i,2}^{u,d}\right)$ if we define $A = h_W$. Thus asymmetry in backorder cost decreases equations A18 and A19. Combined with the above results, we find that both stocking levels and system costs are decreasing in backorder cost asymmetry.

Proof of Proposition 4.

We show the effects of holding cost asymmetry on stocking levels and total system costs. Beginning with the effects on stocking levels, we consider two cases, the collapsed and the decomposed systems.

(a) For the collapsed system, with $h_1 = h_i(1+\Delta)$, and $h_2 = h_i(1-\Delta)$, consider that the weighted holding cost $h_i = \dfrac{1}{\mu}\left(\sum_{j\neq 1,2} h_j + \left(h_1+\Delta\right)+\left(h_2-\Delta\right)\right)$ is independent of Δ. Hence $\Theta_W^{l,c}$ and $\Theta_W^{u,c}$, and thus the collapsed system stocking levels are independent of Δ.

(b) For the decomposed system, consider $2s_{W,i}^{l,d}-s_{W,1}^{l,d}-s_{W,2}^{l,d}$ where $\alpha \neq 1,2$. Letting A = $h_W + h_i$,

$$
\begin{aligned}
&2s_{W,i}^{l,d}-s_{W,1}^{l,d}-s_{W,2}^{l,d}\\
&=2\mu_i\tilde{L}_W+2z_{W,i}^{l,d}\sqrt{\sigma_i^2\tilde{L}_{W,i}}-\mu_1\tilde{L}_W-z_{W,1}^{l,d}\sqrt{\sigma_i^2\tilde{L}_{W,i}}-\mu_2\tilde{L}_W-z_{W,2}^{l,d}\sqrt{\sigma_i^2\tilde{L}_{W,i}}\\
&=2z_{W,i}^{l,d}\sqrt{\sigma_i^2\tilde{L}_{W,i}}-z_{W,1}^{l,d}\sqrt{\sigma_i^2\tilde{L}_{W,i}}-z_{W,2}^{l,d}\sqrt{\sigma_i^2\tilde{L}_{W,i}}\\
&=\sqrt{\sigma_i^2\tilde{L}_{W,i}}\left(2z_{W,i}^{l,d}-z_{W,1}^{l,d}-z_{W,2}^{l,d}\right)\\
&=\sqrt{\sigma_i^2\tilde{L}_{W,i}}\left(2\frac{b_i}{b_i+h_W+h_i}-\frac{b_i}{b_i+h_W+h_i+\Delta}-\frac{b_i}{b_i+h_W+h_i-\Delta}\right)\\
&=\sqrt{\sigma_i^2\tilde{L}_{W,i}}\left(2\frac{b_i}{b_i+A}-\frac{b_i}{b_i+A+\Delta}-\frac{b_i}{b_i+A_i-\Delta}\right)\\
&=\sqrt{\sigma_i^2\tilde{L}_{W,i}}\left(\frac{2b_i\left(b_i+A+\Delta\right)\left(b_i+A_i-\Delta\right)-b_i\left(b_i+A\right)\left(b_i+A_i-\Delta\right)-b_i\left(b_i+A\right)b_i+A+\Delta}{\left(b_i+A\right)\left(b_i+A+\Delta\right)\left(b_i+A_i-\Delta\right)}\right)
\end{aligned}
$$

After expansion and collection of terms,

$$
=\frac{-2b_i\Delta^2}{\left(b_i+A\right)\left(b_i+A+\Delta\right)\left(b_i+A-\Delta\right)}
$$

< 0, and decreasing in Δ

Note that the above also holds for $2s_{W,i}^{u,d} - s_{W,1}^{u,d} - s_{W,2}^{u,d}$ if we let A = h_W. Thus the warehouse echelon base stock level is non-decreasing in holding cost asymmetry.

To see the effects of holding cost asymmetry on total system costs, consider

(c) For the collapsed systems, the weighted holding cost $h_i = \frac{1}{\mu}\left(\sum_{j\neq 1,2} h_j + (h_1+\Delta) + (h_2 - \Delta)\right)$ is independent of Δ. Hence equations A20 and A21 are independent of holding cost asymmetry.

(d) For the decomposed systems, let $A = h_W + h_i$. First consider

$$2C_{W,i}^{u,d}\left(s_{W,i}^{l,d}\right) - C_{W,1}^{u,d}\left(s_{W,1}^{l,d}\right) - C_{W,2}^{u,d}\left(s_{W,2}^{l,d}\right)$$

$$= \left(1-\Theta_{W,i}^{l,d}\right)b_i + h_W\mu_i\tilde{L}_i - \frac{1}{2}\left(1-\Theta_{W,1}^{l,d}\right)b_i - \frac{h_W\mu_i\tilde{L}_i}{2} - \frac{1}{2}\left(1-\Theta_{W,2}^{l,d}\right)b_i - \frac{h_W\mu_i\tilde{L}_i}{2}$$

