Convergence in Almost Periodic Fisher and Kolmogorov Models

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Abstract

We study convergence of positive solutions for almost periodic reaction diffusion equations of Fisher or Kolmogorov type. It is proved that under suitable conditions every positive solution is asymptotically almost periodic. Moreover, all positive almost periodic solutions are harmonic and uniformly stable, and if one of them is spatially homogeneous, then so are others. The existence of an almost periodic global attractor is also discussed.

1 Introduction

The current paper is devoted to the study of asymptotic behavior of positive solutions for the following reaction diffusion equation of Fisher or Kolmogorov type,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(t,x,u), \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is smooth, bounded and connected, $f \in C^2$ is almost periodic in t uniformly for x, u in bounded subsets ([9]). More precisely, we assume that f is either of Fisher type

HF) $f(t,x,u) = m(t,x,u)u(1-u)$,

where m is nonincreasing in u for $u \in (0,1)$,

or of Komogorov type

HK) $f(t,x,u) = m(t,x,u)u$,

where $m(t,x,u) < 0$ for $u \gg 1$ and m is nonincreasing in u for $u > 0$.

When f is of Fisher type, (1.1) represents a population model for the spatial spread of an allele (gene) in a migrating diploid species with two type alleles (genes), in which u is the fraction of one of the two alleles at a particular gene locus, m is the fitnesses ([1], [7], [8], [10], [11], [14]). When f is of Kolmogorov type, (1.1) represents a population growth model of a single species, in which u is the density of the population, m is the growth rate ([3]-[5], [33], [35]). The Neumann boundary condition gives a restriction that no migration should across the spatial boundary in both cases. By considering equations of such general forms, we allow the habitat in which the population lives to be a variable. In other words, we have assumed that the fitness or growth rate m may vary in both space and time, which includes various possibilities: 1) certain place inside the habitat may be more favorable to the population than other places; 2) the habitat may be subjected to a seasonal variation which needs not be exactly periodic but rather almost periodic; 3) the population may be of some inherent periodic variation which may be different from the seasonal variation. The almost periodicity in our equations can be also viewed as a deterministic version of a random or stochastic variation.

Our main interest is the long time behavior of the population dynamics depending upon the variable habitat. Throughout the paper, by positive (non-negative) solution, we mean a solution of (1.1) valued in $(0,1)$ ($[0,1]$) in the case of HF) and a solution valued in $(0,\infty)$ ($[0,\infty)$) in the case of HK). A trivial state of (1.1) is referred to as the constant solution $u = 0$ or 1 in the case of HF) and $u = 0$ in the case of HK).

Our main results state as follows.

Main Results. *Consider (1.1).*

1) *(Asymptotic almost periodicity) Either all positive solutions are asymptotically almost periodic or none of them is. Moreover, the following holds:*

   i) *If a trivial state is asymptotically stable, then all positive solutions converge to the same trivial state.*

   ii) *If all trivial states are unstable, then every positive solution is away from all trivial states and also converges to a positive almost periodic solution.*

   iii) *If one trivial state is uniformly stable, then every positive solution is asymptotically almost periodic.*

2) *(Harmonics) Any positive almost periodic solution $u^*(t,x)$ is harmonic, that is,*

$$\mathcal{M}(u^*) \subset \mathcal{M}(f),$$

   where $\mathcal{M}(\cdot)$ denotes the frequency module of an almost periodic function.

3) *(Stability) All positive almost periodic solutions (if exist) are uniformly stable.*

4) *(Global attractor)*

   i) *There is at most one asymptotically stable, non-negative almost periodic solution, and if there is one, then it is a global attractor of positive solutions.*

   ii) *If m is strictly decreasing for $0 < u < 1$ in the case of HF) and for $u > 0$ in the case of HK), and if all trivial states are unstable, then there is a unique positive almost periodic solution which is also a global attractor of positive solutions.*

5) *(Non-uniqueness) If there are two positive almost periodic solutions, then there is a continuous family of them connecting these two.*

6) *(Spatial homogeneity) If one positive almost periodic solution is spatially homogeneous, then so are others.*

Although under some very general conditions (see the above theorem and also section 3), every positive solution of (1.1) converges to an almost periodic one, it is still possible that none of the positive solutions of (1.1) is asymptotically almost periodic (see Example 3.1 in section 3). Such a non-convergence phenomena does not occur in the periodic Fisher and Kolmogorov models since if m is periodic in t, then 1) above together with the generic convergence property of periodic parabolic equations (see [16] and references therein) implies that every positive solution of (1.1) converges to a periodic solution.

We remark that our main results also hold true when f is of more general form, namely, either

$HF)'$ $f(t, x, u) = m(t, x, u)h(u)$,

where m is the same as in HF),

$h(0) = h(1) = 0$, $h'(0) > 0$, $h'(1) < 0$, and h is concave on $[0, 1]$;

or

$HK)'$ $f(t, x, u) = m(t, x, u)h(u)$

where m is the same as in HK),

$h(0) = 0$, $h'(0) > 0$, $h'(u) \geq 0$ and $h'(u)$ is nonincreasing for $u \geq 0$.

There has been many studies on different type of Fisher and Kolmogorov equations(see [1], [3]-[5], [7], [8], [10], [11], [14], [30]-[32], [33], [35] and references therein). Related to the current work, the existence, stability, and bifurcation phenomena of equilibrium solutions in an autonomous Fisher equation was studied by Fleming ([11]). Convergence to periodic solutions in a time periodic Fisher type of equation was proved by Hess and Weinberger ([14]). In [30]-[32], Vuillermot showed the existence of an almost periodic attractor in certain spatially homogeneous, almost periodic Fisher equations. The existence, uniqueness, and stability of positive equilibria in autonomous Kolmogorov equation were considered by Cantrall and Cosner ([3]-[5]). Permanence in periodic Kolmogorov type of equations was discussed by Zhao and Huston ([35]). Wu, Zhao and He ([33]) investigated the global asymptotic behavior of solutions for certain almost periodic Kolmogorov type of equations. Our results in the current paper generalize many of those existing works mentioned above.

The existence of positive almost periodic solutions in (1.1) is of particular interest in applications since it implies the coexistence of both alleles in the case of HF) and the persistence of the population in the case of HK) (see [3]-[5], [8], [10], [14], [33], [35]). In an ecological context, the existence of spatially homogeneous and non-homogeneous almost periodic solutions are referred to as temporal and spatial-temporal clines respectively. Result 6) above implies that if the fitnesses are spatially homogeneous, then only temporal clines exist, but if the fitnesses are not spatially homogeneous and if the two trivial states $u = 0, 1$ are unstable and there are no other constant state, then spatial-temporal clines exist in the Fisher equation. Depending on the positivity or triviality of the almost periodic solution, our convergence results for the Fisher model indicate that both alleles coexist or eventually only one allele exists, and in the Kolmogorov model indicate that the population persists or eventually extincts. Roughly speaking, if a positive solution is neither away from a trivial state nor convergent to it, then in the Fisher model, the existence of only one allele and the coexistence of both alleles are all frequently expected: whenever an allele closes to extinction, it grows and then both alleles coexist; but after a while the allele decays and closes to extinction again. Similarly, in the Kolmogorov model, if a positive solution is neither away from the trivial state nor convergent to it, then the population often closes to extinction but does not extinct: whenever it closes to extinction, it grows and persists; but after a while it decays and closes to extinction again. We note that this kind of phenomena do not occur when the habitat in which the population lives varies periodically in time. However, Example 3.1 in section 3 shows that it does happen when the habitat changes almost periodically in time.

