A SIMPLIFIED RHEOELECTRIC ANALOGY OF PLANE STRESS AND ITS APPLICATION TO THE DETERMINATION OF STRESS CONCENTRATION FACTORS

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The author also wishes to acknowledge the assistance of the Western Union Telegraph Company in the donation of the conducting paper used in the experiments upon which this study is based.

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NOTATION

a	circular hole diameter, mm
c	sheet conductance, mho/sq cm
d	minimum shoulder width of a filleted bar, mm
D	maximum shoulder width of a filleted bar, mm
e	variable normal distance from a discontinuity to successive equipotential lines, at their point of closest approach to the discontinuity, mm
E	variable normal distance from a discontinuity to points corresponding to a uniform equipotential line spacing, at the minimum (net) cross section of a plane conductor, mm
h, h_1	gradients of the principal stresses p and q
k	stress concentration factor based on nominal stress
l	uniform equipotential line spacing at the minimum (net) cross section of a plane conductor, mm
m	uniform distance between equipotential lines at the insulated edge of a plane conductor, mm
n	distance between equipotential lines at an arbitrary point in a plane conductor, mm
p	maximum principal stress, kg/sq cm
q	minimum principal stress, kg/sq cm
r	fillet radius, mm
t	plate thickness, mm
V	electric potential function, volt
w	plate width, mm; also uniform tension load on a section of plate, kg
W	current function, amp

α	a stress concentration factor
∇	operator del, where $\nabla^2 = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$
σ_x, σ_y	normal stress components parallel to the x and y-axes, respectively, kg/sq cm
σ'	normal stress at any arbitrary point in an elastic plate, tangent to an equipotential line, kg/sq cm
σ_{nom}	nominal stress at the minimum (net) cross section of an elastic plate, tangent to an equipotential line, kg/sq cm
τ_{xy}	shearing stress in rectangular coordinates, kg/sq cm
ϕ	angle describing the tangent to the E vs. e curve at the origin, degree
θ_p, θ_q	angles describing the direction of principal stresses p and q, with respect to the direction of σ_x, degree

SUMMARY

Presented herein is a simplified analogy method for the solution of plane stress problems involving thin elastic plates uniformly loaded in tension on two opposite edges. The solutions involved are given in terms of stress concentrations about discontinuities in the plates, in the form of holes and cut-outs.

The mathematical similarities between electric potential fields in plane conductors and plane stress fields in elastic plates is established to reflect the identity between equipotential voltage lines and stress trajectories. By use of an analog device to construct the equipotential lines a graphic representation of a stress field is obtained and points of stress concentration are discerned as points of maximum line contraction. A physical measurement of this contraction, and subsequent comparison to uniformly spaced stress trajectories, permits evaluation of a stress concentration factor.

Several stress concentration problems with previous analytical and photoelastic solutions are re-solved to show compatible results with the previous work. The rheoelectric analogy is then shown to be adaptable to problems without previous solution.

The simplified analog is seen to provide rapid and

inexpensive solutions to certain stress problems, with results compatible with the more established means of solution.

CHAPTER I

INTRODUCTION

Early studies on the strength of materials produced solutions to plane stress problems which neglected discontinuities in the stress pattern at points of load application and at points of rapid change in cross section. The neglect of these stress disturbances was acceptable in the solution of static elasticity problems, but in the dynamic problem the stress disturbance, or stress concentration, became of major importance. Although the theoretical methods of solving stress concentration problems have improved considerably in recent times, there still remain many problems of practical importance for which no rigorous solution has been obtained. Resort to the experimental methods, although approximate, has become the only practical means of solution. Of particular interest among the experimental means are the analogy methods.

Frequently two or more apparently different physical phenomena can be expressed in identical mathematical form. When this condition exists the systems are said to be analogous, and the characteristics of the one system can be determined through inspection of a mathematical model of the other. The similarities between electric potential and stress fields, for example, permit application of this principle to the determination of stress concentrations for problems not readily tractable by analytical means; and the ease with which electric potential fields can be measured gives this particular analogy a high degree of flexibility and simplicity.

To avoid confusion with other electric analogies (electric-network for example) the term "rheoelectric" is adopted, a term coined by French scientists. The word implies the use of a continuous conducting medium for the creation of an electric potential field.

The use of rheoelectric analogies has become classic in the last thirty years (1)*. In 1925 Jacobsen (2) utilized the distribution of electric potential in a continuous conductor to determine the stress concentration factors about a circular fillet in a shaft subjected to torsion. Peres (3) set up the first of several French laboratories in 1931 to explore problems in aerodynamics, fluid mechanics, and mathematical physics using electrolytic tanks. A review of his work was presented to the International Congress of Mechanics in 1938, and since that time rheoelectric apparatus has become accepted media for supplementing theoretical work in many fields. Most recently the use of conducting sheets has come into prominence for the study of torsional stress concentration problems (4), and for supplementing

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*Numbers in parenthesis indicate references found in the Literature Cited section of the Bibliography.

data obtained with photoelastic equipment (5).

Historically, the study of stress concentrations began with the investigation of railroad axle failures. A rapid change in diameter where the journals joined the body of the axle caused fatigue cracks to appear at the re-entrant corners, and subsequent failure. In his early investigations of that problem Wohler (6) recognized the difficulty as stress concentration, and the situation was corrected by fairing the journal joint. Since formal recognition of the concentration problem, considerable effort has been expended toward its solution, but not always with success. For example, during the early use of Liberty Ships, hatch corner fractures developed in the decks of these ships at the rate of 10 per 100 ship-years (7); and the Civil Aeronautics Administration has recently reported aircraft parts failures, attributable to stress fatigue, at the rate of 30 per month (8).

It is the purpose of this paper to demonstrate an additional experimental method of stress study by examining the suitability of a simplified rheoelectric analogy to the solution of stress concentration problems for thin elastic plates, pierced by holes or cut-outs, and subjected to uniform tension on two opposite edges. To this end, comparisons are made with existing solutions for certain shapes, and solutions are obtained for random problems without analytical solution.

CHAPTER II

THE MATHEMATICAL ANALOGY

It may be seen through the use of Mohr's Circle, Fig. 1, that in a plane stress system the magnitude and direction of the principal stresses p and q, in terms of normal and shearing stress components σ_x, σ_y, and τ_{xy}, are given by

$$p = (\sigma_x + \sigma_y)/2 + \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4(\tau_{xy})^2} \qquad (1)$$

$$q = (\sigma_x + \sigma_y)/2 - \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4(\tau_{xy})^2} \qquad (2)$$

$$\tan 2\theta_{pq} = 2\,\tau_{xy}/(\sigma_x - \sigma_y) \qquad (3)$$

where θ_{pq} represents both θ_p and θ_q, measured counterclockwise from the direction of σ_x when θ_{pq} is positive, and clockwise when θ_{pq} is negative (9). Upon solution of equation 3, two roots are obtained, 90 degrees apart, which define the directions of the principal stresses p and q. Curves whose tangents represent these same directions are defined as principal stress trajectories (10). Since the principal stresses at each point in a stress system are mutually perpendicular, it follows that the stress trajectories form a system of orthogonal curves. This orthogonality can be expressed as

$$1/h \cdot \partial p/\partial x = 1/h_1 \cdot \partial q/\partial y \;;\; 1/h \cdot \partial p/\partial y = -1/h_1 \cdot \partial q/\partial x \quad (4)$$

where h and h_1 are the respective gradients of p and q, such that

$$h = \sqrt{(\partial p/\partial x)^2+(\partial p/\partial y)^2} \; ; \; h_1 = \sqrt{(\partial q/\partial x)^2+(\partial q/\partial y)^2} \quad (5)$$

It has been shown (11) that for a system of principal stresses

$$h = h_1 \quad (6)$$

and equation 4 reduces to

$$\partial p/\partial x = \partial q/\partial y \; ; \; \partial p/\partial y = -\partial q/\partial x \quad (7)$$

Since the Cauchy-Riemann Equations are satisfied by p and q, then

$$\nabla^2 p = 0 \; ; \nabla^2 q = 0 \quad (8)$$

In the case of the plane conductor of constant thickness and conductivity subjected to a conservative potential field, the potential function V and the current function W/c are related (12) by the equations

$$\partial V/\partial x = \partial(W/c)/\partial y \; ; \; \partial V/\partial y = -\partial(W/c)/\partial x \quad (9)$$

where c is the conductivity per unit area. These functions satisfy the Laplace Equation

$$\nabla^2 V = 0 \; ; \nabla^2 W = 0 \quad (10)$$

By comparing equations 7 and 9 and noting that p, q, V, and W all satisfy the Laplacian, it is concluded that equipotential and current lines will represent the principal stress trajectories, provided suitable boundary conditions are established.

