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COMPUTATIONAL PROCEDURES FOR THE SYNTHESIS OF LINEAR STATIONARY CONTROL SYSTEMS

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SUMMARY

This dissertation presents an efficient computational procedure for the synthesis of linear stationary control systems. The procedure is generally applicable to the design of linear controllers that minimize the mean squared system error of multi-input linear plants that are subject to stationary stochastic disturbances. The controller is designed to operate only on the basis of information that is physically measurable, and the powers associated with the plant's control inputs are constrained by the design to lie within prescribed bounds.

The dissertation begins with the formulation of the control problem in state space terminology as the minimization of system error subject to constraints on the variances of the components of the control input. The Kuhn-Tucker theorem is used to reduce this problem to a sequence of problems in which the objective is the unconstrained minimization of a Lagrange function defined to be the sum of the system error and the variances of the control input components weighted by undetermined multipliers. The results of modern control and estimation theory, including the certainty equivalence principle, are used to establish that the solution of the unconstrained problem is a controller consisting of a linear dynamical system whose parameters are determined by the solution of two quadratic matrix equations that are referred to as the control and the estimation equations. Both of these equations have the form

$$P F + F^T P + H^T H = P G \Gamma^2 G^T P$$

where the matrices F, G, and H are determined by the system to be controlled and P is the unknown matrix defining the controller. The matrix P has an order equal to that of the system and it is required to be symmetric and positive definite. Γ^2, which appears only in the control equation, is a diagonal matrix containing the undetermined multipliers associated with the control variances. The solution of this matrix equation, which is equivalent to n (n+1)/2 simultaneous quadratic equations, is the crux of the computational aspect of the control synthesis problem.

A novel and computationally attractive scheme for the solution of the crucial matrix equation is developed in this dissertation. Certain transformations devised by Luenberger for the representation of controllable dynamical systems are used to convert the control and estimation equations to a single canonical form. A theorem due to Potter is introduced whereby the solution of the canonical equation may be written in terms of the eigenvectors of a linear Hamiltonian system whose order is twice that of the system to be controlled. Methods based on a set of Special Notational Conventions devised for dealing with systems in canonical form are used to obtain explicit expressions for the eigenvectors of the Hamiltonian system in terms of the solutions of a much simpler reduced homogeneous system whose order is equal to the number of control inputs (or measured outputs, in the case of the estimation equation). A concise solution of the canonical matrix equation is then written in terms of these explicit expressions. It is further shown that the parameters of the optimal controller may be determined directly from steps intermediate to the full solution of the canonical matrix equation.

These results have important theoretical implications: they constitute, together with Potter's theorem, a purely algebraic proof that the control equation has a unique positive definite solution when the underlying system is controllable. The results are exploited in this dissertation, however, primarily as the basis of a computational algorithm for the determination of the parameters of the optimal controller.

The algorithm for determining the parameter of the optimal controller begins with the numerical determination of the transformation that converts the control or estimation equation to canonical form. This is accomplished by an adaptation of Danilevskii's method for determining the characteristic equation of a matrix. The remaining computationally significant steps are the solution of a polynomial equation and the inversion of a matrix, both of order equal to that of the system to be controlled. The number of arithmetic steps required to carry out the algorithm is carefully determined and shown to be significantly less than alternative conventional algorithms for determining the parameters of the optimal controller. Such conventional algorithms depend on calculating the eigenvectors of the Hamiltonian system.

An iterative procedure for the determination of the multipliers which appear as parameters in the control equation is also developed. It is shown the control system determined by the solution of the control equation leads to control input components whose variances are monotonic decreasing functions of their associated multipliers. Experience and physical reasoning lead to the following conclusions:

1). The variance of a given control input component is relatively independent of the multipliers not directly associated with it.

2). The variances of the control input components are approximately exponential functions of their associated multipliers.

These assertions form the basis for the following iterative procedure: Two initial guesses are made for values of the multipliers and the control system parameters are computed. The control input variances which correspond to these guesses are then determined by the solution of a linear matrix equation whose form closely resembles that of the quadratic matrix equations. Successive guesses for values of the multipliers are then determined by logarithmic interpolation until values have been found such that the control input variances meet their prescribed bounds.

The computational algorithm for the solution of the quadratic matrix equations and the iterative procedure for the determination of the control input multipliers together constitute the complete computational procedure for the synthesis of linear stationary control systems. These procedures are tested in their application to a number of test cases and to the synthesis of lateral and longitudinal control systems for an aircraft subject to random gust disturbances. The computational experience gained in these applications is summarized. This experience is generally excellent. The aircraft control system design study also demonstrates the effectiveness of the design procedure as a practical method of designing complex control systems.

CHAPTER I

INTRODUCTION

The objective of this dissertation is to develop computational procedures for the synthesis of linear stationary control systems. The design of linear control systems is the central problem of classical control theory, and the techniques of this theory, which are by and large analytic, have led to many excellent control system designs. As the systems under consideration have become more complex and the performance requirements more stringent, the methods of the classical theory have become inadequate, and efforts have been made to devise more powerful and essentially synthetic design techniques. These efforts have met with considerable success from a theoretical point of view. The results of these efforts, which are a part of modern control and estimation theory, now provide the basis for the rational synthesis of a broad class of optimal linear control systems. There are, however, serious computational problems that arise in connection with the new theory, and this dissertation is an effort to resolve these difficulties by the development of efficient computational procedures, based on the new theory, for the synthesis of linear stationary control systems.

Statement of the Problem

The control synthesis problem considered in the present research is the design of controllers that minimize the mean squared system error of multi-input linear plants operating in a stochastic environment. The problem is broad enough to include a wide variety of practical control systems and has been so formulated as to lead to realizable control system designs.

A block diagram of the most general control problem that falls within the framework of the present formulation, the servo problem, is shown in Figure 1. Disturbance and reference signals which are assumed to be stationary multi-dimensional processes with known second-order statistics emanate from their respective sources. The design objective is the synthesis of a linear controller that causes the plant output to follow the reference source output with minimum mean squared error in spite of the adverse influence of the disturbance signal. When the reference source is absent and the objective is taken as the minimization of the plant output due to the disturbance signal, then the control problem becomes the regulator problem.

An important part of the formulation of the control problem is the requirement that the controller operate only on the basis of information that is physically measurable. These measurements may or may not include the outputs of the plant and the two stochastic sources, but the measurements are, in any event, usually incomplete and imprecise. An other important part of the formulation is the requirement that the power associated with the control inputs and specified plant variables not exceed the bounds imposed by the plant's physical limitations. These two requirements assure that the resulting control system will be physically realizable.

The results of modern control and estimation theory establish that the solution to this formulation of the control problem is a

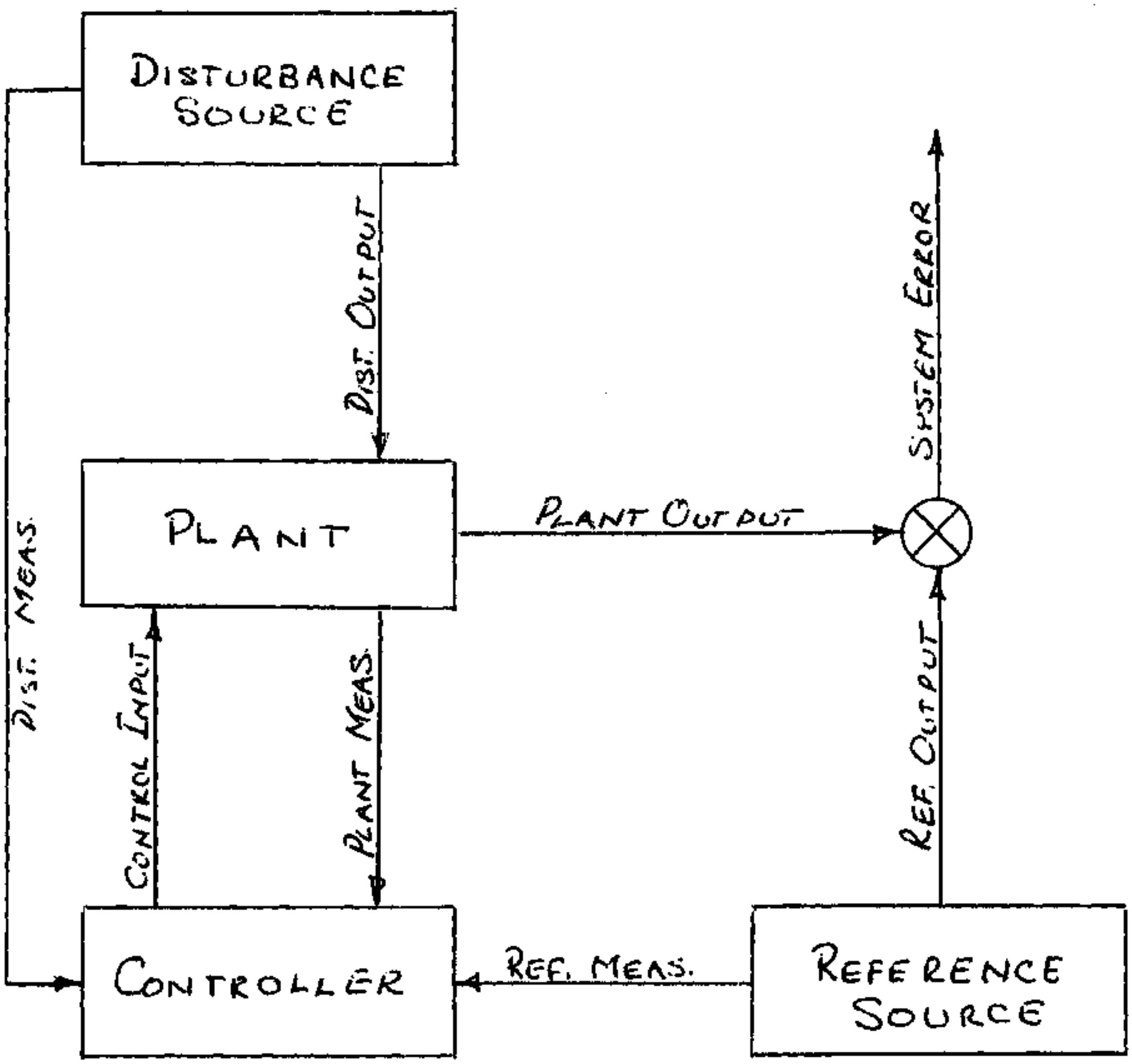

Figure 1. Control System Block Diagram.

controller consisting of a linear dynamical system whose inputs are the physically measurable system variables and whose outputs drive the fixed plant. The parameters of the controller are determined by two quadratic matrix equations and a saddle point condition that are given explicitly in Chapter II. The solution of these equations and the satisfaction of the saddle point condition are the crux of the computational aspects of the control synthesis problem.

History of the Problem

The modern control and estimation theory on which the control synthesis procedure is based began with the well known work of Wiener on the estimation of stochastic signals in the presence of corrupting noise [1]. Subsequent developments fall into two broad catagories: those that grew directly out of Wiener's work and depend on transform techniques, and later developments that grew out of Kalman's work and depend on state space techniques.

Wiener formulated the problem of designing linear filters for the estimation of stationary stochastic signals in the presence stationary stochastic noise with minimum mean squared error. He showed that the impulse response of the optimal filter is determined by an integral equation, now known as the Wiener-Hopf equation, and he gave the widely applicable method of spectral factorization for its solution. At about the same time Wiener, and his co-workers formulated the stochastic control problem and suggested that it was equivalent to the estimation problem. Newton [2] pointed out that this was not strictly true since an essential aspect of the control problem is the presence of fixed components that are subject to saturation. Newton showed, however, that undetermined Lagrange multipliers may be introduced in the formulation of the control problem to bound the saturable variables, and that in this form the problem is identical to the estimation problem. Chang [3] considered the problem of designing controllers for plants with saturable components and devised a computational technique, the root square locus technique, for the solution of the spectral factorization problem and the determination of values for Newton's bounding multipliers.

An important generalization of Wiener's work has been its extension to multivariable* systems. Amara [4], for instance, formulated the problem of controlling plants with multiple inputs and outputs. Other workers have dealt with the corresponding estimation problem. These formulations both lead to matrix equivalents of the Wiener-Hopf equation which may be solved by a generalization of the method of spectral factorization. In its multivariable form, however, spectral factorization is a truly formidable computational task and has begun to receive attention in its own right. Amara presented a method for solving the problem that involves undetermined polynomials. Davis [5] and Youla [6] gave alternative methods. None of these methods is attractive, however, for systems with more than two or three variables. Before considering more recent developments in this area, it is necessary to review the control and estimation theory that grew out of Kalman's work.

*The term "multivariable" is used throughout this dissertation to mean systems with multiple inputs and/or outputs.

The principle motivation for the application of state space techniques to control and estimation theory was its extension to non-stationary problems where transform techniques do not apply. The state space approach has led, however, to results that bear in important ways on the stationary problem. Kalman and Bucy [7] considered the estimation problem from the new point of view. They were able to state that observability of the signal source is a sufficient condition for the existence of a solution to the estimation problem and to characterize the solution in a simple way: the optimal estimator consists of a model of the signal source which is driven by the difference between the measured and the estimated source outputs. The gains of the inputs to the source model are determined from the solution of a non-linear matrix differential equation of the Ricatti type.

Kalman also formulated the deterministic regulator problem in which a control signal is sought that minimizes the mean squared output of a linear plant that is subject to a known initial disturbance [8]. If this problem is naively formulated, the solution leads to impulsive control signals which violate the obvious physical requirement that the control energy be finite. For this reason the objective is modified to be the minimization of the weighted sum of the control energy and the plant output. Although it is rare to see their role explicitly mentioned, the weights on the control energy are in fact equivalent to the multipliers that Newton introduced in the formulation of the control problem. Kalman demonstrated that the deterministic regulator problem is the "dual" of the estimation problem. He showed that the optimal control signal is the feedback of each component of the system state according to gains that do not depend on the initial disturbance and that these gains are determined by the solution of the same Ricatti equation that arises in the estimation problem.

Webb [9] investigated the stochastic servo problem. He showed that this general problem may always be reduced to the problem of regulating a plant subject to uncorrelated stochastic disturbances by the introduction of shaping filters to account for the statistics of the external signals. Webb further showed that the solution of this problem is identical to the solution of the corresponding deterministic problem. This is an important conclusion, but its practical implications are restricted by the fact that the control signal depends on each component of the system state (including those associated with the shaping filters) and these variables are rarely directly measurable.

Kalman and others conjectured, however, that the inaccessible state components could be estimated with the aid of an optimal filter and that a control signal based on these estimates would be optimal in an overall sense. Under appropriate circumstances this conjecture is true; it has become known as the certainty equivalence principle. This result was first obtained by Simon [10] for discrete systems. Joseph and Tou [11] also obtain this result for discrete systems. Roberts [12] proves the principle for continuous stationary systems and Schultz [13] for nonstationary continuous systems. On the basis of this principle, the optimal controller consists of an optimal state estimator whose inputs are the physically measurable variables cascaded with the optimal deterministic controller. The parameters of the overall system are determined by the solution of the Ricatti equations associated with the control and the estimation aspects of the problem.

The foregoing account summarizes the theoretical contributions of the state space approach to the control synthesis problem. In addition to extending the theory to include non-stationary problems, the new approach sheds considerable light on the structure of the optimal control system.

The central computational problem in the synthesis of linear controllers when seen from the state space point of view is the solution of the matrix Ricatti equation. For non-stationary controllers the equation is a non-linear matrix differential equation; for stationary controllers a quadratic matrix equation. It is not surprising that in the stationary case the Ricatti equation may be cast in the form of the Wiener-Hopf equation and solved by spectral factorization. In fact, Anderson [14] has used the reverse procedure to achieve the most recent result in the solution of the spectral factorization problem.

The Ricatti equation arises also in the Calculus of Variations and has been studied by many authors [15-17]. An important result is that the differential Ricatti equation is equivalent to a coupled system of two linear matrix differential equations known as the Hamiltonian system. Sufficient conditions for the existence of a unique, symmetric positive definite equilibrium point of the differential equation have also been developed. The equilibrium point is important in that it is also the solution of the quadratic matrix equation. Until recently, except in the single input case where spectral factorization may be easily used, the quadratic equation was solved by integrating the differential equation numerically until equilibrium was established.

This approach was used, for instance, in the Automatic Synthesis Program that was developed under Kalman at RIAS [18]. Several investigators, including Bass [19] and the author, discovered independently that the solution to the quadratic equation may be written down in terms of the eigenvectors of the Hamiltonian system. Potter [20] obtained the prior and more complete result that when a matrix related to the eigenvectors of the Hamiltonian system is non-singular, then the solution written in terms of these eigenvectors is unique, symmetric and positive definite.

In this dissertation a novel and computationally attractive scheme will be developed for the solution of the quadratic matrix equation. Briefly described, the scheme begins with the transformation of the matrix equations as they arise in connection with the control and the estimation problems to a single canonical form. Explicit expressions which will be developed for the eigenvectors of the associated Hamiltonian system are then used to write down the solution of the canonical matrix equation. An iterative procedure for determination of the Lagrange multipliers that enter in the formulation of the control problem will also be devised. This procedure and the scheme for the solution of the matrix equation, taken together, constitute a completely self-contained computational procedure for the synthesis of optimal linear stationary control systems.

CHAPTER II

THE CONTROL PROBLEM AND ITS FORMAL SOLUTION

In Chapter I the control synthesis problem was stated and the history of the theory underlying the synthesis procedure was outlined. Before proceeding with the development of computational procedures for the solution of the synthesis problem, it is necessary to formulate rigorously the control problem in state space terminology and to organize and precisely state the theoretical results that lead to its formal solution.

The Simplest Regulator Problem

The simplest form of the control problem is the regulation of a plant that is subject to uncorrelated stochastic disturbances. The problem may be formulated mathematically as follows: a linear dynamical system $\mathcal{S}$ is described by the vector differential equation

$$\mathcal{S} : \quad \dot{x} = F\,x + G_v\,v + G_u\,u \qquad z = H_z\,x + w$$

$$y = H_y\,x$$

where x is the system state, u is the control input, v is the (uncorrelated,* zero mean) disturbance input, z is the measurable output which

*The term "uncorrelated" is used here to mean "white."

is corrupted by the (uncorrelated, zero mean) measurement noise w, and y is the significant output. The covariances of v and w are given by

$$\mathcal{E}\left[v(t)\, v^T(t+\tau)\right] = \Upsilon^2 \delta(\tau) \qquad \Upsilon = \operatorname{diag}\left[\nu_i\right]$$

$$\mathcal{E}\left[w(t)\, w^T(t+\tau)\right] = \Omega^2 \delta(\tau) \qquad \Omega = \operatorname{diag}\left[\omega_i\right]$$

The control problem is the determination of the linear dynamical system $\mathcal{S}_c$ with input z and output u

$$\mathcal{S}_c: \qquad \dot{x}_c = F_c x_c + G_z z \qquad\qquad u = H_u x_c$$

which minimizes the stochastic norm $\|y\|$ of the significant output where the stochastic norm is defined by

$$\|y\|^2 = \mathcal{E}\left[y^T(t)\, y(t)\right] = \mathcal{E}\left[y_1^{\,2}(t) + \ldots + y_\ell^{\,2}(t)\right]$$

There is, however, the important restriction that the variances of the components of the control input not exceed prescribed bounds; in other words, the minimization is to be achieved subject to the constraints

$$\|u_i\|^2 = \mathcal{E}\left[u_i^{\,2}(t)\right] \le \gamma_i^{\,2} \qquad i = 1, 2, \ldots, m$$

The motivation for this formulation of the control problem is the following: The objective of the control design is to minimize the significant output, which may or may not be measurable, by operating on the measurable outputs to produce the control input. The controller must function, however, in such a manner that the power associated with the control input components not exceed the bounds imposed by the physical limitations of the fixed plant. It is well known that unless such bounds are imposed the formal solution of the synthesis problem will lead to controllers with infinite control power.

Two Extensions of The Regulator Problem

The formulation of the regulator problem can be extended to cover the case where the disturbance input is not uncorrelated by appending "shaping filters" to the mathematical description of the plant to account for the statistics of the disturbance input. Suppose the disturbance input $\hat{v}$ is correlated with covariance

$$\mathcal{E}\left[\hat{v}(t)\,\hat{v}^T(t+\tau)\right] = R_{\hat{v}\hat{v}}(\tau)$$

It is well known (Webb [9]) that if the corresponding spectrum is rational, it is possible to construct a linear dynamic system $\mathcal{S}_D$

$$\mathcal{S}_D: \qquad \dot{x}_D = F_D\, x_D + G_{Dv}\, v \qquad\qquad \hat{v} = H_v\, x_D$$

whose input v is uncorrelated with covariance

$$\mathcal{E}\left[v(t)\, v^T(t+\tau)\right] = \Upsilon^2 \delta(\tau) \qquad\qquad \Upsilon = \operatorname{diag}\left[\nu_i\right]$$

and whose output $\hat{v}$ has the specified covariance $R_{\hat{v}\hat{v}}(\tau)$.* The disturbance system $\mathcal{S}_D$ may be appended to the plant system $\mathcal{S}$ to form the augmented system $\mathcal{S}_a$.

$$\mathcal{S}\ a:\ \begin{bmatrix} \dot{x}_D \\ \dot{x}_P \end{bmatrix} = \begin{bmatrix} F_D & \phi \\ G_v\ H_v & F_P \end{bmatrix} \begin{bmatrix} x_D \\ x_P \end{bmatrix} + \begin{bmatrix} G_{Dv} \\ G_{Pv} \end{bmatrix} v + \begin{bmatrix} \phi \\ G_{Pu} \end{bmatrix} u$$

$$z = \begin{bmatrix} H_{zD} & H_{zP} \end{bmatrix} \begin{bmatrix} x_D \\ x_P \end{bmatrix} + w \qquad y = \begin{bmatrix} H_{yD} & H_{yP} \end{bmatrix} \begin{bmatrix} x_D \\ x_P \end{bmatrix}$$

where x_D and x_P are the states associated with the disturbance system and the plant system, and the matrix subscripts D and P refer to the disturbance and plant systems respectively. It is important to note that due to the special structure of the augmented system the plant state and the control input can have no effect on the disturbance variables. The significant and the measurable outputs, however, may directly involve the disturbance state. If these facts are kept in mind, then the augmented system may be described by the same vector differential equation used to describe the simpler system $\mathcal{S}$.

In a similar manner the formulation of the regulator problem may

*The construction of the system given the covariance, a non-trivial problem when v is multi-dimensional, is not considered directly in this dissertation. This construction is closely related to the issues dealt with here in that it is the first step in Anderson's solution of the spectral factorization problem.

be extended to cover the servo problem where the objective of the control design is to minimize the error with which the plant's significant output tracks an arbitrary stochastic reference signal. Shaping filters which account for the statistics of the reference signal are appended to the plant description to form the augmented system $\mathcal{S}_a'$

$$\mathcal{S}a': \begin{bmatrix} \dot{x}_R \\ \dot{x}_P \end{bmatrix} = \begin{bmatrix} F_R & \phi \\ \phi & F_P \end{bmatrix} \begin{bmatrix} x_R \\ x_P \end{bmatrix} + \begin{bmatrix} G_{Rv} & \phi \\ \phi & G_{Pv} \end{bmatrix} \begin{bmatrix} v_R \\ v_P \end{bmatrix} + \begin{bmatrix} \phi \\ G_{Pu} \end{bmatrix} u$$

$$z = \begin{bmatrix} H_{zR} & H_{zP} \end{bmatrix} \begin{bmatrix} x_R \\ x_P \end{bmatrix} + w \qquad y = \begin{bmatrix} H_{yR} & H_{yP} \end{bmatrix} \begin{bmatrix} x_R \\ x_P \end{bmatrix}$$

where x_R is the state associated with the reference system and x_P is the state associated with the plant system which may in fact constitute a system already augmented to account for correlated disturbances. Again it is important to note the special structure of the augmented system. The plant state and the control input can have no effect on the reference variables and, in this case, the reference variables can have no direct effect on the plant. As before, the augmented system $\mathcal{S}_a'$ may be described by the same vector differential equation used to describe the system $\mathcal{S}$.