The current work is an application of the theory of almost periodic differential equations recently developed by both authors (see [21]-[26], [34]). In separate papers, we shall consider other applications of our theory in particular to various population problems.

The paper is organized as follows. In section 2, we summarize from [21]-[26], [34] some of the dynamical properties of almost periodic, monotone skew-product semiflows and give an outline of the construction of a skew-product semiflow associated to a given almost periodic parabolic equation. Section 3 is devoted to the study of asymptotic almost periodicity for positive solutions of (1.1). Stability and harmonics of limiting almost periodic solutions are also investigated. We study the uniqueness of almost periodic solutions and the existence of an almost periodic global attractor in section 4. Spatial homogeneity of positive almost periodic solutions are discussed in section 5.

2 Preliminary

2.1. Strongly monotone skew product semiflow

Let Y be a compact metric space and X be a strongly ordered Banach space, that is, there is a closed convex cone $X_+ \subset X$ satisfying that $X_+ \cap (-X_+) = \{0\}$ and $\mathrm{Int}(X_+) \neq \emptyset$. Note that X_+ induces a strong ordering on X as follows:

$$\begin{aligned} x &\geq y \quad \text{iff} \quad x - y \in X_+ \\ x &> y \quad \text{iff} \quad x \geq y \quad \text{and} \quad x \neq y \\ x &\gg y \quad \text{iff} \quad x - y \in \mathrm{Int}\, X_+. \end{aligned}$$

Definition 2.1.

1) *A semiflow* $\Pi_t : X \times Y \to X \times Y$ $(t \geq 0)$ *is a skew-product semiflow if it has the form*

$$\Pi_t(x, y) = (u(t, x, y), y \cdot t) \tag{2.1}$$

for a continuous function $u : \mathbb{R}_+ \times X \times Y \to X$*, where* $(Y, \mathbb{R}) : (y,t) \mapsto y \cdot t$ *is a flow on* Y*.*

2) Π_t *is* <u>$C^{1+\alpha}$</u> $(\alpha \geq 0)$ *if* $u(t,x,y)$ *is* C^1 *in* x*,* u_x *is continuous in* $y \in Y$*,* $t > 0$*, and* C^α *in* x*, moreover,*

$$u_x(t,x,y)v \to v \quad \text{as} \quad t \to 0^+ \tag{2.2}$$

uniformly for (x,y) *in compact subsets of* $X \times Y$ *and* v *in bounded subsets of* X*.*

3) Π_t *is* <u>*almost periodic*</u> *if* $(Y, \mathbb{R})$ *is* <u>*almost periodic minimal*</u> *(that is,* $(Y, \mathbb{R})$ *is minimal and equicontinuous([6])).*

4) Π_t *is* <u>*strongly monotone*</u> *if it is* C^1 *and for any* $(x,y) \in X \times Y$*, any* $v \in X$ *with* $v > 0$*, and any* $t > 0$*,*

$$\Phi(t,x,y)v \gg 0, \tag{2.3}$$

where $\Phi(t,x,y) = u_x(t,x,y)$*.*

In what follows, we assume that Π_t is $C^{1+\alpha}$ for some $\alpha > 0$, and is almost periodic and strongly monotone. Denote $\Phi(t,x,y) = u_x(t,x,y)$ and $p : X \times Y \to Y : (x,y) \mapsto y$ as the natural projection. We let $K \subset X \times Y$ be a compact minimal set of Π_t unless specified otherwise.

Lemma 2.1 (Non-ordering principle) ([25]). *Assume that* $\Pi_t : K \to K$ *has a* <u>*flow extension*</u> *(that is, there is a flow* $\tilde{\Pi}_t : K \to K$ $(t \in \mathbb{R})$ *such that* $\tilde{\Pi}_t = \Pi_t$ *for* $t \geq 0$*). Then there is a residual subset* $Y_0 \subset Y$ *such that* $p^{-1}(y_0) \cap K$ *admits no ordered pairs for any* $y_0 \in Y_0$*, namely, for any* $(x_1, y_0), (x_2, y_0) \in p^{-1}(y_0) \cap K$*,* x_1 *and* x_2 *are not ordered.*

Remark 2.1. If $(x_1, y), (x_2, y) \in K$ is an <u>ordered pair</u>, that is, $x_1 \geq x_2$ or $x_1 \leq x_2$, then it is both a <u>positively</u> and a <u>negatively proximal pair</u>, in other words,

$$\inf_{t \in \mathbb{R}^\pm} d(\Pi_t(x_1,y), \Pi_t(x_2,y)) = 0$$

(see [25]).

Lemma 2.2 (Continuous separation) ([25]). *Assume that* $\Pi_t : K \to K$ *has a flow extension and there is* $t_0 > 0$ *such that* $\Phi(t,x,y)$ *is compact for any* $(x,y) \in K$ *and* $t \geq t_0$*. Then* Π_t *admits a* <u>*continuous separation*</u> *on* K*, that is, there are subspaces* $\{X_1(x,y)\}_{(x,y)\in K}$*,* $\{X_2(x,y)\}_{(x,y)\in K} \subset X$ *with the following properties:*

1) $X = X_1(x,y) \oplus X_2(x,y)$*,* $((x,y) \in K)$ *and* $X_1(x,y), X_2(x,y)$ *vary continuously in* $(x,y) \in K$*.*

2) $X_1(x,y) = span\{v(x,y)\}$*, where* $v(x,y) \in IntX_+$ *and* $\|v(x,y)\| = 1$*,* $((x,y) \in K)$*.*

3) $X_2(x,y) \cap X_+ = \{0\}$, $((x,y) \in K)$.

4) For any $t > 0$, $(x,y) \in K$,

$$\Phi(x,y,t)X_1(x,y) = X_1(\Pi(x,y,t)),$$

$$\Phi(x,y,t)X_2(x,y) \subset X_2(\Pi(x,y,t)).$$

5) There are $M > 0$, $\delta > 0$ *such that for any* $(x,y) \in K$, $w \in X_2(x,y)$ *with* $\|w\| = 1$,

$$\|\Phi(x,y,t)w\| \leq Me^{-\delta t}\|\Phi(x,y,t)v(x,y)\| \quad (t > 0).$$

Definition 2.2. *Let* $(x_0, y_0) \in X \times Y$ *be given.*

1) The set $\omega(x_0,y_0) = \{(x,y)|$ *there is* $t_n \to \infty$ *such that* $(x,y) = \lim_{n\to\infty} \Pi_{t_n}(x_0,y_0)\}$ *is called the* *ω-limit set* *of* $\Pi_t(x_0,y_0)$ *or* (x_0,y_0).

2) $\Pi_t(x_0,y_0)$ *is said to be* *stable* *(asymptotically stable) if for any* $\epsilon > 0$, *there is* $\delta > 0$ *such that for any* $(x,y_0) \in X \times Y$ *with* $d((x,y_0),(x_0,y_0)) < \delta$,

$$d(\Pi_t(x,y_0),\Pi_t(x_0,y_0)) < \epsilon \quad \text{for} \quad t \geq 0$$

$$(d(\Pi_t(x,y_0),\Pi_t(x_0,y_0)) \to 0 \quad \text{as} \quad t \to \infty).$$

3) $\Pi_t(x_0,y_0)$ *is said to be* *uniformly stable* *if for any* $\epsilon > 0$, *there is* $\delta > 0$ *such that for any* $(x,y_0) \in X \times Y$ *and* $\tau \geq 0$ *with* $d(\Pi_\tau(x,y_0),\Pi_\tau(x_0,y_0)) < \delta$,

$$d(\Pi_t(x,y_0),\Pi_t(x_0,y_0)) < \epsilon \quad \text{for} \quad t \geq \tau.$$

Definition 2.3.

