Polyhedral Clinching Auctions and the AdWords Polytope

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Joint work with Vahab Mirrokni and Renato Paes

Designing Truthful Auctions when the agents are budget constrained

Disconnect between

- *how much you value a certain good*

- *how much you are able to pay for it*

Most of the budgets related work in auction theory..

- Analyzes standard auction mechanisms when agents are budget constrained.
- Main motivation was high valued auctions such as spectrum auctions or privatization auctions.

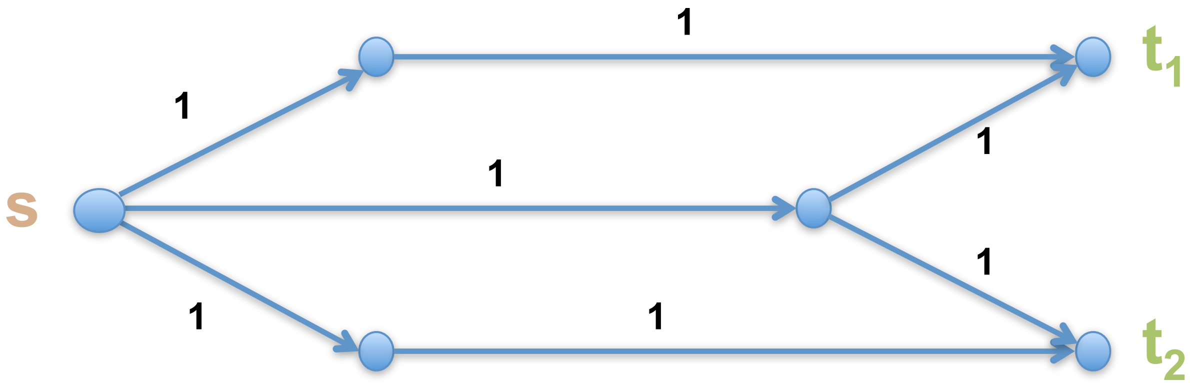

We
Advertising

Ad auctions introduced budgets into the bidding language.

Design mechanisms with budgets as part of the language.

Our setting

- One good
- N players/buyers.



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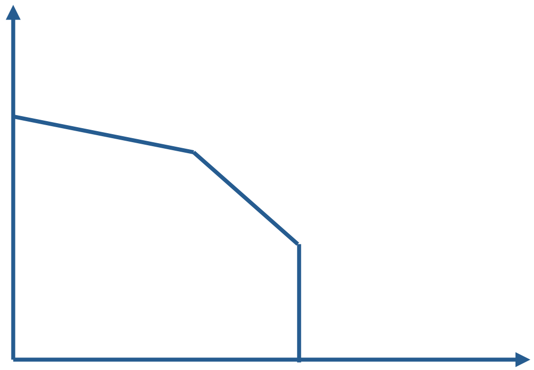
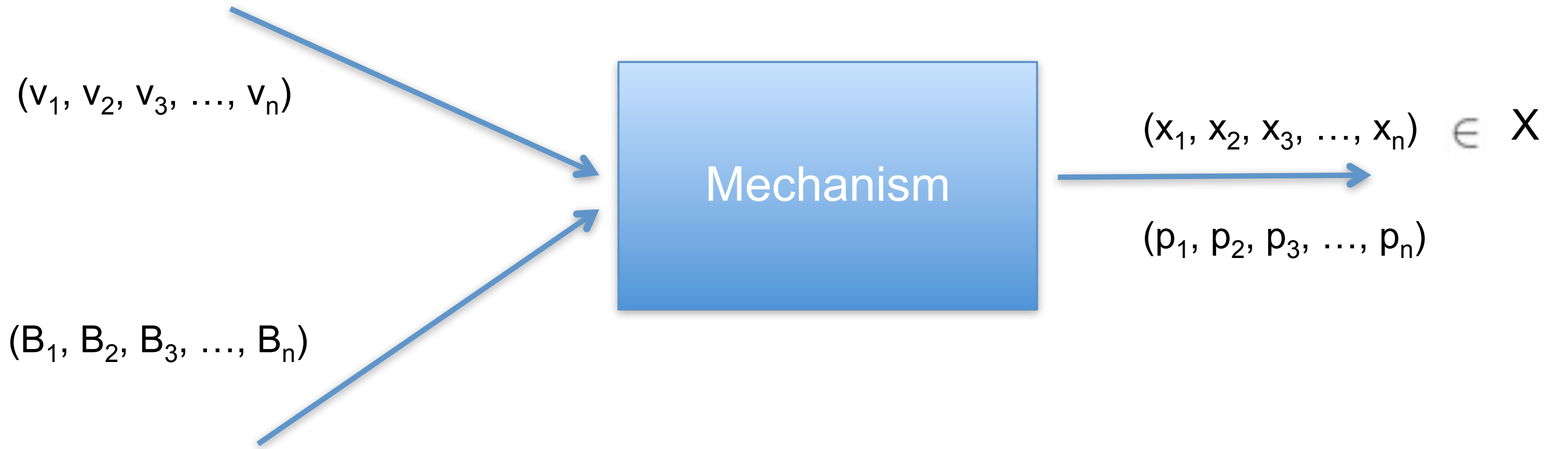
- A set $X \subseteq \mathbb{R}^n_+$ (convex and downward closed)



