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PROJECT ADMINISTRATION DATA SHEET

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roject No. G-37-637 DATE 7/6/82

roject Director: Jeffrey Geronimo & Andrew N. Harrington School/Lab Math

ponsor: National Science Foundation; Washington, D.C.

ype Agreement: Grant No. MCS-8203325

ward Period: From 7/1/82 To ~~12/31/83~~ * 12/31/84 (Performance) _____ (Reports)

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ost Sharing: $5,720 (G-37-340) GTRI/GIT

itle: Orthogonal Polynomials, Julia Sets, and Invariant Measures
(Mathematical Sciences)

ADMINISTRATIVE DATA

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Defense Priority Rating: n/a

Security Classification: n/a

RESTRICTIONS

See Attached NSF Supplemental Information Sheet for Additional Requirements.

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Equipment: Title vests with GIT

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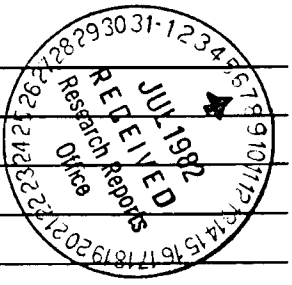
* Includes usual six month unfunded flexibility period.



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SPONSORED PROJECT TERMINATION/CLOSEOUT SHEET

Date 8/23/85

oject No. G-37-637 School/~~Lab~~ Math

cludes Subproject No.(s)

oject Director(s) J. Geronimo and A. Harrington GTRC /~~GIT~~

onsor National Science Foundation

itle Orthogonal Polynomials, Julia Sets, and Invarian~~t~~ & Measures
(Mathematical Sciences)

ffective Completion Date: 12/31/84 (Performance) 3/31/85 (Reports)

rant/Contract Closeout Actions Remaining:

- [ ] None
- [ ] Final Invoice or Final Fiscal Report
- [ ] Closing Documents
- [X] ~~Final Report of Inventions~~ Patent Questionnaire
- [ ] Govt. Property Inventory & Related Certificate
- [ ] Classified Material Certificate
- [ ] Other

ontinues Project No. Continued by Project No.

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Research Communications (2)
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Other A. Jones

Orthogonal Polynomials, Julia Sets, and Invariant Measures.
A progress report.

The principal investigators in collaboration with Professor F. Barnsley have been investigating the nature of Julia sets as ll as the invariant measure and orthogonal polynomials associated th them.The research was initially focused on the quadratic mapping

$= (z-\lambda)^2$, Barnsley et. al. [1]. In this case we were able to explicitly nstruct infinite subsequences of polynomials orthogonal to the natural ıvariant measure when the measure is supported on the real line and exhibit the relation between the orthogonal polynomials and the erates of T. This then allowed us to obtain a number of interesting rmulas on the Padé approximates to the moment generating function, owing that the $[2^n-1/2^n]$ Padé's were none other than the derivative of ıe denominator polynomial divided by 2^n. Some further results contained Barnsley et. al. [1] are relations among the coefficients in the three rm recurrence formula satisfied by the orthogonal polynomials.

Since Barnsley et. al. [1] a number of discoveries have been made. ing the balanced property of the invariant measure we have been able to tend the relations mentioned between the orthogonal polynomials and ıe iterates of the mapping T, to any polynomial mapping of the form $= z^N + k_1 z^{N+1} + \cdots + k_N$, Barnsley et. al. [2]. These relations hold whether ıe support of the measure, i.e. the Julia set, is real or complex. ırthermore using these relations we have been able to show that

$$\lim_{n\to\infty} (P_{N^n}(z))^{1/N^n} = F(z)$$

ere $P_{N^n}(z)$ is the monic orthogonal polynomial of degree N^n and $F(z)$ is e function mapping conformally the exterior of the Julia set to the terior of the unit circle. One of the most interesting points in Barnsley . al. [2] is that one can relate integrals of polynomials to the symmetric nctions associated with Tz and therefore can compute explicit numerical sults. In another paper Barnsley et. al. [3] we were able to compute the neral relations among the coefficients in the three term recurrence rmula satisfied by the orthogonal polynomials when the Julia set is a bset of the real line.

Returning to the quadratic map we have, Barnsley et. al. [4], ploited the nature of the $[2^n-1/2^n]$ Padé approximants to determine

licitly the location of all the zeros and poles of these Padés. Further-
e we have shown that for certain values of $\lambda < 2$ the Julia set has a
e like structure in the complex plane and the points of the tree are
ated at the zeros and poles of the $[2^n-1/2^n]$ Padé approximates. Some
lications to approximation theory are suggested in [5]. Finally
[4] we were able to show that the polynomials $P_{2^n}(Z)$ are also the
bychev polynomials on the Julia set.