1) K *is called* *uniformly stable* *if for any* $\epsilon > 0$, *there is* $\delta > 0$ *such that for any* $(x_0,y_0) \in K$, $(x,y_0) \in X \times Y$, *and* $\tau \geq 0$ *with* $d(\Pi_\tau(x,y_0),\Pi_\tau(x_0,y_0)) < \delta$,

$$d(\Pi_t(x,y_0),\Pi_t(x_0,y_0)) < \epsilon \quad \text{for} \quad t \geq \tau.$$

2) The number

$$\lambda_K = \sup_{(x,y)\in K}\Big(\limsup_{t\to\infty} \frac{\ln\|\Phi(t,x,y)\|}{t}\Big) \tag{2.4}$$

is referred to as the *upper-Lyapunov exponent* *of* K.

3) K *is* *linearly stable* *if* $\lambda_K \leq 0$.

4) K with a flow extension is said to be <u>*distal*</u> *if for any $(x_1, y), (x_2, y) \in K$,*

$$\inf_{t \in \mathbb{R}} d(\Pi_t(x_1, y), \Pi_t(x_2, y)) > 0.$$

Remark 2.2.

1) Let $K \subset X \times Y$ and $v(x, y)$ be as in Lemma 2.2. Then

$$\lambda_K = \sup_{(x,y) \in K} \Big(\limsup_{t \to \infty} \frac{\ln \|\Phi(t, x, y) v(x, y)\|}{t} \Big).$$

2) If $\lambda_K < 0$, then K is uniformly and asymptotically stable. More precisely, there are $\delta > 0$, $\mu > 0$, and $M > 0$ such that for any $(x_0, y_0) \in K$, $(x, y_0) \in X \times Y$ with $d((x, y_0), (x_0, y_0)) < \delta$,

$$d(\Pi_t(x, y_0), \Pi_t(x_0, y_0)) \leq M d((x, y_0), (x_0, y_0)) e^{-\mu t} \quad \text{for} \quad t \geq 0.$$

Lemma 2.3 (Structure of ω-limit sets) ([17], [25]). *Assume that $\Pi_t(x, y)$ is uniformly stable and there is $t_0 > 0$ such that $\{\Pi_t(x, y) | t \geq t_0\}$ is relatively compact. Then $\omega(x, y)$ is nonempty, minimal, uniformly stable, and distal.*

Lemma 2.4 (Structure of minimal sets) ([25]).

1) If K is linearly stable, then there is an integer $N > 0$ such that K is an <u>*almost $N - 1$ extension*</u> *of Y (that is, there is a residual subset $Y_0 \subset Y$ such that card$(p^{-1}(y_0) \cap K) = N$ for any $y_0 \in Y_0$).*

2) If K is also uniformly stable, then it is an <u>*$N - 1$ extension*</u> *of Y (that is, card$(p^{-1}(y) \cap K) = N$ for any $y \in Y$).*

Remark 2.3.

1) If K is as in Lemma 2.4 1), then for any $(x_0, y_0) \in K$ with $y_0 \in Y_0$, $\Pi_t(x_0, y_0)$ is <u>almost automorphic</u> in t (see definition below) and

$$N \mathcal{M}(\Pi_t(x_0, y_0)) \subset \mathcal{M}(y_0 \cdot t)$$

(see [25], [34]).

2) If K is an $N - 1$ extension of Y (for example K is as in Lemma 2.4 2)), then for any $(x, y) \in K$, $\Pi_t(x, y)$ is almost periodic and

$$N \mathcal{M}(\Pi_t(x, y)) \subset \mathcal{M}(y \cdot t)$$

(see [25], [34]).

3) The number N in both Lemma 2.4 1) and 2) indicates a kind of subharmonic phenomena. When $N = 1$, the almost automorphic and almost periodic solutions in 1) and 2) above are said to be harmonic.

Recall that a function $g : \mathbb{R} \to Z$ (Z is a metric space) is almost periodic if for any sequences $\{a_n^{'}\}$, $\{\beta_n^{'}\} \subset \mathbb{R}$, there are subsequences $\{\alpha_n\} \subset \{\alpha_n^{'}\}$, $\{\beta_n\} \subset \{\beta_n^{'}\}$ such that $\lim_{m\to\infty} \lim_{n\to\infty} h(t + \alpha_n + \beta_m) = \lim_{n\to\infty} h(t + \alpha_n + \beta_n)$ ([9]). A function g is almost automorphic if for any sequence $\{\alpha_n^{'}\} \subset \mathbb{R}$, there is a subsequence $\{\alpha_n\} \subset \{\alpha_n^{'}\}$ such that $\lim_{m\to\infty} \lim_{n\to\infty} h(t + \alpha_n - \alpha_m) = h(t)$ ([9]). Clearly, an almost periodic function is almost automorphic, but the converse is not true in general ([15], [21]-[26], [29]). Almost automorphic functions were first introduced by S. Bochner in 1955 in a work of differential geometry ([2]) and were mainly studied by W. A. Veech during the 1960's ([27]-[29]). An almost automorphic function resembles an almost periodic one in many aspects but may behave more irregularly and randomly (see [27], [28], [29], [34] and references therein).

2.2. Almost periodic parabolic equations

Consider the following parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(t, x, u, \bigtriangledown u), \quad x \in \Omega, \quad t > 0, \\ u = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{2.5}$$

where $u \in \mathbb{R}$, f is C^2 and almost periodic in t uniformly in other variables ([9]), $\Omega \subset \mathbb{R}^n$ is smooth, bounded, and connected.

Let

$$Y = H(f) = cl\{f_\tau(\cdot, \cdot, \cdot, \cdot) | \tau \in \mathbb{R}\} \tag{2.6}$$

with compact open topology ([19]), where

$$f_\tau(t, x, u, p) \equiv f(t + \tau, x, u, p).$$

Then

$$(Y, \mathbb{R}) : (g, t) \mapsto g \cdot t = g_t$$

is an almost periodic minimal flow ([19], [26]). Let $X \subset L^p(\Omega)$ ($p > n$) be a fractional power space of $-\Delta : \mathcal{D} \to L^p(\Omega)$ satisfying $X \hookrightarrow C^1(\bar{\Omega})$, where $\mathcal{D} = \{u \in H^{2,p}(\Omega) | u = 0$ or $\dfrac{\partial u}{\partial n} = 0$ for $x \in \partial\Omega\}$. Define $\pi_t : X \times Y \to X \times Y$ by

$$\pi_t(U, g) = (u(t, x, U, g), g \cdot t) \tag{2.7}$$

where $u(t, x, U, g)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + g(t, x, u, \bigtriangledown u), \quad x \in \Omega, \quad t > 0, \\ u = 0 \quad \text{or} \quad \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0 \end{cases} \tag{2.7$_g$}$$

with $u(0, x, U, g) = U(x)$. Then π_t is a (local) skew-product semiflow ([26]). We note that the dynamics of (2.5) is completely reflected by that of (2.6) (see [19]-[26]).

Lemma 2.5 ([26]). *π_t is almost periodic and strongly monotone with respect to the ordering on X induced by the convex cone $X_+ = \{U \in X | U(x) \geq 0\}$.*

Remark 2.4.