A different type of generalization of the control problem, the extension of the formulation to include bounds on the variances of plant variables other than the control input components, is discussed in Appendix I. Further generalizations are possible, but those considered here and in Appendix I cover the majority of the natural situations.

The Constrained Minimization Problem

The formal solution of the control synthesis problem begins with the introduction of (generalized) Lagrange multipliers to reduce the synthesis procedure from a constrained minimization problem to a sequence of unconstrained minimization problems. The stochastic norm of the significant plant output as well as the variances of the control input components depend only on the control system $\mathcal{S}_c$. This functional dependence may be used explicitly in the formulation of the problem. Define

$$f(\mathcal{S}_c) = \|y\|^2 \qquad g_i(\mathcal{S}_c) = \|u_i\|^2$$

and the problem becomes the minimization of $f(\mathcal{S}_c)$ subject to the constraints

$$g_i(\mathcal{S}_c) \leq \gamma_i^2$$

It can be shown that this problem falls within the framework of minimizing a convex function subject to convex inequality constraints and that consequently the Kuhn-Tucker theorem, which justifies an extension of the classical Lagrange multiplier technique, is applicable. As in the classical case, a Lagrange function is defined by

$$\varphi(\mathcal{S}_c, \lambda) = f(\mathcal{S}_c) + \sum \lambda_i^2 \left[g_i(\mathcal{S}_c) - \gamma_i^2\right]$$

where λ is a vector of undetermined multipliers. The Kuhn-Tucker theorem* then asserts that for a control system $\hat{\mathcal{S}}_c$ to be optimal (i.e., minimize f ($\mathcal{S}_c$) subject to the imposed constraints) it is necessary and sufficient that there exist multipliers $\hat{\lambda}$ such that

$$\varphi (\hat{\mathcal{S}}_c, \lambda) \leq \varphi (\hat{\mathcal{S}}_c, \hat{\lambda}) \leq \varphi (\mathcal{S}_c, \hat{\lambda})$$

for all admissible λ and $\mathcal{S}_c$.

This condition, known as a saddle point condition, may be used to obtain a solution to the synthesis problem as follows: For fixed $\hat{\lambda}$ the determination of the system $\hat{\mathcal{S}}_c$ which satisfies the right hand inequality is equivalent to finding the unconstrained system which minimizes

$$\bar{\varphi} (\mathcal{S}_c, \lambda) = \mathcal{E} \left[y^T (t)\ y (t) + u^T (t)\ \Lambda^2\ u (t) \right]$$

where $\bar{\varphi} (\mathcal{S}_c, \lambda)$ is defined to be $\varphi (\mathcal{S}_c, \lambda)$ minus the constant $\sum \lambda_i^{\ 2}\ \gamma_i^{\ 2}$ and $\Lambda = \text{diag} \left[\lambda_i \right]$. This unconstrained minimization problem, which will be discussed in the next section, leads to a controller $\hat{\mathcal{S}}_c$ whose properties depend on the fixed $\hat{\lambda}$. It is possible by methods to be

*Kunzi et al. [21] give a proof of the Kuhn-Tucker theorem for functions defined on a finite dimensional vector space. The extension of this proof to the infinite dimensional vector space required in its present application appears to be straightforward but is beyond the scope of this dissertation. The issue is not critical, however, since it is only the sufficiency of the saddle point condition, which is easily proved without regard to the dimensionality of the space, that is used directly in this dissertation.

discussed in the next section to compute the control variances

$$g_i(\hat{\mathcal{S}}_c) = \mathcal{E}[u_i]^2$$

which result from the particular system $\hat{\mathcal{S}}_c$.

Since the multipliers $\hat{\lambda}_i$ determine the control system $\hat{\mathcal{S}}_c$ which, in turn, determines the control variances, these variances may be regarded as functions of the multipliers λ_i. It is shown in Appendix II that, in fact, each control variance is a monotonic decreasing function of its corresponding multiplier. This fact provides the basis for an iterative solution process whereby multipliers that satisfy the left hand inequality of the saddle point condition may be determined. An initial guess of λ is made; the system $\mathcal{S}_c$ which minimizes $\varphi(\mathcal{S}_c, \lambda)$ is determined and the resulting control variances are computed. If a control variance exceeds its bound, then the corresponding multiplier is increased (or vice versa); the minimization problem is resolved, and the control variances are recomputed. In this manner the multipliers are iteratively adjusted until each control bound is exactly met.

Once this process terminates, the left hand inequality of the saddle point condition is satisfied and the sufficiency of this condition guarantees that the resulting controller is the solution of the constrained minimization problem. It is worth pointing out that the necessity of this condition is not used directly; it rather serves only to guarantee that an optimal solution may in fact be characterized in this manner.

The Unconstrained Minimization Problem

Each step in the solution of the constrained minimization problem involves the determination of the unconstrained system $\mathcal{A}_c$ which minimizes the Lagrange function

$$\bar{\varphi}(\mathcal{A}_c, \lambda) = \mathcal{E}\left[y^T(t)\, y(t) + u^T(t)\, \Lambda^2\, u(t) \right]$$

It is well known* that the minimization of this quadratic form in the plant output and the control input may be achieved by a control input which is a constant linear transformation of the plant state

$$u = -H_u x$$

where the transformation or gain matrix is determined from the symmetric positive definite solution of the quadratic matrix equation in P

$$PF + F^T P + H_y^T H_y = PG_u \Lambda^{-2} G_u^T P \tag{1}$$

by

$$H_u = \Lambda^{-2} G_u^T P$$

A sufficient condition (Kalman [23]) for the existence of a unique

---

*Athans and Falb [22] discuss the deterministic case which by the certainty equivalence principle is applicable to the stochastic case.

positive definite solution of (1), which will be referred to as the control quadratic equation, is that the underlying system be controllable.

It is seen that the optimal control input depends explicitly on each component of the system state. These components, however, are not known in general on account of incomplete and imprecise measurements and for this reason a controller consisting simply of the feedback of a transformation of the system state is not implementable and thus not an admissible solution of the optimization problem. On the other hand, the certainty equivalence principle (Schultz [13]) states that the optimal implementable controller consists of the feedback of the same transformation of the best estimate of the system state that may be made from the available measurements.

It is well known (Kalman [7]) that the best estimate of the state of the system $\mathcal{S}$ as it responds to its stochastic inputs is given, on the basis of the measurements z, by the state of the system $\mathcal{S}_e$

$$\mathcal{S}_e: \quad \dot{x}_e = (F - G_z H_z)\, x_e + G_u\, u + G_z\, z$$

where the gain matrix G_z is determined from the symmetric positive definite solution of the quadratic matrix equation in P

$$P\,F^T + FP + G_v \Upsilon^2 G_v^T = P\, H_z^T \Omega^{-2} H_z\, P \tag{2}$$

by

$$G_z = P\, H_z^T \Omega^{-2}$$

A sufficient condition (Kalman [23], using duality) for the existence of a unique positive definite solution of (2), which will be referred to as the estimation quadratic equation, is that the underlying system be observable.

The solutions of the quadratic control and estimation equations determine the optimal controller which constitutes the linear dynamical system $\mathcal{S}_c$

$$\mathcal{S}_c: \quad \dot{x}_c = (F - G_z H_z - G_u H_u)\, x_c + G_z\, z \qquad u = -H_u\, x_c$$

In order to complete the corresponding step in the constrained minimization problem, it is necessary to compute the control variances that result from the controller $\mathcal{S}_c$. At the same time it is of interest to compute the stochastic norm of the plant output to determine the degree of minimization that has been achieved.

The system $\mathcal{S}$ together with the controller $\mathcal{S}_c$ constitute the composite system $\mathcal{S}_m$

$$\mathcal{S}_m: \quad \begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} F & -G_u H_u \\ G_z H_z & F - G_z H_z - G_u H_u \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} G_v & \phi \\ \phi & G_z \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} H_y & \phi \\ \phi & H_u \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix}$$

which may be more compactly represented in the form

$$\mathcal{S}_m: \quad \dot{x}_m = F_m x_m + G_m V_m \qquad y_m = H_m x_m$$

where

$$x_m = \begin{bmatrix} x \\ x_c \end{bmatrix} \qquad v_m = \begin{bmatrix} v \\ w \end{bmatrix} \qquad y_m = \begin{bmatrix} y \\ u \end{bmatrix}$$

and the covariance matrix

$$\mathcal{E}\left[v_m (t) \, v_m^T (t + \tau) \right] = \Upsilon_m^2 \;\; \delta (\tau)$$

is determined in an obvious way from covariance matrices of v and w. Now it is well known (Kalman [7]) that the covariance of the output of a linear system such as the composite system

$$\mathcal{E}\left[y_m (t) \, y_m^T (t) \right] = R$$

may be determined from the symmetric positive definite solution of the linear matrix equation in P

$$P F_m^T + F_m P + G_m \Upsilon_m^2 G_m^T = 0 \qquad (3)$$

by

$$R = H_m P H_m^T$$

A sufficient condition for the existence of a unique positive definite solution of (3), which will be referred to as the linear variance equation, is that the system be stable. As a matter of fact, the composite systems which arise in this dissertation will also be observable. Now the diagonal elements of the covariance matrix R are

$$r_{ii} = \mathcal{E}\left[y_i\,(t)\right]^2 \qquad i = 1,2,\ldots\ell$$

$$r_{(i+\ell)(i+\ell)} = \mathcal{E}\left[u_i\,(t)\right]^2 \qquad i = 1,2,\ldots m$$

from which the stochastic norm of the plant output and the variances of the control input components may be immediately determined. In summary, the solution of the unconstrained optimization problem and the computation of the resulting variances involves the solution of two quadratic matrix equations and one linear matrix equation, all of substantially the same form. In the next chapter it will be shown that these three equations can be reduced to a single canonical form.

CHAPTER III

SYSTEMS IN CANONICAL FORM

In Chapter II it was shown that the crux of the control synthesis problem is the solution of two quadratic matrix equations and one linear matrix equation. The importance of controllability and observability as sufficient conditions for the existence of solutions of the quadratic equations was emphasized. In this chapter certain canonical forms devised by Luenberger [24] for controllable multi-variable systems will be introduced. It will be shown that the transformations that lead to canonical system representations may be used to reduce the three matrix equations to a single canonical form for which concise solutions will be developed in Chapter IV.

Controllability and Observability

A linear dynamical system $\mathcal{S}$

$$\mathcal{S}: \qquad \dot{x} = Fx + Gu \qquad\qquad y = Hx$$

is said to be controllable if for every initial condition x(0) there is a control input u(t) that carries the state x(t) to the origin in a finite amount of time. The system is observable if the state x(t) at the end of a finite amount of time may be determined from observations of the output y(t) and the input u(t) on this interval of time.

In the case of time invariant systems the issues of controllability and observability are simply decided (Athans and Falb [22]): The system $\mathcal{S}$ is controllable if and only if the controllability matrix

$$\mathcal{C} = \left[G,\ FG, \ldots, F^{n-1}\ G\right]$$

has rank n where n is the order of the system. Similarly the system is observable if and only if the observability matrix

$$\mathcal{O} = \left[H^T,\ F^T\ H^T, \ldots,\ (F^T)^{n-1}\ H^T\right]$$

has rank n. Corresponding to the system $\mathcal{S}$ is the dual system

$$\mathcal{S}_D: \qquad \dot{x} = F^T\ x + H^T\ u \qquad\qquad y = G^T\ x$$

It is important to note that if the system $\mathcal{S}$ is observable, then its dual is controllable.

State Transformations and Canonical Forms

It is usually the case in the study of linear dynamical systems that only the relation between the system's inputs and its outputs is important. In this event the representation of the system by a particular vector differential equation is only one among an infinity of similar representations. Let S be a non-singular linear transformation and define

$$x = S\ \tilde{x}$$

Then the system $\mathcal{S}$ given above may be represented by the alternative vector differential equation

$$\mathcal{S}: \qquad \dot{\tilde{x}} = \tilde{F}\ \tilde{x} + \tilde{G}\ u \qquad\qquad y = \tilde{H}\ \tilde{x}$$

where

$$\tilde{F} = S^{-1}\ FS \qquad \tilde{G} = S^{-1}\ G \qquad\qquad \tilde{H} = HS$$

This transformation of the system representation may be symbolized as follows:

$$\{F,\ G,\ H\} \xrightarrow{S} \{\tilde{F},\ \tilde{G},\ \tilde{H}\}$$

Among the infinity of system representations some are preferred for their particular form. If V is the modal matrix of F, that is, the columns of V are n linearly independent (generalized) eigenvectors of F, then

$$\{F,\ G,\ H\} \xrightarrow{V} \{\Lambda\ ,\Gamma\ \ \Delta\}$$

where Λ is a matrix in Jordan canonical form. The transformation V thus has the effect of decoupling the components of the system state. More important representations for this dissertation are those devised by Luenberger for multi-variable systems. These forms are extensions of the familiar phase variable form for single input systems. They are generally applicable to observable and controllable systems.

Suppose the nth order system $\mathcal{S}$ with m inputs and ℓ outputs is controllable. The input matrix G may be written in terms of its columns as follows:

$$G = \left[g_1,\ g_2, \ldots, g_m\right]$$

where the g_i may be assumed to be linearly independent. Since the controllability matrix has rank n, it is possible to choose from its n x m columns n vectors which include all the g_i and which are linearly independent. It is important to note that this choice is not always unique. Let S be the transformation whose columns are made up of the chosen vectors from the controllability matrix arranged as

$$S = \left[g_1,\ Fg_1, \ldots,\ F^{\nu_1 - 1} g_1,\ g_2, \ldots,\ F^{\nu_2 - 1} g_2, \ldots, F^{\nu_m - 1} g_m\right]$$

It is shown in Appendix III that

$$\left\{F,\ G,\ H\right\} \overset{S}{\rightarrow} \left\{\hat{F},\ \hat{G},\ \hat{H}\right\}$$

Where $\hat{F}$ and $\hat{G}$ have the forms shown in Figure 2 and $\hat{H}$ has a general form. This representation of the system will be referred to as pre-canonical form.

If a system in pre-canonical form is further transformed by the

$$
\hat{F} = \begin{bmatrix}
0 & 0 & \cdots & 0 & -\hat{a}_{11}^{(1)} & 0 & 0 & \cdots & 0 & -\hat{a}_{21}^{(1)} & \cdots & 0 & 0 & \cdots & 0 & -\hat{a}_{m1}^{(1)} \\
1 & 0 & & 0 & -\hat{a}_{11}^{(2)} & 0 & 0 & & 0 & -\hat{a}_{21}^{(1)} & & 0 & 0 & & 0 & -\hat{a}_{m1}^{(2)} \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & \cdots & 1 & -\hat{a}_{11}^{(v_1)} & 0 & 0 & \cdots & 0 & -\hat{a}_{21}^{(v_1)} & & 0 & 0 & \cdots & 0 & -\hat{a}_{m1}^{(v_1)} \\
0 & 0 & \cdots & 0 & -\hat{a}_{12}^{(1)} & 0 & 0 & \cdots & 0 & -\hat{a}_{22}^{(1)} & & 0 & 0 & \cdots & 0 & -\hat{a}_{m2}^{(1)} \\
0 & 0 & & 0 & -\hat{a}_{12}^{(2)} & 1 & 0 & \cdots & 0 & -\hat{a}_{22}^{(2)} & & 0 & 0 & & 0 & -\hat{a}_{m2}^{(2)} \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & \cdots & 0 & -\hat{a}_{12}^{(v_2)} & 0 & 0 & \cdots & 1 & -\hat{a}_{22}^{(v_2)} & & 0 & 0 & \cdots & 0 & -\hat{a}_{m2}^{(v_2)} \\
& & & \vdots & & & & & & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & -\hat{a}_{1m}^{(1)} & 0 & 0 & \cdots & 0 & -\hat{a}_{2m}^{(1)} & & 0 & 0 & \cdots & 0 & -\hat{a}_{mm}^{(1)} \\
0 & 0 & & 0 & -\hat{a}_{1m}^{(2)} & 0 & 0 & & 0 & -\hat{a}_{2m}^{(2)} & & 1 & 0 & & 0 & -\hat{a}_{mm}^{(2)} \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & \cdots & 0 & -\hat{a}_{1m}^{(v_m)} & 0 & 0 & \cdots & 0 & -\hat{a}_{2m}^{(v_m)} & \cdots & 0 & 0 & \cdots & 0 & -\hat{a}_{mm}^{(v_m)}
\end{bmatrix}
\qquad
\hat{G} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
& \vdots & \vdots \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & 0 & 0
\end{bmatrix}
$$

Figure 2. Precanonical Form Matrices.

matrix R that is given in Appendix III, then it is shown there that

$$\left\{\hat{F},\ \hat{G},\ \hat{H}\right\} \xrightarrow{R} \left\{\mathcal{A},\ BD,\ C\right\}$$

where $\mathcal{A}$, B, D, and C have the forms shown in Figure 3. It will be noted that $\mathcal{A}$ is naturally partitioned into blocks of dimension $\nu_i \times \nu_j$ (where the ν_i are the exponents appearing in the transformation S) and that the diagonal blocks are companion matrices. Another important feature of these matrices is that D is an mth order upper triangular matrix with unit diagonal elements and thus non-singular. The symbols used to designate these matrices form part of a set of Special Notational Conventions that are defined in their entirety in Appendix IV and will always be used to designate matrices of the form given in Figure 3. The representation of the system $\mathcal{S}$ in terms of these matrices will be referred to as canonical form. Note that the transformation T = SR converts the original system directly to canonical form.

Suppose the system $\mathcal{S}$ is observable. The dual system is thus controllable and may be transformed to canonical form by a transformation T = SR. An easy calculation shows that the transformation T^{-T} gives

$$\left\{F,\ G,\ H\right\} \xrightarrow{T^{-T}} \left\{\mathcal{A}^T,\ C^T,\ D^TB^T\right\}$$

$$
\mathcal{A} = \begin{bmatrix}
0 & 1 & \dots & 0 & & 0 & 0 & \dots & 0 \\
0 & 0 & & 0 & & 0 & 0 & & 0 \\
\vdots & & & \vdots & & \vdots & & & \vdots \\
0 & 0 & & 1 & & 0 & 0 & & 0 \\
-a_{11}^{(1)} & -a_{11}^{(2)} & & -a_{11}^{(\nu_1)} & \dots & -a_{1m}^{(1)} & -a_{1m}^{(2)} & \dots & -a_{1m}^{(\nu_m)} \\
\vdots & & & & & & & & \vdots \\
0 & 0 & \dots & 0 & & 0 & 1 & \dots & 0 \\
0 & 0 & & 0 & & 0 & 0 & & 0 \\
\vdots & & & \vdots & & \vdots & & & \vdots \\
0 & 0 & & 0 & & 0 & 0 & & 1 \\
-a_{m1}^{(1)} & -a_{m1}^{(2)} & \dots & -a_{m1}^{(\nu_1)} & \dots & -a_{mm}^{(1)} & -a_{mm}^{(2)} & \dots & -a_{mm}^{(\nu_m)}
\end{bmatrix}
\qquad
\mathcal{B} = \begin{bmatrix}
0 & \dots & 0 \\
0 & & 0 \\
\vdots & & \vdots \\
0 & & 0 \\
1 & \dots & 0 \\
\vdots & & \vdots \\
0 & \dots & 0 \\
0 & & 0 \\
\vdots & & \vdots \\
0 & & 0 \\
0 & \dots & 1
\end{bmatrix}
$$

$$
\mathcal{C} = \begin{bmatrix}
C_{11}^{(1)} & C_{11}^{(2)} & \dots & C_{11}^{(\nu_1)} & \dots & C_{1m}^{(1)} & C_{1m}^{(2)} & \dots & C_{1m}^{(\nu_m)} \\
\vdots & & & & & & & & \\
C_{m1}^{(1)} & C_{m1}^{(2)} & \dots & C_{m1}^{(\nu_1)} & \dots & C_{mm}^{(1)} & C_{mm}^{(2)} & \dots & C_{mm}^{(\nu_m)}
\end{bmatrix}
\qquad
\mathcal{D} = \begin{bmatrix}
1 & d_{12} & \dots & d_{1m} \\
0 & 1 & & d_{2m} \\
\vdots & & & \vdots \\
0 & 0 & \dots & 1
\end{bmatrix}
$$

Figure 3. Canonical Form Matrices.

This representation of the system will be referred to as dual canonical form.[*]

Transformations for Augmented Systems

A special difficulty presents itself in the case of a plant system that has been augmented to account for correlated disturbances. These systems are controllable in terms of the inputs u and v and consequently may be put into canonical form. The canonical transformations are not unique, however, and an arbitrary transformation does not preserve the special structure of the augmented system which will be important in the solution of the corresponding control equation. The difficulty is resolved as follows: Let the augmented system be represented by $\{F, G, H\}$ where

$$F = \begin{bmatrix} F_D & \phi \\ F_{PD} & F_P \end{bmatrix} \qquad G = \begin{bmatrix} G_{Dv} & \phi \\ G_{Pv} & G_{Pu} \end{bmatrix} \qquad H = \begin{bmatrix} H_{yD} & H_{yD} \end{bmatrix}$$

The disturbance system $\{F_D, G_{Dv}, H_{yD}\}$ and the plant system $\{F_P, G_{Pu}, H_{yP}\}$ are both controllable and may be transformed to precanonical form by the transformations S_D and S_P. It is shown in Appendix III that a matrix S_{PD} can be constructed such that the transformation

[*]T as a superscript is used in this dissertation to indicate transposition. Thus T^{-T} is used to denote $(T^T)^{-1} = (T^{-1})^T$.

$$S = \begin{bmatrix} S_D & \phi \\ S_{PD} & S_P \end{bmatrix}$$

transforms the system to the pre-canonical form $\{\hat{F}, \hat{G}, \hat{H}\}$ while preserving the structure of the system; that is

$$\hat{F} = \begin{bmatrix} \hat{F}_D & \phi \\ \hat{F}_{PD} & \hat{F}_P \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} \hat{G}_{Dv} & \phi \\ \phi & \hat{G}_{Pu} \end{bmatrix}$$

It can be shown that the transformation R introduced above to transform an un-augmented system from pre-canonical to canonical form preserves the structure when constructed in the usual manner.