- A point $x \in X$ represents a feasible allocation of some abstract good to all the

- Player i has
  - value v_i per unit of the good (private information)
  - budget B_i (public information)

Mechanism Design



Desired Properties

- Individual Rationality

$$u_i = v_i.x_i - p_i \geq 0$$

- Incentive Compatibility (Truthfulness)

$$v_i.x_i(v_i', v_{-i}) - p_i(v_i', v_{-i}) \leq v_i.x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})$$

- Maximize Social Welfare

$$max_{x \in X} v.x$$

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$$max_{x \in X} v.x$$

Pareto-Optimality

A feasible outcome (**x**,**p**) is P.O.

if and only if

There is no other feasible outcome (**x'**, **p'**) where either

1) one of the players' utility is increased, or

2) revenue of the seller is increased,

without hurting anyone.

Properties of P.O. solution

- Allocation **x** must always be on the Pareto boundary of X.

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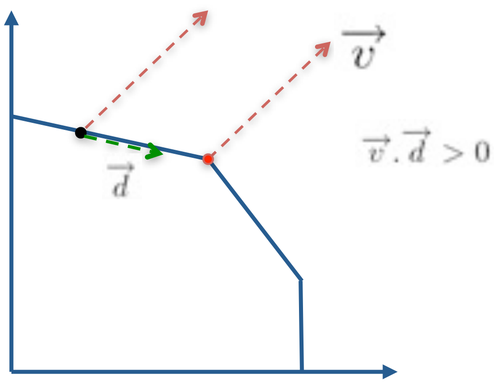
- Allocation **x** must always be on the Pareto boundary of X.
- If the budgets are very large, Pareto-optimal solution achieves social welfare.



$$x_i' = x_i + \epsilon d_i$$
$$p_i' = p_i + \epsilon v_i d_i$$

$$\Longrightarrow$$

$$u_i' = v_i x_i' - p_i' = v_i x_i - p_i = u_i$$
$$\sum p_i' = \sum p_i + \epsilon \sum (v_i . d_i)$$

Properties of P.O. solution

- Lemma:

A feasible outcome$(x, p)$is Pareto-optimal

if and only if

1) $\overrightarrow{x}$ is on the Pareto boundary of X.

2) if $\exists$ a feasible direction $\overrightarrow{d}$ such that $\overrightarrow{d} . \overrightarrow{v} > 0$,

then $\exists i$ with $d_i > 0$ and $p_i = B_i$

What is known..

- [Dobzinski, Lavi, Nisan ‘08]
  - Single divisible good.
- [Fiat, Leonardi, Saia, Sankowski ‘11]
  - Coverage-type functions
- [Baldeschi, Henzinger, Leonardi, Starnberger ‘11]
  - Scheduling for sponsored-search slots.

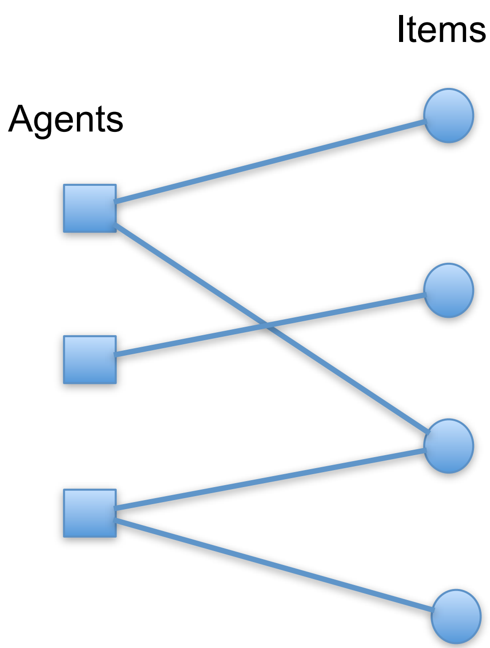
Dobzinski, Lavi, Nisan [FOCS'08]

- Design a Pareto-optimal auction for single divisible good.

X = $\{x \in \mathbb{R}^n_+; \sum_i x_i \leq 1\}$

- Also show that if budgets are private information, there is no Pareto optimal auction.