CHAPTER III

EQUIPMENT

To obtain quick and relatively inexpensive representations of a plane stress field, in terms of stress trajectories, a special conducting paper is used. This paper, called "Teledeltos," is manufactured by the Western Union Telegraph Company and is used in electric transmission communication systems. Teledeltos is formed by introducing a conductive material in the pulp beating stage of the paper making process. The paper is coated on one side with a non-conductor, and on the other side with a fine aluminum powder, such that the resultant product is a nearly homogeneous plane conductor relatively insensitive to atmospheric changes of temperature and humidity. Two types of Teledeltos are presently available: type H, with a rated resistance on the order of 20,000 ohms per square, taken between parallel sides; and type L, with a rated resistance of 3,000 ohms per square. For this experiment the latter type was used in 21 centimeter widths, with a measured resistance on the order of 1300 ohms, measured along parallel sides 30 centimeters in length.

To insure contact between electrodes and the paper, and to represent various holes and cut-outs, a silver paint of the type used for printing radio circuits (General Cement

No. 21-2) was applied to the paper. The paint, which is almost pure silver, has a negligible resistance compared to that of the paper. The model thus prepared was clamped to an insulating backboard of plastic (Plexiglas) by means of spring loaded brass electrodes across which a potential was impressed. The electric potential was supplied by dry cell batteries of the electric lantern variety and it was found that 18 volts provided suitable current for the conduct of the experiment.

To trace equipotential lines on the conducting sheet a balancing method was used requiring a slide-wire resistance of order similar to that of the model, and a zero indicator. In the interest of simplicity, the slide-wire resistor was fabricated using multiple strips of Teledeltos, one inch in width, placed one atop another. A sliding contact was then used to vary the resistance. The resistance was placed in parallel connection with the sides of the model and the electric feed. A zero indicator, connected between the sliding wire contact and a detecting probe is shown schematically in Fig. 2.

Equipotential lines were traced by means of a detecting probe, an insulated wire with the end shaped to a slightly rounded stylus point. Contact was maintained by slight pressure of the probe on the paper, and once the zero point was found, it was only necessary to press slightly harder to mark the equipotential point located.

A galvanometer was used as a zero indicator. When this device was used in conjunction with the detecting probe, determination of equipotential points to within half a millimeter was possible, depending upon the width of the probe point used.

A general view of this equipment is shown in Fig. 3, which includes the model with electrodes, the probe, the zero indicator, the electrical supply, and the measuring bridge.

CHAPTER IV

PROCEDURE

In considering a rectangular flat plate loaded uniformly on two opposite edges, it will be noted that the internal stresses will vary from point to point within the plate in the presence of a discontinuity. However, on the boundaries of the plate the external forces must be in equilibrium with the internal stresses, so that these external loads may be regarded as a continuation of the internal stress distribution.

In a plane conductor, no current may flow in or out of an insulated boundary. Such a boundary is a current flow line and is crossed at right angles by equipotential lines. Then

$$\partial V/\partial n = 0 \qquad (11)$$

where n is taken normal to the boundary. The insulated edge of the model then corresponds to a loaded edge of the elastic plate where the maximum principal stress acts in the direction of the equipotential lines, and

$$\partial p/\partial n = 0 \qquad (12)$$

since the local stress is a continuation of the load intensity.

On an equipotential boundary the current flow is normal to that border and the boundary itself is described as

$$V = \text{Constant} \tag{13}$$

Correspondingly, on an unloaded edge of an elastic plate and on a hole in the plate all normal stress is considered of zero magnitude and the tangential stress describes a stress trajectory on which

$$p = \text{Constant} \tag{14}$$

Hence, by imposing a constant potential difference on the edges of the Teledeltos model, proportional to the uniform tension load on an elastic plate, the lines of equipotential will represent the maximum principal stress trajectories while the current lines will correspond to the minimum principal stress trajectories. The insulated edge of the model then simulates the loaded edge of the plate under study.

In order to trace the equipotential lines, the insulated edge of the paper model is divided into equal increments such that each increment represents a unit of uniform linear loading. A null reading is then established for each division point by placing the exploring probe in contact with the conducting paper at that point and balancing the slide-wire resistance. Points of equal potential are established by locating those points with an identical null while progressing down the length of the model. A white blueprint marking pencil may be used to join these points in a smooth curve for a graphic picture of the principal stress trajectories, as shown in Fig. 4.

On an insulated edge of the paper, each equally spaced division corresponds to a stress σ, where

$$\sigma = w/mt = \text{Constant} \qquad (15)$$

and w is the total load over each division in kilograms, m is the width of each division in centimeters, and t is the thickness of the plate in centimeters. At any other point in the model the stress is σ', where

$$\sigma' = w/nt \qquad (16)$$

and n is the width between equipotential lines at that point. By defining a concentration factor α as

$$\alpha = \sigma'/\sigma = (w/nt)/(w/mt) = m/n \qquad (17)$$

then α is the factor by which the uniform load must be multiplied to determine local stress. However, it has become customary to base the stress concentration factor on the nominal stress where

$$\sigma_{nom} = w/lt \qquad (18)$$

and l corresponds to the width of the minimum (net) cross section of the model divided by the original number of subdivisions (in effect a uniform equipotential line spacing at the minimum section). The stress concentration factor based upon this nominal stress is k, where

$$k = 1/n \qquad (19)$$

In order to compute the maximum value of k, the variable normal distances E and e (See Fig. 5-a) are measured - e between the discontinuity and successive equipotential lines, at the point of closest approach of these lines to the hole; and E between the hole and points indicating uniform equipotential line spacing, at the minimum cross section of the model. The stress concentration factor k is then determined by plotting the curve E/e vs. the distance from the hole x, and extrapolating to the boundary of the hole. Unfortunately, this method of computing k can be affected by a rather large extrapolation error. The error is reduced, however, by a second graphical method. Recalling that the boundary of the hole is itself an equipotential line, the stress concentration factor may be described as

$$k = \lim_{e \to 0} (E/e) = \tan\phi \qquad (20)$$

This limit is the slope of the tangent to the E vs. e curve at the origin of x and can be evaluated graphically as shown in Fig. 5-b. A sample problem utilizing this method of determining k is shown in the Appendix.

In order to compare the accuracy of solutions obtained with this analog to those obtained by other means, tests were made using circular holes and filleted bars. Values of k obtained from these tests compare very favorably with those obtained by mathematical analysis and photoelastic process.

In concluding the experiment, three random cut-outs were examined to determine the suitability of this analogy to arbitrary shapes. No difficulty was experienced in evaluating k.

CHAPTER V

RESULTS

To compare values of k obtained with the conducting paper analog with those values obtained by theoretical means, the problem of a circular hole in a finite width plate was used. Four models were prepared with ratios of hole diameter to plate width (a/w) of 0.053, 0.162, 0.269, and 0.402. Values of k for these models were determined using equation 20, and were compared to those found by the theoretical methods of Howland (12) for models of identical geometry. Fig. 6 presents a comparison of results, and it will be noted that a maximum deviation from Howland's method of 1.4 % was realized, the largest deviation incident to the smallest hole.

For comparison with the photoelastic method a filleted bar was selected for study. Four models were prepared representing bars having a maximum width of 370 milimeters, a shoulder width of 240 milimeters, and ratios of fillet radius to shoulder width (r/d) of 0.030, 0.080, 0.100, and 0.167. A comparison of k values found using these models with those obtained through the photoelastic process of Frocht (13) is illustrated in Fig. 7. It may be noted that a maximum deviation of 3.3 % from the method of Frocht was realized, the largest deviation incident to the smallest fillet radius.

Examination of the suitability of this analogy to arbitrary holes and cut-outs having no theoretical solution was made using three random shapes. Models were prepared representing (a) a yoke-shaped section of plate, (b) a finite width plate pierced by two circular holes connected with a slot, and (c) a filleted cut-out with circular holes in a finite width plate. The geometry of these models is shown in Figures 8, 9, and 10, along with the associated graphical solution for k. In each case the preparation of the model, the location of the point of maximum stress concentration, and the evaluation of k was accomplished in a matter of minutes.

CHAPTER VI

CONCLUSIONS

On the basis of tests run using this rheoelectric analogy, the following conclusions are reached.

1. A graphic representation of a two-dimensional tension stress field, in terms of principal stress trajectories, may readily be obtained by using the conducting paper analogy.

2. The point of maximum stress concentration is easily determined as the point of closest approach of a principal stress trajectory to a hole or cut-out.

3. The stress concentration factor k can be evaluated by measuring the contraction of the principal stress trajectories.

4. Values of k obtained through use of this analogy compare favorably with those derived by theoretical and photoelastic means.