Transformations of The Matrix Equations

The same transformations devised by Luenberger for system representations may be used to cast the three matrix equations into a single canonical form. Consider first the control equation (1). Let

$$H = H_y \qquad G = G_u \qquad \Gamma = \Lambda^{-1}$$

and the control equation becomes

$$PF + F^T P + H^T H = PG\Gamma^2 G^T P \tag{4}$$

Assume that the system $\{F, G, H\}$ is controllable and let T be a transformation that transforms the system to canonical form

$$\{F,\ G,\ H\} \overset{T}{\rightarrow} \{\mathcal{A},\ BD,\ C\}$$

Define $\hat{P} = T^T PT$ and note that since T is non-singular, $\hat{P}$ is unique, symmetric, and positive definite if P is. Substitute $T^{-T}\,\hat{P}T^{-1}$ into (4), premultiply by T^T, and postmultiply by T to obtain

$$\hat{P}\ (T^{-1}\ FT) + (T^T\ F^T\ T^{-T})\ \hat{P} + (T^T\ H^T)\ (HT)$$

$$= \hat{P}\ (T^{-1}G) \quad \Gamma^2\ (G^T\ T^{-T})\ \hat{P}$$

But using the properties of the transformation T, this equation may be reduced to the canonical quadratic equation

$$\hat{P}\mathcal{A} + \mathcal{A}^T\ \hat{P} + C^T\ C = \hat{P}\ BD\,\Gamma^2\ D^T\ B^T\ \hat{P} \tag{5}$$

The gain matrix H_u may be written in terms of the solution $\hat{P}$ of the canonical quadratic equation as follows

$$H_u = \Lambda^{-2}\ G_u{}^T\ P = \Gamma^2\ (G^T\ T^{-T})\ \hat{P}\ T^{-1} = \Gamma^2 (D^T\ B^T\ \hat{P})\ T^{-1} = \Gamma^2\ K\ T^{-1}$$

where K which will be referred to as the canonical gain matrix is defined by

$$K = D^T B^T \hat{P}$$

The estimation equation (2) may be converted to precisely the same canonical quadratic equation. Let

$$G = G_v \Upsilon \qquad H = H_z \qquad \Gamma = \Omega^{-1}$$

and the estimation equation becomes

$$P F^T + F P + G G^T = P H^T \Gamma^2 H P \tag{6}$$

Assume that the system $\{F, G, H\}$ is observable and let T^{-T} be a transformation that transforms the system to dual canonical form

$$\{F, G, H\} \xrightarrow{T^{-T}} \{A^T, C^T, D^T B^T\}$$

Again let $\hat{P} = T^T PT$. Subsbitute $P = T^{-T} \hat{P}T^{-1}$ into (6) and perform the same multiplications as before to obtain

$$\hat{P} (T^{-1} F^T T) + (T^T FT^{-T}) \hat{P} + (T^T G) (G^T T) = \hat{P} (T^{-1} H^T) \Gamma^2 (HT^T) \hat{P}$$

But using the properties of the transformation T^{-T} this equation reduces to the canonical quadratic equation (5). The gain matrix G_z may be written in terms of the solution P of the canonical quadratic equation as follows

$$G_z{}^T = \Omega^{-2} H_z \hat{P} = \Gamma^2 (H_z S^{-T}) \hat{P} T^{-1}$$

$$= \Gamma^2 (D^T B^T \hat{P}) \quad S^{-1} = \Gamma^2 K T^{-1}$$

where K is defined as above.

Finally the variance equation (3) may be similarly dealt with. Let

$$F = F_m \qquad G = G_m \Upsilon_m \qquad H = H_m$$

and the variance equation becomes

$$P F^T + F P + G G^T = 0 \tag{7}$$

Assume that the system $\{F, G, H\}$ is observable and is transformed to dual canonical form by the transformation T^{-T}. Substituting $P = T^{-T} \hat{P} T^{-1}$ into (7) and using the properties of the transformation, the variance equation may be reduced to the canonical linear equation

$$\hat{P} \mathcal{A} + \mathcal{A}^T \hat{P} + C^T C = 0 \tag{8}$$

The covariance matrix R may be written in terms of the solution of the canonical linear equation as follows

$$R = H\,P\,H^T = (HT^{-T})\,\hat{P}\,(T^{-1}\,H^T) = D^T\,B^T\,\hat{P}\,B\,D \;=\; K\,B\,D$$

where K is defined as above.

Again the augmented systems present special difficulties. The transformation of the control equation (1) depended upon the controllability of the system $\{F,\ G_u,\ H_y\}$, but if the system is an augmented system, the control input u can have no effect on either the disturbance or the reference states, and thus the system is not controllable. The resolution of this difficulty will be discussed in terms of a system augmented to account for correlated distrubances. Let

$$F = \begin{bmatrix} F_D & \phi \\ F_{PD} & F_P \end{bmatrix} \qquad G = \begin{bmatrix} G_{Dv} & \phi \\ G_{Pv} & G_{Pu} \end{bmatrix} \qquad \Gamma = \begin{bmatrix} \phi & \phi \\ \phi & \Lambda^{-1} \end{bmatrix}$$

and the control equation (1) is equivalent to (4) as before since $G\Gamma^2\,G^T = G_u\,\Lambda^{-2}\,G_u{}^T$. With these substitutions the system $\{F,\ G,\ H\}$ is controllable and the control equation may be reduced to the canonical quadratic equation. The fact that the matrix $D\,\Gamma$ is singular in this case will present later difficulties, however, and should be kept in mind.

In summary, the canonical transformations devised by Luenberger have led to the reduction of the three crucial matrix equations to what is essentially a single canonical form. The next chapter will be devoted to the solution of the canonical matrix equation.

CHAPTER IV

SOLUTION OF THE CANONICAL MATRIX EQUATION

In Chapter III it was shown that Luenberger's canonical transformations may be used to convert the crucial matrix equations to a single canonical form. In this chapter a concise solution of the canonical matrix equation will be developed. The chapter begins with the presentation of a theorem by Potter [20] that leads to the solution of the matrix equation in terms of the eigenvectors of an associated Hamiltonian system. Methods based on a set of Special Notational Conventions that have been devised for dealing with systems in canonical form are then used to obtain explicit expressions for the eigenvectors of the Hamiltonian system in terms of the solutions of a much simpler reduced homogeneous system. These expressions are then used to write down the solution of the canonical matrix equation.

It will be recalled from Chapter II that the solutions of the matrix equations are required only because the gain and variance matrices discussed in that chapter are expressed in terms of those solutions. In this chapter it will also be shown that these matrices may be obtained without the full solution of the matrix equation.

The Hamiltonian System

The solution of the matrix equations (1), (2), and (3) is equivalent in each case to the solution of $n(n+1)/2$ simultaneous quadratic or linear equations. In this form the solution of these equations is an extremely difficult task even for moderately large n. A considerable reduction in the effort required to obtain these solutions is afforded, however, by a theorem due to Potter.

Associated with the nth order quadratic matrix equation

$$F^T P + P F + H^T H = P G \Gamma^{-2} G^T P$$

is the 2nth order Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} -F & G \Gamma^{-2} G^T \\ H^T H & F^T \end{bmatrix}$$

It is well-known that the eigenvalues of this matrix are symmetrically located about and distant from the imaginary axis. There are thus exactly n eigenvalues with negative real parts (i.e., Hurwitz eigenvalues). Let $\Lambda = \operatorname{diag}\left[\lambda_i\right]$ be the nth order diagonal matrix whose elements are these eigenvalues and suppose that V and U are nth order square matrices that satify

$$\begin{bmatrix} -F & G\Gamma^{-2} G^T \\ H^T H & F^T \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} V \\ U \end{bmatrix} \Lambda$$

In other words, the columns of V and U give the first and second halves of the eigenvectors of $\mathcal{H}$ that are associated with the n Hurwitz eigenvalues. Potter's theorem asserts that if V^{-1} exists, then

$$P = U V^{-1}$$

is the unique, symmetric positive definite solution to the corresponding quadratic matrix equation. When $G = \phi$ the quadratic equation reduces to the linear equation, and an easy modification of Potter's theorem gives an analogous result for the linear system.

Potter did not resolve the matter of the non-singularity of V. In the remainder of this chapter, however, concise expressions for the eigenvectors of the Hamiltonian matrix associated with the canonical matrix equation will be developed, and it will be shown that the resulting V matrix is non-singular. Since a matrix equation can be transformed to canonical form precisely when the sufficiency conditions (controllability or observability of the underlying system) for the existence of a solution are met, the remainder of this chapter constitutes a constructive proof that the non-singular matrix required by Potter's theorem does in these cases exist.

Modal Matrices and Transfer Functions

In order to introduce a number of the Special Notational Conventions and also to establish certain results that will be used in subsequent sections of this chapter, methods based on these conventions will be applied here to the determination of the modal matrix and the transfer function of a system in canonical form.

Suppose that $\mathcal{S}$ is an nth order system with m inputs and ℓ outputs in canonical form

$$\dot{x} = \mathcal{A} x + B\, Du \qquad y = Cx$$

Associated with the system $\mathcal{S}$ are the m x m matrix $\mathcal{F}(\lambda)$ and the ℓ x m matrix $\mathcal{G}(\lambda)$ whose polynomial elements

$$f_{ij}(\lambda) = a_{ij}^{(1)} + a_{ij}^{(2)} \lambda + \dots + a_{ij}^{(\nu_j)} \lambda^{\nu_j - 1} + \delta_{ij} \lambda^{\nu_j}$$

$$g_{ij}(\lambda) = c_{ij}^{(1)} + c_{ij}^{(2)} \lambda + \dots + c_{ij}^{(\nu_j)} \lambda^{\nu_j - 1}$$

are defined in terms of the scalar elements $a_{ij}^{(k)}$ and $c_{ij}^{(k)}$ of the matrices $\mathcal{A}$ and C. Also associated with the system is the m x m matrix

$$\Xi(\lambda) = \text{diag}\left[\xi(\lambda)\right]$$

whose diagonal elements are the ν_j -vectors that are defined in the glossary of Special Notational Conventions given in Figure 10. It is shown in Appendix IV that

$$v = \Xi(\lambda) p$$

is an eigenvector of $\mathcal{A}$ when p is a non-zero solution of the reduced homogeneous sytem

$$\mathcal{F}(\lambda)\, p = 0$$

where p is an m-vector with scalar elements. A solution of the reduced homogeneous system will exist if and only if

$$\text{Det}\left[\mathcal{H}(\lambda)\right] = F(\lambda) = 0$$

where $F(\lambda)$ is an nth order polynomial which is in fact the characteristic polynomial of the matrix $\mathcal{A}$. If the roots λ_j of $F(\lambda)$ are distinct, then there are n solutions $p^{(j)}$ of the reduced homogeneous systems in terms of which an eigenvector $v^{(j)}$ of the matrix $\mathcal{A}$ may be explicitly expressed. It should be pointed out that the determination of the solutions $p^{(j)}$ is inherently simpler than the direct determination of these eigenvectors since the order of the reduced system is equal to the number of system inputs rather than the number of system states.

The modal matrix of the system $\mathcal{S}$ is given by

$$V = \left[v^{(1)}, v^{(2)}, \ldots, v^{(n)}\right]$$

which will be referred to as a composite Vandermonde matrix. V is nonsingular by virtue of the fact that its columns are eigenvectors associated with distinct eigenvalues, but this fact is proved directly in Appendix V where the case of repeated eigenvalues is also discussed.

The transfer function $\mathcal{H}(s)$ of the multi-variable system $\mathcal{S}$ is an $\ell \times m$ matrix of rational functions which gives the relation between the Laplace transforms of the inputs and the outputs of the system.

$$y(s) = \mathcal{H}(s)\, u(s)$$

In contrast to the vector differential equation representation of a system the transfer function representation is unique. It is shown in Appendix IV that

$$\mathcal{H}(s) = \mathcal{G}(s)\, \mathcal{F}^{-1}(s)\, D$$

The simplicity of this expression indicates the close relation between the canonical form and the transfer function representation of the system.

Solution of the Canonical Quadratic Equation

The same methods used to determine the modal matrix and transfer function of a system in canonical form may be used to obtain the eigenvectors of the Hamiltonian matrix corresponding to the canonical quadratic equation. Let V and U be the square matrices required by Potter's theorem; then the columns v and u of these matrices must satisfy the homogeneous Hamiltonian system

$$(\lambda I - \mathcal{A})\, v + B\, D\, \Gamma^{2}\, D^{T} B^{T} u = 0$$

$$C^{T} C\, v + (\lambda I + \mathcal{A}^{T})\, u = 0$$

Set $v = \Xi(\lambda)p$ and define $q = B^{T}u$ where p and q are m-vectors with scalar elements. Note that the premultiplication of u by B^{T} has the effect of "picking out" certain elements of u. It is shown in Appendix IV that v as defined above satisfies the Hamiltonian system when p and q satisfy the reduced Hamiltonian system

$$\mathfrak{F}(\lambda)\, p + D\,\Gamma^{2}\, D^{T} q = 0 \tag{9}$$

$$\mathfrak{G}^{T}(-\lambda)\; \mathfrak{G}(\lambda)\, p - \mathfrak{F}^{T}(-\lambda) q = 0$$

Let

$$q = -D^{-T}\,\Gamma^{-2}\, D^{-1}\,\mathfrak{F}(\lambda) p$$

and it is easy to verify that (9) is satisfied when p is a non-zero solution of the reduced homogeneous system

$$\mathfrak{I}(\lambda)\, p = \left[\mathfrak{G}^{T}(-\lambda)\;\mathfrak{G}(\lambda) + \mathfrak{F}^{T}(-\lambda)\, D^{-T}\Gamma^{-2}\, D^{-1}\,\mathfrak{F}(\lambda)\right] p = 0$$

Note that by defining

$$\mathcal{E}(\lambda) = \begin{bmatrix} (D\,\Gamma)^{-1}\,\mathfrak{F}(\lambda) \\ \mathfrak{G}(\lambda) \end{bmatrix} \tag{10}$$

$\mathfrak{I}(\lambda)$ may be written as

$$I(\lambda) = \mathcal{E}^T(-\lambda) \quad \mathcal{E}(\lambda) \tag{11}$$

For this system to have a solution it is required that

$$\mathrm{Det}\left[I(\lambda) \right] = F(\lambda^2) = 0$$

where $F(\lambda^2)$ is a 2nth order polynomial with exactly n Hurwitz roots λ_j. If these roots are distinct, then there are n solutions $p^{(j)}$ and $q^{(j)}$ to the reduced Hamiltonian system in terms of which corresponding columns $v^{(j)}$ and $u^{(j)}$ of V and U may be explicitly expressed. Here again it is inherently simpler to determine the solutions of the reduced system than to directly determine the eigenvector components $v^{(j)}$ and $u^{(j)}$. In this case $p^{(j)}$ is the solution of a homogeneous system whose order is equal to the number of system inputs and $q^{(j)}$ is determined easily from $p^{(j)}$. On the other hand, the direct determination of the eigenvector components involves the solution of a homogeneous system whose order is twice the number of system states.

The definition given above specifies each element of $v^{(j)}$ in terms of λ_j and $p^{(j)}$. V is seen to be a composite Vandermonde matrix and thus non-singular. A method for computing the elements of $u^{(j)}$ is discussed in Appendix IV, but these computations are not necessary since the canonical gain matrix K is given by

$$K = D^T B^T P = D^T (B^T U) \ V^{-1} = D^T Q \, V^{-1}$$

where

$$Q = \begin{bmatrix} q^{(1)} & q^{(2)} & \dots & q^{(n)} \end{bmatrix}$$

This completes the solution of the quadratic equation.

Solution of the Canonical Linear Equation

The solution of the canonical linear equation involves the determination of v and u that satisfy the simpler Hamiltonian system

$$(\lambda I - \mathcal{A}) v = 0$$

$$C^T C v + (\lambda I + \mathcal{A}^T) u = 0$$

It is clear that v is simply an eigenvector of $\mathcal{A}$ and consequently V is the modal matrix that was determined above. It is shown in Appendix IV that u may be determined from the solution of the reduced system

$$\mathcal{F}(\lambda) p = 0 \qquad (9')$$

$$\mathcal{B}^T(-\lambda) \mathcal{B}(\lambda) \quad - \quad \mathcal{F}^T(-\lambda) q = 0$$

where p and q are m-vectors. Again, however, it is not necessary to complete the solution for u since the gain matrix

$$K = D^T Q V^{-1}$$

where the columns of Q are the n solutions $q^{(j)}$ to the reduced system (9′) which is discussed in Appendix IV.

Solution of the Augmented Quadratic Equation

An essential step in the solution of the quadratic equation was the expression of the m-vector q in terms of the m-vector p by

$$q = - D^{-T} \Gamma^{-2} D^{-1} \mathfrak{J}(\lambda) p$$

This is not possible in the case of augmented systems since Γ^2 is singular, but a solution to the Hamiltonian system may nevertheless be obtained without difficulty.

The treatment of augmented systems is discussed in its entirety in Appendix VI. On account of the notational complexity, however, only the main features of the solution will be discussed here. It is shown in Appendix VI that the problem separates into three parts. The first part involves the determination of the feedback matrix H_{Pu} which is identical to the gain matrix determined above from the solution of the un-augmented control equation. The other two parts involve the determination of the disturbance feedforward matrix H_{Du} and the reference feedforward matrix H_{Ru}. These two problems may be considered separately since the disturbance matrix does not depend on the reference system and vice-versa.

This completes the development of concise solutions of the linear and quadratic matrix equations. These solutions are important theoretically because their simplicity affords insight into the nature of the control problem and because they constitute a constructive proof of the existence of the non-singular matrix required by Potter's theorem. This constructive proof takes on added significance for the following reasons: the demonstration that controllability is a sufficient condition for the existence of the solution of the control equation (Kalman [23]) is difficult; it is analytic in nature and depends upon the behavior of the differential Ricatti equation as the interval of integration becomes infinite. The constructive proof, on the other hand, together with Potter's theorem forms a simple, direct algebraic proof of the sufficiency of this condition. The primary importance of the solutions obtained in this chapter are, for the purposes of this dissertation, their computational implications. It will be shown in the next chapter that they lead to significant reductions, beyond those afforded by Potter's theorem, in the computational effort associated with the control synthesis problem.

CHAPTER V

COMPUTATIONAL PROCEDURES

In the previous chapters of this dissertation it has been shown that the control synthesis problem may be reduced to a sequence of unconstrained minimization problems that involve the solution of three matrix equations. It has been shown that these equations may be reduced to a single canonical form, and concise solutions have been developed for the equations in this form. Taking advantage of these solutions it has been shown that the gain and variance matrices that are required for the solution of the unconstrained minimization problem may be obtained without the full solution of the matrix equations.

In this chapter the entire synthesis problem will be discussed from a computational point of view. A computational procedure for the determination of the gain and variance matrices which is based on the results of Chapters III and IV will be presented. It will be shown that the procedure leads to important computational savings beyond those afforded by Potter's theorem. An iterative procedure that has been devised for the determination of the bounding Lagrange multipliers that enter as parameters in the matrix equations will also be presented. Together these two procedures may be used for the solution of the control synthesis problem. In the course of this chapter comments will be made on the experience that has been accumulated in the use of the computational methods that are presented.

Computation of the Gain Matrices

The concise solutions for the matrix equations provide the basis for an efficient computational procedure for the determination of the optimal control and estimation gain matrices and also for the determination of the variance matrix that are required in the solution of the control problem. The computational procedure will be discussed here as it applies to the determination of the control gain matrix of an unaugmented system with m inputs and outputs and an order $n = m\nu$, and whose canonical form involves a system matrix that is naturally partitioned into blocks of dimension x .

Computation of the Canonical Transformations

The first two steps in the procedure are the numerical determination of the transformations S and R which are defined in Equations (12) and (15) of Appendix III and which convert the system $\{F, G, H\}$ underlying the matrix equation (4) to precanonical and canonical form. The product transformation $T = SR$ is also required.

Step 1: The precanonical transformation S is determined by a modification of Danilevskii's method for the calculation of the characteristic equation of a matrix. The procedure, which is described in detail in Appendix VII, involves the performance of n successive, simply determined transformations. The required precanonical transformation S is given by the product of the simple transformations. The procedure might be considered a departure from the straightforward approach, but it has two important advantages:

1). A non-trivial aspect of the determination of the precanonical transformation S is the choice of its columns so as to guarantee their linear independence. The modification of Danilevskii's method deals with this problem automatically.

2). The generation of the precanonical transformation from its constructed definition (12) is possible, but the determination of the coefficients $a_{ij}^{(k)}$ of the precanonical form would require a matrix inversion and two matrix multiplications. The modification of Danilevskii's method leads directly to these coefficients and to the inverse of the transformation.

By counting the number of operations required to compute the transformation S, its inverse, and the elements of the precononical matrices $\hat{F}$, $\hat{G}$, and $\hat{H}$, it can be verified that the total is less than $2\ m^3 \nu^3 + 3\ m^3 \nu^2$.

<u>Step 2</u>: The transformation R which carries the system from precanonical to canonical form is computed directly from its definition (15), a process that is also discussed in Appendix VII. The total number of operations required to compute the transformation R, its inverse and the product transformation T = SR and its inverse as well as the elements of the canonical matrices $\mathcal{A}$, C, and D is $(m + 1)\ m^2 \nu^3$.

<u>The Reduced Hamiltonian System</u>

The results of these two steps are the transformation that converts the matrix equation (4) to canonical form, its inverse, and the elements of the matrices $\mathcal{A}$, C, and D which appear in the canonical matrix equation (5). The elements of these matrices also constitute by simple rearrangement the coefficients of the polynomial matrices $\mathcal{F}(\lambda)$ and $\mathcal{G}(\lambda)$ which enter in the reduced Hamiltonian system. The required n solutions to this system are determined in the following four steps:

<u>Step 3</u>: Compute $\mathcal{E}(\lambda)$ and $\Phi(\lambda)$ which are defined in Equations (10) and (11) of Chapter IV. The matrix $\bar{D}_{-}$ whose inverse is required for the computation of $\mathcal{E}(\lambda)$ is an mth order diagonal matrix and thus trivially inverted. The computation of $\mathcal{E}(\lambda)$ involves a sequence of ν scalar matrix multiplications and may be accomplished in $m^3 \nu$ operations. The formation of the product

$$\Phi(\lambda) = \mathcal{E}^T(-\lambda) \quad \mathcal{E}(\lambda)$$

involves m^2 (m + 1) multiplications of the polynomial elements of $\mathcal{E}(\lambda)$ defined by

$$\varphi_{ij}^{(k)}(\lambda) = e_{ki}(-\lambda)\, e_{kj}(\lambda)$$

each of which requires ν^2 operations for a total of m^2 (m + 1) ν^2 operations.

<u>Step 4</u>: Determine the characteristic equation $F(-\lambda^2)$ and its n Hurwitz roots. In contrast to the practice in computing scalar determinants the characteristic equation

$$F(-\lambda^2) = \text{Det}\left[\ \Phi(\lambda)\right]$$

is computed on the basis of its definition as the sum of certain products of the polynomial elements of $\Phi(\lambda)$. The computation requires m! m^2 [illegible]$^2/2$ operations.

The determination of the n roots $x_i = \lambda_i^2$ of the polynomial equation

$$F(-x_i) = F(-\lambda_i^2) = 0$$

is a classical computational problem which, according to Bareiss (25), may be accomplished in only 8 m^2 [illegible]^2 operations. The extraction of the Hurwitz roots λ_i of the roots x_i of F (x) involves negligible operations.