Fiat, Leonardi, Saia, Sankowski [EC'11]



$$\forall S, \sum_{i \in S} x_i \leq N(S)$$

[Baldeschi, Henzinger, Leonardi, Starnberger ’11]

Our Results

Our Results

- We give a mechanism for any polymatroid **P**.

$$\mathbf{P} = \{x \in \mathbb{R}^n_+; \sum_{i \in S} x_i \leq f(S); \forall S \subseteq [n]\}$$

- Only need a value oracle.

- Subsumes all the previous results.

Applications

- Besides numerous applications of polymatroids, we show a new application to Ad auctions.
- Adwords Polytope: An allocation scenario capturing multiple keywords and multiple slots per keyword.

Impossibility Result

- There are examples of more general polytopes where one cannot achieve the following three properties simultaneously:
  - Individual Rationality
  - Incentive Compatibility
  - Pareto-optimality

Sketch of our auction

Clinching Auctions

- Due to [Ausubel 04]
- Ascending price auctions that implement VCG for various settings.

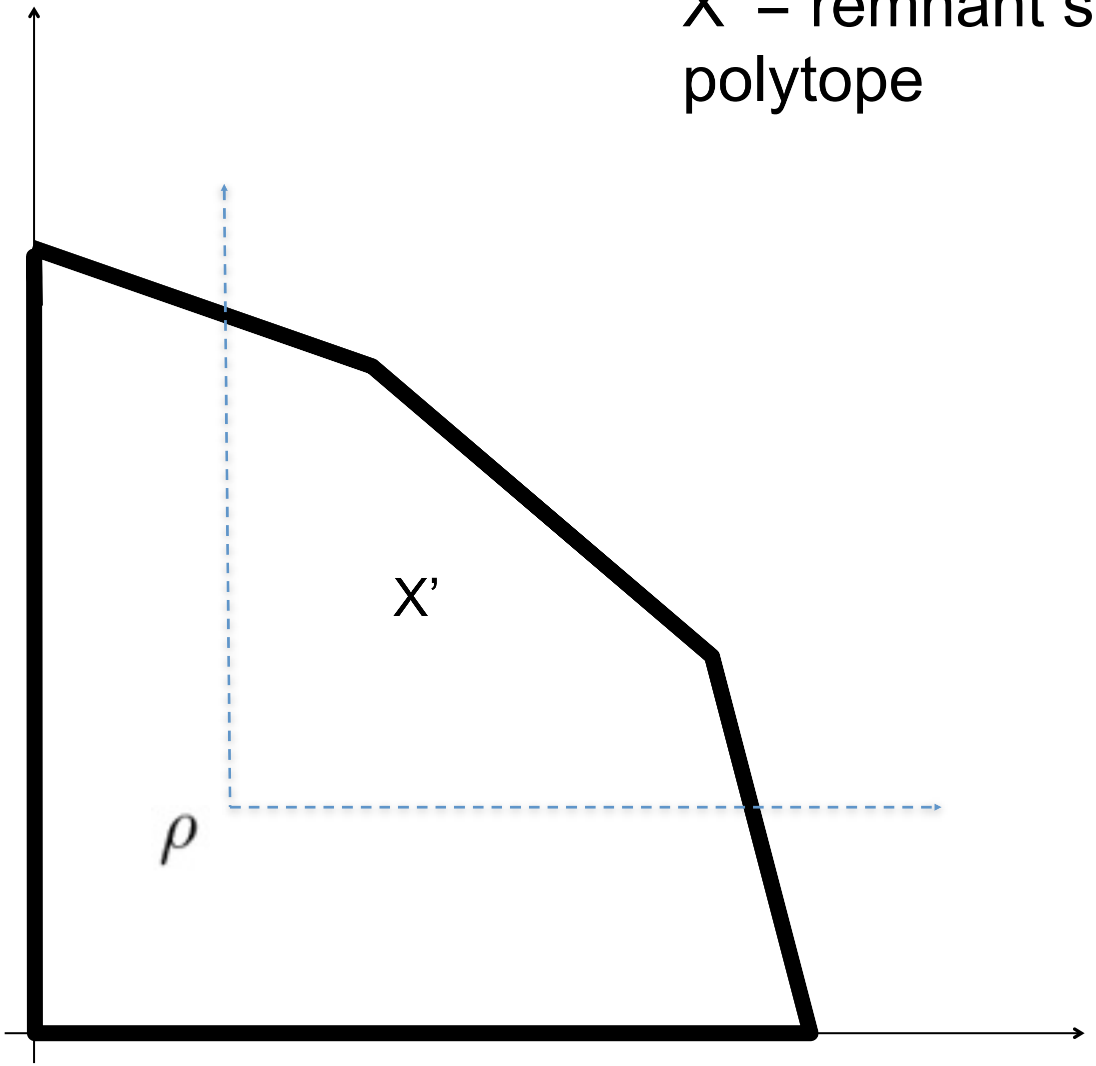
Multiple round auction

	Price	Allocations
Round 4	p_4	
Round 3	p_3	$(z_1, z_2, z_3, \ldots, z_n)$
Round 2	p_2	$(y_1, y_2, y_3, \ldots, y_n)$
Round 1	p_1	$(x_1, x_2, x_3, \ldots, x_n)$