1) By regularity theory and *a priori* estimates for parabolic equations, for any positively bounded motion $\pi_t(U, g)$ and $\delta > 0$, the set $\{\pi_t(U, g) | t \geq \delta\}$ is relatively compact. Therefore, $\omega(U, g) \neq \emptyset$ and is compact.

2) By backward extension properties of parabolic equations ([12],[13]), for any compact invariant set K, $\pi_t : K \to K$ admits a flow extension and it satisfies the following: for any $\delta > 0$, $\Phi(t, U, g)$ is compact for $(x, y) \in K$ and $t \geq \delta$, where $\Phi(t, U, g) = u_U(t, \cdot, U, g)$. Thus, the compactness and flow extension conditions in Lemmas 2.1-2.4 are automatically satisfied for the skew-product semiflow π_t generated by the parabolic equation (2.5).

3) If K is a minimal set of (2.6), then

$$\lambda_K = \sup_{(U,g)\in K} \Big(\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mu(s, U, g) ds \Big),$$

where

$$\begin{aligned} \mu(s, U, g) \quad = \quad & \frac{1}{\displaystyle\int_\Omega [v(\pi_s(U, g))(x)]^2 dx} \int_\Omega \{-[\bigtriangledown v(\pi_s(U, g))(x)]^2 \\ & + \langle a(s, x) v(\pi_s(U, g))(x), \bigtriangledown v(\pi_s(U, g))(x) \rangle \\ & + b(s, x)[v(\pi_s(U, g))(x)]^2\} dx, \end{aligned}$$

and

$$a(s, x) = g_p(s, x, u(s, x, U, g), \bigtriangledown u(s, x, U, g)),$$

$$b(s, x) = g_u(s, x, u(s, x, U, g), \bigtriangledown u(s, x, U, g)),$$

$v(\pi_s(U,g))$ is as in Lemma 2.2. In particular, when f is spatially homogeneous (that is, $f(t,x,u,\bigtriangledown u) = f(t,u,\bigtriangledown u)$) and (2.5) assumes the Neumann boundary condition, if K is a single constant solution c, then

$$\lambda_{\{c\}} = \limsup_{t\to\infty} \frac{1}{t}\int_0^t f_u(s,a,0)ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t f_u(s,a,0)ds.$$

Moreover, in this case, $\{\lambda_{\{c\}}\}$ is a degenerate Sacker-Sell spectrum interval of c ([18], [25], [26]). If $\lambda_{\{c\}} > 0$, then $u = c$ is unstable and for any $u_0 \in X$ with $u_0 > c$, and

$$\inf_{t\in\mathbb{R}^+} \|u(t,\cdot,u_0,f) - c\| > 0.$$

If f is periodic or autonomous, then $\lambda_{\{c\}}$ is actually the first eigenvalue of the linearized equation of (2.5) around $u = c$.

We now let $u(t,x,U,g)$ be a solution of (1.1) with f replaced by $g \in H(f)$ and $u(0,x,U,g) = U(x)$, where $H(f)$ is defined as in (2.6). Denote $u = c$ as a trivial state of (1.1), namely $c = 0$ or 1 when f is of Fisher type and $c = 0$ when f is of Kolmogorov type. Clearly, when f is of Fisher type, any solution with values in $(0,1)$ exists and stays in $(0,1)$ for all $t > 0$. When f is of Kolmogorov type, we have the following.

Proposition 2.6. *Consider (1.1) and assume that f satisfies HK). Then any positive solution $u(t,x) \in X$ exists for all $t > 0$ and is ultimately bounded, that is, there is $M_0 > 0$ such that $u(t,x) < M_0$ for $t \gg 1$.*

Proof. Consider

$$\begin{cases} v_t \geq \Delta v + m(t,x,v)v, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial v}{\partial n} \geq 0, \quad x \in \partial\Omega, \quad t \geq 0. \end{cases} \tag{2.8}$$

Obviously, any solution of (2.8) is a supper-solution of the Kolmogorov equation (1.1). Let $M_0 > 0$ be such that

$$m(t,x,u) \leq -\delta_0$$

for some $\delta_0 > 0$ and any $u \geq M_0$. Then for any $M \geq M_0$,

$$v(t,x,M) = \begin{cases} M - \delta_0 M t, \quad 0 \leq t < \dfrac{M - M_0}{\delta_0 M}, \\ M_0, \quad t \geq \dfrac{M - M_0}{\delta_0 M} \end{cases}$$

is a solution of (2.8).

Now, let $u(t,x)$ be any positive solution of the Kolmogorov equation (1.1). Then there is $M \geq M_0$ such that $u(0,x) < M$ for $x \in \Omega$. It follows from the comparison principle of parabolic equation that

$$u(t,x) \leq v(t,x,M) \quad \text{for} \quad t \geq 0.$$

This implies that $u(t,x)$ exists for all $t \geq 0$ and $u(t,x) \leq M_0$ for $t \geq \dfrac{M - M_0}{\delta_0 M_0}$. ■

3 Asymptotic almost periodicity

Definition 3.1

1) A solution $u = u(t,x)$ of (1.1) converges to a solution $u = u^*(t,x)$ if

$$\lim_{t\to\infty} \|u(t,\cdot) - u^*(t,\cdot)\| = 0.$$

$u = u(t,x)$ is said to be asymptotically almost periodic if it converges to an almost periodic solution $u^*(t,x)$.

2) A trivial state $u = c$ of (1.1) is unstable if there is a positive solution $u(t,x)$ of (1.1) such that

$$\inf_{t\in\mathbb{R}^+} \|u(t,\cdot) - c\| > 0.$$

Let $u_1(t,x)$, $u_2(t,x)$ be two positive solutions of (1.1). We define $z(t,x,u_1,u_2)$ by

$$z(t,x,u_1,u_2) = \left(\int_{u_1(t,x)}^{u_2(t,x)} \frac{1}{h(\eta)} d\eta\right)^2, \tag{3.1}$$

where

$$h(\eta) = \begin{cases} \eta(1-\eta), & f \quad \text{satisfies} \quad HF), \\ \eta, & f \quad \text{satisfies} \quad HK). \end{cases} \tag{3.2}$$

Then $z(t,x,u_1,u_2)$ is well-defined, $\dfrac{\partial z}{\partial n}(t,x,u_1,u_2) = 0$ for $x \in \partial\Omega$, and $z(t,x,u_1,u_2) \geq 0$ for $x \in \Omega$. Moreover, by direct computations, $z(t,x,u_1,u_2)$ satisfies

$$\begin{aligned} &\frac{\partial z}{\partial t} - \Delta z - h'(u_1)\left(\frac{\nabla u_2}{h(u_2)} + \frac{\nabla u_1}{h(u_1)}\right) \nabla z \\ &= 2[m(t,x,u_2) - m(t,x,u_1)] \int_{u_1}^{u_2} \frac{d\eta}{h(\eta)} \\ &+2\left|\frac{\nabla u_2}{h(u_2)}\right|^2 [h'(u_2) - h'(u_1)] \int_{u_1}^{u_2} \frac{d\eta}{h(\eta)} - 2\left|\nabla \int_{u_1}^{u_2} \frac{d\eta}{h(\eta)}\right|^2 \end{aligned} \tag{3.3}$$

(see [14]).