<u>Step 5</u>: Determine the n non-zero solutions $p^{(i)}$ of the reduced homogeneous system

$$\Phi(\lambda_i)\, p^{(i)} = 0$$

and the corresponding

$$q^{(i)} = D^{-T}\, \Gamma^{-2}\, D^{-1}\, \mathfrak{F}(\lambda_i)\, p^{(i)}$$

It is, in fact, more convenient to compute

$$\hat{q}^{(i)} = \Gamma^{2}\, D^{T}\, q^{(i)} = D^{-1}\, \mathfrak{F}(\lambda_i)\, p^{(i)} = \hat{\mathfrak{F}}(\lambda_i)\, p^{(i)}$$

since the gain matrix may then be written

$$H = \left[\hat{q}^{(1)}, \ldots, \hat{q}^{(n)}\right] V^{-1}\, T^{-1} = \hat{Q}\, V^{-1}\, T^{-1}$$

The computation of the $p^{(i)}$ requires first the evaluation of $\Phi(\lambda_i)$ which may be accomplished in $2\, m^2\, \nu$ operations and then the solution of the resulting m x m homogeneous system which may be accomplished in less than $m^3/3$ operations. The total number of operations required for the n solutions $p^{(i)}$ is $m^3\, \nu\, (m + 6\, \nu)/3$ and the $q^{(i)}$ may be determined in $m^3\, \nu^2$ operations.

<u>Step 6</u>: The final step in the computational procedure is the calculation of the gain matrix H from the matrix Q and the composite Vandermonde matrix by standard matrix operations. The Vandermonde matrix is trivially generated from the solutions of the reduced homogeneous system; its inversion and premultiplication by Q consumes $m^3\, (2\nu^3 + \nu^2)$ operations.

<u>Comparison with Conventional Procedures</u>

The number of operations required for the computational procedure based on the concise solutions of the canonical matrix equation is given in Table 1. If it is assumed that ν is large relative to m, then the total number of operations is given by

$$N = 5\, m^3\, \nu^3 + (4\, m^3 + (m-1)!/2)\, m^3 + 8\, m^2)\, \nu^2$$

On the other hand, the solution for the gain matrix on the basis of Potter's theorem alone requires the solution of an eigenvalue eigenvector

Table 1. Arithmetic Steps

Step	Exact No. of Operations	Approx. No. of Operations	No. of Critical Operations
1	$m^3\ (2\nu^3 + 3\ \nu^2)$	$2\ m^3\nu^3$	$2\ m^3\ \nu^3$
2	$(m + 1)\ m^2\ \nu^3$	$m^3\ \nu^3$	---
3	$(m + 1)\ m^2\ \nu^2$	$m^3\ \nu^2$	---
4a	$m!\ (m-1)\ m\ \nu^2/2$	$m!\ m^2\ \nu^2/2$	---
4b	$8\ m^2\ \nu^2$	$8\ m^2\ \nu^2$	$8\ m^2\ \nu^2$
5a	$m^3\ \nu\ (m + 6\nu)/3$	$2\ m^3\ \nu^2$	$2\ m^3\ \nu^2$
5b	$m^3\ \nu^2$	$m^3\ \nu^2$	
6	$m^3\ (2\ \nu^3 + \nu^2)$	$2\ m^3\ \nu^3$	$2\ m^3\ \nu^3$

problem of order 2n. Using a reliable general purpose procedure* for this purpose the computation of the gain matrix requires 130 $m^3\nu^3$ operations. If $m = 4$ and $\nu = 10$, there is a 25 to 1 advantage for the new procedure.

The advantage of the new procedure is even more pronounced when considered from two different points of view:

*In the computational studies performed in connection with this dissertation a procedure programmed by J. M. Varah [27] was used. The procedure begins with the transformation of the matrix to Hessenburg form and the use of the QR algorithm to determine the matrix eigenvalues. The eigenvectors are determined by inverse iteration. Parlett [26] states that the eigenvalue solution requires 16 n^3 operations. Varah's timing data show that the eigenvector solution requires an equal number of operations.

1). In contrast to situations such as the computation of the discrete fourier transform where execution time is of controlling importance, the number of operations required to carry out the present computational procedure is also important in determining the manner in which round-off error effects the stability of the calculation. A reduction in the number of operations associated with the critical phases of a calculation can mean the difference between the success and the failure of the calculation. The critical phases of the new computational procedure are a part of the determination of the canonical transformation requiring $m^3 \lambda^3$ operations, the solution of the polynomial equation requiring $8\ m^2 \lambda^2$ operations, and the inversion of the composite Vandermonde matrix requiring $m^3 \lambda^3$ operations. The critical phases of the conventional method are the solution of the eigenvalue eigenvector problem requiring $128\ m^3 \lambda^3$ operations and the inversion of the eigenvector component matrix requiring $m^3 \lambda^3$ operations. Thus for m large the new method affords a 60 to 1 reduction in the number of operations associated with the critical phases of the calculation.

2). When the computational procedure is used repetitively in conjunction with the iterative process to be discussed in the next section for the adjustment of the Lagrange Multipliers even more important advantages accrue. The only steps in the new method which require operations proportional to $m^3 \lambda^3$ are the determination of the canonical transformation, the inversion of the Vandermonde matrix and several matrix multiplications. Since the canonical transformation does not depend on the values of the multipliers it may be determined once before the iterative process begins, and the matrix multiplications can, by proper arrangement of the calculation, be deferred until the process converges. This leaves only the inversion of the Vandermonde matrix to be performed repetitively. On this basis the new method affords a reduction of approximately 130 to 1 in the computational effort required to calculate the gain matrix.

Computational Experience

The computational procedure described in this section has been implemented as a computer program for the Burroughs B-5500 and a great deal of experience has been accumulated in its application to the practical examples that are discussed in the next chapter and to various test cases.

The test cases were designed to determine the accuracy of the algorithm as it applied to the solution of a quadratic matrix equation in canonical form. An addition to the algorithm which provided for the determination of the eigenvector component matrix U was used to compute P and to check the accuracy with which the canonical matrix equation (5) was satisfied.

A fifth order example in which the coefficients of the polynomial elements of $\mathfrak{F}(\lambda)$ and $\mathfrak{G}(\lambda)$ and the elements of the matrix

$$H = D\Gamma$$

were chosen as integers is given in Appendix VIII. (This choice, it should be noted, determines all the parameters of the canonical matrix equation.) In this example the canonical equation was satisfied to within a part per ten million. Results equally as good were obtained with similar seventh and eighth order examples.

In another series of test cases of order 10, 15, and 20 the roots of the polynomial elements of $\mathcal{F}(\lambda)$ and $\mathcal{G}(\lambda)$ were chosen randomly from the unit square centered in the complex plane and the elements of the matrix H were chosen randomly on the interval (-1, 1). In the twentieth order example the canonical matrix equation was satisfied to within a part per ten thousand.

The complete computational procedure for the determination of the gain matrix was also used extensively in connection with the practical examples that are discussed in the next chapter. Many of the runs were checked against results obtained from a general purpose eigenvalue eigenvector program. The agreement between the two methods has been within a few tenths of a per cent. For example the matrix giving the covariance of the vertical velocity α and the acceleration a of the optimally controlled longitudinal system was computed to be

$$\mathcal{E}\left[a,\ \alpha\right] \begin{bmatrix} a \\ \alpha \end{bmatrix} = \begin{bmatrix} 22.207 & 14.637 \\ 14.638 & 17.072 \end{bmatrix},\ \begin{bmatrix} 22.207 & 14.637 \\ 14.637 & 17.072 \end{bmatrix} \times 10^{-4}$$

by the new and conventional methods respectively.

Cases have been encountered, however, in which either or both of the methods have failed. The failures of the new method have always been associated with two aspects of the procedure:

1). All complete failures of the new method have been due to the subroutine used to solve the polynomial equation. The solution of this equation is inherently simpler than the solution of the eigenvalue problem required in the conventional method, but it is nevertheless a non-trivial computational problem. The computer subroutines that were available for the studies conducted in this dissertation did not represent the state of the art (as exemplified, for instance, by the work of Bareiss [25]) and for this reason the difficulties encountered here should not reflect on the efficacy of the new method.

2). A number of marginal failures of the new method were due to inaccuracies in the computation of the canonical transformation: errors on the order of five per cent in the calculation of the coefficients $a_{ij}^{(k)}$ of the canonical form were encountered in some cases where the number of states associated with a given input exceeded seven. Forwarning of this difficulty is contained in a paper by Frank [33] where examples are cited of the failure of Danilevskii's method when used to compute the characteristic equation of a tenth order matrix using eight figure arithmetic. The author feels that the usefulness of the algorithm in its present form could be greatly extended by the use of double precision arithmetic and that a modification of the algorithm to improve the conditioning of the transformation would extend its usefulness even more. Should these expedients prove inadequate the transformation could be computed on the basis of the eigenvectors of the system matrix as described by Mufti [34].

In spite of occasional failures the overall experience with the new method has been excellent.

Application to Variations of the Problem

The computational procedure for the solution of the matrix equations has been described here as it applies to the determination of the control and estimation gain matrices of an un-augmented system. Modifications of this procedure may also be used to handle augmented systems and to solve the linear matrix equation that determines the variances of the outputs of a linear system.

Determination of The Bounding Multipliers

The computational procedure given in the previous section provided the basis for the solution of the unconstrained minimization problem that was stated in Chapter II. In order to complete the synthesis procedure it is necessary to devise a method for choosing the multipliers λ_i of the Langrangian function so as to satisfy the left-hand inequality of the saddle-point condition that was also stated in Chapter II.

It will be recalled that this condition is satisfied when the control variances g_i ($\mathcal{S}_c$) satisfy their bounds and that these variances may be regarded as functions of the multipliers since their specification leads, through the solution of the unconstrained minimization problem to the determination of the controller $\mathcal{S}_c$. Making explicit use of this functional dependence the control variances may be written

$$h_i \ (\lambda_1, \ldots, \lambda_m)$$

The determination of the multipliers thus involves the solution of the m inequalities

$$h_i(\lambda) \le \gamma_i^2$$

in the m unknowns λ_i.

It is difficult to make rigorous inferences into the nature of the functions $h_i(\lambda)$ and thus to devise a scheme for the solution of the inequalites which can be guaranteed without reservation to be effective. An examination of the functions as they arise in connection with a control problem where complete and precise measurements of the components of the plant state are available, however, does provide considerable insight into the nature of the functions. In this case the dynamical system that is associated with the estimator portion of the controller is absent and the control inputs are given by

$$u_i = (1/\lambda_i)^2 \, g_{ui}^T \, P \, x$$

where P is the solution of the control equation (1). A number of assertions can be made, with varying degrees of rigor, concerning the variances of these control inputs:

1). It is well-known that as a multiplier λ_i approaches zero, the corresponding control variance becomes unbounded. It is also clear that so long as the resulting system is stable a control variance will approach zero as its multiplier approaches infinity.

2). No attempt will be made to prove the statement, but it is also clear that the control variances are continuous functions of the multipliers.

3). As the multiplier associated with a control variance is decreased and the other multipliers are held fixed, the control variance will increase due to its monotonicity and the variance of the plant state will decrease due to the increased disposable control power. The solution P of the control equation will also change since the multiplier appears as a parameter in this equation. There is reason to believe, however, that the predominant effect is due to the factor $(1/\lambda_i^2)$ and that the control variance is given approximately by the expression

$$h_i(\lambda) = k_i \lambda_i^{-\alpha_i}$$

4). The control inputs to a physical system are usually devised so as to render the system controllable, and it is not unreasonable to suppose that when multiple inputs are available that they affect strongly distinct parts of the system. This assumption leads to the conclusion that a control variance is relatively independent of the multipliers that are not associated with it.

Together these assertions may be taken to mean that the functions $h_i(\lambda)$ are approximated by

$$h_i(\lambda) = k_i(\lambda) \lambda_i^{-\alpha_i(\lambda)}$$

where the $k_i(\lambda)$ and the $\alpha_i(\lambda)$ do not vary radically with λ. Assertions 1). and 2). also assure that the inequalities will be satisfied as equalities. On the basis of these assumptions the following iterative computational procedure was devised: Two initial guesses

$$\lambda^{(0)} = \left\{\lambda_1^{(0)}, \ldots, \lambda_m^{(0)}\right\}$$

$$\lambda^{(1)} = \left\{\lambda_1^{(1)}, \ldots, \lambda_m^{(1)}\right\}$$

are made for the multipliers and the functions $h_i\ (\lambda^{(0)})$ and $h_i\ (\lambda^{(1)})$ are evaluated. Successive values of the multipliers are then determined by logarithmic interpolation. In other words, let

$$x_i^j = \log \lambda_i^{(j)}$$

$$y_i^j = \log h_i\ (\lambda^{(j)}) \qquad \bar{y}_i = \log \gamma_i^2$$

and for $j = 2, 3, \ldots$ set

$$x_i^{j+1} = x_i^j - (x_i^j - x_i^{j-1})\ (y_i^j - \bar{y}_i)/(y_i^j - y_i^{j-1})$$

until convergence is obtained.

Computational Experience

The efficacy of the procedure described in this section, based as it is on intuitive assertions, can only be judged experimentally. In fact, the procedure has been used extensively in connection with the practical examples discussed in the next chapter, and the experience has been, without exception, excellent. Only one case requiring six iterations was encountered; all other cases converged to within two per cent in five iterations or less. A typical example, that of the dual input lateral control problem, is given in Table 2. In this example the logs of the bounds were taken as $\bar{y}_1 = -8.0$ and $\bar{y}_2 = -10.0$. The plots given in Figure 4 of the logs of the variances versus the logs of the associated multipliers demonstrates that the assumptions of logarithmic linearity and independence are well justified.

Table 2. Multiplier Convergence.

Iteration	x_1	y_1	x_2	y_2
1	1.00	-7.44	1.00	-8.59
2	0.00	-8.12	0.00	-9.76
3	0.17	-7.94	-0.21	-10.02
4	0.06	-8.04	-0.19	-9.99
5	0.11	-7.99	-0.19	-10.00

In summary the computational procedures discussed in the two sections of this chapter may be put together to form a combined algorithm for the synthesis of optimal control systems. The algorithm begins with the calculation of the estimation gain matrix, which need not be recalculated, and the initial choice of the bounding multipliers. The iterative loop

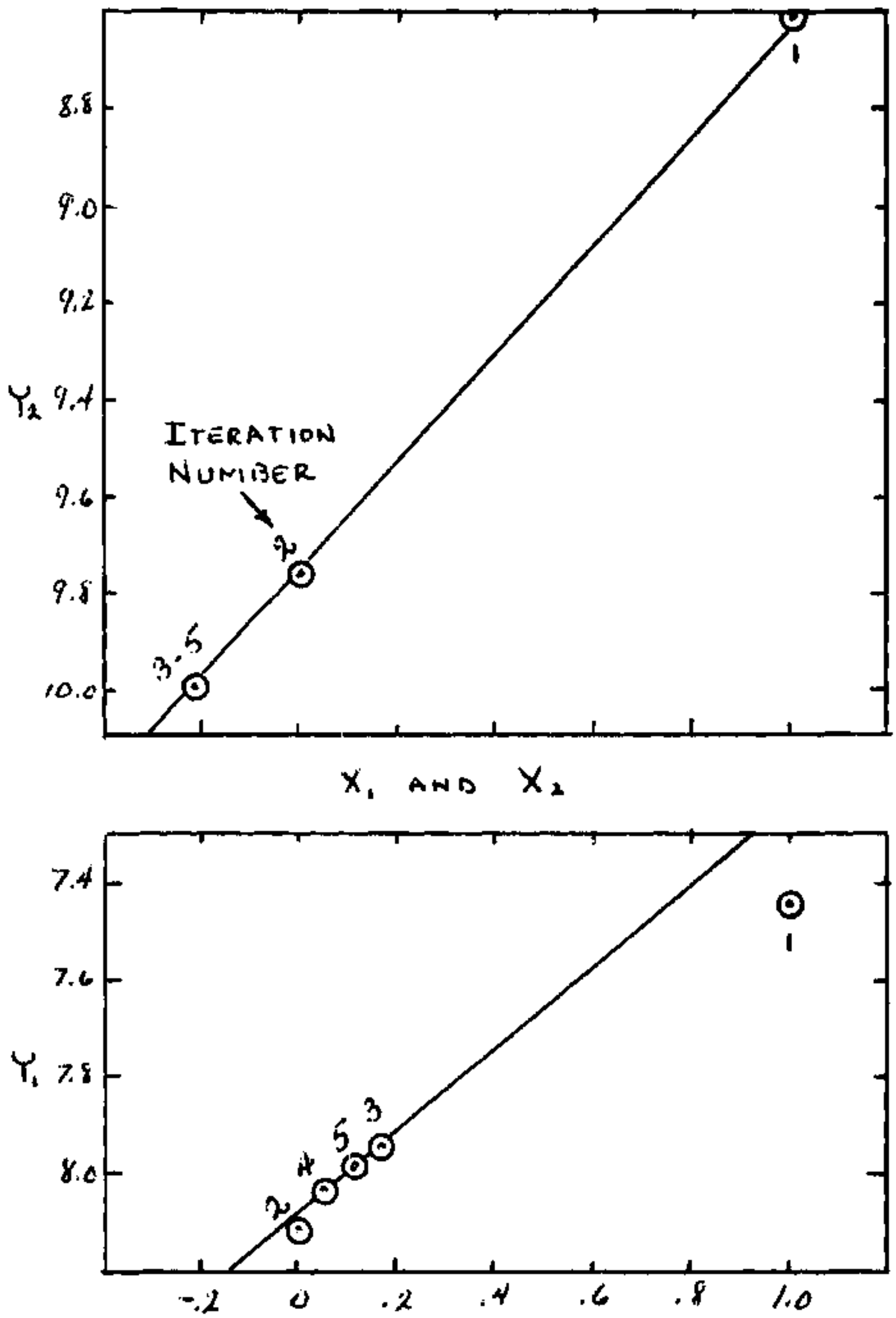

Figure 4. Multiplier Convergence.

which adjust the multipliers so that the control bounds are met is then entered. Each execution of the iterative loop requires the solution of the control and variance equations. When the iterative process terminates the parameters of the optimal control system have been determined. An example of the application of the computational procedure to the synthesis of a practical control system is given in the next chapter.

CHAPTER VI

A PRACTICAL EXAMPLE

The computational procedure for the synthesis of linear stationary control systems developed in this dissertation was presented in Chapter V. In this chapter the procedure is applied to the design of lateral and longitudinal autopilots for a jet transport subject to gust disturbances. The study was undertaken, first of all, to test the computational procedure in its application to a real and non-trivial design problem. The computational experience gained in this application forms a part of the experience used in the evaluation of the computational procedure that was presented in Chapter V.

The study was also undertaken to demonstrate the effectiveness of the procedure and the underlying theory as a practical method for designing optimal control systems. The complete design study is presented in this chapter. The longitudinal control system designed by the methods of this dissertation is compared with a system designed by classical methods and shown to result in significantly improved performance.

The Aircraft Control Problem

The design of an autopilot for an aircraft subject to gusts disturbances was chosen for consideration on account of its moderate complexity and the extent to which the assumptions which underlie the design procedure are fulfilled. Blakelock [28] gives an account of classical design procedures as applied to the design of aircraft autopilots. The methods of this dissertation have been applied to the same aircraft, whose behavior is typical of a jet transport, that Blakelock considered, and the resulting longitudinal control system has been compared with that designed by Blakelock.

The aircraft when considered as a rigid body has three rotational and three translational degrees of freedom each leading to an equation of motion. When the perturbations about an equilibrium flight path are small, the equations of motion separated in to two uncoupled linear systems. The longitudinal system determines the motion of the aircraft in a vertical plane. This motion is specified in terms of the vertical and forward horizontal velocities, α and u, and rotations about the pitch axis ϑ as shown in Figure 5. The longitudinal motion of the aircraft is disturbed by the vertical gust velocity α_g and is controllable by the elevator deflection δ_ϵ.

The lateral system determines the motion of the aircraft in a horizontal plane. This motion is specified in terms of the transverse horizontal velocity β and rotations about the roll and yaw axes, φ and ψ, as shown in Figure 5. The lateral motion is disturbed by the transverse gust velocity β_g and the transverse derivative of the vertical gust velocity φ_g. The motion is controllable by the rudder and aileron deflections, δ_r and δ_a.

The gust disturbances of the aircraft motion are regarded in current aerodynamic practice as due to homogeneous, isotropic, stochastic turbulence (Etkin [29]). The gust inputs to the equations of motion are thus stochastic processes. The spectra of the vertical and transverse gust velocities are found to be well represented by the expression

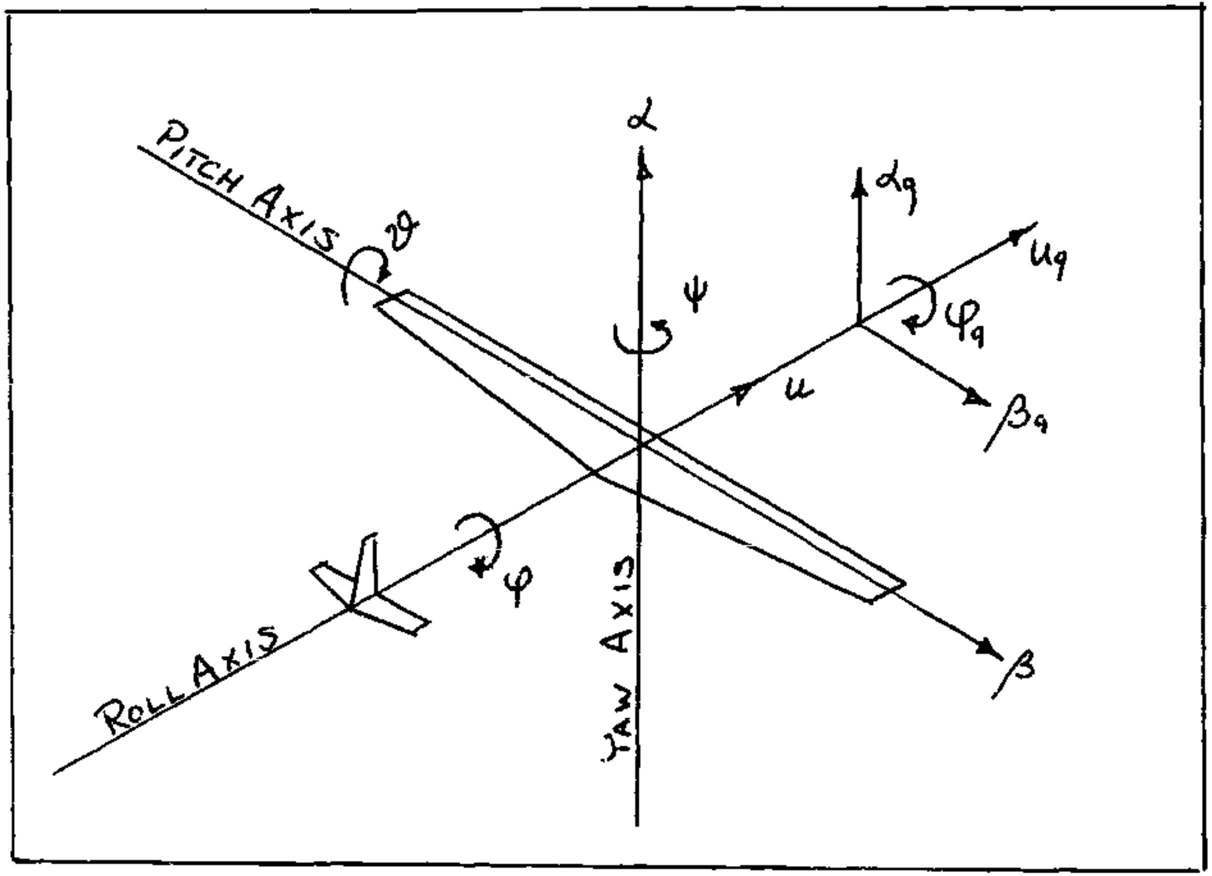

Figure 5. Aircraft Schematic Diagram.

$$\Phi(\omega) = \frac{\sigma^2 \omega_o (\omega_o^2 + 3\omega^2)}{(\omega_o^2 + \omega^2)^2}$$

where σ is the rms gust velocity and ω_o is a constant that is related to the scale of the turbulence and the speed of the aircraft. Arguments involving isotropy lead to the conclusion that the spectrum of φ_g is identical to that of the derivative of α_g and thus given by

$$\Phi'(\omega) = \omega^2 \; \Phi(\omega)$$

These spectra together with the equations of motion determine the free or uncontrolled motion of the aircraft. The purpose of the control system is to modify this free motion through the control inputs to meet certain objectives.