5. This analog permits rapid and inexpensive solutions to stress problems involving a hole or cut-out of arbitrary shape in a thin elastic plate subjected to uniform tension on two opposite edges.

APPENDIX

SAMPLE PROBLEM

For the purposes of this sample problem, it is desired to find the stress concentration factor for a slotted bar in tension. The slot, at right angles to the axes of tension, has the dimensions shown in Fig. 11.

Because of symmetry, only one-half of the bar is prepared in model form. The slot was drawn to size on Teledeltos paper and then painted with silver paint, as were strips on each edge of the model which come into contact with the electrodes. The minimum (net) width of the model at the slot was 125.0 milimeters, each division point representing a point on an equipotential line, if there were no contraction of these lines. The insulated edge of the model, representing the loaded edge of the elastic plate, was also divided into ten equal divisions. Using these latter division points as starting places, the equipotential lines were drawn around the slot, and were observed to deviate from the equally spaced division points previously drawn at the slot. The actual distances e were then measured from the hole along with the equally spaced distances E, and were plotted as a curve E vs. e, as shown in Fig. 12. The tangent to this curve at the origin, which is the stress concentration factor at the slot, was determined as

$$k = \tan\phi = 3.45 \qquad (21)$$

By plotting E/e vs. x, as shown in Fig. 13, and extrapolating to the boundary of the hole, the value of k was found to be 3.47, which is slightly in error from the above value of equation 21, and deviates slightly from that value obtained by Frocht (14) through photoelastic means, for a model of identical geometry. However, the value of k obtained through use of equation 20, is identical to that of Frocht, and is not subject to the extrapolation error of the latter method.

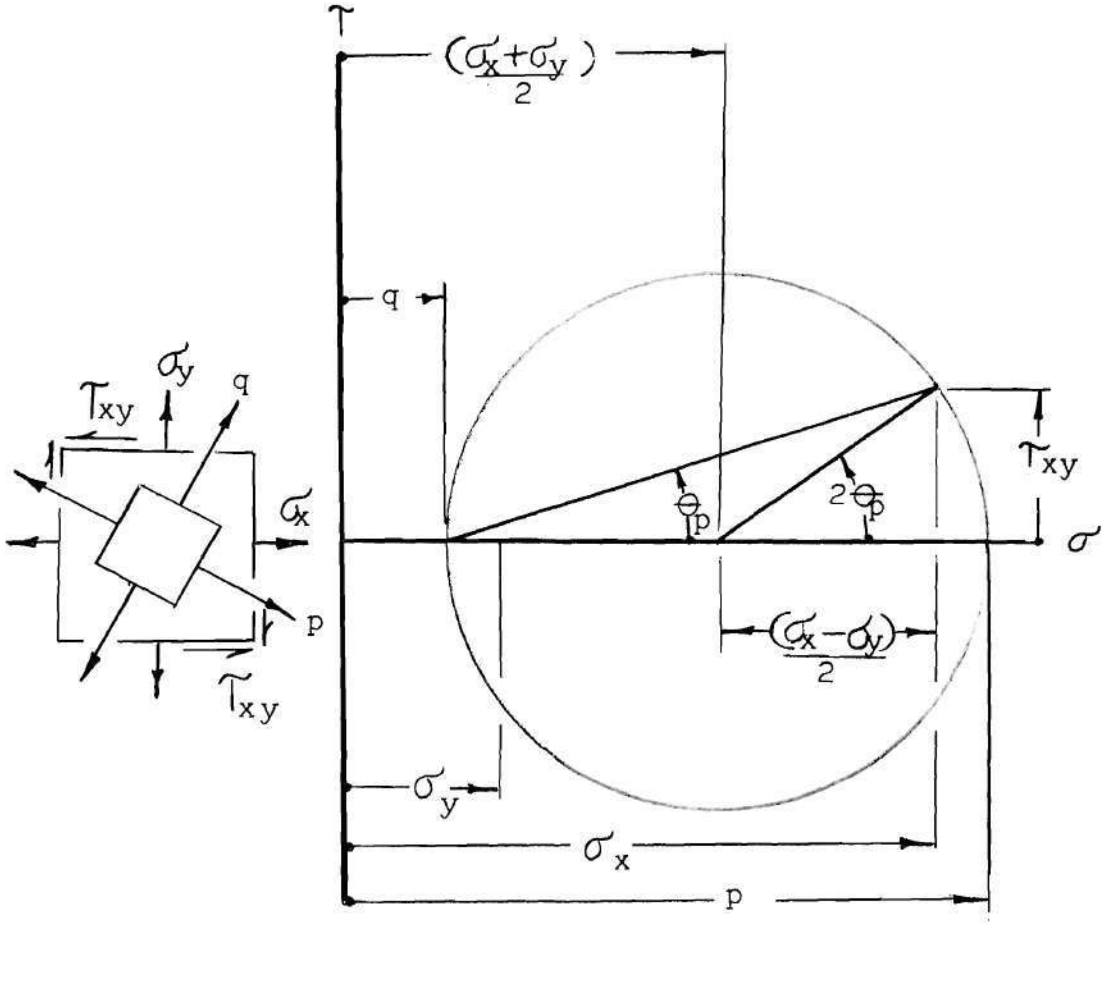

Figure 1. Sketch Showing the Determination of the Magnitudes and Directions of the Principal Stresses for a General System of Two-Dimensional Stresses, by Means of Mohr's Circle.

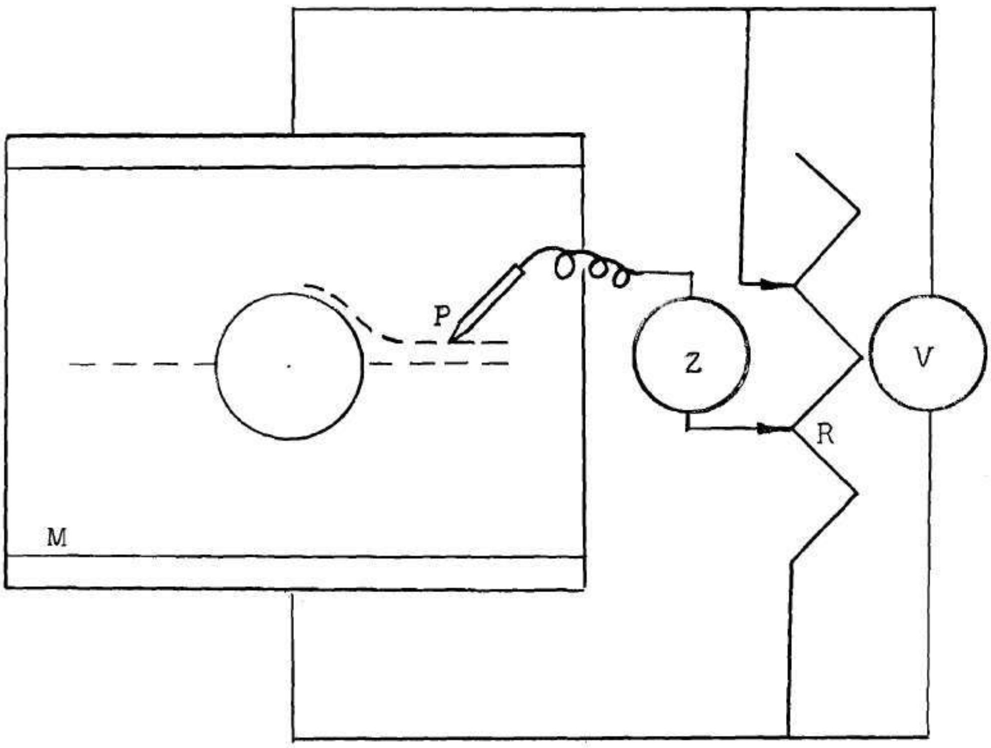

Figure 2. Schematic Diagram Showing Connection of Electrical Supply (V), Slide-Wire Resistance (R), Zero Indicator (Z), Probe (P), and Conducting Model (M).