Lemma 3.1. *Let $u_1(t,x)$, $u_2(t,x)$ be two positive solutions of (1.1). Then the function $\tilde{z}(t) = \max_{x\in\bar{\Omega}} z(t,x,u_1,u_2)$ is non-increasing in t for $t \geq 0$.*

Proof. Since $m(t,x,u)$ and $h^{'}(u)$ ($h^{'}(u) = 1-2u$ or 1) are nonincreasing in u, the right hand side of (3.3) is non-positive. The lemma then follows from the maximum principle of parabolic equations.■

Lemma 3.2. *Assume that there is one positive solution $u^*(t,x)$ of (1.1) which is not away from a trivial state $u = c$, that is, there is $x_n \in \Omega$ and $t_n \to \infty$ such that $u^*(t_n, x_n) \to c$ as $n \to \infty$. Then any positive solution $u(t,x)$ satisfies $\lim_{n\to\infty} u(t_n, x) = c$ for $x \in \Omega$.*

Proof. Let $u(t,x)$ be any positive solution of (1.1). By Lemma 3.1,

$$\max_{x\in\Omega} z(t,x) \le \max_{x\in\Omega} z(0,x) \quad \text{for} \quad t > 0,$$

where $z(t,x) = z(t,x,u^*,u)$. This implies that

$$z(t_n, x_n) \le \max_{x\in\Omega} z(0,x) \tag{3.4}$$

for any n. Since $u^*(t_n, x_n) \to c$ as $n \to \infty$ and $h(c) = 0$, $h^{'}(c) \neq 0$, by (3.1) and (3.4), we must have $u(t_n, x_n) \to c$ as $n \to \infty$.

Now without loss of generality, we assume that $\lim_{n\to\infty} x_n = \tilde{x} \in \bar{\Omega}$, $\lim_{n\to\infty} f(t + t_n, x, u) = \tilde{f}(t,x,u)$ for any $t \in \mathbb{R}$, $x \in \bar{\Omega}$, $u \in \mathbb{R}$, and $\lim_{n\to\infty} u(t_n, x) = \tilde{u}(x)$ for $x \in \bar{\Omega}$. Then $u(t,x,\tilde{u},\tilde{f})$ exists for all $t \in \mathbb{R}$, and

$$u(t,x,\tilde{u},\tilde{f}) = \begin{cases} \ge c, & \text{if} \quad c = 0, \\ \le c, & \text{if} \quad c = 1 \end{cases}$$

for any $t \in \mathbb{R}$, $x \in \bar{\Omega}$, and $u(0,\tilde{x},\tilde{u},\tilde{f}) = c$. By the maximum principle of parabolic equations, $\tilde{u}(x) \equiv c$, that is, $\lim_{n\to\infty} u(t_n, x) = c$ for any $x \in \Omega$. ■

Lemma 3.3. *Any positive solution of (1.1) which is away from all trivial states is uniformly stable.*

Proof. Suppose that $u = u(t,x)$ is a positive solution of (1.1) which is away from all trivial states. Then there is a $\delta_0 > 0$ such that both $u(t,x) + \delta_0$ and $u(t,x) - \delta_0$ are positive. This together with Lemma 3.1 implies that for any $\epsilon > 0$, there is $\delta > 0$ such that for any $U(\cdot) \in X$ with $|u(\tau,x) - U(x)| < \delta$ ($x \in \Omega$) for some $\tau \ge 0$, one has that $U(\cdot)$ is positive and

$$|u(t,x) - u(t,x,U,f_\tau)| < \epsilon \tag{3.5}$$

for $x \in \bar{\Omega}$ and $t \ge \tau$. Therefore, by (3.5) and *a priori* estimates for parabolic equations, $u = u(t,x)$ is uniformly stable.■

Lemma 3.4. *Suppose that $u_1(t,x)$, $u_2(t,x)$ are two positive solutions of (1.1) with $u_1(t_n,x) \to u_1^*(x)$ and $u_2(t_n,x) \to u_2^*(x)$ as $n \to \infty$ for a sequence $t_n \to \infty$. Then*

$$u_1^*(x) \le u_2^*(x) \quad \text{or} \quad u_1^*(x) \ge u_2^*(x) \quad (x \in \Omega).$$

Proof. By Lemma 3.2, either $u_1^* = u_2^*$ is a trivial state, or both u_1^* and u_2^* are positive. Assume that $u_1^*(x)$ and u_2^* are positive. Let $z(t,x) = z(t,x,u_1,u_2)$ and

$$z^*(t,x) = (\int_{u(t,x,u_1^*,f^*)}^{u(t,x,u_2^*,f^*)} \frac{1}{h(\eta)} d\eta)^2, \tag{3.6}$$

where h is as in (3.2). By Lemma 3.1,

$$\max_{x \in \bar{\Omega}} z^*(t,x) = const \quad \text{for} \quad t \in \mathbb{R}. \tag{3.7}$$

Without loss of generality, we assume that $m^*(t,x,u) = \lim_{n\to\infty} m(t+t_n,x,u)$ exists. Then $z^*(t,x)$ satisfies (3.3) with m replaced by m^*, and $u_1(t,x)$, $u_2(t,x)$ replaced by $u(t,x,u_1^*,f^*)$, $u(t,x,u_2^*,f^*)$ respectively. By the maximum principle and (3.7), $z^*(t,x) = const$ for $t \in \mathbb{R}_+$ and $x \in \Omega$. It follows that

$$\int_{u(t,x,u^*,f^*)}^{u(t,x,u^*,f^*)} \frac{1}{h(\eta)} d\eta = const, \tag{3.8}$$

which implies that $u(t,x,u_1^*,f^*) \geq$ (or $\leq$)$u(t,x,u_2^*,f^*)$ for $x \in \Omega$, $t \in \mathbb{R}$, in particular, $u_1^*(x) \geq$ (or $\leq$)$u_2^*(x)$ for $x \in \Omega$. ■

We now give sufficient conditions which guarantee the asymptotic almost periodicity of positive solutions.

Theorem 3.1 (Asymptotic almost periodicity). *Consider (1.1).*

1) *If one positive solution converges to a trivial state, then all positive solutions converge to the same trivial state;*

2) *If one positive solution is away from all trivial states, then every positive solution is away from all trivial states and converges to a positive almost periodic solution.*

Proof. Let $\pi_t : X \times Y \to X \times Y$ be the skew-product semiflow associated to (1.1). Again, denote $u(t,x,U,g)$ as the solution of (1.1) with f being replaced by g and $u(0,x,U,g) \equiv U(x)$.