The Longitudinal Control System

The objective of the longitudinal control system was taken as the minimization of the vertical velocity α subject to bounds on the vertical acceleration

$$a = \alpha - \vartheta$$

and the elevator deflection δ_e.

The Augmented System

The first step in the implementation of the design procedure is the expression of the longitudinal system in state variable form and the construction of a linear dynamical system to account for the statistics of the vertical gust velocity as was discussed in the second section of Chapter II. It is easily verified that the system

$$\mathcal{S}_D: \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_o^2 & -2\omega_o \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$\alpha_g = \begin{bmatrix} \omega_o & \sqrt{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has an output α_g with spectrum $\Phi(\omega)$ as defined above when its input v has the flat spectrum

$$\psi(\omega) = \omega_o \sigma^2$$

This system (modified as explained in Appendix IX) is appended to the plant description to form the augmented system that is shown in Figure 6. The components of the state variable of the augmented system are given by

$$x^T = \begin{bmatrix} x_1 & x_2 & x_3 & u & \alpha & \vartheta & \dot{\vartheta} \end{bmatrix}$$

where the x_i are states associated with the shaping filter.

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -4.32 & -13.32 & -13.08 & 0 & 0 & 0 & 0 \\ 0.0099 & 0.0284 & 0 & 0.0064 & 0.0284 & -0.0537 & 0 \\ -0.1121 & 0.3208 & 0.0034 & -0.1074 & -0.3237 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.405 & -1.147 & 2.658 & 0.0115 & -1.169 & 0 & 0.481 \end{bmatrix}$$

$$G_v = \begin{bmatrix} 0 \\ 0 \\ 0.266 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad G_u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.018 \\ 0 \\ 1.379 \end{bmatrix} \quad H_z^T = \begin{bmatrix} -0.1121 & 0 & 0 \\ 0.3208 & 0 & 0 \\ 0.0034 & 0 & 0 \\ -0.1074 & 0 & 0 \\ 0.3237 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad H_y^T = \begin{bmatrix} 0 & -0.1121 \\ 0 & 0.3208 \\ 0 & 0.0034 \\ 0 & -0.1074 \\ 1 & 0.3237 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_u^{BL} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.41 & 1.19 \end{bmatrix}$$

$$H_u = \begin{bmatrix} 0.597 & -1.187 & -0.028 & 0.014 & 8.671 & 0.950 & 3.288 \end{bmatrix}$$

$$G_z = \begin{bmatrix} -0.373 & -4.48 & 1.63 \\ 0.084 & 32.1 & 104.2 \\ -0.040 & 1110.8 & 3904.5 \\ -0.068 & -0.200 & 0.121 \\ 0.201 & 0.330 & 2.085 \\ 0.273 & 0.592 & 0.191 \\ 0.054 & 13.1 & 23.2 \end{bmatrix}$$

Figure 6. Longitudinal System Matrices.

The Conventional Control System

The system matrices shown in Figure 6 correspond to the same aircraft for which Blakelock designed a control system by classical methods. Blakelock chose a controller consisting of the uncompensated feedback of pitch angle and pitch rate according to gains that were adjusted by means of the root locus procedure to give satisfactory damping. The gains chosen are given by the gain matrix H_u^{BL} that is shown in Figure 6.

Blakelock did not, of course, consider that control problem in the stochastic setting in which it is considered here, but as a further step in this study the statistical response of his system to flight in vertical gusts with an rms velocity of 10 fps was determined. Using the variance equation (3) as it applies to the aircraft under the control of Blakelock's system the rms values of the vertical velocity, the acceleration, and the elevator deflection were computed.* These results are shown in Table 3. At the same time the velocity and the acceleration of the free system were also computed.

The computational procedure developed in this dissertation was applied to two formulations of the longitudinal control problem. In the first formulation it was assumed that exact measurements of all of the components of the system state were available; in the second formulation it was assumed that only limited and imprecise measurements were available. In both cases the control power and the vertical acceleration

*It should be emphasized that these rms values are given directly by the solution of the variance equation. No "Montecarlo" simulation of the system is required.

Table 3. Control System Performance.

	$\bar{\alpha}$ (ft/sec)	$\bar{a}$ (ft/sec^2)	$\bar{\delta}_c$ (degrees)
Free	12.0	4.1	---
Blakelock	3.9	2.9	0.85
Bounded Acc.	2.6	2.9	0.85
Unbounded Acc.	1.4	4.3	0.85

were limited by the design to those of Blakelock's system.

<u>The Optimal Exact Measurement Control System</u>

When exact measurements of all the components of the system state are available the optimal control system consists simply of the feedback of a linear transformation of these states. The gain matrix which minimizes the vertical velocity subject to the bounds imposed on the control power and the vertical acceleration was determined by the computer process to be the matrix H_u shown in Figure 6. It is seen from Table 3 that the optimal system results in a 30 per cent reduction in the vertical velocity.

The spectra of the velocities of the free system, Blakelock's system, and the optimal system are shown in Figure 7. The spectra of the control input for Blakelock's system and the optimal system are

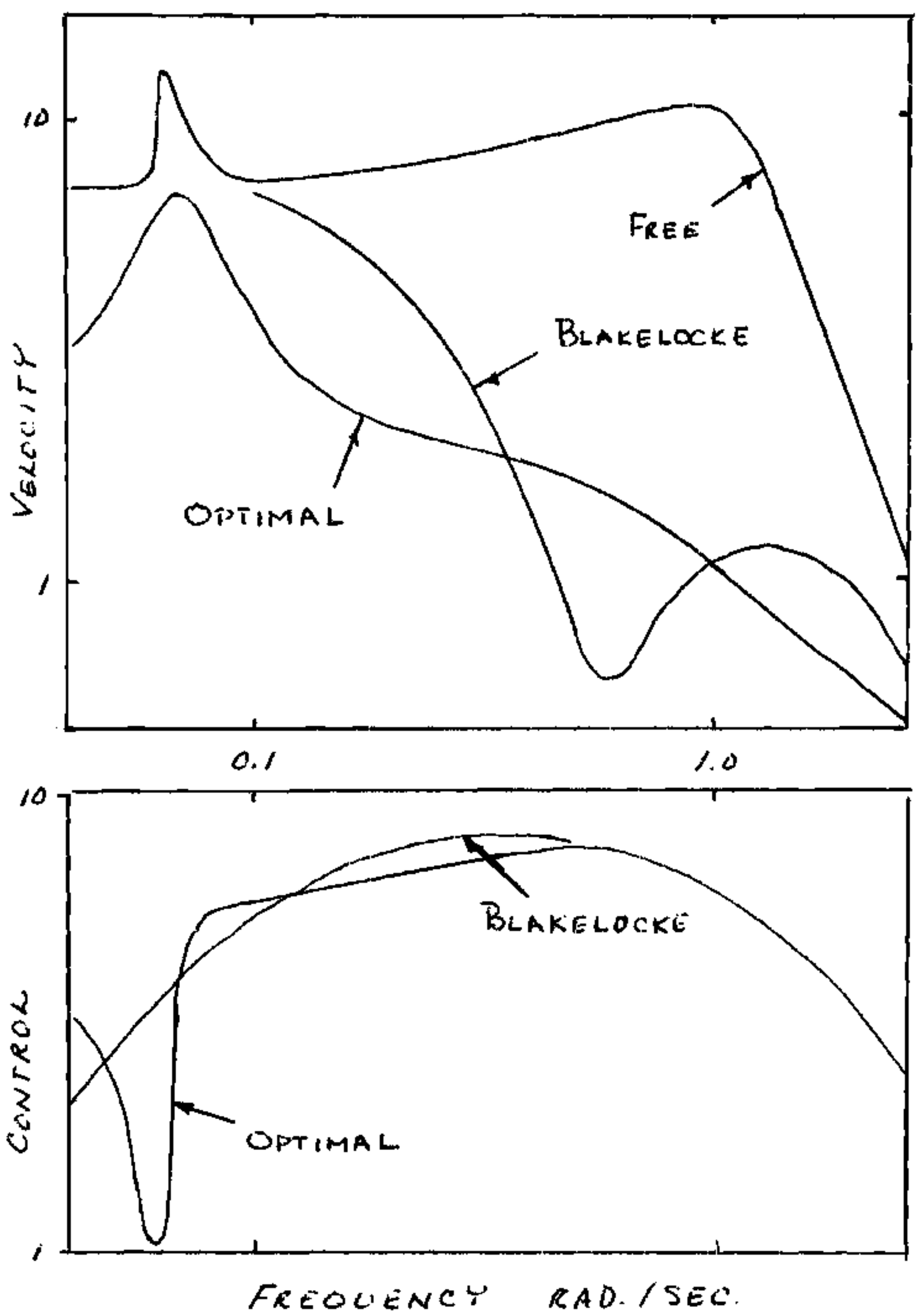

Figure 7. Velocity and Control Spectra.

also shown. It is observed that at high frequencies the magnitude (but not necessarily the phase) of the two control inputs are almost identical. This implies that the optimal control input does not impose unreasonable demands on the control servo's high frequency response.

In this study the multiplier associated with the control input was adjusted automatically by means of the procedure described in Chapter V. The multiplier associated with the acceleration was determined as follows: The functional

$$f\left(\mathcal{S}_c, \delta\right) = \delta^2 \|\alpha\|^2 + (1 - \delta)^2 \|a\|^2$$

was minimized, subject to the control constraints, for $\delta = 0.0$, 0.2, 0.4, 0.6, 0.8, 1.0 and the resulting rms values of the velocity and the acceleration were plotted as a function of the parameter δ as shown in Figure 8. From this figure it was determined that a value of $\delta = 0.6$ leads to the minimization of the velocity subject to the bound on the acceleration. This procedure has the advantage that it establishes the trade-off that exists between the minimization of the velocity and the acceleration. It is seen, for instance, that if the acceleration bound is removed, that a 60 per cent improvement can be obtained in the velocity, but the acceleration exceeds that of the free system. The effect of a 50 per cent increase in the control power is also shown in Figure 8.

The Optimal Limited Measurement Control System

A control system based on limited and imprecise measurements was then designed. In this case the control system contains a dynamical

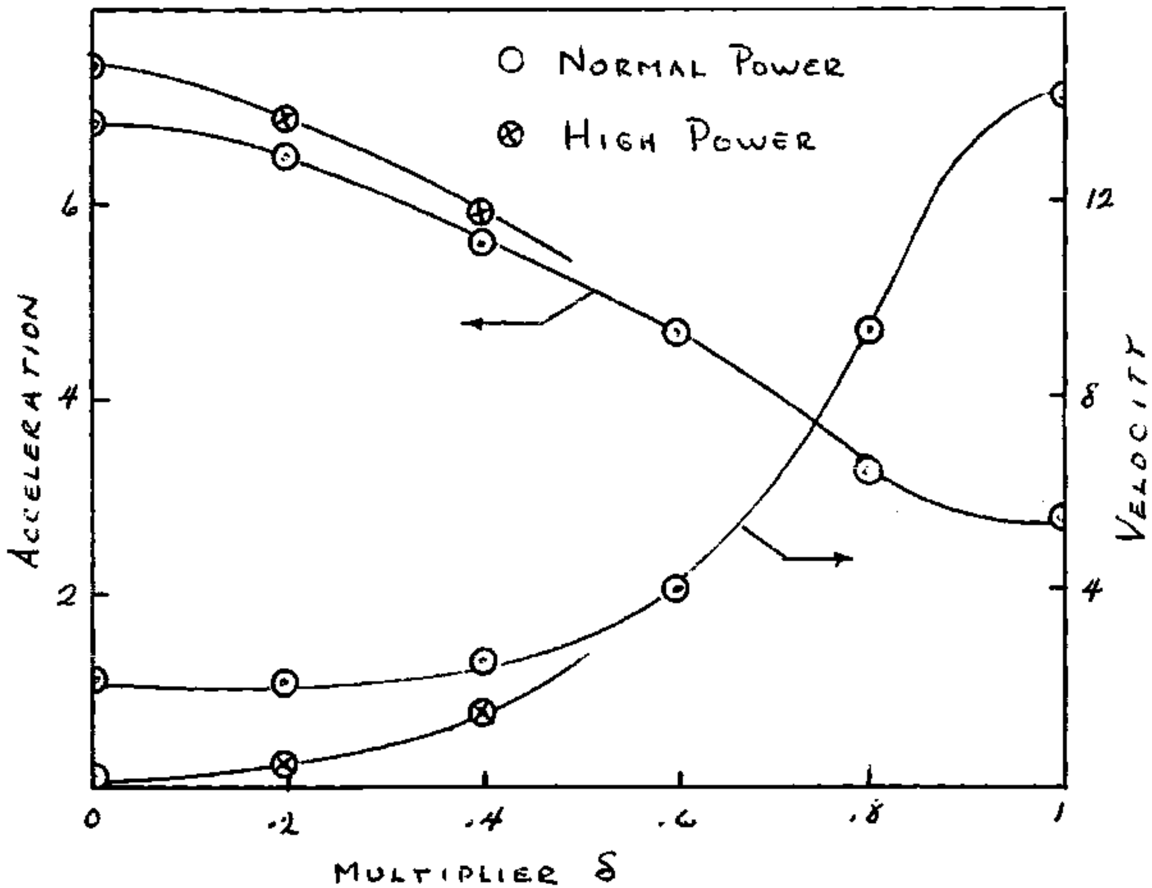

Figure 8. RMS Vertical Velocity & Acceleration.

portion whose function is to provide the best estimate of the optimal control signal on the basis of the available measurements. The measurable outputs were taken as the pitch angle, the pitch rate and the vertical acceleration. The measurements are given by the output matrix H_z shown in Figure 6.

The inaccuracies that are to be expected in these measurements are due to such sources as mechanical vibrations, electronic noise, and instrument non-linearities. The difficult problem of modeling these inaccuracies was handled by imposing a white corrupting noise whose level reflects the grade of measuring instrument on each measurement. The noise levels were chosen as follows: the energy of the measured variables is known to be concentrated in the frequency range between zero and 1 radian per second. If it is assumed that the spectrum of each variable is given by

$$\Phi(\omega) = 2\,\sigma^2/(1+\omega^2)$$

and that a white measurement noise with level $v = 2\,\beta^2\,\sigma^2$ is superimposed, then an optimally designed instrument can measure the variable with an rms eroor of $\sqrt{2}\,\beta\,\sigma$. Thus a 5 per cent instrument with a 3 σ full scale reading which is typical of those currently available corresponds roughly to $\beta = 0.01$.

Calculations were made for $\beta = 0.02$, 0.01, and 0.005 to determine the gains associated with the estimator portion of the controller and the effects the limited measurements have on the control system performance. The results which are summarized in Table 4 lead to the

Table 4. Estimator Performance

β	Approx. Grade (%)	$\bar{\alpha}^2$ $(ft/sec)^2$	$\bar{a}^2$ $(ft/sec^2)^2$	$\bar{\delta}_\epsilon^{\,2}$ $(degrees)^2$
0.020	7	7.17	8.76	0.714
0.010	5	6.95	8.79	0.714
0.005	3	6.99	8.80	0.715
Exact Meas.		6.78	8.82	0.717

the important conclusion that with instruments of the grade currently available the controller with limited measurements is essentially as good as the controller that operates on the basis of complete and exact measurements. Besides its own importance this fact is computationally important in that the estimator portion of the controller need not be considered in the iterative determination of the bounding multipliers.

Another interesting result of these calculations is the manner in which the estimator poles vary with the parameter β. It is seen in Table 5 that there are five poles whose magnitudes are comparable to the aircraft's cut-off frequency of approximately 1 radian per second and that these poles do not change radically as β varies. On the other hand, there is a pair of complex poles with extremely large magnitudes that move out pronouncedly along the 45 degree line of the second and third quadrants of the complex plane as β decreases. The presence of

Table 5. Estimator Poles

β					
0.020	-0.004,	-0.009,	-0.241,	$-1.61 \pm j\ 0.66$,	$-24.5 \pm j\ 22.0$
0.010	-0.002,	-0.010,	-0.241,	$-1.61 \pm j\ 0.66$,	$-34.8 \pm j\ 30.9$
0.005	-0.001,	-0.010,	-0.241,	$-1.61 \pm j\ 0.66$,	$-50.8 \pm j\ 42.0$

these poles and the corresponding high estimator gains, as exemplified by the gain matrix G_z shown in Figure 6 for $\beta = 0.01$, make the implementation of the controller difficult. There is the possibility, however, that the high frequency components of the control signal that are associated with the large magnitude poles have no sensible effect on the aircraft response and that the controller complexity and the difficulties associated with its implementation could be reduced by the removal of these poles. Should this be the case, it would also mean that the approximate manner in which the instrument inaccuracies were handled is adequate since the remaining poles of the estimator system do not vary radically with the noise level.

The Lateral Control System

The lateral control system was studied primarily to determine the behavior of the iterative multiplier solution as it applied to a two input system. The control objective was taken as the minimization of the sum of the mean-squared yaw rate, roll rate, and transverse velocity subject only to bounds on the rudder and aileron deflections. The augmented system shown in Figure 9 is eighth-order and has two disturbance inputs as well as the two control inputs. The filters which account for the statistics of β_g and φ_g are simple second-order systems which are included in the augmented system shown. The iterative calculation of the bounding multipliers and the optimal control gains as it applied to the lateral control system is the example that was cited in Chapter V. The calculation consumed less than two minutes of computer time. The calculated gain matrix which is shown in Figure 9 resulted in a well damped, satisfactory controller.

This concludes the study of lateral and longitudinal aircraft control systems. In addition to providing a realistic test of the computational aspects of the design procedure, the study has demonstrated the effectiveness of the procedure as a practical design method. It is particularly significant that once the control problem is formulated, the optimal control system is arrived at by the methods of this dissertation without the intuitive intervention required when classical methods are employed. The new method is thus applicable to systems whose complexity precludes the use of classical methods.

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.194 & -0.792 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.194 & -0.968 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.622 & 2.34 & 0.399 & 1.47 & 0 & -2.029 & 0.470 & -2.092 \\ -0.473 & 1.80 & 0.011 & 0.042 & 0 & -0.058 & -0.272 & 1.655 \\ -0.032 & 0.127 & 0 & 0 & 0.073 & 0 & -1 & -0.127 \end{bmatrix}$$

$$G_v = \begin{bmatrix} 0 & 0 \\ 0.0248 & 0 \\ 0 & 0 \\ 0 & 0.0274 \\ 0 & 0 \\ -0.0116 & -0.0564 \\ 0.0067 & -0.0016 \\ 0 & 0 \end{bmatrix} G_u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.481 & 22.00 \\ -1.380 & -0.17 \\ 0.036 & 0 \end{bmatrix} \quad H_z^T = H_y^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_u = \begin{bmatrix} 0.160 & 0.582 & -0.005 & -0.018 & 0.076 & 0.025 & -1.107 & 0.348 \\ 0.032 & 0.126 & 0.016 & 0.057 & 0.063 & 0.741 & -0.034 & -0.018 \end{bmatrix}$$

Figure 9. Lateral System Matrices.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

In this dissertation a complete computational procedure for the synthesis of a broad class of optimal multi-input linear stationary control systems is developed. The computational procedure is based on modern control and estimation theory which establishes that the optimal controller is a linear dynamical system whose parameters are determined by two quadratic matrix equations and a saddle point condition. New and improved numerical procedures are developed for the solution of the matrix equations and the satisfaction of the saddle point condition.

The scheme for the solution of the quadratic matrix equations begins with the use of certain transformations devised by Luenberger to convert the equations to a single canonical form. A theorem due to Potter is used to write the solution of the canonical equation in terms of the eigenvectors of an associated Hamiltonian system whose order is twice the number of system states. Explicit expressions are then obtained for these eigenvectors in terms of the solutions of a reduced homogeneous system whose order equals the number of system inputs. A concise solution of the canonical matrix equation is written in terms of these explicit expressions. A computational algorithm based on this scheme reduces the number of operations required to solve the quadratic matrix equation by a factor of 25.

An iterative procedure for the satisfaction of the saddle point condition has also been developed. This iterative procedure requires the repetitive solution of one of the quadratic matrix equations. Taking advantage of the concise solutions developed for the matrix equation the only computationally significant operations which must be performed repetitively as part of the iterative process are the solution of a polynomial equation and the inversion of a matrix. In this setting the number of operations required to solve the matrix equation is reduced by a factor of 130.

The procedures developed in this dissertation have been implemented as computer programs and applied to a number of test cases and to the design of lateral and longitudinal control systems for an aircraft subject to gust disturbances. The computational experience accumulated in these applications has been excellent. The algorithm for the solution of the quadratic matrix equation has been demonstrated to be effective for equations of order as large as 20. Many of the matrix equation solutions have been checked against those obtained by conventional methods and found to be in close agreement.

The convergence of the iterative procedure for the satisfaction of the saddle point condition has been shown experimentally to be quite rapid: an eighth order, two input example converged in 5 iterations and required less than two minutes of computer time.

A complete design study of the aircraft control systems was presented in this dissertation. The longitudinal control system was compared with a control system designed by classical methods and shown to result in a 30 per cent improvement in performance. An interesting result of the study is the conclusion, based on the mathematical models used, that currently available instruments are of sufficiently high grade that their errors lead to negligible degredation in control system performance.

A particularly significant aspect of the computational procedure developed in this dissertation is demonstrated by the aircraft control system study: once the control problem is formulated the procedure constitutes a step-by-step, rationally based process for arriving at the optimal control system without the intuitive intervention required when classical methods are used.

The concise solutions of the quadratic matrix equations that have been developed in this dissertation have theoretical importance in addition to their computational importance. The application of Potter's theorem requires that a matrix containing components of the eigenvectors of the Hamiltonian system be non-singular. It is shown in this dissertation that when the matrix equation is in canonical form, the eigenvector component matrix is a generalization of the Vandermonde matrix; it is proved in an appendix that this matrix is non-singular. This fact together with Potter's theorem constitute a simple algebraic proof that the quadratic matrix equation has a unique, positive definite solution when the underlying system is controllable.

Recommendations

The results obtained in this dissertation have resolved some issues and raised others. Computationally the most important issue that has been raised is the possibility that the parameters of the optimal control system can be determined without the inversion of the eigenvector component matrix V. It will be recalled from the first section of Chapter V that this is the only step requiring operations proportional to n^3 that is performed repetitively as part of the iterative multiplier adjustment. In Appendix V it was shown that a certain unique factorization

$$\Phi(\lambda) = \mathfrak{F}_o^T(-\lambda)\,\mathfrak{F}_o(\lambda)$$

exists for the polynomial matrix appearing in the reduced homogeneous system. The existence of this factorization rests on a result concerning polynomial matrices which is constructively proved. It turns out that the canonical gain matrix K can be written easily in terms of the polynomial matrix $\mathfrak{F}_o(\lambda)$ and the question arises of whether or not it is computationally feasible to determine $\mathfrak{F}_o(\lambda)$ directly. The constructive nature of the result underlying the factorization guarantees that it is possible to do so, but the number of steps required and the stability of the calculation remain open questions that bear further investigation.