A unifying theme in our study of the Julia set B_λ for $T(z) = (z-\lambda)^2$
the relationship with λ-chains, $\lambda \pm \sqrt{(\lambda \pm \sqrt{(\lambda \pm \cdots)})}$, which we have
mined for $-1/4 \leq \lambda < \infty$ with the aid of the Böttcher equation and Cremer's
orem [6].

In this content we have studied the geometry and topology of B_λ, the
st cascade and the Feigenbaum functional equation, and the order of
earance of individual cycles on the real line as λ varies from $-1/4$ to
We show the choice of the branch cut to be important in simplifying these
cussions.

Recent Developments and Continuing Research (4)

We have made substantial new progress in investigating the properties the orthogonal polynomials associated with Julia sets. In Barnsley et. al.] we have been able to generalize the Chebychev property to any even or d polynomial mapping. Furthermore we have shown that the Julia set for y polynomial mapping will have a tree like structure for certain values its parameters. We have been able to generalize the Pade formulas arbitrary polynomial transformations which gives us a nice formula

r the $[N^n-1/N^n]$ Padé approximants. However in the general case the analysis the location of the zeros and poles is not as simple as in the quadratic ıse. As far as the asymptotics of the polynomials is concerned, we have ıown that the $P_{N^n}(z)$ are none other than the Faber polynomials associated

ıth the function mapping conformally the exterior of the Julia set to the ıterior of the unit circle. This result is interesting because it shows ıat the asymptotics of these classes of polynomials are very different rom those given by Szĕgo's formula [9]. Part of our continuing research ffort will be to investigate the asymptotic properties of these polynomials.

In [10] Bessis et. al. were able to show that for the quadratic apping referred to above the coefficients in the recurrence formula satisfy he following remarkable relation

$$\lim_{k\to\infty} a((N)^k+s) \to a(s) \qquad \lambda \geq 2$$

Using this result Bessis et. al. have been able to show [11] that for $\lambda \geq 3$ .he $(a(n))$ are quasi-periodic. This is very interesting since they have shown .11] that the infinite tridiagonal matrix formed from the $(a(n))$ can be related :o an important problem in solid state physics having to do with the Schrödinger equation with a quasi-periodic potential. We have recently been able to show that the above relations hold even in the complex plane when $|\lambda| \geq 4$. Since it is clear from the recurrence relations satisfied by the coefficients that they are rational functions of λ, the above result says that all their zeros and poles are localized in the circle of radius 4. Therefore the three term recurrence formula can be analytically continued off the real line without any problems for $|\lambda| \geq 4$. This then allows one to compute upper bounds on the polynomials satisfying the recurrence formula (these may not be the same as the orthogonal polynomials associated with the invariant measure).

In [3] we found that the coefficients in the recurrence formula

ciated with the cubic map $Tz = z^3 - \lambda z$ also satisfy similar relations hose given above for the quadratic case. Furthermore using perturbation ory arguments one can show that these relations will hold for aysmmetric ic polynomials. One of the questions we are actively investigating is, the quasi-periodic relations found for the quadratic and cubic case e for other polynomial maps that are not trivially derived from quadratic cubic maps listed above and can one determine the analytic properties the coefficients in the recurrence formula associated with other polynomial pings.

Using the theory of continued fraction we are in the process of pleting some work [8] investigating how the periodic continued fraction order 2^n approaches the full continued fraction as $n \to \infty$. This work is ecially interesting in the light of the recent quasi-periodicity results their relation to physical problems.

Addendum

Just this week Harrington and Barnsley cracked the problem of cursively calculating moments of the balanced measure for a rational action. This is a large step. We expect to be able to generalize many our results for Julia sets of polynomial transformations to all Julia ts.

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FINAL PROJECT REPORT
NSF FORM 98A

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PART I–PROJECT IDENTIFICATION INFORMATION

titution and Address	2. NSF Program	3. NSF Award Number
orgia Institute of Tech. hool of Mathematics lanta, GA 30332	Classical analysis	MCS-8203325
	4. Award Period From 6/82 To 12/83	5. Cumulative Award Amount $41,650

oject Title
thogonal Polynomials, Julia Sets and Invariant Measures

PART II–SUMMARY OF COMPLETED PROJECT *(FOR PUBLIC USE)*

The primary purpose of this project was to investigate the properties invariant measures and the orthogonal polynomials associated with them . Julia sets J. Beginning with the Julia sets associated with poly- omial mappings, $Tz = z^N + k_{N-1}z^{N-1} \ldots + k_0$, $N \geq 2$, we studied the pro- erties of the balanced measures satisfying the relation

$$\int_J f(x)\,d\mu(x) = \sum_{i=1}^{N} \frac{1}{N} \int_J f(T_i^{-1}(x))\,d\mu(x), \qquad f \in L^1(J,\mu),$$