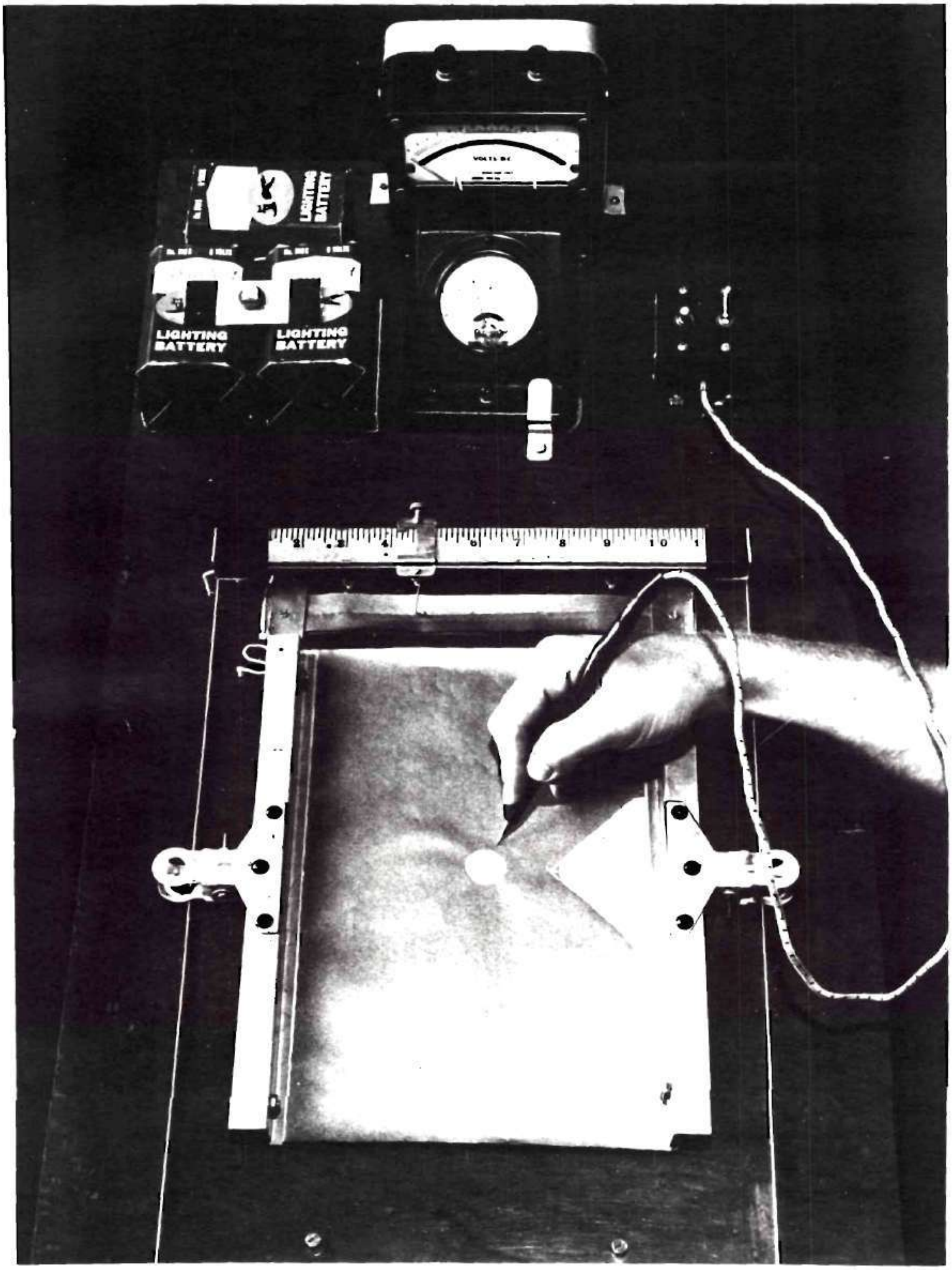

Figure 3. A General View of the Rheoelectric Analog with a Conducting Model.

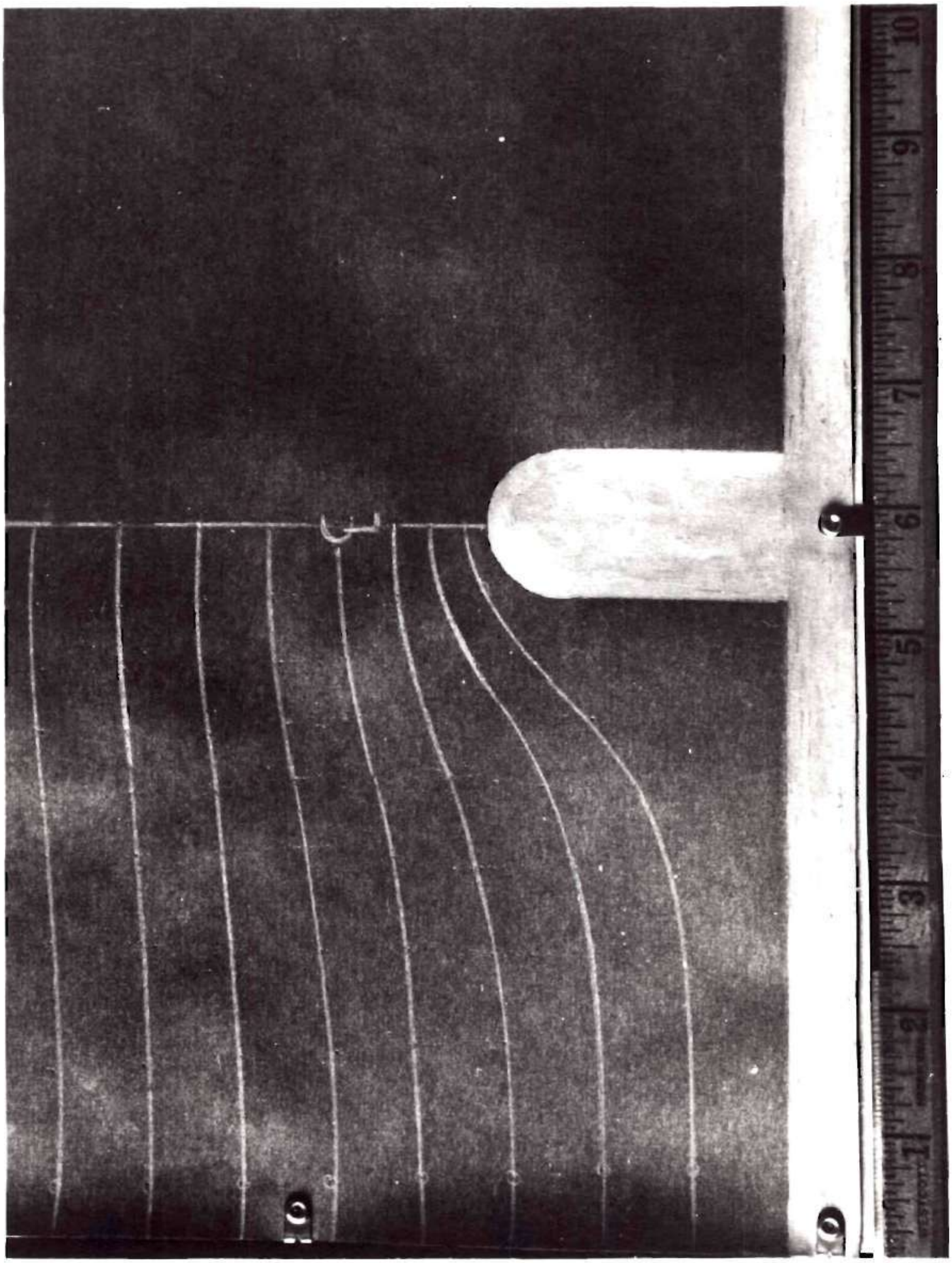

Figure 4. A Teledeltos Model of a Slot in an Elastic Plate on which Maximum Principal Stress Trajectories Have Been Drawn.