A somewhat different issue raised by the dissertation is the possibility of extending the results obtained, which apply only to stationary systems, to non-stationary systems. The explicit expressions developed for the eigenvectors of the Hamiltonian system lead to concise solutions of the differential Ricatti equation as it arises in connection with time invariant systems considered on finite time intervals. It should be possible to extend the results to time varying systems by means of time-varying canonical transformations.

Another issue is the following: in its present form the computational procedure presented in this dissertation leads to control systems that are more complex than classically designed systems, and that are difficult to implement because of the high gains associated with the estimator portion of the control system. Intuitive considerations strongly suggest, however, that these problems can be eliminated by approximating the transfer functions of the optimal control system with that of a simpler system.

Finally, it is pointed out that the methods of this dissertation are ideally suited to the study of the finite difference approximations of systems with distributed parameters. It is suggested that these methods and the associated computer programs afford an excellent opportunity for numerical experimentation with systems of this type.

APPENDIX I

BOUNDING THE PLANT VARIABLES

In this appendix the generalization of the control problem to include bounds on the plant variables is discussed. The only difficulties that arise in this generalization are notational complexity and certain differences that arise in the satisfaction of the saddle point condition.

The bounding of the plant variables begins with the expansion of the definition of the significant plant output to include the variables to be bounded. Let

$$y_1 = H_{y1}\, x$$

be the plant variable to be minimized and

$$y_i = H_{yi}\, x \qquad\qquad i = 2, 3, \ldots, r$$

be the variables to be bounded, where each of these outputs has dimension l_i. The control problem may then be stated as the minimization of

$$f_1\,(\mathcal{S}_c) = \|y_1\|^2 = \mathcal{E}\left[y_1^T\,(t)\; y_1\,(t)\right]$$

subject to the additional constraints

$$f_i(\mathcal{S}_c) = \|y_i\|^2 \leq \beta_i^2 \qquad i = 2, \ldots, r$$

This constrained minimization problem may be dealt with by defining the Lagrangian to be

$$\varphi(\mathcal{S}_c, \delta, \lambda) = f_1(\mathcal{S}_c) + \sum_{2}^{r} \delta_i^2 \left[f_i(\mathcal{S}_c) - \beta_i^2 \right]$$

$$+ \sum_{1}^{n} \lambda_i^2 \left[g_i(\mathcal{S}_c) - \gamma_i^2 \right]$$

where δ is an additional vector of undetermined multipliers. Let Δ be the diagonal matrix

$$\Delta = \mathrm{diag}\left[\overbrace{1, \ldots, 1}^{\ell_1}, \quad \overbrace{\delta_2, \ldots, \delta_2}^{\ell_1}, \ldots, \quad \overbrace{\delta_r, \ldots, \delta_r}^{\ell_r} \right]$$

and with

$$y^T = \left[y_1^T, \quad y_2^T, \ldots, y_r^T \right] \Delta$$

the Lagrangian may be written as before as a quadratic form in the plant output and the control input

$$\varphi\,(\mathcal{S}_c,\ \delta,\ \lambda) = \left[y^T \Delta^2\, y + u^T\, \Lambda^2\, u\right] - \left[\text{a constant}\right]$$

If H_y is defined to be

$$H_y^{\,T} = \left[H_{y1}^{\,T}, \ldots, H_{yr}^{\,T}\right] \Delta$$

then no modification of the matrix equations that arise in the solution of the un-constrained minimization problem is required.

Although the proof that is given in Appendix II concerning the monotonicity of a constraint function applies to plant variable as well as the control input constraints, there is an important difference between the two types of constraints: it is well known that if the bound on a control input component is removed, the variance of that component will be infinite. It is quite possible, however, that the minimization of one plant variable will cause in itself another plant variable to lie within a prescribed bound. When this occurs the later variable is referred to in the language of non-linear programming as a "slack" variable. On account of difficulties associated with slack variables the iterative procedure presented in Chapter V for the determination of control input multipliers cannot be used for the determination of plant variable multipliers.

When there are one or two constrained plant variables, however, the following simple procedure may be used for the determination of the corresponding multipliers: take the significant output to be $\hat{y} = \Delta\, y$

and minimize this output subject to the control input bounds. The variances of the constrained variables which result from this minimization are then functions of the multiplier components of Δ

$$\|y_i\|^2 = \rho_i \, (\delta_2, \ldots, \delta_r)$$

By choosing the multipliers over a range of values the regions where the constraints are met may be mapped out graphically and the multipliers thereby determined. This procedure, which is illustrated in Chapter VI, is awkward compared to the automatic procedure developed for the control input multipliers, but it has the compensating advantage that is establishes the trade off that exists between the minimization of one variable and the constraint of others.

APPENDIX II

MONOTONICITY OF THE CONSTRAINT FUNCTIONS

In this appendix it is shown that the control variances g_i ($\mathscr{S}_c$) are monotonic decreasing functions of their multipliers λ_i. Let

$$F\ (x) = f\ (x) + \sum_{i \neq k} \lambda_1^{\ 2} \left[g_i\ (\ x \ ) - \gamma_1^{\ 2} \right]$$

$$G\ (x) = \left[g_k\ (x) - \lambda_k^{\ 2} \right] \qquad\qquad \mu = \lambda_k^{\ 2}$$

where the symbol x has been used in place of $\mathscr{S}_c$. Then

$$\varphi\ (\mathscr{S}_c, \lambda) = F\ (x) + \mu\ G\ (x) = \psi\ (x,\ \mu)$$

Let h (μ) be a function defined as follows: for fixed μ determine $\hat{x}$ such that ψ ($\hat{x}$, μ) $\leq$ ψ (x, μ) for all admissible x. Then set h (μ) = G ($\hat{x}$). Under these conditions, h (μ) is then a monotonic decreasing function of μ.

To see this choose $\mu_1 < \mu_2$ and determine $\hat{x}_1$ and $\hat{x}_2$ such that

i) $\quad \psi\ (\hat{x}_1,\ \mu_1) \leq \psi\ (x,\ \mu_1)$

ii) $\quad \psi\ (\hat{x}_2,\ \mu_2) \leq \psi\ (x,\ \mu_2)$

Using the definition of ψ and substituting $\hat{x}_2$ in i) and $\hat{x}_1$ in ii), it follows that

$$\text{i')} \quad F(\hat{x}_1) + \mu_1 G(\hat{x}_1) \leq F(\hat{x}_2) + \mu_1 G(\hat{x}_2)$$

$$\text{ii')} \quad F(\hat{x}_2) + \mu_2 G(\hat{x}_2) \leq F(\hat{x}_1) + \mu_2 G(\hat{x}_1)$$

Adding i') and ii') and subtracting

$$F(\hat{x}_1) + F(\hat{x}_2) + \mu_1 \left[G(\hat{x}_1) + G(\hat{x}_2)\right]$$

from both sides, one obtains

$$(\mu_2 - \mu_1) \; G(\hat{x}_2) \leq (\mu_2 - \mu_1) \; G(\hat{x}_1)$$

But since $(\mu_2 - \mu_1) > 0$, this implies that $G(\hat{x}_2) \leq G(\hat{x}_1)$ and consequently that $h(\mu_2) \leq h(\mu_1)$, which establishes the monotonicity of the constraint functions.

APPENDIX III

CANONICAL TRANSFORMATIONS

In this appendix the state transformations that carry a controllable system into precanonical and canonical forms are discussed. Luenberger, who devised these transformations, states only that "simple matrix bookkeeping" verifies that they have the desired properties. By making use of the Special Notational Conventions discussed in their entirety in Appendix IV and defined in the glossary given in Figure 10 it is possible to deal with these matters explicitly.

Let the non-singular transformation S that is given in Chapter III be written as follows:

$$S = \left[S_1,\ S_2, \ldots, S_m \right] \tag{12}$$

$$S_i = \left[g_i,\ Fg_2, \ldots, F^{\nu_i - 1} g_i \right] = \left[s_i^{(1)},\ s_i^{(2)}, \ldots, s_i^{(\nu_i)} \right]$$

It is easy to see that the matrix F shown in Figure 2 may be written in terms of the Special Notational Conventions as

$$F = (\daleth^T - A^T\ B^T)$$

To verify that the transformation S has the desired effect on F it is sufficient to show that

$$S\hat{F} = FS \qquad \text{or} \qquad (S\mathcal{J}^T - FS) = S\hat{A}^T B^T \tag{13}$$

But

$$(S\mathcal{J}^T - FS) = \left[S_1,\dots,S_m\right] \begin{bmatrix} J^T & \cdots & \phi \\ \vdots & & \vdots \\ \phi & & J^T \end{bmatrix} - F\left[S_1,\dots S_m\right]$$

$$= \left[S_1 J^T - F S_1,\dots,S_m J^T - F S_m\right]$$

Taking note of the "shifting" properties of J^T, it is seen that

$$S_i J^T - F S_i = \left[0,\dots,0,-F^{\nu_i} g_i\right] = -F^{\nu_i} g_i b^T$$

and thus that

$$(S\mathcal{J}^T - FS) = \left[F^{\nu_1} g_1,\dots, F^{\nu_m} g_m\right] B^T$$

Equation (13) is therefore satisfied if

$$\left[F^{\nu_1} g_1,\dots,F^{\nu_m} g_m\right] = -S\hat{A}^T$$

or

$$F^{\nu_i} g_i = \left[S_1,\dots,S_m\right] \begin{bmatrix} \hat{a}_{i1}^T \\ \vdots \\ \hat{a}_{im}^T \end{bmatrix}$$

which is equivalent to

$$F^{\nu_i} g_i = \hat{a}_{i1}^{(1)} s_1^{(1)} + \ldots + \hat{a}_{i1}^{(\nu_1)} s_1^{(\nu_1)}$$

$$+ \ldots + \hat{a}_{im}^{(\nu_m)} s_m^{(\nu_m)} \tag{14}$$

But $F^{\nu_i} g_i$ is simply an n-vector and since the columns of S, being linearly independent, span the space, it is possible to find scalars $\hat{a}_{ij}^{(k)}$ such that (14) is satisfied. This shows that S has the desired effect on F.

To see that $S^{-1} G = \hat{G}$ where $\hat{G}$ is also given in Figure 2, it is only necessary to note that all the rows of S^{-1} are orthogonal to g_1 except for the first row whose inner product with g_1 is unity. Similar statements may be made with respect to the remaining columns of G. Thus the transformation S does convert the representation $\{F, G, H\}$ of the system $\mathcal{S}$ to precanonical form.

A transformation R that carries a system from precanonical to canonical form may be constructed as follows: Let the transformation $\hat{R}$ be defined by

$$\hat{R} = \left[R_1, R_2, \ldots, R_m\right] \tag{15}$$

$$R_i = \left[\beta_i, \hat{F}^T \beta_i, \ldots, (\hat{F}^T)^{\nu_i - 1} \beta_i\right]$$

where $\hat{F}$ is a system matrix in precanonical form and the β_i are the columns of the matrix B as defined in the glossary.

Let

$$\mathcal{A} = (\mathcal{J} \quad - BA)$$

and consider the equation

$$\hat{F}^T \hat{R} = \hat{R} \mathcal{A}^T \qquad\qquad (\hat{R} \mathcal{J}^T - \hat{F} \hat{R}) = \hat{R} A^T B^T \qquad (16)$$

It was shown above that an equation of this form may be satisfied by appropriate choice of the scalar elements of A, so long as the columns of $\hat{R}$ are linearly independent. Assuming that they are, take the transpose of equation (16) and post-multiply by $\hat{R}^{-T}$

$$\hat{R}^T F \hat{R}^{-T} =$$

to see that the transformation $R = \hat{R}^{-T}$ transforms $\hat{F}$ to canonical form.

To prove that $\hat{R}$ is non-singular and that $\hat{R}^T G = BD$ as required, it is finally necessary to resort to "simple matrix bookkeeping" which establishes that $\hat{R}^T$ has the form given in Figure 11. It is clear from this form that Det $[\hat{R}] = 1$ implying that $\hat{R}$ is non-singular and that $\hat{R}^T G = BD$.

To conclude this appendix the problem of constructing transformations that preserve the structure of augmented systems will be discussed.

Let an augmented system be represented by $\{F, G, H\}$ where

$$F = \begin{bmatrix} F_D & \phi \\ F_{DP} & F_P \end{bmatrix} \qquad G = \begin{bmatrix} G_D & \phi \\ G_{DP} & G_P \end{bmatrix} \qquad H = \begin{bmatrix} H_D & H_P \end{bmatrix}$$

and suppose that S_D and S_P transform the subsystems $\{F_D, G_D, H_D\}$ and $\{F_P, G_P, H_P\}$ to precanonical form. These transformations will then have the forms

$$S_D = \left[g_{D1}, F_D\, g_{D1}, \ldots, F_D^{(\nu_1 - 1)} g_{D1}, \ldots, F_D^{(\nu_{mD} - 1)} g_{DmD} \right]$$

$$S_P = \left[g_{P1}, F_P\, g_{P1}, \ldots, F_P^{(\mu_1 - 1)} g_{P1}, \ldots, F_P^{(\mu_{mP} - 1)} g\, P_{mp} \right]$$

If S is taken to be the transformation

$$S = \left[g_1, Fg_1, \ldots, F^{(\nu_1 - 1)} q_1, \ldots, F^{(\nu_{mD} - 1)} q_{mD}, \ldots, q_{mD+1}, \ldots, \right.$$

$$\left. F^{(\mu_1 - 1)} g_{mD+1}, \ldots, F^{(\mu_{mP} - 1)} g_m \right]$$

then since

$$\begin{bmatrix} F_D & \phi \\ F_{DP} & F_P \end{bmatrix}^k \begin{bmatrix} g_D \\ g_P \end{bmatrix} = \begin{bmatrix} F_D^k g_D \\ X \end{bmatrix} \qquad \begin{bmatrix} F_D & \phi \\ F_{DP} & F_P \end{bmatrix}^k \begin{bmatrix} \phi \\ G_P \end{bmatrix} = \begin{bmatrix} \phi \\ F_P^k a_P \end{bmatrix}$$

S will have the form

$$S = \begin{bmatrix} S_D & \phi \\ S_{DP} & S_P \end{bmatrix}$$

which clearly has the same effect on the two subsystems as the separate transformation S_D and S_P. But the overall transformation S is a canonical transformation that transforms the augmented system to pre-canonical form, and it is easily verified that when constructed in this manner the transformation S preserves the structure of the augmented system.

APPENDIX IV

SPECIAL NOTATIONAL CONVENTIONS AND THEIR APPLICATIONS

The purpose of this appendix is to introduce certain Special Notational Conventions that have been devised for dealing with systems in canonical form, to establish certain preliminary lemmas relating to these conventions, and then to apply the conventions to the determination of the modal matrix and transfer function of a system in input canonical form and also to the solution of the crucial canonical matrix equations.

The idea of the Special Notational Conventions grew out of a paper by Brand on the companion matrix (30). To motivate their introduction, the conventions will be discussed first as they relate to a single input system in phase variable form. The system $\mathcal{S}$

$$\mathcal{S}: \qquad \dot{x} = A\,x + bu \qquad\qquad y = cx \qquad\qquad A = (J - ba)$$

is in phase variable form when the row vectors a and c, the column vector b and the square matrix J have the forms given in the Glossary of Special Notational Conventions shown in Figure 10. Associated with the system $\mathcal{S}$ are the polynomials $f(\lambda)$ and $g(\lambda)$ and the column vector $\xi(\lambda)$ whose forms are also given in the glossary.

Four lemmas concerning the notational conventions may be established immediately by inspection:

$$J = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad \xi(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \cdot \\ \cdot \\ \cdot \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} \quad a^T = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \quad C^T = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix}$$

$$f(\lambda) = a_1 + a_2\lambda + \dots + a_n\lambda^{n-1} + \lambda^n$$

$$g(\lambda) = c_1 + c_2\lambda + \dots + c_n\lambda^{n-1}$$

$$\mathfrak{J} = \begin{vmatrix} J & \emptyset & \dots & \emptyset \\ \emptyset & J & \dots & \emptyset \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \emptyset & \emptyset & \dots & J \end{vmatrix} \quad B = \begin{vmatrix} b & \emptyset \dots & \emptyset \\ \emptyset & b \dots & \emptyset \\ \vdots & & \vdots \\ \emptyset & \emptyset \dots & b \end{vmatrix} \quad \Xi(\lambda) = \begin{vmatrix} \xi(\lambda) & \emptyset & \dots & \emptyset \\ \emptyset & \xi(\lambda) & \dots & \emptyset \\ \vdots & & & \vdots \\ \emptyset & \emptyset & \dots & \xi(\lambda) \end{vmatrix}$$

$$a_{ij}^T = \begin{bmatrix} a_{ij}^{(1)} \\ a_{ij}^{(2)} \\ \vdots \\ a_{ij}^{(\nu_j)} \end{bmatrix} \quad c_{ij}^T = \begin{bmatrix} c_{ij}^{(1)} \\ c_{ij}^{(2)} \\ \vdots \\ c_{ij}^{(\nu_j)} \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & \dots & c_{1m} \\ \vdots & & \vdots \\ c_{\ell 1} & \dots & c_{\ell m} \end{bmatrix}$$

$$f_{ij}(\lambda) = a_{ij}^{(1)} + a_{ij}^{(2)}\lambda + \dots + a_{ij}^{(\nu_j)}\lambda^{\nu_j - 1} + \delta_{ij}\lambda^{\nu_j}$$

$$g_{ij}(\lambda) = c_{ij}^{(1)} + c_{ij}^{(2)}\lambda + \dots + c_{ij}^{(\nu_j)}\lambda^{\nu_j - 1}$$

$$\mathfrak{F}(\lambda) = \begin{bmatrix} f_{11}(\lambda) & f_{12}(\lambda) & \dots & f_{1m}(\lambda) \\ \vdots & & & \vdots \\ f_{m1}(\lambda) & f_{m2}(\lambda) & \dots & f_{mm}(\lambda) \end{bmatrix} \quad \mathcal{G}(\lambda) = \begin{bmatrix} g_{11}(\lambda) & \dots & g_{1m}(\lambda) \\ \vdots & & \vdots \\ g_{\ell 1}(\lambda) & \dots & g_{\ell m}(\lambda) \end{bmatrix}$$

Figure 10. Glossary of Special Notational Conventions.

Lemma: ia) $f(\lambda) = a\,\xi(\lambda) + \lambda^n$

ib) $f(\lambda) = c\,\xi(\lambda)$

iia) $(\lambda I - J)\,\xi(\lambda) = b\,\lambda^n$

iib) $(\lambda I - A)\,\xi(\lambda) = b\,f(\lambda)$

Lemma iib) asserts that when λ_i is a root of $f(\lambda)$ then $\xi(\lambda_i)$ is an eigenvector of A corresponding to the eigenvalue λ_i. $f(\lambda)$ is thus seen to be the characteristic polynomial of A. If the roots of $f(\lambda)$ are distinct, then the modal matrix of A is

$$V = \left[\xi(\lambda_1),\ \xi(\lambda_2), \ldots, \xi(\lambda_n)\right]$$

which will be recognized as a Vandermonde matrix.

The Laplace transform of the vector differential equation that describes the system may be written

$$(sI - A)\,x(s) = b\,u(s) \qquad y = c\,x(s)$$

Using lemmas ib) and iib) it follows immediately that

$$x(s) = \xi(s)\left[u(s)/f(s)\right] \qquad y(s) = \left[g(s)/f(s)\right]u(s)$$

and consequently that the transfer function of the system is

$$h\ (s) = g\ (s)/f\ (s)$$

The nth order multivariable system with m inputs and ℓ outputs in canonical form that is shown in Figure 3 may be written in terms of the special notational conventions as follows:

$$\dot{x} = \mathcal{A}\, x + BDu \qquad\qquad y = Cx$$

$$\mathcal{A} = (\mathfrak{J} - BA)$$

where the matrices $\mathfrak{J}$,A, B, C, and D have the forms shown in the glossary. Associated with the multivariable system are the polynomial matrices $\mathfrak{F}(\lambda)$ and $\mathfrak{G}(\lambda)$ and the matrix $\Xi(\lambda)$ whose forms are also shown in the glossary.

Four lemmas concerning the multivariable conventions may be established by inspection on the basis of their single variable analogs:

Lemma: Ia) $\mathfrak{F}(\lambda) = A\, \Xi(\lambda) + \mathrm{diag}\left[\lambda^{\nu_j}\right]$

Ib) $\mathfrak{G}(\lambda) = C\, \Xi(\lambda)$

IIa) $(\lambda I - \mathfrak{J})\, \Xi(\lambda) = B\, \mathrm{diag}\left[\lambda^{\nu_j}\right]$

IIb) $(\lambda I - \mathcal{A})\, \Xi(\lambda) = B\, \mathfrak{F}(\lambda)$

These lemmas may be used to determine the eigenvectors of the canonical matrix $\mathcal{A}$ as follows: Suppose that p is a non-zero solution of the reduced homogeneous system

$$\mathfrak{F}(\lambda)\ p = 0 \qquad p^T = \left[p_1,\ p_2, \ldots, p_m\right]$$

where the p_i are scalars. Such a solution exists if and only if

$$\mathrm{Det}\left[\ \mathfrak{F}(\lambda)\right] = \quad f\ (\lambda) = 0$$

where f (λ) is an nth order polynomial. Let $v = \Xi\ (\lambda)\ p$; then

$$(\lambda I - \mathcal{A}\ )\ \Xi\ (\lambda)\ p = B\ \mathfrak{F}(\lambda)\ p = 0$$

by lemma IIb), and v is seen to be an eigenvector of $\mathcal{A}$. If the roots of f (λ) are distinct, then the modal matrix of $\mathcal{A}$ is

$$V = \left[\ \Xi\ (\lambda_1)\ p^{(1)},\ \Xi\ (\lambda_2)\ p^{(2)}, \ldots,\ \Xi\ (\lambda_n)\ p^{(n)}\right]$$

$$= \begin{bmatrix} p_{11}\ \xi\ (\lambda_1) & \cdots & p_{1n}\ \xi\ (\lambda_n) \\ p_{21}\ \xi\ (\lambda_1) & \cdots & p_{2n}\ \xi\ (\lambda_n) \\ \vdots & & \vdots \\ p_{m1}\ \xi\ (\lambda_1) & \cdots & p_{mn}\ \xi\ (\lambda_n) \end{bmatrix}$$

which will be referred to as a composite Vandermonde matrix.

The transfer function of the system $\mathcal{S}$ may be determined in a similar manner. The Laplace transform of the vector differential equation that describes the system may be written

$$(s\,I - \mathcal{A})\; x\;(s) = BDu\;(s) \qquad y = c\;x\;(s)$$

Let

$$x\;(s) = \Xi\;(s)\;\mathfrak{F}^{-1}(s)\;D\;u\;(s)$$

and by lemma IIb)

$$(sI - \mathcal{A})\;\Xi\;(s)\;\;Du\;(s) = BD\;u\;(s)$$

it is seen that the transform equation is satisfied. But

$$y\;(s) = c\;x\;(s) = c\;\Xi\;(s)\;\mathfrak{F}^{-1}(s)\;\;D\;u(s)$$

and by lemma Ib)

$$y\;(s) = \mathcal{G}\;(s)\;\mathfrak{F}^{-1}\;(s)\;\;D\;u\;(s)$$

which proves that

$$\mathfrak{H}\;(s) = \mathcal{G}\;(s)\;\mathfrak{F}^{-1}\;(s)\;D$$

where $\mathcal{H}(s)$ is the system transfer function.