Clinching in Round k

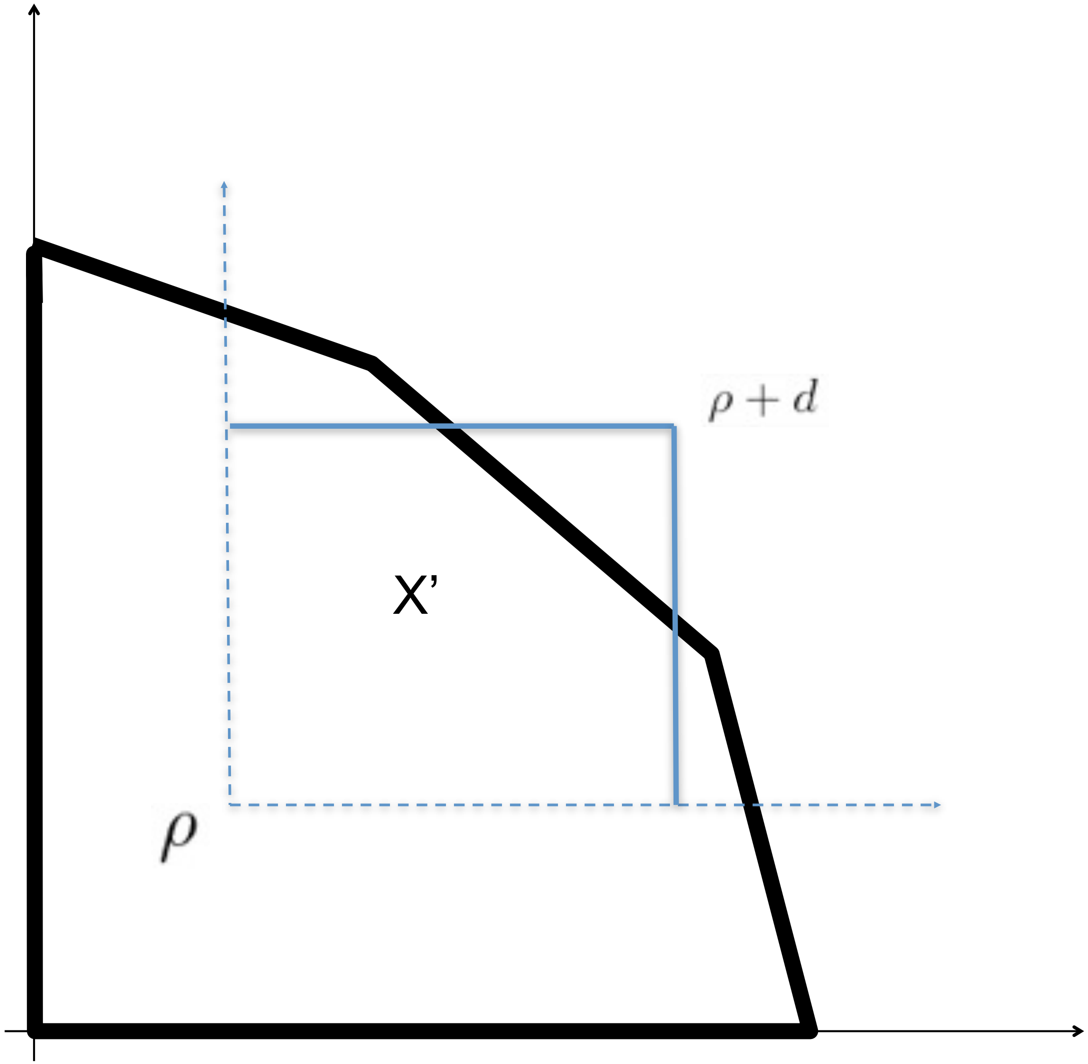
- Say so far will have allocated $(\rho_1, \rho_2, \rho_3, \cdots, \rho_n)$
- Current price is p_k
- How much should each player "clinch" in round k?