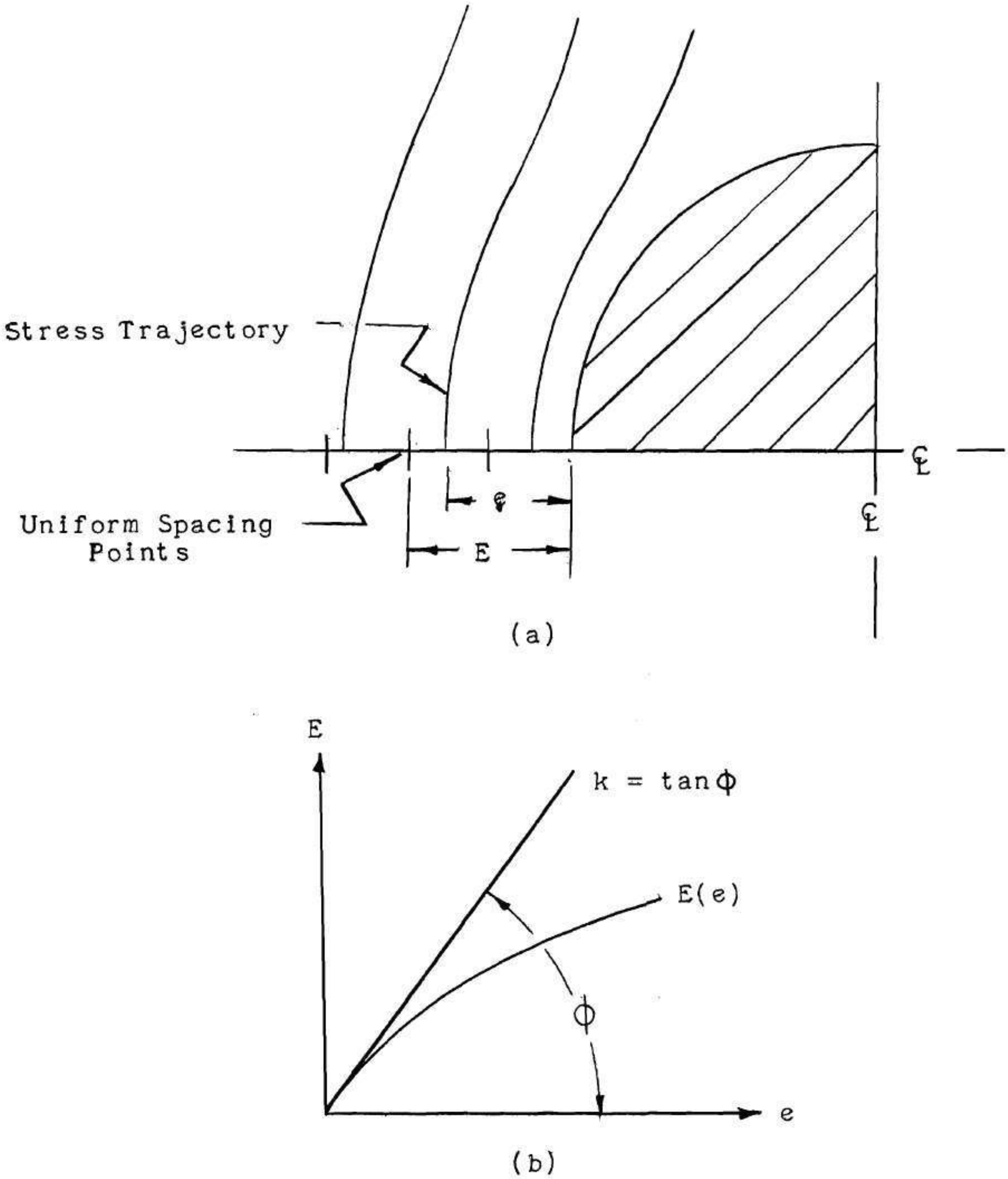

Figure 5. Sketches Showing: (a) The Measurement of Normal Distances E and e; (b) The Method of Determining the Stress Concentration Factor k as the Tangent to the E vs. e Curve at the Origin.

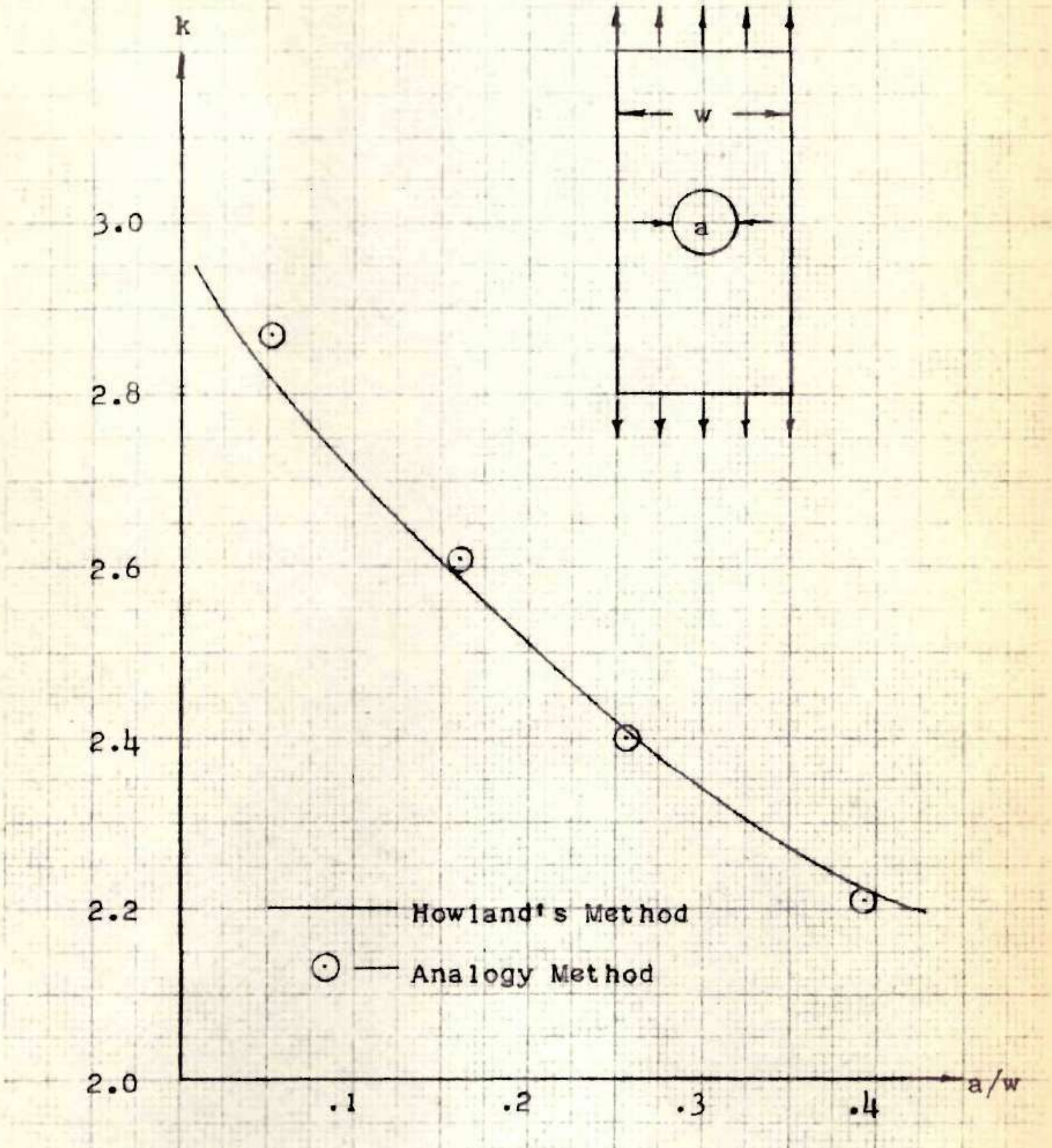

Figure 6. The Stress Concentration Factor k vs. Non-Dimensional Hole Diameter a/w for Circular Holes. A Comparison of k, as Found by the Analogy Method, to Those Values Determined by Howland's Analytical Method.