The Special Notational Conventions and their related lemmas may also be used to determine the eigenvector components of the Hamiltonian matrix that corresponds to a canonical matrix equation. Suppose that V and U are the matrices required by Potter's theorem for the solution of the canonical quadratic equation. Then the columns v and u of these matrices must satisfy the homogeneous Hamiltonian system

$$(\lambda I - \mathcal{A})\, v + B\, D\, \Gamma^{2}\, D^{T}\, B^{T}\, u = 0 \tag{17}$$

$$C^{T}\, C\, v + (\lambda I + \mathcal{A}^{T})\, u = 0 \tag{18}$$

A concise solution to these equations may be obtained as follows:

Let

$$v = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_m \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \end{bmatrix}$$

where the v_i and the u_i are ν_i-vectors; Set $v = \Xi(\lambda)p$ and define $q = B^{T}u$, where p and q are m-vectors with scalar elements. Note that the premultiplication of u by B^{T} has the effect of "picking out" the last element of each of the u_i. Substitute v as defined into (17) and (18) and premultiply (18) by $\Xi^{T}(-\lambda)$ to obtain

$$(\lambda I - \mathcal{A})\,\Xi(\lambda)\,p + B D \Gamma^2 D^T B^T u = 0 \tag{19}$$

$$\Xi^T(-\lambda)\,C^T C\,\Xi(\lambda)\,p + \Xi^T(-\lambda)\,(\lambda I + \mathcal{A}^T)\,u = 0 \tag{20}$$

But by lemmas ib) and IIb)

$$(\lambda I - \mathcal{A})\,\Xi(\lambda) = B\,\mathfrak{F}(\lambda) \qquad C\,\Xi(\lambda) = \mathfrak{G}(\lambda)$$

and by these same lemmas (transposed, $-\lambda$ replacing λ)

$$\Xi^T(-\lambda)\,(\lambda I + \mathcal{A}^T) = -\,\mathfrak{F}^T(-\lambda)\,B^T \qquad \Xi^T(-\lambda)C^T = \mathfrak{G}^T(-\lambda)$$

Thus (19) and (20) become

$$B\left[\mathfrak{F}(\lambda)\,p + D\,\Gamma^2 D^T q\right] = 0 \tag{21}$$

$$\mathfrak{G}^T(-\lambda)\,\mathfrak{G}(\lambda)\,p - \mathfrak{F}^T(-\lambda)\,q = 0 \tag{22}$$

where q has been substituted for $B^T u$. Equations (17) and (18) are therefore satisfied if and only if p and q satisfy the reduced Hamiltonian system.

$$\mathfrak{F}(\lambda)\,p + D\,\Gamma^2 D^T q = 0 \tag{23}$$

$$\mathfrak{G}^T(-\lambda)\,\mathfrak{G}(\lambda)\,p - \mathfrak{F}^T(-\lambda)\,q = 0 \tag{24}$$

Letting

$$q = - D^{-T} \Gamma^{-2} D^{-1} \mathcal{J} (\lambda) \; p$$

it is easily verified that (23) and (24) are satisfied when p is a non-zero solution of

$$\Phi (\lambda) \; p = \left[\mathcal{G}^{T} (-\lambda) \; \mathcal{G} (\lambda) + \mathcal{J}^{T} (-\lambda) \; D^{-T} \Gamma^{-2} D^{-1} \mathcal{J} (\lambda) \right] p = 0$$

which, in turn, requires that

$$\text{Det} \left[\Phi (\lambda) \right] = F (-\lambda^{2}) = 0$$

where $F (-\lambda^2)$ is a 2nth order polynomial that contains only even powers of λ. (To see this note that $\text{Det} \left[\Phi (-\lambda) \right] = \text{Det} \left[\Phi^{T}(\lambda) \right] = \text{Det} \left[\Phi (\lambda) \right]$ and consequently $F (-\lambda^2)$ is an even function of λ.) This guarantees that there are exactly n Hurwitz roots of $F (-\lambda^2)$. To each of these roots λ_i there corresponds a solution $p^{(i)}$ and $q^{(i)}$ to the system (23) and (24) and an eigenvector component $v^{(i)}$. If the roots are distinct then the matrix V is seen to be a composite Vandermonde matrix which is shown to be non-singular in Appendix V. When the roots are not distinct, a solution for V may nevertheless be found by the methods also outlined in Appendix V.

It will turn out that there is no need to know explicitly the component u corresponding to the component v, but for completeness a method for calculating u will be briefly described. Equation (18) written

$$(\lambda I - \mathfrak{J}^T)\, u = A^T B^T u - C^T C\, v$$

But

$$B^T u = q = - D^{-T}\, \Gamma^{-2}\, D^{-1}\, \mathfrak{F}(\lambda)\; p$$

$$C\, v = C\, \Xi(\lambda)\; p = \mathfrak{G}(\lambda)\; p$$

and thus

$$(\lambda I + \mathfrak{J}^T)\, u = - \left[A^T D^T\, \Gamma^{-2}\, D^{-1}\, \mathfrak{F}(\lambda) + C^T\, \mathfrak{G}(\lambda) \right] p$$

Defining

$$E = \begin{bmatrix} D^{-1} A \\ C \end{bmatrix} \qquad \mathcal{E}(\lambda) = \begin{bmatrix} \Gamma^{-1} D^{-1}\, \mathfrak{F}(\lambda) \\ \mathfrak{G}(\lambda) \end{bmatrix}$$

this equation may be written

$$(\lambda I + \mathfrak{J}^T)\, u = - E^T\, \mathcal{E}(\lambda)\; p$$

But the inverse of $(\lambda I + \mathfrak{J}^T)$ is easily written down in closed form.

It will be recalled that the objective of the solution of the quadratic matrix equation was the determination of the gain matrix $K = D^T B^T P$. It is not necessary to complete the solution of the matrix equation, however, in order to determine K. Let

$$Q = \left[q^{(1)}, \ldots, q^{(m)}\right] = \left[B^T u^{(1)}, \ldots, B^T u^{(m)}\right] = B^T U$$

Then since $P = UV^{-1}$ and $K = D^T (B^T U) V^{-1} = D^T QV^{-1}$, the gain matrix may be calculated directly from the matrix Q whose columns contain the solutions $q^{(i)}$ of the reduced Hamiltonian system and the matrix V whose columns are the corresponding eigenvector components.

The solution of the canonical linear equation involves the determination of v and u that satisfy the simpler Hamiltonian system

$$(\lambda I - \mathcal{A}) \, v = 0 \tag{25}$$

$$C^T C \, v + (\lambda I + \mathcal{A}^T) \, u = 0 \tag{26}$$

Let v, p, and q be defined as before and the simpler system will be satisfied if p and q satisfy the reduced system

$$\mathfrak{F}(\lambda) \, p = 0 \tag{27}$$

$$\mathcal{G}^T(-\lambda) \, \mathcal{G}(\lambda) \, p = \mathfrak{F}^T(-\lambda) \, q = 0 \tag{28}$$

It is immediately apparent that p and thus v are identical to the solutions that were obtained above in the determination of the modal matrix of a system in input canonical form. q is given by

$$q = \mathcal{F}^T(-\lambda)^{-1}\,\mathcal{G}^T(-\lambda)\,\mathcal{G}(\lambda)\,p$$

The system $\mathcal{S}$ which underlies the linear equation may be assumed to be stable. This fact has two implications. First of all it implies that the roots λ_i of the characteristic equation

$$\mathrm{Det}\left[\mathcal{F}(\lambda)\right] = F(\lambda) = 0$$

are Hurwitz roots as required by Potter's theorem. Secondly, it guarantees that $\mathcal{F}^T(-\lambda)$ has an inverse since λ_i cannot be a root of

$$\mathrm{Det}\left[\mathcal{F}^T(-\lambda)\right] = F(-\lambda) = 0$$

As in the case of the quadratic equation a solution for u may be written down in closed form from the equation

$$(\lambda I + \mathcal{J}^T)\,u = A^T q - C^T \mathcal{F}(\lambda)\,p$$

but this is not necessary since the gain matrix K may be written

$$K = D^T B^T U V^{-1} = D^T Q V^{-1}$$

where Q is defined as above.

It is convenient to end this appendix by pointing out a ramification of Potter's theorem which obviates a great deal of complex arithmetic. Suppose that the ith eigenvector components $v^{(i)}$ and $u^{(i)}$ are complex; then the (i + 1) st components may be taken as the complex conjugates of the ith components. Let X be the matrix formed from the unit matrix by replacing the ith 2 x 2 diagonal block with

$$\begin{bmatrix} (1/2) & -(j/2) \\ (1/2) & (j/2) \end{bmatrix}$$

Then since, for example

$$\left[v^{(i)}, v^{(i)*}\right] \begin{bmatrix} (1/2) & -(j/2) \\ (1/2) & (j/2) \end{bmatrix} = \left[\mathcal{R}e\left[v^{(i)}\right], \mathrm{Im}\left[v^{(i)}\right]\right]$$

it is seen that the ith and (i + 1) st columns of the matrices

$$\hat{V} = V X \qquad \hat{U} = U X$$

contain real and imaginary parts of the ith columns of the matrices V and U. But since X is clearly non-singular and

$$P = (U X) (V X)^{-1} = \hat{U} \hat{V}^{-1}$$

the solution of the matrix equation may be written in terms of the real matrices $\hat{U}$ and $\hat{V}$. When these real matrices are used in place of the complex ones, the matrix Q in terms of which the canonical gain matrix is written is similarly handled.

APPENDIX V

COMPOSITE VANDERMONDE MATRICES

The purpose of this appendix is to demonstrate the non-singularity of a composite Vandermonde matrix when the eigenvalues of the associated reduced homogeneous system are distinct and also to treat the related issue of the modifications required of this matrix when these eigenvalues are not distinct. It will be recalled that a composite Vandermonde matrix is given by

$$V = \left[\Xi(\lambda_1)\ p^{(1)}, \ldots, \Xi(\lambda_n)\ p^{(n)} \right]$$

where the $p^{(i)}$ are non-zero solutions of the reduced homogeneous system

$$\mathfrak{F}(\lambda_i)\ p^{(i)} = 0 \qquad\qquad i = 1, 2, \ldots, n$$

$\mathfrak{F}(\lambda)$ is an m x m matrix with polynomial elements and the λ_i, which are the eigenvalues of this system, are roots of the polynomial equation

$$\mathrm{Det}\left[\mathfrak{F}(\lambda)\right] = F(\lambda) = 0$$

In order to simplify the notation it will be assumed that the degrees of the elements of $\mathfrak{F}(\lambda)$ are given by

$$\deg \left[f_{ij}\ (\lambda) \right] = \nu \qquad i = j$$
$$\leq \nu - 1 \qquad i \neq j$$

$\mathfrak{F}(\lambda)$ is thus the matrix arising from a system in canonical form of order $n = m\nu$. In this case

$$\deg \left[F\ (\lambda) \right] = n$$

It will now be shown that if the roots of $F\ (\lambda)$ are distinct, then there exist exactly n solutions of the reduced homogeneous system, that each solution is associated with a distinct root, and that these solutions may be written as

$$p^{(i)} = p\ (\lambda_i) = \left[p_1\ (\lambda_i), \ldots, p_m\ (\lambda_i) \right]^T$$

where the $p_i\ (\lambda)$ are polynomials. The proof depends on a well known result from the theory of polynomial matrices (see Perlis [31]): Let $\mathfrak{F}(\lambda)$ be an m x m polynomial matrix. Then there exist unique m x m polynomial matrices $\mathcal{S}\ (\lambda)$, $\mathcal{R}\ (\lambda)$ and $\mathfrak{D}\ (\lambda)$ such that

i) $\mathcal{S}\ (\lambda)$ and $\mathcal{R}\ (\lambda)$ are products of elementary matrices. As a consequence their determinants are non-zero constants, k_r and k_s, and their inverses are also polynomial matrices.

ii) $$\mathcal{S}(\lambda)\ \mathfrak{F}(\lambda)\ \mathcal{R}\ (\lambda) = \mathfrak{D}\ (\lambda)$$

where

$$\mathfrak{D}(\lambda) = \mathrm{diag}\left[f_i(\lambda), \ldots, f_r(\lambda), 0, \ldots, 0\right]$$

and the $f_i(\lambda)$ are monic polynomials (i.e., unity high order coefficient) each of which divides its successor.

iii) $\mathfrak{F}(\lambda)$ and $\mathfrak{D}(\lambda)$ have the same rank and the same determinant.

Using this result it is easy to construct the polynomial solution of the reduced homogeneous system. Since

$$\mathrm{Det}\left[\mathfrak{F}(\lambda)\right] = F(\lambda)$$

and the roots of $F(\lambda)$ are distinct $\mathfrak{D}(\lambda)$ is required to be

$$\mathfrak{D}(\lambda) = \mathrm{diag}\left[1, \ldots, 1, F(\lambda)\right]$$

Let

$$e = \left[0, \ldots, 0, 1\right]^T$$

then

$$p(\lambda_i) = \mathfrak{R}(\lambda_i)\, e = \left[r_{1m}(\lambda_i), \ldots, r_{mm}(\lambda_i)\right]^T \quad i = 1, 2, \ldots, n$$

comprise all the non-zero solutions of the reduced system. To see this, note that

$$\mathcal{A}(\lambda_i)\ \mathfrak{F}(\lambda_i)\ \mathcal{R}(\lambda_i)e = \mathfrak{D}(\lambda_i)e = F(\lambda_i)e = 0$$

As defined $p(\lambda_i)$ is not identically zero since this would imply that each element of the last column of $\mathcal{R}(\lambda)$ contained the factor $\lambda - \lambda_i$, thus contradicting the assertion that

$$\mathrm{Det}\left[\mathcal{R}(\lambda)\right] = k_r$$

On the other hand, the rank of $\mathfrak{D}(\lambda)$ is m if $\lambda \neq \lambda_i$ and $m - 1$ if $\lambda = \lambda_i$. Thus e is the only solution of

$$\mathfrak{D}(\lambda_i)\ x = 0 \qquad i = 1,\ldots,n$$

and since

$$\mathrm{Det}\left[\mathcal{A}(\lambda_i)\right] = k_s \neq 0$$

$\mathcal{R}(\lambda_i)e$ is a solution of

$$\mathfrak{F}(\lambda_i)\ y = 0$$

This establishes that $p(\lambda)$ has the stated properties.

The question of the non-singularity of the composite Vandermonde matrix may be very concisely posed in terms of this polynomial solution of the reduced homogeneous system. The matrix may now be written

$$V = \left[\Xi(\lambda_1)\, p(\lambda_1), \ldots, \Xi(\lambda_n)\, p(\lambda_n)\right]$$

V is singular if and only if there exists a non-zero row vector

$$b = \left[b_1^{(1)}, \ldots, b_1^{(\nu)},\ b_2^{(1)}, \ldots, b_2^{(\nu)}, \ldots, b_m^{(\nu)}\right]$$

such that $bV = 0$. This condition will be referred to as the singularity condition. Defining

$$q_i(\lambda) = b_i^{(1)} + b_i^{(2)}\lambda + \ldots + b_i^{(\nu)}\lambda^{\nu-1}$$

and noting that

$$b\,\Xi(\lambda) = \left[q_1(\lambda),\ q_2(\lambda), \ldots, q_m(\lambda)\right] = q(\lambda)$$

it is seen that the singularity condition may be written as

$$q(\lambda_i)\, p(\lambda_i) = p_1(\lambda_i)\, q_1(\lambda_i) + \ldots + p_m(\lambda_i)\, q_m(\lambda_i) = 0$$

It will now be shown that the singularity condition cannot hold. Let $\mathfrak{J}(\lambda)$ be written as

$$\mathfrak{F}(\lambda) = \begin{bmatrix} f_1 (\lambda) \\ f_2 (\lambda) \\ \vdots \\ f_m (\lambda) \end{bmatrix}$$

where the $f_i(\lambda)$ are the rows of $\mathfrak{F}(\lambda)$ and define $\mathfrak{F}_i(\lambda)$ to be the matrix obtained by replacing the ith row of $\mathfrak{F}(\lambda)$ with $q(\lambda)$. For example

$$\mathfrak{F}_2 (\lambda) = \begin{bmatrix} f_1 (\lambda) \\ q (\lambda) \\ \vdots \\ f_m (\lambda) \end{bmatrix}$$

If the singularity condition holds, then the m homogeneous systems

$$\mathfrak{F}_k (\lambda_i) \, p (\lambda_i) = 0 \qquad k = 1, 2, \ldots, m$$

are satisfied non-trivially for $i = 1, 2, \ldots, n$. These systems may be satisfied in only one of two ways:

i) $\quad \mathrm{Det} \left[\mathfrak{F}_k (\lambda_i) \right] = F_k (\lambda_i) = 0 \qquad k = 1, \ldots, m \quad i = 1, \ldots, n$

ii) $\quad \mathrm{Det} \left[\mathfrak{F}_k (\lambda) \right] \equiv 0 \qquad k = 1, \ldots, m$

If i) holds, then each $F_k(\lambda)$ must have degree not less than n. From their definition the polynomial elements of $q\ (\lambda)$ have degrees not greater than $\nu - 1$. The degree of $F_k\ (\lambda)$ is thus seen to be not greater than $m\nu - 1 = n - 1$ and i) cannot hold. On the other hand, if ii) holds it is easy to show that

$$\text{Det}\left[\mathfrak{F}(\lambda)\right] \equiv 0$$

which is a clear contradiction. This establishes that the singularity condition cannot hold and that the composite Vandermonde matrix is non-singular.

The modifications required of the composite Vandermonde matrix when repeated roots occur will now be considered. The discussion will be limited to roots occurring with multiplicity two, but the results can be extended in an obvious manner to roots occurring with arbitrary multiplicity. If

$$F\ (\lambda) = (\lambda - \lambda_1)^2\ \ F\ (\lambda)$$

then the matrix $\mathfrak{D}\ (\lambda)$ may have one of the two forms

a) $\mathfrak{D}\ (\lambda) = \text{diag}\left[1,\ldots,1,\ (\lambda - \lambda_1),\ (\lambda - \lambda_1)\ F\ (\lambda)\right]$

b) $\mathfrak{D}(\lambda) = \text{diag}\left[1,\ldots,1,\ 1,\ (\lambda - \lambda_1)^2\ F\ (\lambda)\right]$

If $\mathfrak{D}(\lambda)$ has the form a) then $\mathfrak{D}(\lambda_1)$ and $\mathfrak{F}(\lambda_1)$ have rank m-2 and two independent solutions to the reduced homogeneous system may be found. The corresponding columns of the composite matrix are also independent and thus constitute true eigenvectors of the canonical system. If $\mathfrak{D}(\lambda)$ has the form b) then only one solution of the reduced system may be found. The corresponding column of the composite matrix is the only true eigenvector associated with the eigenvalue λ_1.

A generalized eigenvector associated with this eigenvalue may nevertheless be obtained. Let

$$v(\lambda) = \Xi(\lambda)\, p(\lambda)$$

where $p(\lambda)$ is the polynomial solution of the reduced system. In Appendix IV it was shown that

$$(\lambda I - \mathcal{A})\, v(\lambda) = B\, \mathfrak{F}(\lambda)\, p(\lambda)$$

Taking the derivative of this equation gives

$$v(\lambda) + (\lambda I - \mathcal{A})\, v'(\lambda) = B \left[\mathfrak{F}(\lambda)\, p(\lambda)\right]'$$

which shows that $v'(\lambda)$ is a generalized eigenvector if

$$\left[\mathfrak{F}(\lambda_1)\, p(\lambda_1)\right]' = 0$$

To see that this is in fact so recall that

$$\mathcal{L}(\lambda)\,\mathcal{F}(\lambda)\; p\,(\lambda) = F\,(\lambda) e \qquad \mathcal{F}(\lambda)\; p\,(\lambda) = F\,(\lambda)\,\mathcal{L}^{-1}(\lambda)\; e$$

Taking the derivative gives

$$\left[\mathcal{F}(\lambda)\; p\,(\lambda)\right]' = \left[F\,(\lambda)\left\{\mathcal{L}^{-1}(\lambda)\right\}' + F'\,(\lambda)\,\mathcal{L}^{-1}(\lambda)\right] e$$

But since λ is a repeated root of $F\,(\lambda)$ it is also a root of $F'\,(\lambda)$ and the required condition holds. The generalized eigenvector

$$v'\,(\lambda) = \Xi\,(\lambda)\; p'\,(\lambda) + \Xi'\,(\lambda)\; p\,(\lambda)$$

is seen upon examination of the elements of $\Xi\,(\lambda)$ and $\Xi'\,(\lambda)$ to be independent of the true eigenvector $v\,(\lambda)$. This concludes the discussion of the question of repeated roots.

The results obtained thus far in this appendix are restricted to the composite Vandermonde matrix as the modal matrix of a system in canonical form. These results can be extended, however, to the composite Vandermonde matrix as a solution of the Hamiltonian system. It will be recalled that in this case the $p^{(i)}$ are solutions of the reduced system

$$\Phi\,(\lambda_i)\; p^{(i)} = \mathcal{E}^T\,(-\lambda_i)\;\mathcal{E}\,(\lambda_i)\; p^{(i)} = 0$$

Using the result concerning polynomial matrices it can be asserted that there are unique matrices $\mathcal{P}\,(\lambda)$ and $\mathcal{Q}\,(\lambda)$ such that

$$\mathcal{P}(\lambda)\,\Phi(\lambda)\,\mathcal{Q}(\lambda) = \mathcal{D}(-\lambda)\,\mathcal{D}(\lambda)$$

where

$$\mathcal{D}(\lambda) = \text{diag}\left[1,\ldots,1,\ F_o\ (\lambda)\right]$$

Taking the transpose of this equation, replacing λ by $-\lambda$, and recalling that

$$\Phi^T(-\lambda) = \Phi(\lambda)$$

shows that

$$\mathcal{Q}^T(-\lambda)\ \ \Phi(\lambda)\,\mathcal{P}^T(-\lambda) = \mathcal{D}(-\lambda)\ \ \mathcal{D}(\lambda)$$

But the matrices $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ are unique which requires that

$$\mathcal{Q}(\lambda) = \mathcal{P}^T(-\lambda)$$

Thus

$$\Phi(\lambda) = \left[\ \mathcal{P}^T(-\lambda)^{-1}\mathcal{D}(-\lambda)\right]\left[\mathcal{D}(\lambda)\,\mathcal{P}(\lambda)^{-1}\ \right]$$

Define

$$\mathfrak{F}_o(\lambda) = \mathfrak{D}(\lambda)\,\mathfrak{P}(\lambda)^{-1}$$

and since

$$\mathrm{Det}\left[\mathfrak{F}_o(\lambda)\right] = \mathrm{Det}\left[\mathfrak{D}(\lambda)\,\mathfrak{P}^{-1}(\lambda)\right] = F_o(\lambda)/k_p$$

it is seen that

$$\Phi(\lambda) = \mathfrak{F}_o^T(-\lambda)\,\mathfrak{F}_o(\lambda)$$

is the unique factorization of $\Phi(\lambda)$ of this form such that

$$\mathrm{Det}\left[\mathfrak{F}_o(\lambda)\right] = k\,F_o(\lambda)$$

Now it is clear that a polynomial solution constructed as above for the reduced homogeneous system

$$\mathfrak{F}_o(\lambda)\,p_o(\lambda) = 0$$

is also a solution of the system

$$\Phi(\lambda)\,p_o(\lambda) = 0$$

Using $\mathfrak{F}_o(\lambda)$ in place of $\mathfrak{F}(\lambda)$ in the proof given above of the non-singularity of the composite Vandermonde matrix shows that the proof is applicable to this matrix as a solution of the Hamiltonian system. An entirely similar argument extends the results on repeated roots to this case.