$$= \left(1-\Theta_{W,i}^{l,d}\right)b_i - \frac{1}{2}\left(1-\Theta_{W,1}^{l,d}\right)b_i - \frac{1}{2}\left(1-\Theta_{W,2}^{l,d}\right)b_i$$

$$= \left(1-\frac{b_i}{b_i+h_W+h_i}\right)b_i - \frac{1}{2}\left(1-\frac{b_i}{b_i+h_W+h_1}\right)b_i - \frac{1}{2}\left(1-\frac{b_i}{b_i+h_W+h_2}\right)b_i$$

$$= -\frac{b_i^2}{b_i+h_W+h_i} + \frac{1}{2}\frac{b_i^2}{b_i+h_W+h_1} + \frac{1}{2}\frac{b_i^2}{b_i+h_W+h_2}$$

$$= -\frac{b_i^2}{b_i+h_W+h_i} + \frac{b_i^2}{b_i+h_W+h_i+\Delta} + \frac{1}{2}\frac{b_i^2}{b_i+h_W+h_i-\Delta}$$

$$= -\frac{b_i^2}{b_i+A} + \frac{1}{2}\frac{b_i^2}{b_i+A+\Delta} + \frac{1}{2}\frac{b_i^2}{b_i+A-\Delta}$$

$$= \frac{-2b_i^2(b_i+A+\Delta)(b_i+A-\Delta) + b_i^2(b_i+A)(b_i+A-\Delta) + b_i^2(b_i+A)(b_i+A+\Delta)}{2(b_i+A)(b_i+A+\Delta)(b_i+A-\Delta)}$$

$$= \frac{b_i^2\left((b_i+A)(b_i+A-\Delta) + (b_i+A)(b_i+A+\Delta) - 2(b_i+A+\Delta)(b_i+A-\Delta)\right)}{2(b_i+A)(b_i+A+\Delta)(b_i+A-\Delta)}$$

$$= \frac{b_i^2\left((b_i+A)(b_i+A-\Delta) + (b_i+A)(b_i+A+\Delta) - 2(b_i+A+\Delta)(b_i+A-\Delta)\right)}{2(b_i+A)(b_i+A+\Delta)(b_i+A-\Delta)}$$

$$= \frac{b^2\left(\left(b^2 - b\Delta + Ab + Ab - A\Delta + A^2\right) + \left(b^2 + b\Delta + Ab + Ab + A\Delta + A^2\right)\right)}{2(b+A)(b+\Delta+A)(b+A-\Delta)}$$

$$- \frac{2b^2\left(b^2 + Ab - b\Delta + Ab + A^2 - A\Delta + b\Delta + A\Delta - \Delta^2\right)}{2(b+A)(b+\Delta+A)(b+A-\Delta)}$$

and collecting like terms gives

$$= \frac{b^2\Delta^2}{(b+A)(b+\Delta+A)(b+A-\Delta)}$$

> 0, and increasing in Δ.

Note that the above also holds for $2C_{W,i}^{l,d}\left(s_{W,i}^{u,d}\right) - C_{W,1}^{l,d}\left(s_{W,1}^{u,d}\right) - C_{W,2}^{l,d}\left(s_{W,2}^{u,d}\right)$ if we let $A = h_W$. Thus the total system costs are non-decreasing in holding cost asymmetry.

We next consider asymmetry in demand. Because demand does not appear in the critical fractiles, we revert to normal distributions.

Proof of Proposition 5.

We consider two cases, the collapsed and decomposed systems.

(a) For the collapsed system, note that $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ are independent of Δ. Thus the collapsed stocking levels $s_W^{l,c} = \mu\tilde{L}_W + z_W^{l,c}\sqrt{\sigma^2\tilde{L}_W}$ and $s_W^{u,c} = \mu\tilde{L}_W + z_W^{u,c}\sqrt{\sigma^2\tilde{L}_W}$ are likewise independent of asymmetry in μ_i and σ_i^2. Also, the cost equations $C_W^{l,c}\left(s_W^{u,c}\right) = (b+h_W)\phi\left(z_W^{u,c}\right)\sqrt{\sigma^2\tilde{L}_W} + h_W\mu\tilde{L}_i$ and $C_W^{u,c}\left(s_W^{l,c}\right) = (b+h_W+h_i)\phi\left(z_W^{l,c}\right)\sqrt{\sigma^2\tilde{L}_W} + h_W\mu\tilde{L}_i$ are likewise independent of asymmetry in μ_α and σ_α^2.