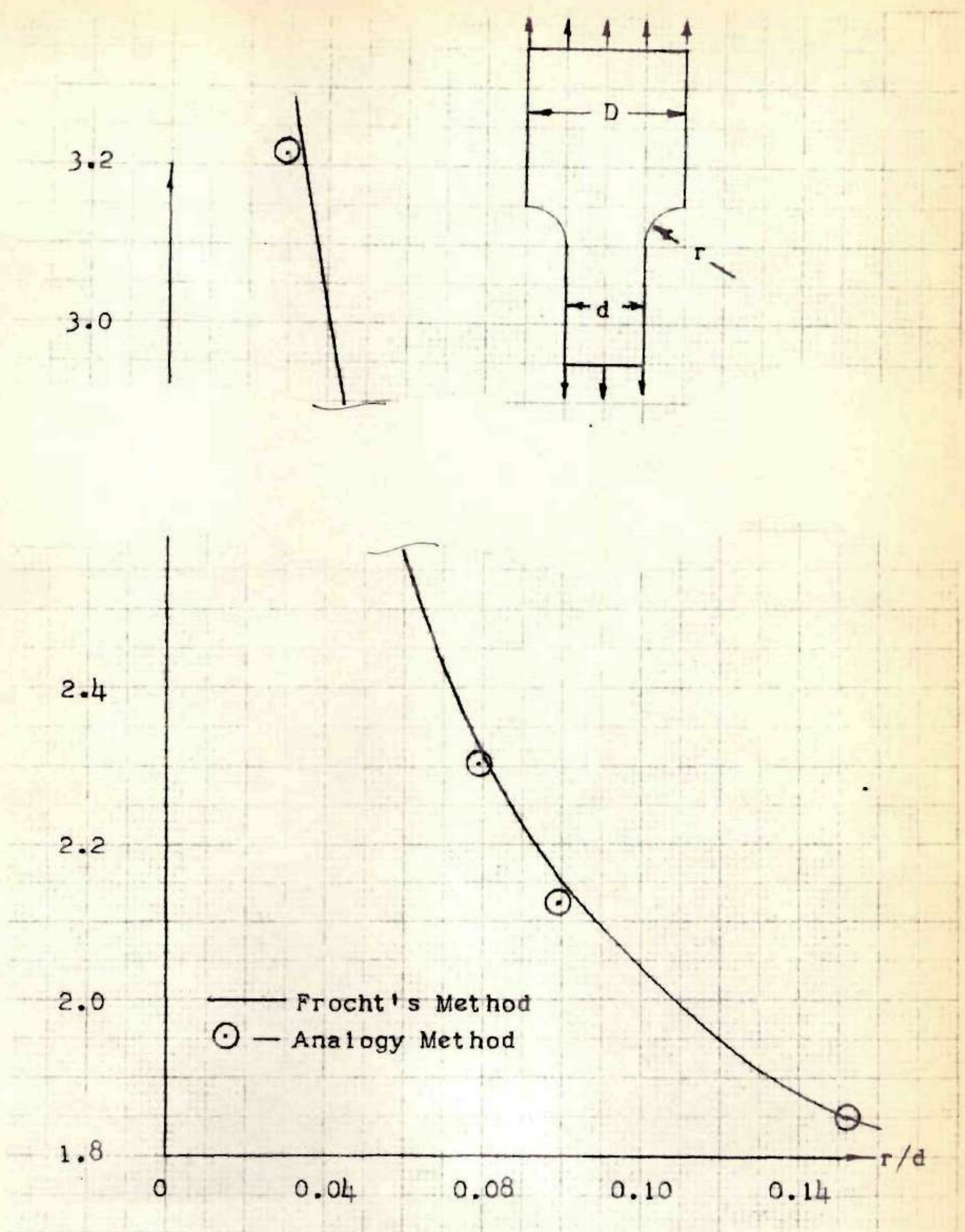

Figure 7. The Stress Concentration Factor k vs. Non-Dimensional Fillet Radius r/d for Filleted Bars. A Comparison of k, as Found by the Analogy Method, to Those Values Determined by Frocht's Photoelastic Method.

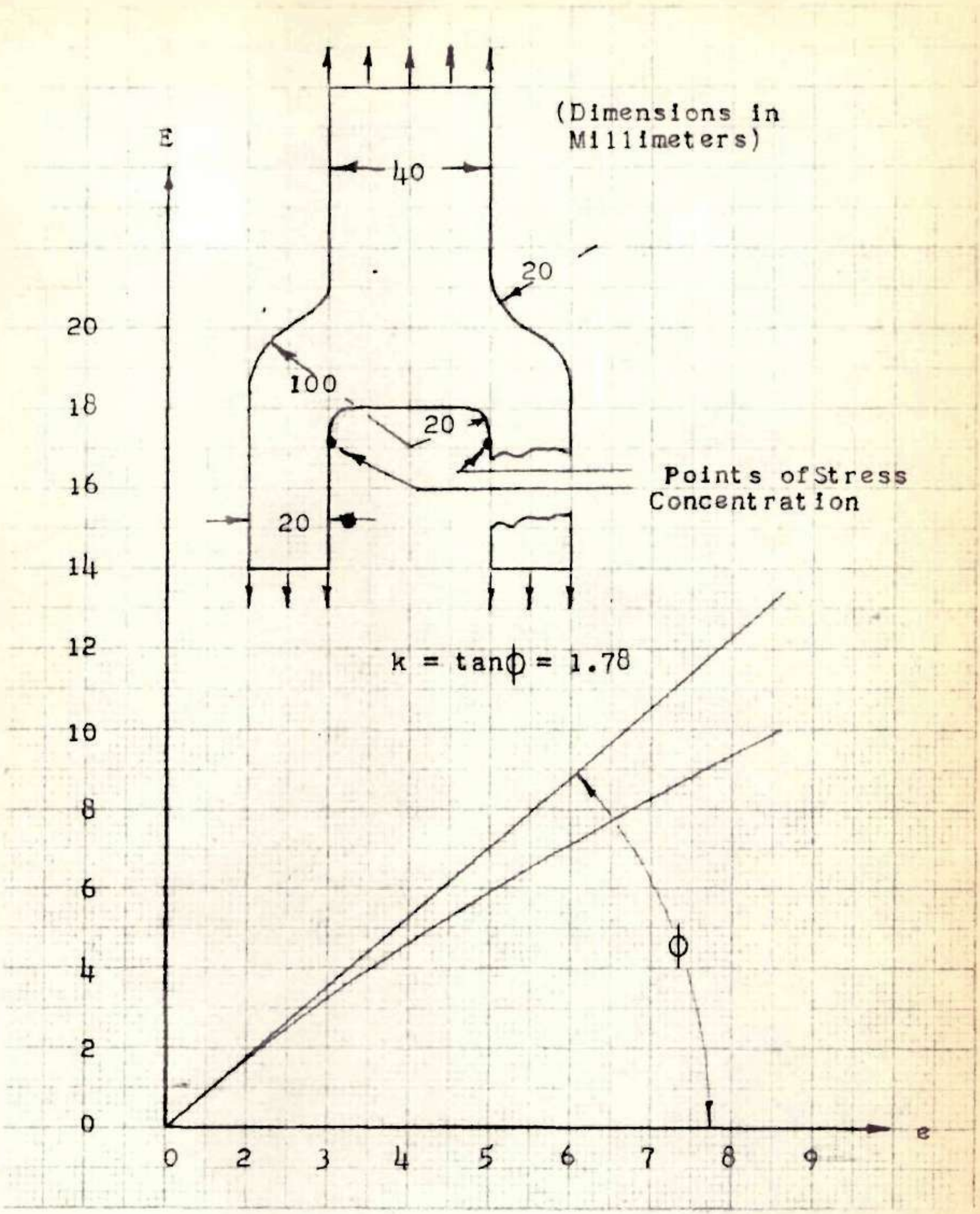

Figure 8. Determination of the Stress Concentration Factor k for a Yoke-Shaped Plate.