APPENDIX VI

AUGMENTED CANONICAL EQUATIONS

Because of the relative complexity of the calculations associated with the solution of the canonical quadratic equation that arises from augmented systems, this separate appendix will be devoted to a discussion of these solutions. The solution proceeds as in the case of the simpler systems to the point of the reduced homogeneous system

a) $$\mathcal{F}(\lambda)\; p + D\,\Gamma^2\,D^T q = 0$$

b) $$\mathcal{G}^T(-\lambda)\;\mathcal{G}(\lambda)\; p - \mathcal{F}^T(-\lambda)\; q = 0$$

Here it is necessary to examine the structure of the matrices $\mathcal{F}(\lambda)$, $\mathcal{G}(\lambda)$, and $D\,\Gamma$. The transformation T that reduced the quadratic equation to canonical form was chosen to preserve the structure of the augmented system. Considering the manner in which the polynomial elements of $\mathcal{F}(\lambda)$ and $\mathcal{G}(\lambda)$ are defined in terms of the canonical matrices $\mathcal{A}$ and C, it is seen that the structure of the augmented system carries through to $\mathcal{F}(\lambda)$ and $\mathcal{G}(\lambda)$ and that these matrices may be written as follows

$$\mathfrak{F}(\lambda) = \begin{bmatrix} \mathfrak{F}_{11}(\lambda) & \phi & \phi \\ \phi & \mathfrak{F}_{22}(\lambda) & \phi \\ \mathfrak{F}_{31}(\lambda) & \phi & \mathfrak{F}_{33}(\lambda) \end{bmatrix}$$

$$(\lambda) = \left[\mathfrak{G}_1(\lambda) \quad \mathfrak{G}_2(\lambda) \quad \mathfrak{G}_3(\lambda) \right]$$

where the subscripts 1, 2, and 3 now refer to the disturbance, the reference and the plant systems respectively. Now the diagonal matrix Γ was chosen as

$$= \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \Gamma_3 \end{bmatrix}$$

where $\Gamma_3 = \Lambda^{-1}$. The reduced Hamiltonian system may therefore be written as

$$\begin{bmatrix} \mathfrak{F}_{11} & \phi & \phi \\ \phi & \mathfrak{F}_{22} & \phi \\ \mathfrak{F}_{31} & \phi & \mathfrak{F}_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & D_3 \Gamma_3^2 D_3^T \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} \mathfrak{G}_1^* \mathfrak{G}_1 & \mathfrak{G}_1^* \mathfrak{G}_2 & \mathfrak{G}_1^* \mathfrak{G}_3 \\ \mathfrak{G}_2^* \mathfrak{G}_1 & \mathfrak{G}_2^* \mathfrak{G}_2 & \mathfrak{G}_2^* \mathfrak{G}_3 \\ \mathfrak{G}_3^* \mathfrak{G}_1 & \mathfrak{G}_3^* \mathfrak{G}_2 & \mathfrak{G}_3^* \mathfrak{G}_3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} - \begin{bmatrix} \mathfrak{F}_{11}^* & \phi & \mathfrak{F}_{31}^* \\ \phi & \mathfrak{F}_{22}^* & \phi \\ \phi & \phi & \mathfrak{F}_{33}^* \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = 0$$

where the arguments of the polynomial matrices have been omitted, $\mathfrak{F}_{ij}{}^*$ indicates $\mathfrak{F}_{ij}{}^T(-\lambda)$ and the p_i and q_i are m_i-vectors. If the order of the augmented system is

$$n = n_1 + n_2 + n_3$$

where the n_i are the orders of the three subsystems, then n solutions of the reduced Hamiltonian system may be obtained by considering three cases:

Case a): Define

$$\Phi_{33}(\lambda) = \left[\mathcal{G}_3{}^*\mathcal{G}_3 + \mathfrak{F}_{33}{}^* D_3{}^{-T} \Gamma_3{}^{-2} D_2{}^{-1}\mathfrak{F}_{33}\right]$$

and suppose that λ_i is a root of

$$F_3(-\lambda^2) = \mathrm{Det}\left[\Phi_{33}(\lambda)\right]$$

Take

$$p_{13}{}^{(j)} \equiv 0 \qquad p_{23}{}^{(j)} \equiv 0$$

and let $p_{33}{}^{(j)}$ be a non-zero solution of

$$\Phi_{33}(\lambda_j)\, p_{33}{}^{(j)} = 0$$

It is then possible to determine $q_{i3}{}^{(j)}$ such that

$$q^{(j)} = \left[q_{13}^{(j)},\ q_{23}^{(j)},\ q_{33}^{(j)} \right]^T \qquad p^{(j)} = \left[p_{13}^{(j)},\ p_{23}^{(j)},\ p_{33}^{(j)} \right]^T$$

satisfy a) and b). The component $q_{33}^{(j)}$ is given by

$$q_{33}^{(j)} = - D_3^{-T}\, \Gamma_3^{-2}\, D_3^{-1}\, \mathfrak{F}_{33}\, (\lambda_j)\, p_{33}^{(j)}$$

Case b): Suppose λ_j is a root of

$$F_2\ (\lambda) = \mathrm{Det} \left[\mathfrak{F}_{22}\ (\lambda) \right] = 0$$

Take

$$p_{12}^{(j)} \equiv 0$$

and let $p_{22}^{(j)}$ be a non-zero solution of

$$\mathfrak{F}_{22}\ (\lambda_j)\ p_{22}^{(j)} = 0$$

It is then possible to determine $p_{32}^{(j)}$ and $q_{22}^{(j)}$ such that

$$q^{(j)} = \left[q_{12}^{(j)},\ q_{22}^{(j)},\ q_{32}^{(j)} \right]^T \qquad p^{(j)} = \left[p_{12}^{(j)},\ p_{22}^{(j)},\ p_{32}^{(j)} \right]^T$$

satisfy a) and b). The components $p_{32}^{(j)}$ and $q_{32}^{(j)}$ are given by

$$p_{32}^{(j)} = \Phi_{33}^{-1} \Phi_{32} \, p_{22}^{(j)}$$

$$q_{32}^{(j)} = (\mathfrak{F}_{33}^{*})^{-1} \mathcal{G}_3^{*} \left[\mathcal{G}_2 \, p_{22}^{(j)} + \mathcal{G}_3 \; p_{32}^{(j)} \right]$$

where

$$\Phi_{32}(\lambda) = \left[\mathcal{G}_3^{*} \mathcal{G}_2 \right]$$

Case c): Suppose λ_j is a root of

$$F_1(\lambda) = \mathrm{Det} \left[\mathfrak{F}_{11}(\lambda) \right]$$

Take

$$p_{21}^{(j)} \equiv 0$$

and let $p_{11}^{(j)}$ be a non-zero solution of

$$\mathfrak{F}_{11}(\lambda_j) \, p_{11}^{(j)} = 0$$

It is then possible to determine $p_{31}^{(j)}$ and $q_{i1}^{(j)}$ such that

$$q^{(j)} = \left[q_{11}^{(j)}, \; q_{21}^{(j)}, \; q_{31}^{(j)} \right]^T \qquad p^{(j)} = \left[p_{11}^{(j)}, \; p_{21}^{(j)}, \; p_{31}^{(j)} \right]^T$$

satisfy a) and b). The components $p_{31}^{(j)}$ and $q_{31}^{(j)}$ are given by

$$p_{31}^{(j)} = \Phi_{33}^{-1} \Phi_{31}\, p_{11}^{(j)}$$

$$q_{31}^{(j)} = (\mathfrak{F}_{33}^{*})^{-1} \mathfrak{g}_{3}^{*} \left[\mathfrak{g}_{1}\, p_{11}^{(j)} + \mathfrak{g}_{3}\, p_{31}^{(j)} \right]$$

where

$$\Phi_{31}(\lambda) = \left[\mathfrak{g}_{3}^{*} \mathfrak{g}_{1} + \mathfrak{F}_{33}^{*} D_{3}^{-T} \Gamma_{3}^{-2} D_{3}^{-1} \mathfrak{F}_{31} \right]$$

Assuming the λ_j are distinct these three cases provide n solutions to the reduced homogeneous system. The solutions $p^{(j)}$ may be displayed as follows

$$\begin{vmatrix} p_{11} & \phi & \phi \\ \phi & p_{22} & \phi \\ p_{31} & p_{32} & p_{33} \end{vmatrix}$$

where

$$p_{ij} = \left[p_{ij}^{(1)},\ p_{ij}^{(2)}, \ldots, p_{ij}^{(n_j)} \right]$$

The structure of these solutions requires the composite Vandermonde matrix which satisfies the Hamiltonian system to have the similar structure

$$V = \begin{bmatrix} V_{11} & \phi & \phi \\ \phi & V_{22} & \phi \\ V_{31} & V_{32} & V_{33} \end{bmatrix}$$

where V_{11} and V_{22} are the modal matrices of the disturbance and the reference systems and V_{33} satisfies the Hamiltonian system that arises from the unaugmented plant system.

The gain matrix H_u for the augmented system may be determined as follows: Define

$$Q_i = \left[q_{3j}^{(1)},\ q_{3j}^{(2)}, \ldots, q_{3j}^{(n_j)} \right]$$

Then

$$H = \left[H_{u1},\ H_{u2},\ H_{u3} \right] = \Lambda_3^{-2}\ D_3^T \left[Q_1\ Q_2\ Q_3 \right] V^{-1}\ T^{-1}$$

where H_{ui} has dimension $n_3 \times n_i$. It will be noted first of all that only those components $q_{ij}^{(k)}$ for which explicit expressions were given above are required for the determination of the optimal gain matrix. Further examination of the solutions of the reduced Hamiltonian system and the matrices V and T reveals that H_{u3} depends only on the plant system, that H_{u2} depends only on the reference and the plant systems, and that H_{u1} depends only on the disturbance and the plant systems. For this reason H_{u3}, which is identical to the gain matrix obtained from the solution of the unaugmented system, may be referred to as the plant feedback matrix. Similarly H_{u2} and H_{u3} may be referred to as the reference and the disturbance feedforward matrices.

APPENDIX VII

AN ADAPTATION OF DANILEVSKII'S METHOD

Faddeev [32] describes in detail a method due to Danilevskii for the determination of the characteristic polynomial of a matrix F. The method in its more general form involves performing the similarity transformation

$$A = S^{-1} F S$$

where S is the transformation

$$S = \left[x, Fx, \ldots, F^{n-1} x \right]$$

and x is an arbitrary vector that is chosen, if possible, to insure the non-singularity of S. It is immediately apparent that if F is the system matrix of a single input controllable system $\{F, g, h\}$, then x = g does insure the non-singularity of S and that S is in this case a precanonical transformation.

The method devised as part of the present research for the determination of the precanonical transformation of a multi-variable system is an adaptation of Danilevskii's method. The transformation is determined in n steps each of which involves the transformation

$$\left\{F^{(k)},\ G^{(k)},\ H^{(k)}\right\} \quad \xrightarrow{T_k S_k} \quad \left\{F^{(k+1)},\ G^{(k+1)},\ H^{(k+1)}\right\}$$

where

$$F^{(o)} = F \qquad G^{(o)} = G \qquad H^{(o)} = H$$

and

$$T_k = \left[e_1, \ldots, e_{k-1},\ e_j \quad e_{k+1}, \ldots, e_{j-1},\ e_k,\ e_{j+1}, \ldots, e_n\right]$$

$$S_k = \left[e_1, \ldots, e_{k-1},\ x_k,\ e_{k+1}, \ldots, e_n\right]$$

and e_k is the unit vector whose only non-zero element is the kth. The transformation T_k rearranges the system matrices so that the pivotal element x_{kk} in the transformation S_k is maximized. The column x_k of the transformation S_k is chosen to be $g_k^{(k)}$ for the first m steps and $f_{(k-m)}^{(k)}$ for the remaining n - m steps where $g_i^{(k)}$ and $f_i^{(k)}$ are the ith columns of $G^{(k)}$ and $F^{(k)}$. Note that the matrices $H^{(k)}$ are not involved in determining the successive transformations. It can be verified that the product of the transformations

$$S = T_1\ S_1\ \ldots\ T_n\ S_n = \left[g_1, \ldots, g_m,\ Fg_1, \ldots, Fg_m,\right.$$

$$\left.\ldots F^{(\ \ -1)}\ g_1, \ldots, F^{(\ \ -1)} g_k\right]$$

is a transformation which by simple rearrangement of its columns can be made a precanonical transformation.

It is convenient for the purpose of carrying out the successive transformations to arrange the elements of the matrices $G^{(k)}$ and $F^{(k)}$ as the single array

$$\left[G^{(k)},\ F^{(k)}\right]$$

Each step of the calculation involves replacing this array with the array

$$\left[S_{k+1}^{-1}\ G^{(k)},\ S_{k+1}^{-1}\ F^{(k)}\ S_{k+1}\right]$$

An analysis shows that these n steps may be carried out with less than $n^3 + 2\,m\,n^2$ arithmetic operations. At the completion of the n steps the last m columns of $F^{(n)}$ contain (a rearrangement of) the elements $\hat{a}_{ij}^{(k)}$ of the precanonical matrix $\hat{F}$, but only the elements of the matrices S_k^{-1} and S_k are available; the calculation of the product matrices S^{-1} and S requires $n^3 - n^2$ operations. Finally the calculation

$$\hat{H} = H\ S$$

requires $m\,n^2$ operations for a total of less than $2\,n^3 + 3\,m\,n^2$ operations.

At the kth step the matrix $\left[G^{(k)},\ F^{(k)}\right]$ will have the form

$$\begin{bmatrix} 1 & 0 & & 0 & f_{1k}^{(k)} & \cdots & f_{1n}^{(k)} \\ 0 & 1 & & \vdots & & & \\ 0 & 0 & & 0 & & & \\ \vdots & & & & & & \\ & & 0 & 1 & f_{kk}^{(k)} & & f_{kn}^{(k)} \\ & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & f_{nk}^{(k)} & \cdots & f_{nn}^{(k)} \end{bmatrix}$$

It may turn out that the (k+1)st to the nth elements of $f_{(k-m)}^{(k)}$ are all zero thus precluding the carrying out of the (k+1)st step. This situation, which may be expected to occur, indicates only that one of the terms

$$\left[F^{(k)}\right]^j \; g_i^{(k)}$$

is a linear combination of preceeding similar terms. This difficulty is avoided simply by skipping the column $f_{(k-m)}^{(k)}$ and using the next instead. The controllability of the system insures that there are always sufficient columns remaining to complete the n steps of the transformation.

The transformation R which transforms the system from precanonical to canonical form is computed directly from its constructed definition. It can be verified that R and R^{-1} have the forms shown in Figure 11. When the system has m inputs and its canonical form is naturally partitioned into blocks of dimension $\nu \times \nu$, the elements $\alpha_{ij}^{(k)}$ and $\beta_{ij}^{(k)}$ are determined by

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & \alpha_{11}^{(1)} & 0 & 0 & \alpha_{21}^{(1)} \\
0 & 1 & \alpha_{11}^{(1)} & \alpha_{11}^{(2)} & 0 & \alpha_{21}^{(1)} & \alpha_{21}^{(2)} \\
1 & \alpha_{11}^{(1)} & \alpha_{11}^{(2)} & \alpha_{11}^{(3)} & \alpha_{21}^{(1)} & \alpha_{21}^{(2)} & \alpha_{21}^{(3)} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \alpha_{12}^{(1)} & 0 & 1 & \alpha_{22}^{(1)} \\
0 & 0 & \alpha_{12}^{(1)} & \alpha_{12}^{(2)} & 1 & \alpha_{22}^{(1)} & \alpha_{22}^{(2)}
\end{bmatrix}
$$

$$
\begin{bmatrix}
\beta_{11}^{(3)} & \beta_{11}^{(2)} & \beta_{11}^{(1)} & 1 & \beta_{21}^{(3)} & \beta_{21}^{(2)} & \beta_{21}^{(1)} \\
\beta_{11}^{(2)} & \beta_{11}^{(1)} & 1 & 0 & \beta_{21}^{(2)} & \beta_{21}^{(1)} & 0 \\
\beta_{11}^{(1)} & 1 & 0 & 0 & \beta_{21}^{(1)} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_{12}^{(2)} & \beta_{12}^{(1)} & 0 & 0 & \beta_{22}^{(2)} & \beta_{22}^{(2)} & 1 \\
\beta_{12}^{(1)} & 0 & 0 & 0 & \beta_{22}^{(1)} & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
$$

Figure 11. Precanonical to Canonical Transformation.

$$\alpha_{ij}^{(o)} = \delta_{ij} \qquad\qquad \beta_{ij}^{(o)} = \delta_{ij}$$

$$\alpha_{ij}^{(k)} = \sum_{p=1}^{m} \sum_{q=1}^{k} \hat{a}_{ip}^{(q)} \quad \alpha_{pj}^{(k-q)} \qquad k = 1,\ldots,\nu - 1$$

$$\beta_{ij}^{(k)} = \sum_{p-1}^{m} \sum_{q=1}^{k} \alpha_{ip}^{(q)} \quad \beta_{pj}^{(k-q)} \qquad k = 1,\ldots,\nu - 1$$

Similar expressions hold for the general care of $\nu_i \times \nu_j$ blocks. By proper arrangement of the calculation it is possible to solve these equations by summations alone in $m^3 \nu^2$ operations. The coefficients $a_{ij}^{(k)}$ of the canonical matrix $\mathcal{A}$ are given by $\alpha_{ij}^{(k)}$ where

$$\alpha_{ij}^{(\nu)} = \sum_{p=1}^{m} \sum_{q=1}^{\nu} a_{ip}^{(q)} \quad \alpha_{pj}^{(\nu - k)}$$

The product transformation T = SR, its inverse, and the elements of the canonical matrix C may then be determined in approximately $(m+1)\, m^2 \nu^3$ operations.

APPENDIX VIII

AN ILLUSTRATIVE EXAMPLE

In this appendix the computational procedure developed in Chapter V for the solution of the canonical matrix equation is illustrated by its application to a simple example. Consider the canonical equation which arises from a 5th order two input system

$$\mathcal{A}^T P + P\mathcal{A} + C^T C = P\ B\ D\ \Gamma^2\ D^T\ B^T\ P$$

where the matrices $\mathcal{A}$, B, D, C, and Γ are shown in Figure 12. The polynomial matrices associated with the underlying system are

$$\mathfrak{F}(x) = \begin{bmatrix} (-x + 2x^2 + x^3) & (1) \\ (2 + 2x) & (1 + x + x^2) \end{bmatrix}$$

$$\mathfrak{G}(x) = \begin{bmatrix} (-1 - x) & (-2x) \end{bmatrix}$$

The matrix $\Phi(x)$ which appears in the reduced homogeneous system is computed to be

$$\Phi(x) = \begin{bmatrix} (21 - 26x^2 + 30x^4 - 5x^6) & (10 + 7x + 8x^2 - 15x^3) \\ (10 - 7x + 8x^2 + 15x^3) & (10 + x^2 + 5x^4) \end{bmatrix}$$

The determinant of this matrix is given by the polynomial

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2 & -2 & 0 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & -1 & 0 & 0 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\Gamma^{-1} = (1/5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Figure 12. Canonical Equation Matrices.

$$F(-x^2) = 110 - 350x^2 + 105x^4 + 75x^6 + 145x^8 - 25x^{10}$$

whose 5 Hurwitz roots are computed to be

$$\left\{-.0.613,\ -0.9238,\ -2.515,\ -5.454 \pm j\,1.084\right\}$$

To each of these roots there corresponds a solution $p^{(j)}$ of the reduced homogeneous system. For example, when $\lambda_1 = -0.613$

$$\Phi(\lambda_1)\, p^{(1)} = \begin{bmatrix} 15.20 & 12.17 \\ 13.84 & 11.08 \end{bmatrix} \begin{bmatrix} -0.8 \\ 1.0 \end{bmatrix} = 0$$

The complete set of solutions is given by

$$\begin{vmatrix} -0.80 & 1.00 & 1.00 & .00 & 0.28 \\ 1.00 & -0.79 & 0.74 & 1.00 & .00 \end{vmatrix}$$

The matrices V and U which are required by Potters theorem may be written explicitly in terms of the solutions of the reduced homogeneous system. These matrices as well as the solution of the canonical equation

$$P = U V^{-1}$$

are shown in Figure 13. When this solution is substituted in the canonical matrix equation the difference between the elements of the left and right-

$$V = \begin{bmatrix} -0.801 & 1.000 & 1.000 & .000 & 0.277 \\ 0.491 & -0.924 & -2.515 & 0.300 & -0.151 \\ -0.301 & 0.853 & 6.327 & -0.328 & -0.243 \\ 1.000 & -0.791 & 0.741 & 1.000 & 0.000 \\ -0.613 & 0.731 & -1.865 & -0.545 & -1.084 \end{bmatrix}$$

$$U = \begin{bmatrix} -0.172 & -4.647 & 0.048 & 1.432 & -0.185 \\ -1.203 & -15.370 & 0.073 & 1.335 & -1.738 \\ 0.460 & -5.256 & 0.016 & 00.198 & -0.767 \\ 1.918 & 2.533 & 1.060 & 2.106 & 0.424 \\ -0.715 & 2.916 & -2.682 & -0.886 & -1.747 \end{bmatrix}$$

$$P = \begin{bmatrix} 7.376 & 20.750 & 6.844 & -3.848 & -2.374 \\ 20.750 & 67.810 & 23.240 & -15.630 & -7.761 \\ 6.844 & 23.240 & 8.046 & -5.549 & -2.588 \\ -3.848 & -15.630 & -5.549 & 6.096 & 2.050 \\ -2.374 & -7.761 & -2.588 & 2.050 & 2.668 \end{bmatrix}$$

Figure 13. Solution Matrices.

hand sides is, in each case, less than 10^{-7} times the corresponding element of the left-hand side.

APPENDIX IX

GUST DISTURBANCE SHAPING FILTERS

The shaping filters that were appended to the lateral and longitudinal systems to account for the statistics of the gust disturbances of these systems were modified in two respects that are discussed in this appendix. In current aerodynamic practice transform techniques are used for the analysis of flight in turbulent air. In the course of this analysis certain difficulties arise on account of the divergence of intergrals involving the derivative spectra such as that of φ_q. It is known, however, that the aerodynamic transfer functions that appear in the analysis are inaccurate for wave lengths shorter than the relevant dimensions of the aircraft, and this fact is used to justify the truncation of the spectra, a practice which eliminates the troublesome divergent integrals. In the study of the longitudinal system the equivalent technique of introducing the "roll-off factor"

$$r(s) = \eta\omega_o / (\eta\omega_o + s) \qquad \eta = 20$$

in the gust shaping filter was employed. This modification changes the filter to a third order system. It was discovered that the "roll-off factor" had a negligible effect on the aircraft response and since there are no problems with divergent spectra when the methods of this dissertation are employed the factor was omitted from the shaping filters for the lateral system.

On the other hand, the repeated roots associated with the gust spectra are inconvenient when the present methods are employed. They can be handled but at great expense in programming complexity. For this reason the roots were perturbed slightly with negligible affect on the aircraft response.

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