1) Suppose that $u = u^*(t,x)$ is a positive solution which converges to a trivial state $u = c$. By Lemma 3.2, any positive solution $u = u(t,x)$ satisfies

$$\lim_{t\to\infty} u(t,x) = c \quad (x \in \Omega).$$

2) By Lemma 3.2, we see that any positive solution $u = u(t,x)$ is away from all trivial states. Moreover, by Lemma 3.3, $u = u(t,x)$ is uniformly stable. Therefore, by Lemma 2.3, $\omega(U,f)$ is minimal, distal, and uniformly stable, where $U(x) = u(0,x)$. It

follows from Lemma 2.1, Remark 2.1, and the distality of $\omega(U,f)$ that $p^{-1}(g)\cap\omega(U,f)$ admits no ordered pair for any $g\in H(f)$. We now claim that $p^{-1}(g)\cap\omega(U,f)$ is a singleton for any $g\in H(f)$. Suppose for contradiction that there is a $g\in H(f)$ and $(U_1,g),(U_2,g)\in p^{-1}(g)\cap\omega(U,f)$ such that $U_1\neq U_2$. By the distality of $\omega(U,f)$,

$$\inf_{t\in\mathbb{R}} d(\pi_t((U_1,g),\pi_t(U_2,g))>0. \tag{3.9}$$

Let $t_n\to\infty$ be such that

$$\pi_{t_n}(U_i,g)\to(U_i^*,g^*)\quad(i=1,2)$$

as $n\to\infty$. By Lemma 3.4, $U_1^*\geq U_2^*$ or $U_1^*\leq U_2^*$. But there is no ordered pair on $p^{-1}(g^*)\cap\omega(U,f)$, and hence $U_1^*=U_2^*$, which implies that

$$d(\pi_{t_n}(U_1,g),\pi_{t_n}(U_2,g))\to 0\quad\text{as}\quad n\to\infty,$$

a contradiction to (3.9). Therefore, for any $g\in H(f)$, $p^{-1}(g)\cap\omega(U,f)$ is a singleton, that is, $\omega(U,f)$ is a $1-1$ extension of $H(f)$. Thus, $\pi_t(u_\omega,f)$ is almost periodic and

$$\lim_{t\to\infty} d(\pi_t(U,f),\pi_t(u_\omega,f))\to 0$$

as $t\to\infty$, where $(u_\omega,f)=p^{-1}(f)\cap\omega(U,f)$, or equivalently, $u(t,x,u_\omega,f)$ is almost periodic and

$$\|u(t,\cdot,U,f)-u(t.\cdot,u_\omega,f)\|\to 0$$

as $t\to\infty$, that is, $u=u(t,x)=u(t,x,U,f)$ converges to a positive almost periodic solution. ■

Corollary 3.2 (Dichotomy of asymptotic almost periodicity) *Either all positive solutions of (1.1) are asymptotically almost periodic or none of them is.*

Proof. Suppose that $u(t,x)$ is a positive solution of (1.1) which is asymptotic to an almost periodic solution $u^*(t,x)$. If $u^*=c$ is a trivial state, then Theorem 3.1 1) implies that all positive solutions of (1.1) converge to c. Otherwise, u^* must be a positive almost periodic solution, and is away from all trivial states. By Theorem 3.1 2), all positive solutions are asymptotically almost periodic. ■

The case of non-asymptotically almost periodicity in Corollary 3.2 happens when every positive solution of (1.1) is neither away from all trivial states nor convergent to a trivial state (that is, trivial states admit weaker stability or instability). To see this, let us consider the following example.

Example 3.1. Consider the following almost periodic scalar ODE

$$u^{'}=a(t)u, \tag{3.10}$$

where $a(t) = -\sum_{k=1}^{\infty} 2^{-k}\pi \sin(2^{-k}\pi t)$. Let $u(t)$ be a non-zero solution of (3.10). Then it is easy to see that $u(t)$ is positively bounded, and there is $t_n \to \infty$ such that $u(t_n) \to 0$ as $n \to \infty$ but $u(t) \not\to 0$ as $t \to \infty$ (see [17]). That is, $u = u(t)$ is neither away from $u = 0$ nor converges to $u = 0$.

Now let $m(t, x, u) \equiv a(t)$ and $h(u)$ satisfy $HF)'$ with $h(u) = u$ for $u \ll 1$. Consider

$$\begin{cases} u_t = \Delta u + a(t)h(u), \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases} \tag{3.11}$$

Then for any given $0 < u_0(x) = u_0 \ll 1$, the solution $u(t, u_0)$ of (3.10) with $u(0, u_0) = u_0$ is also a solution of (3.11), but it is neither away from $u = 0$ nor converges to $u = 0$. We conclude that any positive solution of (3.11) (which is of Fisher type) is not asymptotically almost periodic.

Next, let $m(t, u) = a(t) - b(u)$, where $b'(u) \geq 0$, $b(u) = 0$ for $u \ll 1$, and $b(u) = u$ for $u \gg 1$. Consider

$$\begin{cases} u_t = \Delta u + m(t, u)u, \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases} \tag{3.12}$$

Similar to the above, any positive solution of (3.12) (which is of Kolmogorov type) is not asymptotically almost periodic.

Corollary 3.3 (Asymptotic almost periodicity). *Consider (1.1).*

1) *If a trivial state is asymptotically stable, then all positive solutions converge to the same trivial state.*

2) *If all trivial states are unstable, then every positive solution is away from all trivial states and converges to a positive almost periodic solution.*

3) *If one trivial state is uniformly stable, then every positive solution is asymptotically almost periodic.*

Proof. 1) is an immediate consequence of Theorem 3.1 1). 2) follows from Lemma 3.2 and Theorem 3.1 2). To prove 3), we suppose that $u = c$ is a trivial state which is uniformly stable. Then for any positive function $u_0 \in X$ with $\|u_0 - c\| \ll 1$, we have $\|u(t, \cdot, u_0, f) - c\| \ll 1$ for $t \geq 0$. Therefore, $u(t, x, u_0, f)$ is away from any other trivial state (if exists). Now if $u(t, x, u_0, f)$ is also away from $u = c$, by Theorem 3.1 2), $u(t, x, u_0, f)$ converges to a positive almost periodic solution. If $u(t, x, u_0, f)$ is not away from $u = c$, by the uniformly stability of $u = c$, we must have $u(t, x, u_0, f) \to c$ as $t \to \infty$. In any case, $u(t, x, u_0, f)$ converges to an almost periodic solution. By Corollary 3.2, every positive solution is asymptotically almost periodic.■

Remark 3.1 The stability conditions in 1) 2) of Corollary 3.3 can be verified directly. Let $u = c$ be a trivial state of (1.1). Consider the variational equation of (1.1) around $u = c$,

$$\begin{cases} v_t = \Delta v + f_u(t,x,c)v, \quad x \in \Omega, \quad t > 0 \\ \dfrac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases} \tag{3.13}$$

By Lemma 2.2 and Remark 2.4, there is $v^*(t,x)$ with $\|v^*\| = 1$, $v(t,\cdot) \gg 0$ such that

$$\lambda_{\{c\}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{-\int_\Omega |\bigtriangledown v^*(t,x)|^2 dx + \int_\Omega f_u(t,x,c)[v^*(t,x)]^2 dx}{\int_\Omega [v^*(t,x)]^2 dx} dt. \tag{3.14}$$

Then by Remark 2.2 2) and 2.4 3), $u = c$ is asymptotically stable if $\lambda_{\{c\}} < 0$ and is unstable if $\lambda_{\{c\}} > 0$.

Note that if $\lim_{T\to\infty} \frac{1}{T} \int_0^T (\int_\Omega f_u(t,x,c)dx)dt < 0$, we must have $\lambda_{\{c\}} < 0$ and hence $u = c$ is asymptotically stable. If f is periodic in t or is time-independent, then $\lambda_{\{c\}}$ is the first eigenvalue of (3.13).

Next, we give an alternative asymptotic almost periodicity condition.

Corollary 3.4 (Asymptotic almost periodicity). *Assume that either $m(t,x,u) \geq m^*(t)$ or $m(t,x,u) \leq m^*(t)$ for $t \in \mathbb{R}_+, x \in \Omega$, and $0 < u < 1$ in the case of HF), $0 < u \leq M$ in the case of HK) (here $M > 0$ is a constant), where $m^*(t)$ is an almost periodic function with $\int_\tau^{t+\tau} m^*(s)ds = O(1)$ as $t \to \infty$ uniformly for $\tau \in \mathbb{R}$. Then every positive solution of (1.1) is asymptotically almost periodic.*

Proof. We only prove the case HF) with $m(t,x,u) \leq m^*(t)$. The other cases can be proved similarly.