(b) For the decomposed system, first consider $2s_{W,i}^{l,d} - s_{W,1}^{l,d} - s_{W,2}^{l,d}$

$$\begin{aligned}
&= 2\mu_i\tilde{L}_W + 2z_{W,i}^{l,d}\sqrt{\sigma_i^{\,2}\tilde{L}_{W,i}} - \mu_1\tilde{L}_W - z_{W,1}^{l,d}\sqrt{\sigma_1^{\,2}\tilde{L}_{W,i}} - \mu_2\tilde{L}_W - z_{W,2}^{l,d}\sqrt{\sigma_2^{\,2}\tilde{L}_{W,i}} \\
&= 2z_{W,i}^{l,d}\sqrt{\sigma_i^{\,2}\tilde{L}_{W,i}} - z_{W,1}^{l,d}\sqrt{\sigma_1^{\,2}\tilde{L}_{W,i}} - z_{W,2}^{l,d}\sqrt{\sigma_2^{\,2}\tilde{L}_{W,i}} \\
&= z_{W,i}^{l,d}\left(\sqrt{4\sigma_i^{\,2}\tilde{L}_{W,i}} - \sqrt{\sigma_1^{\,2}\tilde{L}_{W,i}} - \sqrt{\sigma_2^{\,2}\tilde{L}_{W,i}}\right) \\
&= z_{W,i}^{l,d}\sqrt{\tilde{L}_{W,i}}\left(\sqrt{4\sigma_i^{\,2}} - \sqrt{\frac{\mu+\Delta}{\mu}\sigma_i^{\,2}} - \sqrt{\frac{\mu-\Delta}{\mu}\sigma_i^{\,2}}\right) \\
&= z_{W,i}^{l,d}\sqrt{\sigma_i^{\,2}\tilde{L}_{W,i}}\left(\sqrt{4} - \sqrt{\frac{\mu+\Delta}{\mu}} - \sqrt{\frac{\mu-\Delta}{\mu}}\right) \\
&\geq z_{W,i}^{l,d}\sqrt{\sigma_i^{\,2}\tilde{L}_{W,i}}\left(2 - \sqrt{\frac{\mu}{\mu}} - \sqrt{\frac{\Delta}{\mu}} - \sqrt{\frac{\mu}{\mu}} + \sqrt{\frac{\Delta}{\mu}}\right)
\end{aligned}$$

$$= z_{W,i}^{l,d}\sqrt{\sigma_i{}^2\tilde{L}_{W,i}}\left(2-\sqrt{\frac{\mu}{\mu}}-\sqrt{\frac{\mu}{\mu}}\right)=0$$

Note that the above follows for $2s_{W,i}^{u,d}-s_{W,1}^{u,d}-s_{W,2}^{u,d}$ as well if we substitute $z_{W,i}^{u,d}$ for $z_{W,i}^{l,d}$. Thus the echelon base stock level at the warehouse is non-increasing in demand rate asymmetry.

(c) Next, consider $2C_{W,i}^{u,d}\left(s_{W,i}^{l,d}\right)-C_{W,1}^{u,d}\left(s_{W,1}^{l,d}\right)-C_{W,2}^{u,d}\left(s_{W,2}^{l,d}\right)$

$$= 2\left(b_i+h_W+h_i\right)\phi\left(z_{W,i}^{l,d}\right)\sqrt{\sigma_i{}^2\tilde{L}_{W,i}}+2h_W\mu_i\tilde{L}_{W,i}-\left(b_i+h_W+h_i\right)\phi\left(z_{W,1}^{l,d}\right)\sqrt{\sigma_1{}^2\tilde{L}_{W,i}}-h_W\mu_1\tilde{L}_{W,i}$$

$$-\left(b_i+h_W+h_i\right)\phi\left(z_{W,2}^{l,d}\right)\sqrt{\sigma_2{}^2\tilde{L}_{W,i}}-h_W\mu_2\tilde{L}_{W,i}$$

$$=\left(b_i+h_W+h_i\right)\left(2\phi\left(z_{W,i}^{l,d}\right)\sqrt{\sigma_i{}^2\tilde{L}_{W,i}}-\phi\left(z_{W,1}^{l,d}\right)\sqrt{\sigma_1{}^2\tilde{L}_{W,i}}-\phi\left(z_{W,2}^{l,d}\right)\sqrt{\sigma_2{}^2\tilde{L}_{W,i}}\right)$$

$$=\left(b_i+h_W+h_i\right)\phi\left(z_{W,i}^{l,d}\right)\sqrt{\sigma_i{}^2\tilde{L}_{W,i}}\left(\sqrt{4}-\sqrt{\frac{\mu+\Delta}{\mu}}-\sqrt{\frac{\mu-\Delta}{\mu}}\right)$$

$$\geq\left(b_i+h_W+h_i\right)\phi\left(z_{W,i}^{l,d}\right)\sqrt{\sigma_i{}^2\tilde{L}_{W,i}}\left(2-\sqrt{\frac{\mu}{\mu}}-\sqrt{\frac{\Delta}{\mu}}-\sqrt{\frac{\mu}{\mu}}+\sqrt{\frac{\Delta}{\mu}}\right)=0$$

Note that the above follows for $2C_{W,i}^{l,d}\left(s_{W,i}^{u,d}\right)-C_{W,1}^{l,d}\left(s_{W,1}^{u,d}\right)-C_{W,2}^{l,d}\left(s_{W,2}^{u,d}\right)$ as well if we substitute $z_{W,i}^{u,d}$ for $z_{W,i}^{l,d}$. Thus the total system inventory costs are non-increasing in demand rate asymmetry.