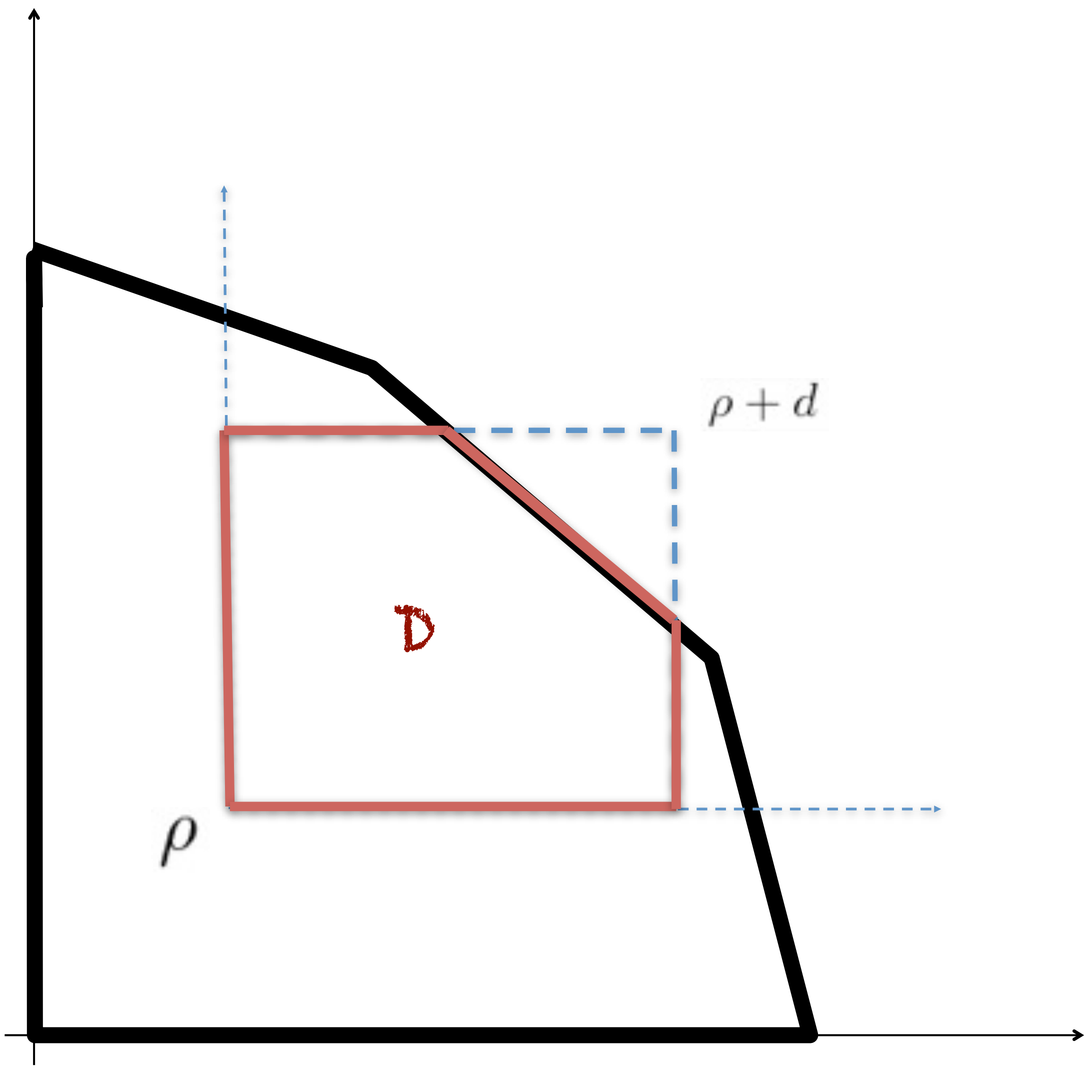
X' = remnant supply polytope



Demand of player i

- Let remaining budget of player i in k^{th} round is R_i^k
- If $p_k \leq v_i$, then the maximum amount player i can afford is $\underbrace{R_i^k / p_k}_{d_i}$