Consider

$$\begin{cases} v_t = \Delta v + m^*(t)v(1-v), \quad x \in \Omega, \quad t > 0, \\ \dfrac{\partial v}{\partial n} = 0, \quad x \in \Omega, \quad t > 0. \end{cases} \tag{3.15}$$

For any $\tau > 0, 0 < v_0 < 1$, we denote $v(t,x,\tau,v_0)$ the solution of (3.15) with $v(\tau,x,\tau,v_0) \equiv v_0$. By direct computations,

$$v(t,x,\tau,v_0) = \frac{v_0 e^{\int_\tau^{t+\tau} m^*(s)ds}}{1 - v_0 + v_0 e^{\int_\tau^{t+\tau} m^*(s)ds}}. \tag{3.16}$$

Now for any $u_0 \in X$ with $0 < u_0(x) \ll 1$ $(x \in \Omega)$, there is a $0 < v_0 \ll 1$ such that $u_0(x) < v_0$ $(x \in \Omega)$. Since $m(t,x,u) \leq m^*(t)$, $u(t,x,u_0,f_\tau)$ is a subsolution of (3.15) with $m^*(t)$ replaced by $m^*(t+\tau)$. It then follows from the comparison principle that

$$u(t,x,u_0,f_\tau) \leq v(t+\tau,x,\tau,v_0) \quad \text{for} \quad t \geq 0. \tag{3.17}$$

By (3.16), (3.17), and $\int_{\tau}^{t+\tau} m^*(s)ds = O(1)$ as $t \to \infty$, the trivial state $u = 0$ of (1.1) is uniformly stable. By Corollary 3.3 3), every positive solution is asymptotically almost periodic. ■

We end this section by a stability and harmonic property of limiting almost periodic solutions.

Theorem 3.5. *Consider (1.1).*

1) *(Stability). Any positive almost periodic solution (if exists) is uniformly stable.*

2) *(Harmonics). Any almost periodic solution $u = u(t, x)$ is harmonic, that is,*

$$\mathcal{M}(u) \subset \mathcal{M}(f).$$

Proof. 1) By Lemma 3.2, any positive almost periodic is away from the trivial state(s). Thus by Lemma 3.3, it is uniformly stable.

2) Suppose that $u = u(t, x)$ is an almost periodic solution. Clearly, the result holds if it is a trivial state. Now, suppose u is not a trivial state. By Lemma 3.2, it must be positive, and by 1) it is also uniformly stable. It follows from the arguments in Theorem 3.1 that $\omega(U, f)$ is a $1-1$ extension of $H(f)$, where $U(x) = u(0, x)$. Note that $(U, f) \in \omega(U, f)$. By Remark 2.3, we have $\mathcal{M}(u) \subset \mathcal{M}(f)$. ■

4 Global attractor

Lemma 4.1. *If $u_1(t, x)$ and $u_2(t, x)$ are two non-negative almost periodic solutions of (1.1), then $u_1(t, x) \geq$ (or $\leq$)$u_2(t, x)$ for $x \in \Omega$ and $t \in \mathbb{R}_+$. Moreover, if both $u_1(t, x)$ and $u_2(t, x)$ are positive solutions, then*

$$\int_{u_1(t,x)}^{u_2(t,x)} \frac{1}{h(\eta)} d\eta = const, \tag{4.1}$$

where $h(\eta)$ is defined in (3.2).

Proof. Note, by minimality of almost periodic functions, there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} u_i(t + t_n, x) = u_i(t, x) \quad (i = 1, 2).$$

The proposition then follows from Lemma 3.4 and its proof.■

Theorem 4.1 (Global attractor). *Consider (1.1). The following holds.*

1) *There is at most one asymptotically stable non-negative almost periodic solution and if there is one, it is a global attractor of positive solutions.*

2) If m is strictly decreasing for $0 < u < 1$ when f is of Fisher type and for $u > 0$ when f is of Kolmogorov type, and if all trivial states are unstable, then there is a unique positive almost periodic solution which is also a global attractor of positive solutions.

Proof. 1) Suppose that there are two non-negative, asymptotically stable, almost periodic solutions, say $u_1(t,x)$ and $u_2(t,x)$. By Lemma 3.2, u_1, u_2 are positive solutions, and by Lemma 4.1,

$$u_1(t,x) > u_2(t,x) \quad \text{or} \quad u_1(t,x) < u_2(t,x)$$

for $x \in \Omega$ and $t \in \mathbb{R}_+$, and

$$\int_{u_1(t,x)}^{u_2(t,x)} \frac{1}{h(\eta)} d\eta = const.$$

Without loss of generality, we assume that $u_1(t,x) < u_2(t,x)$ ($t \in \mathbb{R}_+$, $x \in \Omega$). Let u_0 be such that $u_1(0,x) < u_0(x) < u_2(0,x)$ ($x \in \Omega$) and $u(t,x,u_0,f) - u_1(t,x) \to 0$ as $t \to \infty$. Let $t_n \to \infty$ be such that $u_i(t_n,x) \to u_i(0,x)$ ($i = 1,2$). Then we have

$$\begin{aligned}
\left(\int_{u_0(x)}^{u_2(0,x)} \frac{1}{h(\eta)} d\eta\right)^2 &< \left(\int_{u_1(0,x)}^{u_2(0,x)} \frac{1}{h(\eta)} d\eta\right)^2 \\
&= \lim_{n\to\infty} \left(\int_{u(t_n,x,u_0,f)}^{u_2(t_n,x)} \frac{1}{h(\eta)} d\eta\right)^2 \\
&\le \left(\int_{u_0(x)}^{u_2(0,x)} \frac{1}{h(\eta)} d\eta\right)^2,
\end{aligned}$$

where h is as in (3.2), a contradiction. Therefore, there is at most one asymptotically stable non-negative almost periodic solution, and if there is one, it is clearly a global attractor of positive solutions.

2) Suppose that m is strictly decreasing and $u = 0, 1$ are unstable when f is of Fisher type and $u = 0$ is unstable when f is of Kolmogorov type. Then by Corollary 3.3, there is a positive almost periodic solution. Assume that there are two positive almost periodic solutions $u_1^*(t,x)$ and $u_2^*(t,x)$. Let $z^*(t,x) = z(t,x,u_1^*,u_2^*)$. Then $z^*(t,x) = const.$ By (3.3), we must have $m(x,t,u_2^*) - m(x,t,u_1^*) = 0$. This is a contradiction. ■

Theorem 4.2 (Non-uniqueness). *Consider (1.1). If there are two positive almost periodic solutions, then there is a continuous family of them connecting these two.*