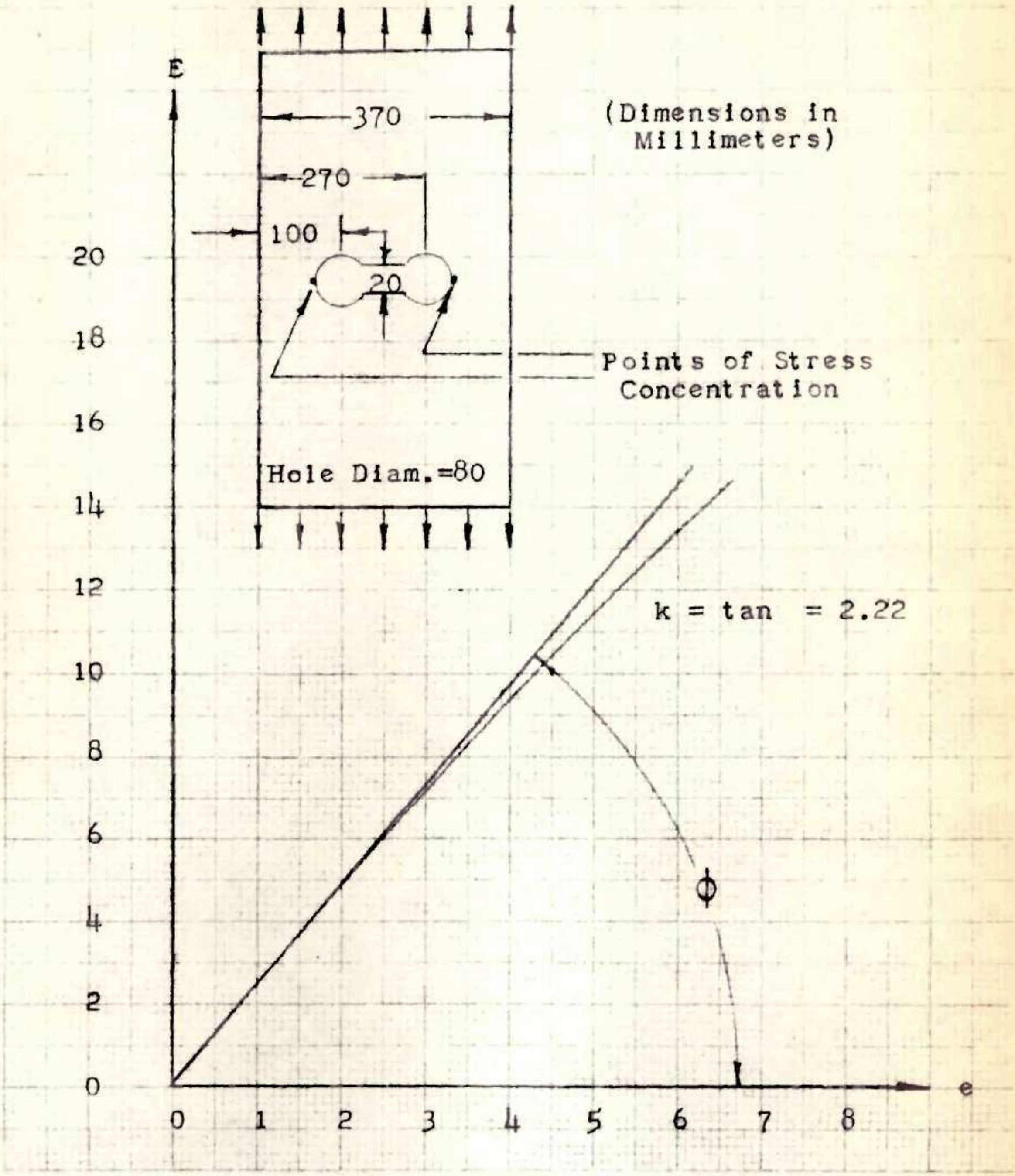

Figure 9. Determination of the Stress Concentration Factor k for Two Circular Holes Connected by a Slot.

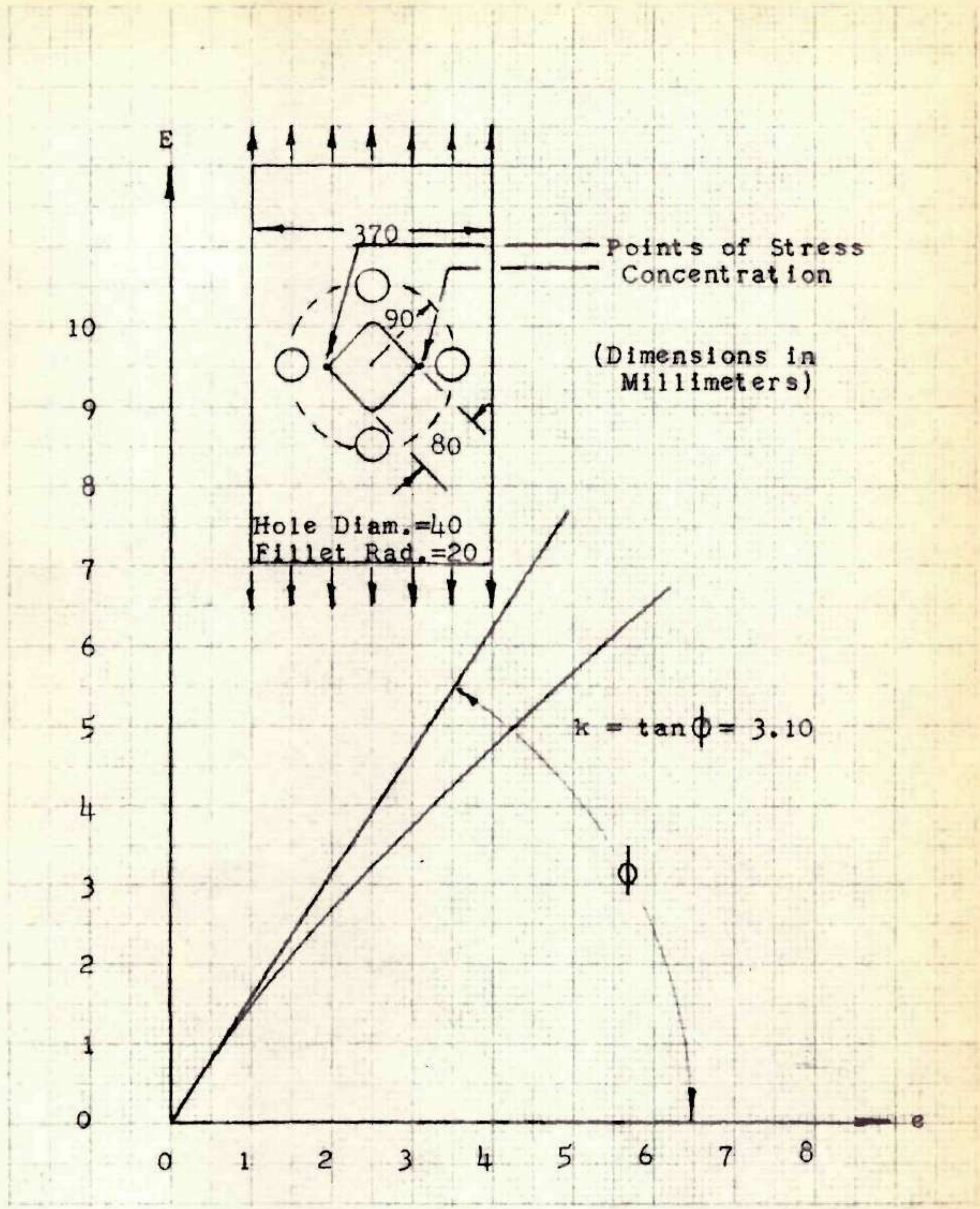

Figure 10. Determination of the Stress Concentration Factor k for a Filleted Cut-Out with Circular Holes.

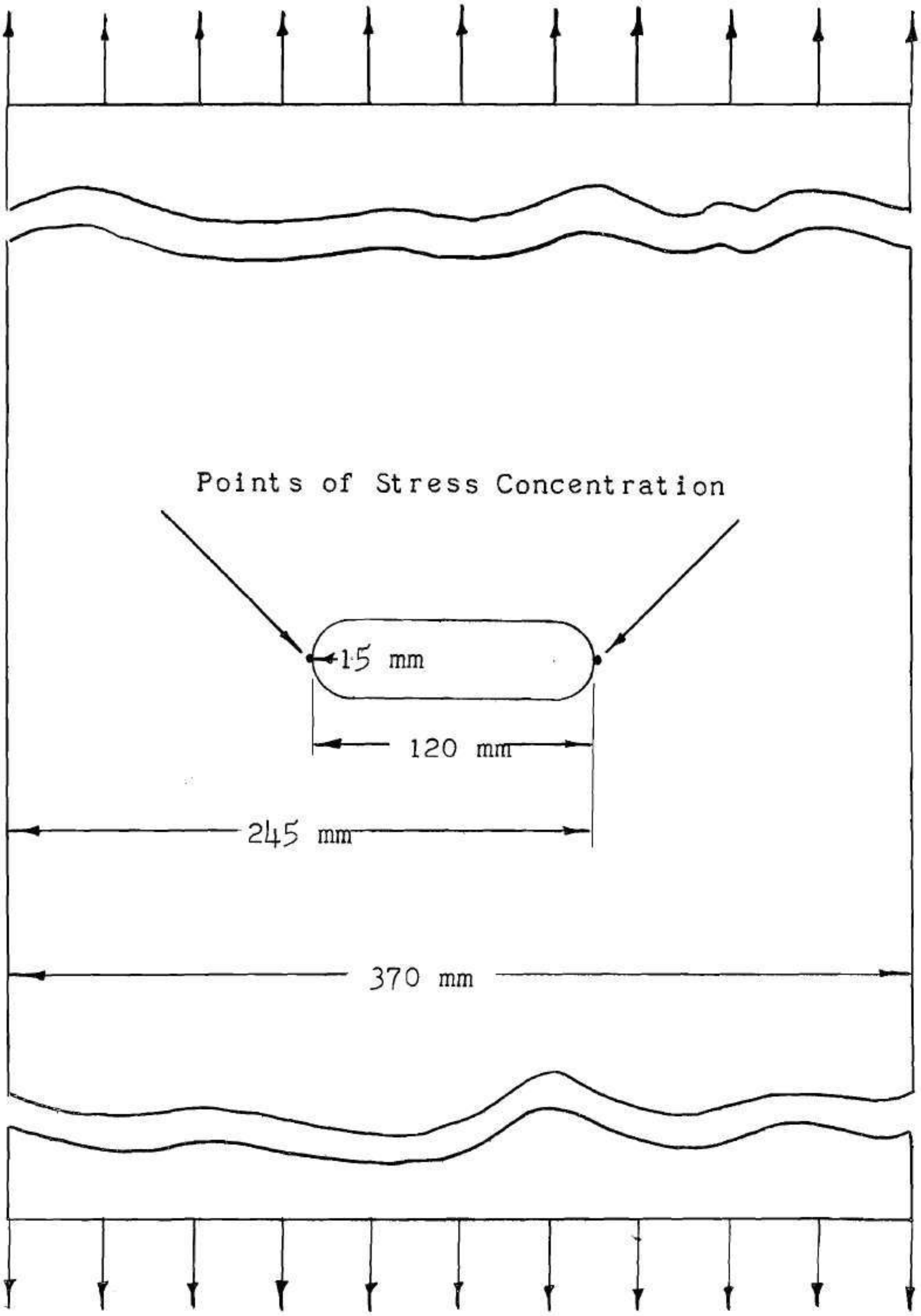

Figure 11. Sketch Showing the Geometry of a Slotted Bar Under Tension, Used in the Sample Problem. Scale 1:3