$\rho + d$
X'
ρ



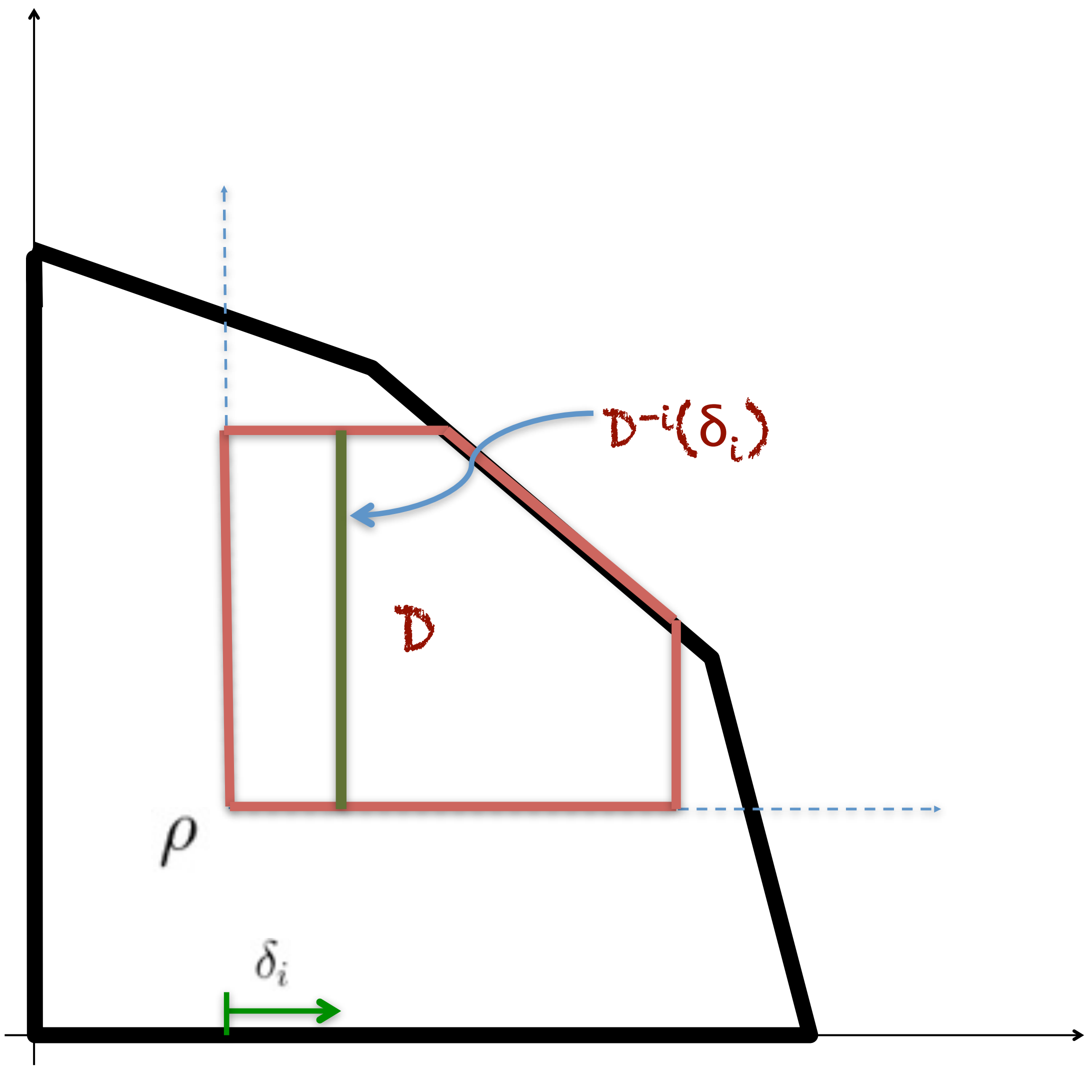
$\rho + d$
D
ρ

How much should player *i* clinch?

- Define

$$D^{-i}(\delta_i) = \{ (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \mid (y, \delta_i) \in D \}$$

How much should player i clinch?



How much should player i clinch?