Proof. Suppose that there are two positive solutions $u_1^*(t,x)$ and $u_2^*(t,x)$. Without loss of generality, we assume that $u_1^*(t,x) < u_2^*(t,x)$. By Lemma 4.1,

$$\int_{u_1^*(t,x)}^{u_2^*(t,x)} \frac{1}{h(\eta)} d\eta = K \tag{4.2}$$

for some $K > 0$. Let $U(t, x, L)$ be defined by

$$\int_{u_1^*(t,x)}^{U(t,x,L)} \frac{1}{h(\eta)} d\eta = L, \tag{4.3}$$

where $0 \le L \le K$. Clearly, $u_1^*(t, x) \le U(t, x, L) \le u_2^*(t, x)$ for any $t \in \mathbb{R}_+$, $x \in \Omega$, and $0 \le L \le K$. By the almost periodicity of u_1^*, $U(t, x, L)$ is also almost periodic in t. Moreover, by direct computations, we have

$$\frac{U_t(t, x, L)}{h(U(t, x, L))} = \frac{u_{1t}^*(t, x)}{h(u_1^*(t, x))},$$

$$\bigtriangledown U(t, x, L) \frac{1}{h(U(t, x, L))} = \bigtriangledown u_1^*(t, x) \frac{1}{h(u_1^*(t, x))}, \tag{4.4}$$

and

$$\Delta U \frac{1}{h(U)} - \bigtriangledown U \cdot \bigtriangledown U \Big(\frac{h'(U)}{h^2(U)} \Big) = \Delta u_1^* \frac{1}{h(u_1^*)} - \bigtriangledown u_1^* \cdot \bigtriangledown u_1^* \Big(\frac{h'(u_1^*)}{h^2(u_1^*)} \Big).$$

Therefore, $U(t, x, L)$ satisfies

$$\begin{cases} U_t \Big(\frac{h(u_1^*)}{h(U)} \Big) = \Delta U \Big(\frac{h(u_1^*)}{h(U)} \Big) - \bigtriangledown U \cdot \bigtriangledown U \Big(\frac{h'(U)}{h^2(U)} \Big) + \bigtriangledown u_1^* \cdot \bigtriangledown u_1^* \Big(\frac{h'(u_1^*)}{h^2(u_1^*)} \Big) \\ \qquad\qquad + m(t, x, u_1^*) h(U) \Big(\frac{h(u_1^*)}{h(U)} \Big), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial U}{\partial n} = 0, \quad x \in \partial\Omega, \quad t > 0. \end{cases} \tag{4.5}$$

Now by (3.3) and (4.2), we must have $m(t, x, u_1^*(t, x)) = m(t, x, u_2^*(t, x))$ and $\bigtriangledown u_1^*(t, x) = 0$, $\bigtriangledown u_2^*(t, x) = 0$ or $h'(u_1^*(t, x)) = h'(u_2^*(t, x))$. This together with (4.4) implies that

$$m(t, x, U) = m(t, x, u_1^*(t, x)) \quad \text{for} \quad u_1^*(t, x) \le U \le u_2^*(t, x) \tag{4.6}$$

and

$$- \bigtriangledown U \cdot \bigtriangledown U \Big(\frac{h'(U)}{h^2(U)} \Big) + \bigtriangledown u_1^* \cdot \bigtriangledown u_1^* \Big(\frac{h'(u_1^*)}{h^2(u_1^*)} \Big) = 0. \tag{4.7}$$

By (4.5)-(4.7), $U(t, x, L)$ is a solution of (1.1) for any $0 \le L \le K$. As L varies, $\{U(t, x, L) | 0 \le L \le K\}$ gives a continuous family of almost periodic solutions which connect u_1^* and u_2^*. ■

Remark 4.1. If there are two almost periodic solutions of (1.1) in which one is a uniformly stable trivial state and the other one is positive, then there is also a family of almost periodic solutions connecting them. To see this, suppose that $u = 0$ (or $u = 1$ when f is of Fisher type) is uniformly stable and $u^*(t, x)$ is a positive almost periodic solution of (1.1). Then for any $U(x)$ with $0 < U(x) \ll 1$ (or with $U(x) < 1$ and $1 - U(x) \ll 1$), the limiting almost periodic solution $\tilde{u}^*(t,x)$ corresponding to the positive solution $u(t,x,U,f)$ is positive, and $0 < \tilde{u}^*(t,x) \ll 1$ (or $\tilde{u}^*(t,x) < 1$ and $1 - \tilde{u}^*(t,x) \ll 1$). By Theorem 4.2, there is a continuous family of almost periodic solutions connecting u^* and $\tilde{u}^*$. Note that $\tilde{u}^*$ can be as close to $u = 0$ (or $u = 1$) as one wishes. Therefore, there is a family of almost periodic solutions connecting u^* and $u = 0$ (or $u = 1$).

We now give an example in which (1.1) admits a continuous family of almost periodic solutions.

Example 4.1. Consider (1.1) of Fisher type with $f(t,x,u) = m(t)u(1-u)$, where m is an almost periodic function satisfying $\int_0^t m(s)ds = O(1)$ as $t \to \infty$. Then for any constant $u_0 \in [0,1]$,

$$u(t,x,u_0,f) = \frac{u_0 e^{\int_0^t m(s)ds}}{1 - u_0 + u_0 e^{\int_0^t m(s)ds}}.$$

Since $\int_0^t m(s)ds = O(1)$ as $t \to \infty$, $u(t,x,u_0,f)$ is almost periodic. Therefore, there is a continuous family of almost periodic solutions connecting the two trivial states $u = 0$ and $u = 1$.

5 Spatial homogeneity

When a population lives in a spatially homogeneous habitat, it is important to know whether the population dynamics is also spatially homogeneous. The following theorem gives a characterization of the spatial homogeneity.

Theorem 5.1 (Spatial homogeneity). *If one of the positive almost periodic solutions of (1.1) is spatial homogeneous, so are all of them.*

Proof. Suppose that $u = u^*(t)$ is a positive, spatially homogeneous, almost periodic solution of (1.1). Let $\tilde{u}^*(t,x)$ be an arbitrary positive almost periodic solutions. By Lemma 4.1, we have

$$\int_{u^*(t,x)}^{\tilde{u}^*(t,x)} \frac{1}{h(\eta)} d\eta = const.$$

Therefore, we must have $\tilde{u}^*(t,x) \equiv \tilde{u}^*(t)$, that is, $\tilde{u}^*(t,x)$ is spatially homogeneous. ■

Corollary 5.2. *If $f(t,x,u)$ is spatially homogeneous, then every almost periodic solution of (1.1) is spatially homogeneous.*

Proof. Suppose for contradiction that (1.1) admits a non-spatially homogeneous, almost periodic solution. Clearly, such a solution must be positive. By Theorem 3.1 2), every positive solution of (1.1) converges to a positive almost periodic solution. But any spatially homogeneous, positive solution of (1.1) converges to a spatially homogeneous, positive almost periodic solution. By Theorem 5.1, all the positive almost periodic solutions are spatially homogeneous, which is a contradiction.■

In the case that $f(t, x, u)$ is spatially homogeneous, stability conditions of trivial states of Corollary 3.3 can be easily verified to guarantee the existence of a positive, spatially homogeneous almost periodic solution, that is, a positive temporal cline.

Theorem 5.3 (Temporal cline). *Consider (1.1) with $m(t, x, u) \equiv m(t, u)$ and define*

$$\lambda_{\{0\}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T m(t, 0)dt,$$

$$\lambda_{\{1\}} = -\lim_{T\to\infty} \frac{1}{T} \int_0^T m(t, 1)dt.$$

If $\lambda_{\{0\}}, \lambda_{\{1\}} > 0$ in the case of HF) and $\lambda_{\{0\}} > 0$ in the case HK), then all trivial states are unstable and there exists a positive temporal cline.

Proof. We only note that in the case that f in (1.1) is spatially homogeneous, the character $\lambda_{\{c\}}$ of Remark 3.1 1) reads

$$\lambda_{\{c\}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f_u(t, a)dt.\blacksquare$$

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