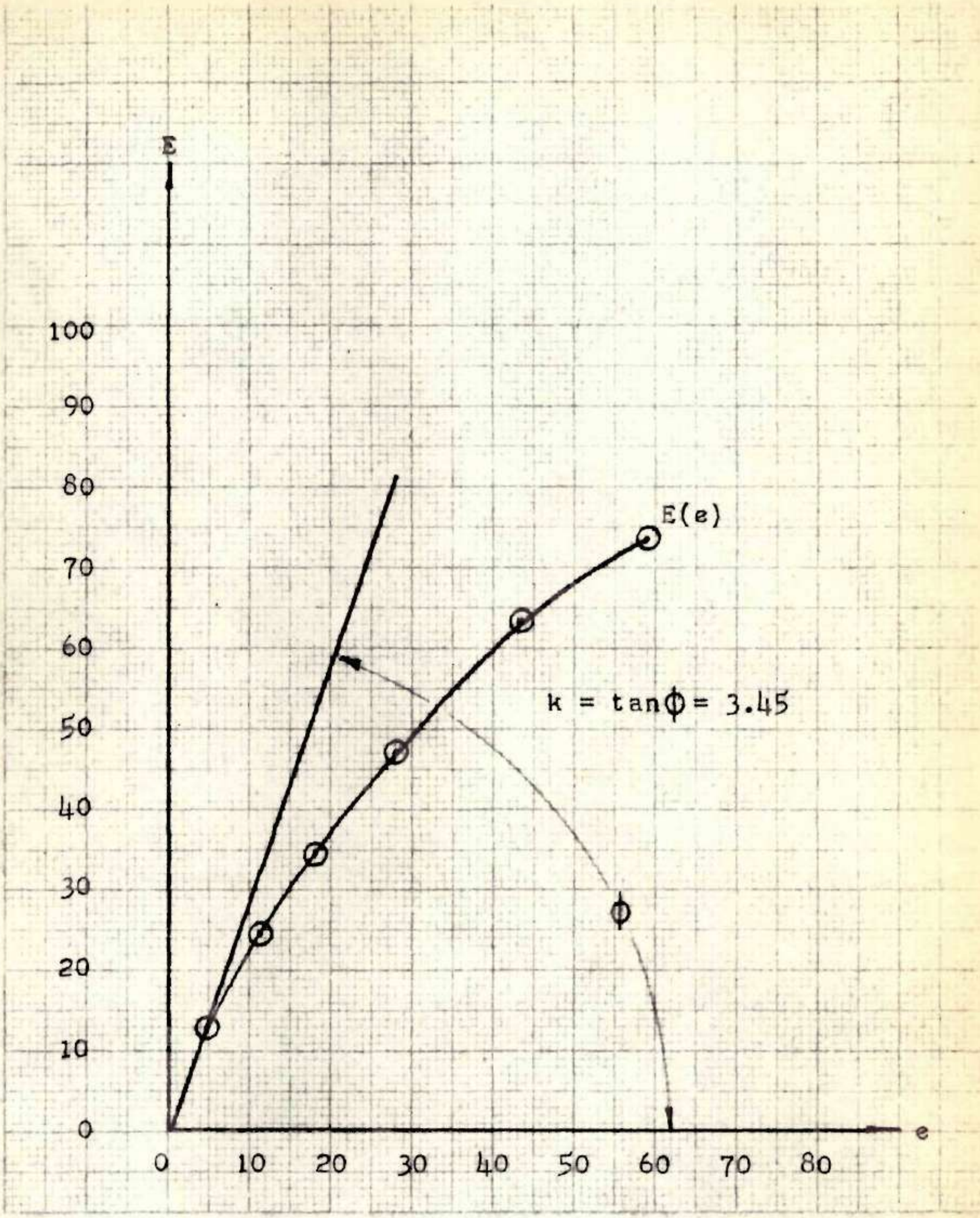

Figure 12. Determination of the Stress Concentration Factor k for the Sample Problem by Plotting Normal Distances E vs. e and Taking k = tanϕ, the Slope of a Tangent at the Origin.

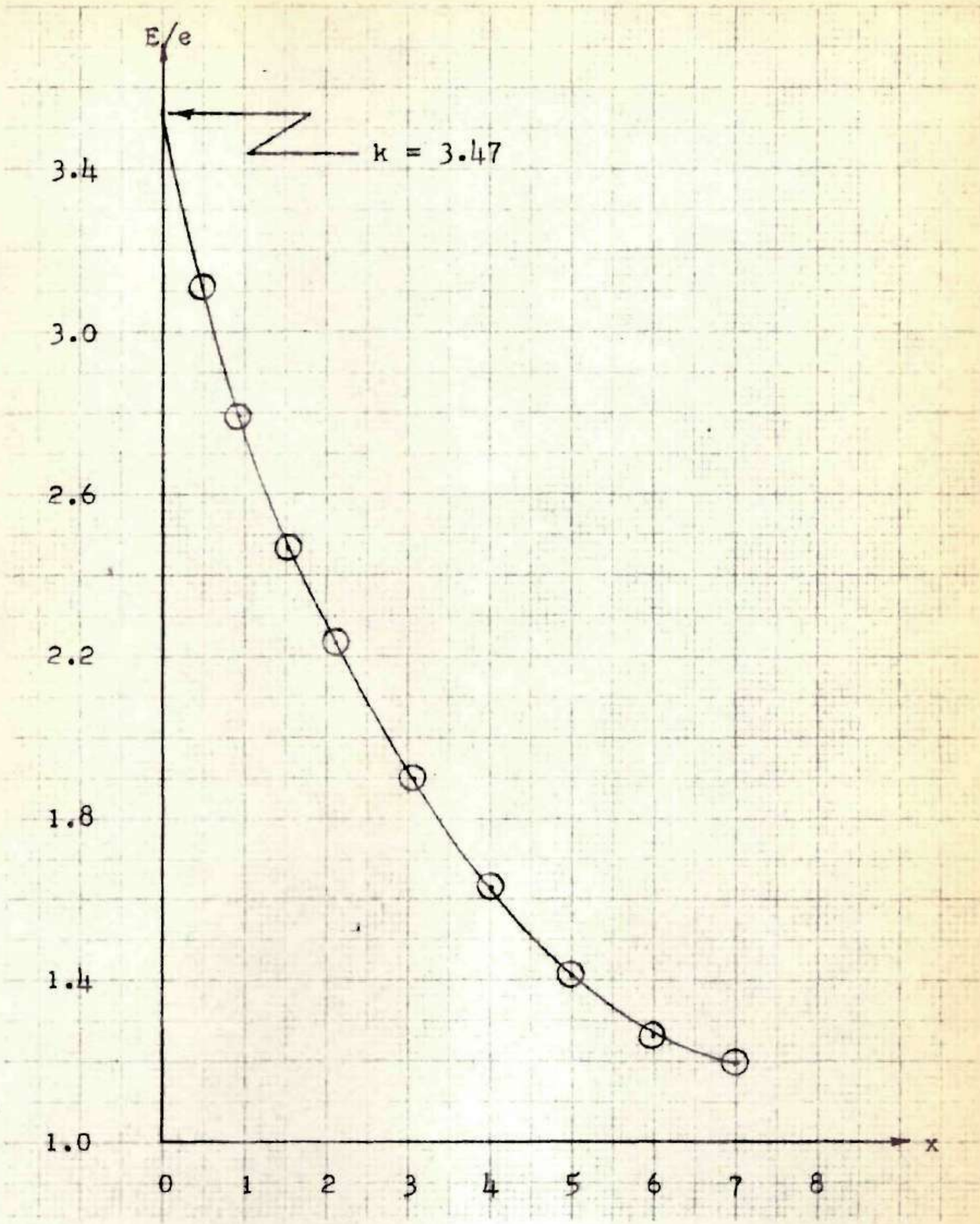

Figure 13. Determination of the Stress Concentration Factor k for the Sample Problem by Plotting the Ratio of Normal Distances E/e vs. Linear Distance From the Hole x, and Extrapolating to the Boundary.

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