$\text{Clinch}_i :=$

$$\max \delta_i \text{ s.t. } D^{-i}(\delta_i) = D^{-i}(0)$$

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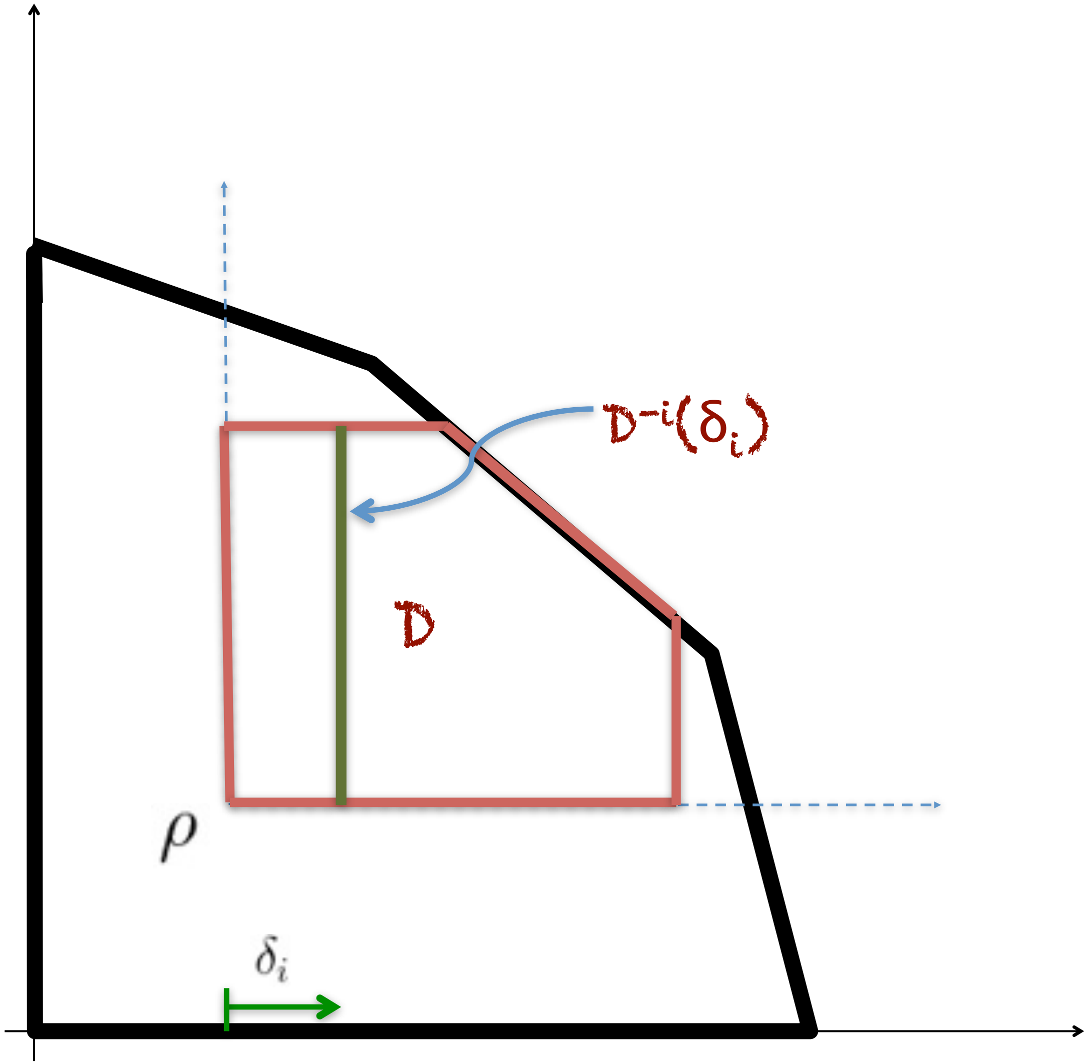

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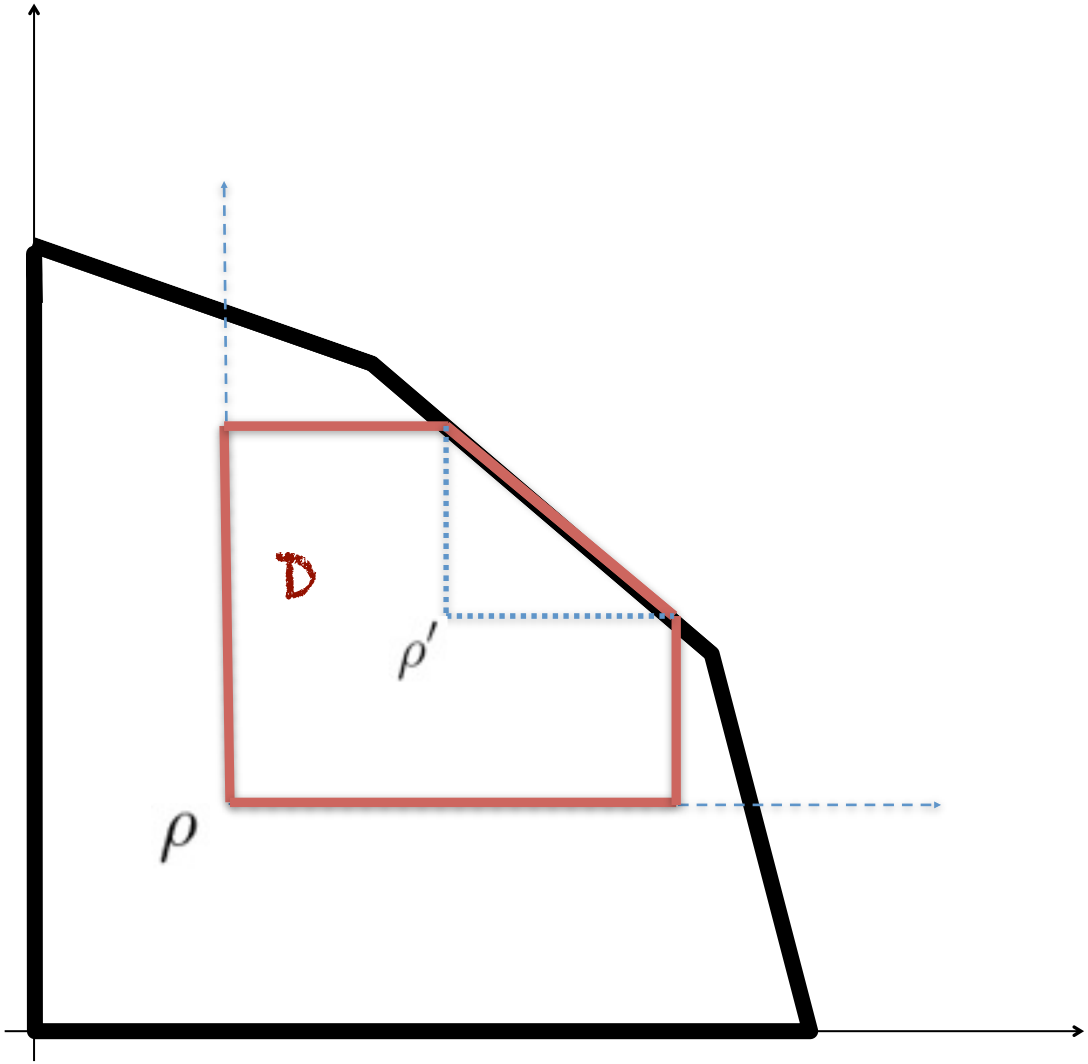