here T_i^{-1}, $i = 1,2,\ldots,N$ are a complete set of inverse branches of T. In articular it was shown that these measures have the "quick mixing" pro- erty

$$\int_J f(T^n x)Q(x)\,d\mu = \int_J f\,d\mu \int_J Q\,d\mu,$$

here $f \in L^1(J,\mu)$ and Q is a polynomial of degree less than N^n. The onic polynomials of the second kind were investigated and shown to obey he relation

$$p^{(1)}_{mN^n-1}(z) = \frac{T'(z)}{N}\, p^{(1)}_{mN^{n-1}-1}(Tz).$$

Whereas μ is a singular continuous measure supported on the Julia set J (now assumed to be a real Cantor set) we have shown that $\mu^{(1)}$ (the mea- sure with respect to which the $p_n^{(1)}$'s are orthogonal) is purely discrete and supported nowhere on J. This was a rather unexpected result. Finally, a new class of measures were discovered which we called condensed

(continued on attached sheet)

PART III–TECHNICAL INFORMATION *(FOR PROGRAM MANAGEMENT USES)*

ITEM *(Check appropriate blocks)*	NONE	ATTACHED	PREVIOUSLY FURNISHED	TO BE FURNISHED SEPARATELY TO PROGRAM	
				Check (✓)	Approx. Date
. Abstracts of Theses	✓				
. Publication Citations		✓			
. Data on Scientific Collaborators		✓			
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. Technical Description of Project and Results		✓			
. Other *(specify)*					

2. Principal Investigator/Project Director Name *(Typed)*	3. Principal Investigator/Project Director Signature	4. Date
Jeffrey Geronimo		8/8/85

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easures. These measures may be associated with the mappings
$\rightarrow R(z) + \varepsilon \sum_{i=1}^{k} (z-a_i)^{-r_i}$ in the limit as $\varepsilon \rightarrow 0$. Here R(z) is a rational
ransformation of degree greater than one. The functional equations
onnected to the condensed measures typifies those encountered in
sing hierarchical models found in statistical physics.

Data on Scientific Collaborators

1) M. F. Barnsley is a Professor of Mathematics at Georgia Tech.

2) A. N. Harrington is an Assistant Professor of Mathematics at Loyola University.

The primary purpose of this project was to investigate the properties of invariant measures and the orthogonal polynomials associated with them on Julia sets J. Beginning with the Julia sets associated with polynomial mappings, $Tz = z^N + k_{N-1}z^{N-1} \ldots + k_0$, $N \geq 2$, we studied the properties of the balanced measures satisfying the relation

$$\int_J f(x)d\mu(x) = \sum_{i=1}^{N} \frac{1}{N} \int_J f(T_i^{-1}(x))d\mu(x) \qquad f \in L'(J,\mu)$$

where T_i^{-1}, $i = 1,2,\ldots,N$ are a complete set of inverse branches of T. In particular it was shown that these measures have the "quick mixing" property [1]

$$\int_J f(T^n x)Q(x)d\mu = \int_J f\, d\mu \int_J Q\, d\mu,$$

where $f \in L^1(J,\mu)$ and Q is a polynomial of degree less than N^n. The monic polynomials of the second kind were investigated and shown to obey the relation

$$p^{(1)}_{mN^n-1}(z) = \frac{T'(z)}{N}\, p^{(1)}_{mN^{n-1}-1}(Tz).$$

Whereas μ is a singular continuous measure supported on the Julia set J (now assumed to be a real Cantor set) we have shown [2] that $\mu^{(1)}$ (the measure with respect to which the $p_n^{(1)}$'s are orthogonal) is purely discrete and supported nowhere on J. This was a rather unexpected result. Finally, a new class of measures were discovered [3] which we called condensed measures. These measures may be associated with the

mappings $z \to R(z) + \varepsilon \sum_{i=1}^{k} (z-a_i)^{-r_i}$ in the limit as $\varepsilon \to 0$. Here $R(z)$ is a rational transformation of degree greater than one. The functional equations connected to the condensed measures typifies those encountered in Ising hierarchical models found in statistical physics.

1) M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, "Geometry, electrostatic measure and orthogonal polynomials on Julia sets for polynomials," J. Ergod. Th. and Dynam. Sys. (1983), 3, 509.

2) M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, "Almost periodic Jacobi matrices associated with Julia sets for polynomials," Commun. Math. Phys. (1985) 99, 303.

3) M. F. Barnsley, J. S. Geronimo, and A. N. Harrington, "Condensed Julia sets, with an application to a fractal lattice model Hamiltonian," Trans. Amer. Math. Soc. (1985), 288, 537.