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Q: Can we compute clinch_i ?

Theorem: If P is a polymatroid, clinch_i can be computed efficiently using submodular minimization.

We can show the following for our clinching auction:

Individually rational

Incentive compatible

} True for any polytope **P**

Pareto-optimality } Polymatroids

Characterization of P.O. solution

- Lemma:

A feasible outcome$(x, p)$is Pareto-optimal

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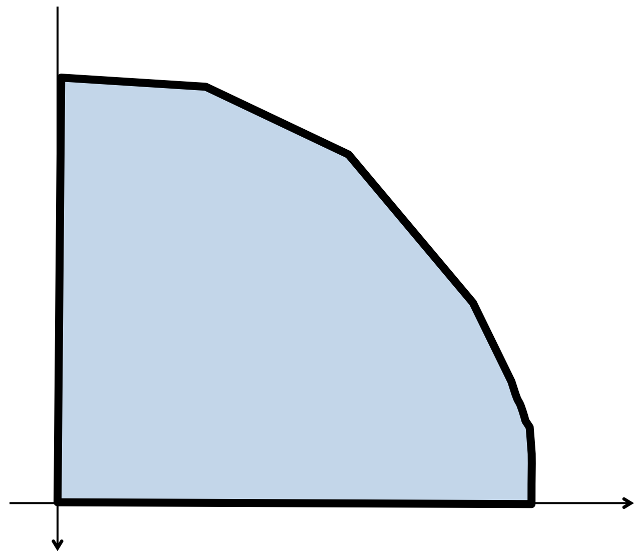
Going beyond polymatroids…

Extensions and Impossibility Results

General convex environment
One budget-constrained player

For a single budget constrained player, it is possible do design an auction for any convex environment.



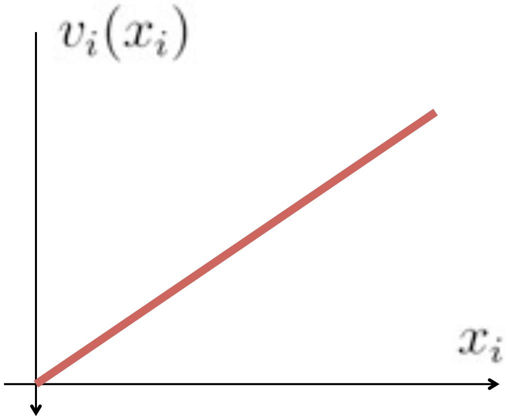
What about 2 budget constrained players?

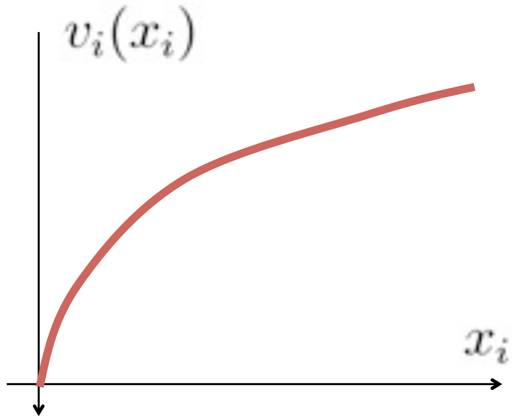
Impossibility Result: There exists a class of polytopes, for which **no auction** can satisfy all the three properties simultaneously.

Impossibility for decreasing marginals

Single divisible good: $\{x \in \mathbb{R}^n_+ ; \sum_i x_i \leq 1\}$

Decreasing marginal valuations





Summary of our results

- Incentive-compatible, budget feasible and Pareto-optimal auction for polymatroidal environments
- New application of polymatroids for modelling sponsored search environments
- Auction for a generic environment with a single budget constrained player
- Impossibility results for environments beyond polymatroids
- Impossibility for single divisible-item setting decreasing marginal valuations

Thanks !